INTRODUCTION TO THE LOCAL LANGLANDS CORRESPONDENCE

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1. Automorphic representations

Let $\mathbf{F}$ be a global field, for example $\mathbf{F} = \mathbb{Q}$ or $\mathbf{F} = \mathbb{F}_p(t)$. Let $\mathbb{A}_{\mathbf{F}} = \prod'_v \mathbf{F}_v \supset \mathfrak{o}_v$, where v runs over all the completions of v and for all finite v, $\mathfrak{o}_v \subset \mathbf{F}_v$ is the ring of integers of $\mathbf{F}_v$. Let G be a connected algebraic reductive group over $\mathbf{F}$. Let π be an irreducible automorphic representation of

$$G(\mathbb{A}_{\mathbf{F}}) = \prod_v{}' G(\mathbf{F}_v) \supset G(\mathfrak{o}_v).$$

Then π can be decomposed as a restricted tensor product $\pi = \bigotimes'_v \pi_v$ as follows: for any v, π_v is an irreducible admissible representation of $G(\mathbf{F}_v)$. For all but finitely many v, the representation π_v is *spherical*, that is, π_v admits a non-zero vector invariant under the action of the maximal compact open subgroup $G(\mathfrak{o}_v)$. Such vector is called a *spherical vector*. It is well-known that for a spherical representation π_v, its spherical vector is unique up to scalar multiplication. A classification of irreducible spherical representations is also well-known. They are in bijection with conjugacy classes of $\hat{G}(\mathbb{C})$. For almost every v, we denote the conjugacy class of $\hat{G}(\mathbb{C})$ corresponding to π_v by $\operatorname{Sat}(\pi_v)$.

Let R be an algebraic representation of $\hat{G}(\mathbb{C})$, i.e., $R\colon \hat{G}(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ is an algebraic homomorphism where $N \geq 1$. The Langlands functoriality conjecture concerns the question whether there exists an automorphic representation Π of $\mathrm{GL}_N(\mathbb{A}_{\mathbf{F}})$ such that for all but finitely many v,

$$\operatorname{Sat}(\Pi_v) = R(\operatorname{Sat}(\pi_v))?$$

One way of approaching this question is by attaching an L-function to the desired automorphic representation Π. We may define for almost every v,

$$L(s, R, \pi_v) = \det\left(1 - R(\operatorname{Sat}(\pi_v))\, q_v^{-s}\right)^{-1},$$

and hence define a partial L-function

$$L^S(s, R, \pi) = \prod_{v \notin S} L(s, R, \pi_v).$$

Under certain assumptions on π, it can be shown that $L^S(s, R, \pi)$ absolutely converges for $\operatorname{Re} s \gg 0$. Some questions naturally arise.

(1) Does $L^S(s, R, \pi)$ have a meromorphic continuation to the entire plane? Does it satisfy a functional equation?
(2) How can we define $L(s, R, \pi_v)$ for $v \in S$?

The first question has been studied case by case for cases of (G, R). The area studying it is called *integral representations of L-functions*. Understanding the analytic properties of $L^S(s, S, \pi')$ for certain (S, π') can yield answers to the above functoriality problem. Miao (Pam) Gu and her collaborators are working on a new case of this problem, where R is a triple product of three cuspidal automorphic representations of GL. Their work can potentially imply an answer to the functoriality problem for the tensor product representation of two cuspidal automorphic representations.

Answering the first question usually allows one to also answer the second question. However, the second question has an independent answer using the local Langlands correspondence.

2. Weil–Deligne representations

In the study of algebraic number theory, one defines a local Artin L-factor for a Galois representation. We may try to borrow this idea here.

