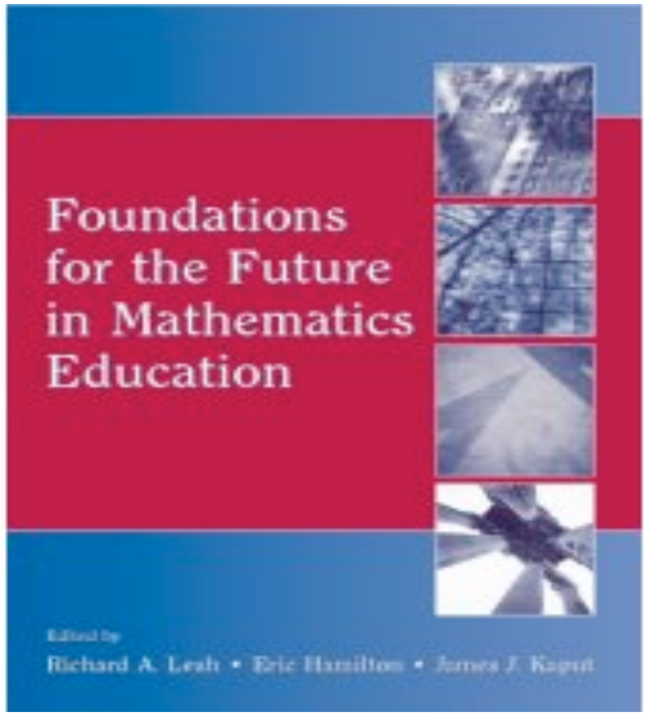
A Readers' Digest Sized Introduction to Models & Modeling Perspectives (MMP) on Mathematics Teaching, Learning, and Problem Solving

(excerpts from several recent publications by R. Lesh)



The central goal of MMP research is to develop useful models of teachers' and students' models and modeling abilities. In particular, MMP was developed explicitly to investigate the following kinds of questions. What is the nature of typical problem-solving situations where elementary-but-powerful mathematical constructs and conceptual systems are needed for success in a technology-based age of information? …. What kind of "mathematical thinking" is emphasized in these situations? …. What does it mean to "understand" the most important of these ideas and abilities? How do these competencies develop? ….What can be done to facilitate development? … How can we document and assess the most important (deeper, higher-order, more powerful) achievements that are needed: (i) for informed citizenship, or (ii) for successful participation in the increasingly wide range of professions that are becoming heavy users of mathematics, science, and technology? … And how can we identify students who have exceptional potential which are not measured on standardized tests.

When investigating the development of students' and teachers' thinking and learning, one of the most important distinguishing characteristics of MMP research is its emphasis on the fact that, in virtually every field where learning science researchers have investigated what it means to develop competence, it has become clear that highly competent individuals not only *do* things differently but they also *see* (or interpret, or conceptualize) things differently. For example, some of the most important interpretation systems that mathematics teachers need to develop involve making sense of students' ways of thinking about mathematical learning and problem solving activities. Therefore, because of the interacting nature of the conceptual systems that are developed by students, teachers, and other educational decision makers, MMP research often involves *multi-tier studies* in which:

- Students go through sequences of cycles in which they iteratively express, test and revise their interpretations or conceptualizations of mathematical learning or problem solving situations.

- Teachers or developers' go through sequences of cycles in which they iteratively express, test and revise their interpretations or conceptualizations of students' interpretation development activities.

- Researchers go through sequences of cycles in which they iteratively express, test and revise their interpretations or conceptualizations of students' and teachers' interpretation development activities.

According to Models & Modeling Perspectives on Mathematical Thinking & Learning, Knowledge is …

- Focused on SEEING as much as on DOING.
  - In virtually every area of learning or problem solving where researchers have investigated differences between effective and ineffective learners or problem solvers (e.g., between experts and novices), results have shown that the most effective people not only *do* things differently, but they also *see* (or *interpret*) things differently.
  - Even the "process objectives" (and problem solving strategies) that we develop often are descriptive instead of prescriptive in nature – and are integral parts of the interpretation systems that we develop.
- Situated:
  - Knowledge is organized around experience at least as much as it is organized around abstractions.
  - Nonetheless, even though models (and underlying conceptual systems) are shaped by the situations in which they are developed, they are like other types of conceptual tools in the sense that they generally are not worth developing unless they are intended to be used more than a single time and in more than a single situation. So, in general, they are not just powerful (in a specific situation) but they also are designed to be sharable (with others) and reuseable (in other situations).
- Socially Shared & Shaped:
  - In the 21st century, where countries have knowledge economies, where companies involve learning organizations, and where learners and problem solvers may consist of teams of diverse specialists, it is naïve to imagine that thinking goes on exclusively within the minds of isolated individuals.
  - The nature of knowledge development is influenced by the fruits as well as by the roots of ideas and feelings – and by sociocultural constraints and affordances related to the availability of capability enhancing tools.
- Connected
  - Realistically complex problem solving situations usually involve more than a single actor – and decisions that involve trade-offs (such as low costs & high quality). So, productive ways of thinking usually need to integrate ideas and abilities drawn from a variety of textbook topic areas.
- Systemic/Emergent:
  - Many of the most important things that we need to understand and explain are complex adaptive systems whose most obvious distinguishing characteristics are that they involve emergent properties of the systems-as-a-whole - which cannot be explained using only a single function (or input-output rule).
- Distributed – and Expressed or Embodied using a Variety of External Media
  - In the 21st century, our abilities to store, retrieve, manipulate, and use information is continually being off-loaded using tools ranging from spell checkers to internet based search engines; and, the conceptual tools that we develop at shaped as much by our continually evolving purposes as by currently existing artifacts or events.

