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Predictive Econometrics and Big Data

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Preface

Econometrics is a branch of economics that uses mathematical (especially statistical) methods to analyze economic systems, to forecast economic and financial dynamics, and to develop strategies for achieving desirable economic performance.

Traditional econometric techniques have been focused on the quantitative description of economic phenomena. However, the ultimate goal of econometrics—as well as the ultimate goal of science in general—is to predict future development of economics and to develop strategies that optimize the future state of economics. It is therefore desirable to develop techniques that are specifically aimed at predicting economic phenomena. Such predictive econometric techniques—and their applications to real-life economic and financial situations—are one of the main foci of this volume.

Another focus of this book is related to the fact that in the modern world, in which computers are ubiquitous, the amount of economic-related data generated and processed by these computers has grown exponentially. The amount of available economic data is so huge that many traditional statistical data processing algorithms are no longer capable of processing all these data in real time. To process this data, we need to utilize "big data" techniques specifically developed for processing such huge amounts of data and we need to develop big data versions of the state-of-the-art econometric techniques and algorithms. This is a new and promising direction in econometrics. Big data is the main subject of this volume's keynote paper by Dr. Chaitanya Baru from the US National Science Foundation.

In addition to papers on predictive econometric techniques and on big data applications, this book also contains applications of more traditional statistical techniques to econometric problems.

We hope that this volume will help practitioners to learn how to apply new predictive and big data econometric techniques and help researchers to further improve the existing predictive and big data techniques and to come up with new ideas on how econometric techniques can utilize large amounts of data to make more accurate predictions.

We want to thank all the authors for their contributions and all anonymous referees for their thorough analysis and helpful comments.

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Contents

Joint Plausibility Regions for Parameters of Skew Normal Family

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Abstract. The estimation of parameters is a challenge issue for skew normal family. Based on inferential models, the plausibility regions for two parameters of skew normal family are investigated in two cases, when either the scale parameter σ or the shape parameter δ is known. For illustration of our results, simulation studies are proceeded.

Keywords: Skew normal distribution · Inferential models
Plausibility regions

1 Introduction

Skew data sets occur in many diverse fields, such as economics, finance, biomedicine, environment, demography, and pharmacokinetics, just to name a few. In conventional procedure, practitioners assume that the data are normally distributed to proceed statistical analysis. This restrictive assumption, however, may result in not only a lack of robustness against departures from the normal distribution and but also in invalid statistical inferences, especially when data are skewed. One solution to analyze skewed data is to extend the normal family by introducing an extra parameter, thus getting skew normal distributions which have location, scale and shape parameters (see Azzalini and his collaborators' work [2,6] and reference therein). In many practical cases, skew normal distribution is suitable for the analysis of data which is unimodal empirical distributed but with some skewness (see Arnold [1], Hill [8]). In past three decades, the family of skew-normal distributions, including multivariate skew-normal distributions, has been studied by many authors, e.g. Azzalini [3–5], Wang et al. [17], Ye et al. [18].

However, estimating parameters of skew normal family is a challenge, especially in relation to estimating shape parameter (see Azzalini [4] and Pewsey [15]). Liseo and Loperfido [9] pointed out the likelihood function is

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increasing with positive probability, which leads to an infinite maximum likelihood estimate for shape parameter, and the method of moments can give even worse results. New methods are needed to solve this problem. Some methods for estimation of parameters were studied by many authors, see Azzalini and Capitanio [4], Sartori [16], Liseo and Loperfido [9], Debarshi [7] and Mameli et al. [14]. In particular, Zhu et al. [19] applied inferential models (IMs) to construct plausibility interval for shape parameter.

In this study, we construct plausibility regions for two parameters of skew normal population in two cases, with either shape parameter or scale parameter is known, by inferential models (IMs). IMs are new methods of statistical inference introduced by Martin and Liu [10,12]. Comparing with Fisher's fiducial inference, Dempster-Shafer theory of belief functions and Bayesian inference, IMs have several advantages: (i) IMs are free of prior distributions; (ii) IMs depend only on the observed data. For more details of IMs, see Martin and his collaborators' work [10–13].

