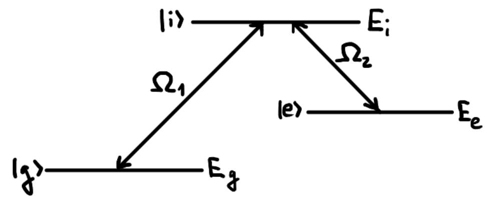
Sheet 1 - The Rule of Two, Episode I
(Due: 01.11.2021, 18:00)

Exercise 1 - Tricks of the Trade I (12 points)

(a) Show that on a Hilbert space $\mathcal{H}$, any unitary operator $\hat{U}:\mathcal{H}\mapsto\mathcal{H}$ preserves norms, i.e. $\forall\,|\psi\rangle\in\mathcal{H}$ the relation $\|\hat{U}\,|\psi\rangle\,\|=\|\,|\psi\rangle\,\|$ holds. Moreover, show that all eigenvalues of $\hat{U}$ lie on the complex unit circle. [2P]

(b) Consider the so-called Λ-system illustrated in the figure on the right (for simplicity all displayed quantities can be assumed to be real-valued). Write the Hamiltonian $\hat{H}$ of this system both as an abstract Hilbert-space operator involving the sum of dyadic products and as a marix with respect to the basis $\{|g\rangle\,,|i\rangle\,,|e\rangle\}$. [4P]
Hint: Horizontal lines indicate (field-free) energy levels, e.g., $\langle g|\hat{H}|g\rangle=E_g$. Connecting arrows indicate couplings between these levels, e.g., $\langle g|\hat{H}|i\rangle=\Omega_1$.



(c) Consider the Hamiltonian $\hat{H}=\alpha\,|g\rangle\langle g|+\beta\,|e\rangle\langle e|$ $(\alpha,\beta\in\mathbb{R})$ on a Hilbert space $\mathcal{H}$ with orthonormal basis states $\{|g\rangle\,,|e\rangle\}$. Write an explicit expression for the time-evolution operator $\hat{U}=e^{-\frac{i}{\hbar}\hat{H}t}$ as a sum of dyadic products on $\mathcal{H}$. [2P]

(d) The *Heisenberg equation of motion* allows to find the time evolution of expectation values via an ordinary first-order differential equation. Specifically, given a Hamiltonian $\hat{H}$ and an operator $\hat{A}$ without explicit time dependence, the expectation value $\langle\hat{A}\rangle_{|\psi(t)\rangle}\equiv\langle\psi(t)\,|\hat{A}|\psi(t)\rangle$ behaves as

$$\frac{d}{dt}\langle\hat{A}\rangle_{|\psi(t)\rangle}(t)=\frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle_{|\psi(t)\rangle}\,.$$

Which statements can you make if the commutator $[\hat{H},\hat{A}]$ vanishes? How does the solution for $\langle\hat{A}\rangle_{|\psi(t)\rangle}(t)$ look if $[\hat{H},\hat{A}]=\hbar\kappa\hat{A}$ for some parameter $\kappa\in\mathbb{C}$? [4P]

Exercise 2 - Like a Hamster in the Wheel (22 points)

From a quantum information point of view, the most fundamental quantum object is the two-level system, or *qubit*. Any such qubit can be mathematically described by a Hilbert space $\mathcal{H}_2$, formed by the so-called *canonical basis states* $|0\rangle$ and $|1\rangle$.

(a) Qubits can be physically realised through a wide variety of implementations. Provide two examples for such a realisation and explain which physical states of the system are encoded in the canonical basis. [2P]
Hint: You are welcome to research the internet for examples! That being said, providing references for your sources is a critical part of good scientific practice.

(b) The Hilbert space $\mathcal{H}_2$ contains fewer states than the "naive" vector space $\mathbb{C}^2\equiv\mathrm{span}_{\mathbb{C}}\{|0\rangle\,,|1\rangle\}$. Show that without any loss of physical generality, the state of a qubit can be written as $(\theta,\varphi\in\mathbb{R})$

$$|\psi\rangle=\cos\tfrac{\theta}{2}\,|0\rangle+e^{i\varphi}\sin\tfrac{\theta}{2}\,|1\rangle\,. \tag{1}$$

Compare the number of real-valued parameters in this expression with the number of real-valued parameters required to describe a general vector in $\mathbb{C}^2$. Explain the physical origin for the discrepancy. [3P]
Hint: How would a state's global phase $\chi\in[0,2\pi)$, i.e., $|\psi\rangle\longrightarrow e^{i\chi}\,|\psi\rangle$, affect expectation values?