- Infrastructural:
  - Conceptual tools (and expressive media) that humans develop are not just neutral carriers of thought; nor are they neutral descriptions of experiences. They induce significant changes on thinking that evolves, and they also structure the situations that we need to understand and explain. In other words, they are infrastructural! … The same conceptual systems that we develop to understand the world are also used to mold and shape that world; and, as soon as we understand a situation, we tend to change it; and, as soon as a conceptual system is expressed, it changes.
- Not just Logical & Mathematical in Nature:
  - When we interpret situations, we don't simply engage models that are purely logical and mathematical in nature. They also involve feelings, values, and a variety of metacognitive capabilities.
- Often Tacit:
  - The conceptual systems that we develop to interpret experiences often are more like windows that we look through than objects that we look at. So, we can think WITH them without necessarily thinking ABOUT them; and, in fact, when people need to develop abilities that involve smoothly functioning complex systems, it often is debilitating to think about these system formally and analytically.
- Generally Piecemeal, Undifferentiated, Unintegrated and Unstable.
  - Regardless whether the "problem solver" is an isolated individual or a group, solutions to *model-eliciting activities* tend to involve communities of competing conceptual systems; and, conceptual evolutions tend to occur best when Darwinian factors such as diversity, selection, propagation, and conservation come into play. Thus, we go beyond emphasizing *the mind in society* (and *the mind of society*) to also emphasize *societies of mind.*
- Characterized by Complex Adaptive Systems:
  - When we develop models of students' modeling abilities, it has become necessary to move beyond machine-based metaphors (hardware), and beyond computer-age metaphors (software), toward metaphors grounded in an age of biotechnologies (wetware) and complex adaptive systems – where "agents" within systems often are living organisms which governed by wetware which obeys logics that are distributed, multi-media, and fuzzy (rather than being characterized by simple, linearly combined or concatenated rules or declarative statements). Thus, knowledge development is considered to be less like the construction of a machine or a computer program, and more like the evolution of a community of living, adapting, and continually evolving biological systems.
- Continually Developing in Non-Linear Ways along a Variety of Interacting Dimensions:
  - During *model-eliciting activities*, conceptual systems typically develop along a variety of interacting dimensions: concrete-abstract, intuitions-formalizations, holistic-analytic, external-internal, simple-complex, situated-decontextualized, unstable-stable, and so on. But, the most appropriate model is not necessarily the one that is most abstract, most formal, most complex, most analytic, or most decontextualized; and, develop seldom occurs along linear paths. … Conceptual evolution tends to involve differentiation, integration, adaptation, and elaboration; and, final ways of thinking usually can trace their heritage to a variety of conceptual ancestors.

What are Model-Eliciting Activities?

Model-eliciting activities are activities which require students to explicitly develop a *model* (or an explicit mathematical or scientific description or explanation) of a personally meaningful situation where some important type of mathematical or scientific thinking is needed in order to interpret the available information is productive ways. At its simplest, a model can be thought of a being a system that is used to describe or design some other system.[1] But, because these descriptions or explanations are generated for some specific purpose, a model also can be thought of as a purposeful conceptual tool whose design typically involves a series of development cycles in which current descriptions are iteratively expressed, tested, and revised. Principles for designing have been described in a variety of recent publications (Lesh, et. al., 2002, Hjalmarson & Lesh, in press). In general, they have the following characteristics. However, no activity is model-eliciting unless it elicits a model. Duh!!!!

- They are simulations of problem solving or decision making situations where some significant type of mathematical or scientific thinking is needed beyond school.
- Solutions generally require at lest 45-90 minutes to construct, and they provide powerful prototypes for dealing with issues that are important to the students or others they would like to impress.
- Issues fit the interests and experiences of targeted students, and they encourage students to engage their personal knowledge, experience, and sense-making abilities.
- Solution procedures encourage students to use realistic tools and resources, including calculators, computers, consultants, colleagues, and "how to" manuals.
- Evaluation procedures recognize more than a single type and level of correct response.
- Overall activities contribute to both learning and assessment … because students simultaneously learn and document what they are learning. So, MEA's sometimes have been called thought-revealing activities.

Model-Eliciting Activities represent only a small part of MMP research; and, they were never intended to be instructional "treatments" whose worth depends on demonstrating that "they work" (whatever that means). If anything, MEAs function more to show why, how, and to what extent learning occurs. Nonetheless, byproducts of MMP have demonstrated the following.

- Because MEAs are designed to focus on the ten-or-so "big ideas" in any given course in which they are used, and because they are designed to reliably elicit significant conceptual adaptations during relatively brief (e.g., 60 minute) problem solving episodes, students' thinking often advances (locally) through one or two stages similar to those described by Piaget-inspired research in mathematics education (ref).

- Because students express their thinking in the form of artifacts and tools that they themselves can test and revise multiple times, and because students go through multiple cycles of expressing, testing, and revising their own ways of thinking (instead of being guided along narrow paths toward teachers' or textbook authors' ways of thinking, their final solutions often go beyond first drafts to produce nth drafts which exhibit impressive ways of thinking

- Because MEAs are designed to be simulations of "real life" situations beyond school, and because the products that students produce generally involve tools that students design to be sharable and

[1] This means that a system that is being modeled at one moment may be used a moment later as a model to describe some other system. So, a system is only a model if it is used to describe or explain some other system; and, a mathematical model is one that focuses on structural (as opposed to physical, or chemical, or biological) characteristics of the system that it is being used to describe or explain. Nonetheless, when a system is interpreted mathematically, students do more than simply engaging a system that is purely logical or mathematical in nature. They also engage a variety of values, feelings, beliefs, and metacognitive processes; and, these latter attributes are also important part of the model that are developed.

reuseable, the concepts which are embodied in these tools often exhibit usual levels of durability and transferability.

- Because MEAs are designed to be *thought-revealing activities*, and because insightfulness about students' thinking is one of the primary characteristics of outstandingly effective teachers, MEAs contribute to teacher development – which in turn encourages student development. In many ways, when MEAs are used at the beginning of curriculum units, the kind of information that teachers see is similar to the kind of information they might have obtained if they'd have been able to conduct clinical interviews with each of their students before starting the unit.

- Because MEAs are designed to be *thought-revealing activities*, they also have been used as performance assessment activities which focus on levels and types of abilities that are seldom emphasized on traditional textbooks and tests. So, MEAs not only contribute to students development but they also document important aspects of what develops.

- Because MEAs are designed to focus on a broader range of understandings and abilities than traditional school tasks (textbooks and tests), a broader range of students naturally tend to emerge as being highly capable. And, these abilities tend to be precisely the kind that are needed for success beyond school – but which are seldom addressed on standardized tests.

