Contents

1 Abstract Integration

The Concept of Measurability

1.2 Definition

- $\tau \subseteq \mathscr{P}(X)$, containing both $\emptyset$ and X, is a **topology** if it is closed under finite intersections and arbitrary unions.
- (X, τ) is a **topological space** and the members of τ are **open sets**.
- $f : (X, \tau_X) \to (Y, \tau_Y)$ is **continuous** if open sets have open preimages.

1.3 Definition

- $\mathfrak{M} \subseteq \mathscr{P}(X)$, containing X, is a **σ-algebra** if it is closed under complementation and countable unions.
- $(X, \mathfrak{M})$ is a **measurable space**; elements of $\mathfrak{M}$ are **measurable sets**.
- $f : (X, \mathfrak{M}) \to (Y, \tau)$ is **measurable** if open sets have measurable preimages (τ a topology).

Note: Instead of $(X, \mathfrak{M})$ we just refer to X as the measurable space.

1.6 Comments on Definition 1.3

- $\emptyset \in \mathfrak{M}$.
- Finite unions are in $\mathfrak{M}$.
- $\mathfrak{M}$ is closed under finite and countable intersection.
- $\mathfrak{M}$ is closed under set subtraction.

1.7 Composition with Continuous Functions

- f measurable, g continuous: $g \circ f$ is measurable.

1.8 Continuous Image of Cartesian Product of Measurable Functions.

- u, v real, measurable functions; ϕ continuous on the plane: $\phi(u(x), v(x))$ is measurable.

1.9 Creating Measurable Functions

- If u, v are real measurable then $f = u + iv$ is complex measurable.
- If $f = u + iv$ is complex measurable then u, v, and $|f|$ are real measurable.
- If f and g are complex measurable then so are $f + g$ and fg.
- Characteristic functions of measurable sets are measurable functions.
- If f is complex measurable then there is a complex measurable function α with $|\alpha| = 1$ and $f = \alpha|f|$.

1.10 σ-Algebra Generated by a Set

- $\mathscr{F} \subseteq \mathscr{P}(X)$ is contained in some smallest σ-algebra $\mathfrak{M}^*$.

1.11 Borel Sets

- The **Borel Sets**, $\mathfrak{B}$, is the σ-algebra generated by the topology of a space.
- G_δ sets are countable intersections of open sets.
- F_σ sets are countable unions of closed sets.
- Borel measurable functions are called **Borel mappings** or **Borel functions**.
- *Every* continuous function is Borel measurable.

1.12 σ-Algebras Associated with a Function

$\mathfrak{M}$ a σ-algebra on X, Y a topological space, $f : X \to Y$ a function:

- $\Omega = \{E \subseteq Y : f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra on Y.
- If f is measurable, E Borel in Y, then $f^{-1}(E) \in \mathfrak{M}$.
- If $Y = [-\infty, \infty]$ and $f^{-1}((a, \infty]) \in \mathfrak{M}$ for all $\alpha \in \mathbb{R}$ then f is measurable.
- If f is measurable, Z a topological space, $g : Y \to Z$ Borel, then $g \circ f : X \to Z$ is measurable.

1.14 Supremum and Limit Supremum of Measurable Functions

- If $f_n : X \to [-\infty, \infty]$ are measurable then so are $\sup f_n$ and $\limsup f_n$.
- The limit of pointwise convergent sequence of complex measurable functions is measurable.
- f, g measurable then so are $\max\{f, g\}$ and $\min\{f, g\}$.

1.15 Positive and Negative parts of f

- $f^+ = \max\{f, 0\}$ is the **positive part** of f and $f^- = -\min\{f, 0\}$ is the **negative part**.
- $|f| = f^+ + f^-$ and $f = f^+ - f^-$.
- If $f = g - h$, $g \geq 0$ and $h \geq 0$ then $f^+ \leq g$ and $f^- \leq h$.

Simple Functions

1.16 Definition

- s, complex measurable on X, is **simple** if its range is finite. If $s(X) = \{\alpha_1, \ldots, \alpha_n\}$ then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad A_i = s^{-1}(\alpha_i).$$

- s is measurable if and only if each A_i is.

1.17 Approximation by Simple Functions

- If $f : X \to [0, \infty]$ is measurable then there exists measurable, simple functions s_n on X such that $0 \leq s_1 \leq \ldots \leq f$ and $s_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$.

Elementary Properties of Measures

1.18 Definition

- A **positive measure** is a function from a σ-algebra $\mathfrak{M}$ to $[0, \infty]$ which is **countably additive**: i.e.

$$\mu\left(\bigcup_{n=1}^{\infty} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

  when A_i are pairwise disjoint members of $\mathfrak{M}$.

- A measurable space equipped with a measure is a **measure space**.
- A **complex measure** is a complex-valued countably additive function on a σ-algebra.

1.19 Basic Properties of a Positive Measure μ

- $\mu(\emptyset) = 0$.
- $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if the A_i are pairwise disjoint members of $\mathfrak{M}$.
- $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for $A, B \in \mathfrak{M}$.
- If $A_n \in \mathfrak{M}$ such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ then $\mu(A_n) \to \mu\left(\cup_{n=1}^{\infty} A_n\right)$.
- If $A_n \in \mathfrak{M}$ such that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and $\mu(A_1) < \infty$ then $\mu(A_n) \to \mu\left(\cap_{n=1}^{\infty} A_n\right)$.

1.20 Measure Space Examples

- **counting measure**: $\mu(E) = |E|$ if $|E| < \infty$ and $\mu(E) = \infty$ otherwise.
- **unit mass at** x_0: $\mu(E) = 1$ if $x_0 \in E$ and $\mu(E) = 0$ otherwise.

Arithmetic in $[0, \infty]$

1.22 Definition

- $a + \infty = \infty + a = \infty$
- $a \cdot \infty = \infty \cdot a = \begin{cases} \infty & a \in (0, \infty] \\ 0 & a = 0 \end{cases}$
- With $0 \cdot \infty = 0$ we have commutativity, associativity, and distributivity.
- Cancellation: $a + b = a + c \implies b = c$ only if $a \neq \infty$; $ab = ac \implies b = c$ only if $a \in (0, \infty)$.
- $0 \leq a_1 \leq a_2 \leq \cdots$, $0 \leq b_1 \leq b_2 \leq \cdots$ with $a_n \to a$ and $b_n \to b \implies a_n b_n \to ab$.

Integration of Positive Functions on $(X, \mathfrak{M}, \mu)$

1.23 Definition

- $s : X \to [0, \infty]$ simple and measurable with $s(X) = \{\alpha_1, \ldots, \alpha_n\}$. For $E \in \mathfrak{M}$ define

$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E), \quad A_i = s^{-1}(\alpha_i).$$

- If $f : X \to [0, \infty]$ is measurable then for $E \in \mathfrak{M}$ define the **Lebesgue Integral of f over E** by
$$\int_E f\,d\mu = \sup \int_E s\,d\mu,$$
where the supremum is taken over all nonnegative measurable simple functions dominated by f.

