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# Lecture 6 Model Selection II

## STAT 441/505: Applied Statistical Methods in Data Mining

Linglong Kong

Department of Mathematical and Statistical Sciences
University of Alberta

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# Outline

Ridge Regression

The LASSO

Ridge regression and the LASSO

Summary and Remark

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# Ridge Regression

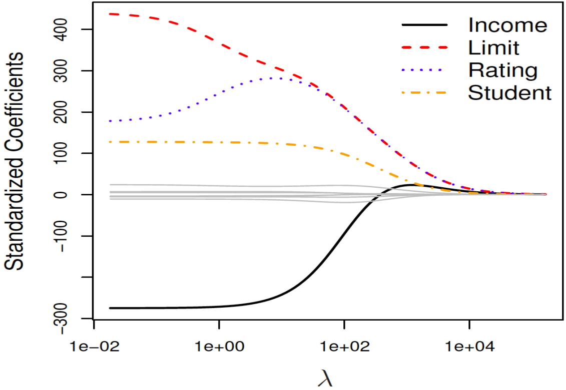
- The ridge regression coefficient estimates $\hat{\beta}^R$ are the values that minimize

$$\sum_i \left( y_i - \beta_0 - \sum_j \beta_i x_{ij} \right)^2 + \lambda \sum_j \beta_j^2,$$

where $\lambda$ is a tuning parameter, to be determined separately.

- The second term $\lambda \sum_j \beta_j^2$ called a shrinkage penalty, is small when $\beta_j,\ j \geq 1$ are close to zero, and so it has the effect t of shrinking the estimates of $\beta_j$ towards zero.
- The tuning parameter $\lambda$ serves to control the relative impact of these two terms on the regression coefficient estimates.
- Selecting a good value for $\lambda$ is critical; cross-validation is used for this.

# Credit data example




As $\lambda$ increases, the coefficients are shrunken to zeros.

# Scaling of predictors

- The standard least squares coefficient estimates are scale equivariant: multiplying $X_j$ by a constant $c$ simply leads to a scaling of the least squares coefficient estimates by a factor of $1/c$. In other words, regardless of how the $j$-th predictor is scaled $X_j\hat{\beta}_j$ will remain the same.
- In contrast, the ridge regression coefficient estimates can change substantially when multiplying a given predictor by a constant, due to the sum of squared coefficient term in the penalty part of the ridge regression objective function.
- Therefore, it is best to apply ridge regression after standardizing the predictors, using the formula

$$\tilde{x}_{ij} = x_{ij} \Big/ \sqrt{\sum_i (x_{ij} - \bar{x}_j)^2 / n}.$$

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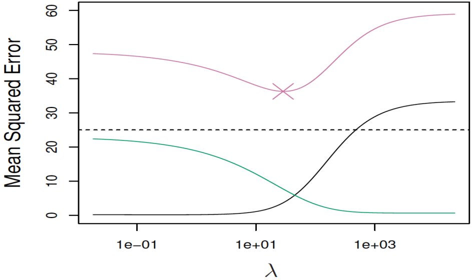
# Credit data example




Simulated data with $n = 50$ observations, $p = 45$ predictors, all having nonzero coefficient. Squared bias (black), variance (green), and test mean squared error (purple) for the ridge regression predictions on a simulated data set. The horizontal dashed lines indicate the minimum possible MSE. The purple crosses indicate the ridge regression models for which the MSE is smallest.

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# The LASSO

- Ridge regression, unlike subset selection, will generally select models that involve just a subset of the variables, ridge regression will include all p predictors in the final model.
- The LASSO is a relatively recent alternative to ridge regression that overcomes this disadvantage. The lasso coefficient $\hat{\beta}^L$ minimize the quantity

$$\sum_i \left( y_i - \beta_0 - \sum_j \beta_i x_{ij} \right)^2 + \lambda \sum_j |\beta_j|,$$

where $\lambda$ is a tuning parameter.

- The LASSO uses $l_1$ penalty instead of $l_2$ (ridge regression).

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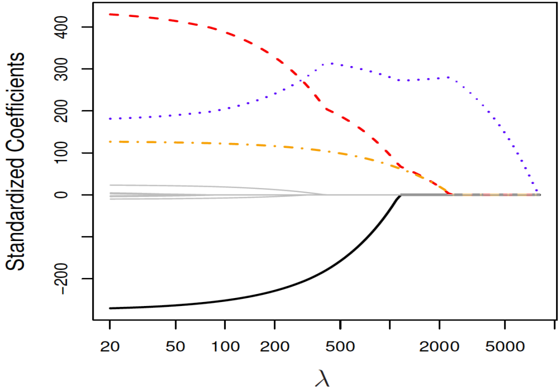
# The LASSO

- As with ridge regression, the lasso shrinks the coefficient estimates towards zero as $\lambda$ increases.
- However, in the case of the lasso, the $l_1$ penalty has the effect of forcing some of the coefficient estimates to be exactly equal to zero when the tuning parameter $\lambda$ is sufficientl large. Thus performs variable selection.
- We say that the lasso yields sparse models — that is, models that involve only a subset of the variables.
- Selecting a good value for $\lambda$ is critical; cross-validation is again used for this.