This paper is organized as following. The basic concepts on skew-normal distributions and IMs are introduced briefly in Sect. 2. Plausibility regions for the parameters of skew normal population are obtained in two cases in Sect. 3. Simulation studies are proceeded for illustration of our main result in Sect. 4.

2 Preliminaries

Throughout of this paper, we use $\phi(\cdot)$ and $\Phi(\cdot)$ to denote the probability density function (pdf) and cumulative distribution function (cdf) of the standard normal distribution, respectively. Let $F(\cdot)$ be the cdf of $\chi^2(0)$ distribution and $G(\cdot)$ be the cdf of skew normal distribution, and $N(0,1)$ and Uniform(0, 1) represent the standard normal and uniform distributions, respectively.

2.1 Brief Review of Skew-Normal Distributions

A random variable Z is said to be *skew-normal* distributed with the *shape parameter* λ, denoted by $Z \sim SN(\lambda)$, if its pdf is given by

$$f(z;\lambda) = 2\phi(z)\Phi(\lambda z), \qquad z, \lambda \in \mathbb{R}.$$

For any $\mu \in \mathbb{R}$ and $\sigma > 0$, the distribution of $X = \mu + \sigma Z$ is said to be *skew-normal* distributed with the *location parameter* μ, the *scale parameter* σ and the *shape parameter* λ, denoted by $X \sim SN(\mu, \sigma^2, \lambda)$ and the pdf of X is

$$f(x;\mu,\sigma^2,\lambda) = \frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{\lambda(x-\mu)}{\sigma}\right).$$

There is an alternative representation of $Z \sim SN(0,1,\lambda)$ given by

$$Z = \delta|Z_0| + \sqrt{1-\delta^2}Z_1, \qquad \delta \in (-1,1) \tag{1}$$

where Z_0 and Z_1 are independent $N(0,1)$ random variables and $\delta = \lambda/\sqrt{1+\lambda^2}$. This stochastic representation plays a vital role in establishing our IMs. See Azzalini and Capitanio's book [5] and reference therein for more details.

2.2 Inference Models

Let X be an observable random sample with a probability distribution $P_{X|\theta}$ on a sample space $\mathbb{X}$, where θ is an unknown parameter, $\theta \in \Theta$, a parameter space. Let U be an unobservable *auxiliary variable* on an auxiliary space $\mathbb{U}$, where although U is unobservable, we assume that U and $\mathbb{U}$ are known. An *association* is a map $a : \mathbb{U} \times \Theta \to \mathbb{X}$ such that

$$X = a(U, \theta).$$

For any given statistical assertion on parameters, an IM consists of following three steps.

Association Step (A-step). Suppose we have an association $X = a(U, \theta)$ and an observation $X = x$, where x could be a scalar or vector, then the unknown θ must satisfy

$$x = a(u^*, \theta)$$

for some unobserved u^* of U. So from the observation $X = x$, we have the set of solutions

$$\Theta_x(u) = \{\theta \in \Theta : x = a(u, \theta)\}, \qquad x \in \mathbb{X}, \qquad u \in \mathbb{U}.$$

Prediction Step (P-step). Since the true u^* is unobservable, to make a valid inference, the key point is to predicate u^*. Let $u \to S(u)$ be a set-value map from $\mathbb{U}$ to $\mathbb{S}$, a collection of P_U-measurable subsets of $\mathbb{U}$. Then the random set $S : \mathbb{U} \to \mathbb{S}$ is called a *predictive random set* of U with distribution $P_S = P_U \circ S^{-1}$. We will use S to predict u^*.

