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Introduction to The Kudla Program

0. General

It has a lot of mixture of arithmetic + geometry
Explain how it arose historically

1. BSD Conjecture

BSD Conjecture
Let E = elliptic curve / $\mathbb{Q}$.

$$\text{rank } E(\mathbb{Q}) = \text{ord}_{s=1} L(E,s) = r,$$

Then

$$\frac{L^{(r)}(E,1)}{\Omega_E} \doteq |Ш_E| \, Reg_E$$

regulator

period

Tate-Shafarevich group order

Means: $\doteq$ = other multiplicative correction terms, Tamagawa numbers etc.

Here take

$P_1, P_2, \ldots, P_r$ generators for $E(\mathbb{Q})/(\text{torsion})$

where

$$Reg_E = \det\left[\langle P_i, P_j\rangle\right]_{1\le i,j\le r = \boxed{\text{rank}}}$$

where

$\langle \, , \rangle$ Neron-Tate height pairing.

Example If rank $r=1$, then

$$L'(E,1) \doteq |Ш_E| \, |\langle P, P\rangle.$$

Here need the height pairing.

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$\boxed{\text{2. Height Pairings}}$

Aside on Height pairings

$\mathbb{P}^n$ = height functi

$\mathbb{P}^n(\mathbb{Q})$ $P = [a_0, a_1, \dots, a_n]$ with $a_i \in \mathbb{Z}$

$\gcd(a_0, a_1, \dots a_n) = 1$.

$H = \max(\log|a_i|)$ is (naive) height

$h = \log H$ is logarithmic height

For an algebraic number field K inside $\mathbb{P}^n(\overline{\mathbb{Q}})$

$P \in \overline{\mathbb{Q}}^{n+1}$.

$$H_K(P) = \prod_{v \in M_K} \max(\|a_0\|_v, \dots, \|a_n\|_v)$$

normalized height

$$h(P) := \frac{1}{[K:\mathbb{Q}]} \log H_K(P)$$

(It is independent of choice of field K, that P belongs to.

Now let X be smooth projective variety. Define height on

$\mathcal{L}$ = very ample line bundle.

Gives map

$$X \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^n$$

Now can define $h_{\mathcal{L}}$ is well defined as $O(1)$

(between two embeddings on the $\overline{\mathbb{Q}}$-points of variety

$|h_{\mathcal{L}} - h_{\mathcal{L}'}| = O(1)$

for all $P \in X(\overline{\mathbb{Q}})$

- $h_{\mathcal{L}\otimes\mathcal{M}} = h_{\mathcal{L}} + h_{\mathcal{M}}$.

- Extend defn to all ample $\mathcal{L}$.
  Then define for all $\mathcal{L}$ (line bundle)

$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$

$\mathcal{L}_1, \mathcal{L}_2$ ample.

<u>Fact</u>. $h_{\mathcal{L}} = \frac{1}{n} h_{\mathcal{L}^n}$.

<u>Defn</u> $H_{\mathcal{L}} = h_{\mathcal{L}_1} - h_{\mathcal{L}_2}$.

Now define <u>canonical height</u>
when $X = A$ is an <u>abelian variety</u>.

$$\boxed{\text{Fact. If } \mathcal{L} \text{ is a symmetric (ample) line bundle.} \quad [m]^* \mathcal{L} = \mathcal{L}^{m^2}}$$

Now <u>Neron-Tate height</u>

$$P \longmapsto \lim_{m\to\infty} \frac{1}{m^2} h_{\mathcal{L}}([m]P) = \hat{h}_{\mathcal{L}}(P)$$

exists.

<u>Remark.</u> For elliptic curve, take line bundle
associated to origin of E, to define Neron-Tate height

$$E(\mathbb{Q}) \otimes \mathbb{R} \xrightarrow{\langle , \rangle} \mathbb{R}$$

$$\langle P, Q\rangle := \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$$

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3. Gross–Zagier Thm

Theorem (Modularity theorem)

Suppose $E/\mathbb{Q}$ is elliptic curve of conductor N, squarefree,
Assuming modularity, there is a map

$$\begin{array}{ll} X_0(N) & \text{differential of a modular form.} \\ \big\downarrow \mu & \\ E & \text{canonical differential} \end{array}$$

(requires Faltings isogeny theorem) [Exists by Taylor–Wiles]

Now the moduli interpretation of $X_0(N)$:, N squarefree
space

Points on $X_0(N)$ correspond to cyclic isogeny

$$A \xrightarrow{\phi} A$$

where ϕ is cyclic isogeny of degree N.

