Final review Exam Mon, 12/18, 2017 MATH 70100

Name ....

1. Let X be a linear space with a countably infinite dimension. Then there is no norm on X with respect to which X is complete. [Hint: Baire Category Theorem].

2. A linear subspace of a normed linear space X is norm closed iff it is weakly closed. [Hint for 'only if': Assume a net $\{x^\alpha\}_{\alpha\in A} \rightharpoonup x$. Fix a base $\{e_\beta\}_{\beta\in B}$ for the subspace V and express $x_\alpha = \sum a^\alpha_\beta e_\beta$ and $x = \sum a_\beta e_\beta$ in this base. The weak convergence implies $a^\alpha_\beta \to a_\beta$. Justify why the net can be replaced by a sequence, and prove $x \in V$.]

3. Let H be a Hilbert space. Show that $T \in \mathcal{L}(H)$ is compact iff whenever $\{u_n\} \rightharpoonup u$ and $\{v_n\} \rightharpoonup v$, then $\langle Tu_n, v_n\rangle \to \langle Tu, v\rangle$.

4. Let (X, ρ) be a compact metric space and V a closed subspace of $(C(X), \rho_\infty)$. Suppose that each $f \in V$ is Hölderian, i.e., there exists $C_f > 0$ and $0 < \alpha_f < 1$ such that
$$|f(x) - f(y)| \leq C_f \rho(x, y)^{\alpha_f}, \quad x, y \in X.$$
Then V is finite dimensional. [Hint: There are three main ingredients that go into the proof: Arzelà-Ascoli Theorem, Baire Category Theorem and Riesz Theorem. For simplicity (justify), assume that $\rho < 1$, and let $F_n = \cap_{x,y\in X}\{f \in V \ : \ |f(x) - f(y)| \leq n\rho(x, y)^{1/n}\}$. Apply Baire to conclude that there is an open ball in V so that the assumptions of Arzelà-Ascoli are satisfied, and hence has a compact closure. Finish off with Riesz.]

5. Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $||T|| < 1$. Show that $I - T$ is an isomorphism. Show that $\sum_{n=1}^{\infty} T^n$ converges in $\mathcal{L}(X)$, and $(I - T)^{-1} = \sum_{n=1}^{\infty} T^n$.

6. Let $X = \ell^\infty$. Show that the closed unit ball B^* of the dual X^* is not weak-* sequentially compact. How do you reconcile this fact with the Alaoglu and Helly's Theorems?

7. Let $(X, \mathcal{T})$ be a compact Hausdorff topological space. Show that any strictly weaker topology on X is compact but not Hausdorff, and any strictly stronger topology is Hausdorff but not compact.

8. Let H be a Hilbert space. An invertible operator $T \in \mathcal{L}(X)$ is said to be orthogonal provided that $T^{-1} = T^*$. Show that an invertible operator is orthogonal iff it is an isometry.

9. Let H be a Hilbert space, and $T \in \mathcal{L}(X)$ symmetric and positive. Show that
$$\langle Tu, v\rangle^2 \leq \langle Tu, u\rangle\langle Tv, v\rangle, \quad u, v \in H.$$

10. Let X be a Banach space. Let $J(X)$ and $J(X^*)$ be the natural embeddings in X^{**} and X^{***}. Let $J(X)^0 = \{\psi \in X^{***} \ : \ \Psi|_{J(X)} = 0\}$. Show $X^{***} = J(X)^0 \oplus J(X^*)$. Show that X is reflexive iff X^* is reflexive.