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A simple analytical solution to one-dimensional consolidation for unsaturated soils

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SUMMARY

This paper presents a simple analytical solution to Fredlund and Hasan's one-dimensional (1-D) consolidation theory for unsaturated soils. The coefficients of permeability and volume change for unsaturated soils are assumed to remain constant throughout the consolidation process. The mathematical expression of the present solution is much simpler compared with the previous available solutions in the literature. Two new variables are introduced to transform the two coupled governing equations of pore-water and pore-air pressures into an equivalent set of partial differential equations, which are easily solved with standard mathematical formulas. It is shown that the present analytical solution can be degenerated into that of Terzaghi consolidation for fully saturated condition. The analytical solutions to 1-D consolidation of an unsaturated soil subjected to instantaneous loading, ramp loading, and exponential loading, for different drainage conditions and initial pore pressure conditions, are summarized in tables for ease of use by practical engineers. In the case studies, the analytical results show good agreement with the available analytical solution in the literature. The consolidation behaviors of unsaturated soils are investigated. The average degree of consolidation at different loading patterns and drainage conditions is presented. The pore-water pressure isochrones for two different drainage conditions and three initial pore pressure distributions are presented and discussed. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: analytical solution; unsaturated soil; one-dimensional consolidation; ramp loading; boundary condition; initial condition

1. INTRODUCTION

The classical one-dimensional (1-D) consolidation theory proposed by Terzaghi [1] is well known in geotechnical engineering. Some researchers have deduced analytical solutions for saturated soils based on Terzaghi's consolidation theory for different loading and boundary conditions [2, 3]. With problems raised in practice engineering, researchers extended the consolidation theory from saturated soils to unsaturated soils [4–7]. Some scholars have developed and analyzed fully coupled models for water flow and air flow in deformable porous media for unsaturated soils [8–11]. Vu [12] verified that some difference would appear between coupled and uncoupled solutions for swelling clay. In general, the fully coupled solutions are theoretically more accurate than the uncoupled results, but uncoupled approach is sufficiently accurate when it comes to most practical engineering problems [7]. Among uncoupled theories, Fredlund and Hasan's [6] 1-D consolidation theory is well accepted for the consolidation of unsaturated soils. Numerical methods such as finite difference or differential

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quadrature method can be applied to solve the consolidation equations for saturated and unsaturated soil in complex loading, boundary, and initial conditions [7, 13–17]. Nevertheless, analytical solutions can set a comparison to verify the accuracy of the numerical solutions and depict the problems exactly. Qin *et al.* [18] obtained an analytical solution of Fredlund and Hasan's 1-D unsaturated soil consolidation model with special boundary conditions. Shan *et al.* [19] also presented an exact solution of 1-D consolidation for unsaturated soils with different boundary and loading conditions. However, their solutions were obtained from complex mathematical process and difficult to be used by engineers.

This paper introduces a new simple analytical solution to Fredlund and Hasan's [6] consolidation model for unsaturated soils. Two new variables, ϕ_1 and ϕ_2, are introduced to transform the two coupled governing equations of pore-water and pore-air pressures into two conventional diffusion equations so that the analytical solutions are easily obtained. It is shown that the present analytical solution can be degenerated into that of Terzaghi's consolidation for fully saturated soils. The analytical solutions are summarized in tables for 1-D consolidation of unsaturated soils under instantaneous loading, ramp loading and exponential loading, different drainage conditions, and initial pore pressure conditions. In the case studies, the present analytical results show good agreement with the available analytical solution [18]. The consolidation behaviors of unsaturated soils are investigated. The results for pore-water pressure isochrones and average degree of consolidation at different loading patterns, boundary, and initial conditions are presented and discussed.

2. MATHEMATICAL MODEL

Consider an unsaturated soil layer with the depth of h subjected to a uniform loading q on the soil surface. For Fredlund and Hasan's 1-D consolidation theory of unsaturated soils [6], the assumptions are similar to those proposed by Terzaghi for saturated soils [1], with some exceptions and additions. In this paper, in order to obtain a closed-form solution, the main assumptions are summarized as follows:

1. Both air and water phases are continuous.
2. The soil particles and pore water are incompressible.
3. The effects of air diffusing through water, air dissolving in the water, and the movement of water vapor are ignored.
4. The coefficients of permeability with respect to air and water phases remain constant throughout the dissipation process.
5. The coefficients of volume change for the soil remain constant during the consolidation process.

The governing equations for water and air phases are as follows [6, 7]:

$$\frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} - C_v^w \frac{\partial^2 u_w}{\partial z^2} + C_\sigma^w \frac{\partial q}{\partial t} \tag{1}$$

$$\frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_w}{\partial t} - C_v^a \frac{\partial^2 u_a}{\partial z^2} + C_\sigma^a \frac{\partial q}{\partial t} \tag{2}$$

where u_w and u_a mean pore-water and pore-air pressures, respectively. k_w and k_a are the coefficients of water and air permeability, respectively. $C_w = \frac{1-m_2^w/m_{1k}^w}{m_2^w/m_{1k}^w}$, $C_v^w = \frac{k_w}{\gamma_w m_2^w}$, and $C_\sigma^w = \frac{m_{1k}^w}{m_2^w}$, where m_{1k}^w and m_2^w are the coefficients of water volume change with respect to a change in the net normal stress $\sigma - u_a$ and matric suction $u_a - u_w$, respectively; subscript 'k' stands for K_0-loading, and γ_w is the unit weight of water. $C_a = \frac{m_2^a}{m_{1k}^a - m_2^a - n_0(1-S_{r0})/\overline{u}_a}$, $C_v^a = \frac{k_a R T_{abs}}{g \overline{u}_a M_a \left[m_{1k}^a - m_2^a - n_0(1-S_{r0})/\overline{u}_a\right]}$, and $C_\sigma^a = \frac{m_{1k}^a}{m_{1k}^a - m_2^a - n_0(1-S_{r0})/\overline{u}_a}$, where g is the acceleration of gravity, and m_{1k}^a and m_2^a are the coefficients of air volume change with respect to a change in $\sigma - u_a$ and $u_a - u_w$, respectively. S_{r0} and n_0 are the initial degree of saturation

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and initial porosity, respectively. M_a is the molecular mass of air; $\overline{u}_a$ is the absolute pore-air pressure, $\overline{u}_a = u_a + \overline{u}_{atm}$, where $\overline{u}_{atm}$ is the atmospheric pressure. To obtain the analytical solution, it is assumed that $\overline{u}_a = \overline{u}_{atm}$ in this paper. This assumption has little influence to the solution when u_a is much smaller than $\overline{u}_{atm}$ and rapidly dissipates during consolidation. R is the universal gas constant, $R = 8.314$ J/mol/K, and T_{abs} is the absolute temperature.

