MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Energy and Power Flow in State Determined Systems[1]

1 Introduction

1.1 Energy Conservation in Physical Systems

A set of primitive elements which form the basis for construction of dynamic models of a physical system may be defined from the energy flows within the system, and between the system and its environment. In this note we define such primitive elements, which characterize the generation, storage, and dissipation of energy in four energy domains: mechanical, electrical, fluid and thermal.

The principle of energy conservation provides a fundamental basis for characterizing and defining the primitive elements. An idealization is adopted in which it is assumed that a system model exchanges energy with its environment through a finite set of energy or power *ports* [1] as shown in Fig. 1. For systems defined by such a boundary, the law of energy conservation may be written as:

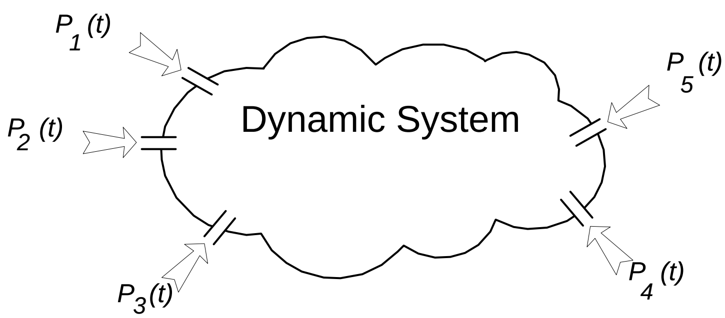

Figure 1: Power flows into a dynamic system.

$$\mathcal{P}(t) = \frac{d\mathcal{E}}{dt} \tag{1}$$

where t is time, $\mathcal{E}(t)$ is the instantaneous stored energy within the system boundary, and $\mathcal{P}(t)$ is the instantaneous net power flow across the system boundary (where power flow is defined as positive into the system and negative out of the system). Equation (1) states that the rate of change of stored energy in the system is equal to the net power flow, $\mathcal{P}(t)$, across the system boundary. It is assumed that the system itself contains no sources of energy. All sources are external to the system and influence the dynamic behavior through the power flows across the system boundary.

For the systems considered in this chapter, the power flow across the system boundary over an incremental time period dt may be written as:

$$\mathcal{P}dt = \Delta W + \Delta H, \tag{2}$$

where ΔW is the increment of work performed on the system by the external sources over the period dt, and ΔH is the increment of heat energy transferred to the system over the same period.

[1] D. Rowell - Revised: 1/30/03

Positive work and heat flow are defined as increasing the system total energy. The work done on the system and heat flow crossing the system boundary results in a change in the total energy level of the system as expressed by Eq. (2.1), which when integrated with respect to time and combined with Eq. (2.2) is a form of the first law of thermodynamics [2].

1.2 Spatial Lumping in Physical Systems

In the formulation of system models it is convenient to consider power flows across the system boundary to be localized at a set of discrete locations on the system boundary as shown in Fig. 2.1. If the power flows are localized at n sites, the net power flow is the sum of the power flows at each location:

$$\begin{aligned} \mathcal{P}(t) &= \mathcal{P}_1(t) + \mathcal{P}_2(t) + \ldots + \mathcal{P}_n(t) \\ &= \sum_{i=1}^{n} \mathcal{P}_i(t) \end{aligned} \tag{3}$$

where $\mathcal{P}_i(t)$ is the power flow at location i. The power flows include power delivered (or extracted) by energy sources in the environment, as well as power flows from the system due to energy dissipation in system elements. The power flows in Eq. (3) do not include the flow of power between system elements contained within the boundary.

In a similar manner the total system energy at any instant may be expressed as the sum of energies stored at m discrete locations within the system:

$$\begin{aligned} \mathcal{E}(t) &= \mathcal{E}_1(t) + \mathcal{E}_2(t) + \ldots + \mathcal{E}_m(t) \\ &= \sum_{i=1}^{m} \mathcal{E}_i(t) \end{aligned} \tag{4}$$

The m energies are associated with a set of m *energy storage elements* in the system model. The energy conservation law may then be expressed in terms of local power flows sites and energy storage elements as:

$$\sum_{i=1}^{n} \mathcal{P}_i(t) = \sum_{i=1}^{m} \frac{d\mathcal{E}_i}{dt} \tag{5}$$

which states that the total power flow across the boundary is distributed among the m energy storage elements.

Equation (5) may be applied directly to systems consisting of *lumped-parameter elements* where elements, which are considered to be "lumped" in space, have locally uniform system variables and parameters. In a lumped-parameter model, the variables at a given spatial location are used to represent the variables of regions in the near vicinity of the point.

In lumped-parameter dynamic models the variables at discrete points in the system are functions of time, and are described by ordinary differential equations. The variables are not considered to be continuous functions of both time and position, as is characteristic of spatially *continuous* or *distributed* models described by partial differential equations [3].

In the following sections, variables are defined for power and energy flow at discrete locations in mechanical, electrical, fluid and thermal systems. These variables provide the basis for defining a set of lumped parameter elements which represent energy sources, energy storage, energy dissipation and energy transformation. Each element is described in terms of a constitutive equation which expresses the functional relationship for the element in terms of material properties and geometry.

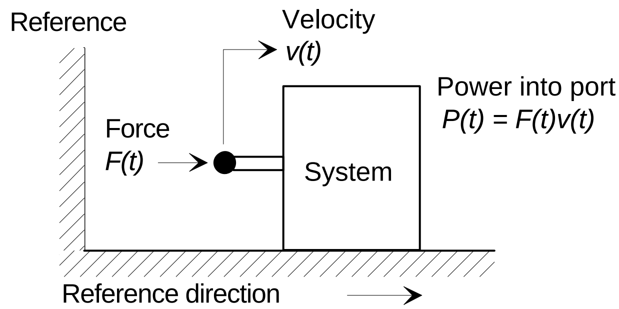

Figure 2: Mechanical system with a single power port.

2 Mechanical Translational System Elements

Mechanical translational systems are characterized by straight-line or linear motion of physical elements. The dynamics of these systems are governed by the laws of mechanical energy conservation, and are described by Newton's laws of motion. The power flow to and from translational systems is through mechanical work supplied by external sources, and energy is dissipated within the system and transferred to the environment through conversion to heat by mechanical friction.

There are two mechanisms for energy storage within a mechanical system:

(1) as *kinetic* energy associated with moving elements of finite mass,

(2) and as *potential* energy stored through elastic deformation of spring-like elements.

Two energy conserving elements, based on these storage mechanisms, together with a third dissipative element representing frictional losses, are used as the basis for lumped-parameter modeling of translational systems. The functional definition of these elements may be developed by considering the mechanical translational power flow into a system.

2.1 Definition of Power Flow Variables

Figure 2 shows an energy port into a mechanical system through which power is transferred by translational motion along a prescribed direction. For any such mechanical system the power flow $\mathcal{P}(t)$ (watts) at any instant is the product of the velocity $v(t)$ (m/sec) and the collinear force $F(t)$ (Newtons):

$$\mathcal{P}(t) = F(t)v(t). \tag{6}$$

The increment in energy expended or absorbed in the form of mechanical work ΔW flowing through a power port in an elemental time period dt is defined as:

$$\Delta W = \mathcal{P}(t)dt = F(t)v(t)dt \tag{7}$$

By convention the external source is said to perform work on the system when the power is positive, that is when $F(t)$ and $v(t)$ have the same sign (or act in the same direction), as shown in Fig. 2. When $\mathcal{P}(t) > 0$ the external source supplies energy to the system, and this energy may be stored within the system and later recovered, or it may be dissipated as heat and thus rendered unavailable to the system. If $F(t)$ and $v(t)$ act in opposite directions the power flow is negative, and the system performs work on its environment. The two variables $F(t)$ and $v(t)$ are the primary variables used to describe the dynamics of mechanical translational systems throughout this book.

Equation (7) may be integrated to determine the total mechanical work W transferred through the port in a time period $0 \leq t \leq T$:

$$W = \int_0^T \mathcal{P}(t)dt = \int_0^T Fvdt. \tag{8}$$

It is useful to introduce two additional variables; the linear displacement $x(t)$ (m), which is the integral of the velocity:

$$x(t) = \int_0^t v(t)dt + x(0) \qquad \text{or} \qquad dx = vdt, \tag{9}$$

and the linear momentum $p(t)$ (N-sec), which is defined as the integral of the force:

$$p(t) = \int_0^t F(t)dt + p(0) \qquad \text{or} \qquad dp = Fdt. \tag{10}$$

The work performed across a system boundary in time dt, may be expressed in terms of the power variables and the integrated power variables in the following three forms:

$$\begin{array}{ccccccc} \Delta W & = & F(vdt) & = & Fdx & = & d\mathcal{E}_{Potential} \\ \Delta W & = & v(Fdt) & = & vdp & = & d\mathcal{E}_{Kinetic} \\ \Delta W & = & (Fv)dt & = & (vF)dt & = & d\mathcal{E}_{Dissipated} \end{array} \tag{11}$$

These three forms of power flow illustrate the origins of the definitions of the three lumped-parameter elements that are used in mechanical system models — the spring, mass and damper elements. The first expression states that the mechanical work may result in a change in the system stored energy through a change in displacement dx, which is usually associated with potential energy storage in a spring-like element. The second expression states that the work may result in a change in momentum dp, which is usually associated with energy storage in a mass-like element. The third expression indicates the work is converted into heat, as occurs with friction in a mechanical system, and is no longer available in mechanical form. The energy is then considered to have been dissipated and transferred to the environment.

Lumped parameter models of physical systems may be defined in terms of these three elements. Often considerable engineering judgement is required in deciding how to represent physical components in terms of the primitive elements. These decisions require knowledge of the function of the component within the system. For example a large coil spring which does not undergo any deflection but has a uniform velocity might be represented as a simple mass because the only energy stored is kinetic. A coil spring in an automobile suspension might be represented as a pure spring for slowly varying applied forces, since the energy stored within it is based on the spring deflection. For very rapidly varying deflections of the coil, however, a significant amount of kinetic energy might be associated with the rapidly moving mass of the coil spring. In high performance mechanical systems a model of a physical spring may need to include both primitive spring and mass effects.

2.2 Primitive Translational Element Definitions

2.2.1 Energy Storage Elements

Energy storage within a mechanical translational system may be described in terms of two lumped-parameter primitive elements, defined in terms of the power and integrated power variables:

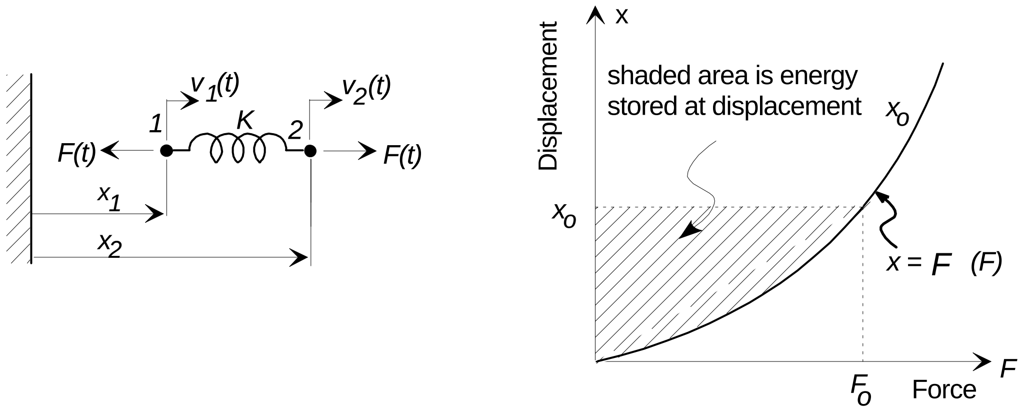

Figure 3: Definition of the pure translational spring element.

Translational Spring: A pure mechanical translational *spring* element is defined as an element in which the displacement x across the element is a single-valued, monotonic function of the force F:

$$x = \mathcal{F}(F) \tag{12}$$

where $\mathcal{F}()$ is a single-valued, monotonic function, as shown in Fig. 3. Equation (12) is known as the *constitutive equation* for a spring. The displacement x is the net spring deflection, as shown in Fig. 3, expressed in terms of the difference of two measurements in the fixed coordinate system, less the spring rest (natural) length ℓ_0, that is $x = x_1 - x_2 - \ell_0$.

From Eq. (11) the energy stored in a spring with displacement x is

$$\mathcal{E} = \int_0^x F dx, \tag{13}$$

and is illustrated as the shaded area in Fig. 3. The energy stored in the spring is a direct function of x and is zero when $x = 0$.

A *pure* spring is any element described by Eq. (12) and may be represented by a nonlinear relationship between x and F. The restriction that the relationship is single-valued and monotonic ensures a unique relationship exists between x and F, so that given a displacement x, the force may be uniquely determined, or given F the displacement may be uniquely determined. An *ideal* or *linear* spring is defined as a pure spring in which the relationship between displacement and force is linear so that Eq. (12) becomes

$$x = CF \tag{14}$$

where the constant of proportionality C is defined to be the spring *compliance* (m/N). In engineering practice linear springs are usually described by the reciprocal of the compliance, and the linear relationship of Eq. (14) is written

$$F = Kx \tag{15}$$

where $K = 1/C$ (N/m) is defined to be the *spring constant*, or *stiffness*. Equation (15) is Hooke's Law for a linear spring [4].

For an ideal spring with a displacement x from the rest position and an applied force $F = Kx$ the energy stored is:

$$E = \int_0^x F dx = \int_0^x Kx dx = \frac{1}{2}Kx^2 = \frac{1}{2K}F^2 \tag{16}$$

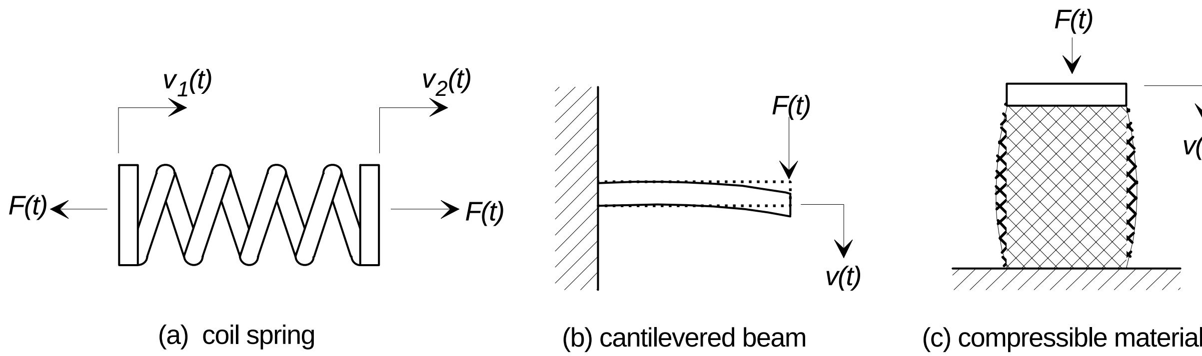

Figure 4: Examples of translational springs.

