Exercises supplementing those in Friedberg, Insel and Spence's
Linear Algebra, 4th Edition.

This is a collection of additional exercises that I have put on homework sheets when teaching from this text (several times when teaching a regular linear algebra course, and most recently, when teaching the honors version of that course), supplementing the exercises in the book. Some I included in the assignments the students were to hand in; most I indicated as for ''students interested in further interesting and/or more challenging problems'' to think about. We covered Chapters 1, 2 and 5 and most of Chapters 6 and 7, and Appendices A, B and D; so these exercises concern those parts only.

I have given the exercises numbers that indicate the relevant section of the text, beginning where the numbering of the exercises in that section leaves off. The numbers of the exercises on these pages are shown in quotation marks to distinguish them from the exercises in the text.

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§1.3, ''**Exercise 32**'': Let F_1 and F_2 be fields, and let ''even'' and ''odd'' elements of $\mathcal{F}(F_1, F_2)$ be defined as in Exercise 22 (p.21). Writing S_e for the set of even elements, and S_o for the set of odd elements, let us assume the result of that exercise: that S_e and S_o are subspaces of $\mathcal{F}(F_1, F_2)$.

Prove that if F_1 and F_2 both have characteristic $\neq 2$, then $\mathcal{F}(F_1, F_2) = \mathsf{S}_e \oplus \mathsf{S}_o$. (Actually, this will be true if we merely assume F_2 has characteristic $\neq 2$; but the picture is somewhat different, and I have given you both assumptions to avoid confusion. But if you prove it only using the assumption on F_2, that's fine.)

§1.4, ''**Exercise 18**'': If S is a subset of a vector space V, show that $S \subseteq \mathrm{span}(S)$.

(The authors should probably have included this in the first sentence of Theorem 1.5, p.30.)

§1.5, ''**Exercise 21**'': Let $x_1, \ldots, x_n$ be elements of a vector-space V.

(a) Show that if $x_i \notin \mathrm{span}(\{x_1, \ldots, x_{i-1}\})$ for $i = 1, \ldots, n$, then $x_1, \ldots, x_n$ are linearly independent.

(Note: For $i = 1$ we understand $\{x_1, \ldots, x_{i-1}\}$ to mean the empty set $\varnothing$. As noted in the Definition on p.30, we understand $\mathrm{span}(\varnothing)$ to be $\{\mathit{0}\}$.)

(b) Let us define

$$B \;=\; \{x_j : x_j \notin \mathrm{span}\,(\{x_i : i < j\})\}.$$

(Here we are not assuming the condition of part (a).) Prove that B is a basis for the subspace $\mathrm{span}\,(\{x_1, \ldots, x_n\})$ of V.

(c) State the contrapositive of (a).

§1.5, ''**Exercise 22**'': (Generalization of §1.5, Exercise 20.) Suppose $r_1, \ldots, r_n$ are distinct real numbers. Show that the elements $f_1, \ldots, f_n \in \mathcal{F}(R, R)$ defined by $f_m(t) = e^{r_m t}$ are linearly independent. In determining what equations they do or do not satisfy, you may use the techniques taught in first-year calculus.

(Suggestion: Look at behavior as $t \to \infty$.)

§1.6, ''**Exercise 36**'': (a) Let F be a field, and in F^4 consider the four elements

$$u_1 = (0,1,1,1), \quad u_2 = (1,0,1,1), \quad u_3 = (1,1,0,1), \quad u_4 = (1,1,1,0).$$

Show that if the characteristic of F is not 3, then $\{u_1, u_2, u_3, u_4\}$ is a basis of F^4, while if the characteristic of F is 3, then that set is neither linearly independent nor a spanning subset of F^4.

The above result is related to §1.4, Exercise 6, and §1.5, Exercise 8. Of those two exercises in the text, the former, like the latter, ought to show different conclusions depending on whether the characteristic of F is 2. (The conclusion that the authors state is the one that is true when the characteristic is not 2.) In the present exercise, by varying things a bit, we get characteristic 3 to be the special case instead of characteristic 2.

(b) Can you get, for every positive integer p which can be the characteristic of a field, a statement of the same sort, for which characteristic p is the exceptional case?

§1.6, ‘‘**Exercise 37**’’: This exercise is essentially §1.6, Exercise 28, turned backward, and I recommend thinking about that one before doing this one. You may assume the results of that exercise in doing this exercise.

Suppose V is a vector space over R, of finite dimension n, and we wish to make it into a vector space over C, in such a way that the operation ‘‘+’’ of the new vector space structure is the same as the operation ‘‘+’’ of the old structure, and such that for every $r \in R \subseteq C$, the operation of multiplication by r is the same in the new vector space structure as it was in the old. (In concise language, we wish to *extend* the structure of vector space over R to a structure of vector space over C.)

(a) Show that this is possible if n is even.

(b) Show that it is not possible if n is odd.

(c) Show that in case (a), if $n \neq 0$, there are infinitely many extensions of the given structure of vector space over R to a structure of vector space over C.

(c) Show, nevertheless, that the dimensions of the vector spaces over C obtained by all extensions of the original structure are the same.

§1.7, ‘‘**Exercise 8**’’: Let F be a field, and let S denote the vector space of all sequences $(a_i) = (a_1, \ldots, a_n, \ldots)$, where each a_i is a member of F. (In the book’s notation, these sequences would be written $\{a_i\} = \{a_1, \ldots, a_n, \ldots\}$.) For each positive integer i, let $e_i \in \mathsf{S}$ denote the sequence $(0, \ldots, 0,1,0, \ldots)$ whose ith coordinate is 1, and whose other coordinates are all 0.

Show that the set $\{e_i\}$ is linearly independent, but is *not* a basis for S.

§1.7, ‘‘**Exercise 9**’’: Show that one can deduce as immediate corollaries of §1.7, Exercise 6, the following results. (You can get credit for this exercise if your deduction is correct, even if you did not do Exercise 6.)

(a) The result of §1.7, Exercise 4.

(b) The following generalization of §1.7, Exercise 7: Let G be a generating set for a vector space V, and let L be a linearly independent subset of V. Then there exists a subset H of G such that $L \cup H$ is a basis of V.

§1.7, ‘‘**Exercise 10**’’: (a) Using the maximal principle, prove the following statement, dual thereto:

> Let $\mathcal{F}$ be a family of sets. If, for each chain $\mathcal{C} \subseteq F$, there exists a member of $\mathcal{F}$ that is *contained* in each member of $\mathcal{C}$, then $\mathcal{F}$ has a *minimal* member (a member M containing no member of $\mathcal{F}$ other than M itself).

(b) Show that a subset β of a vector space V is a basis if and only if β is a *minimal* spanning set of V.

The above two results suggest that we might be able to get a variant proof that every vector space has a basis, by dualizing the proof in our text. This is not so, however:

(c) Show by example that if V is an infinite-dimensional vector space, and $\mathcal{F}$ is the set of all spanning sets of V, and $\mathcal{C}$ is a chain in $\mathcal{F}$, then there need not exist a member of $\mathcal{F}$ that is contained in all members of $\mathcal{C}$.

Thus, we cannot apply the result of (a) to get a minimal spanning set in V. Such sets always exist, since we know that V has a basis. But the only proof we have available is the one based on constructing a maximal linearly independent set.

§1.7, ‘‘**Exercise 11**’’: Prove that the following statement is equivalent to the maximal principle. That is, assuming the maximal principle, prove the statement below, and assuming that statement, prove the maximal principle:

> Every family (i.e., set) $\mathcal{F}$ of sets contains a maximal chain $\mathcal{C}$ (i.e., a chain which is not a subset of any other chain).

§1.7, ‘‘**Exercise 12**’’: Let $\mathcal{F}$ be the set of all subsets X of the natural numbers $Z^{\geq 0}$ which contain no two adjacent integers; i.e., such that for every $n \in Z^{\geq 0}$, if $n \in X$, then $n+1 \notin X$.

Describe in concrete terms what it means for a set X to be a maximal member of $\mathcal{F}$, and show that $\mathcal{F}$ has infinitely many maximal members.

(We haven’t had the concept of countability in this course; but if you have seen it, you might find it interesting to show that $\mathcal{F}$ has uncountably many maximal members.)

§1.7, ‘‘**Exercise 13**’’: The maximal principle, used in §1.7, does not give a way of explicitly finding a maximal chain $\mathcal{C}$ in $\mathcal{F}$, even if $\mathcal{F}$ is described explicitly; hence the proof of Theorem 1.12 does not provide an explicit way of constructing bases for infinite-dimensional vector spaces.

