Lab 3: Applications of Fourier Transform in Signal Filtering and Analysis

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Background

This lab explores the applications of the Fourier Transform, a mathematical tool that enables the decomposition of complex signals into their underlying frequency components. Fourier analysis is essential in fields where signal characteristics, trends, or patterns need to be isolated and examined. In this lab, we will use Fourier Transform methods across three different contexts: audio filtering, stock market analysis, and atmospheric pressure data to demonstrate how Fourier analysis can reveal distinct features within each dataset. Specifically, we will employ the Fast Fourier Transform (FFT) algorithms provided in Python's NumPy library to facilitate efficient computation, an approach that is both practical and widely used in computational analysis.

Fourier Transforms, while mathematically foundational, are computationally intensive due to the large number of calculations required. Here, we use the FFT, which is an optimized algorithm that significantly reduces the time complexity of the Discrete Fourier Transform (DFT). The FFT's performance stems from its ability to take advantage of symmetries in the data, resulting in an $O(N \log_2 N)$ complexity, as opposed to the $O(N^2)$ complexity of a standard DFT. Despite its strengths, the FFT has limitations that impact its accuracy and applicability in certain cases, particularly in achieving correct amplitude and phase representations. These can lead to slight distortions or misinterpretations of the transformed data. We will try to aim to ensure that our FFT analyses maintain a high degree of accuracy.

In this lab, we will utilize the FFT algorithm to filter audio signals by isolating low-frequency components, allowing us to observe how frequency selection impacts sound clarity. For stock market data, we apply a low-pass filter to focus on long-term trends, demonstrating how FFT can extract significant patterns from noisy data. Finally, for atmospheric pressure analysis, we employ the FFT to identify dominant wavenumber components, gaining insight into spatial and temporal atmospheric behaviour.

1. Audio Filtering

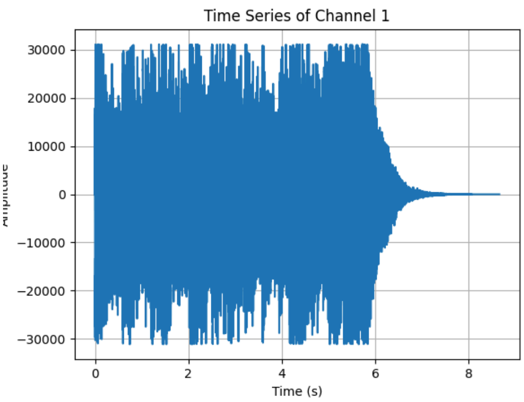
To begin our investigation, we will explore audio filtering by transforming audio signals into the frequency domain, which allows us to selectively isolate and manipulate specific frequency ranges. In this experiment, we apply a low-pass filter to the audio signal, filtering out high-frequency components to create a smoother, bass-enhanced version of the sound. This approach demonstrates the utility of Fourier Transforms in signal processing and noise reduction.

In this experiment, we work with audio files in the .wav format. The specific audio sample we will analyze is a stereo file with two distinct channels: Channel 0 and Channel 1, and each is encoded as

16-bit integers. When saving modified audio data, we'll ensure it is written back in this same format to maintain compatibility.

a) Plotting Stereo Channels of the Audio File

To begin our analysis of audio filtering, we start by loading the stereo wav file (GraviteaTime.wav), containing audio data in two channels, labeled Channel 0 and Channel 1. Using the sampling rate, which is $44100Hz$, we can convert the discrete data points into a time series to observe how the signal evolves. The plot of each channel's data as a function of the time provides a visual representation of the audio signal before applying any filtering. Here's the graph that we plotted:



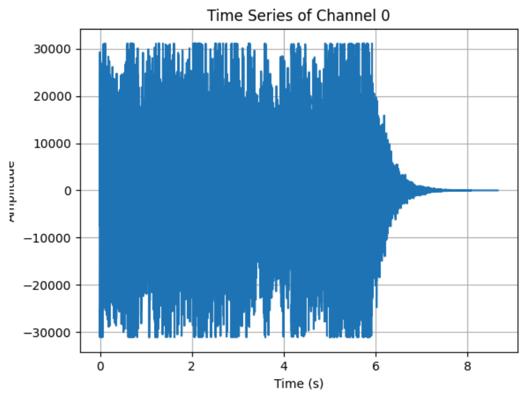


Figure 1 (Left): Time Series of Channel 0 - This plot shows the amplitude of the original audio signal in Channel 0 over time. The high amplitude fluctuations across most of the signal indicate a complex audio waveform, with a noticeable decline of around 6 seconds.
Figure 2 (Right): Time Series of Channel 1 - This plot shows the amplitude of the original audio signal in Channel 1 over time. Similar to Channel 0, the signal exhibits high amplitude variability for the majority of the duration, with a significant reduction in amplitude at around 6 seconds.