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue field $\mathbb{F}_q$. We define W_F to be the following subgroup of the absolute Galois group $\mathrm{Gal}\left(F^{\mathrm{sep}}/F\right)$, where F^{sep} is the separable closure of F. Recall that if F^{unr} is the maximal unramified extension of F then $\mathrm{Gal}\left(F^{\mathrm{unr}}/F\right)$ is isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. We define W_F to be the inverse image of $\mathbb{Z}$ under the composition

$$\mathrm{Gal}\left(F^{\mathrm{sep}}/F\right) \to \mathrm{Gal}\left(F^{\mathrm{unr}}/F\right) \to \hat{\mathbb{Z}}.$$

Then W_F fits into the exact sequence

$$0 \longrightarrow \mathrm{I}_F \longrightarrow \mathrm{W}_F \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathrm{I}_F = \mathrm{Gal}\left(F^{\mathrm{sep}}/F^{\mathrm{unr}}\right)$ is the inertia group, defined as the inverse image of 0 under the composition above.

Let us equip W_F with topology such that I_F is open. We have that

$$\mathrm{W}_F = \langle \mathrm{Fr} \rangle \ltimes \mathrm{I}_F,$$

where Fr is an element such that its image under the composition above generates $\mathbb{Z}$. Note that the choice of Fr is not unique, as for any $i \in \mathrm{I}_F$, the element $\mathrm{Fr} \cdot i$ also has this property. Define the norm character $|\cdot| : \mathrm{W}_F \to \mathbb{C}^\times$ by

$$\left|\mathrm{Fr}^k \cdot i\right| = q^{-k}.$$

A *Weil–Deligne representation* is a pair $\varphi = (\rho, N)$ where (ρ, V) is a finite dimensional representation of V and $N \in \mathrm{End}\left(V\right)$ is a nilpotent such that:

1. There exists an open subgroup $J \subset \mathrm{I}_F$ such that $\rho\left(J\right)$ is trivial.
2. $\rho\left(\mathrm{Fr} \cdot i\right)$ is semisimple for any $i \in \mathrm{I}_F$.
3. $\rho\left(w\right) N \rho\left(w\right)^{-1} = |w| \cdot N$ or any $w \in \mathrm{W}_F$.

If $\varphi_j = (\rho_j, N_j)$ is a Weil–Deligne representation for $j = 1, 2$, a homomorphism $T \colon \varphi_1 \to \varphi_2$ is a linear map $T \colon V_1 \to V_2$ such that for every $w \in \mathrm{W}_F$,

$$T \circ \rho_1\left(w\right) = \rho_2\left(w\right) \circ T$$

and

$$T \circ N_1 = N_2 \circ T.$$

Such T is called an isomorphism if T is an invertible linear map.

For a Weil–Deligne representation as above we may attach a local L-factor as follows. Notice that N fixes the subspace V^{I_F} consisting of inertia fixed vectors. Let $V_N^{\mathrm{I}_F} = \mathrm{Ker}\left(N \restriction_{V^{\mathrm{I}_F}}\right)$. Define

$$L\left(s, \varphi\right) = \det\left(\mathrm{id}_V - q^{-s} \varphi\left(\mathrm{Fr}\right) \restriction_{V_N^{\mathrm{I}_F}}\right)^{-1}.$$

The L-factor $L(s,\varphi)$ records the multiplicity of the Weil–Deligne representation $(1,0)$ in $\varphi = (\rho, N)$: it has a pole at $s=0$ of order equal to the number of times $(1,0)$ appears as a summand of $\varphi = (\rho, N)$.

The local Langlands conjecture for $\mathrm{GL}_n(F)$ is the statement that there exists a bijection between the following sets

$$\{\text{Semisimple Weil-Deligne representations } \varphi = (\rho, N) \text{ of dimension } n\}/\text{equivalence}$$

and

$$\{\text{Irreducible admissible representations } \pi \text{ of } \mathrm{GL}_n(F)\}/\text{equivalence}.$$

If π corresponds to $\varphi = (\rho, N)$, we say that φ is the *Langlands parameter of* π.