1.24 Basic Properties of Lebesgue Integrals

- $0 \le f \le g$ implies $\int_E f\,d\mu \le \int_E g\,d\mu$.
- $A \subseteq B$ and $f \ge 0$ implies $\int_A f\,d\mu \le \int_B f\,d\mu$.
- If $f \ge 0$ and $c \in [0, \infty)$ then $\int_E cf\,d\mu = c\int_E f\,d\mu$.
- If $f \equiv 0$ on E then $\int_E f\,d\mu = 0$ even if $\mu(E) = \infty$.
- If $\mu(E) = 0$ then $\int_E f\,d\mu = 0$ if if $f \equiv \infty$ on E.
- If $f \ge 0$ then $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$.

1.25 Basic Properties of the Lebesgue Integral of Simple Functions

- If s is a nonnegative measurable simple function then $\varphi : \mathfrak{M} \to [0, \infty]$ sending E to $\int_E s\,d\mu$ is a measure.
- If s and t are nonnegative measurable simple functions then $\int_X (s + t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$.

1.26 Lebesgue's Monotone Convergence Theorem

- If $f_n : X \to [0, \infty]$ is a (pointwise) non-decreasing sequence of measurable functions for which $f_n(x) \to f(x)$ for every $x \in X$ then f is measurable and
$$\int_X f_n\,d\mu \to \int_X f\,d\mu.$$

1.27 Interchange of Summation and Integration

- If $f_n : X \to [0, \infty]$ are measurable and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ then
$$\int_X f\,d\mu = \sum_{n=1}^{\infty} f_n\,d\mu.$$

1.28 Fatou's Lemma

- If $f_n : X \to [0, \infty]$ are measurable then
$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_X f\,d\mu.$$

1.29 Change of Measure

- If $f : X \to [0, \infty]$ is measurable then $\varphi : \mathfrak{M} \to [0, \infty]$ sending E to $\int_E f\,d\mu$ is a measure and
$$\int_X g\,d\varphi = \int_X gf\,d\mu.$$
Sometimes this is written as $d\varphi = f\,d\mu$, although no independent meaning is given to these symbols.

Integration of Complex Functions on $(X, \mathfrak{M}, \mu)$

1.30 Definition

- The **Lebesgue Integrable Functions** or **Summable Functions** with respect to μ, denoted by $L^1(\mu)$ is the collection of all complex measurable functions f on X such that $\int_X |f|\, d\mu < \infty$.

1.31 Definition

- If $f = u + iv$ with u, v real measurable functions and $f \in L^1(\mu)$ then for $E \in \mathfrak{M}$:

$$\int_E f\, d\mu = \left(\int_E u^+\, d\mu - \int_E u^-\, d\mu\right) + i\left(\int_E v^+\, d\mu - \int_E v^-\, d\mu\right).$$

- It is useful define the integral of a function $f : X \to [-\infty, \infty]$ to be

$$\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu$$

for $E \in \mathfrak{M}$ and provided only one term on the right is infinite.

1.32 Linearity of $L^1(\mu)$

- For $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$ we have $\alpha f + \beta g \in L^1(\mu)$ and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

1.33 Interchange of Modulus and Integration

- $\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu$ for $f \in L^1(\mu)$.

1.34 Lebesgue's Dominated Convergence Theorem

- f_n are complex measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. If

$$|f_n(x)| \leq g(x), \quad \text{for all } n \in \mathbb{N}$$

for some $g \in L^1(\mu)$ then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

The Role Played by Sets of Measure Zero

1.35 Definition

- If μ is a measure on a σ-algebra $\mathfrak{M}$, $E \in \mathfrak{M}$, then a statement P holds **almost everywhere** (a.e.) on E if there exists $N \subseteq E$ with $\mu(N) = 0$ such that P is true on $E \setminus N$.

Example 1. If f and g are measurable and $\mu(\{x : f(x) \neq g(x)\}) = 0$ then $f = g$ a.e., written as $f \sim g$. $\sim$ is an equivalence relation and if $f \sim g$ then for $E \in \mathfrak{M}$ we have $\int_E f\, d\mu = \int_E g\, d\mu$. Thus sets of measure zero are negligible with respect to integration.

Note: It is *not* the case that a subset of a negligible set is negligible as it may not even be measurable!

1.36 Existence of Completions

- If $(X, \mathfrak{M}, \mu)$ is a measure space, then define $\mathfrak{M}^*$ to be all $E \subseteq X$ such that $A \subseteq E \subseteq B$ for $A, B \in \mathfrak{M}$ such that $\mu(B \setminus A) = 0$. Defining $\mu(E) = \mu(A)$ makes $(X, \mathfrak{M}^*, \mu)$ a measure space,
- The extended μ is **complete** as all subsets of negligible sets are measurable.
- $\mathfrak{M}^*$ is the **μ-completion** of $\mathfrak{M}$.

1.37 Expanding the Definition of What is a Measurable Function

- Since integration is agnostic to functions equal a.e., we now call f defined on $E \in \mathfrak{M}$ **measurable on** X if $\mu(E^c) = 0$ and $f^{-1}(V) \cap E$ is measurable for every open set V.
- In the above, we can define $f \equiv 0$ on E^c to get a measurable function on X.

1.38 Lebesgue's Dominated Convergence Theorem with Negligible Sets

- f_n complex measurable functions defined a.e. on X such that
$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$
Then $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for almost all x and $f \in L^1(\mu)$ with
$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

1.39 Integration and Properties That Hold Almost Everywhere

- If $f : X \to [0, \infty]$ measurable and $E \in \mathfrak{M}$ with $\int_E f \, d\mu = 0$ then $f = 0$ a.e. on E.
- If $f \in L^1(\mu)$ with $\int_E f \, d\mu = 0$ for every $E \in \mathfrak{M}$ then $f = 0$ a.e. on X.
- If $f \in L^1(\mu)$ and
$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu$$
then there exists $\alpha \in \mathbb{C}$ such that $\alpha f = |f|$ a.e. on X.

1.40 Averages Lying in a Closed Set

- If $\mu(X) < \infty$, $f \in L^1(\mu)$, $S \subseteq \mathbb{C}$ is closed, and the averages
$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, d\mu$$
lie in S for every $E \in \mathfrak{M}$ with positive measure then $f(x) \in S$ for almost all $x \in X$.

1.41 Finite Set Membership

- If $E_k \subseteq X$ are measurable with $\sum_{k=1}^{\infty} \mu(E_k) < \infty$ then almost all $x \in X$ lie in finitely many E_k.

2 Positive Borel Measures

Vector Spaces

2.1 Definition

- A **complex vector space** is one with complex scalars.
- A function Λ between vector spaces is a **linear transformation** if $\Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$.
- A **linear functional** is a linear transformation where the codomain is the field of scalars of the domain.

2.2 Integration as a Linear Functional

- For any positive measure μ, $f \mapsto \int_X f \, d\mu$ is a linear functional on $L^1(\mu)$.
- If g is a bounded measurable function then $f \mapsto \int_X fg \, d\mu$ is a linear functional on $L^1(\mu)$.
- A **positive linear functional** is a linear functional Λ such that $\Lambda f \geq 0$ whenever $f \geq 0$.
- If C is the vector space of continuous complex functions on $[0, 1]$ then

$$\Lambda f = \int_0^1 f(x) \, dx$$

  is a positive linear functional on C (with the integral being the Riemann integral).