The time series plots for Channel 0 and Channel 1 show a few key characteristics of the audio signal. For the majority of the signal duration, both channels exhibit high amplitude fluctuations, with values oscillating between approximately -30,000 and 30,000. The patterns in Channel 0 and Channel 1 appear nearly identical, suggesting that the two channels in this stereo file contain synchronized or highly similar information. This initial visualization will serve as a reference for observing the effects of the low-pass filter, as we expect the filter to reduce the high-frequency fluctuations visible in these unprocessed signals.

b) Implementing a Low-Pass Filter at 880 Hz

A low-pass filter is applied to an audio signal to remove high-frequency components, allowing only the lower frequencies to pass through. A low-pass filter works by zeroing out frequencies above a set cutoff frequency, denoted as f_c. This means that if we take the Fourier Transform of a time series signal $s(t)$ and obtain its frequency components, represented as $\hat{s}(f)$, then any frequency f greater than f_c will be suppressed, while frequencies below or equal to f_c will be retained in the signal. Mathematically, the low-pass filter is expressed as

$$\hat{s}_{lp}(f) = \begin{cases} 0, & f > f_c \\ \hat{s}(f), & f \le f_c \end{cases} \quad \text{Eq.1}$$

First, we apply the FFT to convert each channel's time-domain signal into the frequency domain, allowing us to analyze its frequency components. After transforming the signal, we set all frequencies above 880 Hz to zero, effectively filtering out higher frequencies while retaining the lower-frequency content that contributes to a bass-enhanced sound. Finally, we apply the inverse FFT to converge the filtered signal back to the time domain. This allows us to visually compare the original and filtered signals, both in terms of their Fourier amplitude spectra and their time-domain representations.
Here are the graphs that we plotted:

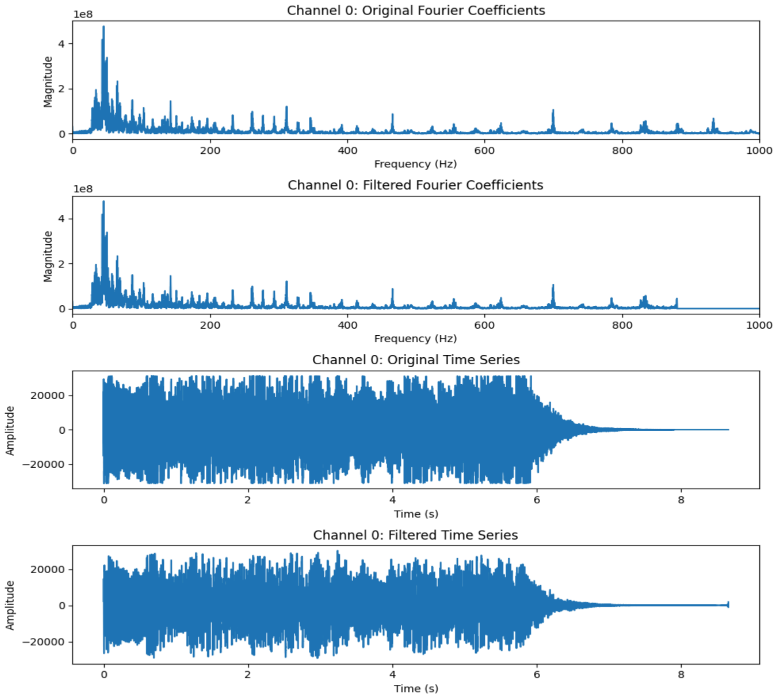


Figure 3: Channel 0 Analysis - The top plot displays the original Fourier coefficients of Channel 0, with a range of frequency components, including peaks above 880 Hz. The second plot shows the filtered Fourier coefficients after applying a low-pass filter, where frequencies above 880 Hz are removed, leaving only lower frequencies. The third and fourth plots compare the original and filtered time series in the time domain, illustrating a smoother signal with reduced high-frequency fluctuations after filtering.