The stored energy is always a positive quantity, since it depends on the square of the displacement or force.

The relationships between force and displacement for springs depend on geometry and material properties. Tabulations of spring constants and derivations of constitutive equations for simple coil and beam-like springs are contained in several references [4,5]. Figure 4 shows several configurations for mechanical springs, including a coil spring, a deflecting beam and a compressible material. The definition of a pure or ideal spring and its stored energy, given in Eq. (16), requires that the displacements be defined as shown in Fig. 4, so that at $x = 0$ the force and energy are both zero, and any finite displacement x is associated with energy storage.

An equation for the ideal spring, expressed directly in terms of the power variables force and velocity, may be derived by differentiating the ideal constitutive equation Eq. (15):

$$\frac{dF}{dt} = Kv. \tag{17}$$

Equation (17) is called an *elemental equation* because it expresses the element characteristic in terms of power variables.

Mass Element: A pure translational *mass* element m is defined as an element in which the linear momentum p is a single-valued, monotonic function of the velocity v, that is

$$p = \mathcal{F}(v). \tag{18}$$

In general for very high velocities, the constitutive relationship for a pure mass is given by the nonlinear relativistic relationship [6]:

$$p = \frac{mv}{\sqrt{1 - (v/c)^2}} \tag{19}$$

where m is the mass at rest (Kg), and c is the velocity of light (m/s). This constitutive relationship is shown in Fig. 5. The energy of the mass is:

$$E = \int_0^p v dp \tag{20}$$

and is indicated by the shaded area in the figure.

At velocities much less than the speed of light, $v \ll c$, Eq. (19) reduces to the linear equation for an ideal mass element:

$$p = mv \tag{21}$$

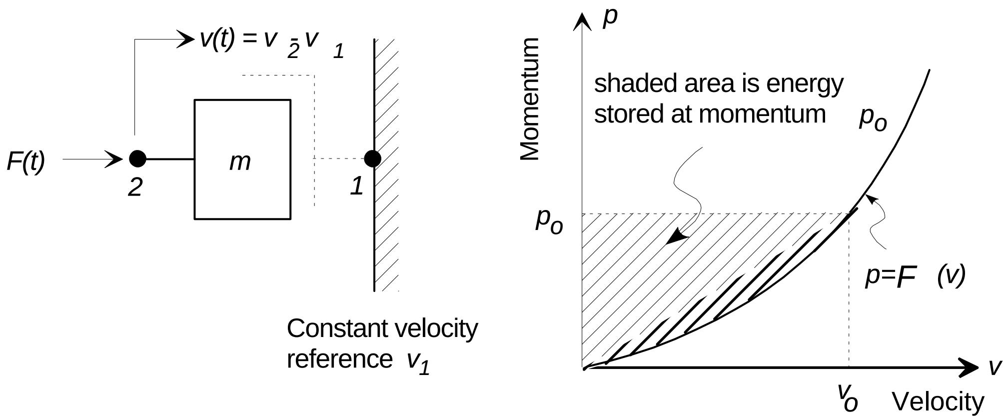

Figure 5: Definition of the pure translational mass element.

where the mass m depends on geometry and material properties, and represents the classical Newtonian mass. The energy stored in an ideal mass is:

$$\mathcal{E} = \int_0^p v dp = \int_0^p \frac{p}{m} dp = \frac{1}{2m} p^2 = \frac{1}{2} m v^2 \tag{22}$$

and is always positive because it depends on the square of the momentum or velocity.

For the ideal mass an elemental equation describing the element in terms of power variables may be derived by differentiating Eq. (21):

$$F = m \frac{dv}{dt}. \tag{23}$$

This form of the elemental equation is a statement of Newton's Law, relating force to mass and acceleration. The schematic symbol for the primitive mass element, shown in Fig. 5, depicts a mass with velocity v referenced to a nonaccelerating coordinate system. The mass symbol has two terminals, one associated with the velocity v of the mass element, and the second connected to the inertial reference frame to indicate that velocity is always measured with respect to the reference.

2.2.2 Energy Dissipation Element - The Damper

A pure mechanical translational *damper* is defined as an element in which the force is a single-valued, monotonic function of the velocity across the element. A relationship for a pure damper is shown in Fig. 6 and is of the form:

$$F = \mathcal{F}(v) \tag{24}$$

where $\mathcal{F}()$ is a single-valued, monotonic function. The symbol for a damper is shown in Fig. 6 where the velocity v is the difference in velocities across the damper terminals, that is $v = v_2 - v_1$.

For an ideal damper the constitutive relationship is linear and is written:

$$F = Bv \tag{25}$$

where B is defined to be the damping constant (N-s/m).

The power flow associated with an ideal damper is:

$$\mathcal{P} = Fv = Bv^2 = \frac{1}{B} F^2 \tag{26}$$

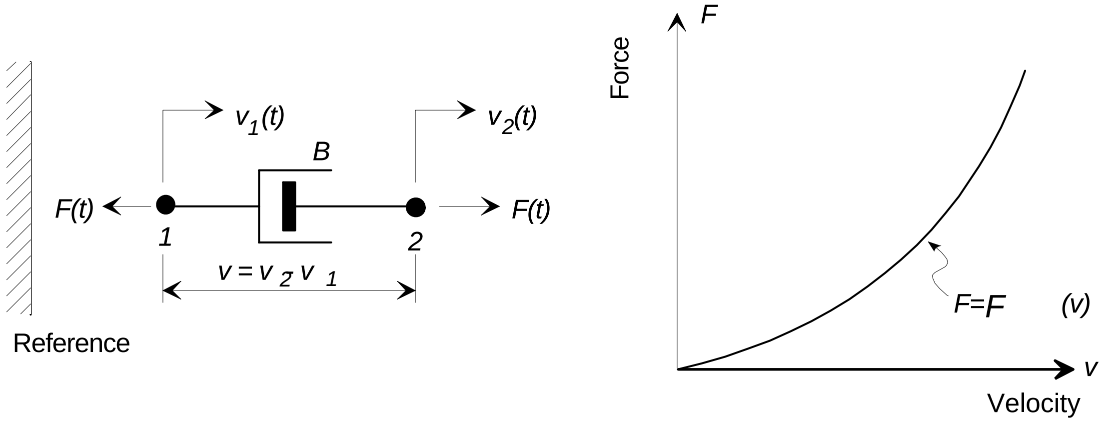

Figure 6: Definition of the pure translational damper element.

and is always positive, so that power always flows into the damper. Energy and power cannot be recovered from the damper; the mechanical work performed on the damper is converted to heat, becoming unrecoverable as mechanical energy, and transferred from the system to its environment.

The constitutive equation for a damper, Eq. (25) is expressed directly in terms of the power variables and is also the elemental equation. The damping constant B is a function of both geometry and material properties. Linear damping phenomena, as described by Eq. (25), are known as viscous damping effects.

In mechanical systems frictional drag forces occur between two members in contact which are moving relative to each other. When the sliding surfaces are sufficiently lubricated, so as to form a hydrodynamic film, the drag force is proportional to the relative velocity between the surfaces and depends on the roughness of the two surfaces, the material properties of the surfaces and lubricant, and the normal load force between the members. When there is very little lubrication, the friction forces tend to be relatively independent of speed, as shown in Fig. 7, with an almost constant force. Such damping is nonlinear since the force is not proportional to velocity.

Also shown in Fig. 7 is the *coulomb friction* characteristic which represents both stiction and sliding friction between two solids. Because this characteristic is not monotonic and a wide range of forces may occur at zero velocity, this characteristic is not strictly a pure damper. If the single-valued, monotonic requirement is relaxed, the coulomb characteristic can be considered as a quasi-pure damper and can be used (albeit with care) in models of physical systems.

An additional form of damping is associated with the drag force on an object traveling through a fluid, such as the aerodynamic drag on an automobile or aircraft. This drag force is nonlinear and is often approximated as being proportional to the square of the velocity:

$$F = c\,|v|\,v, \tag{27}$$

where $|v|$ is the absolute value of velocity, and c is a drag constant that depends on the geometry of the object and properties of the fluid. (The term $|v|\,v$ is known as the absquare, or absolute square, and generates a force which changes sign as the velocity changes direction so that power is always dissipated due to the drag force.)

2.2.3 Source Elements

Two source elements are defined in terms of the power variables for modeling mechanical translational systems;

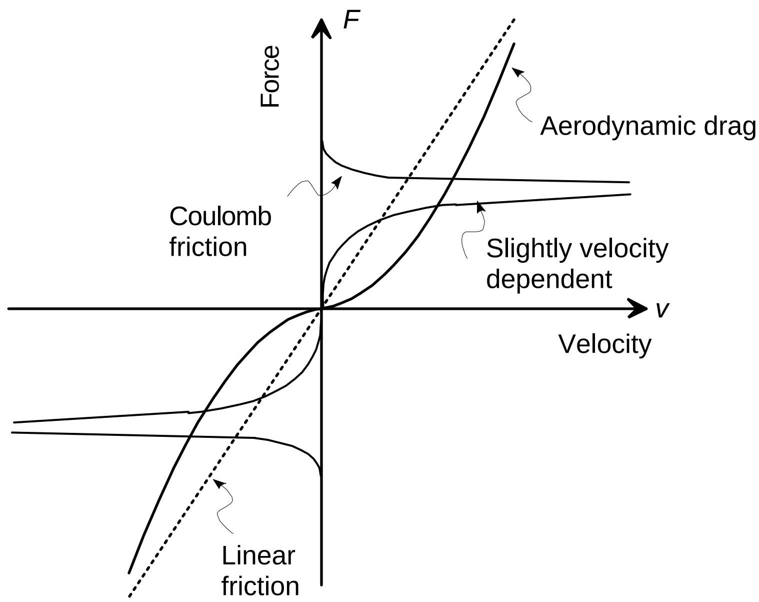

Figure 7: Examples of frictional characteristics representing mechanical damping.

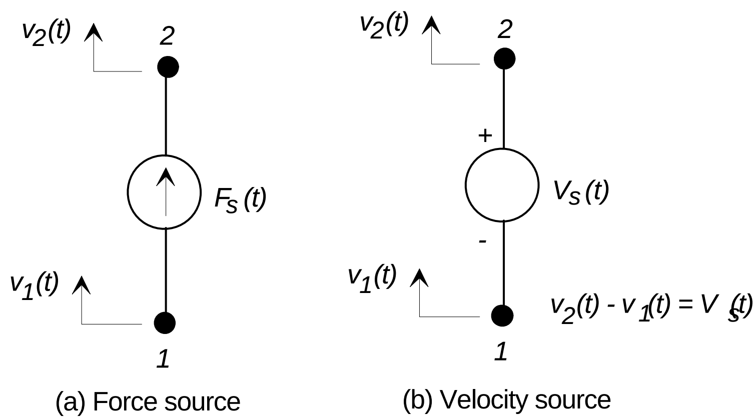

Figure 8: Ideal mechanical translational sources: the force source and the velocity source.

(1) The ideal *force source* is source of energy in which the force exerted is an independently specified function of time $F_s(t)$. The force produced by this element is independent of the velocity at the input port. The velocity produced by the source depends entirely on the system to which it is connected.

(2) Conversely the ideal *velocity source* is an element in which the velocity is an independently specified function of time $V_s(t)$, and is maintained without regard to the force necessary to generate the velocity. The force produced by the source is determined entirely by the reaction of the system to which it is connected.

These sources are illustrated in Fig. 8. These two ideal sources may continuously supply or absorb energy since in each, one power variable is independently specified while the complementary power variable is determined by the system to which the source is coupled. Ideal sources are capable of supplying infinite power and are idealizations of real sources, which have finite power and energy capability.

■ Example

To illustrate the power flow in mechanical systems, consider the distance required for an automobile to stop when the driver suddenly applies the brakes, locking the wheels so that a skid occurs. The system model is sketched in Fig. 9, where the system is defined as the automobile and the tire-pavement interaction, and where the output variable of interest is the car velocity.

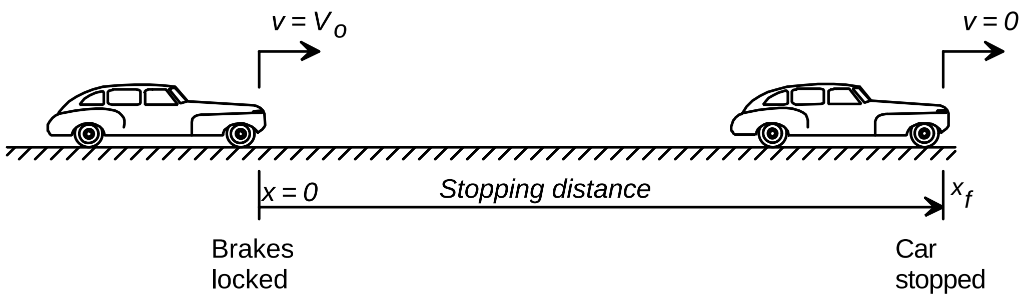

Figure 9: Automobile stopping distance.

A model for the car after the wheels are locked at time $t = 0$ may be formulated by considering:

1. the car as a simple mass element, m, which at time $t = 0$ has velocity v_0,
2. for time $t > 0$, the only forces acting on the car are the tire-pavement resistance force F_r, and the weight of the car which generates the normal force between the car and the pavement.
3. A model for the tire-pavement interaction force after the wheels are locked is represented by a damper with a constant resistance force equal to:

$$F_r = \mu m g \tag{28}$$

where m is the mass of the car, g is the acceleration due to gravity and μ is the coefficient of sliding friction for the tire-pavement interface.