Combination Step (C-step). Define

$$\Theta_x(S) = \bigcup_{u \in S} \Theta_x(u).$$

For any assertion A of θ, i.e., $A \subseteq \Theta$, the *belief function* and *plausibility function* of A with respect to a predictive random set S are defined by,

$$\mathsf{bel}_x(A; S) = P_S\{\Theta_x(S) \subseteq A : \Theta_x(S) \neq \emptyset\};$$

$$\mathsf{pl}_x(A; S) = P_S\{\Theta_x(S) \nsubseteq A^c : \Theta_x(S) \neq \emptyset\}.$$

Note that

$$\mathsf{pl}_x(A; S) = 1 - \mathsf{bel}_x(A^c; S), \qquad \mathsf{bel}_x(A; S) + \mathsf{bel}_x(A^c; S) \leq 1, \qquad \text{for all} \quad A \subseteq \Theta.$$

Based on the plausibility function derived form an IM, the $100(1 - \alpha)\%$ level plausibility region for θ follows

$$\Pi_X(\alpha) = \{\theta : \mathsf{pl}_X(\theta; S) > \alpha\},$$

which is counter part of confidence regions in classical statistics.

3 Plausibility Regions for Parameters of the Skew Normal Family

Suppose that $X_1, \ldots, X_n$ are identical distributed random variables from the population $SN(\mu, \sigma^2, \lambda)$ with stochastic representations

$$X_i = \mu + \sigma(\delta|Z_0| + \sqrt{1-\delta^2} Z_i), \qquad i = 1, \ldots, n, \tag{2}$$

where $(Z_0, Z_1, \cdots, Z_n)' \sim N_{n+1}(0, I_{n+1})$ and $\delta = \lambda/\sqrt{1+\lambda^2}$. See Azzalini and Capitanio [5] for details.

It is clear that $X_i \sim SN(\mu, \sigma^2, \lambda)$ for $i = 1, \cdots, n$, but they are dependent since they share the same component $|Z_0|$ when $\delta \neq 0$. The following result is needed for establishing our IMs.

Theorem 1. *Let $X_1, \ldots, X_n$ be identically distributed with stochastic representations given in (2). Let the sample mean and sample variance be*

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \textit{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2,$$

respectively. Then $\bar{X}$ and S^2 are independent,

$$\bar{X} \sim SN\left(\mu, \frac{n\lambda^2+1}{n(1+\lambda^2)}\sigma^2, \sqrt{n}\lambda\right) \qquad \textit{and} \qquad \frac{(n-1)S^2}{\sigma^2(1-\delta^2)} \sim \chi^2_{n-1}(0).$$

Proof. From the stochastic representations (2), it is easy to obtain

$$\bar{X} = \mu + \sigma\left(\delta|Z_0| + \sqrt{1-\delta^2}\bar{Z}\right) \quad \text{and} \quad S^2 = \frac{\sigma^2(1-\delta^2)}{n-1}\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right)^2,$$

where $\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$. Note that $\bar{Z}$ and $(Z_1 - \bar{Z})$'s are independent so that $\bar{X}$ and S^2 are independent. For the distribution of $\bar{X}$, let $Z_* = \sqrt{n}\bar{Z}$, which is distributed as $N(0,1)$. Then

$$\begin{aligned}\bar{X} &= \mu + \sigma\left(\delta|Z_0| + \sqrt{\frac{1-\delta^2}{n}}Z_*\right)\\ &= \mu + \sigma_*\left(\delta_*|Z_0| + \sqrt{1-\delta_*^2}Z_*\right),\end{aligned}$$

where $\delta_* = \frac{\sqrt{n}\delta}{\sqrt{1+(n-1)\delta^2}}$ and $\sigma_* = \sigma\sqrt{\frac{1+(n-1)\delta^2}{n}}$. Thus by definition, we have $\bar{X} \sim SN(\mu, \sigma_*^2, \lambda_*)$, where

$$\lambda^* = \frac{\delta_*}{\sqrt{1-\delta_*^2}} = \frac{\sqrt{n}\delta}{\sqrt{1-\delta^2}} = \sqrt{n}\lambda,$$

the distribution of $\bar{X}$ is obtained. The distribution of S^2 can be obtained directly from

$$\frac{(n-1)S^2}{\sigma^2(1-\delta^2)} = \sum_{i=1}^{n}(Z_i - \bar{Z})^2 \sim \chi^2_{n-1}(0).$$

□

3.1 Plausibility Function and Plausibility Region of (μ, σ) When δ Is Known

Assume that the skewness parameter λ (or δ) is known. We want to construct the plausibility region for unknown parameters (μ, σ) based on a sample $X_1, \ldots, X_n$ from a skew normal population.