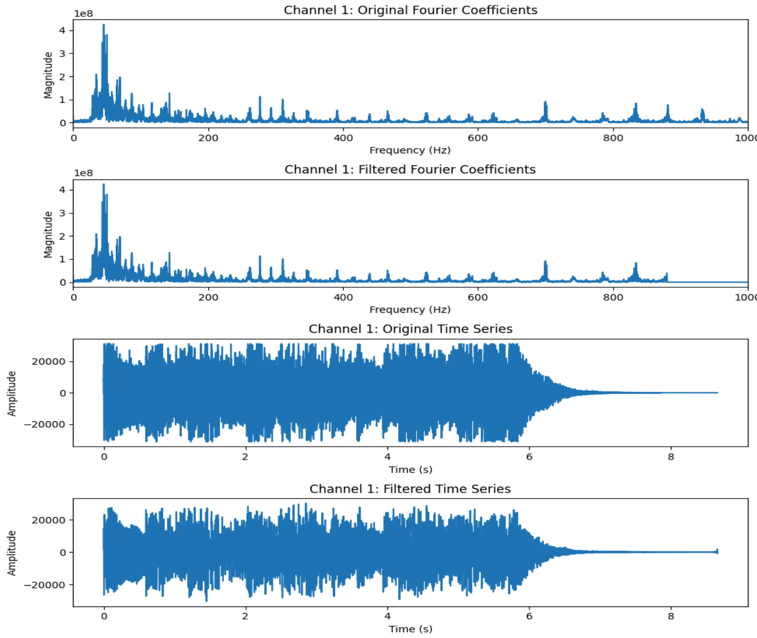


Figure 4: Channel 0 Analysis - The top plot displays the original Fourier coefficients of Channel 0, with a range of frequency components, including peaks above 880 Hz. The second plot shows the filtered Fourier coefficients after applying a low-pass filter, where frequencies above 880 Hz are removed, leaving only lower frequencies. The third and fourth plots compare the original and filtered time series in the time domain, illustrating a smoother signal with reduced high-frequency fluctuations after filtering.

The plots for both Channel 0 and Channel 1 reveal the impact of applying the low-pass filter to the audio signal. In the original Fourier coefficient plots, we see a range of frequency components, with magnitudes peaking at various frequencies below 200 Hz and additional, though lower, peaks extending beyond 880 Hz. These higher-frequency components contribute to the complexity of the original audio signal. However, after applying the low-pass filter, as shown in the filtered Fourier coefficient plots, all frequencies above the 880 Hz cutoff are effectively set to zero, leaving only the lower-frequency components intact. This change isolates the primary, low-frequency elements of the signal, resulting in a cleaner and more bass-focused audio profile.

The time series plots of the original and filtered signals provide further insight. In the original time series, we observe a high degree of fluctuation and variation in amplitude throughout most of the signal duration. After filtering, the time series still retains much of this amplitude structure but appears slightly smoother, particularly in the absence of high-frequency noise. The filtering process reduces sharp oscillations, aligning with our objective to produce a bass-enhanced sound by retaining only the low-frequency components of the signal. This highlights the effectiveness of the low-pass filter in the audio profile by selectively removing higher-frequency elements.

c) Examining the Effect of Filtering on a Short Segment

We further analyze the effect of the low-pass filter by examining a smaller segment of the audio signal. To do this, we select a 30 millisecond window from the time series, beginning at an arbitrary point. By zooming in on this short segment, we can more clearly observe the impact of filtering on the waveform's shape and amplitude variations. This close-up view allows us to see how the high-frequency components are smoothed out in the filtered signal, resulting in a less jagged waveform compared to the original.