The stopping distance may be determined by equating the initial energy of the car $\mathcal{E}_0$, to the energy dissipated $\mathcal{E}_d$ at the tire-pavement interface during the skid. When all of the initial energy has been dissipated, the car velocity has decreased to zero. The initial energy of the mass is:

$$\mathcal{E}_0 = \frac{1}{2} m v_0^2. \tag{29}$$

The energy dissipated by the damper over the time period 0 to t is the integral of the power dissipated by the resistance force:

$$\mathcal{E}_d(t) = \int_0^t \mathcal{P} dt = \int_0^t F_r v(t) dt. \tag{30}$$

Since the resistance force is assumed to be constant, the energy dissipated is:

$$\mathcal{E}_d = F_r \int_0^t v(t) dt = F_r x_f \tag{31}$$

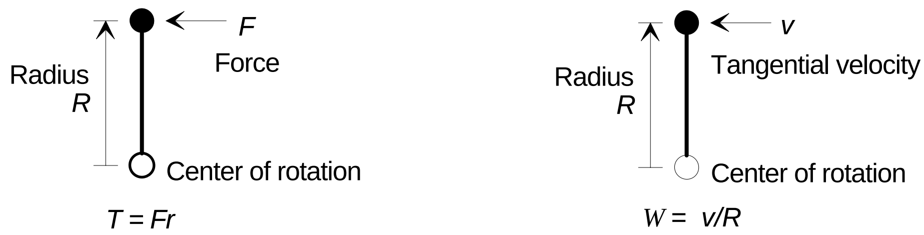

Figure 10: Definition of power variables in a rotational system.

where x_f is the distance traveled (displacement) in time 0 to t.

The stopping distance may be determined by equating the initial energy to the energy dissipated:

$$\frac{1}{2}mv_o^2 = F_r x_f, \tag{32}$$

and using Eq. (i)

$$x_f = \frac{mv_0^2}{2\mu mg} = \frac{v_0^2}{2\mu g} \tag{33}$$

Equation (vi) indicates that, in terms of this model, the stopping distance during a skid is independent of the mass m. For a car traveling at 20 m/sec with a pavement-tire friction coefficient of 0.8, the stopping distance is:

$$\begin{aligned} x_f &= \frac{20^2}{2 \times 0.8 \times 9.81} \quad (\mathrm{m/s})^2/\mathrm{m/s}^2 \\ &= 25.5 \text{ m.} \end{aligned} \tag{34}$$

The stopping distance is proportional to the square of speed and for an initial velocity of 10 m/s the distance is reduced to 6.4 m.

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3 Mechanical Rotational Systems

3.1 Definition of Power Flow Variables

In rotational systems power is transmitted and energy is stored by rotary motion about a single axis. The power flow across a rotational system boundary is through angular motion of a shaft, as illustrated in Figure 11. In a rotational system, the power flow at any instant may be expressed as the product of the *torque* acting about the fixed axis and the *angular velocity* about the axis:

$$\mathcal{P}(t) = T(t)\Omega(t) \tag{35}$$

where T is the applied torque (N-m), and $\Omega(t)$ is the angular velocity in units of radians per second (rad/s).

The torque T acting about a rotational axis is equivalent to a tangential force F at a radius r, that is $T = F \times r$, while the angular velocity Ω about an axis is related to the instantaneous linear velocity v at a radius r as $\Omega = v/r$ as shown in Fig. 10.

Also shown in Fig. 11 is the convention for schematically depicting torques on shafts. An applied torque is shown as an arrow aligned along the shaft, with a length proportional to the magnitude of

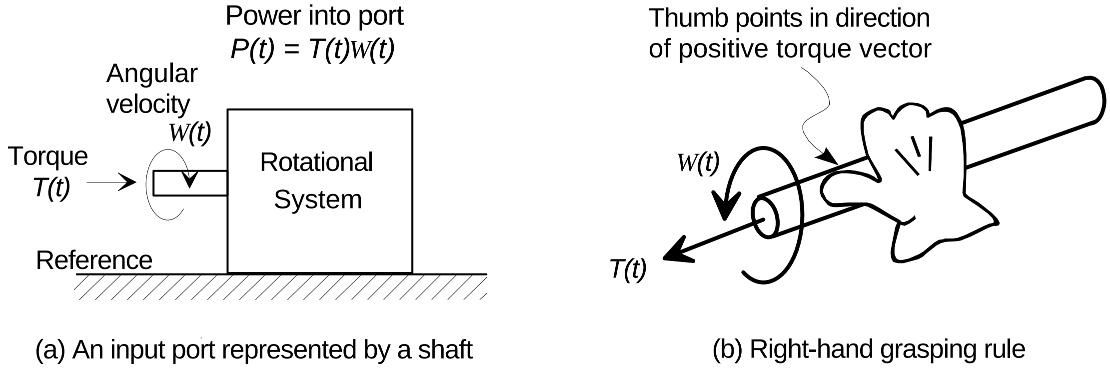


Figure 11: Power flow and torque sign convention in a rotational system.

the torque and with a direction defined by the "right-hand grasping rule". If the shaft is grasped in the right hand with the fingers curling in the direction of the assumed positive angular velocity, the direction of the arrow representing a positive torque is in the direction pointed to by the thumb.

The rotational work crossing the system boundary over a time increment dt is:

$$\Delta W = T(t)\Omega(t)dt \tag{36}$$

In a manner analogous to mechanical translational systems, the energy may be expressed directly in terms of integrated power variables by defining angular displacement Θ as the integral of the angular velocity:

$$\Theta(t) = \int_0^t \Omega(t)dt \qquad \text{or} \qquad d\Theta = \Omega(t)dt, \tag{37}$$

with units of radians, and angular momentum h as the integral of the torque:

$$h(t) = \int_0^t T(t)dt \qquad \text{or} \qquad dh = T(t)dt \tag{38}$$

with units of N-m-sec.

The increment in work, Eq. (36), across a boundary may then be expressed in the following three forms:

$$\begin{array}{lllllll} \Delta W & = & T(\Omega dt) & = & T d\Theta & = & d\mathcal{E}_{Potential} \\ \Delta W & = & \Omega(T dt) & = & \Omega dh & = & d\mathcal{E}_{Kinetic} \\ \Delta W & = & T\Omega dt & = & \Omega T dt & = & d\mathcal{E}_{Dissipated} \end{array} \tag{39}$$

Mechanical rotational power flow into a system may result in a change in stored energy through changes in angular displacement $d\Theta$ associated with a rotational spring, or changes in angular momentum dh associated with a rotational mass or inertia, or may be dissipated by conversion of rotational work to heat in a rotational damper.

3.2 Primitive Rotational Element Definitions

3.2.1 Energy Storage Elements

Rotational Spring: A pure rotational spring, also known as a torsional spring, is defined as an element in which the angular displacement Θ is a single-valued, monotonic function of the torque as illustrated in Figure 12. A pure spring has a general constitutive relationship:

$$\Theta = \mathcal{F}(T). \tag{40}$$

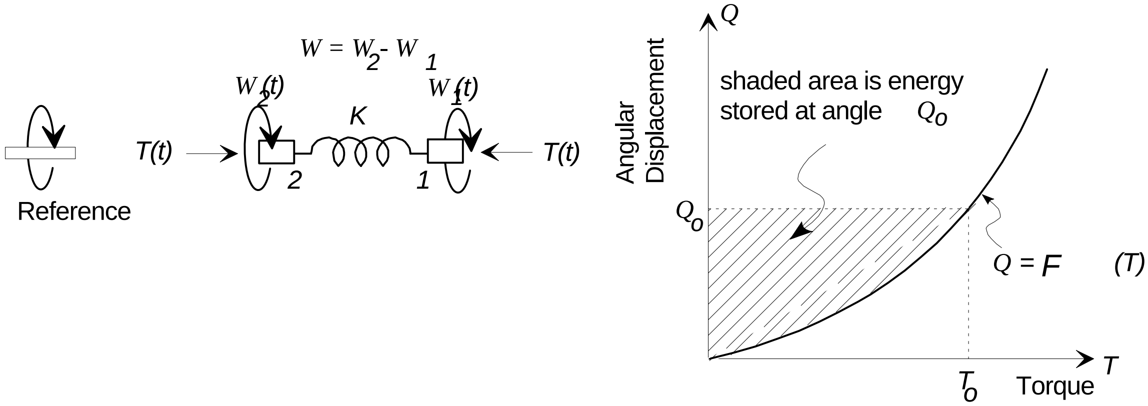


Figure 12: Definition of the pure rotational spring element.

The energy stored in a pure rotational spring is:

$$\mathcal{E} = \int_0^{\Theta} T d\Theta \tag{41}$$

and is illustrated in the shaded area of Fig. 12. An ideal spring is one in which the constitutive relationship Eq. (40) is linear and may be written:

$$\Theta = \frac{1}{K_r} T \tag{42}$$

where K_r is the *rotational spring constant*, or rotational stiffness, with units of N-m/rad.

For an ideal spring the energy E stored with an applied torque T is:

$$\mathcal{E} = \int_0^{\Theta} T d\Theta = \int_0^{\Theta} K_r \Theta d\Theta = \frac{1}{2} K_r \Theta^2 = \frac{1}{2K_r} T^2 \tag{43}$$

The symbol for the rotational spring, shown in Figure 12, is similar to that for the translational spring. The rotational displacement Θ is defined as the displacement from a rest displacement, resulting from a finite torque and the resultant energy storage. As shown in Fig. 12, the relative displacement between the ends of the spring is the difference between displacements at each end measured with respect to a fixed reference, that is $\Theta = \Theta_2 - \Theta_1$.

The rotational spring constant K_r in the constitutive equation is a function of material properties and geometry. In rotational systems, shafts and couplings often have rotational stiffness which can be determined analytically or have been measured experimentally and tabulated [4,5].

An elemental equation for the ideal rotary spring may be derived by differentiating Eq. (42) to give an expression in terms of power variables:

$$\Omega = \frac{1}{K_r} \frac{dT}{dt}, \tag{44}$$

Rotational Inertia Element: A pure rotational inertia element is defined as an element in which the angular momentum h is a single-valued, monotonic function of angular velocity:

$$h = \mathcal{F}(\Omega). \tag{45}$$

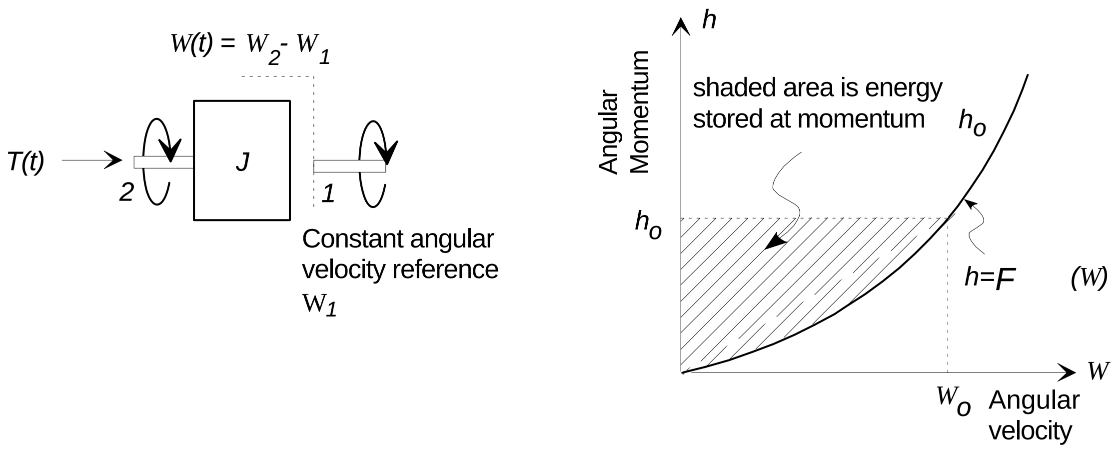


Figure 13: Definition of the pure rotational inertia element.

The energy stored in a pure inertia is illustrated in Figure 13 and may be computed directly from Eq. (39). When the constitutive equation is linear, the ideal inertia may be defined as:

$$h = J\Omega \tag{46}$$

where J is defined as the rotational *moment of inertia*, with units of Kg-m^2.

The value of the moment of inertia of a rotating body depends on the geometry and mass distribution, and has been tabulated for a variety of bodies in [4,5]. The moment of inertia of a single lumped mass m rotating at a constant radius r about an axis is $J = mr^2$, and for a collection of n discrete mass elements the net moment of inertia is the sum of each elemental moment of inertia:

$$J = \sum_{i=1}^{n} m_i r_i^2. \tag{47}$$

For a general body with spatially distributed mass rotating about a fixed axis, the inertia J may be determined by integrating over the body as shown in Figure 14.

The energy stored in an ideal inertia is:

$$\mathcal{E} = \int_0^{h_0} \Omega dh = \frac{1}{J}\int_0^{h_0} h dh = \frac{1}{2J}h_0^2 = \frac{1}{2}J\Omega^2 \tag{48}$$

The symbol for a rotary inertia is shown in Fig. 13, where the angular velocity is referenced to a fixed axis to indicate that the angular velocity of the inertia must be referenced to a fixed or nonaccelerating reference frame. The elemental equation for the ideal rotary inertia may be derived by differentiating Eq. (46) to yield:

$$T = J\frac{d\Omega}{dt}. \tag{49}$$

3.2.2 Energy Dissipation Element

A pure rotational damper is defined as an element in which the torque is a single-valued, monotonic function of the angular velocity across the element:

$$T = \mathcal{F}(\Omega) \tag{50}$$

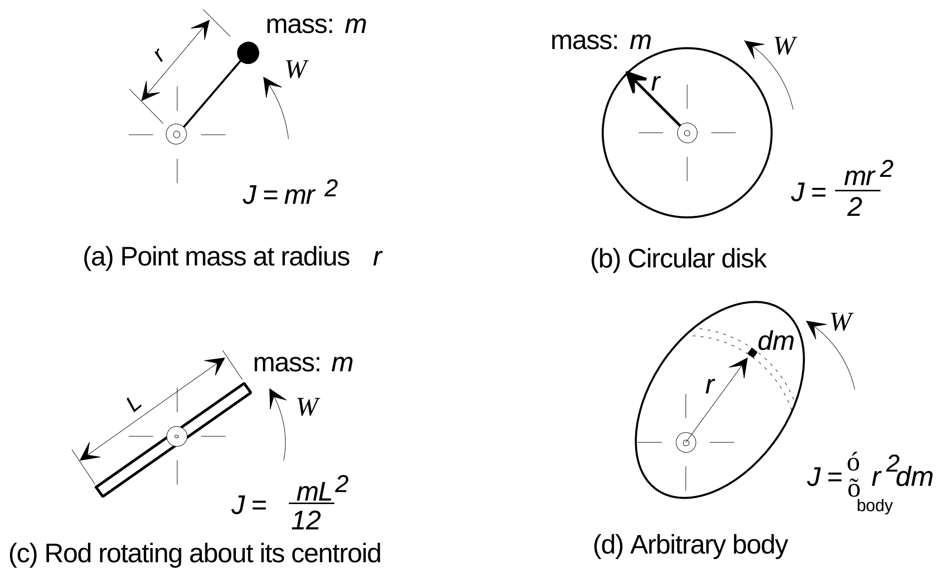


Figure 14: Moments of inertia of typical bodies.