Because of the preceding kinds of factors, research on MEAs has produced significant amounts of evidence showing that: (a) without guidance or scaffolding from teachers which go beyond those specified in the design principles for MEAs, average ability children are capable of developing (or making significant adaptations to) powerful elementary-but-deep mathematical concepts and abilities, (b) in order for the preceding concepts and abilities to be useful beyond school, a variety of new levels and types of understanding are needed beyond those that have been emphasized in even the most innovative and future-oriented current statements of curriculum standards, (c) model development tends to be one of the most important components of what it means to "understand" the small number of "big ideas" that are most important to develop in any given mathematics course in the K-16 curriculum, (d) deeper and higher-order understandings of these "big ideas" generally go beyond computation-related abilities to also include abilities involving description, communication, conceptualization, representation, and mathematization – and other concepts and abilities related to models and modeling, (e) the understandings and abilities that develop during MEAs often exhibit extraordinary levels of transferability, adaptability, and durability, and (f) when assessment instruments document and assess a broader range of mathematical concepts and abilities, a broader range of students tend to emerge as having exceptional potential. However, in spite of the extraordinary success that MEAs have exhibited for goals related to learning, asking MMP researchers whether MEAs "work" (when they are used as teaching and learning activities) is like asking Piaget whether his famous conservation tasks and clinical interviews work (when they are used as the basis for teaching and learning activities). If either Piaget's tasks or MEAs are used to in teaching and learning situations, it seems likely that their ability to "work" depends to a great extent on the fact that teaching and learning tends to be more effective when: (a) teachers develop sound conceptions of what it means to "understand" the concepts they are trying to help students learn, and (b) teachers know as much as possible about the strengths and weaknesses of their students' thinking.

What are Principles for Designing Effective Model-Eliciting Activities?

Principles for developing *model eliciting* activities have been described in a number of past publications. An early version of these principles was developed during a three-year series of studies in which more than 300 outstanding K-12 teachers gradually developed guidelines for creating problems which reflected their own current understandings about the nature of problem solving situations in which new levels and types of "mathematical thinking" are needed for success beyond school (Lesh et al, 2002). These principles have now been used productively by teachers in primary grades through high school – as well as with college students and professionals in fields ranging from Engineering (Diefes-dux, Bowman &

Zawojewski, in press; Lesh, Hamilton & Kaput, 2007) to Teacher Development (Zawojewski, Lewis, Hjalmarson, in press).

The Personal Meaningfulness Principle (sometimes called the "reality" principle) focuses on the question: *Could this really happen in a student's "real life" beyond school? …* Of course, when asking such as question, we is important to recognize that no problem is meaningful for all students. Nor is any situation problematic for all students. Nor is student's interpretation of "reality" likely to be the same as that of a given adult. For example, fanciful situations (or "what if" scenarios) may be quite real for a child or an adolescent. Nonetheless, when students work on a problem, it often is clear that they are using their "school heads" rather than their "real heads" – and that they are not trying to make sense of the situation based on extensions of their own "real life" knowledge and experiences. So, it is these latter kinds of situations that MMP researchers need to avoid.

The Model Construction Principle focuses on the question: *Does the task put students in a situation where they clearly recognize the need to construct, or modify, or extend, or refine a specific type of model (and underlying conceptual system)? …* In a given course, *model-eliciting activities* are intended to focus on one of the 8-10 "big ideas" in a given course; and, one of the foundation-level assumptions underlying MMP research is that most "big ideas" in mathematics involve the development of a conceptual system which is used to generate purposeful interpretations (descriptions, explanations) of relevant situations. Of course, no situation is *model-eliciting* unless it actually elicits a model for a given student. So, the *model-construction principle* requires researchers to ask themselves: *What is it that creates the need for a specific type of model (and its underlying concepts or conceptual system)?* Responses to this question tend to be expressed in a form similar to design "specs" for students to develop a conceptual tool that must serve some specific purpose in some specific situation. Thus, in order to optimize the chances that the conceptual tools that students produce will embody the targeted conceptual system, researchers need to develop design "specs" that are something like the photographic negative of the conceptual system whose development they wish to investigate. They should enable students to focus on the desired quantities, relationships, actions, and patterns; and, they should enable students to identify strengths and weaknesses of alternative ways of thinking; but, they should allow alternative products (and underlying ways of thinking).

Note: During early stages of MMP research, Piaget-style clinical interviews were used to try to determine the nature of the conceptual systems that students needed to develop in order to make judgments about important "big ideas" in elementary mathematics. But, we often noticed that a bit of probing often led to significant changes in students thinking. So, by employing Vygotsky's notion of a *zone of proximal development* we often shifted toward research methodologies called *teaching experiments* (Lesh & Kelly, 2000). But, in these teaching experiment, the kind of student thinking that we observed was always shaped by both the structure of situations that we presented as well as by the students sense-making schemes; and, we became dissatisfied with the extent to which we were only able to observe how close students progressed toward our preconceive way of thinking about what it means to "understand" the relevant concepts. So, we gradually shifted toward situations in which students were able to go through a series of iterative cycles in which they repeatedly expressed→tested→revised their own ways of thinking rather than being guided by questions that were driven by our ways of thinking about their ways of thinking. These studies are called multi-tier design studies – because researchers are able to create situations that optimize the chances the significant conceptual adaptations will occur without needlessly limiting the exact nature of these adaptations.

The Self-Evaluation Principle (sometimes called the "usefulness" principle) focuses on the question: *Will students be aware of criteria to judge for themselves when their responses are good enough? And will they be able to assess the strengths and weaknesses of alternative responses?* (e.g., Will students be likely to believe that their ideas will be taken seriously? Or will they believe that the responses they are expected to give should conform to the teacher's (or author's) notion of the (only) "correct" way to think about the problem situation?) Beyond school, when mathematical thinking is needed, the mathematical product usually needs to serve some purpose outside of mathematics – such as enabling the problem solver to accomplish some goal or make some decision. Therefore, the "math product" is a tool; and, its usefulness depends on how well it serves its intended purpose. Consequently, if it isn't clear who needs the tool (and when, why, and for what purpose), then there tends to be no way to judge whether a trial tool is good enough – or to judge the strengths and weaknesses of alternative tools (and/or underlying ways of thinking).

Note: (a) Because models involve descriptions or explanations, and because there nearly always exist several different levels and types of descriptions and explanations that are appropriate for a given situation, the most effective model-eliciting activities tend to be those that encourage the development of several alternative ways of thinking. So, this provides a useful criteria for assessing whether the model-construction principle has been satisfied. … Sometimes, MMP researchers have elevated this assessment criteria to the status of a seventh principle for designing *model-eliciting activities*; and, in these cases, they have tended to call it *The Diversity Principle*.

