Sheet 4 - From A Different Angle
(Due: 22.11.2021, 18:00)

Exercise 1 - Tricks of the Trade IV (12 points)

On this sheet - just like in the lecture - we denote the three cartesian components of an angular momentum operators for a general spin-j system by $\hat{J}_x^{(j)}$, $\hat{J}_y^{(j)}$, and $\hat{J}_z^{(j)}$. As it is often done in the literature, we use the word "spin" in this context for both integer and noninteger values of j. The canonical basis for the Hilbert space $\mathcal{H}_j$ of a spin-j system is given by the states $|j; m_j\rangle$ where $\hbar m_j$ is the eigenvalue of the state $|j; m_j\rangle$ with respect to the $\hat{J}_z^{(j)}$ operator.

(a) The y component of angular momentum for $j = 1$, $\hat{J}_y^{(1)}$, takes on the following form in the canonical basis $\{|1;1\rangle, |1;0\rangle, |1,-1\rangle\}$,

$$\hat{J}_y^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} .$$

Show that [6P]

$$e^{-\frac{i}{\hbar}\beta \hat{J}_y^{(1)}} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} .$$

Hint: The eigenstates $\{|+\rangle, |0\rangle, |-\rangle\}$ of $\hat{J}_y^{(1)}$, corresponding to the eigenvalues $\{\hbar, 0, -\hbar\}$, are given by

$$|\pm\rangle = \frac{1}{2}\left(|1;1\rangle \pm \sqrt{2}i\,|1,0\rangle - |1;-1\rangle\right), \qquad |0\rangle = \frac{1}{\sqrt{2}}\left(|1;1\rangle + |1;-1\rangle\right) .$$

(b) Using the formula for the matrix elements $D^{(1)}_{mm'}(\alpha,\beta,\gamma) = \langle 1; m'| e^{-\frac{i}{\hbar}\alpha \hat{J}_z^{(1)}} e^{-\frac{i}{\hbar}\beta \hat{J}_y^{(1)}} e^{-\frac{i}{\hbar}\gamma \hat{J}_z^{(1)}} |1; m\rangle$ of the rotation matrix $D^{(1)}(\alpha,\beta,\gamma)$, show that the following relation holds, [2P]

$$D^{(1)}(\alpha,\beta,\gamma) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} e^{i(\gamma-\alpha)} & \frac{1-\cos\beta}{2} e^{2i(\gamma-\alpha)} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i(\gamma-\alpha)} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} e^{i(\gamma-\alpha)} \\ \frac{1-\cos\beta}{2} e^{-2i(\gamma-\alpha)} & \frac{\sin\beta}{\sqrt{2}} e^{-i(\gamma-\alpha)} & \frac{1+\cos\beta}{2} \end{pmatrix} . \tag{1}$$

(c) Prove the following statement: "A state $|\psi\rangle \in \mathcal{H}$ is an eigenstate to the observable $\hat{A} : \mathcal{H} \mapsto \mathcal{H}$ if and only if the expectation value of the standard deviation in this state, $\langle \Delta \hat{A}\rangle_{|\psi\rangle}$, vanishes." [4P]

Exercise 2 - The Sum of its Parts (28 points)

For the first part of this exercise, we consider the Hilbert space of two spin-$\frac{1}{2}$ particles, which is given by the tensor product Hilbert space $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$. Furthermore, we use the popular notation $|\frac{1}{2}; \frac{1}{2}\rangle \equiv |\uparrow\rangle$ and $|\frac{1}{2}; -\frac{1}{2}\rangle = |\downarrow\rangle$ for the canonical basis of $\mathcal{H}_{\frac{1}{2}}$. We use the small letter s for the angular momentum operators on an individual spin-$\frac{1}{2}$ system, i.e., $\hat{J}^{(\frac{1}{2})}_{x/y/z} \equiv \hat{s}_{x/y/z}$, and the capital letter S for operators on the two-particle system.

(a) Show, that the product states $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ are eigenstates to the total spin operator $\hat{S}_z$ in z-direction given by $\hat{S}_z = \hat{s}_z \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \hat{s}_z$. Determine the corresponding eigenvalues. [2P]
Hint: The $\hat{s}_i$ operators for a spin-$\frac{1}{2}$ system are rescaled Pauli matrices, $\hat{s}_x = \frac{\hbar}{2}\hat{\sigma}_x$, $\hat{s}_y = \frac{\hbar}{2}\hat{\sigma}_y$, and $\hat{s}_z = \frac{\hbar}{2}\hat{\sigma}_z$.

