BEZRUKAVNIKOV'S EQUIVALENCE

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Abstract. These are the notes for RTG learning seminar on Bezrukavnikov's equivalence at University of Michigan in 2024 Fall. The notes taker is the only one who is responsible for all the mistakes and typos.

Contents

1. Introduction to Hecke algebra - Elad Zelingher

1.1. **Bi-invariant K-Hecke algebra of G.** Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, uniformizer ϖ_F, residual field $\mathbb{F}_q$. Let G be a split connected reductive group. Fix a Borel subgroup $B = AN$.

Definition. The *co-characters* of A is

$$X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A).$$

Fix $\kappa = G(\mathcal{O}_F)$ to be a maximal compact open subgroup. Then $A(\mathcal{O}_F) = A \cap \kappa$. Let $A(\varpi) = A/A(\mathcal{O}_F)$.

Example 1.1. Let $G = \mathrm{GL}_n$ and A be the diagonal torus. Then we have identification

$$A(\varpi) = \left\{ \mathrm{diag}(\varpi^{k_1}, \cdots, \varpi^{k_n}) \mid k_1, \cdots, k_n \in \mathbb{Z} \right\}.$$

We have an isomorphism $X_*(A) \cong A(\varpi)$ given by $\check{\mu} \mapsto \check{\mu}(\varpi)$.

We will talk about the algebra $\mathcal{H}(G /\!\!/ K) = \mathcal{H}(K\backslash G/K)$for some compact open subgroup $K \subseteq G(F)$, where

$$\mathcal{H}(G /\!\!/ K) = \{f \in C_c^\infty(G) \mid f(k_1 g k_2) = f(g), \forall k_1, k_2 \in K, g \in G\}.$$

Today we will be interested in the cases

- $K = G(\mathcal{O})$;
- $K =$ the Iwahori subgroup.

Why study this? We want to study representations π of $G(F)$ that have a fixed K-vector. Every such π admits a representation of $\mathcal{H}(G /\!\!/ K)$.

1.2. **Spherical Hecke algebra.** This is the case $K = G(\mathcal{O}_F)$. $\mathcal{H}(G /\!\!/ F)$ is an algebra, the product being convolution

$$(f_1 * f_2)(g) = \int_{G(F)} f_1(x^{-1}) f_2(xg) dx.$$

For $K = G(\mathcal{O}_F)$, we have the Cartan decomposition

$$G(F) = \sqcup_{\check{\mu} \in X_*(A)_{\mathrm{dom}}} G(\mathcal{O}_F) \varpi_F^{\check{\mu}} G(\mathcal{O}_F).$$

Here, $X_*(A)_{\mathrm{dom}}$ consists of *dominant* co-characters. By applying the Gelfand's trick to involution, we have that

Proposition 1.1. *$\mathcal{H}(G /\!\!/ K)$ is commutative if $K = G(\mathcal{O}_F)$.*

Example 1.2. For $G = \mathrm{GL}_n$, the involution $g \mapsto {}^t g$ fixes the diagonal matrices and sends $\mathrm{GL}(\mathcal{O}_F)$ to itself.

Remark. In general, $\mathcal{H}(G /\!\!/ K)$ is not commutative. However, if $\check{\mu}_1, \check{\mu}_2$ are dominant characters, then we have

$$\chi_{K\varpi^{\check{\mu}_1}\varpi^{\check{\mu}_2}K} = \chi_{K\varpi^{\check{\mu}_1}K} * \chi_{K\varpi^{\check{\mu}_2}K}.$$

Therefore, R_K^+, the subalgebra of $\mathcal{H}(G /\!\!/ K)$ generated by $\chi_{K\varpi^{\check{\mu}}K}$, then R_K^+ is commutative and we have

$$\mathcal{H}(G /\!\!/ K) = \mathcal{H}(\kappa /\!\!/ K) * R_K^+ * \mathcal{H}(\kappa /\!\!/ K).$$

Here $\mathcal{H}(\kappa /\!\!/ K)$ is non-commutative but finite dimensional, and R_K^+ is commutative but infinite dimensional.