A-step. From Theorem 1, we can have associations

$$\frac{(n-1)S^2}{\sigma^2(1-\delta^2)} = F_{n-1}^{-1}(U_1) \qquad \text{and} \qquad \bar{X} = \mu + G^{-1}(U_2), \tag{3}$$

where $F_{n-1}(\cdot)$ and $G(\cdot)$ are the cdf's of $\chi^2_{n-1}(0)$ and $SN\left(0, \frac{n\lambda^2+1}{n(1+\lambda^2)}\sigma^2, \sqrt{n}\lambda\right)$, respectively, and U_1, U_2 are independent uniformly distributed in interval $(0, 1)$. Thus for any observations $\bar{x}$ and s^2, and $u_1, u_2 \in (0, 1)$, we have the solution set

$$\begin{aligned}\Theta_{(\bar{x},s^2)}(\mu, \sigma) &= \left\{(\mu, \sigma) : \bar{x} = \mu + G^{-1}(u_2), \frac{(n-1)\,s^2}{\sigma^2(1-\delta^2)} = F_{n-1}^{-1}(u_1)\right\} \\ &= \left\{(\mu, \sigma) : G(\bar{x} - \mu) = u_2, F_{n-1}\left(\frac{(n-1)\,s^2}{\sigma^2(1-\delta^2)}\right) = u_1\right\}.\end{aligned}$$

P-step. To predict auxiliary variables U_1 and U_2, we use the default predictive random set

$$S(U_1, U_2) = \{(u_1, u_2) : \max\{|u_1 - 0.5|, |u_2 - 0.5|\} \leq \max\{|U_1 - 0.5|, |U_2 - 0.5|\}\}.$$

C-step. By the P-step, we have the combined set

$$\begin{aligned}\Theta_{(\bar{x},S^2)}(S) = \Bigg\{(\mu, \sigma) : \max\left\{\left|G(\bar{x} - \mu) - 0.5\right|, \left|F_{n-1}\left(\frac{(n-1)\,s^2}{\sigma^2(1-\delta^2)}\right) - 0.5\right|\right\} \\ \leq \max\left\{\left|U_1 - 0.5\right|, \left|U_2 - 0.5\right|\right\}\Bigg\}.\end{aligned}$$

Theorem 2. *For any singleton assertion* $A = \{(\mu, \sigma)\}$,

$$bel_{(\bar{x},s^2)}(A; S) = 0,$$

$$pl_{(\bar{x},s^2)}(A; S) = 1 - \max\left\{\left|2G(\bar{x} - \mu) - 1\right|, \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right) - 1\right|\right\}^2,$$

and the $100(1 - \alpha)\%$ *plausibility region*

$$\Pi_{\bar{X},S^2}(\mu, \sigma) = \{(\mu, \sigma) : pl_{(\bar{x},s^2)}(\mu, \sigma) \geq \alpha\}.$$

Proof. It is clear that $\{\Theta_{(\bar{x},s^2)}(S) \subseteq A\} = \emptyset$, so $\mathsf{bel}_{(\bar{x},s^2)}(A;S) = 0$.

$$\begin{aligned}\mathsf{pl}_{(\bar{x},s^2)}(A;S) &= 1 - \mathsf{bel}_{(\bar{x},s^2)}(A^c;S) = 1 - P_S\left(\Theta_{(\bar{x},S^2)}(S) \subseteq A^c\right) \\ &= 1 - \max\left\{\left|2G(\bar{x}-\mu) - 1\right|, \left|2F_{n-1}\left(\tfrac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right) - 1\right|\right\}^2.\end{aligned}$$

Thus we can obtain the $100(1-\alpha)\%$ plausibility region by its definition. □

The following example is used for the illustration of Theorem 2.