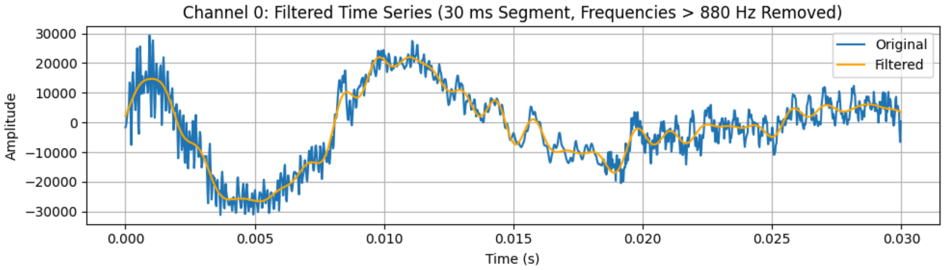


Figure 5: Channel 0 - Zoomed 30 ms segment of the time series for Channel 0, comparing the original signal with the filtered signal after removing frequencies above 880 Hz. The filtered signal exhibits a smoother curve, with high-frequency oscillations attenuated.

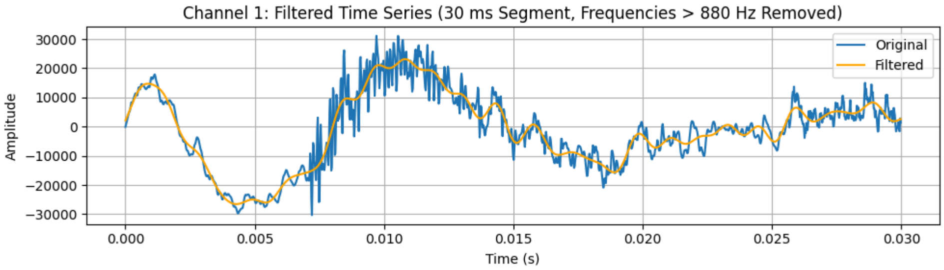


Figure 6: Channel 1 - Zoomed 30 ms segment of the time series for Channel 1, showing the original signal and the filtered signal. The low-pass filter reduces high-frequency fluctuations, resulting in a smoother waveform that emphasizes the lower-frequency content.

Based on the 30 ms segment plots for Channel 0 and Channel 1, we can observe the effect of the low-pass filter. In the original signal, there are numerous high-frequency oscillations and sharp variations, indicating the presence of both low- and high-frequency components in the audio. After filtering, the waveform appears smoother, with many of the sharp peaks and small fluctuations removed. This smoothing effect results from the removal of frequencies above 880 Hz, as the filter preserves only the lower frequencies that contribute to the broader waveform pattern. The filtered signal closely follows the general shape of the original but without the high-frequency noise, which demonstrates the low-pass filter's effectiveness in isolating lower-frequency content.

d) Creating a Filtered Audio File

We take the filtered audio signal from both channels and save it as a new file (GraviteaTime_filtered.wav). When we listen to the resulting file, we can distinctly hear the bass frequencies more prominently. This effect is due to the low-pass filter removing higher frequencies above 880Hz, which isolates the lower frequencies and creates a bass-enhanced version of the original sound.

2. S&P 500 Long-Term Analysis

Next experiment, we analyze the S&P 500 index to remove high-frequency variations, enabling a focus on long-term market trends and revealing behaviour in stock values. This approach can be particularly insightful in financial analysis, as it allows us to observe underlying trends that are often obscured by short-term fluctuations. The data, provided in the file sp500.csv, contains the opening values of the S&P 500 index for each business day from late 2014 to 2019. By applying the Fourier Transform, we decompose this time-series financial data into its frequency components, facilitating the identification of key trends and patterns in the stock market.

a) Plotting S&P 500 Opening Values

To begin our analysis of the S&P 500 index, we load the data from the sp500.csv file, which contains the opening values of the index for each business day from late 2014 to 2019. We focus on plotting the opening values against the business day number, which represents consecutive trading days without gaps for weekends or holidays. This approach provides a continuous view of the market trends over time. By visualizing the data, we can observe fluctuations in the S&P 500's value and identify any overarching trends. Here is the graph that we plotted:

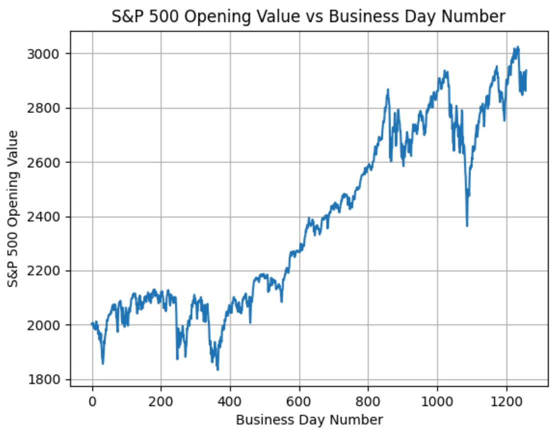


Figure 7: S&P 500 Opening Value vs. Business Day Number - This plot displays the opening values of the S&P 500 index for consecutive business days from late 2014 to 2019. The general upward trend indicates market growth over this period, while intermittent dips and recoveries highlight periods of volatility in the stock market.