CM case [CM = complex multiplication]

Let K be an imaginary quadratic field.

Condition K is $\boxed{\text{split}}$ at all primes ~~the~~ dividing N

Given $\mathcal{O}_K$, $\mathfrak{a}$ an ideal in $\mathcal{O}_K$.

$N = \eta\bar{\eta}$ for η ideal in $\mathcal{O}_K$ of norm N

$(\mathbb{C}/\mathfrak{a} \longmapsto \mathbb{C}/\eta^{-1}\mathfrak{a})$ gives a $\boxed{\text{point}}$ on $X_0(N)(H)$ Hilbert class field of K

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The different choices of $\mathfrak{N}$ relate to action of the Atkin-Lehner operator acting on $X_0(N)$
(switches the p-part of ~~$\mathfrak{N}$~~ ~~[illegible]~~ p-part of $\overline{\mathfrak{N}}$)

Here
$$\begin{tikzcd} X_0(N) \arrow[d, "\varphi"] & P_{\mathfrak{a}} \arrow[d] \\ E & E(H) \end{tikzcd}$$

Here defines a point
$$P_K = \sum_{\substack{[\mathfrak{a}] \text{ ideal} \\ \text{class in } \mathcal{O}_K}} \varphi(P_{\mathfrak{a}}) \in E_K$$

Theorem (Gross-Zagier 1984). Let E_K base change of $E_{\mathbb{Q}}$ to K.
Then
$$L'(K,1) \doteq \langle P_K, P_K \rangle \times \left(\begin{matrix}\text{multiplicative} \\ \text{correction} \\ \text{terms}\end{matrix}\right)$$

Remark.
$$\underbrace{L(E_K, s)}_{\left(\begin{matrix}\text{sign of functional equation} \\ \text{must be } -1 \\ \text{for application}\end{matrix}\right)} = L(E_{\mathbb{Q}}, s)\, L(E_d, s)$$

$\boxed{\text{has a zero at } s=1}$

quadratic twist of $E_{\mathbb{Q}}$ at $d = |d_K|$

Now assume $L(E,s)$ has sign -1.
$L(E_d,s)$ has sign $+1$.

Functional eqn: $s \longrightarrow 2-s$ (so $s=1$ is center of critical strip)

Can also study action of complex conjugation on K.

Assuming that $L(E,s)$ has sign -1, then

$$P_K \in E(\mathbb{Q}).$$

So we get:

$$\langle P_K, P_K \rangle \doteq L'(1,E)\, L(1, E_d) \times \begin{pmatrix}\text{nonzero} \\ \text{multiplicative} \\ \text{terms}\end{pmatrix}$$

4. Varying the imaginary Quadratic field K

Question. What happens if one varies the quadratic field hence varies $d = |d_K|$ in Gross–Zagier while holding $E = E_{\mathbb{Q}}$ fixed?

Theorem (Waldspurger (1980))

There exists a modular form h of weight $3/2$ whose d^{th} Fourier coefficient $a_d(h)$ satisfies

$$a_d(h)^2 \sim \text{proportional to } L(1, E_d)$$

~~(The square roots of $L(1, E_d)$ appear in $a_d(h)$,~~ and there is ambiguity of the square root sign.

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Theorem. (Gross-Kohnen-Zagier (1987))
Assume that N is prime (Avoids Atkin-Lehner sign ambiguity ±1 choice of η.)

Then P_K = point is well-defined (in $\mathbb{Q}$) independent of the choice of η, $N = \eta\bar{\eta}$ in K.

$$P_K \in E(\mathbb{Q}) \otimes \mathbb{R}.$$

[Theorem answers How are the P_K related as K varies.]

(i) the P_K live on a line as K varies, they are proportional to each other.

(ii) There is a modular form H of wt. $3/2$ in the Mordell-Weil group $MW(E) \otimes \mathbb{R}$ such that Fourier coefficients $a_d(H)$ is proportional to P_K.

$MW(E) \otimes \mathbb{R}$ means every coordinate is modular vector space.

