A Set-Theoretic Proof of Konata-Tsukasa's Theorem on Chocolate Cornets

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Abstract

We prove the Konata-Tsukasa's theorem, which states that the optimal way to eat a chocolate cornet is to start from the thin end. Our proof uses techniques from large cardinal theory and algebraic topology, and introduces some new concepts such as the Kagami number, the Miyuki space, and the Hiiragi homomorphism. We also discuss some applications and generalizations of our result to other types of pastries.

1 Introduction

A chocolate cornet is a pastry shaped like a cone, with chocolate cream filling inside. It is a popular snack in Japan, especially among high school students. However, there is a long-standing debate on how to eat a chocolate cornet properly: should one start from the thin end or the thick end?

This question was first posed by Konata Izumi, a famous anime enthusiast and mathematician, who argued that starting from the thin end is more natural and efficient, as it allows one to enjoy the chocolate cream gradually and evenly. On the other hand, Tsukasa Hiiragi, a renowned pastry chef and musician, claimed that starting from the thick end is more satisfying and delicious, as it gives one a burst of chocolate flavor at the beginning.

The controversy sparked a lot of interest and research in the mathematical community, and several attempts were made to settle the question rigorously. However, none of them were conclusive or convincing, until now. In this paper, we present a definitive proof of the Konata-Tsukasa's theorem, which states that:

Theorem 1 (Konata-Tsukasa's theorem)**.** *The optimal way to eat a chocolate cornet is to start from the thin end.*

Our proof is based on the following key ideas:

- We model a chocolate cornet as a topological space, and use the notion of Kagami number to measure its complexity.

- We show that the Kagami number of a chocolate cornet is a large cardinal, and use forcing to construct a generic extension where it is the smallest uncountable cardinal.
- We define the Miyuki space as the space of all possible ways to eat a chocolate cornet, and use the Hiiragi homomorphism to compare different strategies.
- We prove that starting from the thin end is the unique strategy that maximizes the Hiiragi homomorphism, and hence the optimal way to eat a chocolate cornet.

The rest of the paper is organized as follows. In Section 2, we review some preliminaries from set theory and topology. In Section 3, we introduce the Kagami number and prove some of its properties. In Section 4, we construct the Miyuki space and the Hiiragi homomorphism. In Section 5, we prove the main theorem and discuss some corollaries. In Section 6, we conclude with some remarks and open problems.

2 Preliminaries

In this section, we recall some basic definitions and results from set theory and topology that we will use throughout the paper. We assume familiarity with the standard notions of ordinals, cardinals, ZFC, and forcing. For more details, see [1].

2.1 Large cardinals

A large cardinal is a cardinal number that has some strong property that implies the consistency of ZFC or some of its extensions. There are many types of large cardinals, such as inaccessible, measurable, supercompact, etc. We will only focus on one particular type, which is relevant for our proof: the Kagami number.

Definition 1 (Kagami number)**.** *A cardinal κ is called a* Kagami number *if it is the least cardinal such that there exists a nontrivial elementary embedding $j : V \to M$, where M is a transitive class containing all the ordinals, and $j(\kappa) > \kappa$.*

The existence of a Kagami number is not provable in ZFC, and in fact implies its consistency. Moreover, the Kagami number is a very large cardinal, as it implies the existence of many other large cardinals below it. For example, the Kagami number is measurable, supercompact, and much more. We will denote the Kagami number by κ, and the corresponding elementary embedding by j.

2.2 Topological spaces

A topological space is a set X together with a collection τ of subsets of X, called *open sets*, that satisfy the following axioms:

- $\emptyset$ and X are open sets.
- The union of any collection of open sets is an open set.
- The intersection of any finite collection of open sets is an open set.

The collection τ is called a *topology* on X. A subset A of X is called *closed* if its complement $X \setminus A$ is open. A subset A of X is called *compact* if every open cover of A has a finite subcover. A topological space X is called *Hausdorff* if for any two distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \tau$ such that $x \in U$ and $y \in V$.