The upper boundary is considered to be permeable to both air and water phases:

$$u_a(0,t) = 0, \;\; u_w(0,t) = 0 \tag{3}$$

For double drainage condition, the lower boundary is permeable to both air and water phases:

$$u_a(h,t) = 0, \;\; u_w(h,t) = 0 \tag{4}$$

For single drainage condition, the lower boundary is impermeable to both air and water phases:

$$\frac{\partial u_a(h,t)}{\partial z} = 0, \;\; \frac{\partial u_w(h,t)}{\partial z} = 0 \tag{5}$$

The initial pore-water and pore-air pressure distributions along depth can be expressed as follows:

$$u_a(z,0) = u_a^i(z), \;\; u_w(z,0) = u_w^i(z) \tag{6}$$

Three patterns of time-dependent loading are considered:

(a) instantaneous loading: $q = q_u$,
(b) ramp loading: $q = \frac{q_u}{t_c} \cdot t$ for $t < t_c$, $q = q_u$ for $t \geq t_c$,
(c) exponent loading: $q = q_u(1 - \exp(-bt))$.

Defining $T = \frac{k_w t}{\gamma_w m_{1k}^s h^2}$, $\overline{z} = \frac{z}{h}$, $\overline{q} = \frac{q}{q_u}$, $T_c = \frac{k_w t_c}{\gamma_w m_{1k}^s h^2}$, $\overline{b} = \frac{\gamma_w m_{1k}^s h^2 b}{k_w}$, $A_a = \frac{-C_v^a \gamma_w m_{1k}^s}{(1 - C_w C_a) k_w}$, $A_w = \frac{C_a C_v^w \gamma_w m_{1k}^s}{(1 - C_w C_a) k_w}$, $A_\sigma = \frac{\left(C_\sigma^a - C_a C_\sigma^w\right) \gamma_w m_{1k}^s h^2}{(1 - C_w C_a) k_w}$, $W_a = \frac{C_w C_v^a \gamma_w m_{1k}^s}{(1 - C_w C_a) k_w}$, $W_w = \frac{-C_v^w \gamma_w m_{1k}^s}{(1 - C_w C_a) k_w}$, and $W_\sigma = \frac{\left(C_\sigma^w - C_w C_\sigma^a\right) \gamma_w m_{1k}^s h^2}{(1 - C_w C_a) k_w}$, Equations 1 and 2 can be rewritten as follows:

$$\frac{\partial \overline{u}_a}{\partial T} = A_a \frac{\partial^2 \overline{u}_a}{\partial \overline{z}^2} + A_w \frac{\partial^2 \overline{u}_w}{\partial \overline{z}^2} + A_\sigma \frac{\partial \overline{q}}{\partial T} \tag{7}$$

$$\frac{\partial \overline{u}_w}{\partial T} = W_a \frac{\partial^2 \overline{u}_a}{\partial \overline{z}^2} + W_w \frac{\partial^2 \overline{u}_w}{\partial \overline{z}^2} + W_\sigma \frac{\partial \overline{q}}{\partial T} \tag{8}$$

The boundary and initial conditions of Equations 3–6 become the following:

Upper boundary :

$$\overline{u}_a(0,T) = 0, \;\; \overline{u}_w(0,T) = 0 \tag{9}$$

Lower boundary:

Double drainage

$$\overline{u}_a(1,T) = 0, \;\; \overline{u}_w(1,T) = 0 \tag{10}$$

Single drainage

$$\frac{\partial \overline{u}_a(1,T)}{\partial \overline{z}} = 0, \;\; \frac{\partial \overline{u}_w(1,T)}{\partial \overline{z}} = 0 \tag{11}$$

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$$\text{Initial condition}: \quad \overline{u}_a(\overline{z},0)=\overline{u}_a^i(\overline{z}),\ \ \overline{u}_w(\overline{z},0)=\overline{u}_w^i(\overline{z}) \tag{12}$$

3. ANALYTICAL SOLUTION

Equations 7 and 8 can be transformed into equivalent set partial differential equations of ϕ_1 and ϕ_2, as shown in Equation A.8. As the two equations in Equation A.8 are in the same form, they can be expressed by the following general equation for simplification:

$$\frac{\partial \phi}{\partial T}-Q\left(\frac{\partial^2 \phi}{\partial \overline{z}^2}\right)=\beta\left(\frac{\partial \overline{q}}{\partial T}\right) \tag{13}$$

where ϕ represents ϕ_1 or ϕ_2, and correspondingly, Q and β denote Q_1 or Q_2 and β_1 or β_2, respectively. The formulas of Q_1, β_1 and Q_2, β_2 are shown in Appendix A.

By using Equations A.9, A.10, A.11, A.12, the boundary and initial conditions in terms of pore-water and pore-air pressures can be transformed to those of ϕ_1 or ϕ_2 and are written as the following general forms:

$$\text{Double drainage condition}: \quad \phi(0,T)=0,\ \ \phi(1,T)=0 \tag{14}$$

$$\text{Single drainage condition}: \quad \phi(0,T)=0,\ \ \frac{\partial \phi(1,T)}{\partial z}=0 \tag{15}$$

$$\text{Initial distribution}: \quad \phi(\overline{z},0)=\phi^i(\overline{z}) \tag{16}$$

The process of solving Equation 13 under different boundary and initial conditions are presented as follows.

3.1. *Instantaneous loading*

When the unsaturated soil stratum is subjected to instantaneous loading condition, $q=q_u$, Equation 13 becomes the following:

$$\frac{\partial \phi}{\partial T}=Q\left(\frac{\partial^2 \phi}{\partial \overline{z}^2}\right) \tag{17}$$

The solution of Equation 17 can be obtained as follows [20]:

$$\phi=(A_1\cos(B\overline{z})+A_2\sin(B\overline{z}))\cdot\exp\left(-B^2QT\right) \tag{18}$$

where A_1, A_2, and B are constant parameters, which can be determined by boundary and initial conditions.

For double drainage condition, substituting Equation 14 into Equation 18, one can obtain that $A_1=0$ and $Bh=m\pi$. Therefore,

$$\phi=\sum_{m=1}^{m=\infty}A_m\sin(m\pi\overline{z})\exp\left(-(m\pi)^2TQ\right) \tag{19}$$

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Substituting Equation 16 into Equation 19, it follows that

$$\phi^i(\bar{z}) = \sum_{m=1}^{m=\infty} A_m \sin(m\pi\bar{z}) \tag{20}$$

Equation 20 is a Fourier sine series with $A_m = 2\int_0^1 \phi^i(\bar{z}) \sin(m\pi\bar{z})d\bar{z}$. Therefore, the solution of ϕ for double drainage condition is obtained as follows:

$$\phi = \sum_{m=1}^{m=\infty} \left(2\int_0^1 \phi^i(\bar{z}) \sin(m\pi\bar{z})d\bar{z}\right) \sin(m\pi\bar{z}) \exp\left(-(m\pi)^2 TQ\right) \tag{21}$$

In the same way, substituting Equations 15 and 16 into Equation 18, the solution of ϕ for single drainage condition is obtained as follows:

$$\phi = \sum_{m=1}^{m=\infty} \left(2\int_0^1 \phi^i(\bar{z}) \sin(K\bar{z})d\bar{z}\right) \sin(K\bar{z}) \exp\left(-K^2 TQ\right), \;\; K = (2m+1)\pi/2 \tag{22}$$

The solutions of ϕ for three different initial conditions (constant, trapezium, and sine) are listed in Table I.