(c) The three Pauli matrices (here and in the following interpreted as matrices written with respect to the canonical basis),

$$\hat{\sigma}_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\hat{\sigma}_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\hat{\sigma}_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

allow to completely characterise the state of a qubit. Show that for the state in Eq. (1) one obtains [4P]

$$\langle\hat{\sigma}_x\rangle_{|\psi\rangle}=\sin\theta\cos\varphi\,,\quad\langle\hat{\sigma}_y\rangle_{|\psi\rangle}=\sin\theta\sin\varphi,\quad\langle\hat{\sigma}_z\rangle_{|\psi\rangle}=\cos\theta\,. \tag{2}$$

(d) Due to Eq. (2), the state of a qubit can be bijectively mapped onto a point on the surface of the three-dimensional unit sphere, where - as it is usually done in physics - θ is the polar angle and φ is the azimuthal angle. This is known as the *Bloch sphere representation.* Which (sets of) states correspond to (i) the north pole, (ii) the south pole, and (iii) the equator of the Bloch sphere? [2P]

(e) Consider the following Hamiltonian,
$$\hat{H} = \frac{\hbar\Omega_0}{2}\hat{\sigma}_x + \frac{\hbar\Delta}{2}\hat{\sigma}_z \,. \tag{3}$$
Write this Hamiltonian as a matrix with respect to the canonical basis and determine its eigenstates and eigenvalues. [3P]

(f) Show that, given the Hamiltonian $\hat{H}$ in Eq. (3) for $\Delta = 0$, a state with $|\psi(t=0)\rangle = |0\rangle$ transforms under time evolution as follows,
$$|\psi(t)\rangle = \cos\left(\frac{\Omega_0 t}{2}\right)|0\rangle - i\sin\left(\frac{\Omega_0 t}{2}\right)|1\rangle \,.$$
Sketch and describe the state's evolution on the Bloch sphere. For which times t is the initial state $|0\rangle$ transformed to the state $|1\rangle$. At which times t does the initial state $|0\rangle$ return to itself? [8P]

Exercise 3 - Like a Cat on a Swing (26 points)

In this exercise we consider a one-dimensional harmonic oscillator with frequency ω, corresponding to the Hamiltonian
$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{4}$$
where $\hat{a}$ is the so-called annihilation operator, and $\hat{a}^\dagger$ is its Hermitian adjoint, the so-called creation operator.

(a) Show that the so-called number operator, $\hat{n} \equiv \hat{a}^\dagger\hat{a}$, is Hermitian and all its eigenvalues are non-negative. [2P]

(b) The Hamiltonian $\hat{H}$ acts on the infinite-dimensional Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. An orthonormal basis of $\mathcal{H}$ is given by the infinite set $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$, the so-called *Fock states*, with $\hat{a}|0\rangle = 0$ and
$$\forall n \neq 0: \;\; \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \,. \tag{5}$$
Show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. [2P]
Hint: Multiply Eq. (5) *from the left with* $|l\rangle$ *for an arbitrary* $l \in \mathbb{N}$.

(c) Show that the expectation values of position and momentum in a Fock state vanish, i.e., $\langle\hat{x}\rangle_{|n\rangle} = \langle\hat{p}\rangle_{|n\rangle} = 0$. [2P]
Hint: In suitable units (which we will be using in this exercise) the relations $\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$ *and* $\hat{p} = \frac{1}{\sqrt{2}i}\left(\hat{a} - \hat{a}^\dagger\right)$ *hold.*

(d) Show that for an arbitrary $\alpha \in \mathbb{C}$ the state $|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}C_\alpha|n\rangle$ is an eigenstate of $\hat{a}$ - these states are called *coherent states*. Determine the corresponding eigenvalue λ_α and the normalization constant C_α. Why are the eigenvalues of the annihilation operator not necessarily real-valued? [5P]

(e) Show that the expectation values of position and momentum operator in a coherent state with $\alpha = |\alpha|e^{i\varphi}$ take on the values [3P]
$$\langle\hat{x}\rangle_{|\alpha\rangle} = \sqrt{2}\,|\alpha|\cos(\varphi) \,, \quad \langle\hat{p}\rangle_{|\alpha\rangle} = \sqrt{2}\,|\alpha|\sin(\varphi) \,.$$

(f) Consider the time evolution of the harmonic oscillator according to the Hamiltonian in Eq. (4) via the Heisenberg equation of motion for an initial coherent state, $|\alpha(t=0)\rangle = |\alpha\rangle$, with $\alpha = |\alpha|e^{i\varphi}$. Show that the time-dependent expectation values of position and momentum in this state obey the following equations, [5P]
$$\langle\hat{x}\rangle_{|\alpha(t)\rangle} = \sqrt{2}\,|\alpha|\cos(\varphi - \omega t) \,, \quad \langle\hat{p}\rangle_{|\alpha(t)\rangle} = \sqrt{2}\,|\alpha|\sin(\varphi - \omega t) \,.$$

(g) A superposition of two coherent states with opposite sign is called a *coherent cat state*,

$|$$_\alpha\rangle = N_\alpha\left(|\alpha\rangle + |-\alpha\rangle\right)$.

Usually in quantum mechanics, a cat state is a superposition of two orthogonal states. Is this also true for coherent cat states? If not, is it approximately true for some values of α? Finally, determine the normalisation constant N_α. [4P]

(h) Consider a cat state in the harmonic oscillator according to Eq. (4) at time zero, i.e., $|\psi(t=0)\rangle = |$$_\alpha\rangle$. Calculate the time-dependent expectation values of position and momentum operator in this state. [3P]
Hint: The calculation and result for this exercise is comparatively boring. Looking only at the two expectation values $\langle\hat{x}\rangle$ *and* $\langle\hat{p}\rangle$ *evidently does not reveal what is actually going on in this continuous-variable system with its infinite-dimensional Hilbert space. On the next sheet, we will learn about an interesting way to visualise the state of a harmonic oscillator!*