INTRODUCTION TO HECKE ALGEBRAS

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Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, uniformizer $\varpi \in \mathfrak{p} \setminus \mathfrak{o}$ and residue field $\mathbb{F}_q$. Let G be a split connected algebraic reductive group over F, with split maximal torus A and Borel subgroup $B = AN$. Let $\mathcal{K} = G(\mathfrak{o})$ be a hyperspecial maximal open compact subgroup of G. Let W be the Weyl group of G, that is, the normalizer of A in G modulo A.

Let $X_*(A)$ denote the cocharacter group

$$X_*(A) = \operatorname{Hom}(\mathbb{G}_m, A) = \left\{\mu^\vee \colon F^\times \to A \mid \mu^\vee \text{ algebraic character}\right\}.$$

For an element $\mu^\vee \in X_*(A)$ we denote $\varpi^{\mu^\vee} \coloneqq \mu^\vee(\varpi)$. The map $\mu \mapsto \varpi^{\mu^\vee}$ defines an isomorphism

$$X_*(A) \to A(\varpi) \coloneqq A/A(\mathfrak{o}),$$

where $A(\mathfrak{o}) = A \cap \mathcal{K}$.

Example 1. When $G = \mathrm{GL}_n$,

$$A = \left\{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mid a_1, \dots, a_n \in F^\times\right\}$$

and

$$X_*(A) \cong \mathbb{Z}^n$$

by the isomorphism sending

$$\mathbb{Z} \ni (k_1, \dots, k_n) \mapsto \mu^\vee_{(k_1,\dots,k_n)}(x) = \begin{pmatrix} x^{k_1} & & \\ & \ddots & \\ & & x^{k_n} \end{pmatrix}.$$

In this case,

$$A(\varpi) = A/A(\mathfrak{o}) \cong \left\{\begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix} \mid k_1, \dots, k_n \in \mathbb{Z}\right\},$$

where

$$A(\mathfrak{o}) = \left\{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mid a_1, \dots, a_n \in \mathfrak{o}^\times\right\}.$$

It is clear that the map $X_*(A) \to A(\varpi)$ given by $(k_1, \dots, k_n) \mapsto \mu^\vee_{(k_1,\dots,k_n)}(\varpi)$ is an isomorphism.

For any compact open subgroup $K \subset G$ let

$$\mathcal{H}(G \mathbin{/\!\!/} K) = \{f \in C_c^\infty(G) \mid f(kgk') = f(g), \forall g \in G, k, k' \in K\}.$$

A basis for $\mathcal{H}(G \mathbin{/\!\!/} K)$ is given by the characteristic functions $\{\chi_{KgK}\}_g$ where g runs over all the representatives of cosets of $K \backslash G / K$.

The space $\mathcal{H}(G \mathbin{/\!\!/} K)$ is an algebra with convolution as multiplication:

$$(f_1 * f_2)(g) = \int_G f_1\left(x^{-1}\right) f_2(xg)\, dx.$$

1. Spherical Hecke algebra

Recall the Cartan decomposition: we have that

$$G = \bigcup_{\substack{\mu^\vee \\ \mu^\vee \text{ is dominant}}}^{\cdot} \mathcal{K} \cdot \varpi^{\mu^\vee} \cdot \mathcal{K}.$$

It follows from applying Gelfand's trick to an involution based on the Chevalley data, that $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{K})$ is commutative (the involution should act trivially on $\varpi^{\mu^\vee}$ for any $\mu^\vee$ and send $\mathcal{K}$ to itself).