Topological Preliminaries

2.3 Definitions

- E is **closed** if its complement is open.
- The **closure** of E, denoted $\overline{E}$, is the smallest closed set containing E.
- $K \subseteq X$ is **compact** if every open cover of K contains a finite subcover.
- A **neighborhood** of $p \in X$ is any open set containing p.
- X is **Hausdorff** if any two $p \neq q$ can be separated by open sets.
- X is **locally compact** if every points has a neighborhood with compact closure.

Recall: Heine-Borel: Subsets of Euclidean space are compact exactly when they are closed and bounded. Thus $\mathbb{R}^n$ is locally compact.

Recall: Every metric space is Hausdorff.

2.4 Closed Subsets of Compact Sets

- $F \subseteq K$, F closed, K compact. Then F is compact.
- If $A \subseteq B$ and B has compact closure then so does A.

2.5 Separating a Compact Set from a Point

- If X is Hausdorff with K compact in X then any $p \notin K$ can be separated from K by open sets.

2.6 Intersections of Compact Sets

- If $K_\alpha \subseteq X$ are compact, X Hausdorff, and $\cap_\alpha K_\alpha = \emptyset$ then some finite subset has empty intersection.

2.7 Sandwiching Sets

- If U is open in X, Hausdorff, and $K \subseteq U$ is compact then there exists V, open, with $K \subseteq V \subseteq \overline{V} \subseteq U$.

2.8 Definition

- f is **lower semicontinuous** if $f^{-1}((\alpha, \infty])$ is open.
- f is **upper semicontinuous** if $f^{-1}([\infty, \alpha))$ is open.
- χ_U is lower semicontinuous if U is open.
- χ_F is upper semicontinuous if F is closed.
- The supremum of any collection of lower semicontinuous functions is again lower semicontinuous.
- The infimum of any collection of upper semicontinuous functions is again upper semicontinuous.

2.9 Definition

- The **support** of $f : X \to \mathbb{C}$ is the closure of $f^{-1}(\mathbb{C} \setminus \{0\})$.
- $C_c(X)$ is the vector space of functions with compact support.

2.10 Image of a Compact Set

- The continuous image of a compact set is compact.
- The range of $f \in C_c(X)$ is compact subset of $\mathbb{C}$.

2.11 Notation

- $K \prec f$ means K is compact, $0 \leq f(x) \leq 1$ and $f = 1$ on K.
- $f \prec V$ means V is open and f's support lies in V.
- $K \prec f \prec V$ combines the above.

2.12 Urysohn's Lemma

- For $K \subseteq V \subseteq X$ with K compact, V open, and both X locally compact and Hausdorff, there exists $f \in C_c(X)$ with $K \prec f \prec V$.
- In terms of characteristic functions this means there is a continuous f with $\chi_K \leq f \leq \chi_V$.

2.13 Partition of Unity

- For X locally compact and Hausdorff and $V_1, \ldots, V_n$ open in X and $K \subseteq V_1 \cup \cdots \cup V_n$ is compact, there exists functions $h_i \prec V_i$ with $h_1 + \cdots + h_n = 1$ on K.
- This is called a **partition of unity on** K subordinate to the cover $\{V_1, \ldots, V_n\}$.

The Riesz Representation Theorem

2.14 The Riesz Representation Theorem

- For X locally compact and Hausdorff with Λ a positive linear functional on $C_c(X)$ there exists:
  1. a σ-algebra $\mathfrak{M}$ containing all the Borel sets of X;
  2. a *unique* positive measure μ on $\mathfrak{M}$ representing Λ:
     - $\Lambda f = \int_X f \, d\mu$ for all $f \in C_c(X)$
     - $\mu(K) < \infty$ for any compact K
     - $\mu(E) = \inf \{\mu(V) : E \subseteq V,\ V \text{ open}\}$ for all $E \in \mathfrak{M}$
     - $\mu(E) = \sup \{\mu(K) : K \subseteq E,\ K \text{ compact}\}$ for every $E \in \mathfrak{M}$ open or $\mu(E) < \infty$
     - $E \in \mathfrak{M}$, $\mu(E) = 0$ implies every subset of E is in $\mathfrak{M}$.

Regularity Properties of Borel Measures

2.15 Definition

- μ is a **Borel measure** if it is defined on the Borel sets.

  Let μ be a positive Borel measure and $E \subseteq X$ be Borel. Then:
- E is **outer regular** if the infimum property in 2.14 holds.
- E is **inner regular** if the supremum property in 2.14 holds.
- E is **regular** if both hold.

2.16 Definition

- E is **σ-compact** if it is the countable union of compact sets.
- E is **σ-finite** if it is the countable union of sets with finite measure.

2.17 Regularity of σ-Compact Spaces

- X a locally compact, σ-compact Hausdorff space and $\mathfrak{M}, \mu$ are as in 2.14. Then:
  - For $E \in \mathfrak{M}$, $\epsilon > 0$, there is $F \subseteq E \subseteq V$ with F closed, V open, and $\mu(V \setminus F) < \epsilon$.
  - μ is a regular Borel measure on X.
  - There exists an $F_\sigma A$ and a $G_\delta B$ with $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

    Thus, every $E \in \mathfrak{M}$ is the union of an F_σ and a negligible set.

2.18 Regularity in the Presence of σ-Compact Open Sets

- X a locally compact Hausdorff space in which every open set is σ-compact. Then any positive Borel measure that is finite on compact sets is regular.

Lebesgue Measure

2.19 Euclidean Spaces

- $\mathbb{R}^k$ is the **k-dimension Euclidean space** with all the familiar operations.
- If $E \subseteq \mathbb{R}^k$ and $x \in \mathbb{R}^k$ then $E + x = \{y + x : y \in E\}$ is a **translate** of E.
- A **k-cell** is a set of the form $\{(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k : \alpha_i < \xi_i < \beta_i, 1 \leq i \leq k\}$. Either inequality may be replaced with $\leq$. The **volume** of a k-cell is $\mathrm{vol}\,(W) = \prod_{i=1}^k (\beta_i - \alpha_i)$.
- If $a \in \mathbb{R}^k$ and $\delta > 0$ then a **δ-box with corner at** a is

$$Q(a, \delta) = \{(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k : \alpha_i \leq \xi_i < \alpha_i + \delta, 1 \leq i \leq k\}.$$

- If P_n are points whose coordinates are multiples of 2^{-n} and Ω_n are the 2^{-n} boxes with corners at the elements of P_n then we use the following properties:
  - Ω_n covers $\mathbb{R}^k$ disjointly.
  - If $r < n$ and $Q' \in \Omega_n$, $Q'' \in \Omega_r$ then either $Q' \subseteq Q''$ or $Q' \cap Q'' = \emptyset$.
  - vol $Q = 2^{-rk}$ for $Q \in \Omega_r$ and if $n > r$ then $|P_n \cap Q| = 2^{(n-r)k}$.
  - Any non-empty open set is the countable disjoint union of elements of $\cup_{n=1}^{\infty} Q_n$.