(b) Check, whether the four product states from exercise (a) are eigenstates to the total spin operator $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$, where $\hat{S}_x$ and $\hat{S}_y$ are defined analogously to $\hat{S}_z$. [6P]

(c) Consider a basis of $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$ given by the following four states,

$$|\mathfrak{s}\rangle \equiv \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right], \qquad |\mathfrak{t}_+\rangle \equiv |\uparrow\uparrow\rangle, \qquad |\mathfrak{t}_0\rangle \equiv \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right], \qquad |\mathfrak{t}_-\rangle = |\downarrow\downarrow\rangle .$$

Check, whether these four states are eigenstates to the operators $\hat{S}^2$ and $\hat{S}_z$ and determine the corresponding eigenvalues. [4P]

(d) The action of a general spin-j operator is given by

$$\hat{J}_z^{(j)} |j; m_j\rangle = \hbar m_j |j; m_j\rangle \,, \qquad \left[\hat{\vec{J}}^{(j)}\right]^2 |j; m_j\rangle = \hbar^2 j\,(j+1)\,|j; m_j\rangle \,.$$

Compare the action of the operators $\hat{S}^2$ and $\hat{S}_z$ on the basis $\{|\mathfrak{s}\rangle, |\mathfrak{t}_+\rangle, |\mathfrak{t}_0\rangle, |\mathfrak{t}_-\rangle\}$ to the action of the operators $[\hat{J}^{(1)}]^2 \oplus [\hat{J}^{(0)}]^2$ and $\hat{J}_z^{(1)} \oplus \hat{J}_z^{(0)}$ on the basis $\{|1;1\rangle, |1;0\rangle, |1;-1\rangle, |0;0\rangle\}$ in the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_0$. [6P]

A combined system of a spin-j_1 and a spin-j_2 particle allows for several possible values for the total spin quantum number j_{tot} of the two-particle system, namely all integers with $|j_1 - j_2| \leq j_{\text{tot}} \leq j_1 + j_2$.

(e) Compare the dimension of the Hilbert space given by the tensor sum of all spin-j_{tot} spaces, $\bigoplus_{j_{\text{tot}}} \mathcal{H}_{j_{\text{tot}}}$, to the dimension of the tensor product space formed by the spin-j_1 Hilbert space for the first particle and the spin-j_2 Hilbert space for the second particle, i.e., $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. [4P]

Consider now a three-particle system consisting of a spin-j_1, a spin-j_2 and a spin-j_3 particle. Analogously to exercise (e), the corresponding tensor product Hilbert space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$ can be decomposed into a tensor sum of Hilbert spaces $\mathcal{H}_{j_{\text{tot}}}$.

(f) Provide an expression for the permissable values for j_{tot} as a function of j_1, j_2 and j_3 and explain how you arrived at your result. Compare the dimension of the tensor product Hilbert space with the dimension of the tensor sum space $\bigoplus_{j_{\text{tot}}} \mathcal{H}_{j_{\text{tot}}}$. [6P]

Hint: Be wary - this exercise is a little tricky!

Exercise 3 - H H H H H H (20 points)

An experimental device $\mathcal{P}$ prepares hydrogen atoms in a state $|\psi\rangle$. Measurements show, that hydrogen atoms coming out of this aparatus are in a state with principal quantum number $n = 2$ and the following expectation values are obtained with respect to a laboratory-fixed coordinate system,

$$\left\langle \hat{\vec{L}}^2 \right\rangle_{|\psi\rangle} = 2\hbar^2, \qquad \left\langle \Delta\hat{\vec{L}}^2 \right\rangle_{|\psi\rangle} = 0, \qquad \left\langle \hat{L}_z \right\rangle_{|\psi\rangle} = \hbar, \qquad \left\langle \Delta\hat{L}_z \right\rangle_{|\psi\rangle} = 0 \,.$$

Remark: In the following, we use $|nlm\rangle$ to indicate a state with principal quantum number n, orbital quantum number l and magnetic quantum number m.

(a) Explain, how one can conclude from these measurement results with certainty, that the hydrogen atom prepared by the device $\mathcal{P}$ are in the eigenstate $|\psi\rangle = |211\rangle$. [2P]

(b) The hydrogen atoms obtained from device $\mathcal{P}$ now go through another device $\mathcal{R}_{\alpha\beta\gamma}$ where a set of Euler angles (α, β, γ) is provided and the atoms will be rotated accordingly. Provide an expression for the state $|\psi_{\alpha\beta\gamma}\rangle$ of the hydrogen atoms after going through the device $\mathcal{P}$ followed by device $\mathcal{R}_{\alpha\beta\gamma}$ ("setup A", see figure at the bottom left) as a function of the three Euler angles. [4P]
Hint: Write the state $|\psi\rangle$ as a vector and use the rotation matrix from Eq. (1).