Example 2. If $G = \mathrm{GL}_n(F)$, then $\mu^\vee_{(k_1,\ldots,k_n)}$ is dominant if and only if $k_1 \geq k_2 \geq \cdots \geq k_n$. The involution in this case is $x \mapsto {}^t x$. In this case, under the isomorphism above, every dominant cocharacter can be written as a product of the form

$$a_1^{i_1} \cdot a_2^{i_2} \cdot \cdots \cdot a_n^{i_n},$$

where $i_1, \ldots, i_n \in \mathbb{Z}$ and $i_1, \ldots, i_{n-1} \geq 0$, and for $1 \leq j \leq n$,

$$a_j = \begin{pmatrix} \varpi I_j & \\ & I_{n-j} \end{pmatrix}.$$

The spherical Hecke algebra $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{K})$ can be realized as compactly supported $\mathbb{C}$-valued functions on $A(\varpi)$, that are invariant under the action of the Weyl group W.

If $K \subset \mathcal{K}$ is an arbitrary compact open subgroup, then $\mathcal{H}(G \mathbin{/\!\!/} K)$ is not commutative when $K \neq \mathcal{K}$. However, we have the following relation. If $\mu_1^\vee$ and $\mu_2^\vee$ are dominant cocharacters then

$$\chi_{K \cdot \varpi^{\mu_1^\vee} \cdot K} * \chi_{K \cdot \varpi^{\mu_2^\vee} \cdot K} = \chi_{K \cdot \varpi^{\mu_1^\vee} \varpi^{\mu_2^\vee} \cdot K} = \chi_{K \cdot \varpi^{\mu_1^\vee \mu_2^\vee} \cdot K}.$$

Let R_K^+ be the algebra generated by the elements $\chi_{K \cdot \varpi^{\mu^\vee} \cdot K}$, where $\mu^\vee$ goes over all the dominant cocharacters. Then R_K^+ is a commutative subalgebra of $\mathcal{H}(G \mathbin{/\!\!/} K)$. Using the Cartan decomposition, we may decompose

$$\mathcal{H}(G \mathbin{/\!\!/} K) = \mathcal{H}(\mathcal{K} \mathbin{/\!\!/} K) * R_K^+ * \mathcal{H}(\mathcal{K} \mathbin{/\!\!/} K).$$

Here, $\mathcal{H}(\mathcal{K} \mathbin{/\!\!/} K)$ is a finite-dimensional algebra, consisting of functions $\mathcal{K} \to \mathbb{C}$ bi-invariant under K, and R_K^+ is abelian. This shows that $\mathcal{H}(\mathcal{K} \mathbin{/\!\!/} K)$ breaks into two pieces: a small (finite-dimensional) non-commutative one, and a large (infinite-dimensional) abelian one.

Remark 3. An important feature that we will not discuss is the Jacquet functor. Suppose that $P = MU \subset G$ is a proper parabolic subgroup with Levi part M, and U^- is the radical opposite to U. Suppose that we have a Iwahori-type decomposition[1, Lemma 3.11]

$$K = \left(K \cap U^-\right)(K \cap M)(K \cap U).$$

(for instance, this holds for $K = \mathcal{K}$ and $K = \mathcal{I}$ from the next section). Let π be an admissible representation of G. Let

$$J_U(\pi) = \pi / \mathrm{span}_{\mathbb{C}} \{\pi(u) v - v \mid v \in \pi, u \in U\}.$$

Then the quotient map

$$\pi \to J_U(\pi)$$

defines a surjection for the subspaces of K-fixed vectors:

$$\pi^K \to J_U(\pi)^{K \cap M}.$$

This can be used to show that certain representations can be embedded as subrepresentations of principal series representations. See for example [4, Section 12].

2. Iwahori–Matsumoto hecke algebra

Consider the quotient map

$$\nu \colon \mathcal{K} \to G(\mathbb{F}_q).$$

The inverse image of $B(\mathbb{F}_q)$ under this map is called the *Iwahori subgroup* of $G(F)$. We denote it by $\mathcal{I}$. Assume henceforth that the Haar measure is normalized so that $\mathcal{I}$ has measure 1.