For some infinite-dimensional vector spaces that arise naturally in mathematics, it is easy to explicitly describe bases. For instance, the text noted earlier that $\mathsf{P}(F)$ has a basis consisting of the powers of x. Each element of $\mathsf{P}(F)$ is determined by the sequence of coefficients of the powers of x, and that sequence can be any sequence having only finitely many nonzero terms. So it is not surprising to find that the vector space $\mathsf{S}_{\mathrm{fin}}$ of those sequences of members of F having only finitely many nonzero terms also has an explicit basis: the set $\{e_i\}$ of the preceding exercise.

At the opposite extreme, I suspect that logicians can prove that the full vector space S of all sequences of elements of F has no basis that can be described explicitly.

This exercise will consider a case in between these extremes: a subspace of S for which an explicit basis can be found, but where the construction of such a basis is quite challenging.

Let us call a sequence (a_i) *periodic* if there is some positive integer k such that for all $i > 0$, we have $a_{i+k} = a_i$.

(a) Show that the set $\mathsf{S}_{\mathrm{per}}$ of periodic sequences forms a subspace of S.

(b) Construct an explicit basis β of $\mathsf{S}_{\mathrm{per}}$. (You should give an explicit description of members of β, so that given a sequence $s \in \mathsf{S}$ one can easily say whether it belongs to β.)

§2.1, ‘‘**Exercise 41**’’: (*Dimension Theorem for infinite-dimensional vector spaces.*) Let V and W be vector spaces, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. Show that if V is infinite-dimensional, then either $\mathsf{N}(\mathsf{T})$ or $\mathsf{R}(\mathsf{T})$ is infinite-dimensional (without using results from the optional §1.7).

(Note: one can prove a stronger form of the Dimension Theorem, which distinguishes different cardinalities of infinite bases – countable, and the infinitely many types of uncountability. However this course does not assume familiarity with these concepts; moreover, to prove that form *would* require using the results of §1.7.)

§2.1, ‘‘**Exercise 42**’’: Suppose V and W are finite dimensional vector spaces over a field F, let $\mathsf{V}_0 \subseteq \mathsf{V}$ be a subspace, and let $\mathsf{T}: \mathsf{V}_0 \to \mathsf{W}$ be a linear transformation. Show that there exists a linear transformation $\mathsf{T}': \mathsf{V} \to \mathsf{W}$ whose restriction to V_0 is T. (The mathematician’s way of saying this is ‘‘T can be extended to a linear transformation $\mathsf{V} \to \mathsf{W}$’’.)

§2.1, ‘‘**Exercise 43**’’: Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a function between vector spaces over a field F. Show that the following conditions are equivalent:

(i) T is a linear transformation.

(ii) For every positive integer n, every family of n vectors $x_1, \ldots, x_n \in \mathsf{V}$, and every family of n scalars $a_1, \ldots, a_n \in F$, if $\Sigma_{i=1}^{n} a_i x_i = 0$, then $\Sigma_{i=1}^{n} a_i \mathsf{T}(x_i) = 0$. (Intuitively: Every linear relation satisfied by elements $x_1, \ldots, x_n$ in V is also satisfied by their images $\mathsf{T}(x_1), \ldots, \mathsf{T}(x_n)$ in W.)

Suggestion for proving (ii) $\Rightarrow$ (i): Apply (ii) to linear relations of the two forms $1\cdot(a\,x) + (-a)\cdot x = 0$ and $1\cdot x + 1\cdot y + (-1)\cdot(x+y) = 0$.

(The above exercise will be called on in §6.5, ''Exercise 34'' below.)

§2.1, ''**Exercise 44**'': Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation of vector spaces, let V' be a subspace of V, and let $\mathsf{T}_{\mathsf{V}'}: \mathsf{V}' \to \mathsf{W}$ be the restriction of T to V'; i.e., the linear map $\mathsf{V}' \to \mathsf{W}$ defined by $\mathsf{T}_{\mathsf{V}'}(x) = \mathsf{T}(x)$ for all $x \in \mathsf{V}'$.

Show that the following conditions are equivalent:

(i) $\mathsf{R}(\mathsf{T}_{\mathsf{V}'}) = \mathsf{R}(\mathsf{T})$.

(ii) Every vector $x \in \mathsf{V}$ can be written as a sum $x = y + z$, where $y \in \mathsf{V}'$ and $z \in \mathsf{N}(\mathsf{T})$.

(In notation which will be introduced in §5.2, condition (ii) says that $\mathsf{V} = \mathsf{V}' + \mathsf{N}(\mathsf{T})$.)

(The implication (i)$\Rightarrow$(ii) states in general form an observation used in a particular case at the end of the proof of Theorem 7.3.)

§2.1, ''**Exercise 45**'': If $f: X \to Y$ is a function, then the *graph* of f means the set

$$\{(x, f(x)) : x \in X\} \;\subseteq\; X \times Y.$$

Let V and W be vector spaces, and $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ a function. Let us make the product set $\mathsf{V} \times \mathsf{W}$ into a vector space as in §1.2, Exercise 21 (where it was called Z).

Show that a function $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ is a *linear transformation* if and only if the graph of T is a *subspace* of this vector space $\mathsf{V} \times \mathsf{W}$.

§2.1, ''**Exercise 46**'': Let V be the vector space over R of all continuous functions $R \to R$, and let W denote the subspace of all *constant* functions.

Assume as known, from calculus, that every function in V has an antiderivative, and that any two antiderivatives of a given function differ by a constant.

(a) Show that for each $f \in \mathsf{V}$, the set of antiderivatives of f forms a coset of W in V.

(b) Show that the function sending each $f \in \mathsf{V}$ to the set of all its antiderivatives is a *linear* map $\mathsf{V} \to \mathsf{V}/\mathsf{W}$.

(c) Let $\mathsf{U} \subseteq \mathsf{V}$ denote the subspace of *continuously differentiable* functions (differentiable functions whose derivatives are continuous). Assuming the Fundamental Theorem of Calculus, show that the operation of integration induces a linear transformation $\mathsf{V} \to \mathsf{U}/\mathsf{W}$, which is one-to-one and onto.

(d) Writing $\mathsf{D}: \mathsf{U} \to \mathsf{V}$ for the operation of differentiation, describe relation between D and the inverse to the map of part (c).

§2.1, ''**Exercise 47**'': Suppose V is a vector space, and V_1 and V_2 are subspaces such that $\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2$. Let T_1 denote the projection of V onto V_1 along V_2, and T_2 the projection of V onto V_2 along V_1.

Show that $\mathsf{T}_1 + \mathsf{T}_2 = \mathsf{I}_{\mathsf{V}}$, $\mathsf{T}_1^2 = \mathsf{T}_1$, $\mathsf{T}_2^2 = \mathsf{T}_2$, and $\mathsf{T}_1\mathsf{T}_2 = \mathsf{T}_2\mathsf{T}_1 = \mathsf{T}_0$.

§2.1, ''**Exercise 48**'': Suppose V is a vector space, and V_1 and V_2 are subspaces such that $\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2$, and let T_1 denote the projection of V onto V_1 along V_2. Likewise let W be another vector space, W_1 and W_2 subspaces such that $\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2$, and U_1 the projection of W onto W_1 along W_2.

Show that the map $\mathcal{L}(\mathsf{V}, \mathsf{W}) \to \mathcal{L}(\mathsf{V}, \mathsf{W})$ given by $\mathsf{T} \mapsto \mathsf{U}_1\mathsf{T}$ is also a projection. Onto what subspace and along what subspace? Prove the same, and answer the same question, for the map $\mathcal{L}(\mathsf{V}, \mathsf{W}) \to \mathcal{L}(\mathsf{V}, \mathsf{W})$ given by $\mathsf{T} \mapsto \mathsf{T}\mathsf{T}_1$.

§2.1, ''**Exercise 49**'': Show by example that for V a vector space, and $\mathsf{T}, \mathsf{U}: \mathsf{V} \to \mathsf{V}$ two projection maps, the composite $\mathsf{T}\mathsf{U}$ need not be a projection map.

§2.2, ‘‘**Exercise 21**’’: (Extension of §2.2, Exercise 13.)

Let V and W be vector spaces, and let T and U be nonzero linear transformations $\mathsf{V} \to \mathsf{W}$.