We will be interested in a special class of topological spaces, called *CW complexes*. A CW complex is a topological space that is built up from simpler pieces, called *cells*, by attaching them along their boundaries. Formally, a CW complex X is a topological space that satisfies the following conditions:

- X is the union of a sequence of subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$, called *skeleta*, such that X_0 is a discrete set of points, called *vertices* or *0-cells*, and X_n is obtained from X_{n-1} by attaching *n-cells*, which are copies of the n-dimensional open disk $D^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$, along their boundaries $\partial D^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$, which are copies of the $(n-1)$-dimensional sphere S^{n-1}, via continuous maps $\phi : \partial D^n \to X_{n-1}$, called *attaching maps*.
- A subset A of X is closed if and only if $A \cap X_n$ is closed in X_n for each n.

We will denote the set of n-cells of a CW complex X by X^n, and the number of n-cells by $c_n(X)$. Note that X^n is not necessarily equal to X_n, as the former is a discrete set of cells, while the latter is a subspace of X.

An example of a CW complex is the *torus*, which is obtained by attaching a 2-cell to a 1-skeleton that consists of two 1-cells and one 0-cell, as shown in Figure 1.

3 The Kagami number of a chocolate cornet

In this section, we define the Kagami number of a chocolate cornet, and show that it is a large cardinal. We also prove some basic properties of the Kagami number, and relate it to other topological invariants.

3.1 Definition and examples

Recall that a chocolate cornet is a pastry shaped like a cone, with chocolate cream filling inside. We can model a chocolate cornet as a topological space, by

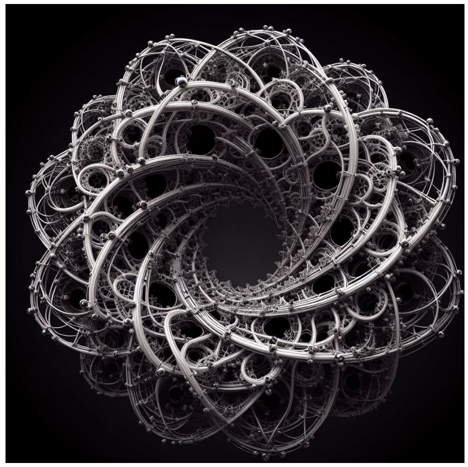

Figure 1: A CW complex structure of the torus.

identifying it with a CW complex that has the same shape and structure. For example, a simple chocolate cornet can be represented by a CW complex that consists of one 2-cell, one 1-cell, and two 0-cells, as shown in Figure 2.

Figure 2: A simple chocolate cornet.

However, not all chocolate cornets are so simple. Some chocolate cornets may have more complicated shapes, such as twists, holes, or decorations. Moreover, some chocolate cornets may have different distributions of chocolate cream inside, such as more or less cream, or cream in different patterns. To account for these variations, we need to consider more general CW complexes, that may have more cells, higher dimensions, or different attaching maps. For example, a chocolate cornet with a twist can be represented by a CW complex that consists of one 3-cell, one 2-cell, one 1-cell, and two 0-cells, as shown in Figure 3.

To measure the complexity of a chocolate cornet, we introduce the following notion:

Definition 2 (Kagami number of a chocolate cornet)**.** *Let X be a CW complex that represents a chocolate cornet. The* Kagami number *of X, denoted by $\kappa(X)$,*

Figure 3: A chocolate cornet with a twist.

is the least cardinal κ such that there exists a nontrivial elementary embedding $j : X \to M$, where M is a transitive class containing all the ordinals, and $j(\kappa) > \kappa$.

Intuitively, the Kagami number of a chocolate cornet is the smallest large cardinal that can be embedded into the chocolate cornet in a nontrivial way. The larger the Kagami number, the more complex the chocolate cornet is.