Example 3.1. *For a sample $X_1, \cdots, X_n$ from skew normal population as described above, graphs of plausibility function and the 95% plausibility region of (μ, σ) are listed in Figs. 1, 2 and 3.*

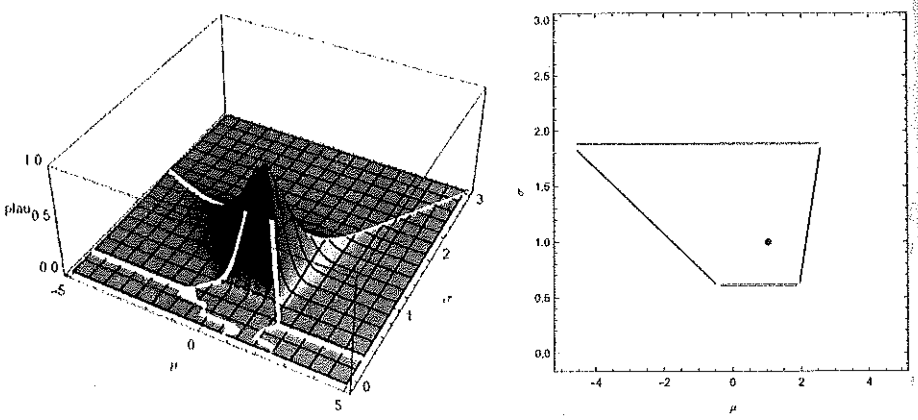

Fig. 1. Graphs of plausibility function and the 95% plausibility region of (μ, σ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 10$.

3.2 Plausibility Function and Plausibility Region for (μ, δ) When σ Is Known

Assume that the scale parameter σ is known. We want to construct the plausibility region for unknown parameters (μ, δ) based a sample $X_1, \ldots, X_n$ from a skew normal population.

A-step. From Theorem 1, we obtain the associations

$$\frac{(n-1)S^2}{\sigma^2(1-\delta^2)} = F_{n-1}^{-1}(U_1) \qquad \text{and} \qquad \bar{X} = \mu + G^{-1}(U_2), \tag{4}$$

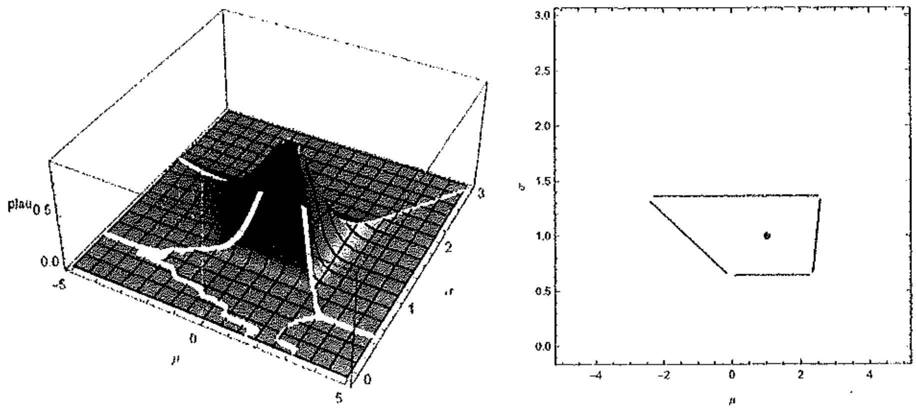

Fig. 2. Graphs of plausibility function and the 95% plausibility region of (μ, σ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 20$.

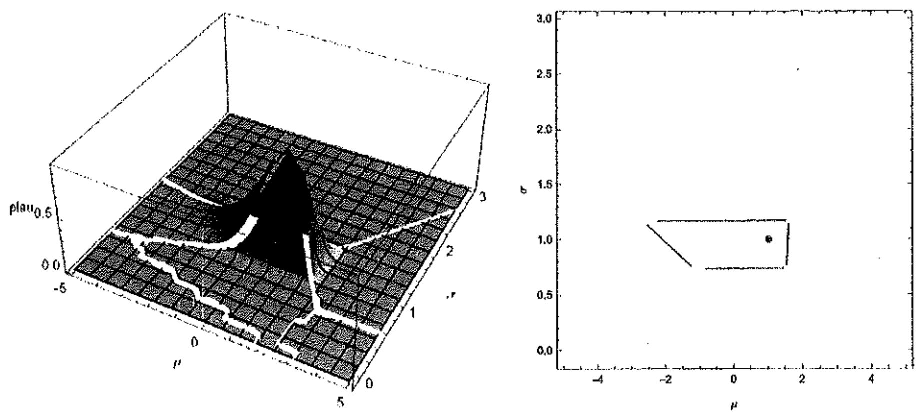

Fig. 3. Graphs of plausibility function and the 95% plausibility region of (μ, σ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 50$.