(c) Show that measurement of the rotated hydrogen atoms in setup A yields the expectation values [5P]

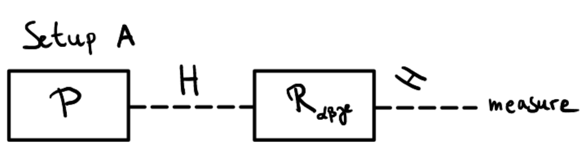
$$\left\langle \hat{\vec{L}}^2 \right\rangle_{|\psi_{\alpha\beta\gamma}\rangle} = 2\hbar^2, \qquad \left\langle \hat{L}_z \right\rangle_{|\psi_{\alpha\beta\gamma}\rangle} = \hbar\cos\beta \,. \tag{2}$$

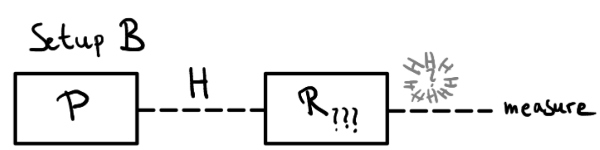
(d) Imagine a situation where the device $\mathcal{R}_{\alpha\beta\gamma}$ is broken and instead rotates the atoms by a completely random set of Euler angles (which can be different every time an atom passes the aparatus). We call this broken device $\mathcal{R}_{???}$. Show that for hydrogen atoms going through the device $\mathcal{P}$ followed by $\mathcal{R}_{???}$ ("setup B", see figure at the bottom right) the following relations hold for the average measurement results for $\hat{\vec{L}}^2$ and $\hat{L}_z$, [4P]

$$\left[\!\left[\left\langle \hat{\vec{L}}^2 \right\rangle_{|\psi_{\alpha\beta\gamma}\rangle} \right]\!\right]_{\text{avg}} = 2\hbar^2, \qquad \left[\!\left[\left\langle \hat{L}_z \right\rangle_{|\psi_{\alpha\beta\gamma}\rangle} \right]\!\right]_{\text{avg}} = 0 \,. \tag{3}$$

Hint: The average $[\![f]\!]_{avg}$ of a function $f(\alpha, \beta, \gamma)$ is given by $[\![f]\!]_{avg} = \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma\, f(\alpha, \beta, \gamma)$.

(e) Show, that in setup B the averages $[\![\hat{L}_x]\!]_{\text{avg}}$ and $[\![\hat{L}_y]\!]_{\text{avg}}$ also vanish. Explain how it is possible that $[\![\hat{\vec{L}}]\!]_{\text{avg}} = \vec{0}$ but $[\![\hat{\vec{L}}^2]\!]_{\text{avg}} \neq 0$. [5P]





[Bonus] Exercise X - From a Different Angle (5 extra points)

An alternative way to describe the hydrogen atoms from exercise 3 is to use the density matrix formalism. The state of the hydrogen atoms is then described by a density matrix $\hat{\rho}$. Expectation values of observables $\hat{A}$ are calculated via $\langle \hat{A} \rangle_{\hat{\rho}} = \mathrm{Tr}[\hat{A}\hat{\rho}]$.

(a) The hydrogen atoms from setup A can be represented by a pure density matrix, $\hat{\rho}_A = |\psi_{\alpha\beta\gamma}\rangle \langle \psi_{\alpha\beta\gamma}|$. Show that the results for $\langle \hat{L}_z \rangle_{\hat{\rho}_A}$ and $\langle \hat{L}^2 \rangle_{\hat{\rho}_A}$ agree with Eqs. (2). [1XP]

(b) The hydrogen atoms from setup B can be represented by a mixed density matrix,

$$\hat{\rho}_B = \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \; D^{(1)}(\alpha,\beta,\gamma) \, |211\rangle \langle 211| \left[D^{(1)}(\alpha,\beta,\gamma) \right]^{\dagger} .$$

Calculate the matrix representation of the density matrix $\hat{\rho}_B$ for the $n = 2, l = 1$ subspace and show that the results for $\langle \hat{\vec{L}}_z \rangle_{\hat{\rho}_B}$ and $\langle \hat{\vec{L}}^2 \rangle_{\hat{\rho}_B}$ agree with Eqs. (3). [4XP]