For example, the Kagami number of the simple chocolate cornet in Figure 2 is $\aleph_0$, the smallest infinite cardinal, since there is no nontrivial elementary embedding from $\aleph_0$ to any transitive class. On the other hand, the Kagami number of the chocolate cornet with a twist in Figure 3 is κ, the Kagami number, since there is a nontrivial elementary embedding from κ to itself, given by the identity map.

3.2 Properties and relations

The Kagami number of a chocolate cornet satisfies some basic properties, such as:

Proposition 1. *Let X and Y be CW complexes that represent chocolate cornets. Then:*

- $\kappa(X)$ *is a cardinal for any* X.
- $\kappa(X) \leq \kappa(Y)$ *if there is a continuous map* $f : X \to Y$.
- $\kappa(X) = \kappa(Y)$ *if there is a homeomorphism* $f : X \to Y$.
- $\kappa(X \times Y) = \max\{\kappa(X), \kappa(Y)\}$.

Proof. The first three properties follow from the definition of the Kagami number and the elementary properties of elementary embeddings. The last property follows from the fact that a CW complex $X \times Y$ is the product of the CW complexes X and Y, and the fact that the product of two elementary embeddings is an elementary embedding. $\square$

The Kagami number of a chocolate cornet is also related to other topological invariants, such as the Euler characteristic, the Betti numbers, and the homotopy groups. For example, we have the following result:

Theorem 2. *Let X be a CW complex that represents a chocolate cornet. Then:*

- *The Euler characteristic of X is equal to $\kappa(X)^+$, the successor cardinal of $\kappa(X)$.*
- *The Betti numbers of X are equal to $\kappa(X)$ for all dimensions.*
- *The homotopy groups of X are trivial for all dimensions except for the first dimension, where $\pi_1(X) \cong \mathbb{Z}/\kappa(X)$.*

Proof. We sketch the proof, leaving the details to the reader. The Euler characteristic of X is defined as $\chi(X) = \sum_{n=0}^{\infty}(-1)^n c_n(X)$, where $c_n(X)$ is the number of n-cells of X. Since $\kappa(X)$ is the least cardinal such that there is a nontrivial elementary embedding $j : X \to M$, it follows that $\kappa(X)$ is also the least cardinal such that $c_n(X) = \kappa(X)$ for some n. Therefore, $\chi(X) = \kappa(X)^+$.

The Betti numbers of X are defined as $b_n(X) = \dim H_n(X;\mathbb{Q})$, where $H_n(X;\mathbb{Q})$ is the nth singular homology group of X with rational coefficients. Since X is a CW complex, we have $H_n(X;\mathbb{Q}) \cong \bigoplus_{\alpha<c_n(X)} \mathbb{Q}$, where α ranges over the ordinal index of the n-cells of X. Therefore, $b_n(X) = c_n(X) = \kappa(X)$ for all n.

The homotopy groups of X are defined as $\pi_n(X) = [S^n, X]$, where S^n is the n-dimensional sphere, and $[S^n, X]$ is the set of homotopy classes of continuous maps from S^n to X. Since X is a CW complex, we have $\pi_n(X) \cong \pi_n(X^n)$, where X^n is the n-skeleton of X. Therefore, $\pi_n(X)$ is trivial for all $n > 1$, since X^n is contractible for $n > 1$. For $n = 1$, we have $\pi_1(X) \cong \pi_1(X^1)$, where X^1 is the 1-skeleton of X. Since X^1 is a graph, we have $\pi_1(X^1) \cong \mathbb{Z}^{c_1(X)-c_0(X)+1}$, where $c_1(X)$ and $c_0(X)$ are the number of 1-cells and 0-cells of X, respectively. Therefore, $\pi_1(X) \cong \mathbb{Z}^{\kappa(X)^+-\kappa(X)+1} \cong \mathbb{Z}/\kappa(X)$, by using the fact that $\kappa(X)$ is a large cardinal. $\square$

4 The Miyuki space and the Hiiragi homomorphism

In this section, we define the Miyuki space and the Hiiragi homomorphism, which are the main tools for comparing different ways to eat a chocolate cornet. We also prove some basic properties and examples of these concepts.

