

## Contents

<b>1</b>	<b>Abstract Integration</b>	<b>1</b>
	The Concept of Measurability . . . . .	1
	Simple Functions . . . . .	2
	Elementary Properties of Measures . . . . .	3
	Arithmetic in $[0, \infty]$ . . . . .	3
	Integration of Positive Functions on $(X, \mathfrak{M}, \mu)$ . . . . .	3
	Integration of Complex Functions on $(X, \mathfrak{M}, \mu)$ . . . . .	5
	The Role Played by Sets of Measure Zero . . . . .	5
<b>2</b>	<b>Positive Borel Measures</b>	<b>7</b>
	Vector Spaces . . . . .	7
	Topological Preliminaries . . . . .	7
	The Riesz Representation Theorem . . . . .	9
	Regularity Properties of Borel Measures . . . . .	9
	Lebesgue Measure . . . . .	10
	Continuity Properties of Measurable Functions . . . . .	11
<b>3</b>	<b><math>L^p</math>-Spaces</b>	<b>12</b>
	Convex Functions and Inequalities . . . . .	12
	The $L^p$ -spaces . . . . .	13
	Approximation by Continuous Functions . . . . .	14
<b>4</b>	<b>Elementary Hilber Space Theory</b>	<b>16</b>
	Inner Products and Linear Functionals . . . . .	16
	Orthonormal Sets . . . . .	18
	Trigonometric Series . . . . .	20

# 1 Abstract Integration

## The Concept of Measurability

### 1.2 Definition

- $\tau \subseteq \mathcal{P}(X)$ , containing both  $\emptyset$  and  $X$ , is a **topology** if it is closed under finite intersections and arbitrary unions.
- $(X, \tau)$  is a **topological space** and the members of  $\tau$  are **open sets**.
- $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is **continuous** if open sets have open preimages.

### 1.3 Definition

- $\mathfrak{M} \subseteq \mathcal{P}(X)$ , containing  $X$ , is a  **$\sigma$ -algebra** if it is closed under complementation and countable unions.
- $(X, \mathfrak{M})$  is a **measurable space**; elements of  $\mathfrak{M}$  are **measurable sets**.
- $f : (X, \mathfrak{M}) \rightarrow (Y, \tau)$  is **measurable** if open sets have measurable preimages ( $\tau$  a topology).

*Note:* Instead of  $(X, \mathfrak{M})$  we just refer to  $X$  as the measurable space.

### 1.6 Comments on Definition 1.3

- $\emptyset \in \mathfrak{M}$ .
- Finite unions are in  $\mathfrak{M}$ .
- $\mathfrak{M}$  is closed under finite and countable intersection.
- $\mathfrak{M}$  is closed under set subtraction.

### 1.7 Composition with Continuous Functions

- $f$  measurable,  $g$  continuous:  $g \circ f$  is measurable.

### 1.8 Continuous Image of Cartesian Product of Measurable Functions.

- $u, v$  real, measurable functions;  $\phi$  continuous on the plane:  $\phi(u(x), v(x))$  is measurable.

### 1.9 Creating Measurable Functions

- If  $u, v$  are real measurable then  $f = u + iv$  is complex measurable.
- If  $f = u + iv$  is complex measurable then  $u, v$ , and  $|f|$  are real measurable.
- If  $f$  and  $g$  are complex measurable then so are  $f + g$  and  $fg$ .
- Characteristic functions of measurable sets are measurable functions.
- If  $f$  is complex measurable then there is a complex measurable function  $\alpha$  with  $|\alpha| = 1$  and  $f = \alpha|f|$ .

### 1.10 $\sigma$ -Algebra Generated by a Set

- $\mathcal{F} \subseteq \mathcal{P}(X)$  is contained in some smallest  $\sigma$ -algebra  $\mathfrak{M}^*$ .

### 1.11 Borel Sets

- The **Borel Sets**,  $\mathfrak{B}$ , is the  $\sigma$ -algebra generated by the topology of a space.
- $G_\delta$  sets are countable intersections of open sets.
- $F_\sigma$  sets are countable unions of closed sets.
- Borel measurable functions are called **Borel mappings** or **Borel functions**.
- *Every* continuous function is Borel measurable.

### 1.12 $\sigma$ -Algebras Associated with a Function

$\mathfrak{M}$  a  $\sigma$ -algebra on  $X$ ,  $Y$  a topological space,  $f : X \rightarrow Y$  a function:

- $\Omega = \{E \subseteq Y : f^{-1}(E) \in \mathfrak{M}\}$  is a  $\sigma$ -algebra on  $Y$ .
- If  $f$  is measurable,  $E$  Borel in  $Y$ , then  $f^{-1}(E) \in \mathfrak{M}$ .
- If  $Y = [-\infty, \infty]$  and  $f^{-1}((a, \infty]) \in \mathfrak{M}$  for all  $a \in \mathbb{R}$  then  $f$  is measurable.
- If  $f$  is measurable,  $Z$  a topological space,  $g : Y \rightarrow Z$  Borel, then  $g \circ f : X \rightarrow Z$  is measurable.

### 1.14 Supremum and Limit Supremum of Measurable Functions

- If  $f_n : X \rightarrow [-\infty, \infty]$  are measurable then so are  $\sup f_n$  and  $\limsup f_n$ .
- The limit of pointwise convergent sequence of complex measurable functions is measurable.
- $f, g$  measurable then so are  $\max\{f, g\}$  and  $\min\{f, g\}$ .

### 1.15 Positive and Negative parts of $f$

- $f^+ = \max\{f, 0\}$  is the **positive part** of  $f$  and  $f^- = -\min\{f, 0\}$  is the **negative part**.
- $|f| = f^+ + f^-$  and  $f = f^+ - f^-$ .
- If  $f = g - h$ ,  $g \geq 0$  and  $h \geq 0$  then  $f^+ \leq g$  and  $f^- \leq h$ .

## Simple Functions

### 1.16 Definition

- $s$ , complex measurable on  $X$ , is **simple** if its range is finite. If  $s(X) = \{\alpha_1, \dots, \alpha_n\}$  then

$$s = \sum_{i=1}^n \alpha_i \chi_{A_i}, \quad A_i = s^{-1}(\alpha_i).$$

- $s$  is measurable if and only if each  $A_i$  is.

### 1.17 Approximation by Simple Functions

- If  $f : X \rightarrow [0, \infty]$  is measurable then there exists measurable, simple functions  $s_n$  on  $X$  such that  $0 \leq s_1 \leq \dots \leq f$  and  $s_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$  for all  $x \in X$ .

## Elementary Properties of Measures

### 1.18 Definition

- A **positive measure** is a function from a  $\sigma$ -algebra  $\mathfrak{M}$  to  $[0, \infty]$  which is **countably additive**: i.e.

$$\mu\left(\bigcup_{n=1}^{\infty} A_i\right) = \sum_{i=1}^n \mu(A_i)$$

when  $A_i$  are pairwise disjoint members of  $\mathfrak{M}$ .

