Sheet 2 - The Rule of Two, Episode II
(Due: 08.11.2021, 18:00)

To accompany the exercises below, we have prepared a Jupyter-Python notebook for you under this link:

https://mybinder.org/v2/git/https%3A%2F%2Fgitlabph.physik.fu-berlin.de%2Fmattkrauss%2Fbinder_repo_aqm_ii_2122/main?urlpath=doc/tree/Sheet2.ipynb

For some of the problems on this sheet you should use this notebook for assistance. These instances are indicated by the keyword "**investigate**". Please do not hand in the Jupyter-Python notebooks - rather, describe the results of your investigations in your submitted solutions, for example by screenshotting and pasting some of the figures you obtain and by explaining your observations with words.

Exercise 1 - Tricks of the Trade II (10 points)

(a) Show that for any $x \in \mathbb{R}$ and linear operator $\hat{A}$ with $\hat{A}^2 = \hat{\mathbb{1}}$, the following relation holds, [4P]

$$e^{i\hat{A}x} = \cos(x)\,\hat{\mathbb{1}} + i\sin(x)\,\hat{A}\,.$$

Hint: On the previous sheet we learned that general functions on operators can be evaluated via the spectral decomposition. In this exercise a different approach is a little easier, namely directly using the definition of the exponential, lifted to the operator level, i.e., $e^{i\hat{A}x} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\hat{A}x\right)^n$.

(b) Show that the operator $\hat{\sigma}_{\vec{n}} \equiv \vec{n}\cdot\hat{\vec{\sigma}} \equiv n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z$ with $\vec{n} \in \mathbb{R}^3,\ \|\vec{n}\| = 1$ posseses the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ for arbitrary $\vec{n}$. [4P]
Hint: Show first that $\hat{\sigma}_{\vec{n}}$ is both Hermitian and unitary. Which eigenvalues are permissable for such operators?

(c) **Investigate** for the three examples $\vec{n}_1 = (0,1,0)$, $\vec{n}_2 = \left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$ and $\vec{n}_3 = \frac{1}{\sqrt{3}}(1,1,1)$, in how far the directions of the eigenvectors of $\hat{\sigma}_{\vec{n}}$, $|\psi_{\pm\vec{n}}\rangle$, correspond to the directions $\pm\vec{n}$ on the Bloch sphere. [2P]

Exercise 2 - Controlled Rotations (28 points)

Remark: For simplicity, we use $\hbar = 1$ throughout this exercise.

(a) Prove that the eigenvectors of $\hat{\sigma}_{\vec{n}}$, $|\psi_{\pm\vec{n}}\rangle$, point along the directions $\pm\vec{n}$ on the Bloch sphere. [5P]

(b) Prove the following general expression for the exponential of a linear combination of Pauli matrices with normalised coefficients,

$$\hat{R}_{\vec{n}}(\phi) \equiv e^{-i\frac{\phi}{2}\left(\vec{n}\cdot\hat{\vec{\sigma}}\right)} = \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)\left(\vec{n}\cdot\hat{\vec{\sigma}}\right), \tag{1}$$

with $\vec{n}\cdot\hat{\vec{\sigma}} \equiv n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z$ and $\|\vec{n}\| = 1$. [2P]

(c) Use the results from $1\,(b)$ and $1\,(c)$ to show that $\hat{R}_{\vec{n}}(\phi)$ rotates a state on the Bloch sphere by an angle ϕ around the $\vec{n}$ direction. [5P]
Hint: An arbitrary state on the Bloch sphere can also be written in rotated spherical coordinates with polar angle $\theta_{\vec{n}}$ and azimuthal angle $\varphi_{\vec{n}}$, where the $\vec{n}$ direction takes the role of the z axis in conventional spherical coordinates,

$$|\psi\rangle = \cos\tfrac{\theta_{\vec{n}}}{2}\,|\psi_{\vec{n}}\rangle + e^{i\varphi_{\vec{n}}}\sin\tfrac{\theta_{\vec{n}}}{2}\,|\psi_{-\vec{n}}\rangle\ .$$

Think about how a rotation around the $\vec{n}$ axis should behave in this representation and show that $\hat{R}_{\vec{n}}(\phi)$ indeed has this property.

(d) Consider a Hamiltonian of the form

$$\hat{H} = \frac{1}{2}\alpha_x\hat{\sigma}_x + \frac{1}{2}\alpha_y\hat{\sigma}_y + \frac{1}{2}\alpha_z\hat{\sigma}_z\,, \tag{2}$$

with $\alpha_x, \alpha_y, \alpha_z \in \mathbb{R}$. Use Eq. (1) to derive an explicit expression for the propagator $\hat{U}(t) = e^{-i\hat{H}t}$. [2P]

(e) We now introduce a time dependence in the Hamiltonian from Eq. (2) by replacing the constants α_i with time-dependent functions $\alpha_i(t)$. Furthermore, we assume that these time-dependent functions are piecewise constant and can only take on the values of 0 or 1 at a given time. At $t = 0$ we start in the state $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Find solutions for the time-dependent functions $\alpha_i(t)$ such that we arrive - for some $T > 0$ - in one of the following two states, respectively, ...

