Advanced Quantum Mechanics

Sheet 1

Tutorial session: Wednesday

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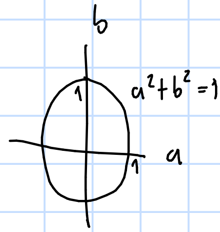
Sheet 1 - Umur Can Kaya

1.1.a $\| \hat{u} |\psi\rangle \| = \underbrace{\langle \psi | \hat{u}^\dagger \hat{u} | \psi \rangle} = \langle \psi | \mathbb{1} | \psi \rangle = \langle \psi | \psi \rangle = \| |\psi\rangle \|$

assume $|\psi\rangle$ is an eigenvec: $u^* u = 1$, let $u = a + bi \rightarrow \boxed{a^2 + b^2 = 1}$



1.1.b $\hat{H} = E_g |g\rangle\langle g| + E_i |i\rangle\langle i| + E_e |e\rangle\langle e| + \Omega_1 \left(|g\rangle\langle i| + |i\rangle\langle g| \right) + \Omega_2 \left(|i\rangle\langle e| + |e\rangle\langle i| \right)$

$$\hat{H} = \begin{bmatrix} E_g & \Omega_1 & 0 \\ \Omega_1 & E_i & \Omega_2 \\ 0 & \Omega_2 & E_e \end{bmatrix} \rightarrow$$ sanity check: looks hermitian ☑

1.1.c H is orthogonal in $\{ |g\rangle$ and $|e\rangle \}$: $H = e^{-iHt/\hbar}$

$$\boxed{H = e^{-i\alpha t/h} |g\rangle\langle g| + e^{-i\beta t/h} |e\rangle\langle e|}$$

1.1.d
$$\frac{d}{dt} \langle \hat{A} \rangle_{|\psi(t)\rangle} (t) = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle_{|\psi(t)\rangle}$$

- if $[\hat{H}, \hat{A}] = 0 \Rightarrow \frac{d}{dt} \langle \hat{A} \rangle_{|\psi(t)\rangle} (t) = 0 \Rightarrow$ $\boxed{\langle \hat{A} \rangle_{|\psi(t)\rangle} \text{ is time independent.}}$
- If $[\hat{H}, \hat{A}] = \hbar k \hat{A} \Rightarrow \frac{d}{dt} \langle \hat{A} \rangle_{|\psi(t)\rangle} (t) = iK \langle \hat{A} \rangle_{|\psi(t)\rangle} \Rightarrow \boxed{\langle \hat{A} \rangle_{|\psi(0)\rangle} (t) = \exp[iKt]}$

1.2.a

- Spin of an electron can be a physical realization of a qubit where spin up and spin down states corresponds to $|1\rangle$ and $|0\rangle$ respectively.

- Spin of an atom in optical lattice can be a physical realization of a qubit in the same manner as spin of electron.

1.2.b

(b) The Hilbert space $\mathcal{H}_2$ contains fewer states than the "naive" vector space $\mathbb{C}^2 \equiv \text{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\}$. Show that without any loss of physical generality, the state of a qubit can be written as $(\theta, \varphi \in \mathbb{R})$

$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle\ . \quad (1)$$

Compare the number of real-valued parameters in this expression with the number of real-valued parameters required to describe a general vector in $\mathbb{C}^2$. Explain the physical origin for the discrepancy. [3P]
Hint: How would a state's global phase $\chi \in [0, 2\pi)$, i.e., $|\psi\rangle \longrightarrow e^{i\chi}|\psi\rangle$, affect expectation values?

$\boxed{1.2.C}$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$\langle\hat{\sigma}_x\rangle_{|\psi\rangle} = \langle\psi|\hat{\sigma}_x|\psi\rangle = \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}e^{i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix} = e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= \underbrace{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}_{=\frac{\sin\theta}{2}}\underbrace{\left(e^{i\varphi}+e^{-i\varphi}\right)}_{2\cos\varphi}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)}$$

$$\boxed{\langle\hat{\sigma}_x\rangle_{|\psi\rangle} = \sin\theta\cos\varphi}$$

$$\langle\hat{\sigma}_y\rangle_{|\psi\rangle} = \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$\boxed{\frac{x}{i} = -ix}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}-ie^{i\varphi}\sin\frac{\theta}{2}\\ i\cos\frac{\theta}{2}\end{bmatrix} = -ie^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + ie^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= \underbrace{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}_{=\frac{\sin\theta}{2}}\underbrace{\left(-ie^{i\varphi}+ie^{-i\varphi}\right)}_{2\frac{e^{i\varphi}-e^{-i\varphi}}{2i}}$$