(b) When the products that students need to produce involve descriptions or explanations, these products usually need to go through several stages of development similar to first drafts, second drafts, and third drafts that occur for other types of documents. But, these cycles only tend to occur if students themselves are able assess the quality of trial drafts. So, during the development of model-eliciting activities, if solutions do not involve multiple cycles, then this is a strong indicator that the *self-assessment principle* has not been satisfied. Consequently, this *multiple-cycle solution criteria* provides a useful indicator for assessing whether the self-assessment principle has been satisfied. Sometimes, MMP researchers have elevated this assessment criteria to the status of a another separate principle for designing model-eliciting activities; and, in these cases, they have tended to call it the *multiple-cycle solution principle*.

(c) When mathematicians develop models, the reasons they do so tend to involve such things as making predictions in the future (or at some remote location), explaining inaccessible situations in the past (where some relevant information has been lost), experimenting with situations that are too dangerous or costly to experiment with directly, simplifying situations that are too complex, or filling in or going beyond information that is given (in situations where not enough is known). So, when students adopt a model without knowing it's purpose, misconceptions are likely to develop.

The Model Generalizability Principle (sometimes called the "sharability and reuseability" principle) focuses on the question: *Will students be likely to recognize the need for the product to be sharable and reuseable?* … Mathematical tools, like other complex artifacts, are seldom worthwhile to develop if they are only going to be used one single time, in one single situation, and for one single purpose? So, in order for students to recognize the need for a given kind of tool (and underlying conceptual system), it usually is important for them to be clear about the fact that the tool not only needs to be useful for a specific situation and purpose – but that it also should be sharable (with other people), and reuseable (for other purposes). Consequently, sharability and reuseability provide important criteria for assessing the quality of results that are produced; and, characteristics such as adaptability tend to be important. … If a conceptual tool is adaptable, sharable, and reuseable, then these characteristics account for a large share of what it means for students to develop transferable or generalizable knowledge.

Note: Taken together, the *diversity principle* and the *self-assessment principle* account for two of the four main Darwinian forces that drive the evolution of nearly any community of complex adaptive systems. These are: diversity and selection. So, in some ways, *sharability* and *reuseability* account for the two remaining driving forces – communication (throughout the community) and preservation (across time, or across cycles or stages of development).

The Model-Documentation Principle focuses on the question: *Will the response require students to explicitly reveal how they are thinking about the situation (givens, goals, possible solution paths)?* Mainly the model-documentation principle ensures that problem solvers will externalize their ways of thinking in forms so that self-assessment is possible – and so that sharability and reuseability are more likely to occur. However, because model-eliciting activities are intended to be used for research and assessment, the model-documentation principle is absolutely essential in order for the activities to automatically generate auditable trails of documentation which enable important aspects of students' evolving models to be observed directly.

Note: Looking at trails of documentation, it is possible for observers (who may include problem solvers themselves during post activity reflections) to identify shifts from one model to another by noticing that different models involve different mathematical objects, relations, operations, patterns, or regularities; and, these often are expressed using different representational media.

The Simplest Prototype Principle focuses on the questions: Is the situation as simple as possible while still creating the need for a significant model? Will the product that is produced provide a useful prototype (or metaphor) for interpreting a variety of other structurally similar situations? … One indicator of whether this principle has been satisfied is reflected in the fact that the most effective model-eliciting activities tend to be remembered by students many months and even years after they have been completed. For example, MMP researchers have recorded many instances where students have given impressively detailed accounts describing their past experiences with model-eliciting activities – sometimes several semesters after the activities were completed. Also, numerous instances have been recorded where students have been in other courses and have said things like *"This is like the Big Foot Problem, or the Paper Airplane Problem, or the Volleyball Problem."*

Note: Interpretation systems often appear to be quite different than production systems. Sometimes overgeneralization is a bigger problem than lack of transfer. … To have an interpretation system is to use it!

One surprising thing about the preceding six principles for developing *model-eliciting activities* is that they seem so sensible that researchers who are unfamiliar with them often assume that most problems in mathematics and science books surely must satisfy most of them. But, this is not the case at all. In most books, it is difficult to find instances of even a single problem solving activity that does not fail to satisfy every one of the six principles. In fact, even in books that are touted as representing models of excellence in problem solving activities (ref. 19xx), most of the problems violate most of the six principles.

What is an Example of a Model-Eliciting Activity?
The Summer Jobs Problem

Each model-eliciting activity is designed to focus on some "big idea" in K-16 mathematics; and, in the case of the following *Summer Jobs Problem,* the main concept that we had in mind was the relationships between averages and rates (or ratios) – plus the idea of "operationally defining" the meanings of constructs like *worker productivity* that cannot be measured directly. However, for many of the same reasons why engineers tend to emphasize the fact that useful solutions to realistically complex problem usually must integrate ideas and procedures drawn from more than a single textbook topic area, the conceptual tools that students develop during the *Summer Jobs Problem* usually sort out and integrate ideas from measurement (i.e., weighting factors), graphing (i.e., trends), statistics (e.g., extrapolating beyond or between available data), and probability (estimating errors due to unknown factors).

In the *Summer Jobs Problem*, like in all MEAs, the product that students are asked to produce is not just a short mathematical answer to a well defined mathematical question. Instead, the problem is presented in a form that engineers might refer to as "design specs" for a tool which needs to be powerful (for some clearly recognized need) as well as being sharable (with other people) and reuseable (for other similar situations, or when other information is given). Thus, students themselves are able to use the preceding "design specs" to make judgments about strengths and weaknesses of alternative products – and underlying ways of thinking. Consequently, because the underlying design principles are important parts of the tool that needs to be designed, important aspects of students' thinking tend to be embodied in trial products and final products that are produced. In other words, MEAs are designed to be *thought-revealing activities*. And, because students are able to assess the quality of their own current tools and underlying ways of thinking, solutions tend to go through a series of testing and revising cycles. So, the understandings that students associate with their final products involve ideas related to discarded ways of thinking – as well as those that are embodied in the artifact that is produced.

The Summer Jobs Problem

Last summer Carla started a concession business at Wild Days Amusement Park. Her vendors carried candy, hot dogs, and drinks around the park, selling wherever they found customers.