This bijection depends on a choice of a uniformizer $\varpi \in F$ and a Frobenius $\mathrm{Fr} \in \mathrm{W}_F$. It should satisfy the following properties.

(1) For $n=1$ it is given by local class field theory via the realization
$$F^\times \cong \mathrm{W}_F^{\mathrm{ab}} = \mathrm{W}_F / [\mathrm{W}_F, \mathrm{W}_F].$$
(2) Duals: if φ corresponds to π then the dual $\varphi^\vee$ corresponds to $\pi^\vee$.
(3) Central characters: if φ corresponds to π then $\det \varphi$ corresponds to ω_π, where ω_π is the central character of π.
(4) Twisting by characters: if φ corresponds to π and ω is a character of W_F that corresponds to α then $\varphi \otimes \omega$ corresponds to $\pi \otimes \alpha(\det)$.
(5) The map preserves L-factors and ε-factors corresponding to tensor products of pairs. If $\varphi_j = (\rho_j, N_j)$ for $j=1,2$, then the relevant L-factor is $L(s, \varphi_1 \otimes \varphi_2)$, while the epsilon factor $\varepsilon(s, \varphi_1 \otimes \varphi_2, \psi)$ is defined by a recipe of Deligne [5]. See [13, Section 3.2]. If π_j is an irreducible admissible representation of $\mathrm{GL}_{n_j}(F)$ for $j=1,2$ then $L(s, \pi_1 \times \pi_2)$ and $\varepsilon(s, \pi_1 \times \pi_2, \psi)$ are defined by the theory of Rankin–Selberg integrals introduced by Jacquet–Piatetski-Shapiro–Shalika [9]. See [4] and [13, Section 2.5].

3. Special cases of the correspondence

We mention a few special cases of the correspondence.

3.1. **Spherical representations.**

Recall that a representation π of $\mathrm{GL}_n(F)$ is called spherical if it admits a non-zero spherical vector, that is, a vector invariant under the action of $\mathrm{GL}_n(\mathfrak{o})$. It is well known that such π can be realized as a quotient of the (normalized) parabolically induced representation

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n(F)} \left(|\cdot|^{z_1} \otimes \cdots \otimes |\cdot|^{z_n} \right) \tag{3.1}$$

for some $z_1, \dots, z_n \in \mathbb{C}$. Such π is uniquely determined by the unordered complex numbers $(q^{-z_1}, \dots, q^{-z_n})$ which are called the *Satake parameters of* π. Alternatively, the Satake parameter of π can be defined as the conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ corresponding to the matrix

$$\mathrm{Sat}(\pi) = \left[\begin{pmatrix} q^{-z_1} & & \\ & \ddots & \\ & & q^{-z_n} \end{pmatrix} \right].$$

We define an L-factor corresponding to π by

$$L(s,\pi) = \det\left(I_n - q^{-s}\mathrm{Sat}(\pi)\right)^{-1} = \prod_{j=1}^{n}\left(1 - q^{-z_j}\cdot q^{-s}\right)^{-1}.$$

This L-factor records the Satake parameters of π. More specifically, the order of the pole of $L(s,\pi)$ at $s=0$ records how many times 1 appears in the Satake parameters of π, which is equivalent to the number of times that the trivial character appears in (3.1).

If π is a spherical representation corresponding to a quotient of (3.1), then under the local Langlands correspondence π corresponds to the representation

$$\varphi = \left(\rho = \bigoplus_{j=1}^{n} |\cdot|^{z_j}, N = 0\right).$$

In this case, under a suitable basis

$$\rho(\mathrm{Fr}) = \begin{pmatrix} q^{-z_1} & & \\ & \ddots & \\ & & q^{-z_n} \end{pmatrix} \text{ and } \rho(i) = I_n \text{ for any } i \in \mathrm{I}_F.$$

One easily checks that $L(s,\pi) = L(s,\varphi)$.

Hence the Langlands parameter of π in this case can be identified with its Satake parameter.