1.3. **Iwahori-Matsumoto Hecke algebra.** Consider the quotient map $G(\mathcal{O}) \twoheadrightarrow G(\mathbb{F}_q)$. Let I be the pre-image of $B(\mathbb{F}_q)$ under this quotient map.

Let W be the Weyl group of G and $\tilde{W} = W \ltimes A(\varpi)$ be the extended affine Weyl group. We have the *Bruhat-Iwahori decomposition*:

$$G(F) = \bigsqcup_{w \in \tilde{W}} IwI.$$

For $w \in \tilde{W}$, $f_w = \chi_{IwI}$ form a basis of $\mathcal{H}(G(F) /\!\!/ I)$. The algebra $\mathcal{H}(G(F) /\!\!/ I)$ is isomorphic to the *Iwahori-Matsumoto algebra.* The latter is defined as follows. $\tilde{W}$ has a decomposition

$$\tilde{W} = W_{\mathrm{aff}} \rtimes \pi_1(G).$$

W_{aff} is a Coxeter group, so it is equipped with a standard presentation with generators and relations. The Iwahori-Matsumoto algebra is the Hecke algebra associated to the Coxeter system (i.e. for each simple reflection s there is a T_s, and for each generator h of $\pi_1(G)$ there is T_h, satisfying all the relations of the Coxeter system and semi-direct product except that $s^2 = 1$ is replaced by $T_s^2 = qT_s + q$). The isomorphism is given by $f_w = h \ltimes (\prod_j s_{i_j}) \mapsto T_h \cdot T_{s_{i_1}} \cdot \dots \cdot T_{s_{i_r}}$.

Let $R_I^+ = \{f_{\varpi^{\check{\mu}}} \mid \check{\mu} \text{ is dominant}\}$. One can show f_w is always invertible. Define

$$R_I = \langle f_{\varpi^{\check{\mu}}}, f_{\varpi^{\check{\mu}}}^{-1} \mid \check{\mu} \text{ is dominant}\rangle.$$

This is a commutative subalgebra of $\mathcal{H}(G /\!\!/ I)$. Tempting to define a map

$$\begin{aligned} A(\varpi) &\to R_I^\times \\ \varpi^\mu &\mapsto f_{\varpi^\mu}. \end{aligned}$$

But this will not be a homomorphism of groups. If $\check{\mu}_1, \check{\mu}_2$ are dominant, then

$$f_{\varpi^{\mu_1}\varpi^{\mu_2}} = f_{\varpi^{\mu_1}} * f_{\varpi^{\mu_2}}.$$

Any co-character $\check{\lambda}$ is of the form $\check{\lambda} = \check{\mu}_1(\check{\mu}_2)^{-1}$ where $\check{\mu}_1$ and $\check{\mu}_2$ are dominant. For $w \in \tilde{W}$, write

$$T_w = q^{-\frac{l(w)}{2}} f_w$$

where $l : \tilde{W} \to W_{\text{aff}} \to \mathbb{Z}_{\geq 0}$ is the length. Define for $\check{\lambda}$

$$\theta(\check{\lambda}) = (T_{\check{\mu}_1}) \cdot (T_{\check{\mu}_2})^{-1}.$$

Then θ is well-defined and is a homomorphism of groups

$$\theta : A(\varpi) \to R_I^\times.$$

1.4. **Bernstein-Zelevinsky relation.**

Theorem 1.1. *Let s be a simple reflection in W. We have the following identity*

$$\theta(\check{\lambda})T_s - T_s\theta(s(\check{\lambda})) = T_s\theta(\check{\lambda}) - \theta(s(\check{\lambda}))T_s = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\frac{\theta(\check{\lambda}) - \theta(s(\check{\lambda}))}{1 - \theta(\check{\alpha})^{-1}}.$$

where $\check{\alpha}$ is a certain co-character $s(\check{\alpha}) = \check{\alpha}^{-1}$.