Example 4. If $G = \mathrm{GL}_n$ then

$$\mathcal{I} = \nu^{-1}(B(\mathbb{F}_q)) = \left\{ \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times & \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \cdots & \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix} \right\}.$$

Let $\tilde{W} = W \rtimes A(\varpi)$ be the extended affine Weyl group. We have the following Bruhat–Iwahori decomposition:

$$G = \bigsqcup_{x \in \tilde{W}} \mathcal{I} \cdot x \cdot \mathcal{I}.$$

It follows that $\{\chi_{\mathcal{I} \cdot x \cdot \mathcal{I}}\}_{x \in \tilde{W}}$ forms a basis for $\mathcal{H}(G /\!\!/ \mathcal{I})$.

The group $\tilde{W}$ is almost a Coxeter group. It has a well-known standard presentation. In general,

$$\tilde{W} = W_{\mathrm{aff}} \rtimes \pi_1(G),$$

where $\pi_1(G)$ is the fundamental group of G and W_{aff} is the affine Weyl group, generated by all affine reflections. W_{aff} has a standard presentation as a Coxeter group.

Example 5. When $G = \mathrm{GL}_n$, we have that W_{aff} is generated by $s_1, \dots, s_n$ satisfying the relations

(1) $s_j^2 = 1$.
(2) $s_j s_k = s_k s_j$ if $j - k \not\equiv 0, \pm 1 \pmod{n}$.
(3) $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$, where $j + 1$ is taken modulo n.

We have that $\pi_1(\mathrm{GL}_n) \cong \mathbb{Z}$ and that if $h \in \pi_1(\mathrm{GL}_n)$ is a generator, then for any j
$$hs_jh^{-1} = s_{j+1},$$
where $j+1$ is taken modulo n.

We may choose the following matrices to represent these elements: for $1 \leq j \leq n-1$, choose for s_j the permutation matrix that swaps the columns at positions j and $j+1$. Choose
$$s_n = \begin{pmatrix} & & \varpi^{-1} \\ & I_{n-2} & \\ \varpi & & \end{pmatrix}$$
and
$$h = \begin{pmatrix} & & & \varpi^{-1} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

There is a notion of the length of an element in $\tilde{W}$: since W_{aff} is a Coxeter group, it comes with a length function $\ell\colon W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ and we denote by $\ell\colon \tilde{W} \to W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ the length function of $\tilde{W}$.

Example 6. If $G = \mathrm{GL}_n$ then any element in $\tilde{W}$ can be written in the form
$$w = h^m \cdot s_{i_1} \cdot s_{i_2} \cdot \dots \cdot s_{i_r},$$
where $m \in \mathbb{Z}$. We denote by $\ell(w)$ the minimal r such that w can be written in such form. Such presentation of w is called a *reduced expression*.

The elements of $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$ corresponding to the generators of $\tilde{W}$ generate $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$ as an algebra. They satisfy the corresponding Hecke algebra relations.

Example 7. If $G = \mathrm{GL}_n$, let us define for any $w \in \tilde{W}$
$$f_w = \chi_{\mathcal{I}\cdot w \cdot \mathcal{I}}.$$
Then we have the following relations:

1. $f_{s_j} * f_{s_j} = (q-1) f_{s_j} + q$. This can also be written as
$$\left(f_{s_j} - q\right) * \left(f_{s_j} + 1\right) = 0.$$
2. $f_{s_j} * f_{s_k} = f_{s_k} * f_{s_j}$ if $j - k \not\equiv 0, \pm 1 \,(\mathrm{mod} n)$.
3. $f_{s_j} * f_{s_{j+1}} * f_{s_j} = f_{s_{j+1}} * f_{s_j} * f_{s_{j+1}}$, where $j+1$ is taken modulo n.
4. $f_{h^{-1}} = f_h^{-1}$.
5. $f_h * f_{s_j} * f_h^{-1} = f_{s_{j+1}}$, where $j+1$ is taken modulo n.

