Chapter VI : Robot dynamics

When a robot moves, a non-linear dynamic effects depending on the robot posture take place. It is necessary to know them in order to dimension the actuators and to calculate the control laws of the axes.

⇒ **Definition** We call Inverse dynamic model, the application that gives the actuators ~~effects~~ joint efforts Γ_m (forces or torques generated by the actuators) according to ~~position~~ joint variable q, velocities $\dot{q}$ and accelerations $\ddot{q}$, ~~visque~~ ~~viscous~~ friction effort Γ_f (viscous/ mdry) and interaction efforts Γ_e between the robot and its environment.

$$\Gamma_m = g(q, \dot{q}, \ddot{q}, \Gamma_f, \Gamma_e)$$

The inverse dynamic model is non-linear, coupled and dependents on the the robot postures and velocities.

The model can be obtained using the Euler-Lagrange ~~formalism~~ formulation or the fundamental principle of dynamics. (for more details about these formalisms, reader can refer to mechanical books).

⇒ Inverse dynamic model:

⇒ Lagrange formalism:

In general the lagrangian ~~formalism~~ formulation is defined as the difference between the kinematic energy K of the system (here robot) and the potential energy P of the system.

$$\boxed{L = K - P}$$

The Euler-Lagrange equations can be written as:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Gamma_i}$$

where Γ_i is the effort at the level of articulation i.

=> <u>kinematic energy calculation</u>:

The kinematic energy of link i can be written as:

$$K_i = \frac{1}{2} m_i \, {}^0V_{i/0}^T(G_i) \, {}^0V_{i/0}(G_i) + \frac{1}{2} \, {}^0\omega_{i/0}^T \, \bar{I}_i \, {}^0\omega_{i/0}$$

<u>where</u>:

- m_i = mass of link i
- G_i = inertial center of link i
- $\bar{I}_i$ inertial tensor according to point G_i

Using the kinematic model, we have:

$$ {}^0\mathcal{V}_{i/0}(P) = \begin{bmatrix} {}^0V_{i/0}(G_i) \\ {}^0\omega_{i/0} \end{bmatrix} = {}^0J_i^c(G_i, q)\dot{q}$$

<u>Therefore</u>:

$$K_i = \frac{1}{2}\dot{q}^T \underbrace{\left[{}^0J_i^c(G_i,q)\right]^T \begin{bmatrix} m_i & 0 & 0 & \phi_{1\times3} \\ 0 & m_i & 0 & \phi_{1\times3} \\ 0 & 0 & m_i & \phi_{1\times3} \\ \phi_{3\times1} & \phi_{3\times1} & \phi_{3\times1} & \bar{I}_i \end{bmatrix} {}^0J_i^c(G_i,q)}_{D_i(q)}\dot{q}$$

$$\boxed{K_i = \frac{1}{2}\dot{q}^T D_i(q)\,\dot{q}}$$

The total kinetic energy is:

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} \frac{1}{2} \dot{q}^T D_i(q) \dot{q}$$

or $$K = \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

where $D(q) = \sum_{i=1}^{n} D_i(q)$

⇒ Potential energy:

The potential energy of link i is:

$$P_i = m_i g \vec{z}_0 \, \vec{O_0 C_i}(q) / R_0 = m_i g z_{Ci} / R_0$$

with
- m_i mass of the link i
- g gravitation constant.
- C_{ni} inertial center of the link i

Total energy can be written as:

$$P = \sum_{i=1}^{n} P_i(q)$$

⇒ Dynamic model of the Robot:

Knowing the kinematic and the potential energy of the robot, we can define the lagrangian as:

$$\boxed{L = \frac{1}{2} \dot{q}^T D(q) \dot{q} - P(q)}$$

Then, we calculate the terms of the Lagrangian equations:

$$\frac{\partial L}{\partial \dot{q}_i} = [D(q)\dot{q}]_i$$

Then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left[D(q)\ddot{q} + \frac{dD(q)}{dt}\dot{q}\right]_i =$$

$$\left[D(q)\ddot{q} + \left(\sum_{k=1}^{n}\frac{\partial D(q)}{\partial q_k}\dot{q}_k\right)\dot{q}\right]_i$$

for the second term:

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\dot{q}^T\frac{\partial D(q)}{\partial q_i}\dot{q} - \frac{\partial P(q)}{\partial q_i}$$

finally, the Euler-Lagrange can be written as:

$$\boxed{D(q)\ddot{q} + C(q,\dot{q}) + Q(q) = \Gamma}$$

where: $D(q)\ddot{q}$ term stands for the inertia contribution.

$$D(q) = \sum_{i=1}^{n}\left[{}^{0}J_i^{c}(c_i,q)\right]^T \begin{pmatrix} m_i & 0 & 0 & \phi_{1\times3} \\ 0 & m_i & 0 & \phi_{1\times3} \\ 0 & 0 & m_i & \phi_{1\times3} \\ \phi_{3\times1} & \phi_{3\times1} & \phi_{3\times1} & \bar{I}_i \end{pmatrix} {}^{0}J_i^{c}(c_i,q)$$

- The term $C(q,\dot{q})$ stands for centrifugal and coriolis effects.