The business was a great success. Next summer, Carla is expecting that all of her vendors will want to work for her again. But, the park managers told her that she won't be allowed to hire as many vendors next summer. So, she needs your help deciding which workers to rehire. If all of last year's vendors apply for a job, she'll only be able to hire about a third of them to work full time, and about a third of tehm to work half time. She won't be able to hire the remaining third of them.

The table below shows a sample of nine people who worked for her last summer. To try to figure out a procedure for deciding who to hire next summer. Carla reviewed her records for the nine vendors who are shown. For each of these vendors, she totaled the number of hours they worked and the amount of money collected – when business in the park was busy (high attendance), steady (average attendance), and slow (low attendance). (See the table that follows.) She wants to rehire the vendors who will make the most money for her. But, she doesn't know how to compare them because they worked different numbers of hours; and, she isn't sure what to do about the fact that it's easier to sell more when the attendance is high.

Write a letter to Carla describing how she can evaluate all of the vendors who worked for her last summer, and how to decide who to hire full-time and part-time. Show how your procedure works for the nine people workers who are shown in the table. Give details so Carla can check your work, and give a clear explanation so she can decide whether your method is a good one for her to use.

HOURS WORKED LAST SUMMER

	JUNE			*JULY*			*AUGUST*		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
MARIA	12.5	15	9	10	14	17.5	12.5	33.5	35
KIM	5.5	22	15.5	53.5	40	15.5	50	14	23.5
TERRY	12	17	14.5	20	25	21.5	19.5	20.5	24.5
JOSE	19.5	30.5	34	20	31	14	22	19.5	36
CHAD	19.5	26	0	36	15.5	27	30	24	4.5
CHERI	13	4.5	12	33.5	37.5	6.5	16	24	16.5
ROBIN	26.5	43.5	27	67	26	3	41.5	58	5.5
TONY	7.5	16	25	16	45.5	51	7.5	42	84
WILLY	0	3	4.5	38	17.5	39	37	22	12

MONEY COLLECTED LAST SUMMER (IN DOLLARS)

	JUNE			*JULY*			*AUGUST*		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
MARIA	690	780	452	699	758	835	788	1732	1462
KIM	474	874	406	4612	2032	477	4500	834	712
TERRY	1047	667	284	1389	804	450	1062	806	491
JOSE	1263	1188	765	1584	1668	449	1822	1276	1358
CHAD	1264	1172	0	2477	681	548	1923	1130	89
CHERI	1115	278	574	2972	2399	231	1322	1594	577
ROBIN	2253	1702	610	4470	993	75	2754	2327	87
TONY	550	903	928	1296	2360	2610	615	2184	2518
WILLY	0	125	64	3073	767	768	3005	1253	253

Figures are given for times when park attendance was high (busy), medium (steady), and low (slow).

Model-Eliciting Activities Involve Local Conceptual Development.
Yet, Development Doesn't Occur Along A Single Linear Path.

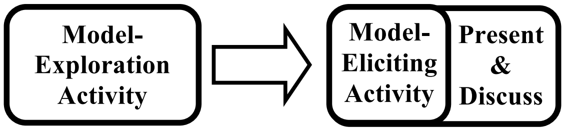
Model-Eliciting Activities Involve *Local Conceptual Development*. During *model-eliciting activities,* the development of useful conceptual systems generally involves going through a series of modeling cycles in which current mathematical interpretations (descriptions, explanations, or conceptualizations) are iteratively expressed, tested, and revised – or rejected. Consequently, in situations where the relevant conceptual systems correspond to concepts that have been investigated by Piaget-inspired researchers, the development cycles that problem solvers go through during 60-90 minute *model-eliciting activities* often bear a striking resemblance to the kind of stages described by Piaget (Lesh & Harel, 2003). For example:

- Distinct "stages" can be identified because they involve somewhat different relations, operations, patterns, or mathematical "objects" (e.g., quantities, shapes, coordinates).
- Relatively primitive ways of thinking tend to be based on less refined, less complex, and less stable relational/organizational systems; and unstable systems tend to be relatively barren and distorted compared with later interpretations.

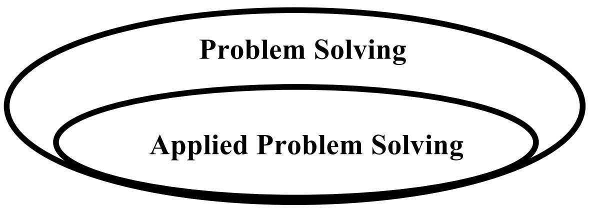
Development Doesn't Occur Along A Single Linear Path. In spite of the preceding similarities, there are significant differences between Piaget's developmental trajectories and the way conceptual systems develop during *model-eliciting activities*. For example:

- Because final conceptualizations of *model-eliciting activities* tend to integrate ideas from multiple-textbook topic areas, resulting chunks of knowledge tend to be organized around experience more than around abstract cognitive structures of the type emphasize by Piaget and his followers.
- During *model-eliciting activities*, construct development is far more situated than Piaget-inspired researchers have suggested (Lesh & Carmona, 2003). For example, when students' thinking evolves through several Piagetian stages during a single 60-90 minute problem solving episode, it is obvious that development is much more a matter of gradually increasing local competence rather than being a manifestation of the evolution of some general cognitive structure. Also, when two tasks are significantly different in difficulty, it is questionable whether students' thinking is organized around these abstractions - even though Piagetians would classify both of them as involving the same cognitive structure.
- In the next section of this chapter, we will describe how development also involves increasing representational fluency as meanings associated with a variety of representational media gradually are sorted out and coordinated - such as those that are expressed using spoken language, written symbols, diagrams, experience-based metaphors, or technical tools (Lesh & Doerr, 2002). Consequently, development involves much more than Piagetian transitions from pre-operational to concrete-operational to formal-operational thinking – and more than Vygotskian transitions in which external functions are gradually internalized.
- During intermediate stages in the solution of *model-eliciting activities*, the productivity of alternative ways of thinking is strongly determined by the purposes that are imposed on the situation by problem solvers or their clients; and, these purposes not only create the need for some kinds of comparisons and transformations (but not others), but they also provide criteria to assess the usefulness of alternative ways of thinking. In fact, if purposes are not apparent, then it is seldom clear whether one interpretation is better than another, or whether mathematical responses are better than responses that require very little mathematical thinking. It probably isn't even clear whether 5-hour responses are better than 5-minute responses or 5-second responses.