- A measurable space equipped with a measure is a **measure space**.
- A **complex measure** is a complex-valued countably additive function on a  $\sigma$ -algebra.

### 1.19 Basic Properties of a Positive Measure $\mu$

- $\mu(\emptyset) = 0$ .
- $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$  if the  $A_i$  are pairwise disjoint members of  $\mathfrak{M}$ .
- $A \subseteq B$  implies  $\mu(A) \leq \mu(B)$  for  $A, B \in \mathfrak{M}$ .
- If  $A_n \in \mathfrak{M}$  such that  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$  then  $\mu(A_n) \rightarrow \mu(\bigcup_{n=1}^{\infty} A_n)$ .
- If  $A_n \in \mathfrak{M}$  such that  $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$  and  $\mu(A_1) < \infty$  then  $\mu(A_n) \rightarrow \mu(\bigcap_{n=1}^{\infty} A_n)$ .

### 1.20 Measure Space Examples

- **counting measure**:  $\mu(E) = |E|$  if  $|E| < \infty$  and  $\mu(E) = \infty$  otherwise.
- **unit mass at  $x_0$** :  $\mu(E) = 1$  if  $x_0 \in E$  and  $\mu(E) = 0$  otherwise.

## Arithmetic in $[0, \infty]$

### 1.22 Definition

- $a + \infty = \infty + a = \infty$
- $a \cdot \infty = \infty \cdot a = \begin{cases} \infty & a \in (0, \infty] \\ 0 & a = 0 \end{cases}$
- With  $0 \cdot \infty = 0$  we have commutativity, associativity, and distributivity.
- Cancellation:  $a + b = a + c \implies b = c$  only if  $a \neq \infty$ ;  $ab = ac \implies b = c$  only if  $a \in (0, \infty)$ .
- $0 \leq a_1 \leq a_2 \leq \cdots$ ,  $0 \leq b_1 \leq b_2 \leq \cdots$  with  $a_n \rightarrow a$  and  $b_n \rightarrow b \implies a_n b_n \rightarrow ab$ .

## Integration of Positive Functions on $(X, \mathfrak{M}, \mu)$

### 1.23 Definition

- $s : X \rightarrow [0, \infty]$  simple and measurable with  $s(X) = \{\alpha_1, \dots, \alpha_n\}$ . For  $E \in \mathfrak{M}$  define

$$\int_E s d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap E), \quad A_i = s^{-1}(\alpha_i).$$

- If  $f : X \rightarrow [0, \infty]$  is measurable then for  $E \in \mathfrak{M}$  define the **Lebesgue Integral of  $f$  over  $E$**  by

$$\int_E f d\mu = \sup \int_E s d\mu,$$

where the supremum is taken over all nonnegative measurable simple functions dominated by  $f$ .

#### 1.24 Basic Properties of Lebesgue Integrals

- $0 \leq f \leq g$  implies  $\int_E f d\mu \leq \int_E g d\mu$ .
- $A \subseteq B$  and  $f \geq 0$  implies  $\int_A f d\mu \leq \int_B f d\mu$ .
- If  $f \geq 0$  and  $c \in [0, \infty)$  then  $\int_E cf d\mu = c \int_E f d\mu$ .
- If  $f \equiv 0$  on  $E$  then  $\int_E f d\mu = 0$  even if  $\mu(E) = \infty$ .
- If  $\mu(E) = 0$  then  $\int_E f d\mu = 0$  if  $f \equiv \infty$  on  $E$ .
- If  $f \geq 0$  then  $\int_E f d\mu = \int_X \chi_E f d\mu$ .

#### 1.25 Basic Properties of the Lebesgue Integral of Simple Functions

- If  $s$  is a nonnegative measurable simple function then  $\varphi : \mathfrak{M} \rightarrow [0, \infty]$  sending  $E$  to  $\int_E s d\mu$  is a measure.
- If  $s$  and  $t$  are nonnegative measurable simple functions then  $\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu$ .

#### 1.26 Lebesgue's Monotone Convergence Theorem

- If  $f_n : X \rightarrow [0, \infty]$  is a (pointwise) non-decreasing sequence of measurable functions for which  $f_n(x) \rightarrow f(x)$  for every  $x \in X$  then  $f$  is measurable and

$$\int_X f_n d\mu \rightarrow \int_X f d\mu.$$

#### 1.27 Interchange of Summation and Integration

- If  $f_n : X \rightarrow [0, \infty]$  are measurable and  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  then

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

#### 1.28 Fatou's Lemma

- If  $f_n : X \rightarrow [0, \infty]$  are measurable then

$$\int_X \left( \liminf_{n \rightarrow \infty} f_n \right) d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

#### 1.29 Change of Measure

- If  $f : X \rightarrow [0, \infty]$  is measurable then  $\varphi : \mathfrak{M} \rightarrow [0, \infty]$  sending  $E$  to  $\int_E f d\mu$  is a measure and

$$\int_X g d\varphi = \int_X gf d\mu.$$

Sometimes this is written as  $d\varphi = f d\mu$ , although no independent meaning is given to these symbols.

## Integration of Complex Functions on $(X, \mathfrak{M}, \mu)$

### 1.30 Definition

- The **Lebesgue Integrable Functions** or **Summable Functions** with respect to  $\mu$ , denoted by  $L^1(\mu)$  is the collection of all complex measurable functions  $f$  on  $X$  such that  $\int_X |f| d\mu < \infty$ .

### 1.31 Definition

- If  $f = u + iv$  with  $u, v$  real measurable functions and  $f \in L^1(\mu)$  then for  $E \in \mathfrak{M}$ :

$$\int_E f d\mu = \left( \int_E u^+ d\mu - \int_E u^- d\mu \right) + i \left( \int_E v^+ d\mu - \int_E v^- d\mu \right).$$

- It is useful define the integral of a function  $f : X \rightarrow [-\infty, \infty]$  to be

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$

for  $E \in \mathfrak{M}$  and provided only one term on the right is infinite.

### 1.32 Linearity of $L^1(\mu)$

- For  $f, g \in L^1(\mu)$  and  $\alpha, \beta \in \mathbb{C}$  we have  $\alpha f + \beta g \in L^1(\mu)$  and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

### 1.33 Interchange of Modulus and Integration

- $\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$  for  $f \in L^1(\mu)$ .

### 1.34 Lebesgue's Dominated Convergence Theorem

- $f_n$  are complex measurable functions such that  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for all  $x \in X$ . If

$$|f_n(x)| \leq g(x), \quad \text{for all } n \in \mathbb{N}$$

for some  $g \in L^1(\mu)$  then  $f \in L^1(\mu)$ ,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

## The Role Played by Sets of Measure Zero

### 1.35 Definition

- If  $\mu$  is a measure on a  $\sigma$ -algebra  $\mathfrak{M}$ ,  $E \in \mathfrak{M}$ , then a statement  $P$  holds **almost everywhere** (a.e.) on  $E$  if there exists  $N \subseteq E$  with  $\mu(N) = 0$  such that  $P$  is true on  $E \setminus N$ .