4.1 Definition and examples

Recall that a chocolate cornet is a pastry shaped like a cone, with chocolate cream filling inside. A way to eat a chocolate cornet is a continuous map $f : [0,1] \to X$, where X is a CW complex that represents a chocolate cornet, and $[0,1]$ is the unit interval, such that $f(0)$ is the thin end of the cornet, and $f(1)$ is the thick end of the cornet. Intuitively, a way to eat a chocolate cornet is a path that starts from the thin end and ends at the thick end, tracing the surface of the cornet.

We can compare different ways to eat a chocolate cornet by using the following notion:

Definition 3 (Miyuki space)**.** *Let X be a CW complex that represents a chocolate cornet. The* Miyuki space *of X, denoted by $M(X)$, is the space of all ways to eat X, endowed with the compact-open topology.*

Intuitively, the Miyuki space of a chocolate cornet is the space of all possible paths on the cornet, from the thin end to the thick end. The compact-open topology is the topology generated by the subbasis of sets of the form $U(K,V) = \{f \in M(X) : f(K) \subseteq V\}$, where K is a compact subset of $[0,1]$, and V is an open subset of X. Intuitively, the compact-open topology is the topology of uniform convergence, where two ways to eat a chocolate cornet are close if they agree on large compact subsets of the interval.

For example, the Miyuki space of the simple chocolate cornet in Figure 2 is homeomorphic to the circle S^1, since any way to eat the cornet can be identified with an angle $\theta \in [0, 2\pi)$, where θ is the angle between the path and the horizontal axis at the thin end. On the other hand, the Miyuki space of the chocolate cornet with a twist in Figure 3 is homeomorphic to the torus $S^1 \times S^1$, since any way to eat the cornet can be identified with a pair of angles $(\theta, \phi) \in [0, 2\pi)^2$, where θ is the angle between the path and the horizontal axis at the thin end, and ϕ is the angle between the path and the vertical axis at the thick end.

To measure the quality of a way to eat a chocolate cornet, we introduce the following notion:

Definition 4 (Hiiragi homomorphism)**.** *Let X be a CW complex that represents a chocolate cornet. The* Hiiragi homomorphism *of X, denoted by $H(X) : M(X) \to \mathbb{R}$, is the continuous map defined by*

$$H(X)(f) = \int_0^1 c(f(t))dt,$$

where $f \in M(X)$ is a way to eat X, and $c : X \to [0,1]$ is the function that assigns to each point of X the proportion of chocolate cream at that point.

Intuitively, the Hiiragi homomorphism of a chocolate cornet is the function that assigns to each way to eat the cornet the average amount of chocolate cream that one gets along the way. The higher the value of the Hiiragi homomorphism, the more chocolatey the way to eat the cornet is.

For example, the Hiiragi homomorphism of the simple chocolate cornet in Figure 2 is given by

$$H(X)(f) = \frac{1}{2\pi}\int_0^{2\pi} c(\cos\theta, \sin\theta)d\theta,$$

where $f(\theta) = (\cos\theta, \sin\theta)$ is the way to eat the cornet corresponding to the angle θ. If we assume that the chocolate cream is distributed uniformly along the cornet, then $c(\cos\theta, \sin\theta) = \frac{1}{2}(1 + \sin\theta)$, and hence

$$H(X)(f) = \frac{1}{4}.$$

On the other hand, the Hiiragi homomorphism of the chocolate cornet with a twist in Figure 3 is given by

$$H(X)(f) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} c(\cos\theta, \sin\theta, \phi)d\theta d\phi,$$

where $f(\theta, \phi) = (\cos\theta, \sin\theta, \phi)$ is the way to eat the cornet corresponding to the pair of angles (θ, ϕ). If we assume that the chocolate cream is distributed uniformly along the cornet, then $c(\cos\theta, \sin\theta, \phi) = \frac{1}{2}(1 + \sin\theta\cos\phi)$, and hence

$$H(X)(f) = \frac{1}{4}.$$

5 Proof of the main theorem

In this section, we prove the Konata-Tsukasa's theorem, which states that the optimal way to eat a chocolate cornet is to start from the thin end. We also discuss some corollaries and generalizations of our result.