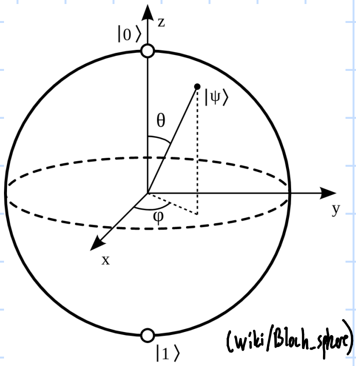
$$\boxed{\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)}$$

$$\boxed{\langle\hat{\sigma}_y\rangle_{|\psi\rangle} = \sin\theta\sin\varphi}$$

$$\langle\hat{\sigma}_z\rangle_{|\psi\rangle} = \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ -e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \Rightarrow \boxed{\langle\hat{\sigma}_z\rangle_{|\psi\rangle} = \cos\theta}$$

$\boxed{1.2.d}$ $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$



(wiki/Bloch_sphere)

i) North pole : $\theta = 0 \Rightarrow |\psi\rangle = e^{i\varphi}|0\rangle$

ii) South pole : $\theta = \pi \Rightarrow |\psi\rangle = e^{i\varphi}|1\rangle$

iii) Euquator : $\theta = \frac{\pi}{2} \Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle$

$\boxed{1.2.e}$ $\hat{H} = \frac{\hbar\Omega_0}{2}\hat{\sigma}_x + \frac{\hbar\Delta}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{bmatrix}\Delta & \Omega_0 \\ \Omega_0 & -\Delta\end{bmatrix}$

Eigenvalues $= \left\{-\frac{\hbar}{2}\sqrt{\Delta^2+\Omega_0^2},\ \frac{\hbar}{2}\sqrt{\Delta^2+\Omega_0^2}\right\}$ Eigenvectors $= \left\{\begin{bmatrix}-\frac{\Omega_0}{\Delta+\sqrt{\Delta^2+\Omega_0^2}} \\ 1\end{bmatrix}, \begin{bmatrix}-\frac{\Omega_0}{\Delta-\sqrt{\Delta^2+\Omega_0^2}} \\ 1\end{bmatrix}\right\}$

```
from sympy import *
init_printing(use_unicode=True)
hbar, omega0, delta = symbols('hbar \Omega_0 \Delta', real=True)
H = Matrix([[delta, omega0],[omega0, -delta]]) * hbar * 1/2
H.eigenvects()
```

$\boxed{1.2.f}$ for $\Delta = 0 \Rightarrow$ eigenvectors $= \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, eigvals $= \left\{\underbrace{-\frac{\hbar}{2}\Omega_0}_{E_-}, \underbrace{\frac{\hbar}{2}\Omega_0}_{E_+}\right\}$

$|\psi(t)\rangle = e^{-iHt/\hbar}|0\rangle$ $\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}, |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\boxed{\langle\pm|0\rangle = \mp\frac{1}{\sqrt{2}}}$ $\quad |0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$

$$= \left[e^{-iE_-t/\hbar}|-\rangle\langle-| + e^{-iE_+t/\hbar}|+\rangle\langle+|\right]|0\rangle$$

$$= -\frac{1}{2}e^{i\Omega_0t/2}|-\rangle + \frac{1}{2}e^{-i\Omega_0t/2}|+\rangle$$

$$= \frac{1}{2}\begin{bmatrix}+e^{i\Omega_0t/2} + e^{-i\Omega_0t/2} \\ -e^{i\Omega_0t/2} + e^{-i\Omega_0t/2}\end{bmatrix} = \begin{bmatrix}\cos\left(\frac{\Omega_0t}{2}\right) \\ -i\sin\left(\frac{\Omega_0t}{2}\right)\end{bmatrix} = \begin{bmatrix}\cos\left(\frac{\Omega_0t}{2}\right) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ -i\sin\left(\frac{\Omega_0t}{2}\right)\end{bmatrix}$$

$$\boxed{|\psi(t)\rangle = \cos\left(\frac{\Omega_0t}{2}\right)|0\rangle - i\sin\left(\frac{\Omega_0t}{2}\right)|1\rangle}$$

$$|\psi(t)\rangle = \cos\left(\frac{\Omega_0 t}{2}\right)|0\rangle - i\sin\left(\frac{\Omega_0 t}{2}\right)|1\rangle \qquad |\psi(0)\rangle = |0\rangle$$