Based on the plot of the S&P 500 opening values against business days, we observe a general upward trend, indicating growth in the index over the period from late 2014 to 2019. Despite this overall increase, there are several notable fluctuations. Periods of volatility, marked by sharp declines, are evident, followed by periods of recovery where the index rises again. These fluctuations likely correspond to economic events or market cycles that impacted stock values during this time. The upward trajectory, along with cyclical dips and rebounds, illustrates both the growth potential and the inherent volatility within the stock market over multi-year spans.

By analyzing the frequency of these fluctuations through Fourier Transform methods, we can attempt to filter out high-frequency variations, allowing for a clearer focus on long-term trends and cycles within the data. Understanding these cycles provides valuable insights into market behaviour, which can inform investment strategies that prioritize long-term gains over short-term volatility.

b) Testing the FFT

We first proceed by applying the FFT to the S&P 500 opening values. This transformation allows us to convert the time-domain data into the frequency domain, where we can analyze the frequency components that make up the stock market trends. To ensure the accuracy of the transformation, we verify the process by applying the inverse FFT, which should reproduce the original time-series data. This step helps us confirm that the Fourier Transform has accurately captured all the components of the original signal.

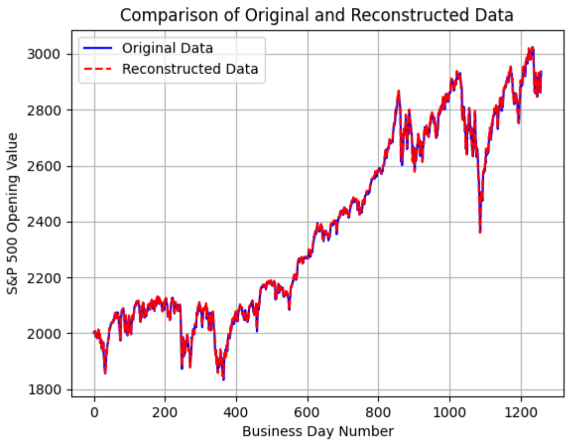


Figure 8: Comparison of Original and Reconstructed Data - This plot shows the S&P 500 opening values for each business day, comparing the original data with the reconstructed data obtained from the inverse FFT. The close alignment of the two lines confirms the accuracy of the Fourier Transform in capturing the essential components of the original time series.

Based on the plot comparing the original data and the reconstructed data for the S&P 500 opening values, we observe that the reconstructed data, obtained by applying the inverse FFT, closely overlaps with the original time series. This near-perfect alignment indicates that the FFT successfully captured all significant components of the original signal and that no information was lost in the transformation process. This comparison between the original and reconstructed data validates the reliability of the Fourier Transform for analyzing time-series financial data. Since the FFT and inverse FFT processes accurately preserved the original data, we can confidently use this transformation to filter out high-frequency noise in subsequent analyses.

c) Long-term Trend Analysis Using Low-Pass Filtering

Finally, we analyze long-term trends in the S&P 500 by applying a low-pass filter to remove high-frequency fluctuations. To focus on variation with periods longer than 63 days (approximately one-quarter of a business day), we set a cutoff in the frequency domain to eliminate components with shorter periods. Specifically, we zero out the Fourier coefficients corresponding to frequencies with periods less than 63 days, leaving only the lower-frequency components that represent longer-term trends in the data. After filtering, we apply the inverse FFT to reconstruct the time series with these low-frequency components. This allows us to observe the S&P 500's behaviour while minimizing short-term noise.