(a) Prove that if $\mathsf{R}(\mathsf{T}) \neq \mathsf{R}(\mathsf{U})$, then $\{\mathsf{T}, \mathsf{U}\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

(b) Show by example that the converse to (a) is not true.

(c) Show by example that the obvious analog of (a) for three linear transformations is not true; that is, one can have nonzero $\mathsf{S}, \mathsf{T}, \mathsf{U} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$ such that all of $\mathsf{R}(\mathsf{S}), \mathsf{R}(\mathsf{T}), \mathsf{R}(\mathsf{U})$ are distinct, but $\mathsf{S}, \mathsf{T}, \mathsf{U}$ are linearly dependent in $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

(d) Given three (or more) elements of $\mathcal{L}(\mathsf{V}, \mathsf{W})$, can you find a statement relating their ranges which *does* imply that they are linearly independent elements of that vector space?

§2.2, ‘‘**Exercise 22**’’: (Dual of the preceding exercise.)

Let V and W be vector spaces, and let T and U be nonzero linear transformations $\mathsf{V} \to \mathsf{W}$.

(a) Prove that if $\mathsf{N}(\mathsf{T}) \neq \mathsf{N}(\mathsf{U})$, then $\{\mathsf{T}, \mathsf{U}\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

(b) Show by example that the converse to (a) is not true.

(c) Show by example that the obvious analog of (a) for three linear transformations is not true; that is, one can have nonzero $\mathsf{S}, \mathsf{T}, \mathsf{U} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$ such that all of $\mathsf{N}(\mathsf{S}), \mathsf{N}(\mathsf{T}), \mathsf{N}(\mathsf{U})$ are distinct, but $\mathsf{S}, \mathsf{T}, \mathsf{U}$ are linearly dependent in $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

(d) Given three (or more) elements of $\mathcal{L}(\mathsf{V}, \mathsf{W})$, can you find a statement relating their null spaces which *does* imply that they are linearly independent elements of that vector space?

§2.2, ‘‘**Exercise 23**’’: Let V and W be finite-dimensional vector spaces over a field F, of dimensions m and n respectively, let β, γ be ordered bases of these spaces, and let A be any $n \times m$ matrix over F. Show that there exists a unique linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ such that $[\mathsf{T}]_\beta^\gamma = A$.

(This will follow immediately from a result in §2.4; the point of this exercise is to see it using the methods of §2.2.)

§2.4, ‘‘**Exercise 26**’’: Suppose V is an n-dimensional vector space with an ordered basis β, and W an m-dimensional vector space with an ordered basis γ. Theorem 2.20 (p.103) implies that $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is mn-dimensional. Use that theorem to find a basis of this space, and describe the elements of that basis in terms of their behavior on the basis β of V.

§2.4, ‘‘**Exercise 27**’’: Let V be the (infinite-dimensional) vector space of all sequences $(a_0, a_1, \ldots, a_n, \ldots)$ with all $a_i \in R$. Let $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ denote the *left shift operator*, defined by

$$\mathsf{T}(a_0, a_1, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_{n+1}, \ldots).$$

I.e., T lops off the first term of a sequence, and shifts each of the remaining terms one step to the left. T is clearly a linear map. (This looks the operator so named in in §2.1, Exercise 21; but note that we are here defining it on the space of *all* sequences of real numbers, not just sequences with all but finitely many terms 0.) In this and some subsequent exercises, we shall study $\mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V})$.

(a) Show that for every pair of real numbers u and v, there exists a unique element $(a_0, a_1, \ldots, a_n, \ldots) \in \mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V})$ such that $a_0 = u$ and $a_1 = v$.

(b) Deduce that $\mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V})$ is 2-dimensional, and in fact has a basis $\{x, y\}$, where x is the unique element of this space which, as a sequence, begins $(1, 0, \ldots)$, and y is the unique element which begins $(0, 1, \ldots)$.

(c) Compute the sequences x and y of part (b) to ten terms. (You may show the results, without writing out any argument.)

In the remaining parts of this exercise we shall write $\mathsf{W} = \mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V})$.

(d) Show that T carries the subspace W into itself, and, regarding the restriction of T to this subspace as a linear operator $T_W : W \to W$, find the matrix of T_W with respect to the basis $\{x, y\}$ of part (b) above.

(e) Show that the linear operator $T_W : W \to W$ is invertible, and find the matrix representing T_W^{-1} in terms of that same basis.

(f) From the original definition of T, we see that for each $x = (a_0, a_1, \ldots, a_n, \ldots) \in W$, the element $T_W^{-1}(x)$ must have the form $(b, a_0, a_1, \ldots, a_{n-1}, \ldots)$ for some $b \in R$. Find a formula for b in terms of a_0 and a_1. (No argument required.)

(The ideas of this exercise will be continued in §5.1, ‘‘Exercise 29’’ and §5.2, ‘‘Exercise 24’’. Some further variants will be looked at in §5.2, ‘‘Exercise 25’’ and §7.1, ‘‘Exercise 17’’.)

§2.6, ‘‘**Exercise 21**’’: Let V be a finite-dimensional vector space, let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be two bases of V, and let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ respectively be the corresponding dual bases of V^*.

(a) Show that $\mathrm{span}(\{f_1\}) = \mathrm{span}(\{g_1\})$ if and only if $\mathrm{span}(\{x_2, \ldots, x_n\}) = \mathrm{span}(\{y_2, \ldots, y_n\})$.

(b) Suppose that $y_i = x_i$ for $i = 2, \ldots, n$, while $y_1 = a_1 x_1 + \ldots + a_n x_n$. Say why we must have $a_1 \neq 0$. Obtain formulas for $g_1, \ldots, g_n$ in terms of $f_1, \ldots, f_n$.

§2.6, ‘‘**Exercise 22**’’: (Extension of §2.6, Exercise 19.) Let V be a vector space, and W any subspace of V. Let $W^0 \subseteq V^*$ be defined as in the paragraph before §2.6, Exercise 13. Prove that

$$\bigcap_{f \in W^0} N(f) = W.$$

§2.7, ‘‘**Exercise 21**’’: (a) Prove that if $T: V \to W$ is a linear transformation between vector spaces, and $N(T)$ and $R(T)$ are finite-dimensional, then so is V. (Once we know that, the dimension theorem says that $\dim V$ is the sum of the dimensions of those spaces.)

(b) Deduce Lemma 2 on p.135 from the result of part (a) above, by taking $N(TU)$ for the V of that result, and U for the T thereof.

§5.1, ‘‘**Exercise 27**’’: Let $T: V \to V$ be a linear operator on a finite-dimensional vector space V. Show that T is diagonalizable if and only if $T + I_V$ is diagonalizable, and determine the relationship between the eigenvalues and eigenvectors of these two operators.

§5.1, ‘‘**Exercise 28**’’: (a) Show that two diagonalizable $n \times n$ matrices over the same field F are similar if and only if their characteristic polynomials are equal.

(b) Show by example that this statement becomes false if the word ‘‘diagonalizable’’ is removed.

§5.1, ‘‘**Exercise 29**’’: This will continue the subject begun in §2.4, ‘‘Exercise 27’’. As in that exercise, we denote by V the vector space of all sequences $(a_0, a_1, \ldots, a_n, \ldots)$ with $a_i \in R$, by $T: V \to V$ the shift operator,

$$T(a_0, a_1, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_{n+1}, \ldots),$$

and by W the subspace $N(T^2 - T - I_V) \subseteq V$.

(a) For every real number λ, determine all eigenvectors of T with eigenvalue λ, if any.

(b) Show that for any λ, the eigenvectors of T_W with eigenvalue λ are precisely those eigenvectors of T with eigenvalue λ which lie in W.

(c) Determine those real numbers λ such that some eigenvector of T with eigenvalue λ lies in W. Hence, determine all eigenvalues of T_W. For each such eigenvalue, describe an eigenvector.

(d) Recall that in part (b) of §2.4, ‘‘Exercise 27’’ we found that W was 2-dimensional. Deduce that W has a basis consisting of eigenvectors of T_W, and give such a basis.

(e) Express in terms of the basis found in (d) the element of W which, as a sequence, begins $(0, 1, \ldots)$.

The *Fibonacci numbers* are the numbers f_n ($n = 0, 1, 2, \ldots$) determined by the formulas $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$. Thus, $f_0, f_1, f_2, f_3, f_4, f_5, f_6$ are $0, 1, 1, 2, 3, 5, 8$.