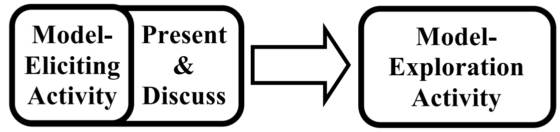
A Comparison of Two Ends of an Instructional Continuum	
Make Mathematics Practical – Teach first. Then, apply what was taught.	**Make Practice Mathematical**: Students express, test, and revise their own ways of thinking. Then, teachers help students "clean up" and empower their results.



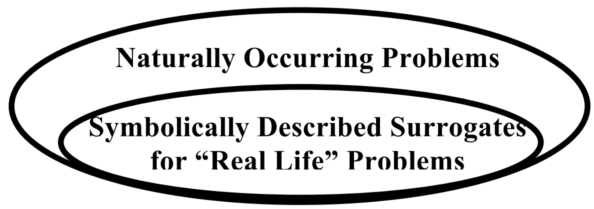
According to this approach, learning to solve "real life" problems is assumed to be more difficult than solving their decontextualized counterparts in text-books and tests - because realistically "messy" problems require students to know context-specific information in addition to knowledge about relevant concepts and processes.



This teach-first-then-apply perspective considers "real life" problem solving to be a special case of decontextualized forms of problem solving - where the messiness has been stripped away.



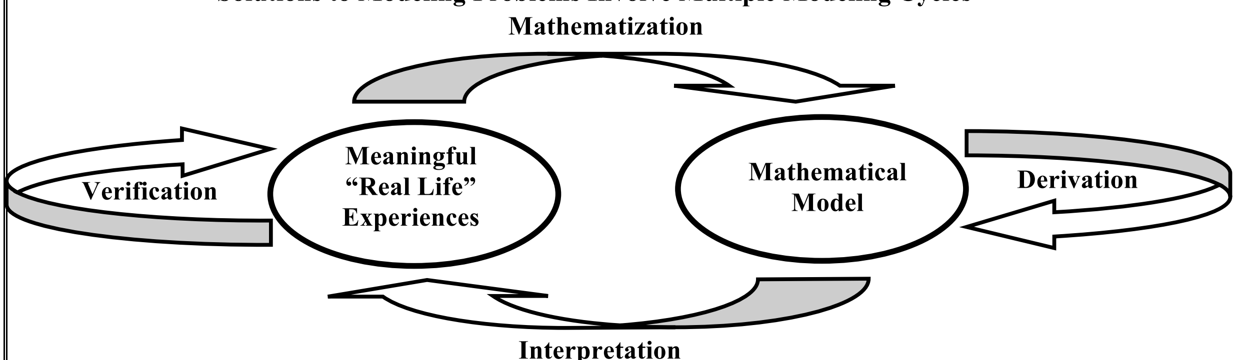
According to this approach, learning to solve meaningful (naturally occurring) "real life" problems is assumed to be easier than solving their decontextualized counterparts in textbooks or tests – because the latter requires students to make meaning out of symbolically described situations before sensible steps can be taken to generate solutions.



This *develop-first-then-harvest* perspective considers traditional conceptions of problem solving to be special cases of meaningful "real life" problems where meaning-ful details and purposes have not been stripped away.

Whereas, model development activities generally involve multiple cycles of the type shown below, traditional word problems tend to involve only getting from (pre-mathematized/computation-ready) givens to (mathematical) goals within a single cycle. Furthermore, even within this single cycle, students usually do not know who needs their mathematical result - or why. Therefore, because there seldom is any need to leave the world of mathe-matics, solution steps tend to emphasize derivation processes (computation or deduction). … Because such problems involve little mathematization, interpretation, or verification, they can be thought of as quarter-cycle (or half-cycle) modeling problems.

Solutions to Modeling Problems Involve Multiple Modeling Cycles



A Significant Assumption About Students' Abilities To Develop Powerful Models

Limitations of instructional methodologies that *teach first and then apply* are not likely to seem significant to anybody who naively accepts the cliché that: *It took brilliant mathematicians hundreds of years to develop most of the most powerful concepts in the elementary mathematics curriculum. So, it's not realistic to expect average ability children or adolescents to come up with such concepts in a few weeks – or during single problem solving episodes.* … In contrast to the preceding point of view, research based on *models & modeling perspectives of learning and problem solving* is filled with transcripts showing that average ability students routinely develop powerful, sharable, and re-useable constructs or conceptual systems (Lesh & Doerr, 2003). In fact, if the goal of instruction is to make significant changes in a student's underlying ways of thinking about important conceptual systems in mathematics, then perhaps the only way to induce significant conceptual change is to engage students in *model-eliciting activities* where they must express their current ways of thinking in forms that lead to testing and revision (or rejection). We reject the notion that only a few exceptionally brilliant students are capable of developing significant mathematical concepts unless step-by-step guidance is provided by a teacher.

Three Confusing Facts About Students' Models

1. Conceptual models that students engage and develop to make sense of their experiences generally should not be thought of as being "toy worlds" that are internal copies of the external systems they are used to describe. Just as equations are not photograph-like copies of the systems they describe, every model (or metaphor) has some properties that the system that it describes does not have; and, it also does not have some properties that the system that it describes does have. Yet, such models clearly must have some "case value" in order to purposefully describe, explain, or predict the behaviors of systems they are intended to describe. So, in spite of the fact that they often have many characteristics that the modeled systems do not have, they also have some properties in common with the systems they are intended to explain.

2. Researchers' models of students' modeling behaviors are not the same thing as students' models. For example, suppose that a student is asked to develop a "conceptual tool" (or a "model") whose design "specs" demand that the underlying design principles must be included in the product that is produced. In particular, acceptable products must explicitly reveal what kind of mathematical objects, relations, operations, and patterns are embodied in the design principles for the tool. Under these circumstances, the products that students produce are indeed the students' tools or models. They are not simply inferred by an observer based on indirect evidence.

3. When students develop models, whatever they explicitly reveal about their thinking is like the tip of an iceberg; a great deal cannot be seen; and, whatever is seen is likely to be in the process of change. For example:

- If the "problem solver" is a team of diverse individuals, then each individual's conception is likely to be somewhat different than the conceptions of other team members – as well as being different than the consensus that was developed by the team-as-a-whole.
- If the "problem solver" is an isolated individual, one of the purposes of externalizing current ways of thinking (e.g, using statements, or drawings, or experience-based metaphors) is to examine and modify these ways of thinking. So, the thinking of a person who draws a diagram cannot be assumed to be the same as the thinking of the same person who looks at the results a moment later.