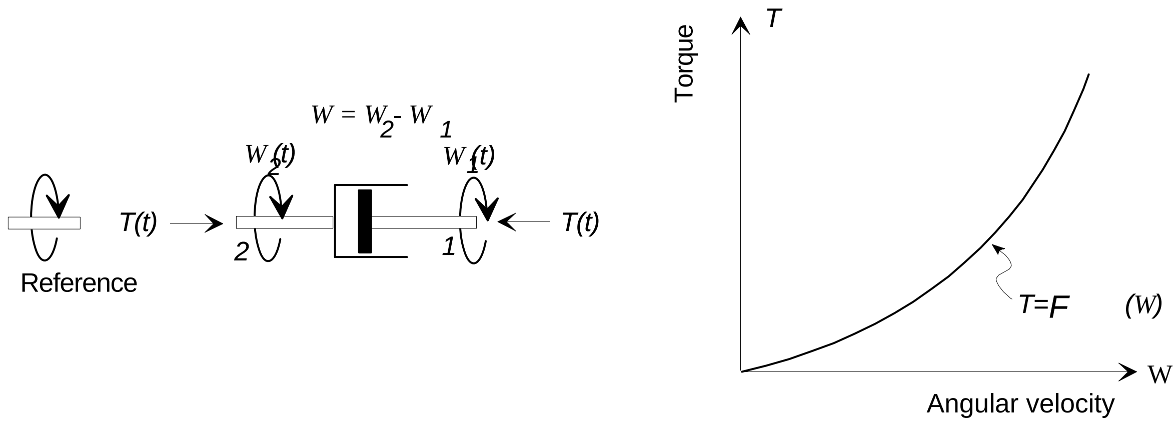


Figure 15: Definition of the pure rotational damper element.

The symbol for the rotary damper is shown in Figure 15 where the angular velocity across the element is defined in terms of the difference in angular velocities at each end of the element $\Omega = \Omega_2 - \Omega_1$. The ideal rotary damper has a linear constitutive equation:

$$T = B_r\Omega \tag{51}$$

where B_r is the rotary damper constant with units of N-m/sec.

The power dissipated by an ideal damper is:

$$\mathcal{P} = T\Omega = B_r\Omega^2 = \frac{T^2}{B_r} \tag{52}$$

and is always positive, that is power is always dissipated. As indicated in Eq. (39), the work performed on a rotary damper element is converted into heat and transferred to the system environment.

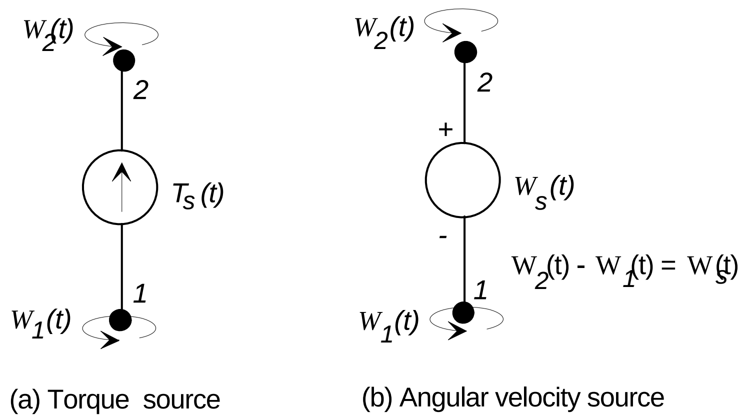


Figure 16: Mechanical rotational sources.

Rotational damping occurs naturally from frictional effects in bearings, and like translational friction, is often highly nonlinear. The ideal or linear form, given by Eq. (51), is often known as viscous rotary damping.

3.2.3 Source Elements

Two energy source elements are defined for mechanical rotational systems:

(1) The ideal *torque source* is an element in which the torque exerted on the system is an independently specified function of time $T_s(t)$. The angular velocity produced by the source is determined by the reaction of the system to which it is connected.

(2) The ideal *angular velocity source* is an element in which the angular velocity at a port is an independently specified function of time $\Omega_s(t)$. The torque produced by the source is determined by the system to which it is connected.

These two ideal sources, illustrated in Fig. 16, are capable of supplying infinite power and thus are idealizations of real power-limited sources.

■ **Example**

A shaft-flywheel system, as shown in Figure 17, is an integral part of many mechanical systems such as pumps, turbines or automobile transmissions. Long light-weight shafts exhibit significant compliance and act as torsional springs. One of the issues in the design of these systems is the amount of "wind-up" or twist which occurs in the drive shaft as the flywheel is accelerated or deaccelerated. A worst-case operating condition occurs when the input end of the drive shaft is stopped suddenly while the flywheel is turning at its normal operating speed. What is the maximum deflection across the shaft? With knowledge of the shaft deflection, the stresses within the shaft and the possibility of failure can be determined.

A model can be formulated to determine the maximum shaft deflection as follows:

1. The flywheel is considered as an ideal rotary inertia J, with an initial angular speed of Ω_o.

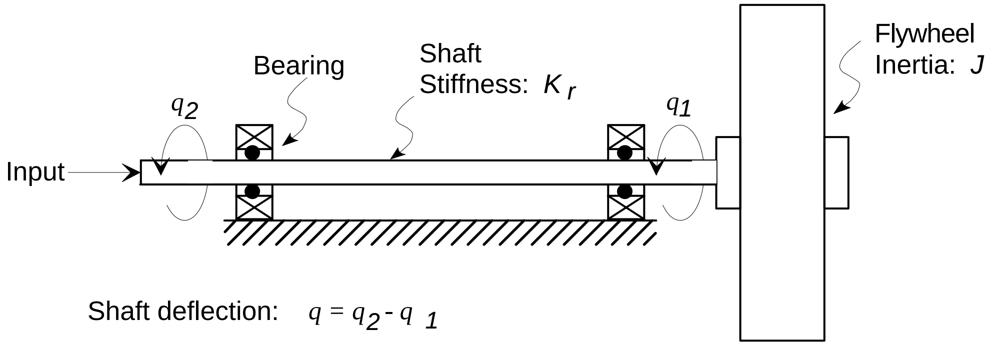


Figure 17: Shaft windup during transients.

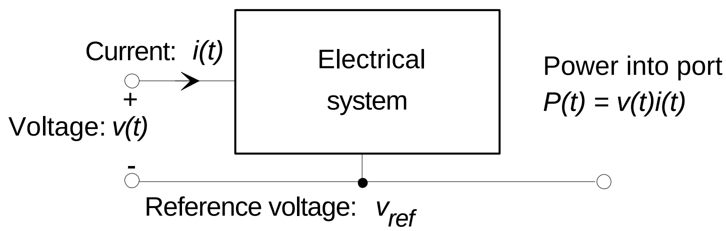


Figure 18: Electrical power port.

2. The shaft is considered as an ideal rotational spring with rotational stiffness K_r.
3. The bearings and all other dissipative features, for example aerodynamic drag on the flywheel, are neglected.

The shaft deflection is estimated by noting that the maximum shaft deflection, Θ_m, occurs when all of the kinetic energy in the flywheel is transferred to potential energy in the shaft with no losses:

$$\mathcal{E} = \frac{1}{2} J\Omega_o^2 = \frac{1}{2} K_r \Theta_m^2 \tag{53}$$

which yields:

$$\Theta_m = \sqrt{\frac{J}{K_r}}\Omega_0 \tag{54}$$

The maximum shaft deflection is therefore directly proportional to the initial flywheel angular velocity, and is proportional to the square root of the inertia to stiffness ratio.

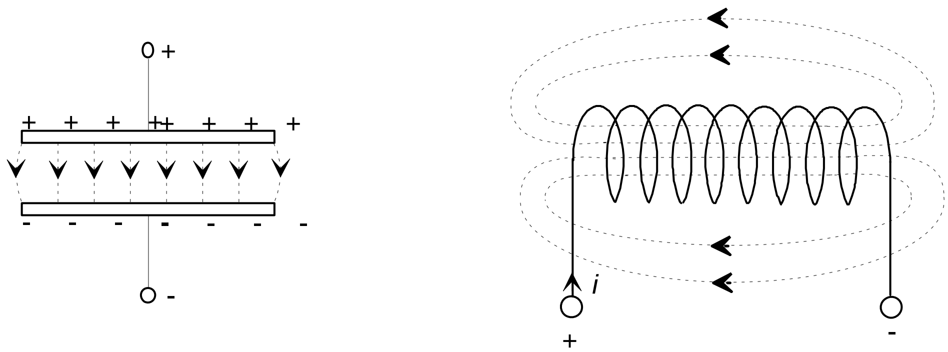
---

4 Electrical System Elements

4.1 Definition of Power Flow Variables

In the electrical domain the power flow through a port represented by a pair of wires is the product of *current* and *voltage drop*, as shown in Fig. 18,

$$\mathcal{P}(t) = i(t)v(t) \tag{55}$$



(a) Energy storage in an electric field (b) Energy storage in a magnetic field

Figure 19: Energy storage mechanisms: (a) electric field between two conducting plates, and (b) magnetic field associated with a conducting coil.

where i (amperes) is the current flowing into the port, and v (volts) is the voltage drop across the terminals. The electrical work flowing across a system boundary in an incremental time period dt is:

$$\Delta W = i(t)v(t)dt \tag{56}$$

The work may be expressed in terms of electrical domain variables by defining the integrals of the two power variables:

$$q = \int_0^t idt \qquad \text{or} \qquad dq = idt \tag{57}$$

where q is the electrical *charge* in units of coulombs (amp-sec), and

$$\lambda = \int_0^t vdt \qquad \text{or} \qquad d\lambda = vdt. \tag{58}$$

where λ is defined to be the *magnetic flux linkage* (v-sec or webers).

The unit of charge, the coulomb, corresponds to the negative of the total charge associated with 6.22×10^{18} electrons, and is equivalent to one ampere-second. The current i is the rate of flow of electrical charge q, and is positive in the direction of positive charge flow (or in the opposite direction to electron flow). The voltage represents the *potential difference* between two points, and is equivalent to the work performed in moving a unit charge between the two points with a potential difference of v. The distribution of charge within a medium leads to the establishment of an electric field as shown in Figure 19a [7].

Flux linkage, the integral of voltage drop, is associated with the magnetic flux generated in a coil of wire carrying a current, as illustrated in Figure 19b. The flux linkage is defined as the total magnetic flux passing through, or linking, the coil. The total flux linkage is a function of the magnetic flux density produced by the coil, the area of the coil and the total number of turns of the coil through which the flux passes [7].

The electrical work passing through a system boundary in time dt may be written in terms of power variables and integrated power variables in the following three forms:

$$\begin{array}{lllllll} \Delta W & = & i(vdt) & = & id\lambda & = & d\mathcal{E}_{Magnetic} \\ \Delta W & = & v(idt) & = & vdq & = & d\mathcal{E}_{Electrical} \\ \Delta W & = & ivdt & = & vidt & = & d\mathcal{E}_{Dissipated} \end{array} \tag{59}$$

Electrical work crossing a system boundary may result in a change in the magnetic flux associated with the system through electromagnetic energy storage in inductors, a change in the total charge

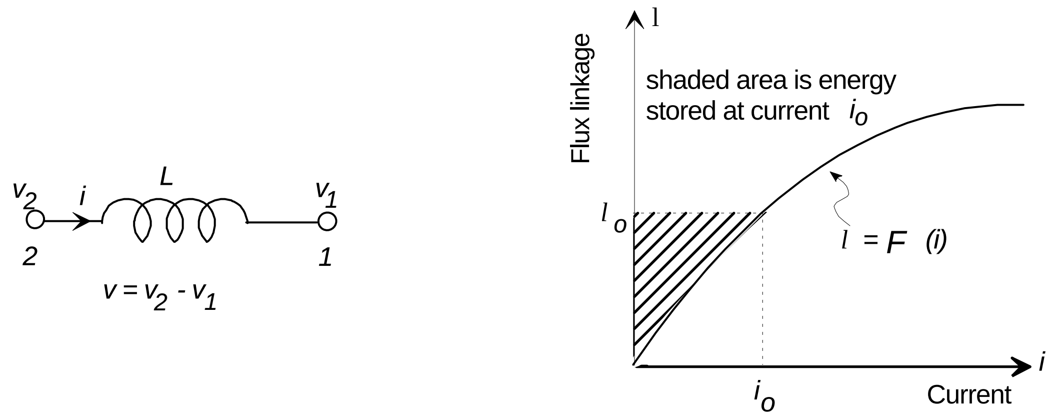


Figure 20: Definition of the pure electrical inductance element.

in the system associated with electrostatic energy storage in capacitors, or dissipation in resistors through the generation of heat with no electrical energy storage.

4.2 Primitive Electrical Element Definitions

4.2.1 Energy Storage Elements

Electrical Inductor: The pure inductor is defined as an element in which the flux linkage is a single-valued, monotonic function of the current:

$$\lambda = \mathcal{F}(i) \tag{60}$$

The constitutive characteristic of an inductor is shown in Fig. 20. When the relationship between flux linkage and current is linear, the ideal constitutive equation is:

$$\lambda = Li \tag{61}$$

where the constant of proportionality L is defined as the inductance, in units of the henry (h) (volt-sec/amp). Practical inductors used in electronic circuits are often specified in units of the millihenry (mh) or 10^{-3} h, or the microhenry (μh) or 10^{-6} h.

The energy stored in an ideal inductor is:

$$\mathcal{E} = \int_0^\lambda i d\lambda = \frac{1}{L}\int_0^\lambda \lambda d\lambda = \frac{1}{2L}\lambda^2 = \frac{1}{2}Li^2. \tag{62}$$

The property of inductance is associated with the magnetic fields generated by currents in a conductor, usually in the form of a coil of wire. The value of the inductance L is a function of the coil geometry and the material properties of the core on which it is wound.