(f) Translate the result of (e) into a formula for the nth Fibonacci number f_n.

§5.2, ''**Exercise 24**'': This is the final part of our series of exercises about $\mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V})$ and the Fibonacci numbers. In part (d) of §2.4, ''Exercise 27'' you found the matrix of T_W with respect to a certain basis, $\{x, y\}$. Now find the characteristic polynomial and the eigenvalues of this matrix. The eigenvalues that you get should be the same as those found in §5.1, ''Exercise 29''(c) above. Why?

If the above exercise is assigned after the two exercises mentioned have been turned in, your instructor should remind the class of what the matrix found in the former exercise and the eigenvalues found in the latter were, to put everyone on an equal footing.

§5.2, ''**Exercise 25**'': Suppose that in our series of exercises about Fibonacci numbers, we replaced the polynomial $\mathsf{T}^2 - \mathsf{T} - \mathsf{I}_\mathsf{V}$ with another polynomial $f(\mathsf{T})$ of degree 2 or higher. Examine (either by general considerations, or by experimenting with different polynomials) to what extent the results obtained in those exercises have analogs for these new operators $f(\mathsf{T})$.

Some features holding when the polynomial $f(t)$ does not have repeated roots will change when it does. We shall examine such an example in §7.1, ''Exercise 17'' below.

§5.2, ''**Exercise 26**'': Suppose $M \in \mathsf{M}_{n\times n}(F)$ is a matrix of the form $M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$, where A and C are square matrices, say $k\times k$ and $(n-k)\times(n-k)$ respectively, B is a $k\times(n-k)$ matrix, and O denotes the $(n-k)\times k$ matrix with all entries zero.

Prove a formula expressing the characteristic polynomial of M in terms of the characteristic polynomials of A and C.

§5.2, ''**Exercise 27**'': Suppose V is a finite-dimensional vector space, and $\mathsf{V}_1, \ldots, \mathsf{V}_k$ are subspaces of V. (Finite-dimensionality of V is not needed for the results below to be true, but the authors assume it in the theorems you will want to call on.)

(a) Show that if $\mathsf{V} = \mathsf{V}_1 \oplus \ldots \oplus \mathsf{V}_k$, then for every vector space W, and every family of linear transformations

$$\mathsf{T}_1 : \mathsf{V}_1 \to \mathsf{W}, \quad \mathsf{T}_2 : \mathsf{V}_2 \to \mathsf{W}, \quad \ldots, \quad \mathsf{T}_k : \mathsf{V}_k \to \mathsf{W},$$

there exists a unique linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ whose restriction to each V_i is T_i.

(b) Prove the converse statement: If for every space W and family of linear transformations T_i as in (a), there exists a unique extension as described there, then $\mathsf{V} = \mathsf{V}_1 \oplus \ldots \oplus \mathsf{V}_k$.

§5.2, ''**Exercise 28**'': Let T be a diagonalizable linear operator on a vector-space V, and U be any linear operator on V. Show that U commutes with T (i.e., $\mathsf{UT} = \mathsf{TU}$) if and only if each eigenspace E_λ of T is U-invariant.

§5.3, ''**Exercise 25**'': (a) Show that if $A \in \mathsf{M}_{n\times n}(C)$ and if $L = \lim_{m\to\infty} A^m$ exists, then $L = L^2$.

A mathematical entity L satisfying the equation $L = L^2$ is called *idempotent* (from Latin roots meaning ''the same [as its] powers'').

(b) Show that if L is any idempotent $n\times n$ matrix, then $\mathsf{N}(L) = \mathsf{R}(I_n - L)$ and $\mathsf{R}(L) = \mathsf{N}(I_n - L)$. (I am writing $\mathsf{N}(L)$ etc. where the book's notation would, strictly, require $\mathsf{N}(\mathsf{L}_L)$ etc..)

(c) Deduce that the only *invertible* idempotent $n\times n$ matrix is I_n.

(d) Deduce from (a) and (c) the result of §5.3, Exercise 4, p.308.

(e) Show that if T is an idempotent linear operator on a finite-dimensional vector space V, then $\mathsf{V} = \mathsf{N}(\mathsf{T}) \oplus \mathsf{R}(\mathsf{T})$, and T is the projection of V onto $\mathsf{R}(\mathsf{T})$ along $\mathsf{N}(\mathsf{T})$. (You may take for granted, without repeating the proof, that the result about *matrices* given in (b) above holds for *linear* *transformations* as well.)

(f) Show that in the situation of (e), if β_1 is an ordered basis of $\mathsf{N}(\mathsf{T})$ and β_2 an ordered basis of $\mathsf{R}(\mathsf{T})$, then $[\mathsf{T}]_{\beta_1 \cup \beta_2}$ is diagonal. Assuming T is neither $\mathsf{I_V}$ nor T_0, what are the eigenvalues of T?

§5.3, ‘‘**Exercise 26**’’: Suppose that an $n\times n$ transition matrix A has the form

$$A = \begin{pmatrix} O & B \\ C & O \end{pmatrix},$$

where B is $r\times(n-r)$ and C is $(n-r)\times r$, for $0 < r < n$.

(a) Show that A is not regular by considering the forms of powers of A.

(b) Show that if λ is an eigenvalue of A, then so is $-\lambda$. (Suggestion: take an eigenvector corresponding to λ, write it as a vector of height r perched on top of a vector of height $n-r$, then look at what A does to the vector gotten by changing the sign of one of these two pieces.)

§5.3, ‘‘**Exercise 27**’’: Find, for as small a value of n as you can, a regular $n\times n$ transition matrix A such that A^{100} has at least one entry equal to 0.

(You and a friend might make it a competition to see which of you can get the smallest n. Writing out the matrices would be tedious and unenlightening, but in place of this, you can draw a diagram with a circle of dots labeled $1, \ldots, n$ representing the states of the Markov chain, and an arrow from the dot labeled i to the dot labeled j whenever $A_{ij} \neq 0$. From this diagram, you should be able to work out which entries of any power A^n of A are nonzero.)

§5.3, ‘‘**Exercise 28**’’: (Simplification of the book’s Exercise 19.)

(a) Suppose A, B, A', B' are matrices of nonnegative real numbers, and that A' has nonzero entries in the positions where A does, and no others, and B' has nonzero entries in the positions where B does, and no others. Prove that $A'B'$ has nonzero entries in the positions where AB does, and no others.

(b) Deduce that if A and A' are transition matrices whose nonzero entries occur in the same positions, then A is regular if and only if A' is.

§5.3, ‘‘**Exercise 29**’’: (Generalization of the book’s Exercise 22.)

(a) For $A \in \mathsf{M}_{n\times n}(C)$, let $\mu(A)$ be n times the maximum of the absolute values of the entries of A. Show that for any $A, B \in \mathsf{M}_{n\times n}(C)$, we have $\mu(AB) \leq \mu(A)\,\mu(B)$.

(a′) (Alternative to part (a).) Prove the same conclusion with ρ (defined on p.295) in place of μ. (The calculation is not much harder than for (a), but takes more insight.)

(b) Deduce from the result of part (a) or (a′) that for any matrix A over C, the matrix e^A introduced on p.312 is defined; i.e., that the limit defining it converges.

§5.3, ‘‘**Exercise 30**’’: In contrast with Exercise 23 of this section, show that if $A, B \in \mathsf{M}_{n\times n}(C)$ commute, i.e., satisfy $AB = BA$, then $e^A e^B = e^{A+B}$.

§5.3, ‘‘**Exercise 31**’’: Letting J denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, compute e^{tJ} for t any real number.

§5.3, ‘‘**Exercise 32**’’: A *permutation matrix* means a square matrix having only entries 0 and 1, with exactly one 1 in each row and exactly one 1 in each column. (If you think about it, you will see that the action of such a matrix *permutes* the entries in each column vector, and that each such $n\times n$ matrix corresponds to a permutation of $\{1, \ldots, n\}$.)

Show that if a transition matrix A is invertible, and if its inverse is also a transition matrix, then A is a permutation matrix.

§5.4, ‘‘**Exercise 43**’’: Suppose T is a linear operator on a vector space V and λ is an eigenvalue of T. Show that every subspace of E_λ is a T-invariant subspace of V.