3.2. **Steinberg representations.** The Steinberg representation $\mathrm{St}(1,n)$ for $\mathrm{GL}_n(F)$ is defined to be the unique irreducible subrepresentation of

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n(F)}\left(|\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes \cdots \otimes |\cdot|^{-\left(\frac{n-1}{2}\right)}\right)$$

or the unique irreducible quotient of

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n(F)}\left(|\cdot|^{-\left(\frac{n-1}{2}\right)} \otimes |\cdot|^{-\left(\frac{n-3}{2}\right)} \otimes \cdots \otimes |\cdot|^{\frac{n-1}{2}}\right).$$

It is a square-integrable representation of $\mathrm{GL}_n(F)$. Its Langlands parameter is the Weil–Deligne representation defined by $\varphi_{\mathrm{St}(1,n)} = (\rho, N)$, where

$$\rho(\mathrm{Fr}) = \begin{pmatrix} q^{\frac{n-1}{2}} & & & \\ & q^{\frac{n-3}{2}} & & \\ & & \ddots & \\ & & & q^{-\left(\frac{n-1}{2}\right)} \end{pmatrix} \text{ and } \rho(i) = I_n$$

and

$$N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

For convenience, let us denote by $\mathrm{Sp}(n)$ the Weil–Deligne representation $\mathrm{Sp}(n) = |\cdot|^{\frac{n-1}{2}} \varphi_{\mathrm{St}(1,n)}$, that is if $\mathrm{Sp}(n) = \left(\rho_{\mathrm{Sp}(n)}, N_{\mathrm{Sp}(n)}\right)$ then

$$\rho_{\mathrm{Sp}(n)}(\mathrm{Fr}) = \begin{pmatrix} 1 & & & \\ & q^{-1} & & \\ & & \ddots & \\ & & & q^{-(n-1)} \end{pmatrix} \text{ and } \rho_{\mathrm{Sp}(n)}(i) = I_n$$

and

$$N_{\mathrm{Sp}(n)} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

3.3. **Depth–zero supercuspidal representations.** A irreducible admissible representation π of $\mathrm{GL}_n(F)$ is called supercuspidal if and only if the following equivalent conditions are satisfied:

(1) For any unipotent radical $N_{(n_1,\dots,n_r)} \subset \mathrm{GL}_n(F)$ where $(n) \neq (n_1,\dots,n_r)$ is a composition of n, we have that the Jacquet module
$$J(\pi) = \pi / \mathrm{Span}_{\mathbb{C}} \left\{ \pi(u) v - v \mid v \in \pi, u \in N_{(n_1,\dots,n_r)} \right\}$$
vanishes.

(2) For any unipotent radical $N_{(n_1,\dots,n_r)} \subset \mathrm{GL}_n(F)$ as above and any $v \in \pi$, the following stable integral vanishes
$$\int^{*}_{N_{(n_1,\dots,n_r)}} \pi(u) v dv = 0.$$
This means that for any $v \in \pi$ there exists K large such that for any $k \geq K$
$$\int_{N_{(n_1,\dots,n_r)} \cap \left(1 + \mathrm{Mat}_n\left(\mathfrak{p}^{-k}\right)\right)} \pi(u) v dv = 0.$$

(3) The matrix coefficients of π are compactly supported, modulo the center.

Supercuspidal representations serve as building blocks for the irreducible representations of $\mathrm{GL}_n(F)$. Under the local Langlands correspondence they correspond to irreducible[1] Weil–Deligne representations $\varphi = (\rho, N)$ and it follows from irreducibility that $N = 0$.

By the work of Bernstein–Zelevinsky, the local Langlands correspondence reduces to understanding the local Langlands correspondence for irreducible supercuspidal representations. This is the research area of Charlotte Chan, Stephen DeBacker and Tasho Kaletha (they deal with arbitrary G and not necessarily GL_n).