where $F_{n-1}(\cdot)$ and $G(\cdot)$ are cdf's of χ^2_{n-1} and $SN\left(0, \frac{n\lambda^2+1}{n(1+\lambda^2)}\sigma^2, \sqrt{n}\lambda\right)$ respectively, and U_1, U_2 are independent uniformly distributed in interval $(0,1)$. So for any observations $\bar{x}$ and s^2, and $u_1, u_2 \in (0,1)$, we have the solution set

$$\begin{aligned}\Theta_{(\bar{x},s^2)}(\mu,\delta) &= \left\{(\mu,\delta) : \bar{x} = \mu + G^{-1}(u_2), \frac{(n-1)s^2}{\sigma^2(1-\delta^2)} = F_{n-1}^{-1}(u_1)\right\}\\ &= \left\{(\mu,\delta) : G(\bar{x}-\mu) = u_2, F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right) = u_1\right\}.\end{aligned}$$

Note that to guarantee the solution set $\Theta_{(\bar{x},s^2)}(\mu,\delta) \neq \emptyset$, we need $u_1 \geq F\left(\frac{(n-1)s^2}{\sigma^2}\right)$.

P-step. To predict auxiliary variables U_1 and U_2, we should use an *elastic predictive random set* (see Martin and Liu [12] Chap. 5) as follow. If $F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) \leq \frac{1}{2}$, we take

$$S(U_1,U_2) = \{(u_1,u_2) : \max\{|u_1 - 0.5|, |u_2 - 0.5|\} \leq \max\{|U_1 - 0.5|, |U_2 - 0.5|\}\},$$

otherwise, we take

$$\begin{aligned}S(U_1,U_2) = \{(u_1,u_2) : \max&\left\{\left|F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) - 0.5\right|, \left|u_1 - 0.5\right|, \left|u_2 - 0.5\right|\right\}\\ &\leq \max\left\{\left|U_1 - 0.5\right|, |U_2 - 0.5|\right\}\}\end{aligned}$$

C-step. By P-step, we have the combined set in two cases. If $F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) \leq \frac{1}{2}$, then

$$\begin{aligned}\Theta_{(\bar{x},S^2)}(S) = \Bigg\{(\mu,\delta) : \max&\left\{\left|G(\bar{x}-\mu) - 0.5\right|, \left|F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right) - 0.5\right|\right\}\\ &\leq \max\left\{\left|U_1 - 0.5\right|, \left|U_2 - 0.5\right|\right\}\Bigg\}.\end{aligned}$$

Otherwise,

$$\begin{aligned}\Theta_{(\bar{x},S^2)}(S) = \Bigg\{(\mu,\delta) : \max&\left\{\begin{array}{c}\left|G(\bar{x}-\mu) - 0.5\right|, \left|F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right) - 0.5\right|,\\ \left|F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) - 0.5\right|\end{array}\right\}\\ &\leq \max\left\{\left|U_1 - 0.5\right|, \left|U_2 - 0.5\right|\right\}\Bigg\}.\end{aligned}$$

Theorem 3. *For any singleton assertion $A = \{(\mu,\delta)\}$,*

$$bel_{(\bar{x},s^2)}(A;S) = 0,$$

If $F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) \le \frac{1}{2}$, *then*

$$pl_{(\bar{x},s^2)}(A;S) = 1 - \max\left\{\left|2G(\bar{x}-\mu)-1\right|, \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right)-1\right|\right\}^2,$$

Otherwise,

$$pl_{(\bar{x},s^2)}(A;S) = 1 - \max\left\{\begin{array}{c}\left|2G(\bar{x}-\mu)-1\right|, \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right)-1\right|, \\ \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right)-1\right|\end{array}\right\}^2,$$

and the $100(1-\alpha)\%$ *plausibility region*

$$\Pi_{\bar{X},S^2}(\mu,\delta) = \{(\mu,\delta) : pl_{(\bar{x},s^2)}(\mu,\delta) \ge \alpha\}.$$

Proof. It is clear that $\{\Theta_{(\bar{x},s^2)}(S) \subseteq A\} = \emptyset$, so $\mathsf{bel}_{(\bar{x},s^2)}(A;S) = 0$. By definition of plausibility function, we can compute plausibility in two cases.