Corollary 1.1. *The center of $\mathcal{H}(G /\!\!/ I)$ is*

$$\left\{ \sum_{\check{\lambda}} a_{\check{\lambda}} \theta(\check{\lambda}) \mid a_{w\check{\lambda}} = a_{\check{\lambda}}, \forall w \in W, \check{\lambda} \in X_*(A)_{\text{dom}} \right\}.$$

2. The nilpotent cone, Springer fibers & resolution and Steinberg variety - Alexander Hazeltine

2.1. **The nilpotent cone.**

- G = complex (semisimple) reductive group, actually we are thinking about $\hat{G}(\mathbb{C})$;
- $\mathfrak{g} = \text{Lie}(G)$

Let $q : \mathfrak{g} \to \mathfrak{g}/G = \mathfrak{h}/W$ be the adjoint quotient map. The *nilpotent cone* is $\mathcal{N} = \bigcup_{0 \in \bar{\mathcal{O}}} \mathcal{O}$

Example 2.1. For $G = \text{GL}_n$, $\mathcal{N} = \{x \in \mathfrak{gl}_n \mid x^n = 0\}$. In this case, one can parametrize the nilpotent elements by

$$\{\text{nilpotent matrices}\}/\text{conjugate} \leftrightarrow \{\text{Jordan normal forms}\} \leftrightarrow \{\text{partitions of } n\}.$$

The same also works for SL_n. Take $n = 2$. There are two conjugacy classes of nilpotent matrices:

$$\begin{aligned} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) &\rightsquigarrow \mathcal{O}_{\lambda_0} = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\} \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) &\rightsquigarrow \mathcal{O}_{\lambda_1} = \{x \in \mathfrak{sl}_2 \mid \text{rank}(x) = 1\}. \end{aligned}$$

Example 2.2. Let $G = \mathrm{Sp}_{2n}$. The nilpotent orbits of Sp_{2n} can also be parametrized by certain partitions:

$\{$nilpotent orbits of $\mathrm{Sp}_{2n}(\mathbb{C})\} \leftrightarrow \{$partition of $2n$ $|$ odd partitions occur with even multiplicities$\}$.

Take $n = 4$, then we have the closure including ordering and dimensions

$$\begin{array}{cc} \mathcal{O}_{[4]} & 8 \\ | & \\ \mathcal{O}_{[2^2]} & 6 \\ | & \\ \mathcal{O}_{[2\ 1^2]} & 4 \\ | & \\ \mathcal{O}_{[1^4]} & 0 \end{array}$$

In general, for $\mathrm{Sp}_{2n}(\mathbb{C})$, let $r_i = |\{j \in \lambda_j = i\}|$ and $s_i = |\{j \mid \lambda_j \geq i\}|$. Then

$$\dim \mathcal{O}_\lambda = 2n^2 + n - \frac{1}{2}\sum s_i^2 - \frac{1}{2}\sum_{\text{odd } i} r_i.$$

We have $\mathcal{O}_{\lambda_1} \leq \overline{\mathcal{O}_{\lambda_2}}$ if and only if $\lambda_1 \leq \lambda_2$ under dominance ordering.

Proposition 2.1. *(1) $\mathcal{N}$ is irreducible, reduced and normal;*
(2) G acts on $\mathcal{N}$ by conjugation, with finitely many orbits, each of which has even dimension.

2.2. **Springer resolution.**

Definition. The *Springer resolution* is the projection to the first factor

$$\pi : \tilde{N} = \{(x, B) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\} \to \mathcal{N}$$

where $\mathcal{B}$ is the variety of Borel subalgebras of $\mathfrak{g}$.
The *Springer fiber* of $x \in \mathcal{N}$ is $\pi_s^{-1}(X)$. For $x \in \mathcal{O}_\lambda$, set $F_\lambda = \pi_s^{-1}(x)$.

Example 2.3. Let $G = \mathrm{SL}_2(\mathbb{C})$. For $x \neq 0$, $\pi_s^{-1}(X) = \{*\}$. For $x = 0$, $\pi_s^{-1}(0) = \mathcal{B} = \mathbb{P}^1(\mathbb{C})$.

Proposition 2.2. $\dim F_\lambda = \frac{1}{2}\operatorname{codim}(\mathcal{O}_\lambda \subseteq \mathcal{N})$.