The circuit symbol for an inductor is shown in Fig. 20. The inductance for two coil geometries is shown in Fig. 21, where it is noted that the inductance is proportional to the permeability μ of the material through which the magnetic flux passes, and to the square of the number of turns of wire. A simple coil of wire with an air core has a relatively low inductance because air has a low magnetic permeability. To achieve a higher inductance for a given coil geometry, ferromagnetic materials with high magnetic permeability are used as a core within the coil. Such materials are however subject to magnetic saturation and losses and are not suitable for all applications.

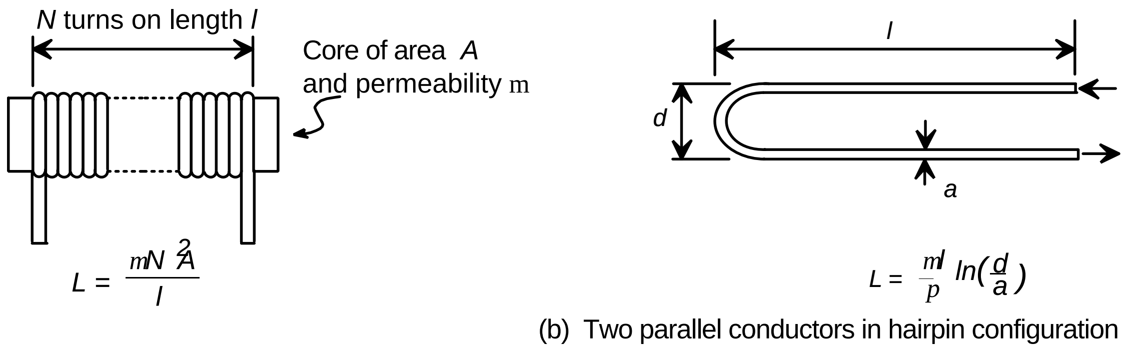


Figure 21: Inductance of two conductor configurations.

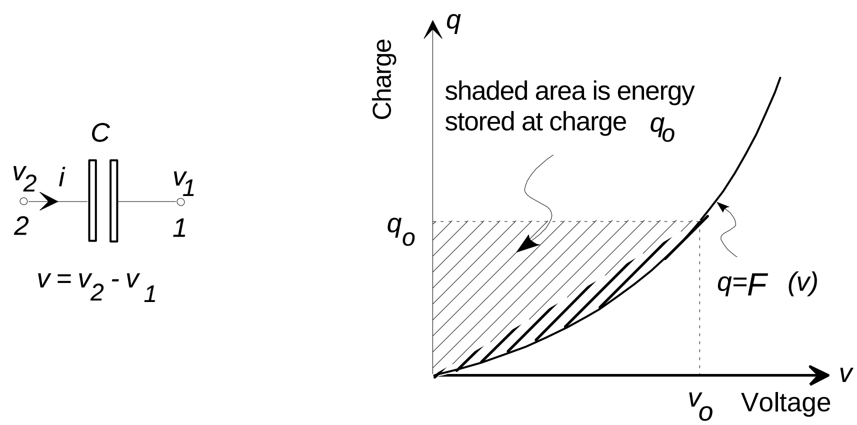


Figure 22: Definition of the pure electrical capacitor element.

An elemental equation for the inductor, expressed in terms of power variables, may be derived by differentiating the constitutive equation, Eq. (61):

$$v = L\frac{di}{dt}. \tag{63}$$

Electrical Capacitor: A pure electrical capacitor is defined as an element in which the electric charge q stored is a single-valued, monotonic function of the voltage v across its terminals:

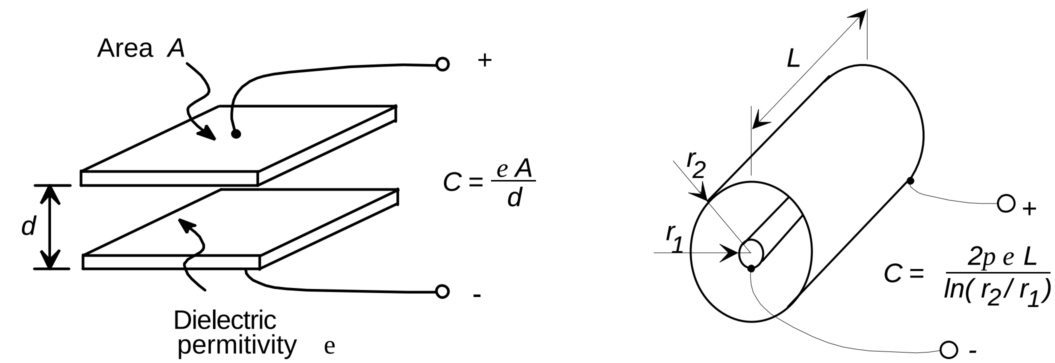
$$q = \mathcal{F}(v). \tag{64}$$

The constitutive relationship for a pure capacitor is illustrated in Fig. 22, where the stored energy is also indicated. When the constitutive relationship for a capacitor is linear:

$$q = Cv \tag{65}$$

the capacitor is defined to be ideal, and C is defined to be the capacitance with units of farads (coulomb/volt).

A capacitor stores energy in an electrostatic field established in a non-conducting dielectric material between two conducting surfaces (plates). The value of capacitance C in the constitutive equation depends on the geometry of the plates and the material properties of the dielectric in the



(a) Parallel plate capacitor (b) Concentric cylinder capacitor

Figure 23: Capacitance of two plate configurations.

gap. For the parallel plate geometric configuration shown in Fig. 23, the capacitance is proportional to the plate area, inversely proportional to the separation of the plates, and directly proportional to the dielectric permittivity, which for air is 8.85×10^{-12} farads/m.

The energy stored in an ideal capacitor is:

$$\mathcal{E} = \int_0^q v dq = \frac{1}{C} \int_0^q q dq = \frac{1}{2C} q^2 = \frac{1}{2} C v^2. \tag{66}$$

The elemental equation for the ideal capacitor may be found by differentiating the constitutive equation, Eq. (65):

$$i = C \frac{dv}{dt}. \tag{67}$$

Electrical capacitors are used in electronic and electrical equipment. They come in many forms with different plate geometries and dielectric materials. In practice the farad (fd) is too large a unit to be of practical use; in electronic circuits, capacitors are usually expressed in units of the microfarad (μfd) or 10^{-6} fd, the nanofarad (nfd) or 10^{-9} fd, or the picofarad (pfd) which is 10^{-12} fd.

4.2.2 Energy Dissipation Element

A pure electrical resistor is defined as an element in which the current is a single-valued, monotonic function of the voltage drop:

$$i = \mathcal{F}(v) \tag{68}$$

Many electrical devices, including semiconductor diodes and resistors, exhibit pure resistance. A typical characteristic is shown in Fig. 24. For standard electrical resistors, the relationship between voltage and current is linear and the ideal elemental equation becomes Ohm's law:

$$i = \frac{1}{R} v \tag{69}$$

where R is the electrical resistance, in ohms (volt/ampere). Electrical resistance is a function of the bulk resistivity ρ of the conducting medium and the size and shape of the conductor. (Resistivity is easily visualized as the resistance between opposite faces of a unit cube of the material, and has units of ohm-m.) Materials with a low resistivity, such as copper which has a resistivity of

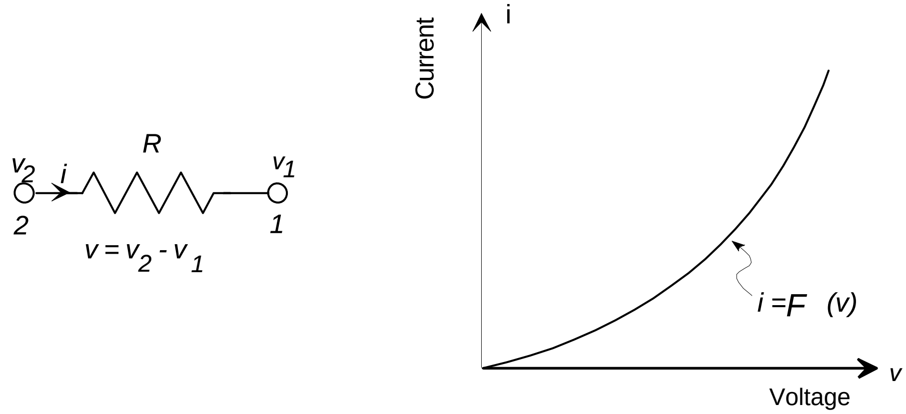


Figure 24: Definition of the pure electrical resistance element.

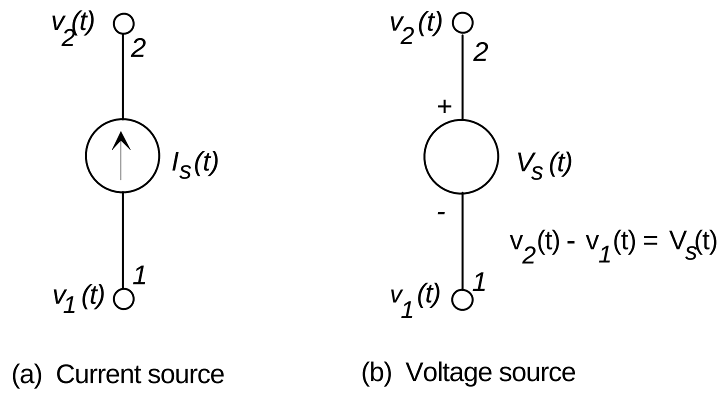


Figure 25: Ideal electrical sources.

1.742×10^{-8} ohm-m at room temperature, are known as *conductors*, while those with very high resistivity are collectively known as *insulators*. A conductor of length l, and uniform cross-section A has a resistance

$$R = \rho l/A. \tag{70}$$

Electrical resistors are important components in electronic circuits. They are typically constructed of bulk carbon or a composite material, a carbon film, or as a coil of resistance wire.

An ideal resistor dissipates power, with the electrical work being converted to heat and made unavailable to the electrical system:

$$P(t) = vi = i^2 R = \frac{v^2}{R}. \tag{71}$$

4.2.3 Source Elements

Two ideal electric source elements may be defined, with symbols as shown in Fig. 25. An ideal *voltage source* is an element in which the voltage across its terminals is an independently specified function of time $V_s(t)$. It is capable of supplying, or absorbing, infinite current in order to maintain the specified voltage. The ideal *current source* is an element in which the current supplied to the system is an independently specified function of time $I_s(t)$. The terminal voltage of a current source is defined by the system to which it is connected. Both ideal sources are capable of supplying or absorbing infinite power, and thus only approximate real power-limited sources.

■ Example

A wire-wound resistor with a nominal resistance of 10 ohms is constructed as a coil as illustrated in Fig. 26. A feature of wire-wound resistors is that they have relatively good heat transfer characteristics, and maintain a relatively constant value of resistance with temperature. However, because of their coil structure, wire-wound resistors also have a small inherent *parasitic* inductance associated with the magnetic field that is generated by the current in the coil. Both the resistive and inductive effects may be important in critical applications. To determine the importance of the inductance effect, the frequency of a sinusoidal current waveform at which the magnitude of the voltage drop associated with the inductance is 10% of that due to the resistance is estimated .

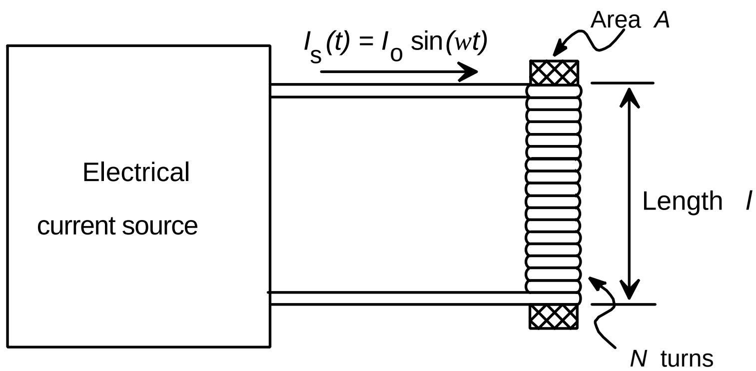


Figure 26: A wire-wound electrical resistor.

First the inductance of the resistance coil is computed assuming an air core coil. The inductance for the configuration illustrated is:

$$L = \frac{\mu A N^2}{l} \tag{72}$$

where μ is the permeability of the core, A is the cross-sectional area and l is the length of the coil, and N is the number of turns distributed along the length.

If the coil has 100 turns, a diameter of 1 cm and, a length 0.02 cm, the inductance (assuming a permeability of air of $\mu = 1.26 \times 10^{-6}$ h/m) is:

$$L = \frac{(1.26 \times 10^{-6})(\pi \times 0.005^2)(100^2)}{0.02} = 49.5 \times 10^{-6} \text{ h} \tag{73}$$

Assume that the resistor is driven by a sinusoidal current source

$$I_s(t) = I_0 \sin(\omega t) \tag{74}$$

where I_0 is the magnitude and ω is the angular frequency. If the resistor is an ideal resistance (without any inductance) the voltage drop v_R is determined from the resistance elemental equation:

$$v_R = R I_0 \sin \omega t, \tag{75}$$

and its magnitude is

$$|v_R| = |RI_0 \sin(\omega t)| = RI_0 \tag{76}$$

which is independent of the frequency ω.

If the coil is an ideal inductor L with no resistance, the associated voltage drop v_L is determined from the elemental equation:

$$v_L = L\frac{di}{dt} = I_0 L\omega \cos(\omega t) \tag{77}$$

and the magnitude of the voltage drop due to the inductance is:

$$|v_L| = I_0 L\omega \tag{78}$$

which is directly proportional to the angular frequency ω. Therefore as the frequency of the applied current waveform is increased the total voltage drop, which is the sum of the two effects, increases. The frequency ω_0 at which the ratio of the magnitude v_L to v_R is 0.1 is therefore given by:

$$\frac{|v_L|}{|v_R|} = \frac{\omega_0 L}{R} = 0.1 \tag{79}$$

and for the values of L and R given above

$$\omega_0 = \frac{0.1 \times 10}{49.5 \times 10^{-6}} = 20,202 \text{ rad/s.} \tag{80}$$

As the applied frequency is increased the inductive voltage drop becomes more significant, and the resistor behaves more like an inductor. At 202,020 rad/s the voltage drops due to resistance and inductance are equal, and at higher frequencies the inductive voltage drop exceeds the resistance voltage drop.