If $F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right) \le \frac{1}{2}$, we have

$$\begin{aligned}\mathsf{pl}_{(\bar{x},s^2)}(A;S) &= 1 - \mathsf{bel}_{(\bar{x},s^2)}(A^c;S) = 1 - P_S\left(\Theta_{(\bar{x},S^2)}(S) \subseteq A^c\right) \\ &= 1 - \max\left\{\left|2G(\bar{x}-\mu)-1\right|, \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right)-1\right|\right\}^2.\end{aligned}$$

Otherwise,

$$\begin{aligned}\mathsf{pl}_{(\bar{x},s^2)}(A;S) &= 1 - \mathsf{bel}_{(\bar{x},s^2)}(A^c;S) = 1 - P_S\left(\Theta_{(\bar{x},S^2)}(S) \subseteq A^c\right) \\ &= 1 - \max\left\{\begin{array}{c}\left|2G(\bar{x}-\mu)-1\right|, \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2(1-\delta^2)}\right)-1\right|, \\ \left|2F_{n-1}\left(\frac{(n-1)s^2}{\sigma^2}\right)-1\right|\end{array}\right\}^2 . \square\end{aligned}$$

Remark. Note that plausibility functions in Theorems 2 and 3 are the same because we use the same association, but the plausibility regions are different, because in Theorem 2, we use this plausibility function to solve plausibility regions for (μ,σ), while in Theorem 3, we use this plausibility function to obtain plausibility regions for (μ,δ).

Similarly, the following example is used for the illustration of Theorem 3.

Example 3.2. *For a sample* $X_1,\cdots,X_n$ *from skew normal population as described above, graphs of plausibility function and the* 95% *plausibility region of* (μ,δ) *are given in Figs. 4, 5 and 6 for sample sizes* $n = 10, 20$ *and* 50, *respectively.*

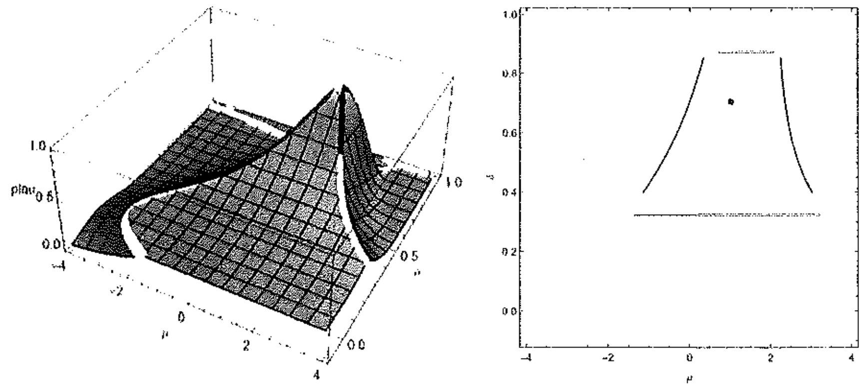

Fig. 4. Graphs of plausibility function and the 95% plausibility region of (μ, δ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 10$.