$$C(q,\dot{q}) = \left(\sum_{k=1}^{n} \frac{\partial D(q)}{\partial q_k}\dot{q}_k\right)\dot{q} - \begin{bmatrix} \frac{1}{2}\dot{q}^T \frac{\partial D(q)}{\partial q_1}\dot{q} \\ \vdots \\ \frac{1}{2}\dot{q}^T \frac{\partial D(q)}{\partial q_n}\dot{q} \end{bmatrix}$$

→ The term $Q(q)$ stands for the gravity contribution

$$Q(q) = \begin{bmatrix} \frac{\partial P(q)}{\partial q_1} \\ \vdots \\ \frac{\partial P(q)}{\partial q_n} \end{bmatrix} = \sum_{i=1}^{n} m_i g \begin{bmatrix} \frac{\partial z_{ci}/R_0}{\partial q_1} \\ \vdots \\ \frac{\partial z_{ci}/R_0}{\partial q_n} \end{bmatrix}$$

→ The joint efforts Γ correspond to the difference between the motors efforts and friction efforts and the interaction efforts Γ_e.

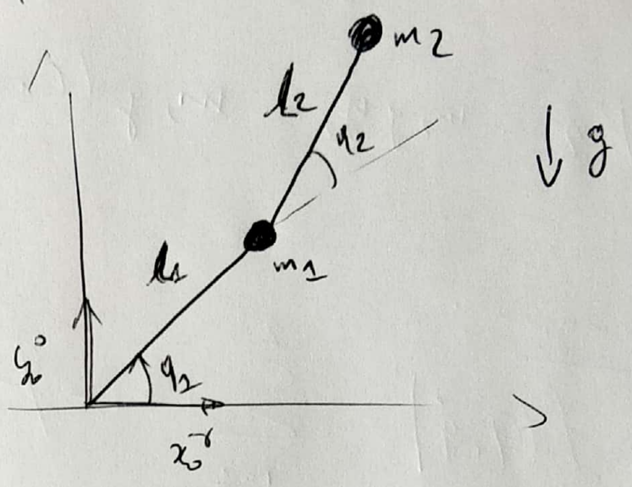
$$\boxed{\Gamma = \Gamma_m - \Gamma_e - \Gamma_f(q,\dot{q})}$$

→ Taking into account Γ we can write the inverse dynamic model:

$$\boxed{\Gamma_m = D(q)\ddot{q} + C(q,\dot{q}) + Q(q) + \Gamma_f(q,\dot{q}) + \Gamma_e}$$

example: planar robot

we assume that the mass is concentrated in points m_1 and m_2.



$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \cancel{\begin{bmatrix} L_1 \cos q_1 \\ L_1 \, s \cdot q_1 \end{bmatrix}}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \cancel{\begin{pmatrix} L_1 \, q_1 \\ L_1 \, S q_1 \end{pmatrix}} = \begin{bmatrix} L_1 \, c q_1 \\ L_1 \, s q_1 \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} -L_1 \, S q_1 \\ L_1 \, C q_1 \end{pmatrix} \dot{q}_1$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{bmatrix} L_1 \, c q_1 + L_2 \, c(q_1+q_2) \\ L_1 \, s q_1 + L_2 \, s(q_1+q_2) \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} -L_1 \, s q_1 - L_2 \, s(q_1+q_2) & -L_2 \, s(q_1+q_2) \\ L_1 \, c q_1 + L_2 \, c(q_1+q_2) & L_2 \, c(q_1+q_2) \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}.$$

Kinematic energy:

$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right) = \frac{1}{2} m L_1^2 \dot{\theta}_1^2$$

$$K_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

$$\boxed{K_2 = \frac{1}{2} m_2 \left((L_2^2 + 2 L_1 L_2 c q_2 + L_2^2) \dot{q}_1^2 + 2 (L_2^2 + L_1 L_2 c q_2) \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_2^2 \right).}$$

$$P_1 = m_1 g y_1 = m_1 g L_1 \sin q_1$$

$$P_2 = m_2 g y_2 = m_2 g \left(L_1 s q_1 + L_2 s(q_1+q_2) \right)$$

Lagrangian:

$$\mathcal{L}(q, \dot{q}) = \sum_{i=1}^{2} (K_i - P_i)$$

$$\Gamma_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \qquad i = 1, 2$$

$$\tau_1 = \boxed{\left(m_1 L_1^2 + m_2 (L_1^2 + 2 L_1 L_2 c q_2 + L_2^2)\right) \ddot{q}_1 + m_2 (L_1 L_2 c q_2 + L_2^2) \ddot{q}_2}$$
$$\boxed{- m_2 L_1 L_2 s q_2 (2 \dot{q}_1 \dot{q}_2 + \dot{q}_2^2)}$$
$$\boxed{+ (m_1 + m_2) L_1 g \cdot c q_1 + m_2 g L_2 c(q_1+q_2)}$$

$M(q, \ddot{q})$ mass matrix

$C(q, \dot{q})$ velocity product term

$g(q)$ gravity term

$$\tau_2 = m_2 (L_1 L_2 c q_2 + L_2^2) \ddot{q}_1 + m_2 L_2^2 \ddot{q}_2$$
$$+ m_2 L_1 L_2 \dot{q}_1^2 s q_2$$
$$+ m_2 g L_2 c(q_1+q_2).$$