# Credit data example




As $\lambda$ increases, the coefficients are shrunken to exact zeros.

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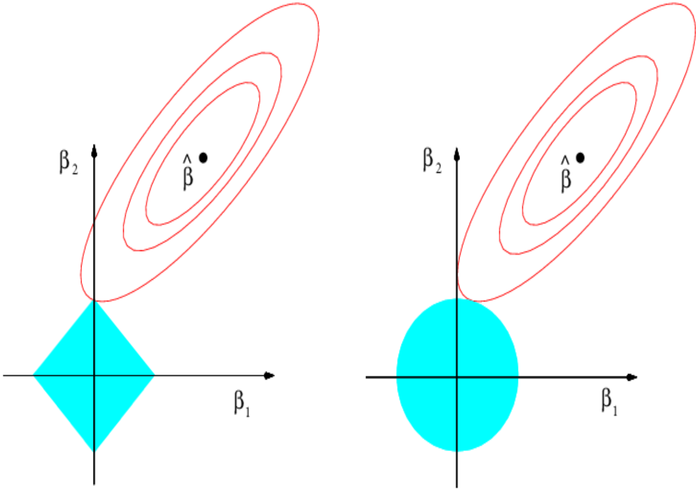
# Ridge regression and the LASSO

- Why is it that the lasso, unlike ridge regression, results in coefficient estimates that are exactly equal to zero?
- One can show that the lasso and ridge regression coefficient estimates solve the problems

$$\min_{\beta} \sum_i \left( y_i - \beta_0 - \sum_j \beta_i x_{ij} \right)^2, \text{ subject to } \sum_j |\beta_j| \le c;$$

$$\min_{\beta} \sum_i \left( y_i - \beta_0 - \sum_j \beta_i x_{ij} \right)^2, \text{ subject to } \sum_j \beta_j^2 \le c;$$

# Ridge regression and the LASSO



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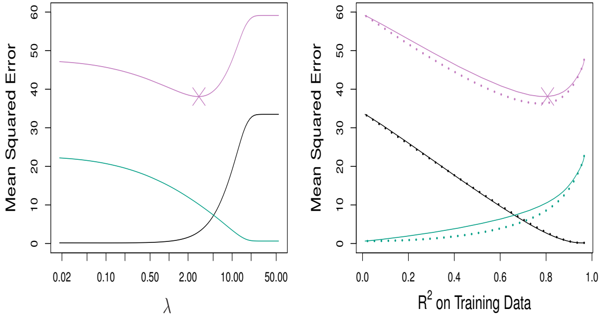
# Credit data example




Left: Plots of squared bias (black), variance (green), and test mean squared error (purple) for the LASSO on a simulated data set. Right: Comparison of squared bias, variance and test MSE between lasso (solid) and ridge (dashed). The purple crosses indicate the LASSO models for which the MSE is smallest.

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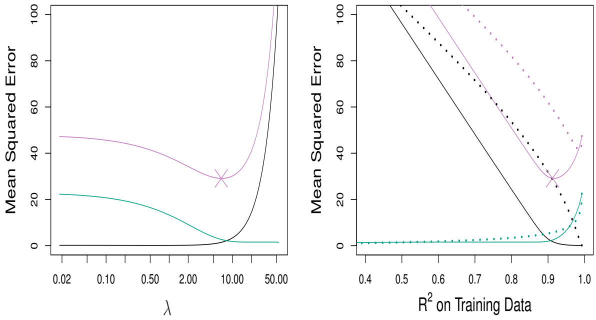
# Credit data example




Left: Plots of squared bias (black), variance (green), and test mean squared error (purple) for the LASSO on another simulated data set. Right: Comparison of squared bias, variance and test MSE between lasso (solid) and ridge (dashed). The purple crosses indicate the LASSO models for which the MSE is smallest.

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# Conclusions

- These two examples illustrate that neither ridge regression nor the lasso will universally dominate the other.
- In general, one might expect the lasso to perform better when the response is a function of only a relatively small number of predictors.
- However, the number of predictors that is related to the response is never known a priori for real data sets.
- A technique such as cross-validation can be used in order to determine which approach is better on a particular data set.

# Summary and Remark

- Ridge Regression
- The LASSO
- Ridge Regression and the LASSO
- Read textbook Chapter 3
- Do R lab