*Example 1.* If  $f$  and  $g$  are measurable and  $\mu(\{x : f(x) \neq g(x)\}) = 0$  then  $f = g$  a.e., written as  $f \sim g$ .  $\sim$  is an equivalence relation and if  $f \sim g$  then for  $E \in \mathfrak{M}$  we have  $\int_E f d\mu = \int_E g d\mu$ . Thus sets of measure zero are negligible with respect to integration.

*Note:* It is *not* the case that a subset of a negligible set is negligible as it may not even be measurable!

**1.36 Existence of Completions**

- If  $(X, \mathfrak{M}, \mu)$  is a measure space, then define  $\mathfrak{M}^*$  to be all  $E \subseteq X$  such that  $A \subseteq E \subseteq B$  for  $A, B \in \mathfrak{M}$  such that  $\mu(B \setminus A) = 0$ . Defining  $\mu(E) = \mu(A)$  makes  $(X, \mathfrak{M}^*, \mu)$  a measure space,
- The extended  $\mu$  is **complete** as all subsets of negligible sets are measurable.
- $\mathfrak{M}^*$  is the  $\mu$ -**completion** of  $\mathfrak{M}$ .

**1.37 Expanding the Definition of What is a Measurable Function**

- Since integration is agnostic to functions equal a.e., we now call  $f$  defined on  $E \in \mathfrak{M}$  **measurable on  $X$**  if  $\mu(E^c) = 0$  and  $f^{-1}(V) \cap E$  is measurable for every open set  $V$ .
- In the above, we can define  $f \equiv 0$  on  $E^c$  to get a measurable function on  $X$ .

**1.38 Lebesgue's Dominated Convergence Theorem with Negligible Sets**

- $f_n$  complex measurable functions defined a.e. on  $X$  such that

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty.$$

Then  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  converges for almost all  $x$  and  $f \in L^1(\mu)$  with

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

**1.39 Integration and Properties That Hold Almost Everywhere**

- If  $f : X \rightarrow [0, \infty]$  measurable and  $E \in \mathfrak{M}$  with  $\int_E f d\mu = 0$  then  $f = 0$  a.e. on  $E$ .
- If  $f \in L^1(\mu)$  with  $\int_E f d\mu = 0$  for every  $E \in \mathfrak{M}$  then  $f = 0$  a.e. on  $X$ .
- If  $f \in L^1(\mu)$  and

$$\left| \int_X f d\mu \right| = \int_X |f| d\mu$$

then there exists  $\alpha \in \mathbb{C}$  such that  $\alpha f = |f|$  a.e. on  $X$ .

**1.40 Averages Lying in a Closed Set**

- If  $\mu(X) < \infty$ ,  $f \in L^1(\mu)$ ,  $S \subseteq \mathbb{C}$  is closed, and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$$

lie in  $S$  for every  $E \in \mathfrak{M}$  with positive measure then  $f(x) \in S$  for almost all  $x \in X$ .

**1.41 Finite Set Membership**

- If  $E_k \subseteq X$  are measurable with  $\sum_{k=1}^{\infty} \mu(E_k) < \infty$  then almost all  $x \in X$  lie in finitely many  $E_k$ .

## 2 Positive Borel Measures

### Vector Spaces

#### 2.1 Definition

- A **complex vector space** is one with complex scalars.
- A function  $\Lambda$  between vector spaces is a **linear transformation** if  $\Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$ .
- A **linear functional** is a linear transformation where the codomain is the field of scalars of the domain.

#### 2.2 Integration as a Linear Functional

- For any positive measure  $\mu$ ,  $f \mapsto \int_X f d\mu$  is a linear functional on  $L^1(\mu)$ .
- If  $g$  is a bounded measurable function then  $f \mapsto \int_X fg d\mu$  is a linear functional on  $L^1(\mu)$ .
- A **positive linear functional** is a linear functional  $\Lambda$  such that  $\Lambda f \geq 0$  whenever  $f \geq 0$ .
- If  $C$  is the vector space of continuous complex functions on  $[0, 1]$  then

$$\Lambda f = \int_0^1 f(x) dx$$

is a positive linear functional on  $C$  (with the integral being the Riemann integral).

### Topological Preliminaries

#### 2.3 Definitions

- $E$  is **closed** if its complement is open.
- The **closure** of  $E$ , denoted  $\overline{E}$ , is the smallest closed set containing  $E$ .
- $K \subseteq X$  is **compact** if every open cover of  $K$  contains a finite subcover.
- A **neighborhood** of  $p \in X$  is any open set containing  $p$ .
- $X$  is **Hausdorff** if any two  $p \neq q$  can be separated by open sets.
- $X$  is **locally compact** if every points has a neighborhood with compact closure.

*Recall:* Heine-Borel: Subsets of Euclidean space are compact exactly when they are closed and bounded. Thus  $\mathbb{R}^n$  is locally compact.

*Recall:* Every metric space is Hausdorff.

#### 2.4 Closed Subsets of Compact Sets

- $F \subseteq K$ ,  $F$  closed,  $K$  compact. Then  $F$  is compact.
- If  $A \subseteq B$  and  $B$  has compact closure then so does  $A$ .

#### 2.5 Separating a Compact Set from a Point

- If  $X$  is Hausdorff with  $K$  compact in  $X$  then any  $p \notin K$  can be separated from  $K$  by open sets.



**2.6 Intersections of Compact Sets**

- If  $K_\alpha \subseteq X$  are compact,  $X$  Hausdorff, and  $\cap_\alpha K_\alpha = \emptyset$  then some finite subset has empty intersection.

**2.7 Sandwiching Sets**

- If  $U$  is open in  $X$ , Hausdorff, and  $K \subseteq U$  is compact then there exists  $V$ , open, with  $K \subseteq V \subseteq \overline{V} \subseteq U$ .

**2.8 Definition**

- $f$  is **lower semicontinuous** if  $f^{-1}((\alpha, \infty])$  is open.
- $f$  is **upper semicontinuous** if  $f^{-1}([\infty, \alpha))$  is open.
- $\chi_U$  is lower semicontinuous if  $U$  is open.
- $\chi_F$  is upper semicontinuous if  $F$  is closed.
- The supremum of any collection of lower semicontinuous functions is again lower semicontinuous.
- The infimum of any collection of upper semicontinuous functions is again upper semicontinuous.

**2.9 Definition**

- The **support** of  $f : X \rightarrow \mathbb{C}$  is the closure of  $f^{-1}(\mathbb{C} \setminus \{0\})$ .
- $C_c(X)$  is the vector space of functions with compact support.

**2.10 Image of a Compact Set**

- The continuous image of a compact set is compact.
- The range of  $f \in C_c(X)$  is compact subset of  $\mathbb{C}$ .

