Complex Analysis Notes
Graduate Center, CUNY
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These notes are drawn from the upcoming text by Saeed Zakeri of Queens College in NY. Some remarks, theorems, etc are paraphrased; some remarks, theorems, etc are verbatim. Notes created with the permission of the professor, who retains all copyright rights regarding his work. This is, explicitly, a derivative work made with permission of the author.

Section and subsection titles exactly follow the chapters and sections of the text as of the time of the writing. There is some small re-ordering of results.

Contents

1 Rudiments of Complex Analysis

1.1 What is a holomorphic function?

- f defined near $p \in \mathbb{C}$ and complex valued, is **complex differentiable** at p if
$$f'(p) = \lim_{z \to p} \frac{f(z) - f(p)}{z - p}$$
exists. $f'(p)$ is called the **complex derivative** of f at p.
- Differentiability implies continuity.
- **Differentiation Rules.** f and g differentiable at p:
  - $(f + g)'(p) = f'(p) + g'(p)$
  - $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$,
  - If $g(p) \neq 0$ then
$$\left(\frac{f}{g}\right)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{(g(p))^2}$$

  If $g'(p)$ and $f'(g(p))$ exist, then $(f \circ g)'(p) = f'(g(p))g'(p)$.
- $z = x + iy \mapsto (x, y)$ gives the canonical isomorphism $\mathbb{C} \to \mathbb{R}^2$.
- $f = u + iv$ can be identified with $f(x, y) = (u(x, y), v(x, y))$ – i.e. $f : \mathbb{C} \to \mathbb{C}$ vs $f : \mathbb{R}^2 \to \mathbb{R}^2$.
$$f_x = u_x + iv_x, \quad f_y = u_y + iv_y$$
- *New* partial operators:
$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y)$$
$$f_x = f_z + f_{\overline{z}}, \quad f_y = i(f_z - f_{\overline{z}})$$
- **Conditions for Differentiability.** For f defined near $p \in \mathbb{C}$, the following are equivalent:
  - Complex derivative $f'(p)$ exists.
  - Real derivative $Df(p)$ exists and $f_{\overline{z}} = 0$.
  - Real derivative $Df(p)$ exists with $u_x = v_y$ and $u_y = -v_x$ at p.

  In such cases $f'(p) = f_z(p) = f_x(p) = -if_y(p)$.
- An angle preserving linear transformation is called **conformal** (shapes preserved, not scales).
- **Conformality of Derivative.** Suppose $f'(p) \neq 0$ exists. Then $Df(p) : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving conformal linear transformation.
- For $U \subseteq \mathbb{C}$ non-empty and open, $f : U \to \mathbb{C}$ is **holomorphic** in U, written $f \in \mathcal{O}(U)$, if $f'(p)$ exists $\forall p \in U$. Elements of $\mathcal{O}(\mathbb{C})$ are called **entire**. $\mathcal{O}(U)$ is closed under sums, products, and composition. Pointwise addition and multiplication makes $\mathcal{O}(U)$ a commutative ring with identity.
- **Conditions for Being Holomorphic.** Suppose $f = u + iv$, defined on U, is real differentiable. Then $f \in \mathcal{O}(U)$ exactly when the following equivalent conditions hold throughout U:
  1. $u_x = v_y$ and $u_y = -v_x$, the **Cauchy-Riemann Equations**;
  2. $f_{\overline{z}} = 0$, the **Complex Cauchy-Riemann Equation**.

  In this case, $f' = f_z = f_x = -if_y$.

1.2 Complex analytic functions

- $\sum_{n=0}^{\infty} a_n(z-p)^n$, where $p, a_0, a_1, \ldots \in \mathbb{C}$, is a **power series**. Each power series has a **disk of convergence** about p: the series converges inside the disk, diverges outside it, and can do anything on the boundary. The disk's radius, possibly 0 or $+\infty$, is the **radius of convergence** of the series.

- **Radius of Convergence.** The radius of convergence for $\sum_{n=0}^{\infty} a_n(z-p)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0, +\infty].$$

  The series converges absolutely and uniformly inside $\mathbb{D}(p, r)$, $r < R$, and diverges for z with $|z-p| > R$.

- For $U \subseteq \mathbb{C}$ non-empty and open, $f : U \to \mathbb{C}$ is **complex analytic** if, whenever $\mathbb{D}(p, r) \subseteq U$, there is a power series $\sum_{n=0}^{\infty} a_n(z-p)^n$ converging to $f(z)$ whenever $z \in \mathbb{D}(p, r)$.

- **Analytic Implies Holomorphic.** A complex analytic function has derivatives of all orders, each of which is complex analytic and can be found via term-by-term differentiation. The coefficients of a complex analytic function's power series are independent of the radius of disk under consideration: $a_n = f^{(n)}(p)/n!$, $n \geq 0$ (p the center of the disk under consideration).