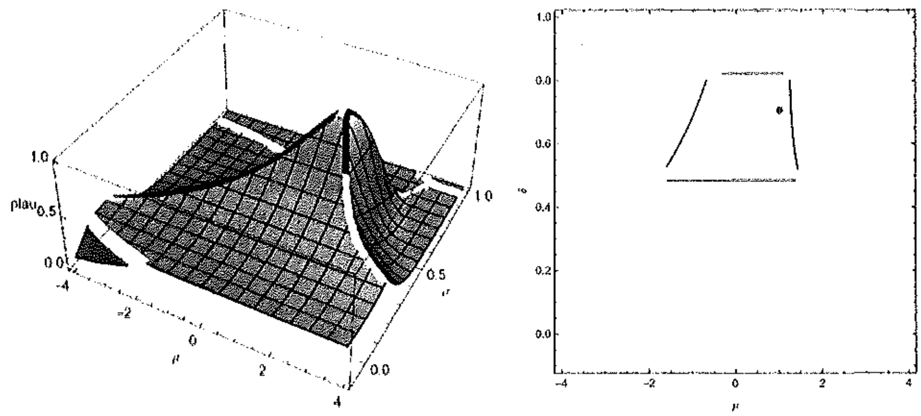

Fig. 5. Graphs of plausibility function and the 95% plausibility region of (μ, δ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 20$.

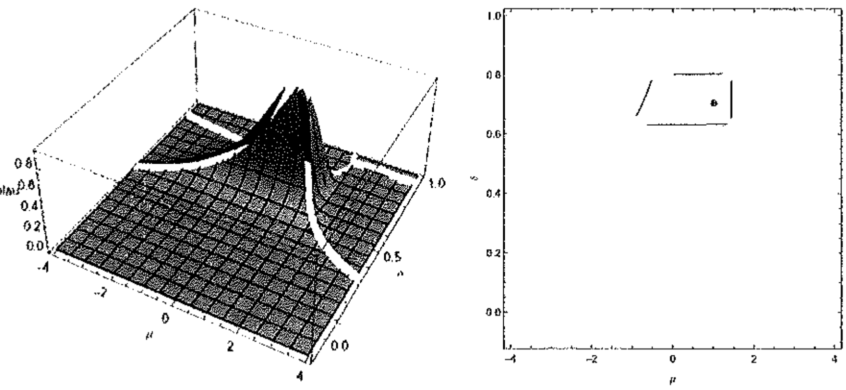

Fig. 6. Graphs of plausibility function and the 95% plausibility region of (μ, δ) based on simulated data with $\mu = 1, \sigma = 1, \delta = 1/\sqrt{2}$ and $n = 50$.

4 Simulation Studies

In this section, we perform two simulation studies on coverage probabilities of 95% plausibility regions for skew normal population of two cases discussed above.

4.1 Coverage Probabilities of the 95% Plausibility Regions for (μ, σ) When δ Is Known

When $\delta = 1/\sqrt{2}$, we choose sample sizes of 10, 20 and 50 and simulate 10,000 runs for different parameters, which is given in Table 1.

Table 1. Simulation results of coverage probabilities of the 95% plausibility regions for (μ, σ) when $\delta = 1/\sqrt{2}$.

n	$\mu = 0$				$\mu = 1$			
	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
10	0.9500	0.9449	0.9493	0.9503	0.9489	0.9540	0.9479	0.9513
20	0.9472	0.9524	0.9514	0.9493	0.9481	0.9502	0.9510	0.9463
50	0.9521	0.9537	0.9482	0.9499	0.9506	0.9485	0.9487	0.9507

4.2 Coverage Probabilities of 95% Plausibility Regions for (μ, δ) When σ Is Known

When $\sigma = 1$, we choose sample sizes of 10, 20 and 50 and run the simulation 10,000 times. Simulation results of coverage probabilities of the 95% plausibility regions for (μ, δ) is listed in Table 2.

Table 2. Simulation results of coverage probabilities of the 95% plausibility regions for (μ, δ) when $\sigma = 1$.

n	$\mu = 0$				$\mu = 1$			
	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.7$	$\delta = 0.9$	$\delta = 0.2$	$\delta = 0.4$	$\delta = 0.7$	$\delta = 0.9$
10	0.9495	0.9490	0.9518	0.9484	0.9502	0.9524	0.9513	0.9520
20	0.9487	0.9488	0.9493	0.9524	0.9515	0.9446	0.9481	0.9479
50	0.9513	0.9513	0.9508	0.9472	0.9489	0.9487	0.9540	0.9527

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