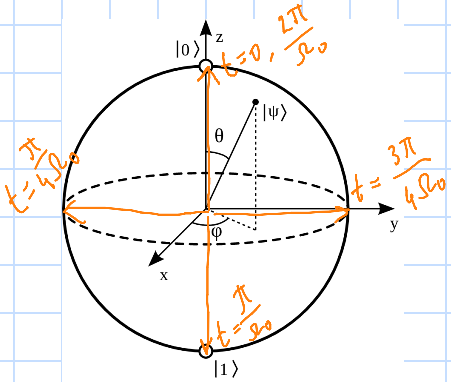
$$|\psi(t)\rangle = |0\rangle$$

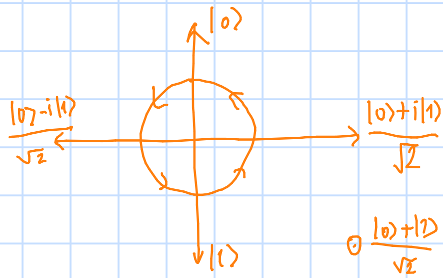
$$\begin{bmatrix} \cos\left(\frac{\Omega_0 t}{2}\right) \\ -i\sin\left(\frac{\Omega_0 t}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow \frac{\Omega_0 t}{2} = n\pi \Longrightarrow \boxed{t = \frac{2n\pi}{\Omega_0}, \quad n = 0,1,2,\dots}$$

$$\rightarrow |\psi(t)\rangle = |0\rangle$$

$$\rightarrow |\psi(t)\rangle = |1\rangle$$

$$\boxed{t = \frac{(2n+1)\pi}{\Omega_0}, \quad n = 0,1,2,\dots}$$





1.3.a $\hat{n} = \hat{a}^\dagger \hat{a}$

$$\hat{n}^\dagger = (\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger \hat{a} = \hat{n} \Rightarrow \boxed{\hat{n} = \hat{n}^\dagger}$$

$$\langle n|\hat{n}|n\rangle = n$$

$$\left.\begin{aligned} \langle n|\hat{n}^\dagger|n\rangle &= n^* \\ \langle \hat{n} n|n\rangle &= n \end{aligned}\right\} \quad n^* = n \Rightarrow n$$

1.3.b $\hat{a}|n\rangle = \sqrt{n-1}\,|n-1\rangle$

$$\langle \ell|\hat{a}|n\rangle = \sqrt{n}\,\langle \ell|n-1\rangle$$

$$\langle \hat{a}^\dagger \ell|n\rangle = \sqrt{n}\,\langle \ell|n-1\rangle$$

$$\langle \hat{a}^\dagger (n-1)|n\rangle = \sqrt{n} \Longrightarrow \langle \hat{a}^\dagger (n-1)| = \sqrt{n}\,\langle n| \Rightarrow \boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

1.3.c

$$\langle n|\hat{x}|n\rangle = \frac{1}{\sqrt{2}}\langle n|\hat{a}+\hat{a}^\dagger|n\rangle = \frac{1}{\sqrt{2}}\left(\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle\right) = \frac{1}{\sqrt{2}}\left(\sqrt{n}\underbrace{\langle n|n-1\rangle}_{0} + \sqrt{n+1}\underbrace{\langle n|n+1\rangle}_{0}\right) = 0$$

$$\langle n|\hat{p}|n\rangle = \frac{1}{\sqrt{2}i}\langle n|\hat{a}-\hat{a}^\dagger|n\rangle = \frac{1}{\sqrt{2}i}\left(\sqrt{n}\underbrace{\langle n|n-1\rangle}_{0} - \sqrt{n+1}\underbrace{\langle n|n+1\rangle}_{0}\right) = 0$$

$\boxed{1.3.d}$ $|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_\alpha |n\rangle = C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$m := n-1$

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_\alpha \hat{a}|n\rangle = \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_\alpha \sqrt{n}|n-1\rangle = C_\alpha \sum_{m=0}^{\infty} \frac{\alpha^{(m+1)}}{\sqrt{(m+1)!}} \sqrt{m+1}|m\rangle = \alpha C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \alpha|\alpha\rangle$$

$$\boxed{\hat{a}|\alpha\rangle = \lambda_\alpha |\alpha\rangle \, , \; \lambda_\alpha = \alpha}$$