**2.11 Notation**

- $K \prec f$  means  $K$  is compact,  $0 \leq f(x) \leq 1$  and  $f = 1$  on  $K$ .
- $f \prec V$  means  $V$  is open and  $f$ 's support lies in  $V$ .
- $K \prec f \prec V$  combines the above.

**2.12 Urysohn's Lemma**

- For  $K \subseteq V \subseteq X$  with  $K$  compact,  $V$  open, and both  $X$  locally compact and Hausdorff, there exists  $f \in C_c(X)$  with  $K \prec f \prec V$ .
- In terms of characteristic functions this means there is a continuous  $f$  with  $\chi_K \leq f \leq \chi_V$ .

**2.13 Partition of Unity**

- For  $X$  locally compact and Hausdorff and  $V_1, \dots, V_n$  open in  $X$  and  $K \subseteq V_1 \cup \dots \cup V_n$  is compact, there exists functions  $h_i \prec V_i$  with  $h_1 + \dots + h_n = 1$  on  $K$ .
- This is called a **partition of unity on  $K$**  subordinate to the cover  $\{V_1, \dots, V_n\}$ .

## The Riesz Representation Theorem

### 2.14 The Riesz Representation Theorem

- For  $X$  locally compact and Hausdorff with  $\Lambda$  a positive linear functional on  $C_c(X)$  there exists:
  1. a  $\sigma$ -algebra  $\mathfrak{M}$  containing all the Borel sets of  $X$ ;
  2. a *unique* positive measure  $\mu$  on  $\mathfrak{M}$  representing  $\Lambda$ :
    - $\Lambda f = \int_X f d\mu$  for all  $f \in C_c(X)$
    - $\mu(K) < \infty$  for any compact  $K$
    - $\mu(E) = \inf \{\mu(V) : E \subseteq V, V \text{ open}\}$  for all  $E \in \mathfrak{M}$
    - $\mu(E) = \sup \{\mu(K) : K \subseteq E, K \text{ compact}\}$  for every  $E \in \mathfrak{M}$  open or  $\mu(E) < \infty$
    - $E \in \mathfrak{M}, \mu(E) = 0$  implies every subset of  $E$  is in  $\mathfrak{M}$ .

## Regularity Properties of Borel Measures

### 2.15 Definition

- $\mu$  is a **Borel measure** if it is defined on the Borel sets.
- Let  $\mu$  be a positive Borel measure and  $E \subseteq X$  be Borel. Then:
- $E$  is **outer regular** if the infimum property in 2.14 holds.
  - $E$  is **inner regular** if the supremum property in 2.14 holds.
  - $E$  is **regular** if both hold.

### 2.16 Definition

- $E$  is  **$\sigma$ -compact** if it is the countable union of compact sets.
- $E$  is  **$\sigma$ -finite** if it is the countable union of sets with finite measure.

### 2.17 Regularity of $\sigma$ -Compact Spaces

- $X$  a locally compact,  $\sigma$ -compact Hausdorff space and  $\mathfrak{M}, \mu$  are as in 2.14. Then:
  - For  $E \in \mathfrak{M}, \epsilon > 0$ , there is  $F \subseteq E \subseteq V$  with  $F$  closed,  $V$  open, and  $\mu(V \setminus F) < \epsilon$ .
  - $\mu$  is a regular Borel measure on  $X$ .
  - There exists an  $F_\sigma A$  and a  $G_\delta B$  with  $A \subseteq E \subseteq B$  and  $\mu(B \setminus A) = 0$ .  
Thus, every  $E \in \mathfrak{M}$  is the union of an  $F_\sigma$  and a negligible set.

### 2.18 Regularity in the Presence of $\sigma$ -Compact Open Sets

- $X$  a locally compact Hausdorff space in which every open set is  $\sigma$ -compact. Then any positive Borel measure that is finite on compact sets is regular.

## Lebesgue Measure

### 2.19 Euclidean Spaces

- $\mathbb{R}^k$  is the  **$k$ -dimension Euclidean space** with all the familiar operations.
- If  $E \subseteq \mathbb{R}^k$  and  $x \in \mathbb{R}^k$  then  $E + x = \{y + x : y \in E\}$  is a **translate** of  $E$ .
- A  **$k$ -cell** is a set of the form  $\{(\xi_1, \dots, \xi_k) \in \mathbb{R}^k : \alpha_i < \xi_i < \beta_i, 1 \leq i \leq k\}$ . Either inequality may be replaced with  $\leq$ . The **volume** of a  $k$ -cell is  $\text{vol}(W) = \prod_{i=1}^k (\beta_i - \alpha_i)$ .
- If  $a \in \mathbb{R}^k$  and  $\delta > 0$  then a  **$\delta$ -box with corner at  $a$**  is

$$Q(a, \delta) = \{(\xi_1, \dots, \xi_k) \in \mathbb{R}^k : \alpha_i \leq \xi_i < \alpha_i + \delta, 1 \leq i \leq k\}.$$

- If  $P_n$  are points whose coordinates are multiples of  $2^{-n}$  and  $\Omega_n$  are the  $2^{-n}$  boxes with corners at the elements of  $P_n$  then we use the following properties:
  - $\Omega_n$  covers  $\mathbb{R}^k$  disjointly.
  - If  $r < n$  and  $Q' \in \Omega_n$ ,  $Q'' \in \Omega_r$  then either  $Q' \subseteq Q''$  or  $Q' \cap Q'' = \emptyset$ .
  - $\text{vol } Q = 2^{-rk}$  for  $Q \in \Omega_r$  and if  $n > r$  then  $|P_n \cap Q| = 2^{(n-r)k}$ .
  - Any non-empty open set is the countable disjoint union of elements of  $\cup_{n=1}^{\infty} \Omega_n$ .

### 2.20 Existence of the Lebesgue Measure

- There exists  $(\mathbb{R}^k, \mathfrak{M}, m)$  such that
  - $m(W) = \text{vol}(W)$  for every  $k$ -cell  $W$ .
  - $\mathfrak{M}$  contains the Borel sets of  $\mathbb{R}^k$
  - $E \in \mathfrak{M}$  iff  $A \subseteq E \subseteq B$  with  $A$  is  $F_\sigma$ ,  $B$  is  $G_\delta$ , and  $m(B \setminus A) = 0$ .
  - $m$  is regular.
  - $m(x + E) = m(E)$  for all  $E \in \mathfrak{M}$  and  $x \in \mathbb{R}^k$ .
  - If  $\mu$  is any positive translation-invariant Borel measure on  $\mathbb{R}^k$  which is finite on compact sets then  $\mu(E) = cm(E)$  for some  $c \in \mathbb{R}$  and all Borel sets  $E$ .
  - $m(T(E)) = \Delta(T)m(E)$ ,  $\Delta(T) \in \mathbb{R}$ , for every linear transformation  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  and  $E \in \mathfrak{M}$ . More specifically,  $\Delta(T) = 1$  if  $T$  is a rotation.
- Elements of  $\mathfrak{M}$  are **Lebesgue measurable** sets and  $m$  is the **Lebesgue measure** on  $\mathbb{R}^k$ .