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5 Fluid System Elements

5.1 Definition of Power Flow Variables

The internal flow of fluids through pipes, vessels, and pumps, and the external flow around vehicles, aircraft, spacecraft, and ships are complex phenomena involving flow variables that are continuous functions of both space and time. As such they generally cannot be represented in terms of pure lumped elements. With some simplifying assumptions, however, a number of significant characteristics of the dynamic behavior of fluid systems, particularly for one dimensional pipe flows, can be adequately modeled with lumped parameter elements. In this section we define a set of lumped parameter elements that store and dissipate energy in network-like fluid systems, that is systems which consist primarily of conduits (pipes) and vessels filled with incompressible fluid. These definitions are analogous to those for mechanical and electrical system networks.

The power flow through a port into a fluid system, shown in Fig. 27, is expressed as the product of two fluid variables:

$$\mathcal{P}(t) = P(t)Q(t) \tag{81}$$

where $Q(t)$ is the fluid *volume flow rate* (m^3/s), and $P(t)$ is the fluid *pressure drop* across the port. The unit of pressure is the Pascal (N/m^2), named after Jacques Pascal who formulated the law

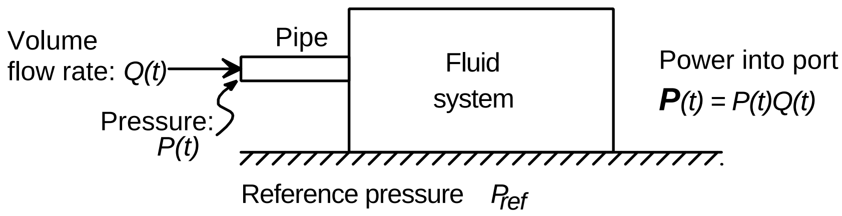

Figure 27: Fluid power port.

that the pressure at any point in a fluid at rest has a single scalar value independent of direction [8].

The fluid pressure is the normal force F per unit area A of the port (pipe) cross-section:

$$P = \frac{dF}{dA} \tag{82}$$

For a pipe with a uniform pressure profile the pressure is the total force acting across a cross-section divided by the area. The volume flow rate Q is the total volume of fluid passing through the port per unit time. It is the integration of the fluid velocity v over the area of the port:

$$Q = \int_A v dA \tag{83}$$

For a port with uniform pressure and fluid velocity profiles, the power flow may be written:

$$\mathcal{P} = PQ = \frac{F}{A}(vA) = Fv \tag{84}$$

which is directly related to mechanical power as defined in Section 2. The incremental fluid work crossing a system boundary at a port in time dt is:

$$\Delta W = P(t)Q(t)dt \tag{85}$$

and may be expressed in terms of fluid system variables by defining a pair of integrated power variables: the total *fluid volume* V (m^3) is the integration of volume flow rate, that is

$$V = \int_0^t Q(t)dt \qquad \text{or} \qquad dV = Q(t)dt \tag{86}$$

and the integral of the pressure is defined to be the *pressure momentum* Γ, with units of N-sec/m^2,

$$\Gamma = \int_0^t P(t)dt \qquad \text{or} \qquad d\Gamma = P(t)dt. \tag{87}$$

The volume V represents the total volume of fluid passing through the port over a given time period. The pressure momentum Γ is the time integral of pressure, analogous to momentum in mechanical systems. The increment in work passing through a fluid port in time dt may be written in terms of the power variables P and Q, and the integrated power variables Γ and V, in the following three forms:

$$\begin{array}{lllllll} \Delta W & = & Q(Pdt) & = & Qd\Gamma & = & d\mathcal{E}_{Kinetic} \\ \Delta W & = & P(Qdt) & = & PdV & = & d\mathcal{E}_{Potential} \\ \Delta W & = & PQdt & = & PQdt & = & d\mathcal{E}_{Dissipated} \end{array} \tag{88}$$

Work done by a fluid crossing a system boundary may result in:

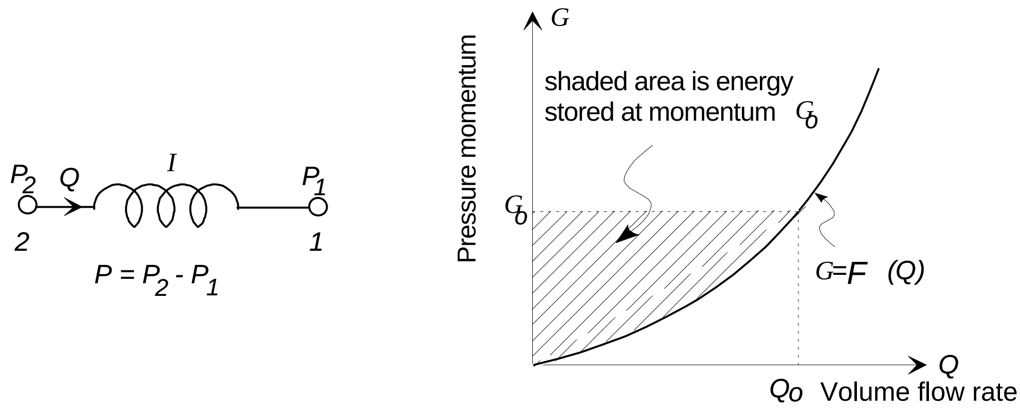

Figure 28: Definition of the pure fluid inertance element.

(i) a change in pressure momentum in the system which is usually associated with energy storage in a fluid inertance,

(ii) a change in the energy stored in a fluid volume which is usually associated with fluid capacitance, or

(iii) simply a change in pressure and flow rate representing dissipation in a fluid resistance in which fluid work is converted into heat.

5.2 Primitive Fluid Element Definitions

5.2.1 Energy Storage Elements

Fluid Inertance: A pure fluid *inertance* is defined as an element in which the pressure momentum is a single-valued, monotonic function of flow rate:

$$\Gamma = \mathcal{F}(Q) \tag{89}$$

A representation of this characteristic is shown in Fig. 28, where the energy stored in the inertance is indicated as determined from Eq. (88). The symbol for a fluid inertance shown in Fig. 28 is similar to that of a electrical inductor. When the relationship between pressure momentum and flow is linear, the constitutive equation (89) for an ideal inertance is:

$$\Gamma = IQ \tag{90}$$

where I is defined as the fluid inertance (N-s^2/m^5).

The energy stored in an ideal fluid inertance is:

$$\mathcal{E} = \int_0^\Gamma Q d\Gamma = \frac{1}{I}\int_0^\Gamma \Gamma d\Gamma = \frac{1}{2I}\Gamma^2 = \frac{1}{2}IQ^2 \tag{91}$$

The elemental equation for a fluid inertance may be determined by differentiating Eq. (90) to obtain a first order differential equation in terms of power variables:

$$P = I\frac{dQ}{dt} \tag{92}$$

The value of fluid inertance depends on the pipe geometry and fluid properties [9]. For an incompressible fluid flowing in a pipe of uniform area A and length l, with a uniform velocity profile, the

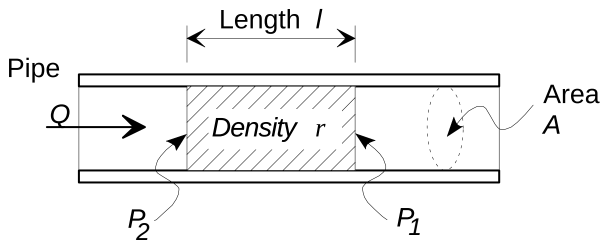

Figure 29: The fluid inertance of a pipe section.

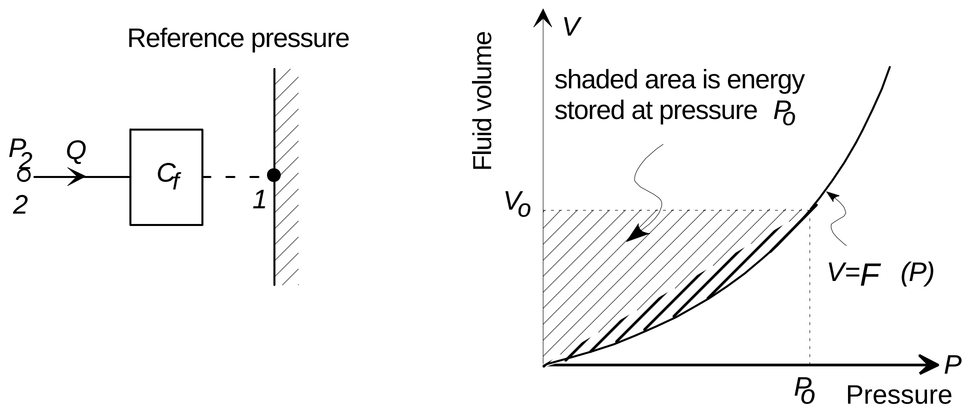

Figure 30: Definition of the pure fluid capacitance element.

element of fluid is accelerated by a force F equal to the pressure difference between the two ends of the pipe multiplied by pipe area, as shown in Fig. 29. The acceleration of this element of fluid is given by Newton's law:

$$F = AP = m\frac{dv}{dt} \tag{93}$$

The mass m of fluid in the pipe is the density of the fluid ρ multiplied by its volume $V = Al$, and the velocity is volume flow rate Q divided by the pipe area A, so that Eq. (93) may be expressed in terms of the fluid variables as:

$$P = \frac{1}{A}(\rho Al)\frac{1}{A}\frac{dQ}{dt} = \frac{\rho l}{A}\frac{dQ}{dt} \tag{94}$$

which is the same form as Eq. (92). The fluid inertance of the pipe is therefore

$$I = \frac{\rho l}{A} \tag{95}$$

The fluid inertance increases with increasing fluid density and pipe length and decreases with increasing area.

Fluid Capacitance: A pure fluid capacitance is defined as an element in which the volume of fluid stored is a single-valued, monotonic function of the pressure:

$$V = \mathcal{F}(P) \tag{96}$$

The constitutive characteristic of a pure fluid capacitor is illustrated in Fig. 30, where the energy stored in the capacitor is shown as the shaded area. The symbol for a fluid capacitor always has one

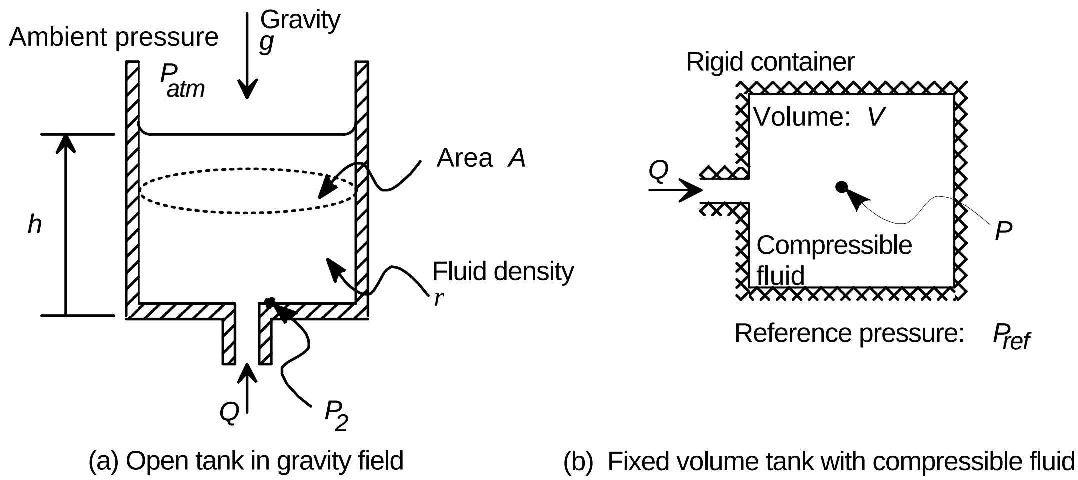

Figure 31: Two forms of fluid capacitance.

terminal connected to a fixed reference pressure because the pressure P and the energy E stored in a fluid capacitor are always measured with respect to a known pressure. For convenience the reference is usually selected as either atmospheric pressure or absolute zero pressure.

An ideal capacitor is one with a linear relationship between volume and pressure:

$$V = C_f P \tag{97}$$

where C_f is defined to be the fluid capacitance (m^5/N).

The energy stored in an ideal fluid capacitor is:

$$\mathcal{E} = \int_0^V P dV = \int_0^V \frac{1}{C_f} V dV = \frac{1}{2C_f} V^2 = \frac{1}{2} C_f P^2 \tag{98}$$

The elemental equation for an ideal fluid capacitor, expressed in terms of power variables, is found by differentiating Eq. (97):

$$Q = C_f \frac{dP}{dt} \tag{99}$$

The most common form of fluid capacitance is an open tank containing an incompressible fluid of density ρ in a gravity field g, as shown in Fig. 31. The pressure at the bottom of the tank is related to the depth, and therefore the volume, of fluid in the tank. The pressure due to the weight of a liquid column of height h is:

$$P = \rho g h \tag{100}$$

where P is the difference between the pressure at the bottom of the tank and atmospheric pressure. For a tank with uniform cross-section A, the volume of fluid is $V = Ah$ so that Eq. (100) may be written:

$$P = \frac{\rho g}{A} h A = \frac{\rho g}{A} V \tag{101}$$

or

$$V = \frac{A}{\rho g} P. \tag{102}$$

The fluid capacitance of the tank is therefore

$$C_f = \frac{A}{\rho g}. \tag{103}$$

Compressible fluids (fluids whose density varies with pressure) in a rigid wall container, as shown in Fig. 31, also exhibit fluid capacitance. For slightly compressible fluids, the relationship between changes in density and pressure may be expressed as:

$$\frac{d\rho}{\rho} = \frac{dP}{\beta} \tag{104}$$

where β is the fluid bulk modulus. The bulk modulus is a fluid property which expresses the degree of compressibility; for liquids such as oil and water it is large, on the order of 2.1×10^9 Pa (300,000 psi) and a significant change in pressure is required to change the fluid density, while for gases such as air the bulk modulus depends on the process employed to change the fluid state. For perfect gases the bulk modulus is:

$$\beta = kP \tag{105}$$

where k is a constant, and P is the absolute pressure of the gas.

The constant k is equal to one if the temperature remains constant as the gas changes state, and is larger (equal to 1.4 for air) when the process is adiabatic, that is with no heat transfer occurring between the gas and its environment [2]. The latter case is usually associated with rapid changes in the state of the gas which do not allow significant heat transfer to occur.