1.3 Complex integration

- For $U \subseteq \mathbb{C}$ non-empty and open, $\gamma : [a, b] \to U$, continuous, is a **curve** in U with **initial point** $\gamma(a)$ and **end point** $\gamma(b)$, and it is **closed** if $\gamma(a) = \gamma(b)$. $|\gamma| = \{\gamma(t) : t \in [a, b]\} \subseteq \mathbb{C}$ is the **image**. γ is **piecewise** C^1 if there is $a = t_0 < t_1 < \cdots < t_n = b$ with γ continuously differentiable on (t_{k-1}, t_k) and the limits $\lim_{t\to t_{k-1}^+} \gamma'(t)$ and $\lim_{t\to t_k^-} \gamma'(t)$ exist.

  If γ is a curve then γ^- is the **reverse** of γ, and if η is another curve starting where γ ends, then $\gamma \cdot \eta$ is the **product** of γ and η and is formed by traversing γ and then η.

- Throughout the section, it will be assumed that all curves are piecewise C^1.

- For $\gamma : [a, b] \to \mathbb{C}$ a curve, $f : |\gamma| \to \mathbb{C}$ continuous, the **integral of f along** γ is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

  where $\gamma' = d\gamma/dt$ is defined at all but finitely any points in $[a, b]$, and is invariant under orientation preserving reparameterizations of γ. Note that

$$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz, \qquad \int_{\gamma\cdot\eta} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\eta f(z)\, dz.$$

  For $f = u + iv$ and $\gamma(t) = x(t) + iy(t)$,

$$\int_\gamma f(z)\, dz = \int_\gamma (u\, dx - v\, dy) + i \int_\gamma (v\, dx + u\, dy).$$

- The **length** of a curve γ is $\int_\gamma |dz| = \int_0^1 |\gamma'(t)|\, dt$, denoted $\text{length}(\gamma)$.

- The **ML-inequality** is:

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z\in|\gamma|} |f(z)| \cdot \text{length}(\gamma).$$

- **Continuous Dependence on Vertices.** Let $f : U \to \mathbb{C}$ be continuous. For T a closed triangle lying in U, $\int_{\partial T} f(z)\,dz$ depends continuously on the vertices of T.
- F is a **primative** of the continuous function $f : U \to \mathbb{C}$ if $F'(z) = f(z)$ in U.
- $\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$.
- **Existence of Primitives.** $f : U \to \mathbb{C}$, continuous, has a primitive in U iff $\int_\gamma f(z)\,dz = 0$ for every closed curve in U.

1.4 Cauchy's theory in a disk

- **Primitives in Disks.** Let $D \subseteq \mathbb{C}$ be an open disk and $f : D \to \mathbb{C}$ continuous. If $\int_{\partial T} f(z)\,dz = 0$ for every closed triangle $T \subseteq D$ then f has a primitive in D.
- **Goursat's Theorem.** If $f \in \mathcal{O}(U)$ then $\int_{\partial T} f(z)\,dz = 0$ for every closed triangle $T \subseteq U$.
- **Primitives in a Disk.** If $D \subseteq \mathbb{C}$ is an open disk and $f \in \mathcal{O}(D)$ then f has a primitive in D.
- **Cauchy's Theorem in a Disk.** If $D \subseteq \mathbb{C}$ is an open disk and $f \in \mathcal{O}(D)$ then $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in D.
- Cauchy's theorem remains true if f is continuous in D and holomorphic in $D \setminus \{p\}$ for some $p \in D$.
- **Cauchy's Integral Formula in a Disk.** For $\mathbb{D}(p, r) \subseteq D \subseteq \mathbb{C}$, D an open disk, and $f \in \mathcal{O}(D)$:
$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in \mathbb{D}(p, r).$$
- **Holomorphic Implies Complex Analytic.** A holomorphic function is complex analytic. If $f \in \mathcal{O}(U)$ and $\mathbb{D}(p, r) \subseteq U$ then
$$f(z) = \sum_{n=0}^{\infty} a_n (z - p)^n, \quad z \in \mathbb{D}(p, r);$$
$$a_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{\mathbb{T}(p,s)} \frac{f(\zeta)}{(\zeta - p)^{n+1}}\,d\zeta, \quad 0 < s < r.$$
- **Derivatives of All Orders.** If $f \in \mathcal{O}(U)$ then $f' \in \mathcal{O}(U)$ and so $f^{(k)} \in \mathcal{O}(U)$ exists for all $k \geq 1$.
- **Morera's Theorem.** If $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\,d(z) = 0$ for every closed triangle $T \subseteq U$, then $f \in \mathcal{O}(U)$.
- If $U \subseteq \mathbb{C}$ is open, L is a line, $f \in \mathcal{O}(U \setminus L)$ but continuous in U, then $f \in \mathcal{O}(U)$.
- **Cauchy's Estimates.** If $f \in \mathcal{O}(\mathbb{D}(p, r))$ is continuous on $\overline{\mathbb{D}}(p, r)$, then for $n \geq 0$:
$$\left| f^{(n)}(p) \right| \leq \frac{n!}{r^n} \sup_{|z-p|=r} |f(z)|.$$
- **Ratio Bound.** If f is holomorphic with $f(\mathbb{D}(p, r)) \subseteq \mathbb{D}(q, R)$, then $|f'(p)| \leq R/r$.
- **Liouville's Theorem.** Every bounded entire function is constant.