Remark 8. We may attach to W_{aff} an Iwahori–Matsumoto Hecke algebra $\mathcal{H}_v(W_{\mathrm{aff}})$. It is an algebra over $\mathbb{Z}[v, v^{-1}]$. The Iwahori–Matsumoto Hecke algebra is generated by the generators T_s for any generator s of W_{aff} and is subject to their relations, where we modify the quadratic relations $s^2 = 1$ to be
$$(T_s - v)\left(T_s + v^{-1}\right) = 0,$$
for every quadratic generator s of W_{aff}. Then Iwahori and Matsumoto proved that if G is semisimple then $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$ is isomorphic to $\mathcal{H}_{q^{\frac{1}{2}}}(W_{\mathrm{aff}})$. More generally, we have that
$$\mathcal{H}(\mathcal{K} \mathbin{/\!\!/} \mathcal{I}) \cong \mathcal{H}_{q^{\frac{1}{2}}}(W_{\mathrm{aff}})$$

and that

$$\mathcal{H}(G /\!/ \mathcal{I}) \cong \mathcal{H}(\mathcal{K} /\!/ \mathcal{I}) \otimes_{\mathbb{Z}\left[q^{\frac{1}{2}},q^{-\frac{1}{2}}\right]} R_{\mathcal{I}},$$

by the isomorphism sending

$$\chi_{\mathcal{I}\cdot w\cdot\mathcal{I}} \otimes \chi_{\mathcal{I}\cdot\varpi^{\mu^\vee}\cdot\mathcal{I}} \mapsto \chi_{\mathcal{I}\cdot w\cdot\mathcal{I}} * \chi_{\mathcal{I}\cdot\varpi^{\mu^\vee}\cdot\mathcal{I}}.$$

2.1. **Bernstein–Zelevinsky relation.** Denote for $w \in \tilde{W}$,

$$f_w = \chi_{\mathcal{I}\cdot w\cdot\mathcal{I}}.$$

Recall that $R_{\mathcal{I}}$ is the subalgebra of $\mathcal{H}(G /\!/ \mathcal{I})$ generated by $f_{\varpi^{\mu^\vee}}$ for every dominant cocharacter $\mu^\vee$. If $\mu_1^\vee$ and $\mu_2^\vee$ are dominant cocharacters then we have that

$$\ell\left(\varpi^{\mu_1^\vee\mu_2^\vee}\right) = \ell\left(\varpi^{\mu_1^\vee}\varpi^{\mu_2^\vee}\right) = \ell\left(\varpi^{\mu_1^\vee}\right) + \ell\left(\varpi^{\mu_2^\vee}\right),$$

and

$$f_{\varpi^{\mu_1^\vee}} * f_{\varpi^{\mu_2^\vee}} = f_{\varpi^{\mu_1^\vee}\varpi^{\mu_2^\vee}} = f_{\varpi^{\mu_1^\vee\mu_2^\vee}}. \tag{2.1}$$

In particular $R_{\mathcal{I}}^+$ is commutative. It can be shown from the relations of the generators of the Hecke algebra that f_w is invertible for any $w \in \tilde{W}$. Let $R_{\mathcal{I}}$ be the algebra generated by $f_{\varpi^{\mu^\vee}}$ and $f_{\varpi^{\mu^\vee}}^{-1}$ for every dominant cocharacter $\mu^\vee$. It seems tempting to define a map

$$A(\varpi) \to R_{\mathcal{I}}^\times$$

by the formula

$$\varpi^{\mu^\vee} \mapsto f_{\varpi^{\mu^\vee}}.$$

However, this map will not be a group homomorphism $A(\varpi) \to R_{\mathcal{I}}^\times$. This can be fixed as follows. First let us define a normalization of f_w:

$$T_w = q^{-\frac{\ell(w)}{2}} f_w.$$

If $\lambda^\vee$ is a cocharacter, we can write $\lambda^\vee = \mu_1^\vee \cdot (\mu_2^\vee)^{-1}$ where $\mu_1^\vee$ and $\mu_2^\vee$ are dominant cocharacters. We define a map

$$\theta\colon A(\varpi) \to R_{\mathcal{I}}^\times,$$
$$\theta(\lambda^\vee) = T_{\mu_1^\vee} * \left(T_{\mu_2^\vee}\right)^{-1}.$$

This is well defined because of (2.1). This map is now a homomorphism.