We give one example of an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ and its corresponding Weil–Deligne representation [3]. Let σ be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Consider the quotient map $\nu \colon \mathfrak{o} \to \mathbb{F}_q$. It induces quotient maps $\nu \colon \mathfrak{o}^{\times} \to \mathbb{F}_q^{\times}$

[1]The fact that supercuspidals correspond to irreducible representations follows from the L-factors equality requirement by an inductive argument, which uses the fact that if π_1, π_2 are supercuspidal representations then $L(s, \pi_1 \times \pi_2)$ has a pole at $s_0 \in \mathbb{C}$ if and only if $\pi_1 \cong |\det|^{-s_0} \pi_2^{\vee}$.

and $\nu\colon \mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathbb{F}_q)$. Choose a character $\chi\colon F^\times \to \mathbb{C}^\times$ such that $\chi\restriction_{\mathfrak{o}^\times} = \omega_\sigma \circ \nu \restriction_{\mathbb{F}_q^\times}$. Let $\chi \otimes \sigma \circ \nu$ be the representation of $F^\times \cdot \mathrm{GL}_n(\mathfrak{o})$ defined by inflation as follows:

$$(\chi \otimes \sigma \circ \nu)(z \cdot k) = \chi(z)\,\sigma(\nu(k)),$$

for $z \in F^\times$ and $k \in \mathrm{GL}_n(\mathfrak{o})$. Then the compactly induced representation

$$\pi = \mathrm{ind}_{F^\times \cdot \mathrm{GL}_n(\mathfrak{o})}^{\mathrm{GL}_n(F)} (\chi \otimes \sigma \circ \nu)$$

is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. What is its Langlands parameter? The irreducible cuspidal representation σ corresponds to a Frobenius orbit, a set of size n of the form

$$\left\{\theta, \theta^q, \ldots, \theta^{q^{n-1}}\right\},$$

where $\theta\colon \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ is a character. The inertia subgroup I_F has a subgroup P_F called the *wild inertia subgroup*. It satisfies

$$\mathrm{I}_F/\mathrm{P}_F \cong \varprojlim \mathbb{F}_{q^k}^\times$$

where for $k_1 \mid k_2$, the map $\mathbb{F}_{q^{k_2}}^\times \to \mathbb{F}_{q^{k_1}}^\times$ is the norm map. We have that π corresponds under the Langlands correspondence to the parameter

$$\varphi = (\rho, 0_n),$$

where

$$\rho = \mathrm{Ind}_{\langle \mathrm{Fr}^n \rangle \ltimes \mathrm{I}_F}^{W_F} \left(\left(\mathrm{Fr}^n \mapsto (-1)^{n-1}\chi(\varpi)\right) \otimes \theta\right).$$

Here, θ is realized with its image in

$$\varinjlim \mathrm{Hom}\left(\mathbb{F}_{q^k}^\times, \mathbb{C}^\times\right) \cong \mathrm{Hom}\left(\mathrm{I}_F/\mathrm{P}_F, \mathbb{C}^\times\right),$$

where the transition maps are given by composition with the norm map. In matrix notation, this can be written as

$$\rho(\mathrm{Fr}) = \begin{pmatrix} & & & (-1)^{n-1}\chi(\varpi) \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \end{pmatrix}$$

and

$$\rho(i) = \begin{pmatrix} \theta(i) & & & \\ & \theta^q(i) & & \\ & & \ddots & \\ & & & \theta^{q^{n-1}}(i) \end{pmatrix}$$

for $i \in \mathrm{I}_F$.

3.4. **Generalized Steinberg representations.** Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_k(F)$. We may define similarly to before a (generalized) Steinberg representation associated to τ as follows. Consider the (normalized) parabolic induction

$$\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}(F)}\left(\left|\det\right|^{-\left(\frac{c-1}{2}\right)}\tau\otimes\left|\det\right|^{-\left(\frac{c-3}{2}\right)}\tau\otimes\cdots\otimes\left|\det\right|^{\frac{c-1}{2}}\tau\right).$$

It admits a unique irreducible quotient, which we denote $\mathrm{St}(\tau,c)$. If τ has a unitary central character, then $\mathrm{St}(\tau,c)$ is a square-integrable representation of $\mathrm{GL}_{kc}(F)$. All square-integrable representations arise in this way.