- **Differentiating Under the Integral.** Let $U \subseteq \mathbb{C}$ be open and $\varphi : U \times [a,b] \to \mathbb{C}$ continuous such that for $t \in [a,b]$, $z \mapsto \varphi(z,t) \in \mathcal{O}(U)$ with derivative $\varphi'(z,t)$. Define $f : U \to \mathbb{C}$ by
$$f(z) = \int_a^b \varphi(z,t)\,dt.$$
Then $f \in \mathcal{O}(U)$ and
$$f'(z) = \int_a^b \varphi'(z,t)\,dt, \quad z \in U.$$
Inductively,
$$f^{(n)}(z) = \int_a^b \varphi^{(n)}(z,t)\,dt, \quad z \in U.$$

- **Differentiating Curve Integral.** If γ is a piecewise C^1 curve, $g : |\gamma| \to \mathbb{C}$ is continuous, and $f : \mathbb{C} \setminus |\gamma| \to \mathbb{C}$ is defined as
$$f(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n}\,d\zeta, \quad n \geq 1$$
then $f \in \mathcal{O}(\mathbb{C} \setminus |\gamma|)$ and
$$f'(z) = n \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \quad z \in \mathbb{C} \setminus |\gamma|.$$

1.5 Mapping Properties of Holomorphic Functions

- Let $f \in \mathcal{O}(U)$ be not identically zero in $\mathbb{D}(p,r) \subseteq U$ with power series $f(z) = \sum_{n=0}^\infty a_n (z-p)^n$ there. The **order** of f at p is the index of the first non-zero a_m, denoted by $\mathrm{ord}(f,p)$. $\mathrm{ord}(f,p) \geq 1$ iff $f(p) = 0$ and p is a **simple zero** of f if $\mathrm{ord}(f,p) = 1$. $\mathrm{ord}(f,p)$ is the unique integer $m \geq 0$ such that $f(z) = (z-p)^m f_1(z)$ with $f_1 \in \mathcal{O}(U)$ and $f_1(p) \neq 0$.

- $U \subseteq \mathbb{C}$ is a **domain** if it is open, non-empty, and connected.

- **Accumulating Zero.** For $f \in \mathcal{O}(U)$ and U a domain, if $f^{-1}(0)$ accumulates in U then $f \equiv 0$ in U. Consequently, a non-constant holomorphic function on a domain has at most countably many zeroes, all of which are isolated.

- **Identity Theorem.** If $f = g$ accumulates in U, a domain, with $f, g \in \mathcal{O}(U)$ then $f = g$ in U.

- **Slope Continuity.** If $f \in \mathcal{O}(U)$ then $g : U \times U \to \mathbb{C}$ is continuous, where g is:
$$g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

- $f : V \to W$ is a **biholomorphism** if f is bijective and holomorphic with holomorphic inverse.

- **Holomorphic Inverse Function Theorem.** Suppose $f \in \mathcal{O}(U)$, $p \in U$, and $f'(p) \neq 0$. Then there are open neighborhoods $V \subseteq U$ of p, $W \subseteq \mathbb{C}$ of $f(p)$ such that $f : V \to W$ is a biholomorphism with
$$\left(f^{-1}\right)'(w) = \frac{1}{f'(f^{-1}(w))}, \quad w \in W.$$

- **Local Normal Form.** For $f \in \mathcal{O}(U)$ non-constant, U a domain, and $p \in U$, there exists positive $m \in \mathbb{N}$ and neighborhoods $V \subseteq U$ of p and $W \subseteq \mathbb{C}$ of $q = f(p)$ and biholomorphisms $\varphi : V \to \mathbb{D}$, $\psi : W \to \mathbb{D}$ such that $\varphi(p) = \psi(q) = 0$ and $(\psi \circ f \circ \varphi^{-1})(w) = w^m$ for $w \in \mathbb{D}$.

- The m in the Local Normal Form is the order of the zero at p of $f - f(p)$ and is called the **local degree** of f at p, denoted $\deg(f,p) = \operatorname{ord}(f - f(p), p)$. p is called a **critical point** when $\deg(f,p) > 1$ (i.e. $f'(p) = 0$) with **critical value** $f(p)$.
- **Open Mapping Theorem.** If $f \in \mathcal{O}(U)$ non-constant with U a domain then $f(U)$ is open in $\mathbb{C}$.
- **Critical Points & Biholomorphisms.** If $f \in \mathcal{O}(U)$ is injective in a domain U, then f has no critical points in U and is biholomorphic with its image.
- **Maximum Principle for Open Maps.** Suppose $f : U \to \mathbb{C}$ is open. Then $|f|$ does not attain a local maximum in U and, if f is non-vanishing in U, does not attain a local minimum.
- **Maximum Principle for Holomorphic Functions.** Let $U \subseteq \mathbb{C}$ be a bounded domain and $f : \overline{U} \to \mathbb{C}$ be continuous with $f \in \mathcal{O}(U)$. Then

  (i) $|f(z)| \leq \sup_{\zeta \in \partial U} |f(\zeta)|$ for all $z \in U$;

  (ii) $|f(z)| \geq \inf_{\zeta \in \partial U} |f(\zeta)|$ for all $z \in U$ provided f does not vanish in U.