2.20 Existence of the Lebesgue Measure

- There exists $(\mathbb{R}^k, \mathfrak{M}, m)$ such that
  - $m(W) = \mathrm{vol}\,(W)$ for every k-cell W.
  - $\mathfrak{M}$ contains the Borel sets of $\mathbb{R}^k$
  - $E \in \mathfrak{M}$ iff $A \subseteq E \subseteq B$ with A is F_σ, B is G_δ, and $m(B \setminus A) = 0$.
  - m is regular.
  - $m(x + E) = m(E)$ for all $E \in \mathfrak{M}$ and $x \in \mathbb{R}^k$.
  - If μ is any positive translation-invariant Borel measure on $\mathbb{R}^k$ which is finite on compact sets then $\mu(E) = cm(E)$ for some $c \in \mathbb{R}$ and all Borel sets E.
  - $m(T(E)) = \Delta(T)m(E)$, $\Delta(T) \in \mathbb{R}$, for every linear transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ and $E \in \mathfrak{M}$. More specifically, $\Delta(T) = 1$ if T is a rotation.
- Elements of $\mathfrak{M}$ are **Lebesgue measurable** sets and m is the **Lebesgue measure** on $\mathbb{R}^k$.

2.21 Remarks

- If m is the Lebesgue measure on $\mathbb{R}^k$ we write $L^1(\mathbb{R}^k)$ instead of $L^1(m)$.
- Instead of $f \in L^1$ on E we write $f \in L^1(E)$ (in the measure space with m restricted to subsets of E).
- If I is an interval in $\mathbb{R}$ and $f \in L^1(I)$ we write $\int_a^b f(x)\,dx$ instead of $\int_I f\,dm$.
- If f is continuous on $[a, b]$ then the Riemann and Lebesgue integrals agree.
- Most sets are *not* Borel sets.

2.22 Sufficient Condition for Measure Zero

- If $A \subseteq \mathbb{R}$ and every subset of A is Lebesgue measurable then $m(A) = 0$.
- Every set of positive measure has unmeasurable subsets.

2.23 Determinants

- The $\Delta(T)$ in 2.20 is $|\det T|$.

Continuity Properties of Measurable Functions

We assume in this section that μ is a measure on a locally compact Hausdorff space with the properties listed in 2.14 – μ could be the Lebesgue measure on some $\mathbb{R}^k$.

2.24 Lusin's Theorem

- f complex measurable, $\mu(A) < \infty$, and $f = 0$ outside A. Then for $\epsilon > 0$ there exists $g \in C_c(X)$ with

  $$\mu(\{x : f(x) \neq g(x)\}) < \epsilon.$$

  We may pick g so that

  $$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$

- If $|f| \leq 1$ then there is a sequence $g_n \in C_c(X)$, $|g_n| \leq 1$, with

  $$f(x) = \lim_{n \to \infty} g_n(x) \quad \text{a.e.}$$

2.25 Vitali-Carathéodory Theorem

- If $f \in L^1(\mu)$ is real valued and $\epsilon > 0$ then there exists u, upper semicontinuous and bounded from above, and v, lower semicontinuous and bounded from below, such that $u \leq f \leq v$ and $\int_X (v - u)\, d\mu < \epsilon$.

3 L^p-Spaces

Convex Functions and Inequalities

3.1 Definition

- φ is **convex** on (a, b) if $x, y \in (a, b)$ and $\lambda \in [0, 1]$ imply
$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y).$$
That is, the segment between $(x, \varphi(x))$ and $(y, \varphi(y))$ lies above the graph of φ.
- The above is equivalent to $a < s < t < u < b$ implying
$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Note: The mean value theorem for differentiation with the above imply that φ, real differentiable, is convex in (a, b) iff $a < s < t < b$ implies $\varphi'(s) \leq \varphi'(t)$.

3.2 Convexity Implies Continuity

- If φ is convex on (a, b) then φ is continuous on (a, b).

Note: This relies on the fact that we are working on an *open* segment.

3.3 Jensen's Inequality

- $\mathfrak{M}$ a σ-algebra on Ω, μ a positive measure on it such that $\mu(\Omega) = 1$. If $f \in L^1(\mu)$ is real with $f(\Omega) \subseteq (a, b)$ and φ is convex on (a, b) then
$$\varphi\left(\int_\Omega f\, d\mu\right) \leq \int_\Omega (\varphi \circ f)\, d\mu.$$

Note: $a = -\infty$ or $b = \infty$ are not excluded values.

Note: If $\varphi \circ f \notin L^1(\mu)$ then the integral has value $+\infty$ (see 1.31).

Example 1. For $\varphi(x) = e^x$ we get
$$\exp\left\{\int_\Omega f\, d\mu\right\} \leq \int_\Omega e^f\, d\mu.$$

Example 2. If $\Omega = \{p_1, \ldots, p_n\}$ and $\mu(\{p_i\}) = 1/n$, $f(p_i) = x_i$ then the example 1 becomes:
$$\exp\left\{\frac{1}{n}\sum_{i=1}^n x_i\right\} \leq \frac{1}{n}\sum_{i=1}^n e^{x_i}$$
for real x_i. Setting $y_i = e^{x_i}$ we can relate the arithmetic and geometric means of n positive numbers:
$$\left(\prod_{i=1}^n y_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n y_i.$$
Given this, it is clear why
$$\exp\left\{\int_\Omega \log g\, d\mu\right\} \leq \int_\Omega g\, d\mu$$
are called the arithmetic and geometric means of the positive function g.

Example 3. If $\mu(\{p_i\}) = \alpha_i > 0$ with $\sum \alpha_i = 1$ then we get a more general version of the above:

$$\prod_{i=1}^{n} y_i^{\alpha_i} \le \sum_{i=1}^{n} \alpha_i y_i.$$

3.4 Definition

- $p, q \in (1, \infty)$ are **conjugate exponents** if $p + q = pq$ (or, equivalently, $p^{-1} + q^{-1} = 1$).
- $p \to 1$ forces $q \to \infty$ and so 1 and ∞ are regarded as conjugate exponents.
- Many denote p's conjugate exponent by p'.

3.5 Hölder and Minkowski's Inequalities

p and q are conjugate exponents with $p \in (1, \infty)$ and f and g are measurable with range in $[0, \infty]$:

- Hölder's inequality:

$$\int_X fg\, d\mu \le \left\{ \int_X f^p\, d\mu \right\}^{1/p} \left\{ \int_X g^q\, d\mu \right\}^{1/q}.$$

  If $p = q = 2$ then this is called Schwarz's inequality.

- Minkowski's inequality:

$$\left\{ \int_X (f+g)^p\, d\mu \right\}^{1/p} \le \left\{ \int_X f^p\, d\mu \right\}^{1/p} + \left\{ \int_X g^p\, d\mu \right\}^{1/p}.$$

Note: Assuming the right hand side of Hölder's inequality has only finite factors, equality holds if and only if there are constants α and β, not both zero, such that $\alpha f^p = \beta g^q$ a.e.

The L^p-spaces

For this section, let X be arbitrary and μ a positive measure.

3.6 Definition

- If $0 \le p \le \infty$ and f is a complex measurable function, then the **L^p-norm** of f is

$$\|f\|_p = \left\{ \int_X |f|^p\, d\mu \right\}^{1/p}.$$

  $L^p(\mu)$ is the collection of all f for which $\|f\|_p < \infty$ and is called the **L^p-space** of X.