Proposition 2.3. $\tilde{N} \cong T^*\mathcal{B} = \{(\mathfrak{b}, v) \in B \times \mathfrak{g}^* \mid v \in \mathfrak{b}^\perp\}$

Proof. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the killing form. By Cartan's 2nd criterion, κ is non-degenerate. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$, consider $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Since $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, $\kappa|_{\mathfrak{n}_-\times\mathfrak{n}_+}$ is also a non-degenerate pairing. $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$ corresponds to $\mathfrak{n}_+$ and hence

$$\begin{aligned} T^*\mathcal{B} &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} \mid x \text{ is nilpotent}\} \\ &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b}\} \\ &= \tilde{N}. \end{aligned}$$

□

2.3. **Steinberg variety.** Let $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ be stratified variety (for example $\mathcal{N} = \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$).

Definition. The *conormal space* is $T^*_\Lambda X = \bigcup_{\lambda\in\Lambda} T^*_\lambda X \subseteq T^*X$ where $T^*_\lambda X = \{(x, \xi) \in X_\lambda \times T^*_x X \mid \xi$ vanishes on $TX_\lambda\}$

Example 2.4. Consider $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$. Then

$$\begin{aligned} T_0^* &= \mathbb{C} = \{(0,y) \in \mathbb{C}^2\} \\ T_x^* &= \{(x,0) \in \mathbb{C}^2\} \text{ for } x \neq 0. \end{aligned}$$

Hence $T_\Lambda^*\mathbb{C} = \{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$.

Proposition 2.4. *(1) $T_\Lambda^* X$ is a closed subvariety of X;*
(2) $\dim T_\lambda^* X = \dim X_\lambda + \operatorname{codim}(X_\lambda \subseteq X) = \dim X$;
(3) Irreducible components of $T_\Lambda^* X$ are in bijection with Λ.

Remark. Intersection pattern of $\overline{T_\Lambda^* X}$ is hard.

Definition. Let H be a group which acts on varieties X and Y on the left and right respectively. The *balanced product* is

$$X \times_H Y = X \times Y/((xh,y) \sim (x,hy)).$$

Remark. $X \times_H Y$ is not always a variety, but for our cases of interest it will be.

Fix a Borel $B \subseteq G$, so $\mathcal{B} = G/B$. Consider

$$\begin{aligned} G \times_B G/B &\cong G/B \times G/B \\ (g, g'B) &\mapsto (gg'B, g'B). \end{aligned}$$

The set of G-orbits on $G \times_B G/B$ is equal to the set of B-orbits on G/B. Therefore, the set of B-orbits are parametrized by the Weyl group W by the Bruhat decomposition, i.e.,

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{x \in W} \mathcal{O}_x.$$

Definition. The *Steinberg variety* is

$$\mathrm{St} = \{(\mathfrak{b}, \mathfrak{b}', x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

In other words, St is the fiber product

$$\begin{tikzcd} \mathrm{St} \arrow[r] \arrow[d] & \tilde{\mathcal{N}} \arrow[d] \\ \tilde{\mathcal{N}} \arrow[r] & \mathcal{N} \end{tikzcd}.$$

Now by Proposition 2.3, we have

$$\begin{aligned} \mathrm{St} &\subseteq T^*\mathcal{B} \times T^*\mathcal{B} \cong T^*(\mathcal{B} \times \mathcal{B}) \\ ((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)) &\mapsto (x_1, \mathfrak{b}_1, -x_2, \mathfrak{b}_2). \end{aligned}$$

Proposition 2.5. $\mathrm{St} = \bigcup_{(\mathfrak{b}_1,\mathfrak{b}_2)} T^*_{\mathcal{O}_x,(\mathfrak{b}_1,\mathfrak{b}_2)}(\mathcal{B} \times \mathcal{B}) = \bigcup_{(\mathfrak{b}_1,\mathfrak{b}_2)} (T_{(\mathfrak{b}_1,\mathfrak{b}_2)}\mathcal{O}_x)^\perp$.

