Homework 4 Exam Wed, 10/25, 2017 MATH 70100

Name ....

1. Let (X, ρ) be a compact metric space and $T : X \to X$, such that $\rho(T(u), T(v)) < \rho(u, v)$, $u, v \in X$. Show that T has a unique fixed point.

2. Show that the unit ball $\{\mathbf{x} \in \ell^2 \ : \ ||\mathbf{x}||_{\ell^2} \leq 1\}$ in the Banach space ℓ^2 is not compact.

3. Let X be a metric space and $\{f_n\}$ a sequence in $C(X)$ that converges uniformly on X to some $f \in C(X)$. Show that $\{f_n\}$ is equicontinuous.

4. Show that if two continuous functions defined on a metric space X take the same values on a dense subset, then they are equal.

5. Show that the product of two separable metric spaces is again separable.

6. Let X be a totally bounded metric space, Y a metric space and $f : X \to Y$ uniformly continuous. Show that $f(X)$ is totally bounded. Show that this doesn't necessarily hold if f is merely continuous.

7. Let X be a topological space and Y be a Hausdorff space. Let $f, g : X \to Y$ be continuous. Prove that $\{x \ : \ f(x) = g(x)\}$ is closed.

8. Let $(X, \mathcal{T})$ be completely regular. Then $\mathcal{T}$ is the weak topology induced by the family of functions $\mathcal{F} = C(X)$.

9. Show that the metric space $(C(\mathbb{R}), \rho_\infty)$ of bounded continuous functions is not separable.

10. The set of rationals is not completely metrizable. (Hint: Use Baire Category Theorem)