3.2. Time-dependent loading

Olson [3] presented a mathematical solution for 1-D consolidation of saturated soil subjected to single ramp load. Here, the solutions for the consolidation of unsaturated soils subjected to two kinds of time-dependent loading are presented. From Table I, the solution of ϕ under instantaneous loading, double drainage, and constant initial condition $\phi^i = \phi_0$ is as follows:

$$\phi = \sum_{m=0}^{m=\infty} \left(\frac{4\phi_0}{M}\right) \sin(M\bar{z}) \exp\left(-M^2 TQ\right) \tag{23}$$

When the applied loading is a function of time, $\bar{q} = f(T_a)$, in which T_a is the time of application of any load. For a differential load $d\bar{q}$, the instantaneous increment will be $d\phi = d\bar{q}$. At time T, the differential increment $d\phi$ after a time period $(T - T_a)$ for double drainage condition is as follows:

$$d\phi = \sum_{m=0}^{m=\infty} \left(\frac{4\beta \cdot d\bar{q}}{M}\right) \sin(M\bar{z}) \exp\left(-M^2 Q(T - T_a)\right), M = (2m+1)\pi \tag{24}$$

In the same way, the differential increment $d\phi$ for the single drainage condition is as follows:

$$d\phi = \sum_{m=0}^{m=\infty} \left(\frac{2\beta \cdot d\bar{q}}{K}\right) \sin(K\bar{z}) \exp\left(-K^2 Q(T - T_a)\right), K = (2m+1)\pi/2 \tag{25}$$

The expression of ϕ can be obtained by integrating Equations 24 and 25 from T_a to T. Two different kinds of time-dependent loading patterns, ramp and exponential loading, are considered; and the solutions of ϕ are summarized in Table I for both double and single drainage conditions.

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Table I. Parameter ϕ in the computation of the pore-air and pore-water pressures.

Loading	Double drainage	Single drainage	Initial condition
$\bar{q}=1$	$\phi=\sum_{m=0}^{m=\infty}\frac{4\phi_0}{M}\sin(M\bar{z})e^{-M^2TQ}$	$\phi=\sum_{m=0}^{m=\infty}\frac{2\phi_0}{K}\sin(K\bar{z})e^{-K^2TQ}$	$\phi^i(\bar{z})=\phi_0$
	$\phi=\frac{2\phi_0}{\pi}\sum_{m=0}^{m=\infty}\left[\left(\frac{\sin((2m-1)\pi/2)}{(2m-1)}-\frac{\sin((2m+1)\pi/2)}{(2m+1)}\right)\cdot\sin(m\pi\bar{z})e^{-(m\pi)^2TQ}\right]$	$\phi=\phi_0\sin\left(\frac{\pi\bar{z}}{2}\right)e^{-\pi^2TQ/4}$	$\phi^i(\bar{z})=\phi_0\sin(\pi\bar{z}/2)$
	$\phi=\sum_{m=0}^{m=\infty}\left[\frac{2}{n\pi}(a-(a+k)\cos(m\pi))\cdot\sin(m\pi\bar{z})e^{-(m\pi)^2TQ}\right]$	$\phi=\sum_{m=0}^{m=\infty}\left[\frac{2}{K^2}(aK+k\sin(K))\cdot\sin(K\bar{z})e^{-K^2TQ}\right]$	$\phi^i(\bar{z})=k\cdot\bar{z}+a$
$\bar{q}=\begin{cases}\frac{T}{T_c}, T<T_c\\ 1,\ T\geq T\end{cases}$	$\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{4\beta}{M^3QT_c}\sin(M\bar{z})\left(1-e^{-M^2QT}\right)\right], T<T_c$ $\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{4\beta}{M^3QT_c}\sin(M\bar{z})e^{-M^2QT}\left(e^{M^2QT_c}-1\right)\right], T\geq T_c$	$\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{2\beta}{K^3QT_c}\sin(K\bar{z})\left(1-e^{-K^2QT}\right)\right], T<T_c$ $\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{2\beta}{K^3QT_c}\sin(K\bar{z})e^{-K^2QT}\left(e^{K^2QT_c}-1\right)\right], T\geq T_c$	$\phi^i(\bar{z})=0$
$\bar{q}=1-e^{-\bar{b}T}$	$\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{4\beta\bar{b}}{M^3Q-M\bar{b}}\sin(M\bar{z})\left(e^{-\bar{b}T}-e^{-M^2QT}\right)\right]$	$\phi(T)=\sum_{m=0}^{m=\infty}\left[\frac{2\beta\bar{b}}{K^3Q-K\bar{b}}\sin(K\bar{z})\left(e^{-\bar{b}T}-e^{-K^2QT}\right)\right]$	$\phi^i(\bar{z})=0$

Note: $M=(2m+1)\pi$, $K=(2m+1)\pi/2$. The solutions of ϕ_1 and ϕ_2 can be obtained by replacing Q and β in ϕ expressions with Q_1, β_1 and Q_2, β_2, respectively. Then the normalized pore-water and pore-air pressures are computed by Equations 27 and 28.

Table II. The average degree of consolidation respective to parameter ϕ.

Loading	Double drainage	Single drainage	Initial condition
$\overline{q}=1$	$U^{\mathrm{m}}=\phi_0-\sum_{m=1}^{m=\infty}\left[\frac{8\phi_0}{M^2}e^{-M^2TQ}\right]$	$U^{\mathrm{m}}=\phi_0-\sum_{m=1}^{m=\infty}\left[\frac{2\phi_0}{K^2}e^{-K^2TQ}\right]$	$\phi^i(\overline{z})=\phi_0$
	$U^{\mathrm{m}}=\frac{2\phi_0}{\pi}-\sum_{m=1}^{m=\infty}\left[\frac{8\phi_0}{(4m^2-1)m\pi^2}e^{-(m\pi)^2TQ}\right], m \text{ is odd}$	$U^{\mathrm{m}}=\frac{2\phi_0}{\pi}\left(1-e^{-\pi^2QT/4}\right)$	$\phi^i(\overline{z})=$ $\phi_0\sin(\pi\overline{z}/2)$
	$U^{\mathrm{m}}=\left(a+\frac{k}{2}\right)-\sum_{m=1}^{m=\infty}\left[\frac{4(2a+k)}{m^2\pi^2}e^{-(m\pi)^2TQ}\right], m \text{ is odd}$	$U^{\mathrm{m}}=\left(a+\frac{k}{2}\right)-\sum_{m=0}^{m=\infty}\frac{2}{K^3}(aK+k\sin(K))e^{-K^2TQ}$	$\phi^i(\overline{z})=$ $k\cdot\overline{z}+a$
$\overline{q}=\begin{cases}\frac{T}{T_c}, T<T_c\\ 1, \quad T\geq T\end{cases}$	$U^{\mathrm{m}}=\frac{\beta T}{T_c}-\sum_{m=0}^{m=\infty}\left(\frac{8\beta}{M^4QT_c}\right)\left[1-e^{-M^2QT}\right], T<T_c$ $U^{\mathrm{m}}=\beta-\sum_{m=0}^{m=\infty}\left[\frac{8\beta}{M^4QT_c}e^{-M^2QT}\left(e^{M^2QT_c}-1\right)\right], T\geq T_c$	$U^{\mathrm{m}}=\frac{\beta T}{T_c}-\sum_{m=0}^{m=\infty}\left(\frac{2\beta}{K^4QT_c}\right)\left[1-e^{-K^2QT}\right], T<T_c$ $U^{\mathrm{m}}=\beta-\sum_{m=0}^{m=\infty}\left[\frac{2\beta}{K^4QT_c}e^{-K^2QT}\left(e^{K^2QT_c}-1\right)\right], T\geq T_c$	$\phi^i(\overline{z})=0$
$\overline{q}=1-e^{-\overline{b}T}$	$U^{\mathrm{m}}=\beta\left(1-e^{-\overline{b}T}\right)-\sum_{m=0}^{m=\infty}\frac{8\beta\overline{b}}{M^4Q-M^2\overline{b}}\left[e^{-\overline{b}T}-e^{-M^2QT}\right]$	$U^{\mathrm{m}}=\beta\left(1-e^{-\overline{b}T}\right)-\sum_{m=0}^{m=\infty}\frac{2\beta\overline{b}}{K^4Q-K^2\overline{b}}\left[e^{-\overline{b}T}-e^{-K^2QT}\right]$	$\phi^i(\overline{z})=0$