Corollary 2.1. $\mathrm{St} = \bigsqcup_{w \in W} T^*_{\mathcal{O}_w}(\mathcal{B} \times \mathcal{B})$ *is a conormal space.*

3. Kazhdan-Lusztig I - Alex Bauman

3.1. **Convolution algebras.** Let X_1, X_2 be varieties over k. A *correspondence* from X_1 to X_2 is a closed immersion

$$\begin{tikzcd} Z_{12} \arrow[r, hook] & X_1 \times_k X_2 \arrow[r, "p_1^{12}"] \arrow[d, "p_2^{12}"] & X_1 \\ & X_1 & \end{tikzcd}.$$

Correspondences induce maps on cohomology:

$$\begin{aligned} H^*(X_1) &\to H^*(X_2) \\ c &\mapsto p^{12}_{2,*}((p^{12,*}_1 c) \cup [Z_{12}]). \end{aligned}$$

More generally, we get

$$\begin{aligned} H^*(X_1) \otimes H^*(Z_{12}) &\to H^*(X_2) \\ (c \otimes d) &\mapsto p^{12}_{2,*}(p^{12,*}_1 c \cup d). \end{aligned}$$

Suppose X_3 is another variety, $Z_{23} \subseteq X_2 \times X_3$, set

$$Z_{12} \circ Z_{23} = Z_{13} = \{(z_1, z_3) \in Z_1 \times_{Z_3} \mid \exists z_2 \in Z_2 \text{ s.t. } (z_1, z_2) \in Z_1 \times Z_2, (z_2, z_3) \in Z_2 \times Z_3\}.$$

The *convolution product in homology* is

$$\begin{aligned} H(Z_{12}) \times H(Z_{23}) &\to H(Z_{12}) \\ (c_{12}, c_{23}) &\mapsto p_{13,*}(p^*_{12} c_{12} \cap p^*_{23} c_{23}). \end{aligned}$$

We have factorization

$$\begin{tikzcd} H(X_1) \otimes H(Z_{12}) \otimes H(Z_{23}) \arrow[r] \arrow[d] & H(X_3) \\ H(X_1) \otimes H(Z_{12}) \arrow[ur] & \end{tikzcd}.$$

Example 3.1. (1) For $X = X_1 = X_2 = X_3$ finite set, $Z_{ij} = X \times X$, we can identify $H(X \times X)$ as functions $X \times X \to k$. The product is given by

$$(f * g)(x, y) = \sum_{z \in X} f(x, z) g(z, y).$$

One sees that $H(X \times X) \cong M_{*X}(k)$ and the product is matrix multiplication.

(2) Say $\tilde{X}$ is smooth with $\tilde{X} \to X$ is proper. For $X_1 = X_2 = X_3 = \tilde{X}$, $Z_{ij} = \tilde{X} \times_X \tilde{X} \subseteq \tilde{X} \times_k \tilde{X}$. Then $H(\tilde{X} \times \tilde{X})$ has a ring structure.

3.2. **Equivariant Grothendieck group.** Let G be an algebraic group action on some variety X, both defined over k. Then we have

$$\begin{tikzcd} G \times X \arrow[r, "p_2"] \arrow[d, "\sigma"] & X \\ X & \end{tikzcd}, \quad \begin{tikzcd} G \times G \times X \arrow[r, "p_{23}"] \arrow[d, "m"] & X \\ G & \end{tikzcd}.$$

An *equivariant sheaf* is a sheaf $\mathcal{F}$ on X together with an isomorphism

$$\varphi : \sigma^* \mathcal{F} \to p_2^* \mathcal{F}$$

such that $(p^*_{23} \varphi) \circ (\mathrm{id} \times \sigma)^* \varphi = (m \times \mathrm{id})^* \varphi$.

Example 3.2. If G is affine, G-action on X is trivial, then equivariant sheaf is just a sheaf of $\mathcal{O}[G] = \mathcal{O}_X \otimes \mathcal{O}_G$-modules.