§5.4, ''**Exercise 44**'': Suppose T is a linear operator on a vector space V, and that $\mathsf{f} \in \mathsf{V}^*$ is an eigenvector of $\mathsf{T}^t\colon \mathsf{V}^* \to \mathsf{V}^*$. Show that $\mathsf{N}(\mathsf{f})$ is a T-invariant subspace of V.

§5.4, ''**Exercise 45**'': Let T be a linear operator on an n-dimensional vector-space V. Show that the following two conditions are equivalent:

(i) There exists an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of V such that the matrix $[\mathsf{T}]_\beta$ is upper triangular.

(ii) There exist distinct T-invariant subspaces $\mathsf{V}_0, \ldots, \mathsf{V}_n$ of V such that

$$\{0\} = \mathsf{V}_0 \subset \mathsf{V}_1 \subset \ldots \subset \mathsf{V}_n = \mathsf{V}.$$

§6.1, ''**Exercise 31**'': Let $F = R$ or C, let V be a 2-dimensional vector space over F, with basis $\{x,y\}$, and let p, q, r be any three elements of F.

(a) Show that there is a unique function $\langle\ ,\ \rangle\colon \mathsf{V}\times\mathsf{V} \to F$ which satisfies conditions (a)-(c) in the definition of an inner product (p.330), and has $\langle x,x\rangle = p$, $\langle x,y\rangle = q$, $\langle y,y\rangle = r$.

(b) Show that the function $\langle\ ,\ \rangle$ of part (a) is an inner product on V (i.e., also satisfies condition (d) on p.330) if and only if $p > 0$, $r > 0$, and $|q|^2 < pr$. (Recall the convention that when one writes an inequality such as $p > 0$, this is understood to imply in particular that p is real.)

§6.2, ''**Exercise 24**'': Suppose V and W are finite-dimensional inner product spaces over the same field (R or C), such that $\dim(\mathsf{V}) = \dim(\mathsf{W})$. Show that there exists an isomorphism $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ such that for all $x, y \in \mathsf{V}$ one has $\langle \mathsf{T}(x), \mathsf{T}(y)\rangle = \langle x, y\rangle$.

§6.2, ''**Exercise 25**'': This exercise is for students familiar with the concepts of countable and uncountable sets.

An infinite-dimensional vector space is called ''countable-dimensional'' if it has a countable basis, and ''uncountable-dimensional'' if it does not. (One can prove that for infinite-dimensional as for finite-dimensional vector spaces, the cardinalities of all bases are the same, so these conditions are equivalent to ''all bases are countable'', respectively, ''all bases are uncountable''; but we will not need this fact.)

(a) Prove that every countable-dimensional inner product space has an orthonormal basis. (Starting point: Express the space as the union of a chain of finite-dimensional subspaces.)

(b) Show, however, that not every maximal orthonormal subset of a countable-dimensional inner product space need be a basis. (Suggestion: Take an orthonormal basis of the subspace W of §6.2, Exercise 23(c), and regard it as a subset of V.)

(c) Let V be the vector space of all bounded sequences of real numbers (*not* restricted to have only finitely many nonzero terms). Prove that V is uncountable-dimensional. (Suggestion: If $S = \{v_1, v_2, \ldots, v_n, \ldots\}$ is any countable subset of V, construct successively elements $a_0, a_1, a_2, \ldots, a_n, \ldots$ such that for each n, no linear combination of $v_1, v_2, \ldots, v_n$, regarded as a sequence of elements of F, can have first $n{+}1$ terms $(a_0, a_1, a_2, \ldots, a_n)$. Deduce that the element $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ of V is not in $\mathrm{span}(S)$. Incidentally, in this construction, each a_n can, if you wish, be chosen from $\{0, 1\}$.)

(d) The space V of part (c) can be made an inner product space by defining, for sequences $a = (a_1, \ldots, a_n, \ldots)$ and $b = (b_1, \ldots, b_n, \ldots)$, their inner product $\langle a, b\rangle$ to be $\Sigma_1^\infty a_n b_n / 2^n$ (a convergent infinite sum). Taking this for granted, show that this inner product space has no orthonormal basis. (Suggestion: Let W be the subspace of V consisting sequences that have only finitely many nonzero terms. Show that every set S that spans V must have a countable subset S_0 whose span contains W; and that $\mathsf{W}^\perp = \{0\}$. How do these facts imply that V can have no orthonormal basis?)

On the other hand, in the theory of infinite-dimensional inner product spaces a modified version of the concept of a basis is often used, which allows vectors to be expressed as ''infinite linear combinations'' of basis elements, such infinite sums being defined by convergence in the norm. The subject is more complicated, because some infinite sums converge and others do not; but in terms of that modified definition, the above space does have an ''orthonormal basis''.

§6.2, ‘‘**Exercise 26**’’: Let W_1, W_2 be subspaces of an inner product space V.

(a) Show that $\mathsf{W}_1 \subseteq \mathsf{W}_2^{\perp}$ if and only if $\mathsf{W}_2 \subseteq \mathsf{W}_1^{\perp}$, and that these equivalent conditions imply $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$.

(b) Show that if the equivalent conditions of (a) hold, and also the relation $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$, then the inclusions of (a) are both equalities.

(c) Show that if V is finite-dimensional then we have a converse to (b): if equality holds in at least one of the inclusions of (a), then $\mathsf{W}_2 \subseteq \mathsf{W}_1^{\perp}$.

(d) On the other hand, show by example that for infinite-dimensional V, the inclusions of (a) can both be equalities without $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$ holding.

(e) Also show by example that for infinite-dimensional V, one of the the inclusions of (a) can be an equality without the other being so.

§6.4, ‘‘**Exercise 25**’’: (a) Suppose A is an $n \times n$ matrix and B an $n \times m$ matrix over F (the real or complex field). Show that if $AA^* + BB^* = A^*A$, then $B = O$. (*Hint.* Apply the trace map to both sides of the given equation. You may use Example 5, p.331, for the case of $\mathsf{M}_{m\times n}(F)$, even though it is only stated for $\mathsf{M}_{n\times n}(F)$; the proof is the same.)

(b) Deduce that if a matrix of the form $\begin{pmatrix} A & B \\ O & C \end{pmatrix}$, where A and C are square matrices, is normal, then $B = O$. (I.e., B is a zero matrix – not in general the same matrix as the ‘‘O’’ appearing opposite it, but, rather, the transpose thereof.)

(c) Deduce that if T is a normal linear operator on a finite-dimensional real or complex inner product space V, and W is a T-invariant subspace of V, then W is also T^*-invariant, and $\mathsf{W}^{\perp}$ is T-invariant. (§6.4, Exercise 8, p.376 proves that W is T^*-invariant in the complex case, but does not cover the real case.)

(d) Show by example that the analog of (c) above fails for infinite-dimensional inner product spaces. (Suggestion: Let V be as in Example 3, p.372, let T be the restriction to V of the operator called T in that example, and let W be the subspace of V spanned by the functions f_n with $n \geq 0$.)

§6.4, ‘‘**Exercise 26**’’: Theorem 6.17, about self-adjoint linear operators on *real* inner product spaces, is proved using the Lemma on p.373, which is proved using Theorem 6.15, which concerns linear operators on *complex* inner product spaces. This is an illustration of the usefulness of the complex numbers in proving results involving only real numbers.

However, one can ask whether there is *some* proof of Theorem 6.17 that avoids using complex numbers. This exercise will give such a proof.

We will call on a few facts from real analysis (and later, a computation from calculus). Namely, we will assume that if V is a nonzero finite-dimensional real inner product space, then the set

$$S = \{x \in \mathsf{V} \mid \|x\| = 1\}$$

is a nonempty closed bounded set; that for a linear operator T on a finite-dimensional real inner product space, the real-valued function $x \mapsto \langle x, \mathsf{T}(x)\rangle$ is continuous; and finally, that every continuous function f on a nonempty closed bounded set E in a finite-dimensional real vector space assumes a maximum value, i.e., that there is a point $p \in E$ such that for all $x \in E$, $f(p) \geq f(x)$.

Putting these assumptions together, we conclude that if T is any linear operator on a nonzero finite-dimensional real inner product space V, then there is a point p in the set S defined above at which the function $x \mapsto \langle x, \mathsf{T}(x)\rangle$ assumes a maximum value. The rest of the argument, which is what you are to complete, will use only this conclusion, together with basic facts about real inner product spaces, and one bit of calculus:

(a) Show that for any points $u, v \in S$ which satisfy $\langle u, v \rangle = 0$, we have $(\cos t)\, u + (\sin t)\, v \in S$ for all real numbers t.