- If u is the Lebesgue measure on $\mathbb{R}^k$ then we write $L^p(\mathbb{R}^k)$ instead of $L^p(\mu)$.
- If μ is the counting measure on a countable set A we denote the L^p-space by $\ell^p(A)$ or just ℓ^p. $x \in \ell^p$ is a sequence $x = \{\xi_n\}$ and

$$\|x\|_p = \left\{ \sum_{n=1}^{\infty} |\xi|^p \right\}^{1/p}.$$

3.7 Definition

- For $g : X \to [0, \infty]$ measurable, let S be the set such that $\mu(g^{-1}((\alpha, \infty])) = 0$. If $S = \emptyset$ then set $\beta = \infty$, else $\beta = \inf S$. Since the countable union of sets of measure zero is a set of measure zero and

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right),$$

  $\beta \in S$. β is the **essential supremum** of g.

- If f is a complex measurable function then $\|f\|_\infty$ is the essential supremum of $|f|$. $L^\infty(\mu)$ is the set of all f with $\|f\|_\infty < \infty$, it's members called the **essentially bounded** measurable functions on X.
- $L^\infty(\mathbb{R}^k)$ is the class of Lebesgue measure essentially bounded functions on $\mathbb{R}^k$.
- $\ell^\infty(A)$ is the class of bounded functions on A.

Note: $|f(x)| \le \lambda$ holds almost everywhere iff $\lambda \ge \|f\|_\infty$.

3.8 Hölder's Inequality With L^p-norms

- If p and q are conjugate exponents, $1 \le p \le \infty$, with $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$.

3.9 Minkowski's Inequality With $L - p$-norms

- If $1 \le p \le \infty$ and $f, g \in L^p(\mu)$ then $f + g \in L^p(\mu)$ and $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

3.10 Remarks

- $L^p(\mu)$ is a complex vector space.
- Triangle inequality holds: $\|f - h\|_p \le \|f - g\|_p + \|g - h\|_p$.

Note: If $f \sim g$ (see 1.35) then $\|f - g\|_p = 0$.

- $L^p(\mu)$ is a complete metric space if we pass to equivalence classes under $\sim$.

3.11 Completeness of $L^p(\mu)$

- $L^p(\mu)$ is complete for every $1 \le p \le \infty$ and every positive measure μ.

3.12 Pointwise Convergence of Cauchy Subsequences

- For $1 \le p \le \infty$ and $f_n \to f$ Cauchy in $L^p(\mu)$, $\{f_n\}$ has a subsequence converging pointwise to f a.e.

3.13 Density of (some) Simple Functions

- For $1 \le p < \infty$, the set of all complex, measurable, simple functions s with $\mu(\{x : s(x) \neq 0\}) < \infty$ is dense in $L^p(\mu)$.

Approximation by Continuous Functions

For this section X is locally compact and Hausdorff, μ a measure on σ-algebra with the features in 2.14.

3.14 Density of $C_c(X)$

- $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$.

3.15 Remarks

- $C_c(\mathbb{R}^k)$ has a metric that does not need to pass to equivalence classes.
- Likewise, the essential supremum there is the same as the supremum: $\|f\|_\infty = \sup_{x\in\mathbb{R}^k} |f(x)|$.
- If $1 \leq p < \infty$ then 3.14 gives $C_c(\mathbb{R}^k)$ is dense in $L^p(\mathbb{R}^k)$, which is complete by 3.11. $L^p(\mathbb{R}^k)$ is the completion of $C_c(\mathbb{R}^k)$ with respect to the $L^p(\mathbb{R}^k)$ metric.

Note: Keep in mind that we are having *different* completions of the same set under different metrics.

- If the distance between $f, g \in C_c(\mathbb{R}^1)$ is given by $\int_{-\infty}^{\infty} |f(t)-g(t)|\, dt$ then the completion of the resulting metric space is the space of equivalence classes (under $\sim$) of Lebesgue integrable functions.

Note: Important that the completion of functions on $\mathbb{R}^k$ are again functions on $\mathbb{R}^k$.

- The L^∞-completion is $C_0(\mathbb{R}^k)$ of functions which vanish at infinity (see below).

Reminder: For 3.16 and 3.17, please remember in this section that X is **locally compact** and **Hausdorff**.

3.16 Definition

- The complex function f **vanishes at infinity** if for $\epsilon > 0$ there is a K, compact, with $|f| < \epsilon$ on K^c.
- $C_0(X)$ is the class of all continuous functions f on X which vanish at infinity.
- $C_c(X) \subseteq C_0(X)$ with equality when X is compact, in which case $C(X)$ is used for either.

3.17 Completion of $C_c(X)$

- $C_0(X)$ is the completion of $C_c(X)$ relative to the supremum norm metric: $\|f\| = \sup_{x\in X} |f(x)|$.

4 Elementary Hilber Space Theory

Inner Products and Linear Functionals

4.1 Definition

- An **inner product** on a complex vector space H is a function $(\cdot,\cdot) : H \times H \to \mathbb{C}$:
  1. $(y, x) = \overline{(x, y)}$
  2. $(x + y, z) = (x, z) + (y, z)$
  3. $(\alpha x, y) = \alpha(x, y)$
  4. $(x, x) \geq 0$
  5. $(x, x) = 0$ only when $x = 0$
- Implications:
  - $(0, y) = 0$ for all $y \in H$.
  - $x \mapsto (x, y)$ is a linear functional for all $y \in H$.
  - $(x, \alpha y) = \overline{\alpha}(x, y)$
  - $(z, x + y) = (z, x) + (z, y)$
- $\|x\|^2 = (x, x)$ is the **norm** of x.
- An **inner product space** (aka **unitary space**) is a vector space with an inner product.

4.2 Schwarz Inequality

- $|(x, y)| \leq \|x\|\|y\|$.

4.3 Triangle Inequality

- $\|x + y\| \leq \|x\| + \|y\|$.

4.4 Definition

- Define the distance between x and y to be $\|x - y\|$.
- Since $\|x - z\| \leq \|x - y\| + \|y - z\|$, H is a metric space.
- If H is a complete metric space, then H is a **Hilbert space**.

Note: Throughout the rest of the chapter, H denotes a Hilbert space.

4.5 Examples

Example 1. $\mathbb{C}^n$ is a Hilbert space (scalar multiplication and vector addition defined component wise). Then

$$(x, y) = \sum_{j=1}^{n} \xi_j \overline{\eta}_j$$

where ξ_j and η_j are the components of x and y.

Example 2. $L^2(\mu)$ is a Hilbert space for a positive measure μ, with

$$(f, g) = \int_X f\overline{g}\, d\mu.$$

Note that

$$\|f\| = (f, f)^{\frac{1}{2}} = \left\{\int_X |f|^2\, d\mu\right\}^{\frac{1}{2}} = \|f\|_2.$$

Example 3. The continuous complex functions on $[0, 1]$ with

$$(f, g) = \int_0^1 f(t)\overline{g(t)}\, dt$$

is an inner product space but *not* a Hilbert space.

4.6 Continuity of the Inner Product

- For $y \in H$ the following are continuous functions on H:

$$x \mapsto (x, y); \quad x \mapsto (y, x); \quad x \mapsto \|x\|.$$

4.7 Subspaces

- $M \subseteq V$, V a vector space, is a **subspace** if M is also a vector space.
- A **closed subspace** of H is a subspace that is a closed set in the metric topology of H.
- If M is a subspace of H then so is it's closure $\overline{M}$.

4.8 Convex Sets

- $E \subseteq V$ is **convex** if $(1-t)x + ty \in E$ for all $x, y \in E$, and $0 < t < 1$.
- E is convex if it contains every segment between any two of its points.
- Every subspace of a vector space is complex, and every translate of a convex set is convex.