Definition. We define the *K-group of G-equivariant coherent sheaves* on X as

$$K^G(X) := \frac{\mathbb{Z}\{[\mathcal{F}] \mid \mathcal{F} = \text{ coherent equivariant sheaf on } Z\}}{\{[\mathcal{G}] - [\mathcal{F}] - [\mathcal{H}] \mid \exists \text{ SES } \mathcal{F} \to \mathcal{G} \to \mathcal{H} \text{ of } G\text{-sheaves}\}}.$$

Remark. K^G behaves steadily to Borel-Moore homology, so K^G is equipped with a convolution algebra structure.

Remark. G-equivariant sheaves are the same as sheaves on the quotient stack $[X/G]$.

3.3. **Kazhdan-Lusztig isomorphism.** Let F be a local field and $\mathbf{G}/F$ split reductive group over F. Let $I \subseteq \mathbf{G}(F)$ be the Iwahori. Recall that the Iwahori-Matsumoto extended Hecke algebra $\mathcal{H}_{\text{ext}}$ is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with

$$\mathcal{H}_{\text{ext}} \otimes_{\mathbb{Z}[v^{\pm 1}], v \mapsto q^{\frac{1}{2}}} \mathbb{C} = \mathcal{H}(I \backslash \mathbf{G}(F)/I, \mathbb{C}).$$

Theorem 3.1 (Kazhdan-Lusztig)**.** *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}_{ext} \arrow[r, "\sim"] \arrow[d, "v=1"'] & K^{\hat{G}\times\mathbb{G}_m}(\mathrm{St}) \arrow[d, "forgetful"] \\
Z[W_{ext}] \arrow[r] & K^{\hat{G}}[\mathrm{St}]
\end{tikzcd} \cdot$$

where the action of $\hat{G} \times \mathbb{G}_m$ is giving by adjoint action and scaling in $\mathcal{N}$.

Remark. Consider the diagonal $\tilde{\mathcal{N}} \subseteq \mathrm{St} = \tilde{N} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Using $\tilde{N} \cong T^*\mathcal{B}$, we have $K^{\hat{G}\times\mathbb{G}_m}(\tilde{\mathcal{N}}) = K^{\hat{G}\times\mathbb{G}_m}(\mathcal{B}) = K^{\hat{B}\times\mathbb{G}_m}(*) = \mathrm{Rep}(\hat{B}) \otimes \mathrm{Rep}(\mathbb{G}_m) = \mathbb{Z}[X^\vee] \otimes \mathbb{Z}[v^{\pm 1}] \cong R_T$. So we have $\mathbb{Z}[v^{\pm 1}][X^\vee]$ sitting inside $\mathcal{H}_{\text{ext}}$. Moreover, $K^{\hat{G}\times\mathbb{G}_m}(*) = (K^{\hat{B}\times\mathbb{G}_m}(*))^{W_{\text{fin}}} = (\mathbb{Z}[v^{\pm 1}][X^\vee])^{W_{\text{fin}}}$ gives the Bernstein center.

4. Kazhdan-Lusztig II - Calvin Yost-Wolff

4.1. **Recap.**

- Alex introduced the convolution product. In particular, form the commutative diagram

$$\begin{tikzcd}
& \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3 \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
\tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2 & \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3 & \tilde{\mathcal{N}}_2 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3
\end{tikzcd}$$

  we get the convolution product given by

$$\mathcal{F}_{12} * \mathcal{F}_{23} = p_{13,*}(p_{12}^*\mathcal{F}_{12} \otimes p_{23}^*\mathcal{F}_{23}).$$

- Let $Z = \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2$, then there is an action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$ given by

$$\mathcal{F} \cdot \mathcal{G} = p_{1,*}(\mathcal{F} \otimes p_2^*\mathcal{G}).$$

There are also two questions remain.

- What equivariant sheaves correspond to which elements in $K^{G\times\mathbb{G}_m}(\mathrm{St})$?
- What is a the approach to Kazhdan-Lusztig isomorphism.

