Sheet 3 - Tens⊕r Spaces and M⊗re
(Due: 15.11.2021, 18:00)

Exercise 1 - Tricks of the Trade III (12 points)

Consider a Hilbert space $\mathcal{H}_1$ with dimension N_1 and orthonormal basis $\{|\phi_i^{(1)}\rangle\}$ as well as another Hilbert space $\mathcal{H}_2$ with dimension N_2 and orthonormal basis $\{|\phi_i^{(2)}\rangle\}$.

(a) We denote an orthornomal basis for the *tensor sum* Hilbert space $\mathcal{H}_{\text{sum}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ by $\{|\phi_i^{(\text{sum})}\rangle\}$, which contains $N_1 + N_2$ basis states. The tensor sum $\hat{A} \oplus \hat{B}$ of two operators $\hat{A} : \mathcal{H}_1 \mapsto \mathcal{H}_1$ and $\hat{B} : \mathcal{H}_2 \mapsto \mathcal{H}_2$ obeys

$$\langle\phi_i^{(\text{sum})}|\hat{A} \oplus \hat{B}|\phi_j^{(\text{sum})}\rangle = \begin{cases} \langle\phi_i^{(1)}|\hat{A}|\phi_j^{(1)}\rangle\,, & \text{if } 1 \leq i \leq N_1 \text{ and } 1 \leq j \leq N_1 \\ \langle\phi_{i-N_1}^{(2)}|\hat{B}|\phi_{j-N_1}^{(2)}\rangle\,, & \text{if } N_1 < i \leq N_1 + N_2 \text{ and } N_1 < j \leq N_1 + N_2 \\ 0, & \text{otherwise.} \end{cases}$$

Write down an expression for the matrix form of the tensor sum $\hat{A} \oplus \hat{B}$ where $(\alpha, \beta, \gamma \in \mathbb{R})$ [2P]

$$\hat{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}. \tag{1}$$

(b) We denote an orthornomal basis for the *tensor product* Hilbert space $\mathcal{H}_{\text{prod}} = \mathcal{H}_1 \otimes \mathcal{H}_2$ by $\{|\phi_i^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle\}$ (short-hand notation: $\{|\phi_i^{(1)}\phi_j^{(2)}\rangle\}$) for all combinations of i and j. This yields $N_1 \cdot N_2$ basis states for $\mathcal{H}_{\text{prod}}$. The tensor product $\hat{A} \otimes \hat{B}$ of two operators $\hat{A} : \mathcal{H}_1 \mapsto \mathcal{H}_1$ and $\hat{B} : \mathcal{H}_2 \mapsto \mathcal{H}_2$ obeys

$$\langle\phi_i^{(1)}\phi_j^{(2)}|\hat{A} \otimes \hat{B}|\phi_k^{(1)}\phi_l^{(2)}\rangle = \langle\phi_i^{(1)}|\hat{A}|\phi_k^{(1)}\rangle\,\langle\phi_j^{(2)}|\hat{B}|\phi_l^{(2)}\rangle\,.$$

Write down an expression for the matrix form of the tensor product $\hat{A} \otimes \hat{B}$ with $\hat{A}, \hat{B}$ from Eq. (1). [2P]

(c) Consider operators $\hat{A}, \hat{C} : \mathcal{H}_1 \mapsto \mathcal{H}_1$ and $\hat{B}, \hat{D} : \mathcal{H}_2 \mapsto \mathcal{H}_2$. Show, that $(\hat{A} \otimes \hat{B})(\hat{C} \otimes \hat{D}) = \hat{A}\hat{C} \otimes \hat{B}\hat{D}$. [6P]

(d) Consider operators $\hat{A} : \mathcal{H}_1 \mapsto \mathcal{H}_1$ and $\hat{B} : \mathcal{H}_2 \mapsto \mathcal{H}_2$. Show, that $[\hat{A} \otimes \hat{\mathbb{1}}, \hat{\mathbb{1}} \otimes \hat{B}] = 0$. [2P]

Exercise 2 - Even and Odd (24 points + 4 extra points)

Remark: For simplicity, we use $\hbar = 1$ throughout this exercise

(a) Consider a particle in one spatial dimension whose state is described by an element of a Hilbert space $\mathcal{H}$. The so-called *parity operator* $\hat{\Pi}$ performs a spatial inversion, i.e., $\hat{\Pi}\,|x\rangle = |-x\rangle$ for every eigenstate $|x\rangle$ of the position operator $\hat{x}$. Is $\hat{\Pi}$ hermitian? Is $\hat{\Pi}$ unitary? Justify your answers! [3P]

(b) Show that $\hat{\Pi}$ can be written as follows,

$$\hat{\Pi} = \hat{P}_+ - \hat{P}_-\,, \tag{2}$$

where $\hat{P}_\pm$ are projectors. Determine the results of the expressions $\hat{P}_+\,|x\rangle$ and $\hat{P}_-\,|x\rangle$ for arbitrary $|x\rangle$. [5P]