  If in either case equality holds at some $z \in U$ then f is constant.

2 Topological Aspects of Cauchy's Theory

2.1 Homotopy of Curves

- $\gamma : [a, b] \to X$, continuous, is a **curve** in a topological space X with **initial point** $\gamma(a)$ and **end point** $\gamma(b)$, and it is **closed** if $\gamma(a) = \gamma(b)$. $|\gamma| = \{\gamma(t) : t \in [a, b]\} \subseteq X$ is the **image**. A closed curve in X can be identified with a continuous map of the circle into X.

  If γ is a curve then γ^- is the **reverse** of γ, and if η is another curve starting where γ ends, then $\gamma \cdot \eta$ is the **product** of γ and η and is formed by traversing γ and then η. ε_p is the **constant curve** at p.

- A **homotopy** between γ_0, γ_1, curves in X from p to q is a continuous map $H : [0, 1] \times [0, 1] \to X$ with:
  - $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$, $t \in [0, 1]$.
  - $H(0, s) = p$ and $H(1, s) = q$, $s \in [0, 1]$.

  When such a function exists, γ_0 and γ_1 are called **homotopic** in X, written $\gamma_0 \simeq \gamma_1$. $\gamma_0 \simeq \gamma_1$ if γ_0 can be continuously deformed *in* X to γ_1.

- **Homotopy of Compositions.** Let γ be a curve in X, $\varphi : [0, 1] \to [0, 1]$ continuous.
  (i) If φ fixes 0 and 1 then $\gamma \circ \varphi \simeq \gamma$.
  (ii) If $\varphi(0) = \varphi(1)$ then $\gamma \circ \varphi \simeq \varepsilon_p$, $p = (\gamma \circ \varphi)(0)$.

- **Homotopy is an Equivalence.** Homotopy $\simeq$ is an equivalence relation on the set of curves with a given initial point and end point. Moreover:
  (i) $\gamma \simeq \eta \iff \gamma^- \simeq \eta^-$.
  (ii) If $\gamma(1) = \eta(0)$ and $\gamma \simeq \hat{\gamma}, \eta \simeq \hat{\eta}$, then $\gamma \cdot \eta \simeq \hat{\gamma} \cdot \hat{\eta}$.
  (iii) If $p = \gamma(0)$ and $q = \gamma(1)$ then $\varepsilon_p \cdot \gamma \simeq \gamma \cdot \varepsilon_q \simeq \gamma$ and $\gamma \cdot \gamma^- \simeq \varepsilon_p$.
  (iv) If $\gamma(1) = \eta(0)$ and $\eta(1) = \xi(0)$ then $(\gamma \cdot \eta) \cdot \xi \simeq \gamma \cdot (\eta \cdot \xi)$.

- The equivalence class of a curve γ under $\simeq$ is called its **homotopy class** and is denoted $[\gamma]$. If $\gamma(1) = \eta(0)$ then $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$ is well defined. For $p \in X$, the set of homotopy classes of closed curves starting at p forms a group, called the **fundamental group** of X with **base point** p, under the aforementioned product and is denoted by $\pi_1(X, p)$.

- If $f : X \to Y$ is continuous then $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ given by $f_*([\gamma]) = [f \circ \gamma]$ is a group homomorphism. If $g : Y \to Z$ is continuous then $(g \circ f)_* = g_* \circ f_*$. If f is a homeomorphism then f_* is an isomorphism with inverse $(f^{-1})_*$ i.e. homemorphic spaces have isomorphic fundamental groups).

- X, path-connected, is **simply connected** if $\pi_1(X, p)$ is trivial for some (and hence every) $p \in X$. If every closed curve γ from $p \in X$ is **null-homotopic**, i.e. $\gamma \simeq \varepsilon_p$, then X is simply connected.

- **Homotopy of Curves in Simply Connected Spaces.** A path-connected space X is simply connected if and only if any two curves with the same initial and end points are homotopic.

- γ_0, γ_1, closed curves in X, are **freely homotopic** in X if there exists a continuous map $H : [0, 1] \times [0, 1] \to X$ with:
  - $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1$ for all $t \in [0, 1]$.
  - $H(0, s) = H(1, s)$ for all $s \in [0, 1]$.

  γ_0 is called **freely null-homotopic** in X if $\gamma_1 = \varepsilon_p, p \in X$. Free homotopy is an equivalence relation.

2.2 Covering properties of the exponential map

- $\exp(z) = \exp(\zeta) \iff z - \zeta = 2k\pi i, k \in \mathbb{Z}$ and $\exp(\mathbb{C}) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
- For $w_0 \in \mathbb{C}^*$, $\exp^{-1}(w_0) = z_0 + 2\pi i\mathbb{Z}$ where z_0 is *some* preimage of w_0.
- Let $w_0 \in \mathbb{C}^*$ and select $z_0 \in \exp^{-1}(w_0)$, and set $z_k = z_0 + 2\pi ik$. Then w_0 has a neighborhood W that is **evenly covered** by exp, that is $\exp^{-1}(W)$ is the disjoint union of open sets $O_k = O_0 + 2\pi ik$ where $z_k \in O_k$ and $\exp : O_k \to W$ is a homeomorphism, yielding a **local inverse** of exp, $L_k : W \to O_k$.
- A **lift** (under exp) of a continuous map $f : X \to \mathbb{C}^*$ is a continuous map $\tilde{f} : X \to \mathbb{C}$ satisfying $\exp(\tilde{f}) = f$. $\tilde{f}$ is also called a **branch of the logarithm of** f.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{C}^* \end{tikzcd}$$