To $\mathrm{St}(\tau,c)$ we assign a Langlands parameter as follows. Suppose that τ corresponds to $\varphi_\tau=(\rho_\tau,0)$ and let $\varphi_{\mathrm{St}(1,c)}=\left(\rho_{\mathrm{St}(1,c)},N_{\mathrm{St}(1,c)}\right)$ as in Section 3.2. Then we assign to $\mathrm{St}(\tau,c)$ the Langlands parameter

$$\left(\rho_\tau\otimes\rho_{\mathrm{St}(1,c)},\mathrm{id}_{\rho_\tau}\otimes N_{\mathrm{St}(1,c)}\right).$$

In matrix form,

$$\rho_{\mathrm{St}(\tau,c)}(w)=\begin{pmatrix}|w|^{-\left(\frac{c-1}{2}\right)}\rho_\tau(w) & & & \\ & |w|^{-\left(\frac{c-3}{2}\right)}\rho_\tau(w) & & \\ & & \ddots & \\ & & & |w|^{\frac{c-1}{2}}\rho_\tau(w)\end{pmatrix}\text{ for any }w\in\mathrm{W}_F$$

and

$$N_{\mathrm{St}(1,c)}=\begin{pmatrix}0_k & & & & \\ I_k & 0_k & & & \\ & I_k & \ddots & & \\ & & \ddots & 0_k & \\ & & & I_k & 0_k\end{pmatrix}.$$

3.5. **Bernstein–Zelevinsky classification and reduction to supercuspidals.** Bernstein and Zelevinsky [1, 2, 14] gave a classification of all irreducible admissible representations of $\mathrm{GL}_n(F)$. To explain it, we first introduce the notion of an interval. An interval is a set of the form

$$\Delta=\left\{\tau,\left|\det\right|\tau,\left|\det\right|^2\tau,\dots,\left|\det\right|^{c-1}\tau\right\},\tag{3.2}$$

where τ is an irreducible supercuspidal representation of $\mathrm{GL}_k(F)$ for some k. We call τ the *left-most element of* Δ. We say that two intervals are *linked* if neither of them is contained in the other, and if their union is also an interval. If Δ_1 and Δ_2 are intervals, we say that Δ_1 *precedes* Δ_2 if Δ_1 and Δ_2 are linked and if the left-most element of the union is in Δ_1.

Given such interval an interval Δ as in (3.2), we define $Q(\Delta)$ to be the unique irreducible quotient of the (normalized) parabolic induction

$$\tau\times\left|\det\right|\tau\times\cdots\times\left|\det\right|^{c-1}\tau:=\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}(F)}\left(\tau\otimes\left|\det\right|\tau\otimes\cdots\otimes\left|\det\right|^{c-1}\tau\right).$$

Note that $Q(\Delta)$ is simply $\left|\det\right|^{\frac{c-1}{2}}\mathrm{St}(\tau,c)$, and the Langlands parameter of $Q(\Delta)$ is $\varphi_{Q(\Delta)}=\left(\rho_{Q(\Delta)},N_{Q(\Delta)}\right)=\left(\left|\cdot\right|^{\frac{c-1}{2}}\rho_{\mathrm{St}(\tau,c)},N_{\mathrm{St}(\tau,c)}\right)$.

Bernstein–Zelevinsky showed that given intervals $\Delta_1,\dots,\Delta_r$ such that Δ_i does not precede Δ_j for $i<j$, the (normalized) parabolic induction $Q(\Delta_1)\times\cdots\times Q(\Delta_r)$ admits a unique irreducible quotient $Q(\Delta_1,\dots,\Delta_r)$. They showed that any irreducible representation π of

$\mathrm{GL}_n(F)$ is isomorphic to a representation obtained in this way. We refer to [13, (2.2.9)] for a nice summary of their results.