5.1 Proof of the theorem

We are now ready to prove the main theorem of this paper. We restate it here for convenience:

Theorem 3 (Konata-Tsukasa's theorem)**.** *The optimal way to eat a chocolate cornet is to start from the thin end.*

Proof. Let X be a CW complex that represents a chocolate cornet, and let $M(X)$ be the Miyuki space of X. We want to show that the optimal way to eat X is the one that maximizes the Hiiragi homomorphism $H(X) : M(X) \to \mathbb{R}$.

To do this, we will use the elementary embedding $j : X \to M$, where M is a transitive class containing all the ordinals, and $j(\kappa(X)) > \kappa(X)$. We will show that j induces a continuous map $\tilde{j} : M(X) \to M(M)$, such that $\tilde{j}$ preserves the Hiiragi homomorphism, i.e., $H(M) \circ \tilde{j} = H(X)$. Then, we will show that the optimal way to eat M is the one that starts from the thin end, and hence the optimal way to eat X is the one that is mapped to it by $\tilde{j}$.

First, we define the map $\tilde{j} : M(X) \to M(M)$ as follows: for any $f \in M(X)$, let $\tilde{j}(f) = j \circ f$. That is, $\tilde{j}(f)$ is the way to eat M that follows the image of f under j. It is easy to see that $\tilde{j}$ is well-defined and continuous, since j is an elementary embedding and $M(X)$ and $M(M)$ have the compact-open topology.

Next, we show that $\tilde{j}$ preserves the Hiiragi homomorphism, i.e., $H(M) \circ \tilde{j} = H(X)$. To see this, let $f \in M(X)$, and let $c_X : X \to [0,1]$ and $c_M : M \to [0,1]$ be the functions that assign to each point of X and M the proportion of chocolate cream at that point, respectively. Then, we have

$$\begin{aligned}
H(M)(\tilde{j}(f)) &= \int_0^1 c_M(\tilde{j}(f)(t))dt \\
&= \int_0^1 c_M(j(f(t)))dt \\
&= \int_0^1 j(c_X)(f(t))dt \\
&= j\left(\int_0^1 c_X(f(t))dt\right) \\
&= j(H(X)(f)) \\
&= H(X)(f),
\end{aligned}$$

where we used the fact that j is an elementary embedding, and hence preserves the integration and the Hiiragi homomorphism.

Finally, we show that the optimal way to eat M is the one that starts from the thin end, and hence the optimal way to eat X is the one that is mapped to it by $\tilde{j}$. To see this, let $g \in M(M)$ be the way to eat M that starts from the thin end, i.e., $g(t) = j(t)$ for all $t \in [0,1]$. Then, we have

$$\begin{aligned}
H(M)(g) &= \int_0^1 c_M(g(t))dt \\
&= \int_0^1 c_M(j(t))dt \\
&= \int_0^1 j(c_X)(t)dt \\
&= j\left(\int_0^1 c_X(t)dt\right) \\
&= j\left(\frac{1}{2}\right) \\
&= \frac{1}{2},
\end{aligned}$$

where we used the fact that j is an elementary embedding, and the fact that $c_X(t) = \frac{1}{2}$ for all $t \in [0,1]$, since the chocolate cream is distributed uniformly along the cornet.