- **Uniqueness of Lifts.** Any two lifts of a continuous map from a connected space X to $\mathbb{C}^*$ differ by an integer multiple of $2\pi i$.
- **Existence of Lifts from Unit Interval & Square.**
  (i) Given a curve $\gamma : [0,1] \to \mathbb{C}^*$ and $p \in \exp^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = p$.
  (ii) Given a continuous map $H : [0,1] \times [0,1] \to \mathbb{C}^*$ and $p \in \exp^{-1}(H(0,0))$, there is a unique lift $\tilde{H}$ of H with $\tilde{H}(0,0) = p$.

  Part (i) is the **curve lifting property** of exp.
- **Lebesgue's Cover Lemma.** For an open cover $\{U_\alpha\}$ of a compact metric space X, there exists a **Lebesgue number** $\delta > 0$ such that any set E with $\operatorname{diam} E < \delta$ lies in some U_α.
- **Lift End Points Corollary.** Let $\gamma_0 \simeq \gamma_1$ with $p \in \mathbb{C}$ such that $\exp(p) = \gamma_i(0)$. Let $\tilde{\gamma}_i$ be the unique lift of γ_i with $\tilde{\gamma}_i(0) = p$. Then $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$. γ_0 is a null-homotopic closed curve if and only if $\tilde{\gamma}$ is a closed curve.
- A space is **locally path-connected** if every neighborhood of every point contains a path-connected neighborhood of that point.
- **Existence of Lifts.** Suppose X is simply connected and locally path-connected and $f : X \to \mathbb{C}^*$ is continuous. Then $f = \exp(g)$ for some continuous $g : X \to \mathbb{C}$ which unique up to addition of an integer multiple of $2\pi i$.
- **Existence of Holomorphic Logarithms and n-th Roots.** Suppose $U \subseteq \mathbb{C}$ is a simply connected domain on which $f \in \mathcal{O}(C)$ does not vanish. Then:
  (i) There is $g \in \mathcal{O}(U)$, unique up to addition of an integer multiple of $2\pi i$, with $f = \exp(g)$.
  (ii) For n a positive integer, there is $h \in \mathcal{O}(U)$, unique up to multiplication by an n-th root of unity, with $f = h^n$.

2.3 The winding number

- All the lifting results of the previous section hold if we replace $\mathbb{C}^*$ and $z \mapsto \exp(z)$ by $\mathbb{C} \setminus \{p\}$ and $z \mapsto \exp(z) + p$, respectively.
- Let γ be a curve, $p \notin |\gamma|$, and $\tilde{\gamma}$ a lift of γ under $z \mapsto \exp(z) + p$. The quantity

$$W(\gamma, p) = \frac{1}{2\pi i} \left(\tilde{\gamma}(1) - \tilde{\gamma}(0)\right)$$

  is the **winding number** of γ with respect to p. $W(\gamma, p) \in \mathbb{Z}$ if and only if γ is closed.

- **Properties of the Winding Number.** Let γ be a curve with $p \notin |\gamma|$, then:
  (i) $W(\gamma, p) = W(\gamma + w, p + w)$ for all $w \in \mathbb{C}$.
  (ii) $W(\gamma^-, p) = -W(\gamma, p)$.
  (iii) If η is a curve with $p \notin |\eta|, \eta(0) = \gamma(1)$ then $W(\gamma \cdot \eta, p) = W(\gamma, p) + W(\eta, p)$.
  (iv) If $\eta \simeq \gamma$ (or they are freely homotopic closed curves) in $\mathbb{C} \setminus \{p\}$ then $W(\gamma, p) = W(\eta, p)$.

  If γ is closed, then

  (v) $W(\gamma, p) = 0$ if and only if γ is freely homotopic to a constant curve in $\mathbb{C} \setminus \{p\}$.
  (vi) $W(\gamma, z)$ is constant on the components of $\mathbb{C} \setminus |\gamma|$ and 0 on the unbounded region.

- The **jump principle**: each time $|\gamma|$ is crossed from right to left, the winding number increments.
- A closed curve is a **Jordan curve** (or **simple closed curve**) if it is injective on $[0, 1)$. If γ is a Jordan curve, then the **Jordan Curve Theorem** says $\mathbb{C} \setminus |\gamma|$ has two components: a bounded **interior** and unbounded **exterior**, denoted $\operatorname{int}(\gamma)$ and $\operatorname{ext}(\gamma)$, respectively. $|\gamma| = \partial \operatorname{int}(\gamma) = \partial \operatorname{ext}(\gamma)$.
- **Winding Number for Jordan Curves.** If γ is a Jordan curve, then $W(\gamma, \cdot)$ vanishes on $\operatorname{ext}(\gamma)$ and is ± 1 on $\operatorname{int}(\gamma)$. γ is **positively** or **negatively oriented** according to the sign of $W(\gamma, \cdot)$ inside $\operatorname{int}(\gamma)$.
- **Analytic Description of the Winding Number.** Let γ be a piecewise C^1 curve, $p \notin |\gamma|$. Then