(c) Any spectral decomposition of a Hermitian operator as a sum of projectors, cf. Eq. (2), allows for a tensor sum decomposition of the underlying Hilbert space. In our case we obtain $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\ker(\hat{P}_+) = \mathcal{H}_-$ and $\ker(\hat{P}_-) = \mathcal{H}_+$. Decide for the following (unnormalised) wave functions $\psi\,(x)$ whether their corresponding Hilbert space state $|\psi\rangle$ is an element of $\mathcal{H}_+$, $\mathcal{H}_-$ or neither: (i) $\psi_1\,(x) = \langle x|\psi_1\rangle = e^{-x^2}$, (ii) $\psi_2\,(x) = \langle x|\psi_2\rangle = \cos\,(x) + i \sin\,(x)$, and (iii) $\psi_3\,(x) = \langle x|\psi_3\rangle = \tanh\,(x)$. Justify your answer! [3P]

(d) Show that $\hat{\Pi}\,|p\rangle = |-p\rangle$ where $|p\rangle$ are the eigenstates of the momentum operator $\hat{p}$. [2P]
Hint: $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ixp}$.

(e) Show that $\hat{\Pi}\hat{x}\hat{\Pi} = -\hat{x}$ and $\hat{\Pi}\hat{p}\hat{\Pi} = -\hat{p}$. [2P]

(f) Consider the Hamiltonian of a single particle (mass m) in a one-dimensional harmonic potential (force constant k) with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$. Show that $[\hat{H}, \hat{\Pi}] = 0$. Discuss whether the following statement is true or false: "Any eigenstate of $\hat{H}$ must be either an element of $\mathcal{H}_+$ or $\mathcal{H}_-$.". [3P]
Hint: Remember a statement on the eigenbases for two commuting observables.

(g) Show that $\hat{\Pi}\,|n\rangle = (-1)^n\,|n\rangle$ where $|n\rangle$ is a Fock state. [3P]
Hint: Looking up the eigenfunctions $\psi_n\,(x)$ is not allowed! However, you may assume that $\hat{\Pi}\,|0\rangle = |0\rangle$.

In the rest of this exercise we will consider a set of two particles with masses m_1 and m_2 in a one-dimensional harmonic potential (force constant k). We assume the second particle's mass to be a quarter of the first's, i.e., $m_2 = \frac{1}{4}m_1$. The two-particle state is described by an element of the Hilbert space $\mathcal{H}_{\text{2P}} = \mathcal{H}\otimes\mathcal{H}$ with the Hamiltonian

$$\hat{H}_{\text{2P}} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}k\hat{x}_1^2 + \frac{1}{2}k\hat{x}_2^2\,, \tag{3}$$

where $\hat{x}_1 = \hat{x}\otimes\mathbb{1}$, $\hat{x}_2 = \mathbb{1}\otimes\hat{x}$ and $\hat{p}_1 = \hat{p}\otimes\mathbb{1}$, $\hat{p}_2 = \mathbb{1}\otimes\hat{p}$ are the vectors for position and momentum of the first and second particle. The total parity operator $\hat{\Pi}_{\text{tot}} = \hat{\Pi}\otimes\hat{\Pi}$ inverts position and momentum for both particles, i.e., $\hat{\Pi}_{\text{tot}}\hat{x}_i\hat{\Pi}_{\text{tot}} = -\hat{x}_i$ and $\hat{\Pi}_{\text{tot}}\hat{p}_i\hat{\Pi}_{\text{tot}} = -\hat{p}_i$ $(i = 1, 2)$. The operator $\hat{\Pi}_{\text{tot}}$ admits a tensor sum decomposition of the Hilbert space, $\mathcal{H}_{\text{2P}} = \mathcal{H}_+^{(\text{2P})}\oplus\mathcal{H}_-^{(\text{2P})}$, similar to what we discussed in (c).

(h) Show that $[\hat{H}_{\text{2P}}, \hat{\Pi}_{\text{tot}}] = 0$. Discuss whether the following statement is true or false: "Any eigenstate of $\hat{H}_{\text{2P}}$ must be either an element of $\mathcal{H}_+^{(\text{2P})}$ or $\mathcal{H}_-^{(\text{2P})}$.". [3P]
Hint: The oscillator frequency ω of a particle with mass m in a harmonic potential with force constant k is given by $\omega = \sqrt{\frac{k}{m}}$.