Note: $M=(2m+1)\pi$, $K=(2m+1)\pi/2$. The solutions of U_1^{m} and U_2^{m} can be obtained by replacing the Q and β in U^{m} expressions with Q_1, β_1 and Q_2, β_2, respectively. The average degrees of consolidation for air and water phases are computed by Equations 28 and 29 for instantaneous loading and by Equations 30 and 31 for time-dependent loading.

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3.3. Solutions for pore pressures and degree of consolidation

Based on the solutions of ϕ_1 and ϕ_2, the pore-water and pore-air pressures can be expressed as follows:

$$\overline{u}_a = \frac{c_{21}\phi_2 - \phi_1}{c_{12}c_{21} - 1} \tag{26}$$

$$\overline{u}_w = \frac{c_{12}\phi_1 - \phi_2}{c_{12}c_{21} - 1} \tag{27}$$

where c_{21} and c_{12} are computed by Equations A.6 and A.7.

The average degrees of consolidation for air and water phases subjected to instantaneous loading are expressed as follows:

$$U_a = \frac{c_{21}U_2^{\mathrm{m}} - U_1^{\mathrm{m}}}{(c_{12}c_{21} - 1)\cdot\overline{u}_a^i}\cdot 100\% \tag{28}$$

$$U_w = \frac{c_{12}U_1^{\mathrm{m}} - U_2^{\mathrm{m}}}{(c_{12}c_{21} - 1)\cdot\overline{u}_w^i}\cdot 100\% \tag{29}$$

where $U_1^{\mathrm{m}} = \int_0^1 \phi_1^i d\bar{z} - \int_0^1 \phi_1 d\bar{z}$ and $U_2^{\mathrm{m}} = \int_0^1 \phi_2^i d\bar{z} - \int_0^1 \phi_2 d\bar{z}$ are the average degree of consolidations with respect to ϕ_1 and ϕ_2, respectively.

The average degrees of consolidation for air and water phases subjected to time-dependent loading are obtained as follows:

$$U_a = \frac{c_{21}U_2^{\mathrm{m}} - U_1^{\mathrm{m}}}{(c_{12}c_{21} - 1)\cdot A_\sigma \overline{q}_u}\cdot 100\% \tag{30}$$

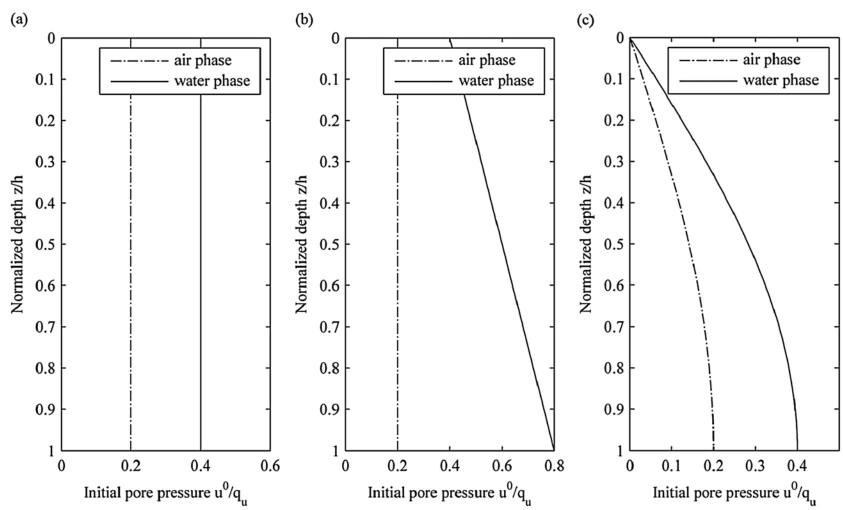

Figure 1. Initial distributions of pore pressures: (a) constant, (b) trapezium, and (c) sine.

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$$U_w = \frac{c_{12}U_1^{\mathrm{m}} - U_2^{\mathrm{m}}}{(c_{12}c_{21} - 1)\cdot W_\sigma \overline{q}_u}\cdot 100\% \tag{31}$$

where $U_1^{\mathrm{m}} = \beta_1\cdot\overline{q} - \int_0^1 \phi_1 d\overline{z}$ and $U_2^{\mathrm{m}} = \beta_2\cdot\overline{q} - \int_0^1 \phi_2 d\overline{z}$.

The average degrees of consolidation U^{m} with respect to ϕ are summarized in Table II, from which U_1^{m} and U_2^{m} can be easily obtained by replacing Q and β with Q_1, β_1 and Q_2, β_2, respectively.

3.4. Solutions degenerated to fully saturated condition

It is noted that the governing equations 1 and 2 can be degenerated to Terzaghi's consolidation equation for saturated soils [1, 6, 7]. For a fully saturated soil, the coefficients of water