4.9 Orthogonality

- x is **orthogonal** to y if $(x, y) = 0 = (y, x)$, written $x \perp y$.
- $x^\perp = \{y \in H : x \perp y\}$ is the closed subspace of H here $y \mapsto (x, y)$ is identically zero.
- If M is a subspace of H then

$$M^\perp = \bigcap_{x \in M} x^\perp$$

is the closed subspace of H of vectors orthogonal to *every* vector in M.

4.10 Unique Element of Minimal Norm

- If $E \subseteq H$ is nonempty, closed, and convex then there is a unique element of minimal norm.

4.11 Orthogonal Decompositions

M is a closed subspace of H:

- Every $x \in H$ decomposes uniquely into a sum $Px + Qx$, $Px \in M$, $Qx \in M^\perp$.
- Px and Qx are the points in M and$M^\perp$ nearest to x.
- $P : H \to M$ and $Q : H \to M^\perp$ are linear.
- $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.
- If $M \neq H$ then there exists $y \in H, y \neq 0$ with $y \perp M$.

4.12 Unique Representation of Continuous Linear Functionals

- If L is a continuous linear functional on H then there is a unique $y \in H$ with $Lx = (x, y)$.

Orthonormal Sets

4.13 Definitions

- If $x_1, \ldots, x_k \in V$ and $c_1, \ldots, c_k$ are scalars then $c_1v_1 + \cdots + c_kv_k$ is a **linear combination** of $x_1, \ldots, x_k$.
- $\{x_1, \ldots, x_k\}$ is **independent** if $c_1x_1 + \cdots + c_kx_k = 0$ implies $c_1 = \cdots = c_k = 0$.
- $S \subseteq V$ is **independent** if every finite subset of S is independent.
- $[S]$, the set of all finite linear combinations of S, is the smallest subspace of V containing S.
- $[S]$ is called the **span** of S.
- $\{u_\alpha\}_{\alpha \in A}$ is **orthonormal** if $\|u_\alpha\| = 1$ for all $\alpha \in A$ and $u_\alpha \perp u_\beta$ for $\alpha \neq \beta$.
- If $\{u_\alpha\}_{\alpha \in A}$ is orthonormal then the **Fourier coefficients** of x (relative to $\{u_\alpha\}_{\alpha \in A}$) are the values of

$$\hat{x} : A \to \mathbb{C}, \quad \hat{x}(\alpha) = (x, u_\alpha).$$

4.14 Fourier Coefficients and Approximations in Finite Subspaces

$\{u_\alpha\}_{\alpha \in A}$ are orthonormal in H, $F \subseteq A$ is finite, and M_F is the span of $\{u_\alpha\}_{\alpha \in F}$.

- If $\varphi : A \to \mathbb{C}$ is 0 outside F then

$$y = \sum_{\alpha \in F} \varphi(\alpha) u_\alpha \in M_F$$

has the properties $\hat{y}(\alpha) = \varphi(\alpha)$ and $\|y\|^2 = \sum_{\alpha \in F} |\varphi(\alpha)|^2$.

- $\|x - s_F(x)\| < \|x - s\|$ for every $s \in M_F \setminus \{s_F(x)\}$, where

$$s_F(x) = \sum_{\alpha \in F} \hat{x}(\alpha) u_\alpha.$$

i.e. $s_F(x)$ is the unique best approximation of x in M_F relative to the norm induced metric.

- For $x \in H$,

$$\sum_{\alpha \in F} |\hat{x}(\alpha)|^2 \leq \|x\|^2.$$

4.15 Infinite Sums

- Assume $0 \leq \varphi(\alpha) \leq \infty$ for $\alpha \in A$. Then
$$\sum_{\alpha \in A} \varphi(\alpha)$$
is the supremum of all finite sums $\varphi(\alpha_1) + \cdots + \varphi(\alpha_n)$ (α_k's distinct).
- This is just the Lebesgue integral of φ with respect to the counting measure on A.
- In this context we write $\ell^p(A)$ instead of $L^p(\mu)$. $\varphi : A \to \mathbb{C} \in \ell^p(A)$ if and only if $\sum_{\alpha \in A} |\varphi(\alpha)|^p < \infty$.
- Example 2 of 4.5 shows that $\ell^2(A)$ is a Hilbert space with inner product:
$$(\varphi, \psi) = \sum_{\alpha \in A} \varphi(\alpha)\overline{\psi(\alpha)}$$
where again the sum is the integral of $\varphi\overline{\psi}$ with respect to the counting measure. Note $\varphi\overline{\psi} \in \ell^1(A)$.
- 3.13 shows that functions identically zero outside some finite subset of A are dense in $\ell^p(A)$.
- If $\varphi \in \ell^2(A)$ then $\{\alpha \in A : \varphi(\alpha) \neq 0\}$ is at most countable.

Note: The above uses ℓ^p in some places above where the text uses ℓ^2.

4.16 Isometry Extension

- Suppose that
  1. X and Y re metric spaces with X complete;
  2. $f : X \to Y$ is continuous;
  3. X_0 is dense in X and f is an isometry on X_0; and
  4. $f(X_0)$ is dense in Y.

  Then f is an isometry of X ***onto*** Y.

Recall: An isometry preserves distances.

4.17 The Bessel Inequality & The Riesz-Fischer Theorem

- Let $\{u_\alpha\}_{\alpha \in A}$ be orthonormal in H and P the span of $\{u_\alpha\}_{\alpha \in A}$. Then
$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leq \|x\|^2$$
holds for all $x \in H$. Moreover, $x \mapsto \hat{x}$ is a continuous linear mapping of H onto $\ell^2(A)$ whose restriction to $\overline{P}$ is an isometry onto $\ell^2(A)$.
- The fact that $x \mapsto \hat{x}$ carries H onto $\ell^2(A)$ is the Riesz-Fischer theorem.

4.18 Maximal Orthonormal Set Conditions

- If $\{u_\alpha\}_{\alpha \in A}$ are orthonormal, then the following are equivalent:
  1. $\{u_\alpha\}_{\alpha \in A}$ is a maximal orthonormal set in H.
  2. The span of $\{u_\alpha\}_{\alpha \in A}$ is dense in H.
  3. $\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = \|x\|^2$ holds for all $x \in H$.
  4. $\sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = (x, y)$ holds for all $x, y \in H$.
- The last formula is Parseval's identity. Note $\hat{x}\hat{y} \in \ell^1(A)$ since $\hat{x}, \hat{y} \in \ell^2(A)$.
- A maximal orthonormal set is usually called a **complete orthonormal set** or an **orthonormal basis**.

4.19 Isomorphisms

- Two vector spaces are **isomorphic** if there is a linear bijection between them.
- Two Hilbert spaces are **isomorphic** if there is a linear bijection between them which preserves inner products: $(x, y) = (\Lambda x, \Lambda y)$ for all x, y.
- With the above, we have the following from 4.17 and 4.18:

  If $\{u_\alpha\}_{\alpha \in A}$ is a maximal orthonormal set in H then $x \mapsto \hat{x}$ is an isomorphism between H and $\ell^2(A)$.
- With 4.22 we show that every nontrivial Hilbert space has a maximal orthonormal set and so is isomorphic to some $\ell^2(B)$.