The capacitance of the rigid chamber may be determined from conservation of mass, by equating the net mass flow rate into the chamber to the change of mass within the chamber:

$$\rho Q = \frac{d}{dt}(\rho V) = V\frac{d\rho}{dt} + \rho\frac{dV}{dt} \tag{106}$$

Since the volume of the chamber is constant, the derivative of chamber volume with respect to time is zero and Eqs. (104) and (106) may be combined to yield:

$$Q = \frac{V}{\beta}\frac{dP}{dt} = C_f\frac{dP}{dt} \tag{107}$$

with the result:

$$C_f = \frac{V}{\beta} \tag{108}$$

5.2.2 Energy Dissipation Element

A pure fluid resistance is defined as an element in which the flow rate is a single-valued, monotonic function of the pressure drop across the element as indicated in Fig. 32.

$$Q = \mathcal{F}(P) \tag{109}$$

The symbol for a fluid resistor, shown in Fig. 32, has a pressure drop P across the terminals and a flow Q through the element. An ideal fluid resistor has a linear elemental relationship:

$$Q = \frac{1}{R_f}P \tag{110}$$

where R_f is defined to be the fluid resistance (N-sec/m^5).

The power dissipated by an ideal fluid resistor is:

$$\mathcal{P} = PQ = R_f Q^2 = \frac{1}{R_f}P^2 \tag{111}$$

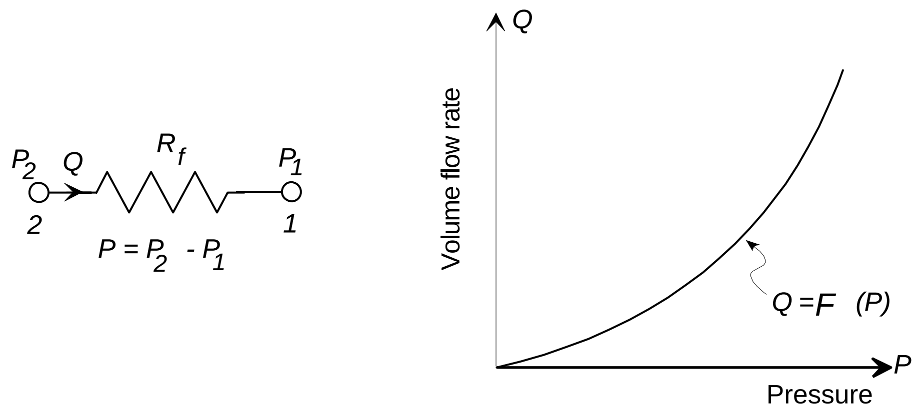

Figure 32: Definition of the pure fluid resistance element.

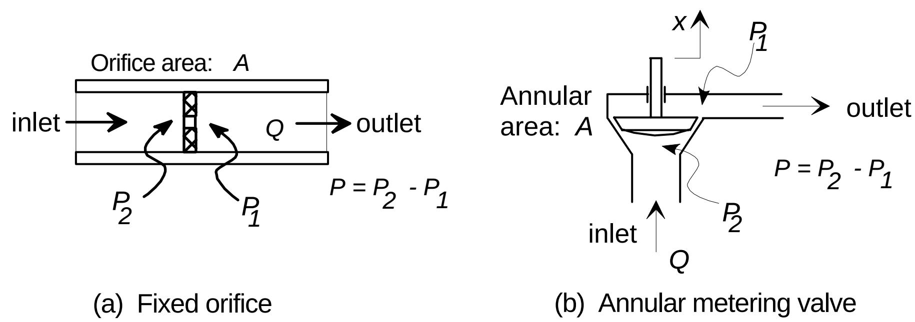

Figure 33: Fluid orifices and valve metering areas.

and is always positive representing the conversion of fluid work into heat.

Fluid resistance is associated with flow through pipes, orifices and valve openings. The value of resistance depends on geometry and fluid properties as well as flow conditions. For a long, uniform area circular pipe with laminar flow, the fluid resistance is given by the Hagen-Poiseuille flow law [8,9]:

$$R_f = \frac{128\mu l}{d^4} \tag{112}$$

where l is the pipe length, μ is the fluid viscosity, and d is the pipe diameter. This law assumes that the flow is laminar, which requires the flow to correspond to a Reynolds number R_y, defined as

$$R_y = \frac{4\rho}{\pi d\mu} Q, \tag{113}$$

be less than 2000. As the flow rate increases so that $R_y > 2000$, the flow in the pipe becomes turbulent and the pipe resistance becomes nonlinear and a function of flow.

For incompressible flows through orifices and valve openings, such as illustrated in Fig. 33, the orifice equation relating pressure and flow is of the form:

$$P = C_R Q\,|Q| \tag{114}$$

where the coefficient C_R is a function of the fluid density ρ, the orifice area A, and the orifice discharge coefficient C_d:

$$C_R = \frac{\rho}{2C_d^2 A^2}. \tag{115}$$

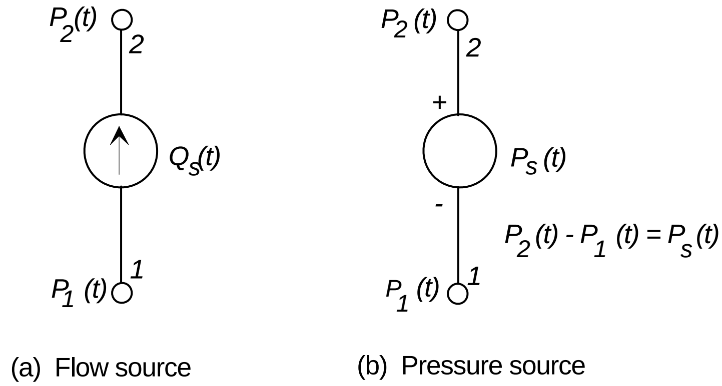

Figure 34: Ideal fluid sources.

The discharge coefficients C_d have been tabulated for orifices of different geometries [9].

5.2.3 Fluid Source Elements

Two ideal fluid sources are defined and are illustrated in Fig. 34. An ideal *pressure source* is an element in which the pressure applied to a port is an independently specified function of time $P_s(t)$. An ideal *flow source* is an element in which the flow through the port is an independently specified function of time $Q_s(t)$. Like the ideal sources in the other energy domains the fluid sources are not power limited.

■ Example

In many hydraulic systems, including automotive power steering and transmission systems, water distribution systems and medical assist systems, fluid is transmitted through pipes and passageways such as shown in Fig. 35. When the flow is time varying, the question of the relative importance of the resistive and inertance effects in the fluid passageway is important. In this example, we compute the fluid resistance and iner-

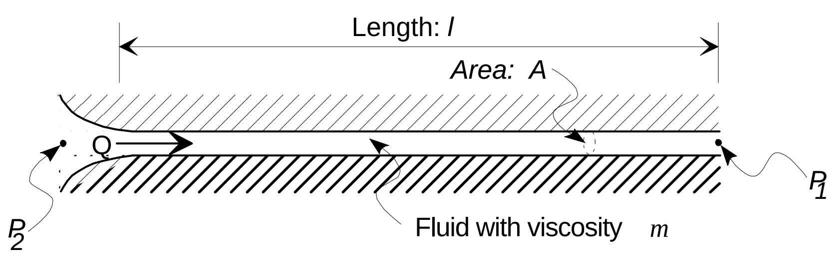

Figure 35: Long fluid passageway.

tance for a typical fluid passageway, and then the pressure drop due to the two effects is compared for sinusoidally varying flows. The angular frequency at which the magnitude of the resistive pressure drop is equal to that due to the passageway inertance is determined.

A pipe of diameter $d = 0.002$m and length of $l = 0.1$m is filled with a liquid which has fluid properties equivalent to water at room temperature, that is a density of 996 Kg/m^3

and a viscosity 7.98×10^{-4} N-s/m^2. The resistance of the passageway is computed from Eq. (112), with the assumption that the flow is laminar and that the effects of flow nonuniformity at the entrance and exit conditions of the passage may be neglected:

$$R_f = \frac{128\mu l}{\pi d^4} = \frac{128 \times 7.98 \times 10^{-4} \times 0.1}{3.14 \times 16 \times 10^{-12}} = 2.03 \times 10^8 \text{ N-s/m}^5. \tag{116}$$

The fluid inertance may be computed directly from Eq. (95) as:

$$I = \frac{\rho l}{A} = \frac{996 \times 0.1}{3.14 \times 10{-}6} = 5.08 \times 10^5 \text{ N-s}^2. \tag{117}$$

If the flow in the passageway is assumed to be sinusoidal with magnitude Q_0 and angular frequency ω, that is:

$$Q = Q_0 \sin(\omega t) \tag{118}$$

then the pressure drop due to resistance is:

$$P_r = R_f Q = R_f Q_0 \sin(\omega t) \tag{119}$$

and its magnitude is

$$|P_r| = R_f Q_0. \tag{120}$$

The pressure drop due to the inertance of the passageway is:

$$P_I = I\frac{dQ}{dt} = I\omega Q_0 \cos(\omega t) \tag{121}$$

and the magnitude is

$$|P_I| = I\omega Q_0 \tag{122}$$

The frequency ω_0 at which the magnitudes of the two pressure drops are equal is found from Eqs. (v) and (vii):

$$\omega_0 = \frac{R_f}{I} = \frac{2.03 \times 10^8}{5.08 \times 10^5} \approx 400\text{rad/s} \tag{123}$$

For sinusoidal flow variations with frequencies above 400 rad/s, the magnitude of the pressure drop due to the inertance effect is greater than the magnitude of the pressure drop due to passageway resistance. For slowly varying flows, with frequencies much less than 400 rad/s, the primary pressure drop is due to the passageway resistance.

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6 Thermal System Elements

6.1 Definition of Power Flow Variables

Thermal systems, in which heat is generated, stored, and transferred across boundaries, have historically been characterized in terms of thermal energy and power flow and the relationships of these variables to temperature. In the lumped parameter thermal system models considered in this text, the temperature T is selected as a fundamental thermal variable. The temperature of an object may be defined using several relative and absolute scales, including the Kelvin absolute temperature K associated with SI units and Rankine R absolute temperature associated with English units [10].

At absolute zero temperature, a body has no kinetic energy associated with its molecules. At standard pressure water freezes at a temperature of 273.2^oK or 0^oC, Celsius (centigrade), while at the boiling point of water the temperature is 373.2^oK or 100^oC.

When bodies at two different temperatures are brought into contact, the basic laws of thermodynamics indicate that heat, H, flows from the body at a higher temperature to the body at a lower temperature. Heat continues to flow until the bodies are at equal temperatures. Heat has been quantitatively defined in terms of pure substances. One *calorie* of heat is the amount of heat required to raise the temperature of one gram of water one degree Celsius (pure water at atmospheric pressure raised from 15^oC to 16^oC). Heat is a measure of thermal energy, and the rate of change of energy with respect to time is heat flow rate q, or power $\mathcal{P}$:

$$\mathcal{P} = q = \frac{dH}{dt} \tag{124}$$

where H is the heat (joules), and q is the heat flow rate, in units of joules/sec or watts. One calorie is equivalent to 4.187 joules.

The law of energy conservation, defined in Section 2, may be written specifically for thermal systems in a form that represents the first law of thermodynamics [2]:

$$q = \frac{d\mathcal{E}}{dt} - \mathcal{P}_w, \tag{125}$$

which states the net flow of heat across a system boundary must equal the sum of change in the internal system stored thermal energy and the heat converted into another form $\mathcal{P}_w$, such as mechanical, electrical or fluid work. The thermal power variable is simply heat flow rate q, and thermal energy is represented by heat H.

Unlike the mechanical, electrical, and fluid systems, power flow in thermal systems is not commonly described as a product of two variables. While mathematically a complementary pair of variables could be defined, physical observation of simple thermal systems of engineering interest has not identified a pair of complementary energy storage mechanisms. Thermal systems are represented with a set of pure elements which includes only one energy storage element.

6.2 Primitive Thermal Element Definitions

Thermal Capacitance: Thermal energy storage may be defined in terms of a pure thermal *capacitance* in which the heat H is a single-valued monotonic function of the temperature T.

$$H = \mathcal{F}(T) \tag{126}$$

A symbolic representation of the pure thermal capacitance and the constitutive relationship is illustrated in Fig. 36. The constitutive relationship is in general a function of geometry and material properties. For an ideal thermal capacitance, the relationship is linear:

$$H = C_t T \tag{127}$$

where C_t is the thermal capacitance (joule/^{o}K). The energy stored in a thermal capacitance is simply the heat H.

The elemental equation for the ideal thermal capacitance is derived by differentiating Eq. (127):

$$q = C_t \frac{dT}{dt} \tag{128}$$

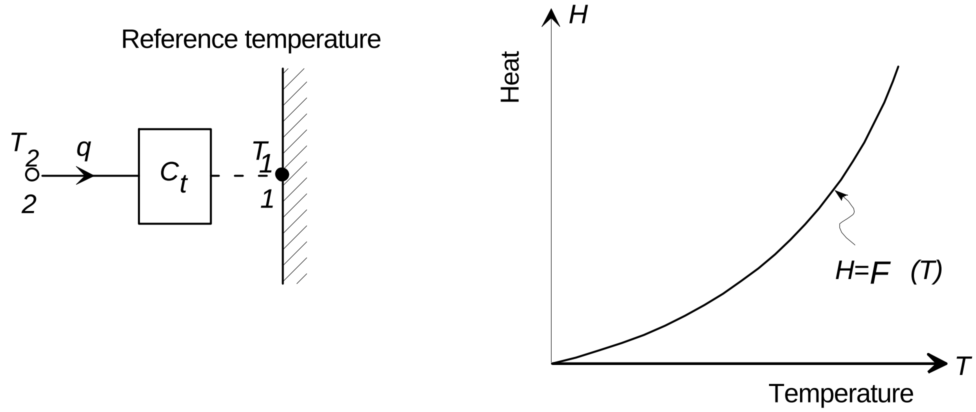

Figure 36: Definition of the pure thermal capacitance.

The symbol for the thermal capacitance is shown in Fig. 36, where the variables associated with the element are identified as heat flow H and temperature difference T across the terminals. Because the temperature associated with thermal energy must be referenced to a fixed temperature, one terminal of the symbol is shown as connected to a fixed reference. The reference temperature nominally is absolute zero (where the stored energy is zero); however in many engineering applications the temperature and energy difference from a specific fixed ambient level is of greater interest. In these cases a fixed non-zero temperature is often used as the reference. For example the temperatures and energy level in many systems is referenced to the ambient environmental temperature.

For an object made of a pure substance the thermal capacitance may be expressed as:

$$C_t = C_p m \tag{129}$$

where C_p is the *specific heat* of the substance, and m is the mass of substance present.

Thermal Inductance: No significant physical phenomenon has been observed which corresponds to energy storage due to heat flow in a "thermal inductor". Thus only one thermal energy storage element, the thermal capacitance, is defined.