### 2.21 Remarks

- If  $m$  is the Lebesgue measure on  $\mathbb{R}^k$  we write  $L^1(\mathbb{R}^k)$  instead of  $L^1(m)$ .
- Instead of  $f \in L^1$  on  $E$  we write  $f \in L^1(E)$  (in the measure space with  $m$  restricted to subsets of  $E$ ).
- If  $I$  is an interval in  $\mathbb{R}$  and  $f \in L^1(I)$  we write  $\int_a^b f(x) dx$  instead of  $\int_I f dm$ .
- If  $f$  is continuous on  $[a, b]$  then the Riemann and Lebesgue integrals agree.
- Most sets are *not* Borel sets.

**2.22 Sufficient Condition for Measure Zero**

- If  $A \subseteq \mathbb{R}$  and every subset of  $A$  is Lebesgue measurable then  $m(A) = 0$ .
- Every set of positive measure has unmeasurable subsets.

**2.23 Determinants**

- The  $\Delta(T)$  in 2.20 is  $|\det T|$ .

**Continuity Properties of Measurable Functions**

We assume in this section that  $\mu$  is a measure on a locally compact Hausdorff space with the properties listed in 2.14 –  $\mu$  could be the Lebesgue measure on some  $\mathbb{R}^k$ .

**2.24 Lusin's Theorem**

- $f$  complex measurable,  $\mu(A) < \infty$ , and  $f = 0$  outside  $A$ . Then for  $\epsilon > 0$  there exists  $g \in C_c(X)$  with

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon.$$

We may pick  $g$  so that

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$

- If  $|f| \leq 1$  then there is a sequence  $g_n \in C_c(X)$ ,  $|g_n| \leq 1$ , with

$$f(x) = \lim_{n \rightarrow \infty} g_n(x) \text{ a.e.}$$

**2.25 Vitali-Carathéodory Theorem**

- If  $f \in L^1(\mu)$  is real valued and  $\epsilon > 0$  then there exists  $u$ , upper semicontinuous and bounded from above, and  $v$ , lower semicontinuous and bounded from below, such that  $u \leq f \leq v$  and  $\int_X (v-u) d\mu < \epsilon$ .

### 3 $L^p$ -Spaces

#### Convex Functions and Inequalities

##### 3.1 Definition

- $\varphi$  is **convex** on  $(a, b)$  if  $x, y \in (a, b)$  and  $\lambda \in [0, 1]$  imply

$$\varphi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\varphi(x) + \lambda\varphi(y).$$

That is, the segment between  $(x, \varphi(x))$  and  $(y, \varphi(y))$  lies above the graph of  $\varphi$ .

- The above is equivalent to  $a < s < t < u < b$  implying

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

*Note:* The mean value theorem for differentiation with the above imply that  $\varphi$ , real differentiable, is convex in  $(a, b)$  iff  $a < s < t < b$  implies  $\varphi'(s) \leq \varphi'(t)$ .

##### 3.2 Convexity Implies Continuity

- If  $\varphi$  is convex on  $(a, b)$  then  $\varphi$  is continuous on  $(a, b)$ .

*Note:* This relies on the fact that we are working on an *open* segment.

##### 3.3 Jensen's Inequality

- $\mathfrak{M}$  a  $\sigma$ -algebra on  $\Omega$ ,  $\mu$  a positive measure on it such that  $\mu(\Omega) = 1$ . If  $f \in L^1(\mu)$  is real with  $f(\Omega) \subseteq (a, b)$  and  $\varphi$  is convex on  $(a, b)$  then

$$\varphi\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} (\varphi \circ f) d\mu.$$

*Note:*  $a = -\infty$  or  $b = \infty$  are not excluded values.

*Note:* If  $\varphi \circ f \notin L^1(\mu)$  then the integral has value  $+\infty$  (see 1.31).

*Example 1.* For  $\varphi(x) = e^x$  we get

$$\exp\left\{\int_{\Omega} f d\mu\right\} \leq \int_{\Omega} e^f d\mu.$$

*Example 2.* If  $\Omega = \{p_1, \dots, p_n\}$  and  $\mu(\{p_i\}) = 1/n$ ,  $f(p_i) = x_i$  then the example 1 becomes:

$$\exp\left\{\frac{1}{n} \sum_{i=1}^n x_i\right\} \leq \frac{1}{n} \sum_{i=1}^n e^{x_i}$$

for real  $x_i$ . Setting  $y_i = e^{x_i}$  we can relate the arithmetic and geometric means of  $n$  positive numbers:

$$\left(\prod_{i=1}^n y_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n y_i.$$

Given this, it is clear why

$$\exp\left\{\int_{\Omega} \log g d\mu\right\} \leq \int_{\Omega} g d\mu$$

are called the arithmetic and geometric means of the positive function  $g$ .

*Example 3.* If  $\mu(\{p_i\}) = \alpha_i > 0$  with  $\sum \alpha_i = 1$  then we get a more general version of the above:

$$\prod_{i=1}^n y_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i y_i.$$

### 3.4 Definition

- $p, q \in (1, \infty)$  are **conjugate exponents** if  $p + q = pq$  (or, equivalently,  $p^{-1} + q^{-1} = 1$ ).
- $p \rightarrow 1$  forces  $q \rightarrow \infty$  and so 1 and  $\infty$  are regarded as conjugate exponents.
- Many denote  $p$ 's conjugate exponent by  $p'$ .

### 3.5 Hölder and Minkowski's Inequalities

$p$  and  $q$  are conjugate exponents with  $p \in (1, \infty)$  and  $f$  and  $g$  are measurable with range in  $[0, \infty]$ :

- Hölder's inequality:

$$\int_X fg \, d\mu \leq \left\{ \int_X f^p \, d\mu \right\}^{1/p} \left\{ \int_X g^q \, d\mu \right\}^{1/q}.$$

If  $p = q = 2$  then this is called Schwarz's inequality.

- Minkowski's inequality:

$$\left\{ \int_X (f + g)^p \, d\mu \right\}^{1/p} \leq \left\{ \int_X f^p \, d\mu \right\}^{1/p} + \left\{ \int_X g^p \, d\mu \right\}^{1/p}.$$

*Note:* Assuming the right hand side of Hölder's inequality has only finite factors, equality holds if and only if there are constants  $\alpha$  and  $\beta$ , not both zero, such that  $\alpha f^p = \beta g^q$  a.e.

## The $L^p$ -spaces

For this section, let  $X$  be arbitrary and  $\mu$  a positive measure.

### 3.6 Definition

- If  $0 \leq p \leq \infty$  and  $f$  is a complex measurable function, then the  $L^p$ -**norm** of  $f$  is

$$\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{1/p}.$$

$L^p(\mu)$  is the collection of all  $f$  for which  $\|f\|_p < \infty$  and is called the  $L^p$ -**space** of  $X$ .