4.20 Partially Ordered Sets

- $\mathcal{P}$ is **partially ordered** by $\leq$ if $\leq$ is reflexive, anti-symmetric, and transitive.
- $\mathcal{L} \subseteq \mathcal{P}$ is **totally ordered** (or **linearly ordered**) if either $a \leq b$ or $b \leq a$ for every $a, b \in \mathcal{L}$.

4.21 Hausdorff Maximality Theorem

- Every non-empty partially ordered set contains a maximal totally ordered subset.

4.22 Existence of Maximal Orthonormal Sets

- Every orthonormal set in a Hilbert space is contained in a maximal orthonormal set.

Trigonometric Series

4.23 Definitions

- The 2π-periodic functions on $\mathbb{R}^1$ can be identified with the functions on the unit circle $T \subseteq \mathbb{C}$.

Note: We can thus switch between writing $f(t)$ and $f(e^{it})$ where $t \in [0, 2\pi]$.

- $L^p(T)$, $1 \leq p < \infty$, consists of all complex, Lebesgue measurable, 2π-periodic functions on $\mathbb{R}^1$ for which

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty.$$

- $L^\infty(T)$ consists of all 2π-periodic members of $L^\infty(\mathbb{R}^1)$ with the essential supremum norm.

- $C(T)$ consists of all continuous complex functions on T with norm $\|f\|_\infty = \sup_{t\in[0,2\pi]} |f(t)|$.
- A **trigonometric polynomial** is a sum of the form $f(t) = \sum_{n=-N}^{N} c_n e^{int}$.
- Define the inner product in $L^2(T)$ by

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt.$$

- $\{u_n\}_{n\in\mathbb{Z}}$, where $u_n(t) = e^{int}$, is an orthonormal set in $L^2(T)$ called the **trigonometric system**.

4.24 Completeness of the Trigonometric System

- If we can prove that $f \in C(T)$ can be approximated by trigonometric polynomials then since $C(T)$ is dense in $L^2(T)$ (see 3.14) the trigonometric system is complete will be complete by 4.18.

4.25 Approximations in $C(T)$ by Trigonometric Polynomials

- If $f \in C(T)$, $\epsilon > 0$, then there exists a trigonometric polynomial P with $|f - P| < \epsilon$.
- By 4.24, the trigonometric system is complete.

4.26 Fourier Series

- For $f \in L^1(T)$ the **Fourier coefficients** of f are defined, for $n \in \mathbb{Z}$, by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}\,dt.$$

- The **Fourier series** of f is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

  with partial sums

$$s_N(t) = \sum_{-N}^{N} \hat{f}(n)e^{int}.$$

- *Riesz-Fischer Theorem* now states if $\{c_n\}$ are complex numbers with $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$ then there exists $f \in L^2(T)$ such that, for $n \in \mathbb{Z}$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}.$$

- *Parseval's theorem* asserts that, for $f, g \in L^2(T)$,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt.$$

- The $L^2(T)$ limit of partial sums s_N of f converges in the $L^2(T)$ norm to f.
- Associating f with $\hat{f}$ is a Hilbert space isomorphism between $L^2(T)$ and $\ell^2(\mathbb{Z})$.

5 Examples of Banach Space Techniques

Banach Spaces

5.2 Definition

- X is a **normed linear space** if for $x \in X$ there is a nonnegative real $\|x\|$ called the **norm** of x such that
  1. $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
  2. $\|\alpha x\| = |\alpha|\|x\|$ for $x \in X$, α scalar;
  3. $\|x\| = 0$ only if $x = 0$.
- By the first property, the **triangle inequality** holds: $\|x-y\| \leq \|x-z\| + \|z-y\|$.
- Every normed linear space is a metric space with distance $d(x,y) = \|x-y\|$.
- A **Banach space** is a complete normed linear space.
- Definition same for *real* Banach spaces (real scalars instead).

Example 1. Every Hilbert space is a Banach space.

Example 2. Every $L^p(\mu)$ space normed by $\|\cdot\|_p$.

Example 3. $C_0(X)$ under the supremum norm.

Example 4. The complex numbers $\mathbb{C}$.

5.3 Definition

- The **operator norm** of a linear transformation $\Lambda : X \to Y$ with X, Y normed linear spaces, is

$$\|\Lambda\| = \sup\{\|\Lambda x\| : x \in X, \|x\| \leq 1\}.$$

- If $\|\Lambda\| < \infty$ then Λ is a **bounded linear transformation**.
- A **unit vector** is a vector x such that $\|x\| = 1$.
- In the definition of $\|\Lambda\|$ we could have considered only unit vectors in X.
- $\|\Lambda\|$ is the smallest number such that $\|\Lambda x\| \leq \|\Lambda\|\|x\|$ holds for all $x \in X$.
- Λ maps the **closed unit ball** $\{x \in X : \|x\| \leq 1\}$ to the closed ball of radius $\|\Lambda\|$ centered at $0 \in Y$.
- If $Y = \mathbb{C}$ and $\|\Lambda\| < \infty$ then Λ is a **bounded linear functional**.

5.4 Relation Between Boundedness and Continuity

- If $\Lambda : X \to Y$ is a linear transformation between normed linear spaces, then the following are equivalent:
  1. Λ is bounded.
  2. Λ is continuous.
  3. Λ is continuous at a point of X.

Consequences of Baire's Theorem

5.6 Baire's Theorem

- In a complete metric space, the countable intersection of open dense sets is dense.
- In a complete metric space, the countable intersection of dense G_δ's is a dense G_δ.

5.7 Cateogry

- Baire's theorem is sometimes called Baire's category theorem.
- $E \subseteq X$ is **nowhere dense** if E's closure has empty interior.
- The countable union of nowhere dense sets is a set of the **first category**.
- Any set *not* of the first category is of the **second category**.
- Baire's theorem says no complete metric space is of the first category.

5.8 Banach-Steinhaus Theorem

- Suppose X is Banach, Y is a normed linear space, and $\{\Lambda_\alpha\}_{\alpha\in A}$ are bounded linear transformations from X into Y. Then either the $\{\Lambda_\alpha\}_{\alpha\in A}$ have a common bound or $\sup_{\alpha\in A}\|\Lambda_\alpha x\| = \infty$ for all x in some dense G_δ of X.
- Geometrically, either all Λ_α map the unit ball in X into some common ball in Y or no ball in Y contains Λx for x in a dense G_δ.
- Sometimes called the **uniform boundedness principle**.

5.9 Open Mapping Theorem

- Let U and V be the open unit balls of the Banach spaces X and Y. For every surjective bounded linear transformation $\Lambda : X \to Y$ there is a $\delta > 0$ such that $\delta V \subseteq \Lambda(U)$.
- By linearity, the image of any ball in X centered at x_0 contains a ball in Y centered at Λx_0, And so every open set has an open image.
- Rewording: If $\|y\| < \delta$ then $y = \Lambda x$ for some x with $\|x\| < 1$.

5.10 Existence of Inverse Bounded Linear Transformations

- If Λ is a bijective bounded linear transformation between Banach spaces then there is a δ such that $\|\Lambda x\| \geq \delta\|x\|$ for all x in the domain, and so Λ^{-1} is a bijective bounded linear transformation.