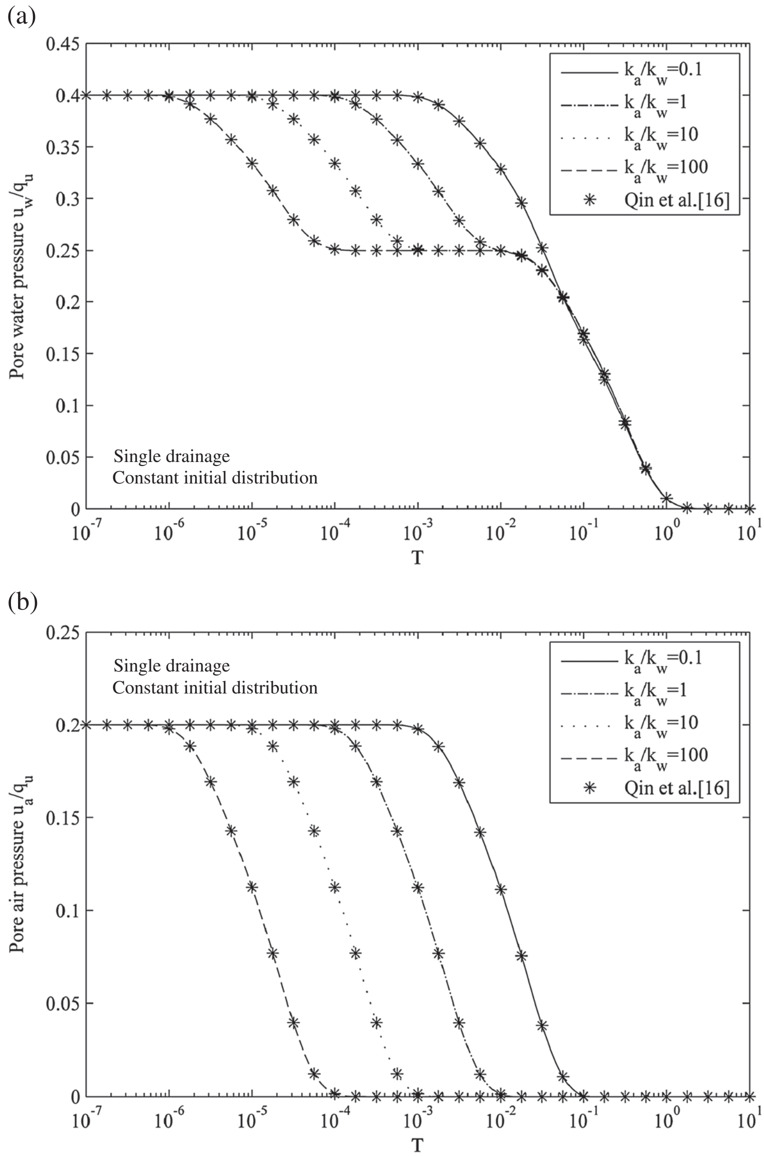

Figure 2. Variations of pore pressure with T at $z = 5$ m for the case of instantaneous loading under different k_a/k_w values (a) pore-water pressure and (b) pore-air pressure.

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volume change m_{1k}^w and m_2^w are equal to the coefficient of volume change m_v and coefficients of air volume change (m_{1k}^a and m_2^a) are zero; the coefficient of permeability for water phase k_w is equal to the saturated permeability coefficient k_s; air pressure may be assumed to be equal to water pressure, that is, $u_a - u_w = 0$. Then the parameters in Equations 1 and 2 become $C_w = 0$, $C_v^w = \frac{k_s}{\gamma_w m_v}$, $C_\sigma^w = 1$, $C_a = 0$, $C_v^a = 0$, and $C_v^a = 0$. The corresponding parameters in Equations 7 and 8 become $W_a = 0$, $W_w = 1$, $W_\sigma = 1$, $A_a = 0$, $A_w = 0$, and $A_\sigma = 0$. Then it can be obtained from Equation A.5 that $Q_1 = 0$ and $Q_2 = 1$. From Equations A.6 and A.7, $c_{11} = 1$, $c_{12} = 0$, $c_{21} = 0$, and $c_{22} = 1$. It follows from Equations 28 and 30 that $\overline{u}_w = \phi_2$ and $U_w = U_2^{\mathrm{m}}/\overline{u}_w^i$. That is to say, the present analytical solutions can be degenerated to the condition of a fully saturated soil. For instance, the pore-water pressure and average degree of consolidation under instantaneous loading, double drainage, and constant initial pore-water pressure distribution can be obtained from Tables I and II as follows:

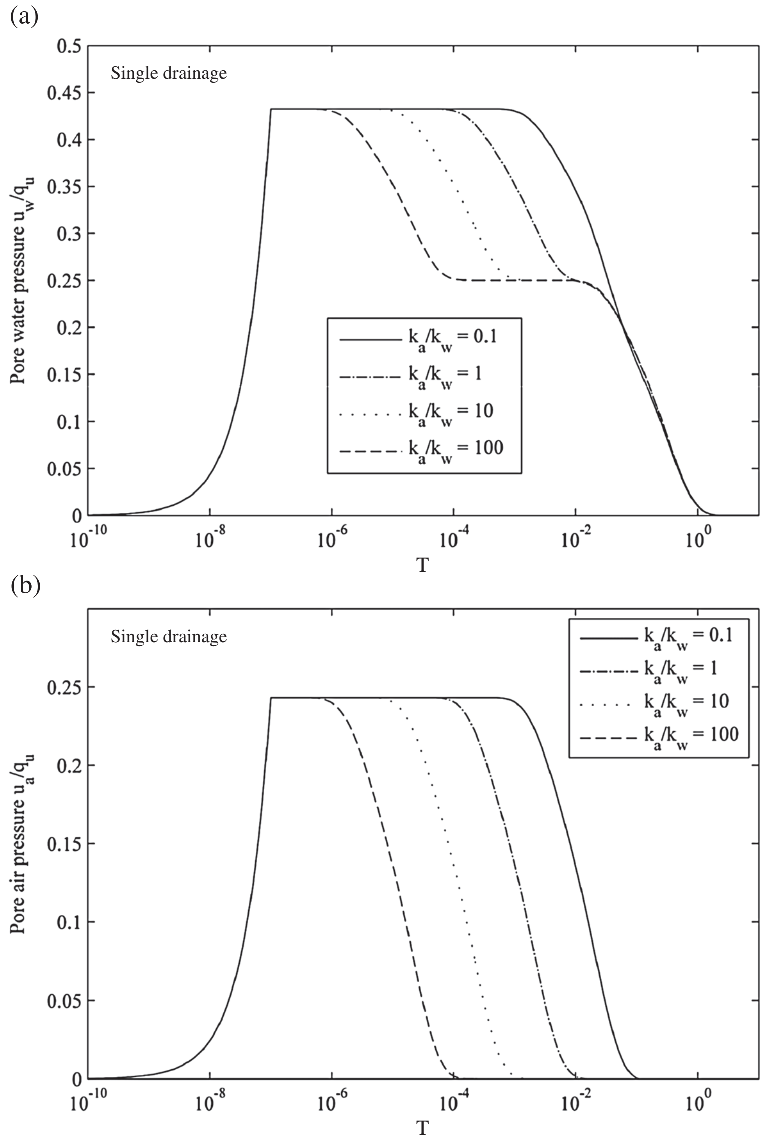

Figure 3. Variations of pore pressure with T at $z = 5$ m for the case of ramp loading under different k_a/k_w values (a) pore-water pressure and (b) pore-air pressure.