Example 9. If $G = \mathrm{GL}_n$ then if we denote

$$b_j = \begin{pmatrix} I_{j-1} & & \\ & \varpi & \\ & & I_{n-j} \end{pmatrix},$$

then

$$b_j = a_j \cdot a_{j-1}^{-1},$$

and

$$\theta(b_j) = T_{a_j} * \left(T_{a_{j-1}}\right)^{-1}.$$

One now needs to express a_{j-1} as a product of h^{-j} and of the generators $s_1, \ldots, s_{n-1}$ in order to be able to write an expression for $\left(T_{a_{j-1}}\right)^{-1}$ as a product of the inverses of the corresponding generators. In turn, these inverses are computed using the quadratic relations of the Hecke algebra.

Theorem 10 (Bernstein–Zelevinsky presentation)**.** *Let s be any one of the generators of W_{aff} (as a Coxeter group). For any cocharacter $\lambda^\vee$ we have that $\theta(\lambda^\vee) - \theta(s(\lambda^\vee))$ is divisible by $1-\theta(\alpha^\vee)^{-1}$ in the ring $R_{\mathcal{I}}$ and the following equality holds:*

$$\begin{aligned}\theta(\lambda^\vee)T_s - T_s\theta(s(\lambda^\vee)) &= T_s\theta(\lambda^\vee) - \theta(s(\lambda^\vee))T_s \\ &= \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\frac{\theta(\lambda^\vee)-\theta(s(\lambda^\vee))}{1-\theta(\alpha^\vee)^{-1}},\end{aligned}$$

where $\alpha^\vee$ is a certain fundamental cocharacter on which s acts by $s(\alpha^\vee) = (\alpha^\vee)^{-1}$.

Example 11. When $G = \mathrm{GL}_n$, recall that $\lambda^\vee$ is parameterized by $\lambda^\vee = \mu^\vee_{(k_1,\ldots,k_n)}$ where $k_1,\ldots,k_n \in \mathbb{Z}$. We have that $W = \langle s_i \mid 1 \le i \le n\rangle \cong S_n$ acts on $\lambda^\vee$ by permuting the coordinates $(k_1,\ldots,k_n) \in \mathbb{Z}$. In this case, $s = s_k$ for some k and

$$\alpha^\vee = \alpha_k^\vee = (0,0,\ldots,0,-1,1,0,\ldots,0)$$

is the cocharacter corresponding to the matrix

$$\begin{pmatrix} I_{k-1} & & & \\ & \varpi^{-1} & & \\ & & \varpi & \\ & & & I_{n-k-1}\end{pmatrix},$$

so $\theta(\alpha^\vee) = T_{a_{k-1}} * T_{a_{k+1}} * T_{a_k}^{-2}$.

2.2. Center of $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$.

From the Bernstein–Zelevinsky relation, the following description of the center of $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$ can be concluded.

Theorem 12. *The center of $\mathcal{H}(G \mathbin{/\!\!/} \mathcal{I})$ is the subspace of $R_{\mathcal{I}}^+$ consisting of elements of the form*

$$\sum_{\mu^\vee} a_{\mu^\vee}\theta(\mu^\vee)$$

such that $a_{w\mu^\vee} = a_{\mu^\vee}$ for every $w \in W$ and cocharacter $\mu^\vee$. In particular, this center is isomorphic to $\mathbb{C}[A(\varpi)]^W$.

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