6.2.1 Pure Thermal Resistance

Resistance to heat flow is characterized in terms of a pure thermal resistance, in which the heat flow is a single-valued monotonic function of the temperature difference T between two points:

$$q = \mathcal{F}(T). \tag{130}$$

A number of thermal phenomena are characterized by thermal resistance, including heat transfer by conduction, convection and radiation [10,11]. In an *ideal* thermal resistance the relationship between heat flow and the temperature difference is linear:

$$q = C_D T = \frac{1}{R_t} T \tag{131}$$

where C_D is the heat conductance (watt/^{o}K), and the reciprocal R_t is the thermal resistance (^{o}K/watt). or where

$$T = R_t q \tag{132}$$

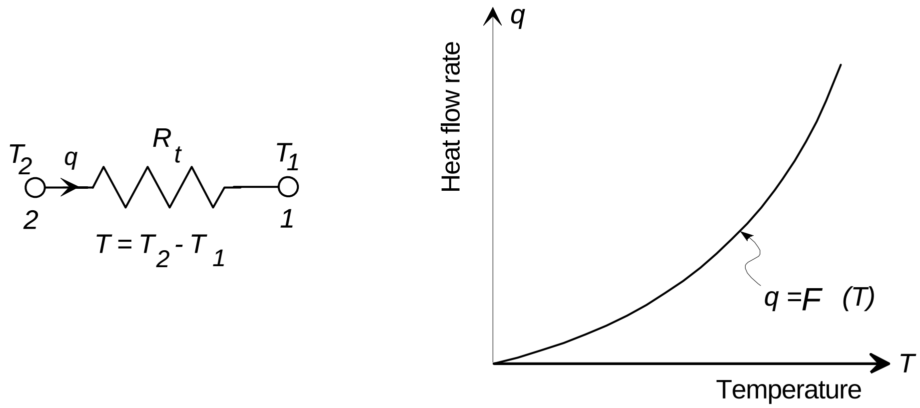

Figure 37: Definition of the pure thermal resistance.

with the ideal thermal resistance simply

$$R_t = \frac{1}{C_D}. \tag{133}$$

The symbol for thermal resistance is shown in Fig. 37 with the temperature difference T across the terminals and heat flow q identified as the variables associated with the element.

A pure thermal resistance neither stores nor dissipates energy since the net heat flow into the element is zero, that is the heat flow into the resistance equals the heat flow out of the element. The thermal resistance simply acts to *impede* heat flow and is associated with a change in system thermodynamic entropy [2]. The values of thermal resistance depend on material properties and geometry and are determined for three heat transfer mechanisms in the following paragraphs:

Conduction Heat Transfer: The conduction of heat through a material such as a metal block, was characterized by Fourier (1768-1830) [11] as:

$$q = C_D(T_2 - T_1) \tag{134}$$

where T_1 and T_2 are the temperatures of the two sides of the block. For a uniform material, the conduction constant C_D for the geometry shown in Fig. 38 is

$$C_D = \frac{\rho_c A}{l} \tag{135}$$

where ρ_c is the thermal conductivity, (watt/^{o}K), A is the area of body (m^2), and l is the length of body (m). The temperature difference $(T_2 - T_1)$ across the body is the driving force for the heat flux. Materials with high thermal conductivities, such as metals are classified as thermal conductors, while those with low conductivity such as wood or plastic foam are known as insulators. The thermal conductivity of various materials is tabulated in [11].

Conduction heat transfer occurs in solid, liquid, and gaseous states. It is usually the dominant form of heat transfer in stationary fluids and gases, and in solids at moderate temperatures. In moving fluids heat transfer by convection may dominate, and at high temperatures radiation heat transfer also may become important.

Convection Heat Transfer: In moving fluids heat may be transferred between spatially separated regions by bulk transport of the fluid. This mechanism is known as *convective heat*

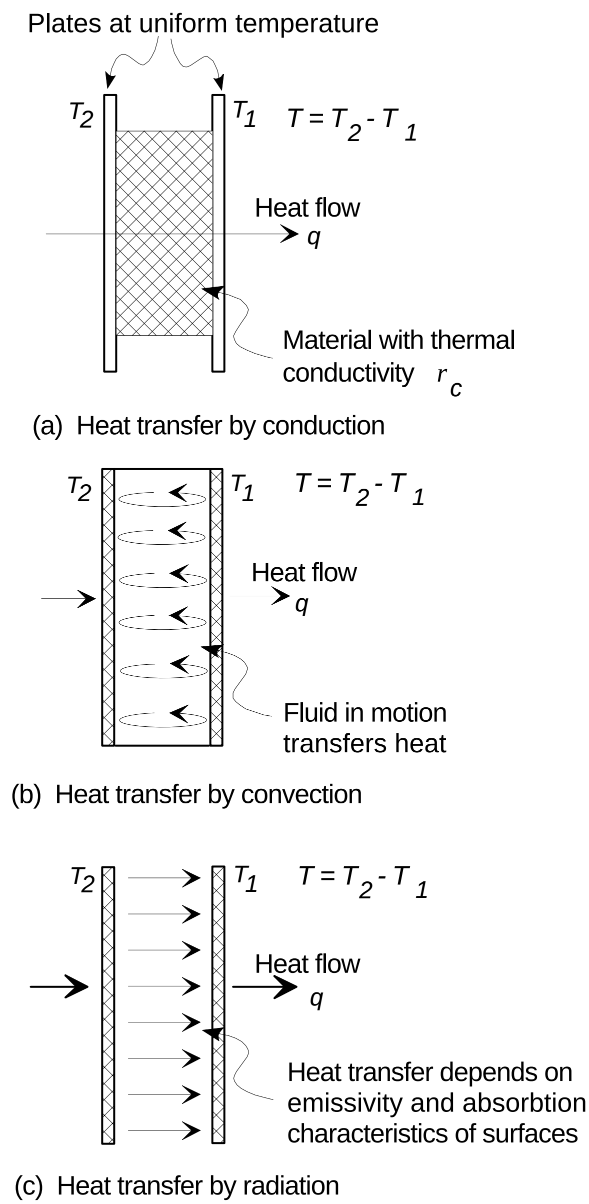

Figure 38: Modes of heat transfer

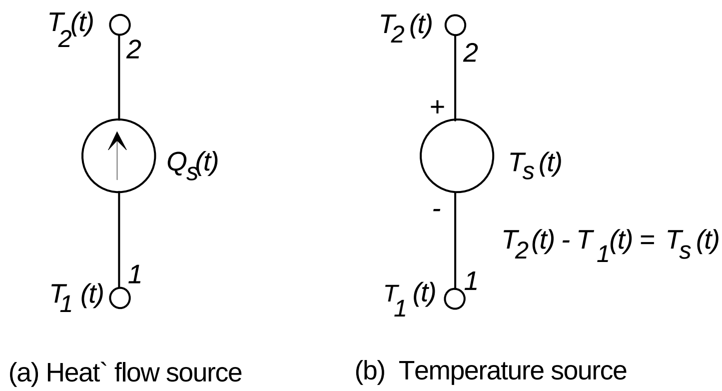

Figure 39: Ideal thermal source elements.

transfer, and is directly related to the fluid motion. Convection may be natural (when the fluid moves on its own as it is heated), or forced (when an external source is used to move the fluid). Heat transfer between a moving fluid and the walls containing the fluid has been studied extensively [11] and characterized by an overall convection heat transfer coefficient and the area of contact between the fluid and the walls:

$$q = C_h A \left(T_2 - T_1\right) \tag{136}$$

where A is the contact area (m^2), and C_h is the convection heat transfer coefficient (watt/m^2-^{o}K).

The equivalent thermal resistance associated with convection is:

$$R_t = \frac{1}{AC_h} \tag{137}$$

The convection heat transfer coefficient is strongly dependent on flow geometry and flow velocities.

Radiation Heat Transfer: Heat may also be transferred from one body to a second body at a distance by thermal radiation. This type of heat transfer is described by the Stefan-Boltzmann law [11]:

$$q = C_r \left(T_1^4 - T_2^4\right) \tag{138}$$

where C_r is the radiation heat transfer constant (watt/^{o}K^4), and T_1 and T_2 are the absolute temperatures of bodies 1 and 2, (^{o}K). The constant C_r depends on the geometry of the two surfaces and material properties which influence the absorption and emission characteristics of the bodies as a function of wavelength.

Radiation heat transfer is highly nonlinear as indicated in Eq. (138), and in fact cannot be defined in terms of pure thermal resistance since the heat transfer depends on the differences between absolute temperatures raised to the fourth power. Thus care must be used in modeling thermal systems in which heat transfer by radiation is significant, since radiation is not a direct function of the difference in temperatures.

6.2.2 Source Elements

Two ideal thermal sources may be defined as shown in Figure 39. An ideal *temperature source* is an element in which the temperature at a system port is an independently specified function of time $T_s(t)$, without regard to the heat flow necessary to maintain the temperature. An ideal *heat flow source* is an element in which heat flow is an independently specified function of time $Q_s(t)$. The temperature at the port is determined by the system connected to the heat flow source.

■ Example

An electrical resistor, with a resistance of 10 ohms and a thermal capacitance of 0.1 joule/^{o}K, is subjected to short duration (0.01 sec) voltage pulses of magnitude 100 volts, as shown in Figure 40. The peak temperature rise of the element, resulting from the

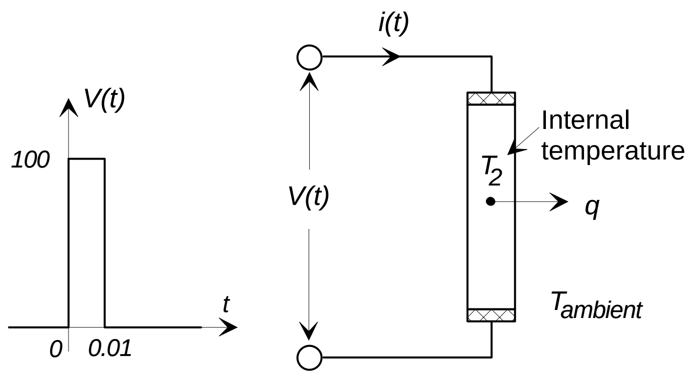

Figure 40: Temperature variation in an electrical resistor.

conversion of electrical energy to heat is to be determined . Also the time required for the element temperature to decay to within 1^{o}K of its original temperature is required, as an estimate of the minimum time period between pulses that will allow the temperature to moderate (so that the element will not burn out after a few pulses).

It is assumed that in the short period of the pulse duration no significant heat is transferred from the element to the environment. The temperature rise due to a single pulse may be then estimated by assuming that all of the electrical energy is converted into heat and stored in the element. During the pulse, the power $\mathcal{P}$ is constant

$$\mathcal{P} = \frac{1}{R}v^2 = \frac{100^2}{10} = 1000 \text{ watts} \tag{139}$$

The power flow occurs over a period of 0.01 seconds which generates a total electrical work of:

$$W = 10 \text{ joules.} \tag{140}$$

The heat transferred is therefore $H = 10$ joules. For the element the temperature rise ΔT with respect to the initial temperature is:

$$\Delta T = \frac{H}{C_t} = 10/0.1 = 100^o\text{K}. \tag{141}$$

The resistor's temperature is therefore raised 100^{o}K above the ambient environmental temperature.

After the rapid temperature rise, heat transfer occurs between the element and the environment. As stored heat is transferred to the environment, the temperature of the element decreases. The rate of the temperature decay may be found from Eq. (128):

$$C_t\frac{dT}{dt} = -q_e \tag{142}$$

where T is the temperature of the resistive element above the environmental temperature. Assume that the thermal resistance characterizing heat transfer to the surroundings is $R_t = 10\ ^o$K/watt:

$$q_e = \frac{T_r}{R_t} \tag{143}$$

Combining equations (iii) and (iv), the equation for the temperature difference as a function of time is obtained as:

$$C_t \frac{dT_r}{dt} = -\frac{T_r}{R_t}. \tag{144}$$

Equation (v) may be rearranged and integrated to determine the time required to reach $T_r = 1^o$K above the initial temperature:

$$\frac{dT_r}{T_r} = -\frac{dt}{R_t C_t} \tag{145}$$

$$\int_{T_0}^{T_f} \frac{dT_r}{T_r} = -\frac{1}{R_t C_t} \int_0^{t_f} 1 dt \tag{146}$$

$$\ln \frac{T_f}{T_0} = -\frac{t_f}{R_t C_t} \tag{147}$$

For $T_0 = 100^o$K and $T_f = 1^o$K, the time is:

$$t_f = -R_t C_t \ln \frac{T_f}{T_0} = -10 \times 0.1 \times \ln(.01) = 4.6 \text{ seconds.} \tag{148}$$

Thus it takes 4.6 seconds for the temperature of the resistor to return to within 1^oK of its original temperature.

■ References:

1. Paynter, H.M., *Analysis and Design of Engineering Systems*, The MIT Press, Cambridge, MA, 1961

2. Cravalho, E.G. and Smith, J.L., Jr., *Engineering Thermodynamics*, Pitman Publishing Co., Marshfield, MA, 1981.

3. Crandall, S.H., Karnopp, D.C., Kurtz, E.F., Jr. and Pridemore-Brown, D.C., *Dynamics of Mechanical and Electromechanical Systems*, McGraw-Hill Book Co., New York, NY, 1968.

4. Baumeister, et.al., *Handbook for Mechanical Engineers*, McGraw-Hill Book Co., New York, NY, 1967.

5. Roark, R.J. and Young, W.C., *Formulas for Stress and Strain*, McGraw-Hill Book Co., New York, NY, 1975.

6. Feyman, R.P., Leighton, R.B., Sareds, M., *The Feyman Lectures on Physics*, Addison-Wesley Publishing Co., Reading, MA, 1977.

7. Halliday, D. and Resnick, R. *Fundamentals of Physics*, Wiley and Sons, New York, NY, 1988.

8. Sabersky, R.H., Acosta, A.J., Hauptmann, E.G., *Fluid Flow: A First Course in Fluid Mechanics*, Macmillan Publishing Company, New York, NY, 1989.

9. Streeter, V.L., *Handbook of Fluid Dynamics*, McGraw-Hill Book Co., New York, NY, 1961.

10. Lienhard, J.H., *A Heat Transfer Textbook*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1981.

11. Ronsenow, W.M. and Hartnett, J.P., *Handbook of Heat Transfer*, McGraw-Hill Book Co., New York, NY, 1973.