(i) $|\psi_1(T)\rangle = |1\rangle$.

(ii) $|\psi_2(T)\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$.

Investigate the states' trajectories on the Bloch sphere and sketch / screenshot your solutions. [4P]

(f) Explain how to obtain an *arbitrary state* $|\psi(T)\rangle \in \mathcal{H}_2$ for some $T > 0$ from $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ with the piecewise-constant time-dependent approach from (e). Describe your general protocol in words and **investigate** your solution for a few examples via their trajectories on the Bloch sphere. [4P]

(g) We now assume that we only have access to two of the three Paulis in Eq. (2) via the additional constraint $\forall t: \ \alpha_y(t) = 0$. In other words, you are not allowed to use the $\hat{\sigma}_y$ term in the Hamiltonian! Repeat exercises (e) and (f) under this constraint. [6P]

Remark: What you have observed in this exercise, is the so-called "state controllability" of a qubit with two of the three Pauli matrices only. Two Paulis are sufficient to reach any state because the so-called "dynamical Lie algebra" is complete as soon as it is generated by two of the three Paulis - their commutator always yields the third. Studying the controllability of quantum mechanical systems is a crucial step to understand whether certain tasks can be achieved with a given physical implementation, see, e.g. the paper "Fundamental bounds on qubit reset", published as Phys. Rev. Research **3**, 013110 (2021), *about how to initialise qubits in a quantum computer.*

Exercise 3 - The Cat in the Hat Came Back (22 points + 5 extra points)

The so-called Wigner distribution $W(x,p)$,

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \psi^*(x-y)\,\psi(x+y)\,e^{\frac{2ipy}{\hbar}}\ dy$$

is a very useful tool to visualise the dynamics of a particle in one dimension by associating any wave function $\psi(x)$ to a distribution with respect to phase space coordinates, $(x,p) \in \mathbb{R}^2$.

(a) Show that the integral of the Wigner distribution over all p yields the probability $P_x(x)$ to find the particle at position x, i.e., $P_x(x) = \int_{-\infty}^{\infty} W(x,p)\ dp$. [3P]

(b) Show that the integral of the Wigner distribution over all x yields the probability $P_p(p)$ to find the particle at position p, i.e., $P_p(p) = \int_{-\infty}^{\infty} W(x,p)\ dx$. [5P]
Hint: Remember that the wave function with respect to position, $\psi(x)$, is connected to the wave function with respect to momentum, $\tilde{\psi}(p)$, via the Fourier transform,

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \tilde{\psi}(p)\,e^{-\frac{i}{\hbar}px}\ dp\,.$$

(c) **Investigate** the Wigner distribution of Fock states $|n\rangle$. Describe your observations, in particular with respect to the role of n. [2P]

(d) **Investigate** the Wigner distribution of an equal superposition between two Fock states, i.e., a state of the form $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|n_1\rangle + e^{i\varphi}|n_2\rangle\right)$. Describe how the relative phase φ affects the Wigner distribution. Would such a distribution have a classical phase space analogue? Justify your answer. [4P]

(e) **Investigate** the Wigner distribution of coherent states $|\alpha\rangle$. Describe in how far the Wigner distribution changes when you modify the modulus and phase of α. [2P]

(f) On the previous sheet we introduced coherent cat states. Their definition can be further generalised as follows,

$|$  $_{\alpha,\theta}\rangle = \frac{1}{\sqrt{2\left(1+e^{-2|\alpha|^2}\cos\theta\right)}}\left(|\alpha\rangle + e^{i\theta}|-\alpha\rangle\right)$.

Investigate the Wigner distribution of coherent cats $|$  $_{\alpha,\theta}\rangle$. Describe how the relative phase $\theta \in [0, 2\pi)$ affects the Wigner distribution. [2P]

(g) Explain why, for fixed $\alpha \neq 0$, all coherent cats $\{|$  $_{\alpha,\theta}\rangle\}$ with $\theta \in [0, 2\pi)$ are elements of a two-dimensional subspace of the harmonic oscillator's Hilbert space. Determine an orthornormal basis of this subspace which consists of two generalised coherent cats. [4P]

(h) `Bonus exercise:` Show that, for $\alpha \neq 0$, for each coherent cat $|$  $_{\alpha,\theta}\rangle$ with angle θ there exists exactly one coherent cat $|$  $_{\alpha,\theta'}\rangle$ with angle θ' such that $\langle$  $_{\alpha,\theta}|$  $_{\alpha,\theta'}\rangle = 0$. Denote an explicit formula for the relation between θ and θ'. **Investigate** your expression for different pairs of θ and θ' to check your result. [5XP]