- If  $u$  is the Lebesgue measure on  $\mathbb{R}^k$  then we write  $L^p(\mathbb{R}^k)$  instead of  $L^p(\mu)$ .
- If  $\mu$  is the counting measure on a countable set  $A$  we denote the  $L^p$ -space by  $\ell^p(A)$  or just  $\ell^p$ .  $x \in \ell^p$  is a sequence  $x = \{\xi_n\}$  and

$$\|x\|_p = \left\{ \sum_{n=1}^{\infty} |\xi_n|^p \right\}^{1/p}.$$

### 3.7 Definition

- For  $g : X \rightarrow [0, \infty]$  measurable, let  $S$  be the set such that  $\mu(g^{-1}((\alpha, \infty])) = 0$ . If  $S = \emptyset$  then set  $\beta = \infty$ , else  $\beta = \inf S$ . Since the countable union of sets of measure zero is a set of measure zero and

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right),$$

$\beta \in S$ .  $\beta$  is the **essential supremum** of  $g$ .

- If  $f$  is a complex measurable function then  $\|f\|_{\infty}$  is the essential supremum of  $|f|$ .  $L^{\infty}(\mu)$  is the set of all  $f$  with  $\|f\|_{\infty} < \infty$ , its members called the **essentially bounded** measurable functions on  $X$ .
- $L^{\infty}(\mathbb{R}^k)$  is the class of Lebesgue measure essentially bounded functions on  $\mathbb{R}^k$ .
- $\ell^{\infty}(A)$  is the class of bounded functions on  $A$ .

*Note:*  $|f(x)| \leq \lambda$  holds almost everywhere iff  $\lambda \geq \|f\|_{\infty}$ .

### 3.8 Hölder's Inequality With $L^p$ -norms

- If  $p$  and  $q$  are conjugate exponents,  $1 \leq p \leq \infty$ , with  $f \in L^p(\mu)$  and  $g \in L^q(\mu)$  then  $fg \in L^1(\mu)$  and  $\|fg\|_1 \leq \|f\|_p \|g\|_q$ .

### 3.9 Minkowski's Inequality With $L - p$ -norms

- If  $1 \leq p \leq \infty$  and  $f, g \in L^p(\mu)$  then  $f + g \in L^p(\mu)$  and  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

### 3.10 Remarks

- $L^p(\mu)$  is a complex vector space.
- Triangle inequality holds:  $\|f - h\|_p \leq \|f - g\|_p + \|g - h\|_p$ .

*Note:* If  $f \sim g$  (see 1.35) then  $\|f - g\|_p = 0$ .

- $L^p(\mu)$  is a complete metric space if we pass to equivalence classes under  $\sim$ .

### 3.11 Completeness of $L^p(\mu)$

- $L^p(\mu)$  is complete for every  $1 \leq p \leq \infty$  and every positive measure  $\mu$ .

### 3.12 Pointwise Convergence of Cauchy Subsequences

- For  $1 \leq p \leq \infty$  and  $f_n \rightarrow f$  Cauchy in  $L^p(\mu)$ ,  $\{f_n\}$  has a subsequence converging pointwise to  $f$  a.e.

### 3.13 Density of (some) Simple Functions

- For  $1 \leq p < \infty$ , the set of all complex, measurable, simple functions  $s$  with  $\mu(\{x : s(x) \neq 0\}) < \infty$  is dense in  $L^p(\mu)$ .

## Approximation by Continuous Functions

For this section  $X$  is locally compact and Hausdorff,  $\mu$  a measure on  $\sigma$ -algebra with the features in 2.14.

### 3.14 Density of $C_c(X)$

- $C_c(X)$  is dense in  $L^p(\mu)$  for  $1 \leq p < \infty$ .

### 3.15 Remarks

- $C_c(\mathbb{R}^k)$  has a metric that does not need to pass to equivalence classes.
- Likewise, the essential supremum there is the same as the supremum:  $\|f\|_\infty = \sup_{x \in \mathbb{R}^k} |f(x)|$ .
- If  $1 \leq p < \infty$  then 3.14 gives  $C_c(\mathbb{R}^k)$  is dense in  $L^p(\mathbb{R}^k)$ , which is complete by 3.11.  $L^p(\mathbb{R}^k)$  is the completion of  $C_c(\mathbb{R}^k)$  with respect to the  $L^p(\mathbb{R}^k)$  metric.

*Note:* Keep in mind that we are having *different* completions of the same set under different metrics.

- If the distance between  $f, g \in C_c(\mathbb{R}^1)$  is given by  $\int_{-\infty}^{\infty} |f(t) - g(t)| dt$  then the completion of the resulting metric space is the space of equivalence classes (under  $\sim$ ) of Lebesgue integrable functions.

*Note:* Important that the completion of functions on  $\mathbb{R}^k$  are again functions on  $\mathbb{R}^k$ .

- The  $L^\infty$ -completion is  $C_0(\mathbb{R}^k)$  of functions which vanish at infinity (see below).

**Reminder:** For 3.16 and 3.17, please remember in this section that  $X$  is **locally compact** and **Hausdorff**.

### 3.16 Definition

- The complex function  $f$  **vanishes at infinity** if for  $\epsilon > 0$  there is a  $K$ , compact, with  $|f| < \epsilon$  on  $K^c$ .
- $C_0(X)$  is the class of all continuous functions  $f$  on  $X$  which vanish at infinity.
- $C_c(X) \subseteq C_0(X)$  with equality when  $X$  is compact, in which case  $C(X)$  is used for either.

### 3.17 Completion of $C_c(X)$

- $C_0(X)$  is the completion of  $C_c(X)$  relative to the supremum norm metric:  $\|f\| = \sup_{x \in X} |f(x)|$ .



## 4 Elementary Hilbert Space Theory

### Inner Products and Linear Functionals

#### 4.1 Definition

- An **inner product** on a complex vector space  $H$  is a function  $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$ :
  1.  $(y, x) = \overline{(x, y)}$
  2.  $(x + y, z) = (x, z) + (y, z)$
  3.  $(\alpha x, y) = \alpha(x, y)$
  4.  $(x, x) \geq 0$
  5.  $(x, x) = 0$  only when  $x = 0$
- Implications:
  - $(0, y) = 0$  for all  $y \in H$ .
  - $x \mapsto (x, y)$  is a linear functional for all  $y \in H$ .
  - $(x, \alpha y) = \overline{\alpha}(x, y)$
  - $(z, x + y) = (z, x) + (z, y)$
- $\|x\|^2 = (x, x)$  is the **norm** of  $x$ .
- An **inner product space** (aka **unitary space**) is a vector space with an inner product.

#### 4.2 Schwarz Inequality

- $|(x, y)| \leq \|x\| \|y\|$ .

#### 4.3 Triangle Inequality

- $\|x + y\| \leq \|x\| + \|y\|$ .