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$$\overline{u}_w = \phi_2 = \sum_{m=0}^{m=\infty} \left(\frac{4\overline{u}_w^i}{M}\right) \sin(M\overline{z})e^{-M^2T}, \quad M = (2m+1)\pi \tag{32}$$

$$U_w = \frac{U_2^{\mathrm{m}}}{\overline{u}_w^i} = 1 - \sum_{m=1}^{m=\infty} \frac{8}{M^2} e^{-M^2T}, \quad M = (2m+1)\pi \tag{33}$$

It is seen that Equations 32 and 33 are identical to the solutions of Terzaghi's consolation theory for saturated soils [1]. It is therefore concluded that the present simple analytical solution is a general solution for a soil from unsaturated to fully saturated state.

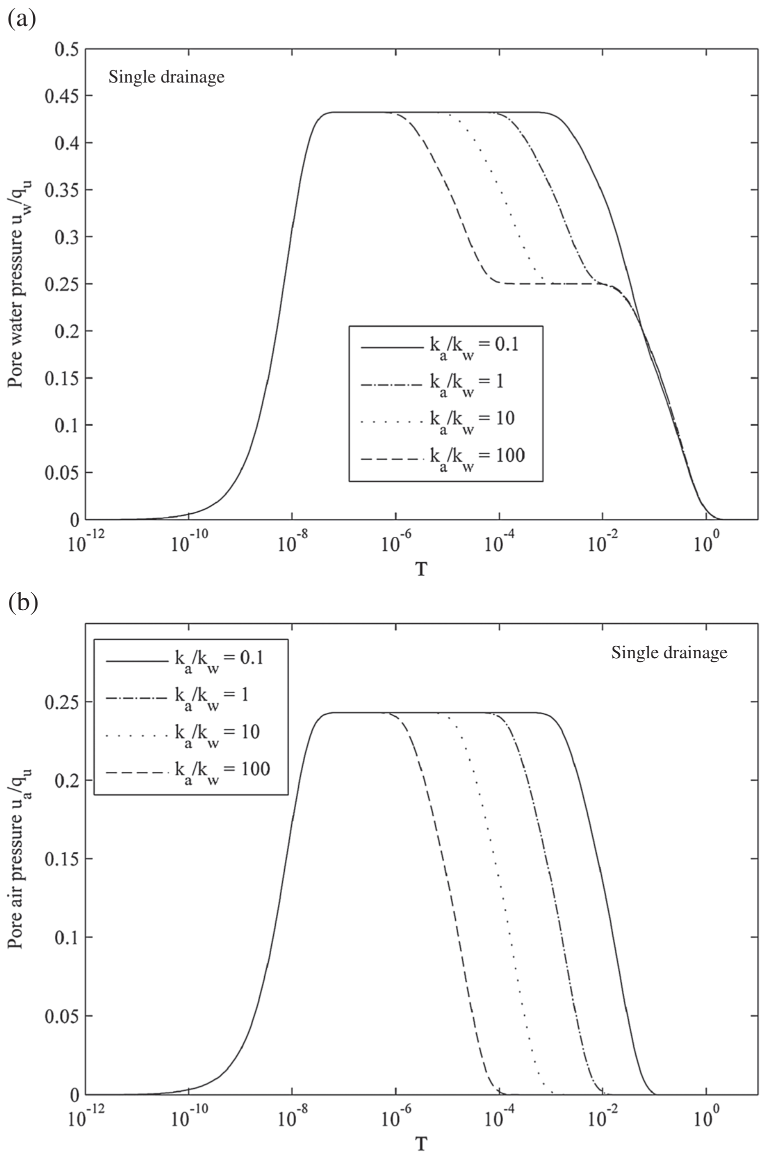

Figure 4. Variations of pore pressure with T at $z = 5$ m for the case of exponent loading under different k_a/k_w values (a) pore-water pressure and (b) pore-air pressure.

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4. CASE STUDY AND DISCUSSIONS

In the case study, the parameters are assumed as follows: $h=10\,\mathrm{m}$, $n_0=0.50$, $S_{r0}=0.80$, $k_w=10^{-10}$ m/s, $m_{1k}^s=-2.5\times10^{-4}\ \mathrm{kPa}^{-1}$, $m_{1k}^w=-0.5\times10^{-4}\ \mathrm{kPa}^{-1}$, $m_2^s=-1.0\times10^{-4}\ \mathrm{kPa}^{-1}$, and $m_2^w=-2.0\times10^{-4}\ \mathrm{kPa}^{-1}$. Three different loading patterns are considered with ultimate extra load $q_u=100$ kPa, construction time $t_c=10^6$ s, and $b=5\times10^{-5}$. Three kinds of initial pore-water and pore-air distributions are considered as shown in Figure 1:

(a) constant: $u_a^i=20$ kPa, $u_w^i=40$ kPa,
(b) trapezium: $u_a^i=20$ kPa, $u_w^i=40+40\times\bar{z}$ kPa,
(c) sine: $u_a^i=20\sin(\pi z/2h)$ kPa, $u_w^i=40\sin(\pi z/2h)$ kPa.

For the single drainage and constant initial pore-water and pore-air pressure distributions, the variations of pore-water and pore-air pressures with time factor T at $z=5$ m are computed. Figure 2 presents the results under instantaneous loading for different k_a/k_w values. The

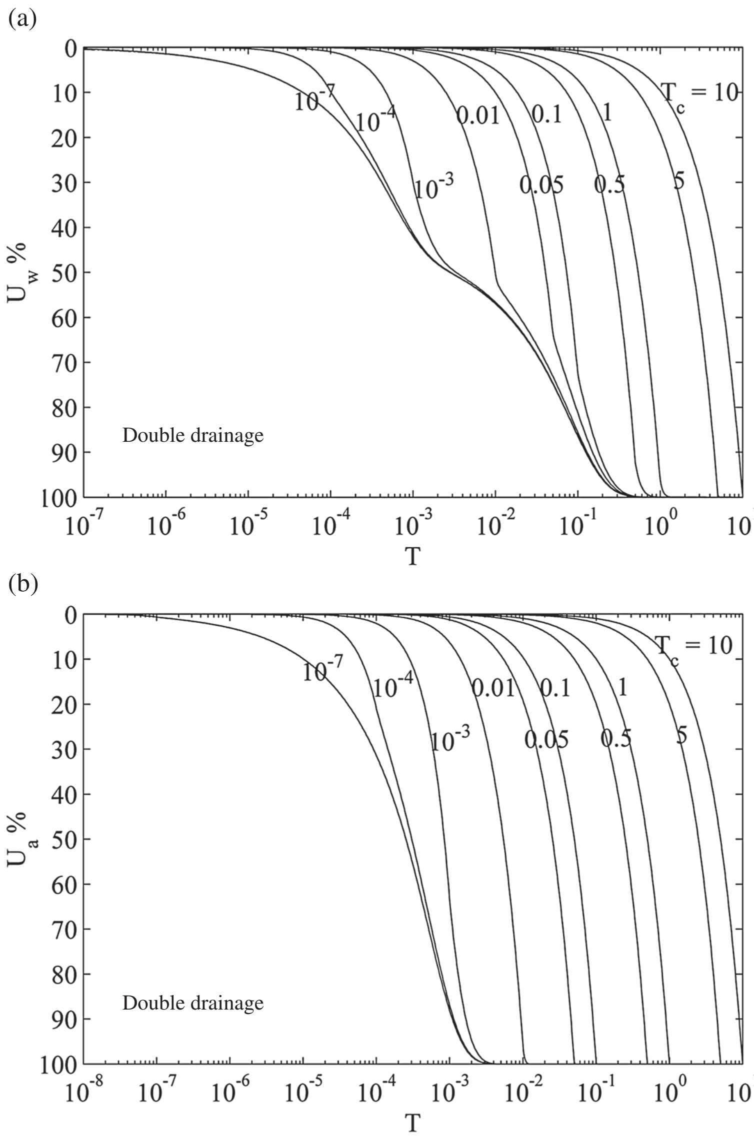

Figure 5. Average degree of consolidation under different T_c values under ramp loading for (a) water phase and (b) air phase.