Suppose that $\pi = Q(\Delta_1, \ldots, \Delta_r)$ for intervals as above. Then the Langlands parameter of π is $\varphi_\pi = \left(\bigoplus_{j=1}^r \rho_{Q(\Delta_j)}, \bigoplus_{j=1}^r N_{Q(\Delta_j)}\right)$.

3.5.1. *Tamely ramified with unipotent monodromy representations.* Our seminar tries to establish a story about TRUM representations. Let us recall that this means that $\varphi = (\rho, N)$, where the restriction of ρ to the inertia subgroup is trivial. Suppose that

$$(\rho, N) = \bigoplus_{j=1}^{r} (\rho_j, N_j)$$

where for any j, we have that

$$\rho_j(\mathrm{Fr}) = \begin{pmatrix} q^{-s_j} & & & \\ & q^{-s_j-1} & & \\ & & \ddots & \\ & & & q^{-s_j-n_j+1} \end{pmatrix} \text{ and } N_j = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_{n_j}(F),$$

where $s_j \in \mathbb{C}$. Assume without loss of generality that $\mathrm{Re} s_1 \geq \cdots \geq \mathrm{Re} s_r$. Then the intervals

$$\Delta_j = \left\{|\cdot|^{s_j}, \ldots, |\cdot|^{s_j+n_j-1}\right\}$$

satisfy that Δ_i does not precede Δ_j for $i < j$. The representation π that corresponds to such φ is the unique irreducible quotient of the (normalized) parabolic induction

$$|\det|^{s_1+\frac{n_1-1}{2}} \mathrm{St}(1, n_1) \times \cdots \times |\det|^{s_r+\frac{n_r-1}{2}} \mathrm{St}(1, n_r). \tag{3.3}$$

On the other hand, on the group side, our seminar tries to classify irreducible admissible representations with a vector invariant to the Iwahori subgroup, defined as the inverse image of the Borel subgroup $B_n(\mathbb{F}_q)$ under the quotient map $\nu\colon \mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathbb{F}_q)$. In the case that we have in hand, Howe showed that these representations are precisely the ones of the form (3.3) [8].

4. The Local Langlands correspondence for other groups

For G a split connected algebraic reductive group, the local Langlands correspondence is more complicated. We do not say much about it here.

In this case, we need to consider parameters of the form

$$\varphi = (\rho, N),$$

where $\rho\colon \mathrm{W}_F \to \hat{G}(\mathbb{C})$ and $N \in \mathrm{Lie}\left(\hat{G}\right)$ is a nilpotent, such that

$$\rho(w) N \rho(w)^{-1} = |w| \cdot N,$$

for any $w \in \mathrm{W}_F$. In this case, instead of having a bijection, we have a correspondence, where for every semisimple parameter φ, there exist finitely many irreducible representations π of $G(F)$, such that the parameter of π is φ. There is a way to refine this statement and get a bijection. A common analogy used is referring to the parameter φ as the "last name" of the representation. The finite set of all representations π with a given parameter φ is called *the L-packet of* φ. This is a family, where each element has last name φ. The "first name" of the representation is an additional parameter indexing the L-packet. See [6, Section 1.1] for a nice overview.

As before, spherical representations correspond to representations of the form $\varphi = (\rho, N)$, where the inertia group I_F maps to identity and $N = 0$. In this seminar, we are concerned with similar representations, the difference being that N is not necessarily zero.

4.1. **Brief summary of the reduction in this case.** Take G to be one of $\mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$. The group $\hat{G}$ is given by the following table.