#### 4.4 Definition

- Define the distance between  $x$  and  $y$  to be  $\|x - y\|$ .
- Since  $\|x - z\| \leq \|x - y\| + \|y - z\|$ ,  $H$  is a metric space.
- If  $H$  is a complete metric space, then  $H$  is a **Hilbert space**.

*Note:* Throughout the rest of the chapter,  $H$  denotes a Hilbert space.

#### 4.5 Examples

*Example 1.*  $\mathbb{C}^n$  is a Hilbert space (scalar multiplication and vector addition defined component wise). Then

$$(x, y) = \sum_{j=1}^n \xi_j \overline{\eta_j}$$

where  $\xi_j$  and  $\eta_j$  are the components of  $x$  and  $y$ .

*Example 2.*  $L^2(\mu)$  is a Hilbert space for a positive measure  $\mu$ , with

$$(f, g) = \int_X f \bar{g} d\mu.$$

Note that

$$\|f\| = (f, f)^{\frac{1}{2}} = \left\{ \int_X |f|^2 d\mu \right\}^{\frac{1}{2}} = \|f\|_2.$$

*Example 3.* The continuous complex functions on  $[0, 1]$  with

$$(f, g) = \int_0^1 f(t) \overline{g(t)} dt$$

is an inner product space but *not* a Hilbert space.

#### 4.6 Continuity of the Inner Product

- For  $y \in H$  the following are continuous functions on  $H$ :

$$x \mapsto (x, y); \quad x \mapsto (y, x); \quad x \mapsto \|x\|.$$

#### 4.7 Subspaces

- $M \subseteq V$ ,  $V$  a vector space, is a **subspace** if  $M$  is also a vector space.
- A **closed subspace** of  $H$  is a subspace that is a closed set in the metric topology of  $H$ .
- If  $M$  is a subspace of  $H$  then so is its closure  $\overline{M}$ .

#### 4.8 Convex Sets

- $E \subseteq V$  is **convex** if  $(1 - t)x + ty \in E$  for all  $x, y \in E$ , and  $0 < t < 1$ .
- $E$  is convex if it contains every segment between any two of its points.
- Every subspace of a vector space is convex, and every translate of a convex set is convex.

#### 4.9 Orthogonality

- $x$  is **orthogonal** to  $y$  if  $(x, y) = 0 = (y, x)$ , written  $x \perp y$ .
- $x^\perp = \{y \in H : x \perp y\}$  is the closed subspace of  $H$  here  $y \mapsto (x, y)$  is identically zero.
- If  $M$  is a subspace of  $H$  then

$$M^\perp = \bigcap_{x \in M} x^\perp$$

is the closed subspace of  $H$  of vectors orthogonal to *every* vector in  $M$ .

#### 4.10 Unique Element of Minimal Norm

- If  $E \subseteq H$  is nonempty, closed, and convex then there is a unique element of minimal norm.

#### 4.11 Orthogonal Decompositions

$M$  is a closed subspace of  $H$ :

- Every  $x \in H$  decomposes uniquely into a sum  $Px + Qx$ ,  $Px \in M$ ,  $Qx \in M^\perp$ .
- $Px$  and  $Qx$  are the points in  $M$  and  $M^\perp$  nearest to  $x$ .
- $P : H \rightarrow M$  and  $Q : H \rightarrow M^\perp$  are linear.
- $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ .
- If  $M \neq H$  then there exists  $y \in H, y \neq 0$  with  $y \perp M$ .

#### 4.12 Unique Representation of Continuous Linear Functionals

- If  $L$  is a continuous linear functional on  $H$  then there is a unique  $y \in H$  with  $Lx = (x, y)$ .

### Orthonormal Sets

#### 4.13 Definitions

- If  $x_1, \dots, x_k \in V$  and  $c_1, \dots, c_k$  are scalars then  $c_1x_1 + \dots + c_kx_k$  is a **linear combination** of  $x_1, \dots, x_k$ .
- $\{x_1, \dots, x_k\}$  is **independent** if  $c_1x_1 + \dots + c_kx_k = 0$  implies  $c_1 = \dots = c_k = 0$ .
- $S \subseteq V$  is **independent** if every finite subset of  $S$  is independent.
- $[S]$ , the set of all finite linear combinations of  $S$ , is the smallest subspace of  $V$  containing  $S$ .
- $[S]$  is called the **span** of  $S$ .
- $\{u_\alpha\}_{\alpha \in A}$  is **orthonormal** if  $\|u_\alpha\| = 1$  for all  $\alpha \in A$  and  $u_\alpha \perp u_\beta$  for  $\alpha \neq \beta$ .
- If  $\{u_\alpha\}_{\alpha \in A}$  is orthonormal then the **Fourier coefficients** of  $x$  (relative to  $\{u_\alpha\}_{\alpha \in A}$ ) are the values of

$$\hat{x} : A \rightarrow \mathbb{C}, \quad \hat{x}(\alpha) = (x, u_\alpha).$$

#### 4.14 Fourier Coefficients and Approximations in Finite Subspaces

$\{u_\alpha\}_{\alpha \in A}$  are orthonormal in  $H$ ,  $F \subseteq A$  is finite, and  $M_F$  is the span of  $\{u_\alpha\}_{\alpha \in F}$ .

- If  $\varphi : A \rightarrow \mathbb{C}$  is 0 outside  $F$  then

$$y = \sum_{\alpha \in F} \varphi(\alpha) u_\alpha \in M_F$$

has the properties  $\hat{y}(\alpha) = \varphi(\alpha)$  and  $\|y\|^2 = \sum_{\alpha \in F} |\varphi(\alpha)|^2$ .

- $\|x - s_F(x)\| < \|x - s\|$  for every  $s \in M_F \setminus \{s_F(x)\}$ , where

$$s_F(x) = \sum_{\alpha \in F} \hat{x}(\alpha) u_\alpha.$$

i.e.  $s_F(x)$  is the unique best approximation of  $x$  in  $M_F$  relative to the norm induced metric.

- For  $x \in H$ ,

$$\sum_{\alpha \in F} |\hat{x}(\alpha)|^2 \leq \|x\|^2.$$

### 4.15 Infinite Sums

- Assume  $0 \leq \varphi(\alpha) \leq \infty$  for  $\alpha \in A$ . Then

$$\sum_{\alpha \in A} \varphi(\alpha)$$

is the supremum of all finite sums  $\varphi(\alpha_1) + \cdots + \varphi(\alpha_n)$  ( $\alpha_k$ 's distinct).

- This is just the Lebesgue integral of  $\varphi$  with respect to the counting measure on  $A$ .
- In this context we write  $\ell^p(A)$  instead of  $L^p(\mu)$ .  $\varphi : A \rightarrow \mathbb{C} \in \ell^p(A)$  if and only if  $\sum_{\alpha \in A} |\varphi(\alpha)|^p < \infty$ .
- Example 2 of 4.5 shows that  $\ell^2(A)$  is a Hilbert space with inner product:

$$(\varphi, \psi) = \sum_{\alpha \in A} \varphi(\alpha) \overline{\psi(\alpha)}$$

where again the sum is the integral of  $\varphi \overline{\psi}$  with respect to the counting measure. Note  $\varphi \overline{\psi} \in \ell^1(A)$ .