Example 4.1. We have isomorphisms

$$\begin{tikzcd}
K^G(\tilde{N}) \arrow[r] & K^G(G/B) \arrow[r] & K^B(*) = \mathbb{C}[X] \\
e^\lambda & G \times^B \lambda := L(\lambda) \arrow[l, mapsto] & \lambda \arrow[l, mapsto]
\end{tikzcd}.$$

Another way to is this is via localization. For nice $T \subseteq G \curvearrowright X$, $K^G(X) = [K^T(X)]^{W_{\text{fin}}}$. For $X = G/B$, $X^T = \{wB/B \mid w \in W_{\text{fin}}\}$. λ should correspond a equivariant vector bundle whose fiber at wB/B is λ^w.

These $L(\lambda)$'s are well-understood and important later. Let s be a simple transposition and α be the corresponding root. Let $\pi_s : G/B \to G/P_s$ be the natural projection. Then we have the following "Weyl character formula" type of formula

$$\pi_s^*\pi_{s,*}e^\lambda = \frac{e^{\lambda+\frac{\alpha}{2}} - e^{s(\lambda)-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}}.$$

4.2. **Strategy for understanding Kazhdan-Lusztig isomorphism.**

(1) On the Bernstein generators of $\mathcal{H}$, define a map to $K^{G\times\mathbb{G}_m}(\mathrm{St})$
   - finite part: $T_s \mapsto Q_s$ (to be defined later),
   - lattice part: $\theta_\lambda \mapsto \mathcal{O}_\lambda =$ pull-back of $\mathcal{O}(\lambda)$ along $T^*\mathcal{B} \xrightarrow{\text{diag}} \mathrm{St}$.
(2) Define the anti-spherical module M of $\mathcal{H}$ and study $\mathcal{H} \curvearrowright M$.
(3) Study the action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$.
(4) Match the actions above at the level of generators introduced in (1). Since both actions are faithful, this implies the Kazhdan-Lusztig isomorphism.

4.3. **Anti-spherical module.** To prove Kazhdan-Lusztig isomorphism, we are going to show that $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) = \mathbb{Z}[v,v^{-1}][X]$ realize an action of Iwahori Hecke algebra on $\mathbb{Z}[v,v^{-1}][X]$.

Recall that Elad defined

$$\mathcal{H} := \mathbb{Z}[v,v^{-1}][\{T_s\},\{\theta_\lambda\}]/\sim$$

where the relation $\sim$ consists of

- quadratic relation: $(T_s+v)(T_s-v^{-1})=0$,
- BZ relation: $T_s\theta_\lambda - \theta_{s(\lambda)}T_s = (v-v^{-1})\left(\frac{\theta_\lambda-\theta_{s(\lambda)}}{1-\theta_{-\alpha}}\right)$.

Recall the sign representation $\mathcal{H}_{\text{fin}} \curvearrowright \mathbb{Z}[v,v^{-1}]$ where all T_s acts as v^{-1}. The *anti-spherical module* is $M := \mathcal{H}\otimes_{\mathcal{H}_{\text{fin}}}\mathbb{Z}[v,v^{-1}]$ equipped with $\mathcal{H}$-action via the first factor. M has a $\mathbb{Z}[v,v^{-1}]$-basis given by $\theta_\lambda\otimes 1 =: [\theta_\lambda]$. Then these $[\theta_\lambda]$'s satisfy the relation

- $\theta_\mu\cdot[\theta_\lambda] = [\theta_{\lambda+\mu}]$, and
- (from BZ relation) $(T_s+v)[\theta_\lambda] = (v[\theta_{-\alpha}]-v^{-1})\left(\frac{[\theta_\lambda]-[\theta_{s(\lambda)}]}{1-[\theta_{-\alpha}]}\right)$.

Moreover, M is a faithful $\mathcal{H}$-module.