$$W(\gamma, p) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - p}.$$

2.4 Cycles and homology

- The **free abelian group generated by** S is the collection $\mathcal{G}(S)$ of functions $\varphi : S \to \mathbb{Z}$ with φ non-zero finitely often. Let φ_x be the characteristic function on $\{x\}$. Then for $\varphi \in \mathcal{G}(S)$ vanishing outside $\{x_1, \ldots, x_m\}$ with $n_k = \varphi(x_k)$ we have $\varphi = n_1\varphi_{x_1} + \cdots n_m\varphi_{x_m}$. Such representations are unique, and identifying φ_x with x we get $\varphi = n_1x_1 + \cdots + n_mx_m$.
- Let $U \subseteq \mathbb{C}$ be non-empty and open. A **chain** in U is an element of the free abelian group generated by curves in U, e.g. a formal sum

$$\gamma = n_1\gamma_1 + \cdots + n_m\gamma_m$$

  with $n_k \in \mathbb{Z}$ and γ_k a curve in U. n_k is the **multiplicity** of γ_k and

$$|\gamma| = |\gamma_1| \cup \cdots \cup |\gamma_m|$$

  is the **image** of γ.

- If U_γ is the set of curves in U, then the **boundary map** $\partial : \mathcal{G}(U_\gamma) \to \mathcal{G}(U)$ is the group homomorphism:

$$\sum_{k=1}^{m} n_k \gamma_k \mapsto \sum_{k=1}^{m} n_k \gamma_k(1) - \sum_{k=1}^{m} n_k \gamma_k(0).$$

- A chain is a **cycle** if every $p \in U$ is an initial point and end point an equal number of times, counting multiplicity; e.g.

$$\sum_{\{k:\gamma_k(0)=p\}} n_k = \sum_{\{k:\gamma_k(1)=p\}} n_k.$$

  A chain γ is a cycle if and only if $\partial\gamma = 0$. Since ∂ is a group homomorphism, the cycles of U are the kernel of the boundary map of U and are thus an additive group.

- **Cycles decompose into closed curves.** If $\gamma = \sum_{k=1}^{m} n_k \gamma_k$ is a cycle, then collecting $|n_k|$ copies of either γ_k or, when $n_k < 0$, γ_k^-, the curves can be partitioned into subcollections each of whose product can be arranged to be a closed curve.

- The **winding number** extends to chains. If $\gamma = \sum_{k=1}^{m} n_k \gamma_k$ and $p \notin |\gamma|$ then

$$W(\gamma, p) = \sum_{k=1}^{m} n_k W(\gamma_k, p).$$

- **Winding numbers of chains and cycles.**
  (i) For any chain γ, $W(-\gamma, p) = -W(\gamma, p)$.
  (ii) For any two chains γ and η, $W(\gamma + \eta, p) = W(\gamma, p) + W(\eta, p)$.
  (iii) If γ is a cycle then $W(\gamma, \cdot)$'s range is $\mathbb{Z}$, is constant on the connected components of its domain, and vanishes on the unbounded component.

- Let $U \subseteq \mathbb{C}$ be non-empty and open. A cycle γ is **null-homologous**, $\gamma \sim 0$, if $W(\gamma, \cdot) \equiv 0$ on $\mathbb{C} \setminus U$.

- Two cycles γ, η in U are **homologous** if $\gamma - \eta \sim 0$, i.e. $W(\gamma, \cdot) = W(\eta, \cdot)$ on $\mathbb{C} \setminus U$.

- $\sim$ is an equivalence relation. The equivalency class of a cycle γ under $\sim$, denoted by $\langle\gamma\rangle$, is called the **homology class** of γ. $\langle\gamma\rangle + \langle\eta\rangle = \langle\gamma + \eta\rangle$ is well-defined. The set of homology classes in U is an abelian group called the **first homology group** of U, denoted by $H_1(U)$.

- $H_1(U)$ is the quotient group of all cycles in U by the all the null-homologous cycles in U.

- **Homotopic implies homologous.**
  (i) If $\gamma \simeq \eta$ in U then $\gamma - \eta \sim 0$ in U.
  (ii) If γ and η are freely homotopic closed curves in U then $\gamma \sim \eta$ in U.

  Thus $\varphi : \pi_1(U, p) \to H_1(U)$, $[\gamma] \mapsto \langle\gamma\rangle$, is well-defined.

- **Homology class of a closed product.** Suppose $\gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_n$ is a closed curve. Then, in U,

$$\gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_n \sim \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

  And so φ defined above is a group homomorphism. φ vanishes on the commutator C of $\pi_1(U, p)$ since $H_1(U)$ is abelian, inducing a homomorphism $\Phi : \pi_1(U, p)/C \to H_1(U)$. Poincaré and Hurewicz showed Φ is an isomorphism.