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analytical results by Qin *et al.* [18] are also shown in the figure for comparison. It is found that identical results are obtained from both analytical solutions. However, the present solution in this paper is much simpler in terms of mathematical expressions. Moreover, the present solution is applicable for different time-dependent loading conditions, boundary conditions, and initial conditions.

Figures 3 and 4 present the variations of pore-water and pore-air pressures with time under ramp loading and exponent loading, respectively, for single drainage condition. Figure 5 depicts the average degree of consolidation for water and air phases of the unsaturated soil stratum subjected to ramp loading with various construction time factors T_c under double drainage condition. It can be found that when T_c is small, the influence of construction time is mainly at the early stage of consolidation. Figure 6 shows the average degree of consolidation for water and air phases under different loading patterns and drainage conditions. It can be seen that the drainage condition has a significant influence on the soil consolidation process. Figure 7 plots the pore-water pressure isochrones for double drainage condition under three different initial pore pressure distributions as shown in Figure 1. Figure 8 shows the pore-water

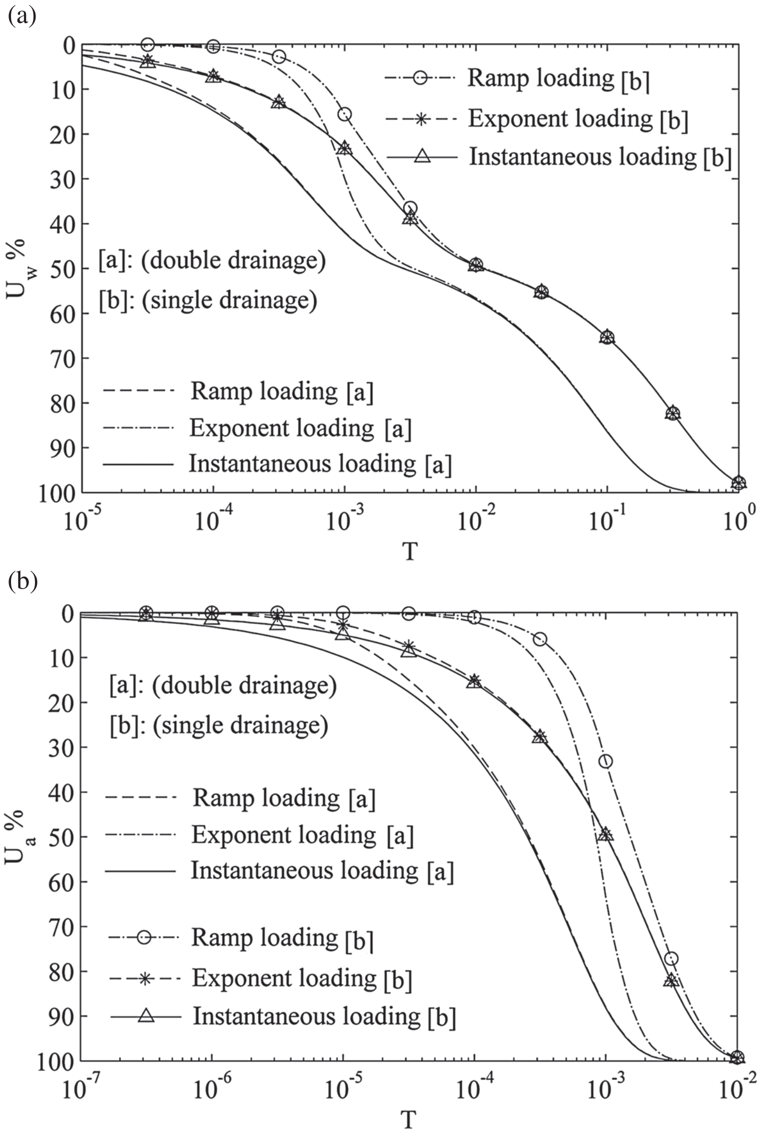

Figure 6. Average degree of consolidation under different loading patterns and drainage conditions for (a) water phase and (b) air phase.

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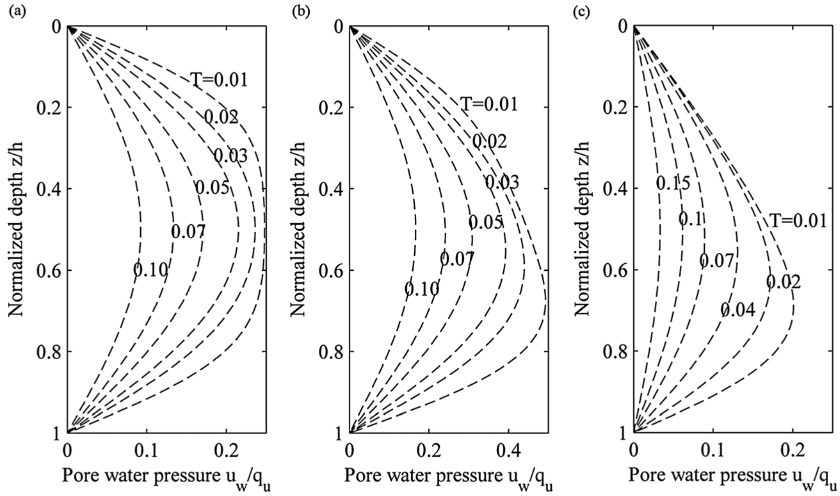

Figure 7. Pore-water pressure isochrones for double drainage condition under the initial distribution: (a) constant, (b) trapezium, and (c) sine.

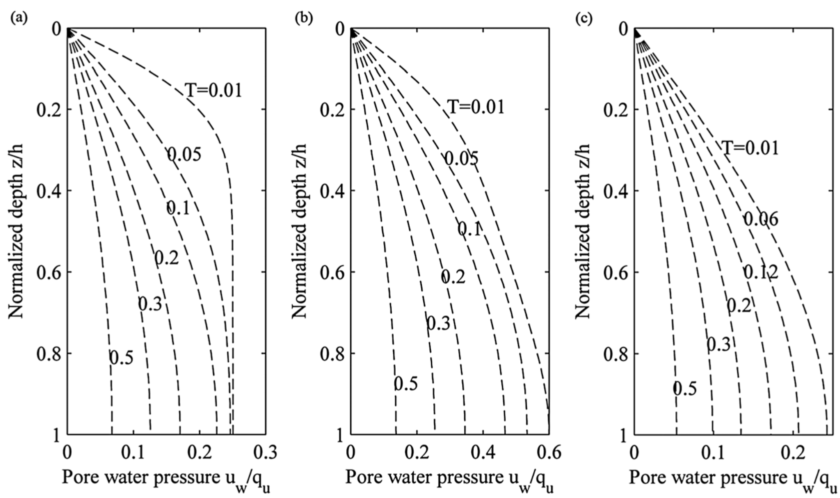

Figure 8. Pore-water pressure isochrones for single drainage condition under the initial distribution: (a) constant, (b) trapezium, and (c) sine.

pressure isochrones for the case of single drainage condition. It can be observed that the initial condition has much effect on the pore-water pressure distribution along depth, especially at the early stage of consolidation.