(b) Assuming $p \in S$ chosen to maximize $\langle p, \mathsf{T}(p) \rangle$, as discussed above, deduce from (a) that for any $v \in S$ which is orthogonal to p, the function $f: R \to R$ given by

$$f(t) \;=\; \langle\, (\cos t)\, p + (\sin t)\, v,\; \mathsf{T}((\cos t)\, p + (\sin t)\, v) \,\rangle$$

assumes a maximum at $t = 0$. Expand the above function as a trigonometric expression in t, involving certain inner products as coefficients, and use calculus to obtain from the above maximality statement a linear relation between those inner products.

In the remaining parts, T will be assumed *self-adjoint*.

(c) The self-adjointness of T can be applied to the formula obtained in (b) in two ways. Show by applying it in one way that p is an eigenvector of T, and by applying it in the other way that $\{p\}^{\perp}$ is T-invariant. You may use §6.2, Exercise 13(c). (Suggestion: apply that exercise with $\mathsf{W} = \mathrm{span}(\{p\})$.)

(d) Letting $\mathsf{V}' = \{p\}^{\perp}$, show that $\mathsf{T}_{\mathsf{V}'}$ is a self-adjoint linear operator on V', and that $\dim(\mathsf{V}') = \dim(\mathsf{V}) - 1$.

(e) If V' is nonzero, then by repeating the above argument we get an eigenvector of $\mathsf{T}_{\mathsf{V}'}$ whose orthogonal complement V'' is $\mathsf{T}_{\mathsf{V}'}$-invariant, and we can repeat this construction as long as it continues to give us nonzero subspaces $\mathsf{V}^{(k)}$. Turn this observation into an inductive proof that V has an orthonormal basis consisting of eigenvectors of T.

§6.4, ‘‘**Exercise 27**’’: (a) Suppose V is any complex vector space, and T, U are two linear transformation from V into a complex inner product space W. Show that if all $x \in \mathsf{V}$ satisfy $\langle \mathsf{T}(x), \mathsf{U}(x) \rangle = 0$, then all $x, y \in \mathsf{V}$ satisfy $\langle \mathsf{T}(x), \mathsf{U}(y) \rangle = 0$; i.e., the subspaces $\mathsf{R}(\mathsf{T})$ and $\mathsf{R}(\mathsf{U})$ of W are orthogonal to one another. (Suggestion: Apply the given condition to $x+y$ and to $x+iy$, and treat the resulting equations as in §6.4, Exercise 11(b).)

(b) Deduce the result of §6.4, Exercise 11(b) by taking $\mathsf{V} = \mathsf{W}$ and $\mathsf{T} = \mathsf{I}_{\mathsf{V}}$ above.

§6.4, ‘‘**Exercise 28**’’: Suppose T is a normal operator on a finite-dimensional inner product space V. Show that V has an orthonormal basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_m\}$ such that the action of T on this basis is described by

$$\mathsf{T}(u_j) = \lambda_j\, u_j, \qquad \mathsf{T}(v_j) = \rho_j \cos(\theta_j)\, v_j + \rho_j \sin(\theta_j)\, w_j, \qquad \mathsf{T}(w_j) = -\rho_j \sin(\theta_j)\, v_j + \rho_j \cos(\theta_j)\, w_j,$$

where $\lambda_1, \ldots, \lambda_k$ are real numbers, $\rho_1, \ldots, \rho_m$ are positive real numbers, and $0 < \theta_j < \pi$ $(j = 1, \ldots, m)$.

Thus, letting $\mathsf{W}_j = \mathrm{span}(\{u_j\})$ for $j = 1, \ldots, k$, and $\mathsf{W}_{k+j} = \mathrm{span}(\{v_j, w_j\})$ for $j = 1, \ldots, m$, we get a decomposition of V into mutually orthogonal 1- and 2-dimensional T-invariant subspaces,

$$\mathsf{V} \;=\; \mathsf{W}_1 \oplus \ldots \oplus \mathsf{W}_k \oplus \mathsf{W}_{k+1} \oplus \ldots \oplus \mathsf{W}_{k+m}.$$

(Hint: Begin by reducing to the case where T has the form L_A. Then regard A as a matrix over C rather than R, and write its eigenvalues as $\lambda_1, \ldots, \lambda_k, \rho_1 e^{\pm i\theta_1}, \ldots, \rho_m e^{\pm i\theta_m}$.)

§6.5, ‘‘**Exercise 33**’’: (a) Suppose A is a 2×2 real symmetric matrix which is *not* of the form cI $(c \in R)$. Show that there exist *exactly eight* (no more, no less) real orthogonal matrices Q such that $Q^t A Q$ is diagonal.

(b) If A *is* of the form cI, how many such matrices Q are there?

(c) If A is a 3×3 real symmetric matrix with distinct eigenvalues, determine the number of real orthogonal matrices Q such that $Q^t A Q$ is diagonal. (The reasoning should be fairly close to that of (a), so it will suffice to briefly sketch how to modify that reasoning to get your answer.)

§6.5, ‘‘**Exercise 34**’’: Let V and W be inner product spaces over F, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a function (not assumed to be a linear transformation!) such that for all $x, y \in \mathsf{V}$ one has

$$\langle \mathsf{T}(x), \mathsf{T}(y) \rangle = \langle x, y \rangle.$$

Show that for every positive integer n, every family of n vectors $x_1, \dots, x_n \in \mathsf{V}$, and every family of n scalars $a_1, \dots, a_n \in F$, one has

$$\Sigma\, a_i x_i = 0 \quad \Rightarrow \quad \langle \Sigma\, a_i x_i, \Sigma\, a_i x_i \rangle = 0 \quad \Rightarrow \quad \langle \Sigma\, a_i \mathsf{T}(x_i), \Sigma\, a_i \mathsf{T}(x_i) \rangle = 0 \quad \Rightarrow \quad \Sigma\, a_i \mathsf{T}(x_i) = 0.$$

(I.e., whenever one equality above holds, so does the next. Note that at the middle step you cannot *assume* T a linear transformation!)

Deduce using §2.1, ''Exercise 43'' that T is in fact a linear transformation.

This gives an alternative to the calculations beginning at the bottom of p.386 and continued at the top of p.387, which complete the proof of Theorem 6.22. The moral is that a proof of something in general may be more transparent than the same proof in a particular case. (The calculation on pp.386-387 is a particular instance of the $n = 3$ case of the above calculation.)

§6.5, ''**Exercise 35**'': (a) Show that the Lemma on p.380 remains true if the phrase ''finite-dimensional'' is deleted. (Suggestion: Apply the given relation with x replaced by $x + \mathsf{U}(x)$. When you expand the resulting equation, two terms will disappear by obvious applications of the same relation. Now apply the self-adjointness assumption.)

(b) Deduce that if U_1 and U_2 are self-adjoint operators on the same inner product space V, such that the functions $q_1,\ q_2\colon \mathsf{V} \to F$ defined by $q_1(x) = \langle x, \mathsf{U}_1(x) \rangle$, $q_2(x) = \langle x, \mathsf{U}_2(x) \rangle$ are equal, then $\mathsf{U}_1 = \mathsf{U}_2$.

§6.4, Exercise 11(b) (or its generalization, §6.4, ''Exercise 26'') gives a different proof of the above result for complex inner product spaces, which shows that in that case, the condition ''self-adjoint'' is also not needed! However, (c) below shows that for real inner product spaces self-adjointness must be assumed.

(c) Show that the corresponding statement without the condition ''self-adjoint'' is not true for real inner product spaces V; namely, that if U_1 is a non-self-adjoint linear operator on such a space, and we take $\mathsf{U}_2 = \mathsf{U}_1{}^*$, then the functions q_1, q_2 defined as in (b) above are equal. Show, moreover, that if we take $\mathsf{U} = (\mathsf{U}_1 + \mathsf{U}_1{}^*)/2$, then U is the unique *self-adjoint* operator inducing the same function $\mathsf{V} \to F$ that U_1 and $\mathsf{U}_1{}^*$ each induce.

The functions $\mathsf{V} \to F$ discussed above are examples of what are called *quadratic forms*. The case $\mathsf{V} = \mathsf{R}^2$ is mentioned on p.389; the general case is studied in §6.8, generally not covered in Math 110.