Fourier Series of Continuous Functions

5.11 A Convergence Problem

Question: Is it true that for every $f \in C(T)$ that the Fourier series of f converges to $f(x)$ at every point x?

Recall: The nth partial sum of the Fourier series of f at x is given by

$$s_n(f;x) = \frac{1}{2\pi} \in_{-\pi}^{\pi} f(t)D_n(x-t)\,dt \quad (n = 0, 1, 2, \ldots),$$

where

$$D_n(f) = \sum_{k=-n}^{n} e^{ikt}.$$

- The question is to determine whether $\lim_{n\to\infty} s_n(f;x) = f(x)$ for every $f \in C(T)$ and $x \in \mathbb{R}$.
- Result: No (from Banach-Steinhouse).

5.12 Dense G_δ's in $C(T)$ and $\mathbb{R}^1$.

Define $s^*(f;x) = \sup_n |s_n(f;x)|$.

- There exists $E \subseteq C(T)$, a dense G_δ, such that for $f \in E$ the set

$$Q_f = \{x : s^*(f;x) = \infty\}$$

is a dense G_δ in $\mathbb{R}^1$.

5.13 Countable and Dense G_δ's

- In a complete metric space without isolated points, a dense G_δ must be uncountable.

Fourier Coefficients of L^1-functions

5.14 Riemann-Lebesgue Lemma

- For $f \in L^1(T)$ we associate $\hat{f} : Z \to \mathbb{C}$ defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$

- $\hat{f}(n) \to 0$ as $n \to \infty$.
- Let c_0 be the space of all functions $\varphi : \mathbb{Z} \to \mathbb{C}$ with $\varphi(n) \to 0$ as $n \to \pm\infty$ with the norm

$$\|\varphi\|_\infty = \sup\{|\varphi(n)| : n \in \mathbb{Z}\}.$$

c_0 is a Banach space; c_0 is nothing but $C_0(\mathbb{Z})$ when $\mathbb{Z}$ has the discrete topology.

5.15 Converse to Riemann-Lebesgue Lemma

- The mapping $f \mapsto \hat{f}$ is a injective bounded linear transformation from $L^1(T)$ into, but not onto, c_0.

The Hahn-Banach Theorem

5.16 Hahn-Banach Theorem

- If M is a subspace of a normed linear space X with f a bounded linear function on M then f can be extended to a bounded linear functional F on X with $\|F\| = \|f\|$.

Note: M need not be closed.

- The norms are calculated relative to the appropriate domains.

5.17 Relation Between Complex-Linear and Real-Linear Functionals

- A **complex-linear functional** is a complex valued linear function on a complex vector space.
- A **real-linear functional** is a real valued linear function on a complex or real vector space where linearity for scalar multiplication is only required for *real* scalars.
- If $f = u + iv$ is a complex-linear functional then u is a real-linear functional.
- For a complex vector space V:
  1. If $f = u + iv$ is a complex-linear functional on V then $f(x) = u(x) - iu(ix)$ for all $x \in V$.
  2. If u is a real-linear functional on V then $f(x) = u(x) - iu(ix)$ is a complex-linear functional on V.
  3. If V is normed and $f(x) = u(x) - iu(ix)$ then $\|f\| = \|u\|$.

5.19 Characterization of Closures of (Normed) Linear Subspaces

- If M is a linear subspace of a normed linear space X, then $x_0 \in \overline{M}$ if and only if any bounded linear functional vanishing on M vanishes at x_0.

5.20 Unit Linear Functionals

- If X is a normed linear space and $x_0 \in X \setminus \{0\}$ then there is a bounded linear functional f on X with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

5.21 Dual Space

- If X is a normed linear space then the **dual space** X^* is the collection of all bounded linear functionals on X. If addition and scalar multiplication are defined in the obvious way, it is a Banach space.
- If X is nontrivial then X^* is nontrivial (5.20).
- X^* separates points: If $x_0 \neq x_1$ then there exists $f \in X^*$ with $f(x_0) \neq f(x_1)$.
- $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$.
- $f \mapsto f(x)$ is a bounded linear functional on X^* with norm $\|x\|$.

An Abstract Approach to the Poisson Integral

5.22 Remarks, Definitions, and Notations

Notation: Let $\|f\|_E = \sup\{|f(x)| : x \in E\}$.

- H is a **boundary** of K corresponding to A if:
  - K is a compact Hausdorff space;
  - $H \subseteq K$ is compact; and
  - A, a subspace of $C(K)$, contains the constant 1 function and $\|f\|_K = \|f\|_H$ for all $f \in A$.
- If $f \in A, x \in K$ then $|f(x)| \leq \|f\|_H$ and so if $f \equiv 0$ on H then $f \equiv 0$ on K.
- If $f_1, f_2 \in A$ agree on H then they agree on K.
- Let M be the H restriction of elements of A. M is a subspace of $C(H)$ and M and A are in a one-to-one norm-preserving correspondence via restriction (or extension).

- $f \mapsto f(x)$ is a bounded linear functional of norm 1 on M (for fixed x). Since M is a subspace of $C(H)$, Hahn-Banach yields a linear functional Λ on $C(H)$ of norm 1 such that $\Lambda f = f(x)$ for $f \in M$.
- Λ is a positive linear functional on $C(H)$. Combined with 2.14 this gives a regular positive Borel measure μ_x such that for $f \in A$,
$$f(x) = \Lambda f = \int_H f\,d\mu_x.$$
Thus, for $x \in K$ there is a positive measure μ_x on the boundary H which represents x in the sense that the above integral holds for $f \in A$.

5.24 The Poisson Kernel

- Let A be a subspace of $C(\overline{U})$, U the open unit disk, such that A contains all polynomials and $\|f\|_U = \|f\|_T$ for $f \in A$. For $z \in U$ we have a positive Borel measure μ_z on T such that for every $f \in A$,
$$f(z) = \int_T f\,d\mu_z.$$
- Fix $z = re^{i\theta} \in U$. If $u_n(w) = w^n$ we get
$$r^n e^{in\theta} = \int_T u_n\,d\mu_z \quad (n = 0, 1, 2, \ldots).$$
Since $u_{-n} = \overline{u}_n$ on T,
$$r^{|n|} e^{in\theta} = \int_T u_n\,d\mu_z \quad (n = 0, \pm 1, \pm 2, \ldots).$$
- The **Poisson Kernel** is the real function (where $t \in \mathbb{R}$):
$$P_r(\theta - t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}.$$
Note the series is dominated by the convergent geometric series $\sum r^{|n|}$, and so we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)e^{int}\,dt = r^{|n|}e^{in\theta} = \int_T f\,d\mu_z$$
for $f = u_n$ and thus for every trigonometric polynomial, and by 4.25 every $f \in C(T)$.
- The above holds if $f \in A$ and so for such f,
$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\,dt.$$
- We also have
$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}.$$

Note: $P_r(\theta - t) \geq 0$ for $0 \leq r < 1$.

5.25 The Poisson Integral

Let U be the open unit disc.

- Suppose A is a subspace of $C(\overline{U})$ containing all polynomials such that for $f \in A$,

$$\sup_{z \in U} |f(z)| = \sup_{z \in T} |f(z)|.$$

Then, for $f \in A, re^{i\theta} \in U$, the Poisson integral representation holds:

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f(e^{it})\, dt.$$

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