When researchers develop models of students' models, there is a big difference between claims that are based on tasks in which students were never asked to produce models versus those where the design "specs" for the task made it clear that students needed to explicitly express, test, and revise their own models, conceptual tools, or underlying ways of thinking. … In general, in this chapter, when we speak of students' models (or other conceptual tools), we are referring to research studies in which students were asked explicitly to produce models (or conceptual tools). In such research studies, the students are enlisted as co-researchers who interact with researchers while investigating their own ways of thinking. Therefore, what we report is what the students themselves reveal about the objects, relations, operations, and patterns that they intended to be embodied in the conceptual tools that they produce.

New Directions for Research on Problem Solving

Traditional Perspectives	Models & Modeling Perspectives
Problem Solving is defined to be *a process of getting from givens to goals when the steps are not immediately obvious.*	*Problematic Situations* are defined to be *goal directed activities in which adaptations need to be made in existing ways of thinking about givens, goals, and possible solution steps.*

<table>
<tr><td>Products that need to be produced tend to be thought of as being short answers to well defined questions about pre-mathematized situations.</td><td>Products that need to be produced often involve complex artifacts (e.g., conceptual tools) which are designed to be:
(a) powerful (for the specific purpose at hand),
(b) sharable (with others) and
(c) reuseable (in other situations).</td></tr>
<tr><td colspan="2">Conceptual tools are situated forms of knowledge. They are molded and shaped by the situations in which they are created or modified. Yet, they are not simply situation-specific knowledge that does not transfer. For example, if they are designed to be sharable and reuseable, then they represent generalizable achievements. Also, in realistically complex situations, useful ways of thinking usually must integrate concepts and conceptual systems that do not fit into a single textbook topic area – or even into a single discipline. One reason this is true is because it is usually the case that solutions to realistically complex problems not only need to be effective, but they also need to be cost-effective, durable, timely, politically acceptable, and so on. Therefore, when we say that knowledge is situated, part of what we mean is that the “chunks” of knowledge that develop tend to be organized around experience at least as much as around discipline-based abstractions.</td></tr>
<tr><td>Problem Solvers are thought of as being processors of information.</td><td>Problem Solvers are thought of a being conceptualizers and creators of (complex) systems. Also, rather than being isolated individuals, “problem solvers” often are teams of diverse specialists with access to powerful tools for conceptualization, computation, & communication.</td></tr>
<tr><td>Relevant Knowledge is thought of as a list of condition-action rules.

Note: Although relatively few cognitive scientists continue to try to explain human thinking using simple-minded lists of condition-action rules, rule-base conceptions of knowledge are alive and well in the behavioral objectives that underlie most standardized tests – as well as in the skills that dominate “back to basics” curriculum standards documents.</td><td>Relevant Knowledge in considered to involve, above all, conceptual systems (or models) for describing or interpreting patterns, regularities, and other systemic properties of problem-atic situations. Because these conceptual systems are expressed in the form of tools, the tools sometimes are referred to as embodiments of the conceptual systems.</td></tr>
<tr><td colspan="2">The development of underlying conceptual systems is not expected to be an all or nothing process. It is assumed that conceptual systems will develop along a variety of dimensions (e.g., concrete-abstract, simple-complex, unstable-stable, internal-external, situated-decontextualized, intuitive-formal). So, when they are needed to solve "real life" problems, most are expected to be at intermediate stages of development. Also, regardless whether the problem solver is an isolated individual or a team of diverse specialists, their thinking usually involves a community of competing (yet perhaps undifferentiated) conceptual systems.</td></tr>
<tr><td>Solutions Processes are thought of as linking together strategies (rules) along a linear path – or trajectory.</td><td>Solution Processes tend to involve iterative cycles in which existing ways of thinking are gradually expressed, tested, and revised (or rejected); and, development involves sorting out, integrating, refining or rejecting conceptual systems that you DO have – not trying to find missing ideas or abilities, and not trying to figure out what to do when you’re stuck (i.e., when you’re not aware of any relevant ways of thinking). So, progress seldom occurs along any single conceptual path. Instead, evolution tends to resemble genetic inheritance trees in which descendents inherit characteristics from a variety of ancestors.</td></tr>
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<table>
<tr><th colspan="2">Two Ways Of Thinking About Problem Solving Strategies</th></tr>
<tr><th>Traditional Perspectives</th><th>Models & Modeling Perspectives</th></tr>
<tr><td>Productive problem solving strategies are intended to provide answers to the question: What should I do when I’m “stuck” (i.e., when I don’t know what to do)?</td><td>Productive problem solving strategies are expected to help problem solvers develop adaptations to conceptual systems that they do have – not to help them function better when none seem to be available (which almost never is the case in</td></tr>
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<tr><td></td><td>meaningful simulations of “real life” situations).</td></tr>
<tr><td>*Higher-order (metacognitive) functions* are those that operate on lower-order skills or processes by specifying which to use, when, and why.</td><td>*Higher-order conceptual systems* serve the function of helping problem solvers develop beyond current ways of thinking.</td></tr>
<tr><td>*Increasing competence* is characterized as becoming more proficient at generalizing prescriptive-level strategies and ways of thinking.</td><td>*Increasing competence* is characterized as becoming more proficient at particularizing descriptive-level strategies and ways of thinking. These proficiencies depend on developing useful models-of-modeling – which include productive conceptions of mathematics, problem solving, and one’s own dynamic and malleable profile of dispositions, competencies, and attributes as a learner or problem solver.</td></tr>
<tr><td>*Problem solving strategies and metacognitive functions* (as well as dispositions, feelings, attitudes, values, and statements of belief about mathematics, problem solving, and one’s own personal identity) are assumed to have positive effects in virtually all situations. So, the goal of productive problem solvers is to adopt a single invariant “productive profile.”</td><td>*Problem solving strategies and metacognitive functions* (as well as dispositions, feelings, attitudes, values, and ways of thinking about mathematics, problem solving, and one’s own personal identity) are expected to have effects that vary from one situation to another, and from one stage of problem solving to another. So, the goal of productive problem solvers is to manipulate their own “profiles” to fit complex and continually changing circumstances.</td></tr>
<tr><td colspan="2">Whether impacts are positive or negative depends on functions that need to be served. For example, drawing a picture can lock problem solvers into their current ways of thinking rather than helping them recognize alternatives – and rather than helping them develop beyond current ways of thinking. Similar statements apply to nearly any rule-based characteristic, capability, or action.</td></tr>
<tr><td>*Problem solving strategies and higher-order functions* are assumed to be learned in two steps. First, learn them in simple situations. Then, learn to use them in a variety of “real life” situations where additional context-specific knowledge and information are required.</td><td>*Higher-order conceptual systems* are viewed as continually developing! And relevant dimensions of development are similar to those that apply to other conceptual systems – and concepts or abilities that depend on them. Thus, higher-order conceptual systems are essentially models-of-modeling – or situated mini-theories of problem solving. But, compared to the models-of-modeling that researchers develop, ordinary problem solvers think WITH the conceptual systems that they develop (intuitively and implicitly) rather than thinking ABOUT them (formally and analytically). The models-of-modeling that ordinary problem solvers create tend to be far more situated than the abstract and decontextualized models developed by researchers.</td></tr>
<tr><td colspan="2">*Models & modeling perspectives* anticipate that, when problem solvers interpret situations, they don’t simply engage systems that are logical and mathematical in nature. Their interpretations also involve problem solving strategies, higher-order understandings and processes, dispositions, feelings, values, and beliefs. Furthermore, these latter aspects of knowledge and ability are not simply learned separately and then added onto students’ interpretations (or models). Instead, they are integral parts of the models (and underlying conceptual systems) that students develop. So, as interpretations change, feelings and other processes and dispositions also change – and vice versa.</td></tr>
<tr><td>*Problem solving strategies and higher-order functions* are expected to function best when they are engaged explicitly during problem solving processes.</td><td>*Problem solving strategies and higher-order functions* are expected to function implicitly in most cases during problem solving processes. Explicit attention focuses on current and continually changing conceptions of sub-goals within the task at hand.</td></tr>
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Six Characteristics that make Students’ Models (and Underlying Conceptual Systems) Difficult to Document and Assess.