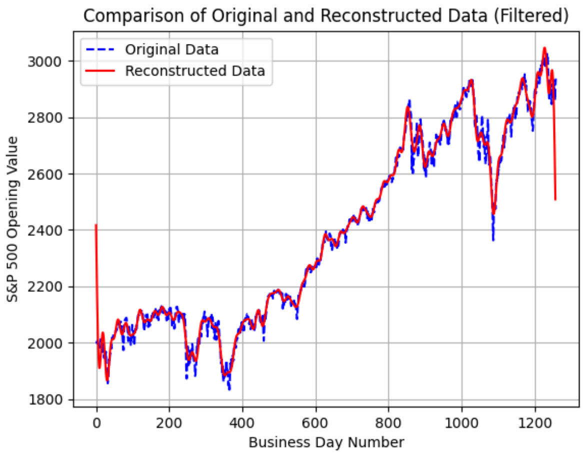


Figure 9: Comparison of Original and Filtered Data (Period > 63 Days) - This plot displays the S&P 500 opening values for each business day, comparing the original data with the filtered data obtained by retaining only components with periods longer than 63 days. The filtered data shows a smoother version of the market's long-term trend by eliminating high-frequency variations.

The filtered data displays a smoother and less volatile version of the original S&P 500 trend, highlighting the broader, long-term movements in the stock market while minimizing day-to-day fluctuations. This smoother shape arises because the low-pass filter has removed high-frequency components associated with short-term variations, such as daily or weekly price swings, which are often driven by immediate market reactions to news, investor sentiment, or other transient factors. What remains are the lower-frequency elements, which retain the general upward trajectory of the market over this period and the major peaks and troughs that correspond to longer-term cycles. These preserved features likely represent broader economic influences or sustained market trends that are less susceptible to short-term volatility.

The filtered data provides a clearer representation of the S&P 500's underlying direction, making it particularly useful for identifying large-scale market cycles and assessing fundamental trends. For instance, this smoothed curve can help reveal seasonal patterns or long-term growth phases that may be masked by noise in the unfiltered data. This approach is valuable for long-term investors or analysts focused on strategic decision-making, as it strips away the distracting short-term fluctuations that might not reflect underlying economic conditions. By focusing on the low-frequency elements, we can gain insights into the cyclical behaviour of the market and potentially forecast sustained trends.

However, it's also important to note that filtering out high-frequency components removes certain details that might be relevant for short-term analysis. Daily or weekly fluctuations, while noisy, can contain valuable signals for traders or analysts seeking to capitalize on short-term opportunities. Therefore, while low-pass filtering is beneficial for long-term trend analysis, it may not be suitable for every type of financial analysis, especially those that rely on high-frequency data to detect quick changes in market sentiment or short-lived opportunities.

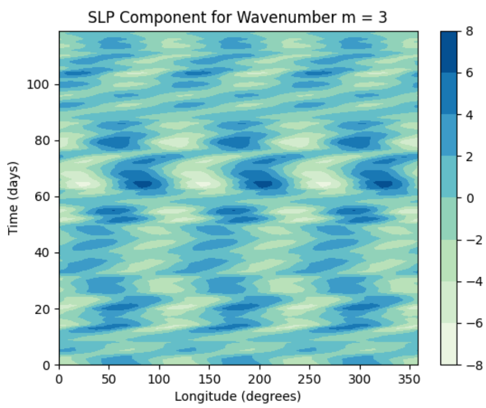
3. Sea Level Pressure Analysis

In the last analysis, we examine Sea Level Pressure (SLP) data collected over time and longitude at a latitude of 50°S, using Fourier decomposition to extract specific wave modes in the longitudinal direction. This approach enables us to study atmospheric wave dynamics, especially the eastward propagation of pressure systems, which aligns with meteorological theories on wave behaviour. The SLP data, obtained from the NCEP Reanalysis dataset, includes values in hectopascals (hPa) and has had the mean pressure value removed to focus on variations. This dataset spans the first 120 days of 2015, with daily readings across longitudes from 0° to 360° at 2.5° intervals.

The Fourier Transform allows us to break down the SLP into a series of waves represented by different wavenumbers, where each wavenumber characterizes different propagation features across longitudes. Our analysis will involve identifying these wavenumbers and interpreting how different wave modes propagate. The coherent wave trains we extract provide insights into atmospheric dynamics in the Southern Hemisphere, where pressure systems often exhibit structured eastward movement. This analysis will give us a deeper understanding of how atmospheric waves can be decomposed into fundamental components, as predicted by atmospheric wave theory.

a) Extracting Longitudinal Components from SLP Data

We begin our analysis of SLP by isolating specific wave components along the longitudinal direction. Using Fourier decomposition, we extract the SLP components corresponding to wavenumbers m = 3 and m = 5. These wavenumbers represent different spatial frequencies in the longitudinal direction, allowing us to examine how particular wave patterns propagate over time. After isolating these components, we create filled contour plots of the extracted SLP data in the time-longitude domain. These plots will help visualize the eastward progression and the structure of the pressure system associated with each wavenumber.