(x) `Bonus exercise:` Imagine you have access to an experimental setup which allows measurement of the total energy E_{2P} for the two particles. Furthermore, you are able to measure the parity of each particle individually, corresponding to the operators $\hat{\Pi}_1 = \hat{\Pi}\otimes\mathbb{1}$ and $\hat{\Pi}_2 = \mathbb{1}\otimes\hat{\Pi}$. Discuss under which conditions this setup allows to uniquely identify a Fock state $|n_1\rangle\otimes|n_2\rangle$ with unknown n_1 and n_2. [4XP]

Exercise 3 - Dancing in the Noonlight (24 points)

A mode of a light field can be described by the Hilbert space of a harmonic oscillator, $\mathcal{H}_{\text{HO}} = \text{span}\,\{|n\rangle\}$, where $|n\rangle$ are the Fock states corresponding to the number of photons in that mode. The Hilbert space of two-mode states for such a light field is given by the tensor product space $\mathcal{H}_{\text{HO}}\otimes\mathcal{H}_{\text{HO}}$. We call $\hat{a}_1$ the annihilation operator for a photon in the first mode, i.e., $\hat{a}_1 = \hat{a}\otimes\mathbb{1}$, and $\hat{a}_2$ the annihilation operator for a photon on the second mode, i.e., $\hat{a}_2 = \mathbb{1}\otimes\hat{a}$. Furthermore, we define a vector operator $\hat{\vec{J}}$ with the following three components,

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger\right), \quad \hat{J}_y = \frac{i}{2}\left(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2\right), \quad \hat{J}_z = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right).$$

(a) An optical element inducing a phase shift of $\varphi = \varphi_1 - \varphi_2$ between the two modes can be implemented by the Hamiltonian $\hat{H} = \hbar\alpha\hat{J}_z$. Show that the equation of motion for $\hat{\vec{J}}(t)$ in the Heisenberg picture is given by [8P]

$$\frac{d}{dt}\hat{J}_x(t) = -\alpha\hat{J}_y(t)\,, \quad \frac{d}{dt}\hat{J}_y(t) = \alpha\hat{J}_x(t)\,, \quad \frac{d}{dt}\hat{J}_z(t) = 0\,. \tag{4}$$

Hint: This is not a difficult calculation, but it is somewhat lengthy, hence the 8 points awarded for this exercise.

(b) Show that for $\hat{\vec{J}}(t=0)\equiv(\hat{J}_x^{(0)}, \hat{J}_y^{(0)}, \hat{J}_z^{(0)})$ one obtains the following solution of Eq. (4), [2P]

$$\hat{J}_x(t) = \cos(\alpha t)\,\hat{J}_x^{(0)} - \sin(\alpha t)\,\hat{J}_y^{(0)}, \quad \hat{J}_y(t) = \sin(\alpha t)\,\hat{J}_x^{(0)} + \cos(\alpha t)\,\hat{J}_y^{(0)}, \quad \hat{J}_z(t) = \hat{J}_z^{(0)}\,. \tag{5}$$

Consider an initial state $|\psi_{\text{in}}\rangle$ subject to a phase shift φ generated by the $\hat{J}_z$ operator. Setting, e.g., $\varphi = \alpha t$ in Eq. (5) we obtain a state $|\psi_{\text{out}}\rangle$ which contains information on this phase shift. The standard deviation $\Delta\varphi$ of estimating φ from measurements on $|\psi_{\text{out}}\rangle$ is then bounded by the so-called *quantum Cramer-Rao bound*,

$$\Delta\varphi \geq \frac{\hbar}{2\Delta J_z}\,, \tag{6}$$

where $\Delta J_z = \sqrt{\left\langle\hat{J}_z^2\right\rangle_{|\psi_{\text{out}}\rangle} - \left\langle\hat{J}_z\right\rangle_{|\psi_{\text{out}}\rangle}^2}$ is the standard deviation of $\hat{J}_z$ evaluated for the state $|\psi_{\text{out}}\rangle$.

(c) Motivate Eq. (6) by using the Heisenberg uncertainty principle. [4P]
Hint: Phase and photon number can be considered as canonically conjugate variables, i.e., their commutator is equal to $\pm\hbar\mathbb{1}$. Which observable of the two-mode light field is connected to $\hat{J}_z$?

(d) A single-mode Fock state is given by $|\psi_n\rangle = |n\rangle\otimes|0\rangle$. A single-mode coherent state is given by $|\psi_\alpha\rangle = |\alpha\rangle\otimes|0\rangle$. A so-called *NOON state* is given by $|\psi_{\text{N}}\rangle = \frac{1}{\sqrt{2}}\left(|N\rangle\otimes|0\rangle + |0\rangle\otimes|N\rangle\right)$. Determine parameters n, α, N for these three states such that their expectation value for the total number of photons in given by N_0. Choosing these states as initial state candidates $|\psi_{\text{in}}\rangle$, calculate the quantum Cramer-Rao bound in Eq. (6) as a function of N_0. Discuss, which of the three states is the most sensitive probe for determining the phase shift φ. [10P]
Hint: To simplify the calculation for ΔJ_z, argue that it can be evaluated for $|\psi_{in}\rangle$ in this setup.