4.4. **Explicit computation of $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$.**

Example 4.2. Let first look at the case where $G=\mathrm{SL}_2$. Recall from Alex's talk that we have $Z = T^*\mathbb{P}^1 \cup (\mathbb{P}^1\times\mathbb{P}^1\backslash\Delta)$. The diagonal part is easy to handle:

$$e^\mu_\Delta\cdot e^\lambda = e^{\mu+\lambda}.$$

The complement is more difficult. Let's compute the action of $Q_s = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,0) = p_1^*\Omega_{\mathbb{P}^1/*}$ where p_1,p_2 are the projections:

$$\begin{tikzcd} \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}^1 \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\pi"] & \{*\} \end{tikzcd}.$$

We want to compute $p_{1,*}(Q_s\otimes p_2^*e^\lambda)$:

$$\begin{aligned} p_{1,*}(Q_s\otimes p_2^*e^\lambda) &= [\Omega_{\mathbb{P}^1}]\otimes p_{1,*}p_2^*(e^\lambda) && \text{(projection formula)} \\ &= [\Omega_{\mathbb{P}^1}]\cdot\left(\frac{e^{\lambda+\frac{\alpha}{2}}-e^{-\lambda-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\right) && \text{(base change formula).} \end{aligned}$$

Now use the following fact: let $\pi: E\to X$ be a vector bundle, $\mathcal{F}$ a quasi-coherent sheaf on X, $\mathcal{G}$ be a free sheaf on X and $i: X\to E$ be the zero section. Then we have

$$i_*\mathcal{F}\otimes\pi^*\mathcal{G} = i_*(\mathcal{F}\otimes\mathcal{G}).$$

Apply this fact to $\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}\to\mathbb{P}^1\times\mathbb{P}^1$. We get $[i_*\Omega_{\mathbb{P}^1}] = (e^{-\alpha}-v^{-2}\cdot e^0)$ via Koszul resolution. Therefore, we get

$$(T_s+v) = v\cdot Q_s.$$

Now let's generalize this method. For general G, the components of Z for simple transposition pairs

$$\begin{tikzcd} \bar{\mathcal{O}}_s \arrow[r] \arrow[d, "p_1"] & G/B \arrow[d, "\pi_s"] & \tilde{\mathcal{Z}}_s \arrow[r] \arrow[d] \arrow[l, "q"'] & \tilde{N}_s \arrow[d] \\ G/B \arrow[r, "\pi_s"] & G/P_s & \tilde{\mathcal{N}}_s \arrow[r] & T^*G/P_s \end{tikzcd}$$

where $\tilde{N}_s = \{(e, B) \mid e \in \text{unipotent radical of } \pi(B)\}$, $\bar{\mathcal{O}}_s \subseteq \mathcal{B} \times \mathcal{B}$ is the G-orbit consisting of pairs of flags in relative position s. As earlier in the case of SL_2, the same trick implies a similar result for $Q_s = q^*\Omega_{\bar{\mathcal{O}}_s/\mathcal{B}}$:

$$p_{1,*}(Q_s \otimes p_2^*\mathcal{O}(\lambda)) = \left(\frac{e^\lambda - e^{s(\lambda)-\alpha}}{1 - e^{-\alpha}}\right)[i_*\mathcal{O}_{\tilde{\mathcal{N}}_s}(-\alpha)]$$

where $i : \tilde{\mathcal{N}}_s \hookrightarrow \tilde{\mathcal{N}}$ is the inclusion. Again, one needs to express $[i_*\mathcal{O}_{\tilde{\mathcal{N}}_s}(-\alpha)] \in K^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}})$ in the basis of line bundles. This can be done using a Koszul resolution for the short exact sequence of B-modules

$$0 \to \mathfrak{p}_s/\mathfrak{b} \to \mathfrak{g}/\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}_s \to 0.$$

4.5. **Completing the proof.** There are two things remained to be showed:

(1) Q_s and e^λ_Δ generate $K^{G\times\mathbb{G}_m}(Z)$, and
(2) the action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$ is faithful.

We already have the lower bound of $\dim_{K^{G\times\mathbb{G}_m}(\tilde{N})} K^{G\times\mathbb{G}_m}(Z)$ is equal to $|W_{\text{fin}}|$. Let's compute the upper bound. There are two ways to get upper bound.

(1) Localization. We can embed $K^{G\times\mathbb{G}_m}(Z) \hookrightarrow (K^{T\times\mathbb{G}_m}(Z^{T\times\mathbb{G}_m}))^W$ with $Z^{T\times\mathbb{G}_m} = \{0, w_1B/B, w_2B/B\}$.
(2) Something like MV sequence?