5. SUMMARY AND CONCLUSIONS

In this paper, a simple analytical solution to Fredlund and Hasan's 1-D consolidation theory for unsaturated soils is presented. Mathematical expressions of the new solution are summarized in tables for three kinds of loading patterns and different drainage and initial conditions. It is shown that the present solutions of pore-water pressure and average degree of consolidation can be degenerated to those of Terzaghi's consolidation for fully saturated condition. In the case studies, the consolidation behaviors of unsaturated soils are presented under different loading, boundary, and

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initial conditions. The average degrees of consolidation under different loading patterns and drainage conditions are presented and discussed.

It shall be pointed out that the present solution is for a linear elastic stress–strain model of unsaturated soil based on Fredlund and Hasan's 1-D consolidation theory. Some additional assumptions are made to obtain the closed-form solution. For example, the coefficients of permeability and volume change of the soil are assumed to remain constant throughout the consolidation process. In real engineering problems, these parameters are not constant, and the solutions can be obtained by using numerical methods. Nevertheless, the present exact solution is a general solution for the soil from unsaturated to fully saturated state and can be easily used by engineers. Caution should be shown when using the present solution to real problems.

APPENDIX A

Multiplying Equations 7 and 8 by arbitrary constants c_1 and c_2, respectively, and adding these two equations together lead to the following:

$$\frac{\partial(\overline{u}_a c_1 + \overline{u}_w c_2)}{\partial T} - (A_a c_1 + W_a c_2)\left(\frac{\partial^2 \overline{u}_a}{\partial \overline{z}^2}\right) - (A_w c_1 + W_w c_2)\left(\frac{\partial^2 \overline{u}_w}{\partial \overline{z}^2}\right) = (A_\sigma c_1 + W_\sigma c_2)\left(\frac{\partial \overline{q}}{\partial T}\right) \quad \text{(A.1)}$$

Equation A.1 could be transformed into a conventional diffusion equation with variable $\phi = \overline{u}_a c_1 + \overline{u}_w c_2$ by introducing a constant Q that satisfies the following relationships:

$$Qc_1 = A_a c_1 + W_a c_2 \quad \text{(A.2)}$$

$$Qc_2 = A_w c_1 + W_w c_2 \quad \text{(A.3)}$$

In order to make Equations A.2 and A.3 hold true, constant Q must satisfy the following condition:

$$(Q - A_a)(Q - W_w) - A_w W_a = 0 \quad \text{(A.4)}$$

Equation A.4 is a quadratic equation in Q that has the following two roots Q_1 and Q_2:

$$Q_{1,2} = \frac{1}{2}\left[A_a + W_w \pm \sqrt{(A_a - W_w)^2 + 4A_w W_a}\right] \quad \text{(A.5)}$$

When $Q = Q_1$, the solutions of Equations A.2 and A.3 are c_{11} and c_{21}, whereas when $Q = Q_2$, the solutions are c_{12} and c_{22}.

Without loss of generality, it is possible to assume that $c_{11} = c_{22} = 1$, and c_{12} and c_{21} can be expressed as follows:

$$c_{12} = \frac{W_a}{Q_2 - A_a} = \frac{Q_2 - W_w}{A_w} \quad \text{(A.6)}$$

$$c_{21} = \frac{A_w}{Q_1 - W_w} = \frac{Q_1 - A_a}{W_a} \quad \text{(A.7)}$$

Equation A.1 can therefore be rewritten in following form:

$$\begin{cases} \dfrac{\partial \phi_1}{\partial T} - Q_1\left(\dfrac{\partial^2 \phi_1}{\partial \bar{z}^2}\right) = \beta_1\left(\dfrac{\partial \bar{q}}{\partial T}\right) \\ \dfrac{\partial \phi_2}{\partial T} - Q_2\left(\dfrac{\partial^2 \phi_2}{\partial \bar{z}^2}\right) = \beta_2\left(\dfrac{\partial \bar{q}}{\partial T}\right) \end{cases} \tag{A.8}$$

where $\phi_1 = \bar{u}_a + c_{21}\bar{u}_w$, $\phi_2 = c_{12}\bar{u}_a + \bar{u}_w$, $\beta_1 = (A_\sigma + W_\sigma c_{21})$, and $\beta_2 = (A_\sigma c_{12} + W_\sigma)$.

Therefore, the transformed upper boundary conditions $\phi_1(0,T)$ and $\phi_2(0,T)$ are as follows:

$$\begin{cases} \phi_1(0,T) = \bar{u}_a(0,T) + c_{21}\cdot\bar{u}_w(0,T) = 0 \\ \phi_2(0,T) = c_{12}\cdot\bar{u}_a(0,T) + \bar{u}_w(0,T) = 0 \end{cases} \tag{A.9}$$

The transformed lower boundary conditions $\phi_1(1,T)$ and $\phi_2(1,T)$ are as follows:

$$\text{Double drainage :} \quad \begin{cases} \phi_1(1,T) = \bar{u}_a(1,T) + c_{21}\cdot\bar{u}_w(1,T) = 0 \\ \phi_2(1,T) = c_{12}\cdot\bar{u}_a(1,T) + \bar{u}_w(1,T) = 0 \end{cases} \tag{A.10}$$

$$\text{Single drainage :} \quad \begin{cases} \dfrac{\partial \phi_1(1,T)}{\partial z} = \dfrac{\partial \bar{u}_a(1,T)}{\partial z} + c_{21}\cdot\dfrac{\partial \bar{u}_w(1,T)}{\partial z} = 0 \\ \dfrac{\partial \phi_2(1,T)}{\partial z} = c_{12}\cdot\dfrac{\partial \bar{u}_a(1,T)}{\partial z} + \dfrac{\partial \bar{u}_w(1,T)}{\partial z} = 0 \end{cases} \tag{A.11}$$

The transformed initial condition $\phi_1(\bar{z},0)$ and $\phi_2(\bar{z},0)$ are as follows:

$$\begin{cases} \phi_1(\bar{z},0) = \phi_1^i(\bar{z}) = \bar{u}_a^i + c_{21}\cdot\bar{u}_w^i \\ \phi_2(\bar{z},0) = \phi_2^i(\bar{z}) = c_{12}\cdot\bar{u}_a^i + \bar{u}_w^i \end{cases} \tag{A.12}$$

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