$\langle\alpha|\alpha\rangle = 1$

$$\sum_n \frac{|\alpha|^{2n}}{n!} C_\alpha^2 = 1 \Rightarrow e^{|\alpha|} C_\alpha^2 = 1 \Rightarrow \boxed{C_\alpha = e^{-\frac{1}{2}|\alpha|}} \Longrightarrow |\alpha\rangle = e^{-\frac{1}{2}|\alpha|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$\boxed{1.3.e}$

$$\hat{a}^+|\alpha\rangle = e^{-\frac{1}{2}|\alpha|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{a}^+|n\rangle = e^{-\frac{1}{2}|\alpha|} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}\, |n+1\rangle \quad , \quad m := n+1$$

$$= e^{-\frac{1}{2}|\alpha|} \sum_{m=1}^{\infty} \frac{\alpha^{m-1}}{\sqrt{(m-1)!}} \sqrt{m}\, |m\rangle = e^{-\frac{1}{2}|\alpha|} \Big[|1\rangle + \alpha \sqrt{2}|2\rangle + \frac{\alpha^2}{\sqrt{2}} \sqrt{3}|3\rangle \ldots$$

$$\langle\beta|\hat{a}^+|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|+|\beta|)} \sum_{m=0} \sum_{n=0} \frac{(\beta^*)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}\, \underbrace{\langle m|n+1\rangle}_{\delta_{m,n+1}} = e^{-\frac{1}{2}(|\alpha|+|\beta|)} \sum_{n=0}^{\infty} \frac{(\beta^*)^{n+1}}{\sqrt{(n+1)!}} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}$$

$$\langle\beta|\hat{a}^+|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|+|\beta|)} \beta \sum_{n=0}^{\infty} \frac{(\beta^*\alpha)^n}{n!}$$

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{1}{\sqrt{2}} \langle\alpha|\hat{a}+\hat{a}^+|\alpha\rangle = \frac{1}{\sqrt{2}} \Big(\underbrace{\langle\alpha|\hat{a}|\alpha\rangle}_{\alpha} + \underbrace{\langle\alpha|\hat{a}^+|\alpha\rangle}_{} \Big) = \sqrt{2}\,\mathrm{Re}[\alpha] \Rightarrow \boxed{\langle\hat{x}\rangle = \sqrt{2}|\alpha| \cos\varphi}$$

$$e^{-|\alpha|} \alpha^* \sum_n \frac{|\alpha|^{2n}}{n!} = \alpha^*$$

$$\langle\alpha|\hat{p}|\alpha\rangle = \frac{1}{\sqrt{2}i} \langle\alpha|\hat{a}-\hat{a}^+|\alpha\rangle = \frac{1}{\sqrt{2}i} (\alpha - \alpha^*) = \sqrt{2}\,\mathrm{Im}[\alpha] \Rightarrow \boxed{\langle\hat{p}\rangle = \sqrt{2}|\alpha| \sin\varphi}$$

$\boxed{1.3.9}$ $|cat_\alpha\rangle = N_\alpha \left(|\alpha\rangle + |-\alpha\rangle\right)$

$$\langle\alpha|-\alpha\rangle = \sum_m \sum_n C_{\alpha^*} C_{-\alpha} \frac{(\alpha^*)^n}{\sqrt{m!}} \frac{(-\alpha)^n}{\sqrt{n!}} \langle m|n\rangle = C_{\alpha^*} C_{-\alpha} \sum_n (-1)^n \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|} e^{-|\alpha|^2} = e^{-|\alpha|^3}$$

for $\alpha >> 1$, $|cat_\alpha\rangle$ is a superposition of two approximately orthogonal states.

$$\langle cat_\alpha | cat_\alpha\rangle = |N_\alpha|^2 \Big(\underbrace{\langle\alpha|\alpha\rangle}_{1} + \underbrace{\langle -\alpha|-\alpha\rangle}_{1} + \underbrace{\langle -\alpha|\alpha\rangle}_{e^{-|\alpha|^3}} + \underbrace{\langle \alpha|-\alpha\rangle}_{e^{-|\alpha|^3}} \Big)$$, assume $\alpha >> 1$:

$$1 = |N_\alpha|^2 4 \Rightarrow \boxed{N_\alpha = \frac{1}{2}}$$