- **Homology group of simply connected domains.** Every cycle in a simply connected domain U is null homologous, and so $H_1(U) = 0$

2.5 The homology version of Cauchy's theorem

- Let $\gamma = \sum_{k=1}^{m} n_k \gamma_k$ be a chain, each γ_k piecewise C^1, and let $f : |\gamma| \to \mathbb{C}$ be continuous. Define

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{m} n_k \int_{\gamma_k} f(z)\, dz.$$

- The special case of $f(z) = 1/(z-p)$, $p \notin |\gamma|$ gives

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p} = \sum_{k=1}^{m} n_k W(\gamma_k, p) = W(\gamma, p).$$

- **Homology version of Cauchy's theorem.** Let $U \subseteq \mathbb{C}$ be open and γ a piecewise C^1 null-homologous cycle in U. For $f \in \mathcal{O}(U)$:

$$\int_\gamma f(z)\, dz = 0,$$

$$f(z) \cdot W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in U \setminus |\gamma|.$$

- **Cauchy's theorem in simple connected domains.** The above equations hold for all piecewise C^1 cycles in a simply connected domain.

- **Integration independent of homotopic curves / homologous cycles.** If $\gamma \simeq \eta$ or $\gamma \sim \eta$ (and are piecewise C^1) then

$$\int_\gamma f(z)\, dz = \int_\eta f(z)\, dz.$$

- **Primitives in simply connected domains.** Every holomorphic function in a simply connected domain has a primitive.

- **Cauchy's integral formula for higher derivatives.** For $f \in \mathcal{O}(U)$, $\gamma \sim 0$ in U, and $n \geq 1$:

$$f^{(n)}(z) \cdot W(\gamma, z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \quad z \in U \setminus |\gamma|.$$

- If γ is not piecewise C^1 or rectifiable, there is a piecewise C^1 curve, η, it is homotopic to. If $f \in \mathcal{O}(U)$ then define $\int_\gamma f(z)\, dz$ to be $\int_\eta f(z)\, dz$ which by the above statements is independent of the choice of η. Extending to cycles by linearity we can always assume paths of integration are piecewise C^1.

3 Meromorphic Functions

3.1 Isolated singularities

- p is an **isolated singularity** of f if there is an open neighborhood U of p such that $f \in \mathcal{O}(U \setminus \{p\})$. p is a **removable singularity** if there exists $g \in \mathcal{O}(U)$ such that $g(z) = f(z)$ for $z \in U \setminus \{p\}$.
- **Riemann's removable singularity theorem.** If a holomorphic function is bounded in a punctured neighborhood of a isolated singularity, then the singularity is removable.
- Boundedness can be replaced by $\lim_{z\to p}(z-p)f(z) = 0$ in the above.
- **Classification of isolated singularities.** An isolated singularity p of f, holomorphic, is one of:
  - (i) Removable if $\lim_{z\to p} f(z)$ exists.
  - (ii) A **pole** if $\lim_{z\to p} f(z) = \infty$. In this case there is a positive integer m such that $f_1(z) = (z-p)^m f(z)$ has a removable singularity at p and $f_1(p) \neq 0$.
  - (iii) An **essential singularity** if $\lim_{z\to p}$ does not exist.
- **Casorati-Weierstrass theorem.** If p is an essential singularity of f then for all small $r > 0$, $f(\mathbb{D}^*(p,r))$ is dense in $\mathbb{C}$. Picard showed that at most one value of $\mathbb{C}$ is missed.
- If f has a pole at p, then the m in the classification theorem is the **order** of f at p, denoted $\operatorname{ord}(f,p)$.
- If $\operatorname{ord}(f,p) = 1$ then p is a **simple pole**.
- If $m = \operatorname{ord}(f,p)$ then

$$f(z) = \sum_{n=0}^{\infty} b_n (z-p)^{n-m}$$

  in a punctured neighborhood about p. The **principle part** of f at p is the rational function

$$\frac{b_0}{(z-p)^m} + \cdots + \frac{b_{m-1}}{z-p}.$$

- The principle part of f at p is the unique polynomial P in $(z-p)^{-1}$ with $P(0) = 0$ such that p is a removable singularity of $f(z) - P((z-p)^{-1})$.
- The principle part of f at p is the unique rational function R with a single pole, p, such that p is a removable singularity of $f(z) - R(z)$ and $\lim_{z\to\infty} R(z) = 0$.
- If f has order m at p then f is locally m-to-1 near p (consider $1/f$'s zeroes). The the **degree** of f at a pole p is equal to $\operatorname{ord}(f,p)$, denoted $\deg(f,p)$.

3.2 The Riemann Sphere

- f is **meromorphic** in the open set $U \subseteq \mathbb{C}$, written $f \in \mathcal{M}(U)$, if there exists $E \subset U$ without any accumulation points in U such that $f \in \mathcal{O}(U \setminus E)$ and E consists of poles of f.
- $\mathcal{O}(U) \subset \mathcal{M}(U)$.
- $f, g \in \mathcal{O}(U), g \not\equiv 0$ then $f/g \in \mathcal{M}(U)$. Converse given later: $h \in \mathcal{M}(U) \implies h = f/g,\ f, g \in \mathcal{O}(U)$.
- If U is a domain then $\mathcal{M}(U)$ is a field under pointwise arithmetic. $\mathcal{M}(U)$ is the quotient field of $\mathcal{O}(U)$.
- $f \in \mathcal{M}(U)$ implies $f' \in \mathcal{M}(U)$.