Recall P_K well defined up ~~sign ±1~~ no sign ambiguity.
Signs must be correct to fall on a line in vector space $E(\mathbb{Q}) \otimes \mathbb{R}$

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5. Hirzebruch-Zagier

Work on Hilbert Modular surface. Let $F = \mathbb{Q}(\sqrt{p})$ real quadratic field

$$X = \mathfrak{h} \times \mathfrak{h} / SL_2(\mathcal{O}_F).$$

They constructed a sequence $T_1, T_2, T_3, \ldots \in \mathrm{Pic}(X) = CH^1(X)$

Such that the generating function

$$\sum_n [T_n] q^n \in M_2(\Gamma_0(p), (\tfrac{p}{\cdot})) \otimes H^2(X, \mathbb{C})$$

$[T_n]$ class $\in H^2(X, \mathbb{C})$
$\cup$
class $\in H^{1,1}(X, \mathbb{C})$ Hodge structure inside.

First kind of result of generating series (of algebraic cycles) giving a classical modular form. (weight 2), on $\Gamma_0(2)$

For later generalization, view this construction: $\dim_{\mathbb{Q}}(V) = 4$,

$$SL_2(F) \sim O(V)$$

$SL_2(F)$: real quadratic; $O(V)$: 4-dim. orthogonal group of sign $(2,2)$

Take vector v with $\langle v, v \rangle > 0$.
Write $V = \mathbb{Q}v \oplus W$, W orthog complement of sign $(1,2)$. $O(1,2)$

Setup has $O(W) \hookrightarrow O(V)$
$O(1,2) \hookrightarrow O(2,2)$

6. Kudla-Millson Generalization

They did n-dim'l generalization

$$O(p,q) \qquad [\text{or} \quad U(p,q).]$$

They formed symmetric matrix $\left[\langle\langle v_i, v_j \rangle\rangle \right]$ gram matrix of positive form

$v_1, v_2, \dots, v_n$.
inside $\perp$ to
$O(p-n, q)$

Theorem (Kudla-Millson)
There is a holomorphic Siegel modular form on $Sp(\mathbb{Z}_n)$ of weight $= \frac{p+q}{2}$ valued in $\underbrace{H^{nq}(X, \mathbb{C})}_{\text{cohomology gp Clatha}}$

where Fourier coefficients
(Indexed by symmetric matrices)
Siegel parabolic
are given by classes of "Special cycles"

Proof
(Uses the theta functions.)
correspondence

(Greens functions are constructed) (not holoidy / weak harmonic Maass form)
Greens functions are Borcherds lifts.

7. Kudla Program

This program ~~combines~~ both theories:

Gross-Kohnen-Zagier Kudla-Millson (~~Siegel~~ modular form)

$\searrow \quad \swarrow$

Kudla program

Conjecture 1.
Modularity holds at the level of cycles $\mathbb{Z}$
"~~Chow~~ groups"

Conjecture 2. (something on heights) [
(Arithmetic Modularity Conjecture): Modularity holds at level of arithmetic ~~Arakelov~~ theory $\widehat{\mathbb{Z}}$
("Arithmetic Chow group")

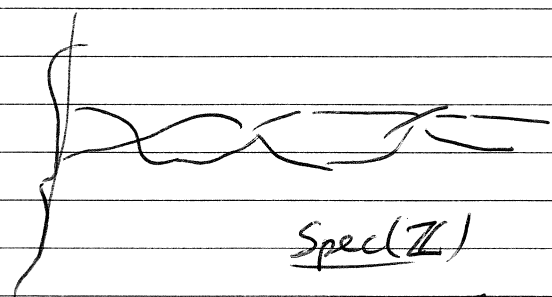
To explain Conjecture 2

Aside on Heights: Tate ~~defn~~ of height:

$$\hat{h} : E(\mathbb{Q}) \xrightarrow{ht} \mathbb{R}.$$

Neron defn. of height: Global ~~Height~~ is a sum of local heights

- At finite places, the Neron pairing corresponds to intersection pairing on a regular model.



- At infinite places Greens functions.

Spec($\mathbb{Z}$)

spread out over Spec

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At infinite place, need some measure how close the points are, you need a Greens function for intersection pairing.

Takes cycles, extend to (integral model. Then at) infinite place, with Greens function

Subject of Arakelov theory: $\begin{cases} \text{Greens function} \\ G \end{cases}$

Have a $\hat{Z}$ (Arakelov cycle)
(special cycles)

The conjecture 2 is the intended culmination of Kudla Program. The idea of the Kudla program is to get a different proof of Gross-Zagier formula (19 by working back (from Kudla conjectures) to recover Gross-Kohnen-Zagier formula.

Using approach method coming from theta correspondence.