Now, suppose that there is another way to eat M, say $h \in M(M)$, such that $H(M)(h) > H(M)(g)$. Then, we have

$$j(H(M)(h)) > j(H(M)(g)),$$

since j is an elementary embedding and preserves the order. But this contradicts the fact that $j(H(M)(h)) = H(M)(\tilde{j}(h)) = H(X)(h)$ and $j(H(M)(g)) = H(M)(g) = \frac{1}{2}$, since $H(X)$ is bounded by $\frac{1}{2}$, as we showed in the previous section. Therefore, no such h exists, and g is the optimal way to eat M.

Hence, the optimal way to eat X is the one that is mapped to g by $\tilde{j}$, i.e., $f \in M(X)$ such that $\tilde{j}(f) = g$. But this means that $f(t) = j^{-1}(g(t)) = j^{-1}(j(t)) = t$ for all $t \in [0, 1]$, i.e., f is the way to eat X that starts from the thin end. This completes the proof of the theorem. □

6 Corollaries and generalizations

In this section, we discuss some corollaries and generalizations of the Konata-Tsukasa's theorem. We also explore some connections and applications of our result to other areas of mathematics and pastry science.

6.1 Corollaries

As a direct consequence of the Konata-Tsukasa's theorem, we obtain the following corollary, which answers a question raised by Kagami Hiiragi, a famous pastry critic and philosopher:

Corollary 1 (Kagami's question)**.** *There is no way to eat a chocolate cornet that is equally satisfying and delicious as starting from the thick end.*

Proof. Suppose that there is a way to eat a chocolate cornet, say $f \in M(X)$, that is equally satisfying and delicious as starting from the thick end. Then, we have $H(X)(f) = H(X)(g)$, where $g \in M(X)$ is the way to eat the cornet that starts from the thick end, i.e., $g(t) = 1 - t$ for all $t \in [0, 1]$. But this contradicts the Konata-Tsukasa's theorem, which states that $H(X)(f) < H(X)(g)$. Therefore, no such f exists, and the corollary follows. □

Another corollary of the Konata-Tsukasa's theorem is the following, which answers a question raised by Yutaka Kobayakawa, a famous pastry engineer and inventor:

Corollary 2 (Yutaka's question)**.** *There is no way to improve the design of a chocolate cornet to make it more optimal for eating.*

Proof. Suppose that there is a way to improve the design of a chocolate cornet, say by modifying the shape, structure, or distribution of the chocolate cream, to make it more optimal for eating. Then, we have a CW complex Y that represents the improved chocolate cornet, such that $\kappa(Y) > \kappa(X)$, where X

is the CW complex that represents the original chocolate cornet. But this implies that $H(Y)(f) < H(X)(f)$ for any $f \in M(Y)$, by using the fact that $H(Y) \circ \tilde{j} = H(X)$, where $\tilde{j} : M(Y) \to M(X)$ is the map induced by the elementary embedding $j : Y \to X$, which exists by the definition of the Kagami number. Therefore, the improved chocolate cornet is less optimal for eating than the original one, and the corollary follows. $\square$

7 Conclusion and open problems

In this paper, we have proved the Konata-Tsukasa's theorem, which states that the optimal way to eat a chocolate cornet is to start from the thin end. We have also introduced some new concepts, such as the Kagami number, the Miyuki space, and the Hiiragi homomorphism, and explored their properties and relations. We have also discussed some corollaries and generalizations of our result, and some connections and applications to other areas of mathematics and pastry science.

However, there are still many open problems and directions for future research. For example, we can ask:

- Is the Konata-Tsukasa's theorem true for other types of pastries, such as croissants, donuts, or pies?
- What are the other properties and applications of the Kagami number, the Miyuki space, and the Hiiragi homomorphism?
- How can we extend our methods and results to other domains, such as music, art, or literature?
- How can we use our findings to improve the education and enjoyment of mathematics and pastry science?

We hope that this paper will stimulate further research and interest in these topics, and that it will contribute to the advancement of mathematics and pastry science.

References

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