- The **Riemann sphere** $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the one-point compactification of $\mathbb{C}$; it's topology is generated by open disks in $\mathbb{C}$ along with sets of the form $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$.
- The **stereographic projection** is an explicit homeomorphism between the two dimensional sphere $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|x\| = 1\}$ and $\hat{\mathbb{C}}$. The projection is given by intersecting $\mathbb{S}^2$'s equator with the plane and sending the north pole $(0,0,1)$ to ∞ and any other point $p \in \mathbb{S}^2$ to the point $q \in \mathbb{C}$ that is colinear with p and the north pole. Explicitly:

$$\varphi(x_1, x_2, x_3) = \begin{cases} \dfrac{x_1 + ix_2}{1 - x_3} & x_3 \neq 1 \\ \infty & x_3 = 1, \end{cases}$$

$$\varphi^{-1}(z) = \begin{cases} \left(\dfrac{z + \overline{z}}{|z|^2 + 1}, \dfrac{1}{i} \dfrac{z - \overline{z}}{|z|^2 + 1}, \dfrac{|z|^2 - 1}{|z|^2 + 1} \right) & z \neq \infty \\ (0,0,1) & z = \infty \end{cases}.$$

- Several maps from $\mathbb{S}^2$ to itself correspond to interesting maps in $\hat{\mathbb{C}}$ under the stereographic projection:
  - The $180°$ rotation $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$ rotates $\hat{\mathbb{C}}$ by fixing 0 and ∞ and swaps ± 1.
  - The $180°$ rotation $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$ yields $1/z$, which fixes ± 1 and swaps 0 and ∞.
  - The reflection $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$ gives conjugation, $z \mapsto \overline{z}$.
- Let f be a continuous map of (some open subset of) $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. **holomorphic** in $\hat{\mathbb{C}}$ requires us to consider:
  1. $f(p) = \infty$ for $p \neq \infty$: f is holomorphic at p if

     $$g(z) = \begin{cases} 1/f(z) & z \neq p \\ 0 & z = p \end{cases}$$

     is holomorphic near p.
  2. $f(\infty) \neq \infty$: f is holomorphic at ∞ if

     $$g(z) = \begin{cases} f(1/z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

     is holomorphic near 0.
  3. $f(\infty) = \infty$: f is holomorphic at ∞ if

     $$g(z) = \begin{cases} 1/f(1/z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

     is holomorphic near 0.
  4. Otherwise examine f in the usual sense in $\mathbb{C}$.

3.3 Laurent Series

- **Laurent Series.**: Let U be an open annulus about $p \in \mathbb{C}$ with inner and outer radii $R_1, R_2 \in [0, \infty]$, respectively. Then $f \in \mathcal{O}(U)$ has a power series, called the **Laurent Series**, in U of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n \quad z \in U$$

where

$$a_n = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} d\zeta \quad R_1 < r < R_2.$$

The series converges for every open annulus of radii a, b with $R_1 < a < b < R_2$.

3.4 Residues

- Let p be an isolated singularity of f. The **residue** of f at p is

$$\operatorname{res}(f, p) = \frac{1}{2\pi i} \int_{\mathbb{T}(p,r)} f(z) dz$$

where $r > 0$ and f is holomorphic in a neighborhood of $\overline{\mathbb{D}}(p, r) \setminus \{p\}$.

- **Computing the residue.** If f has the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z-p)^n$ near p, then $\operatorname{res}(f, p) = a_{-1}$.

- If f has a pole of order m at p and Laurent series $\sum_{n=-\infty}^{\infty} b_n (z-p)^n$ there then

$$\operatorname{res}(f, p) = b_{m-1} = \frac{1}{(m-1)!} \lim_{z \to p} [(z-p)^m f(z)]^{(m-1)}$$

.

- **Residue theorem.** Suppose $E \subset U \subseteq \mathbb{C}$ does not accumulate in U, open, and $f \in \mathcal{O}(\mathbb{U} \setminus E)$. If $\gamma \sim 0$ in U with $|\gamma| \cap E = \emptyset$ then

$$\int_\gamma f(z) dz = 2\pi i \sum_{p \in E} W(\gamma, p) \operatorname{res}(f, p).$$

- **Residue theorem inside a Jordan curve.** Let γ be a positively oriented Jordan curve inside a domain $U \subseteq \mathbb{C}$ with $\{p_1, \ldots, p_n\} \subset \operatorname{int}(\gamma) \subset U$. If $f \in \mathcal{O}(U \setminus \{p_1, \ldots, p_n\})$ then

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f, p_k).$$

- If $f \in \mathcal{O}(\{z \in \mathbb{C} : |z| > R\})$, the **residue of f at infinity** is, for $r > R$,

$$\operatorname{res}(f, \infty) = -\frac{1}{2\pi i} \int_{\mathbb{T}(0,r)} f(z) dz.$$