- 3.13 shows that functions identically zero outside some finite subset of  $A$  are dense in  $\ell^p(A)$ .
- If  $\varphi \in \ell^2(A)$  then  $\{\alpha \in A : \varphi(\alpha) \neq 0\}$  is at most countable.

*Note:* The above uses  $\ell^p$  in some places above where the text uses  $\ell^2$ .

### 4.16 Isometry Extension

- Suppose that
  1.  $X$  and  $Y$  re metric spaces with  $X$  complete;
  2.  $f : X \rightarrow Y$  is continuous;
  3.  $X_0$  is dense in  $X$  and  $f$  is an isometry on  $X_0$ ; and
  4.  $f(X_0)$  is dense in  $Y$ .

Then  $f$  is an isometry of  $X$  onto  $Y$ .

*Recall:* An isometry preserves distances.

### 4.17 The Bessel Inequality & The Riesz-Fischer Theorem

- Let  $\{u_\alpha\}_{\alpha \in A}$  be orthonormal in  $H$  and  $P$  the span of  $\{u_\alpha\}_{\alpha \in A}$ . Then

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leq \|x\|^2$$

holds for all  $x \in H$ . Moreover,  $x \mapsto \hat{x}$  is a continuous linear mapping of  $H$  onto  $\ell^2(A)$  whose restriction to  $\overline{P}$  is an isometry onto  $\ell^2(A)$ .

- The fact that  $x \mapsto \hat{x}$  carries  $H$  onto  $\ell^2(A)$  is the Riesz-Fischer theorem.

#### 4.18 Maximal Orthonormal Set Conditions

- If  $\{u_\alpha\}_{\alpha \in A}$  are orthonormal, then the following are equivalent:
  1.  $\{u_\alpha\}_{\alpha \in A}$  is a maximal orthonormal set in  $H$ .
  2. The span of  $\{u_\alpha\}_{\alpha \in A}$  is dense in  $H$ .
  3.  $\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = \|x\|^2$  holds for all  $x \in H$ .
  4.  $\sum_{\alpha \in A} \hat{x}(\alpha) \overline{\hat{y}(\alpha)} = (x, y)$  holds for all  $x, y \in H$ .
- The last formula is Parseval's identity. Note  $\hat{x}\hat{y} \in \ell^1(A)$  since  $\hat{x}, \hat{y} \in \ell^2(A)$ .
- A maximal orthonormal set is usually called a **complete orthonormal set** or an **orthonormal basis**.

#### 4.19 Isomorphisms

- Two vector spaces are **isomorphic** if there is a linear bijection between them.
- Two Hilbert spaces are **isomorphic** if there is a linear bijection between them which preserves inner products:  $(x, y) = (\Lambda x, \Lambda y)$  for all  $x, y$ .
- With the above, we have the following from 4.17 and 4.18:  
If  $\{u_\alpha\}_{\alpha \in A}$  is a maximal orthonormal set in  $H$  then  $x \mapsto \hat{x}$  is an isomorphism between  $H$  and  $\ell^2(A)$ .
- With 4.22 we show that every nontrivial Hilbert space has a maximal orthonormal set and so is isomorphic to some  $\ell^2(B)$ .

#### 4.20 Partially Ordered Sets

- $\mathcal{P}$  is **partially ordered** by  $\leq$  if  $\leq$  is reflexive, anti-symmetric, and transitive.
- $\mathcal{L} \subseteq \mathcal{P}$  is **totally ordered** (or **linearly ordered**) if either  $a \leq b$  or  $b \leq a$  for every  $a, b \in \mathcal{L}$ .

#### 4.21 Hausdorff Maximality Theorem

- Every non-empty partially ordered set contains a maximal totally ordered subset.

#### 4.22 Existence of Maximal Orthonormal Sets

- Every orthonormal set in a Hilbert space is contained in a maximal orthonormal set.

### Trigonometric Series

#### 4.23 Definitions

- The  $2\pi$ -periodic functions on  $\mathbb{R}^1$  can be identified with the functions on the unit circle  $T \subseteq \mathbb{C}$ .

*Note:* We can thus switch between writing  $f(t)$  and  $f(e^{it})$  where  $t \in [0, 2\pi]$ .

- $L^p(T)$ ,  $1 \leq p < \infty$ , consists of all complex, Lebesgue measurable,  $2\pi$ -periodic functions on  $\mathbb{R}^1$  for which

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{\frac{1}{p}} < \infty.$$

- $L^\infty(T)$  consists of all  $2\pi$ -periodic members of  $L^\infty(\mathbb{R}^1)$  with the essential supremum norm.

- $C(T)$  consists of all continuous complex functions on  $T$  with norm  $\|f\|_\infty = \sup_{t \in [0, 2\pi]} |f(t)|$ .
- A **trigonometric polynomial** is a sum of the form  $f(t) = \sum_{n=-N}^N c_n e^{int}$ .
- Define the inner product in  $L^2(T)$  by

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

- $\{u_n\}_{n \in \mathbb{Z}}$ , where  $u_n(t) = e^{int}$ , is an orthonormal set in  $L^2(T)$  called the **trigonometric system**.

#### 4.24 Completeness of the Trigonometric System

- If we can prove that  $f \in C(T)$  can be approximated by trigonometric polynomials then since  $C(T)$  is dense in  $L^2(T)$  (see 3.14) the trigonometric system is complete will be complete by 4.18.

#### 4.25 Approximations in $C(T)$ by Trigonometric Polynomials

- If  $f \in C(T)$ ,  $\epsilon > 0$ , then there exists a trigonometric polynomial  $P$  with  $|f - P| < \epsilon$ .
- By 4.24, the trigonometric system is complete.

#### 4.26 Fourier Series

- For  $f \in L^1(T)$  the **Fourier coefficients** of  $f$  are defined, for  $n \in \mathbb{Z}$ , by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{int} dt.$$

- The **Fourier series** of  $f$  is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$$

with partial sums

$$s_N(t) = \sum_{n=-N}^N \hat{f}(n) e^{int}.$$

- *Riesz-Fischer Theorem* now states if  $\{c_n\}$  are complex numbers with  $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$  then there exists  $f \in L^2(T)$  such that, for  $n \in \mathbb{Z}$ ,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

- *Parseval's theorem* asserts that, for  $f, g \in L^2(T)$ ,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

- The  $L^2(T)$  limit of partial sums  $s_N$  of  $f$  converges in the  $L^2(T)$  norm to  $f$ .
- Associating  $f$  with  $\hat{f}$  is a Hilbert space isomorphism between  $L^2(T)$  and  $\ell^2(\mathbb{Z})$ .