§6.5, ''**Exercise 36**'': Show that a square matrix A over C is unitary if and only if $A = e^{iB}$ for some self-adjoint matrix B. (Hint: Use §5.3, Exercise 21.)

§6.6, ''**Exercise 11**'': Suppose V is a finite-dimensional vector space over a field F. (Note that we do not assume F is R or C, nor that V is an inner product space.) Show that if $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ is a projection, then V has a basis consisting of eigenvectors of T.

§6.6, ''**Exercise 12**'': Let T be a diagonalizable linear operator on a vector space over any field F (*not* necessarily R or C), with eigenvalues $\lambda_1, \dots, \lambda_k$, and corresponding eigenspaces $\mathsf{E}_{\lambda_1}, \dots, \mathsf{E}_{\lambda_k}$; and for $i = 1, \dots, k$, let T_i denote the projection of V onto E_{λ_i} along $\bigoplus_{j \neq i} \mathsf{E}_{\lambda_j}$.

Verify that all statements of the Spectral Theorem not referring to the inner product structure, i.e., all but (b), remain true in this context, and likewise all parts of §6.6, Exercise 7 that do not refer to the inner product structure, i.e., all but (d) and (g).

So if I were the authors, I would have given these results about diagonalizable matrices in §5.2, and just put in a few addenda about the case of normal matrices on inner product spaces in this section.

§6.8, ''**Exercise 27**'': (a) Show that if invertible $n \times n$ matrices A and B over a field F are congruent, then $\det(A)/\det(B)$ is a square in F. (I.e., there is an element $c \in F$ such that $\det(A)/\det(B) = c^2$.)

(b) Deduce that for every positive integer n, there are infinitely many congruence classes of invertible

$n \times n$ matrices over the field of *rational* numbers.

§6.8, ''**Exercise 28**'': Given H and K as in Exercise 16, state and prove another formula expressing $H(x,y)$ in terms of K, which involves evaluating K at only two elements. (Suggestion: Look at §6.1, Exercise 20.)

§7.1, ''**Exercise 14**'': Show that every matrix of the form $A = \begin{pmatrix} \lambda & a & b \\ 0 & \lambda & c \\ 0 & 0 & \lambda \end{pmatrix}$ with a and c both nonzero has Jordan canonical form $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, and find an explicit invertible matrix Q (depending on a, b and c) such that $QAQ^{-1} = J$. (Suggestion: Start by verifying that the standard basis vector e_3 can be taken as the end-vector of a length-3 cycle for A.)

§7.1, ''**Exercise 15**'': Find the Jordan canonical form of the general matrix of the form $A = \begin{pmatrix} \lambda & a & b \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ where $a \neq 0$.

§7.1, ''**Exercise 16**'': Show that for every positive integer n, one has

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & n(n-1)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

(This is a special case of §7.2, Exercise 19(b).)

§7.1, ''**Exercise 17**'': Let F be any field, let V be the infinite-dimensional vector space of all sequences $(a_0, a_1, \ldots, a_n, \ldots)$ with $a_i \in F$, and, as in §2.4, ''Exercise 27'' and §5.1, ''Exercise 29'', let $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ denote the *shift operator*

$$\mathsf{T}(a_0, a_1, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_{n+1}, \ldots).$$

In those earlier exercises, we studied $\mathsf{N}(\mathsf{T}^2 - \mathsf{T} - \mathsf{I_V})$ for $F = R$; that space, which we called W, turned out to contain the sequence of Fibonacci numbers. Here we shall study $\mathsf{N}(\mathsf{T}^2 - 2\mathsf{T} + \mathsf{I_V})$, so in this exercise let us instead write W for this space.

(a) Show that for every pair of scalars $u, v \in F$, there exists a unique vector $x = (a_0, a_1, \ldots, a_n, \ldots) \in \mathsf{W}$ such that $a_0 = u$ and $a_1 = v$, and give a formula for the term a_n of this vector.

(b) Deduce that W is 2-dimensional.

(c) Show that W is T-invariant, and find the matrix for the restricted operator $\mathsf{T_W}: \mathsf{W} \to \mathsf{W}$ with respect to some basis of W.

(d) Determine the characteristic polynomial of $\mathsf{T_W}$, its Jordan canonical form, and a Jordan canonical basis (of W) for $\mathsf{T_W}$.

§7.1, ''**Exercise 18**'': Let V be an n-dimensional vector space, let T be a linear operator on V whose characteristic polynomial splits, $p(t) = (t-\lambda_1)^{m_1} \ldots (t-\lambda_k)^{m_k}$, let $\mathsf{K}_{\lambda_1}, \ldots, \mathsf{K}_{\lambda_k}$ be the generalized eigenspaces associated with the eigenvalues of T, and let J be a Jordan canonical form for T.

Let us call V ''T-cyclic'' if it is a T-cyclic subspace of itself, as defined on p.313; i.e., if there is some $x \in \mathsf{V}$ such that $\mathsf{V} = \mathrm{span}(\{x, \mathsf{T}(x), \mathsf{T}^2(x), \ldots\})$.

Show that the following conditions are equivalent:

(i) V is T-cyclic.

(ii) Each of $\mathsf{K}_{\lambda_1}, \ldots, \mathsf{K}_{\lambda_k}$ is a T-cyclic subspace of V.

(iii) The matrix J has only one Jordan block for each eigenvalue.

(iv) There is no nonzero polynomial g of degree less than n such $g(\mathsf{T}) = \mathsf{T}_0$.

(v) The operators $\mathsf{I_V}$, T, $\mathsf{T}^2, \ldots, \mathsf{T}^{n-1}$ are linear independent.

How to approach the above problem: Examine these conditions, and note down what implications among them you can easily see how to prove. Then make a diagram with corners the symbols (i), (ii), (iii), (iv), (v), and arrows among these corners indicating these implications. Note which further implications could complete the diagram, i.e., would yield a diagram in which one can get from any corner to any other by following arrows. Then try to prove such implications.

After entering the new implications you have proved in the diagram, you may find that you can drop some arrows and still have a diagram showing all conditions equivalent. If so, do so.

There are many forms your final diagram could take; e.g., a pentagon with arrows from one vertex to the next, a ''star'' with all arrows going into and out of one central vertex, etc.. You should choose the arrangement that makes for the easiest proof. Show the diagram of implications you will prove at the beginning of your write-up.

You can get partial credit for proving some but not all of the required implications. (But added redundancy will not help your score; i.e., if you give a proof of an implication that follows anyway by combining other implications you have proved, this will give no additional credit.)

§7.2, ''**Exercise 25**'': Find two 3×3 matrices A and B over the complex numbers such that for every complex number λ, $\mathrm{rank}(A-\lambda I_3) = \mathrm{rank}(B-\lambda I_3)$, but such that A and B are not similar; and prove that the matrices you have given have these properties.

§7.2, ''**Exercise 26**'': Find, for some integer n, two $n\times n$ matrices A and B over the field of complex numbers such that $A^n = B^n = 0$, and such that $\mathrm{rank}(A) = \mathrm{rank}(B)$, but such that A and B are not similar; and prove that the matrices you have given have these properties.

§7.2, ''**Exercise 27**'': Let n be a positive integer and λ an element of a field F, and suppose J is the $n\times n$ Jordan block over F with eigenvalue λ. Let $\{e_1, \ldots, e_n\}$ be the standard basis of F^n. Show that a vector $a_1 e_1 + \ldots + a_n e_n$ is the end vector of a Jordan canonical basis of F^n for L_J if and only if $a_n \neq 0$.

(Every Jordan canonical basis is determined by its end vector; so from this result one can get all Jordan canonical bases for this linear transformation.)

§7.2, ''**Exercise 28**'': (a) Suppose V is a 5-dimensional vector space over a field F, and suppose T is a linear operator on V, which has characteristic polynomial $-(t-\lambda)^5$ for some $\lambda\in F$, and such that $\mathrm{rank}(\mathsf{T}-\lambda\mathsf{I}) = 2$ while $(\mathsf{T}-\lambda\mathsf{I})^2 = \mathsf{T}_0$.

Show that $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I}) \subseteq \mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$, and that the former is 2-dimensional and the latter 3-dimensional. (You will not need anything beyond the methods of Chapter 2 to get the above facts.)