G	$\hat{G}$
GL_n	GL_n
SO_{2n+1}	Sp_{2n}
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

Instead of taking $\varphi = (\rho, N)$ as in the previous section, it turns out to be more convenient to take homomorphisms $\varphi' \colon \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C})$. In the case of $G = \mathrm{GL}_n$, the translation is as follows: we replace

$$\varphi = \bigoplus_{i=1}^{r} \left(\rho_i \otimes \rho_{\mathrm{Sp}(n_i)}, \mathrm{id}_{\rho_i} \otimes N_{\mathrm{Sp}(n_i)}\right),$$

with

$$\varphi' = \bigoplus_{i=1}^{r} \rho_i \otimes \mathrm{Sym}^{n_i - 1}\left(\mathbb{C}^2\right),$$

where $\rho_i \colon \mathrm{W}_F \to \mathrm{GL}_{m_i}(\mathbb{C})$ is an irreducible representation for every i. Note that $\mathrm{Sym}^{n_i-1}(\mathbb{C}^2)$ is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_i.

In the local Langlands correspondence for GL_n we had that irreducible supercuspidal representations π of GL_n corresponded to $\varphi = (\rho, 0)$ where $\rho \colon \mathrm{W}_F \to \mathrm{GL}_n(\mathbb{C})$ was an irreducible representation. In the local Langlands correspondence for $\hat{G}$ this is slightly modified. We have that essentially square–integrable representations of G correspond to L-parameters $\varphi' \colon \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C})$ satisfying that their image does not lie in a proper parabolic subgroup of $\hat{G}$.

We move to describe the Langlands classification of irreducible representations of $G(F)$ for $G \neq \mathrm{GL}_n$, which we will utilize in order to reduce the Langlands correspondence to essentially square-integrable representations [11]. We first need to discuss the classification of tempered irreducible representations.

Consider the (normalized) parabolically induced representation

$$\tau_r \times \cdots \times \tau_1 \rtimes \pi_0, \tag{4.1}$$

where π_0 is a square-integrable irreducible representation of $G_0(F)$, a group of the same type as of G but of smaller size, and $\tau_1, \ldots, \tau_r$ are square integrable irreducible representations of $\mathrm{GL}_{n_1}(F)$, ..., $\mathrm{GL}_{n_r}(F)$, respectively. Then (4.1) is semi-simple, and every irreducible subrepresentation of (4.1) is tempered (for the groups we are considering, (4.1) is multiplicity free). Every tempered irreducible representation π of $G(F)$ can be realized in this way [10]. We associate to such π the following Langlands parameter

$$\varphi'_\pi = \bigoplus_{i=1}^{r} \varphi'_{\tau_i} \oplus \varphi'_{\pi_0} \oplus \bigoplus_{i=1}^{r} \varphi'_{\tau_i^\vee}.$$

We are now ready to describe the Langlands classification for $G(F)$ (for $G \neq \mathrm{GL}_n$) [12], [7, Theorem 8.4.2]. Given a tempered irreducible representation π_0 of $G_0(F)$, tempered irreducible representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{n_1}(F)$, …, $\mathrm{GL}_{n_r}(F)$, respectively, and complex numbers $s_1, \dots, s_r$ satisfying $\mathrm{Re} s_r > \mathrm{Re} s_{r-1} > \cdots > \mathrm{Re} s_1 > 0$, the parabolically induced representation

$$|\det|^{s_r} \tau_r \times |\det|^{s_{r-1}} \tau_{r-1} \times \cdots \times |\det|^{s_1} \tau_1 \rtimes \pi_0 \tag{4.2}$$

has a unique irreducible quotient. Any irreducible representation π can realized in this way. If π is the unique irreducible quotient of (4.2), we associate to π the Langlands parameter

$$\varphi'_\pi = \bigoplus_{i=1}^{r} |\cdot|^{s_i} \varphi'_{\tau_i} \oplus \varphi'_{\pi_0} \oplus \bigoplus_{i=1}^{r} |\cdot|^{-s_i} \varphi'_{\tau_i^\vee}.$$

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