1. The way a researcher describes a student's thinking may be very different than the way the student herself describes her own thinking. For example, there are many ways to think about problems that mathematics teachers usually describe using the equation A/B=C/D; and, some of these do not involve: (a) co-variation – where one "variable" is thought of a influencing another "variable", or (b) proportional reasoning – by comparing two relationships (A-to-B and C-to-D). … When researchers develop models of students' models, there is a big difference between claims that are based on: (a) tasks in which students were never asked to produce models (any other type of thought-revealing conceptual tool), and (b) tasks where the design "specs" made it clear that students needed to explicitly express, test, and revise their own models, conceptual tools, or underlying ways of thinking.

2. Students often think WITH conceptual systems without thinking ABOUT them. In such situations, requiring students to be explicitly aware of their thinking may be debilitating rather than helpful.

3. Even if a model-eliciting activities put students in situations where they are likely to externalize their thinking in a variety of ways, whatever they explicitly reveal about their thinking is like the tip of an iceberg; a great deal cannot be seen.

4. When problem solvers express their thinking – using language, symbols, diagrams, metaphors, or other representational media - whatever can be seen is likely to be in the process of change. This is because one of the main purposes of externalizing thinking is to examine it so that it can be changed. For example, the thinking of a person who draws a diagram cannot be assumed to be the same as the thinking of the same person who looks at the results a moment later.

5. If the "problem solver" is a team of diverse individuals, then each individual's conception is likely to be somewhat different than the conceptions of other team members – as well as being different than the consensus that was developed by the team-as-a-whole.

6. A fundamental characteristic of conceptual models is that they are dynamic and continually adapting entities. When they are engaged, they change; and, when they function, they evolve. Like subatomic particles in quantum physics, causing conceptual models to appear in a form that is observable usually induces changes in them. So, researchers can never observe them as isolated entities; and, what can be observed is, in general, the residue from past trajectories of interactions and change.

Implications related to Design Assessment

Assessment is about generating useful conceptual tools (and information) for decision makers._So, the following questions arise:

- FOCUSING ON SPECIFIC DECISION MAKERS: Who are the decision makers?
    - Policy makers? Tax Payers? Administrators? Teachers? Care Takers (Parents)? Students? Learning Communities?
    - There no longer is a need to use one-size-fits-all approaches! (Learning Progress Maps)
- ESTABLISHING GOALS: What decisions are priorities for these decision makers? For what purposes? What are their ends-in-view? (Strategic Planning)
    - Sometimes the goal is to help decision makers ask better questions? (*Does "it" work?)*
    - Do people really know what they want? (Adopt "design science" perspectives.) Abandon convergence models and process objectives. (Darwin)
- OPERATIONAL DEFINITIONS: What "subjects" or "objects" need to be described (and/or measured)? What are reasonable assumptions to make about these "subjects" or "objects"?)
    - Most are (or embody) complex adaptive systems. They range from the conceptual systems that underlie curriculum materials or programs of instruction – to the sense-making systems that students or teachers develop to interpret their experiences.
    - Most of these systems are (partly) products of human design. They're continually changing.
        - The conceptual systems that are developed to make sense of such systems also are used to

make, mold, and manipulate new systems.

- The "design specs" for these systems are shaped by human purposes as much as by "given" conditions – and these purposes continually change.
- A fundamental characteristic of conceptual systems is that when they are engaged, they change; and, when they function, they adapt.
- So, nothing happens twice, and no two situations are exactly alike.

- The most important to attributes if these systems that need to be described (documented and measured) involve deeper patterns and regularities, and emergent properties of the systems-as-a-whole – not just surface-level characteristics. … So: How should these attributes by operationally defined?
- Reductionism doesn't work. The most important attributes are systemic (emergent). Most systems cannot be understood by partitioning them into pieces and measuring the pieces.

- DESIGNING TOOLS: What form should these take to be most useful?
  - Designing for usefulness, sharability, and re-useability – not just testing for it.
  - Most decisions involve trade-offs (low costs & high quality) and stake-holders with partly conflicting interests. So, in general, we need to draw on more than a single practical or theoretical perspective.
    - Ta Da: They are MODELS!

Multi-Tier Design Assessments!!!!

The most important outcomes to cultivate (and document and assess) occur because of interactions among agents within the system – not due to characteristics elements in the system.