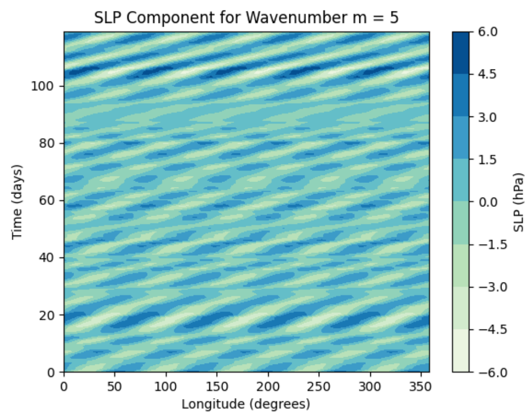


Figure 10 (Left): SLP Component for Wavenumber $m = 3$ — This contour plot shows the sea level pressure variations over time and longitude for wavenumber $m = 3$, with broad and evenly spaced pressure oscillations indicating a lower-frequency wave with a slower eastward propagation.

Figure 11 (Right): SLP Component for Wavenumber $m = 5$ — This contour plot displays the sea level pressure variations for wavenumber $m = 5$, where more frequent oscillations represent a higher-frequency wave pattern that propagates eastward more rapidly.

Based on the contour plots for the SLP components at wave numbers $m = 3$ and $m = 5$, we observe distinct wave patterns that differ in both spatial and temporal characteristics. In the plot for wavenumber $m = 3$, the wave pattern shows broader and more evenly spaced regions of high and low pressure, with wavelengths that extend over larger longitudinal intervals. This structure suggests a lower-frequency wave that propagates eastward over time, likely indicating a larger-scale atmospheric wave pattern with slower periodicity. By contrast, the wavenumber $m = 5$ plot reveals a higher-frequency wave pattern, with more frequent oscillations along the longitude axis. The pressure variations are narrower and occur more frequently over shorter longitudinal distances, indicating a faster propagating wave.

b) Qualitative Analysis

The contour plots for wavenumbers $m = 3$ and $m = 5$ display several important qualitative characteristics that align well with the theoretical understanding of atmospheric wave propagation. The plot for wavenumber $m = 3$ shows broader wave bands with fewer oscillations across the longitudinal axis, indicating a longer wavelength and lower frequency. In contrast, the $m = 5$ plot reveals more frequent, narrow bands, which correspond to a shorter wavelength and higher frequency. This difference in wave structure highlights how different wavenumbers represent distinct scales of atmospheric waves, with lower wavenumbers capturing large-scale, slower oscillations and higher wavenumbers representing smaller-scale, faster oscillations.

Both plots also demonstrate an eastward progression of wave patterns over time, which is typical of atmospheric pressure systems in the Southern Hemisphere and aligns with the general direction of prevailing atmospheric flows. Also, the observed dispersive behaviour, where shorter-wavelength $m = 5$ waves propagate eastward faster than the longer-wavelength waves $m = 3$, is consistent with atmospheric wave theory. This theory suggests that wave disturbances in the atmosphere can depend on the wavelength. The faster eastward progression of the higher-frequency $m = 5$ waves compared to the slower $m = 3$ waves supports this concept, as it demonstrates that shorter waves travel more quickly than longer waves. Overall, these observations provide qualitative support for the theoretical understanding of atmospheric wave dynamics, where phase speed varies with wavelength, confirming the dispersive nature of atmospheric waves.

Understanding this dispersive behaviour is essential for interpreting large-scale weather patterns and their movement across longitudes. So for instance, larger, slower-moving waves associated with lower wavenumbers often represent broader weather systems that evolve over extended periods, while smaller, faster-moving waves correspond to short-term atmospheric fluctuations. These insights can be valuable in predicting the movement and impact of atmospheric pressure systems, as they reveal how different wave components interact and contribute to the overall atmospheric behaviour.