Hence we may choose a basis $\{v_1, v_2\}$ for $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I})$ and extend this to a basis $\{v_1, v_2, v_3\}$ of $\mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$. Further, since we have taken v_1, v_2 in $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I})$, there exist $v_4, v_5\in\mathsf{V}$ such that $v_1 = (\mathsf{T}-\lambda\mathsf{I})(v_4)$ and $v_2 = (\mathsf{T}-\lambda\mathsf{I})(v_5)$.

Verify that Theorem 7.6 is applicable to $\{v_1, v_2, v_3, v_4, v_5\}$, and deduce that this set is a basis for V, and in fact a Jordan basis for T, with dot-diagram $\begin{smallmatrix}\bullet&\bullet&\bullet\\ \bullet&\bullet&\end{smallmatrix}$. In particular, label that diagram with the five symbols, $v_1, \ldots, v_5$ showing how this basis decomposes into cycles under T.

(b) More generally, suppose V is n-dimensional, and T is a linear operator on V with characteristic polynomial $(-1)^n(t-\lambda)^n$, and with $\mathrm{rank}(\mathsf{T}-\lambda\mathsf{I}) = m$, $(\mathsf{T}-\lambda\mathsf{I})^2 = \mathsf{T}_0$. Imitating the above argument, show how to obtain a Jordan basis for T, and give its dot diagram.

Your analysis will show an arithmetic relation stronger than the obvious condition $m \leq n$ that must hold between m and n. Give this relation.

§7.2, ''**Exercise 29**'': Here we will push the idea of the preceding exercise one step further.

(a) Suppose V is a 6-dimensional vector space over a field F, and T a linear operator on V which has characteristic polynomial $(t-\lambda)^6$ for some $\lambda\in F$, and such that $\mathrm{rank}(\mathsf{T}-\lambda\mathsf{I}) = 3$, $\mathrm{rank}((\mathsf{T}-\lambda\mathsf{I})^2) = 1$, and $(\mathsf{T}-\lambda\mathsf{I})^3 = \mathsf{T}_0$.

Show that $\mathsf{R}((\mathsf{T}-\lambda\mathsf{I})^2) \subseteq \mathsf{R}(\mathsf{T}-\lambda\mathsf{I})\cap\mathsf{N}(\mathsf{T}-\lambda\mathsf{I}) \subseteq \mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$, and that these subspaces have dimensions 1, 2, 3 respectively. Deduce that there exists a basis $\{v_1, v_2, v_3\}$ of $\mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$ such that v_1 can be made the initial vector of a cycle of length 3 and v_2 the initial vector of a cycle of length 2, and such that these cycles, together with v_3, give a Jordan basis of V, with dot-diagram $\begin{array}{lll}\bullet&\bullet&\bullet\\ \bullet&\bullet&\\ \bullet&&\end{array}$.

(b) By analogy with part (b) of the preceding exercise, generalize the above argument to the case where V is n-dimensional, T has characteristic polynomial $(-1)^n(t-\lambda)^n$, $\mathrm{rank}(\mathsf{T}-\lambda\mathsf{I}) = m_1$, $\mathrm{rank}((\mathsf{T}-\lambda\mathsf{I})^2) = m_2$, and $(\mathsf{T}-\lambda\mathsf{I})^3 = \mathsf{T}_0$.

§7.2, ‘‘**Exercise 30**’’: Generalize the above two exercises to an algorithm for constructing a Jordan basis for a linear operator T on an n-dimensional vector space V which has characteristic polynomial $(-1)^n(t-\lambda)^n$, and satisfies $\mathrm{rank}(\mathsf{T}-\lambda\mathsf{I}) = m_1$, $\mathrm{rank}((\mathsf{T}-\lambda\mathsf{I})^2) = m_2$, ..., $\mathrm{rank}((\mathsf{T}-\lambda\mathsf{I})^{k-1}) = m_{k-1}$, and $(\mathsf{T}-\lambda\mathsf{I})^k = \mathsf{T}_0$.

Since an *algorithm* means a prescription for applying a sequence of operations to get a desired result, one needs to know what basic operations one is allowed. So let us assume we have the ability to find the range and null space of any linear operator, to find the intersection of two subspaces of V, to find a basis of a given subspace of V, to extend such a basis to a basis of a given larger subspace, and, for any element in the range of an operator, to find an element which the operator maps to it. These, clearly, are the tools used in the two preceding exercises.

(The generalization of the chain of subspaces $\mathsf{R}((\mathsf{T}-\lambda\mathsf{I})^2) \subseteq \mathsf{R}(\mathsf{T}-\lambda\mathsf{I}) \cap \mathsf{N}(\mathsf{T}-\lambda\mathsf{I}) \subseteq \mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$ used in the preceding exercise will consist of the subspaces $\mathsf{R}((\mathsf{T}-\lambda\mathsf{I})^e) \cap \mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$ for $e = k-1,\ k-2,\ \ldots,\ 1,\ 0$. For your own understanding, begin by verifying that for $m = 3$, this is the chain of the preceding exercise.)

§7.2, ‘‘**Exercise 31**’’: Suppose T is a linear operator on a finite-dimensional vector space V, and that the characteristic polynomial of T splits.

Show that if β is a Jordan canonical basis for T, then the basis β^* of V^*, *appropriately reordered*, is a Jordan canonical basis for T^t. How do the elements of β^* have to be reordered to make this true?

§7.2, ‘‘**Exercise 32**’’: Let $F = R$ or C, so that for any square matrix A over F, the matrix e^A is defined (§7.2, Exercise 22).

(a) Show that if A is an $n\times n$ matrix and c is a scalar, then $e^{cI+A} = e^c e^A$ (where e^c is understood as a scalar, and e^A as a matrix).

(b) Let J be a Jordan block matrix over F as in §7.2, Exercise 19. Describe precisely the matrix e^J.

Appendix A, ‘‘**Exercise 1**’’: Label the following statements as true or false. In the first four, ‘‘Z’’ denotes the set of integers (whole numbers), i.e., $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Answers are given at the end of the next page.

(a) $Z \in R$.

(b) $Z \subseteq R$.

(c) $R \cup Z = R$.

(d) $R \cap Z = R$.

(e) If A and B are any sets, then $A \cap B = \{x \in A : x \in B\}$.

(f) If A and B are any sets, then $A \cap B = \{x \in B : x \in A\}$.

(g) If $X \cap Y = \varnothing$, then either $X = \varnothing$ or $Y = \varnothing$.

Appendix A, ‘‘**Exercise 2**’’: Suppose $\overset{1}{\sim}$ and $\overset{2}{\sim}$ are two equivalence relations on the same set A.

(a) Show that the set-theoretic intersection of these equivalence relations (i.e., the intersection of these equivalence relations regarded as sets of ordered pairs of elements of A) is an equivalence relation.

(b) Show by example that the set-theoretic union of these equivalence relations may not be an equivalence relation.

(c) Suppose we define a relation $\stackrel{3}{\sim}$ on A by letting $a \stackrel{3}{\sim} a'$ hold if and only if there exists a positive integer n, and a sequence of elements $a_1, \ldots, a_n \in A$, such that $a_1 = a$, $a_n = a'$, and for each $i = 1, \ldots, n-1$, we have either $a_i \stackrel{1}{\sim} a_{i+1}$ or $a_i \stackrel{2}{\sim} a_{i+1}$. Show that $\stackrel{3}{\sim}$ is an equivalence relation on A.

(d) Show that for every equivalence relation $\stackrel{4}{\sim}$ on A, the relation $\stackrel{4}{\sim}$ contains the union of $\stackrel{1}{\sim}$ and $\stackrel{2}{\sim}$ if and only if it contains $\stackrel{3}{\sim}$.

Appendix B, ''**Exercise 1**'': Suppose A and B are sets and $f: A \to B$ a function. One of the following statements is true (i.e., true in all cases) and the other is false (i.e., false in some cases). Prove the true statement. For **extra credit**, you can show the other statement is false, by giving an explicit example where it fails.

(i) If $\sim_1$ is an equivalence relation on A, then the relation $\sim_2$ on B given by $\sim_2 = \{(f(a), f(a')) : (a, a') \in \sim_1\}$ is an equivalence relation.

(ii) If $\sim_1$ is an equivalence relation on B, then the relation $\sim_2$ on A given by $\sim_2 = \{(a, a') : (f(a), f(a')) \in \sim_1\}$ is an equivalence relation.

Appendix D, ''**Exercise 1**'': (a) Show that if z is a complex number, then $z = w + \bar{w}$ for some complex number w if and only if $z \in R$ (the field of real numbers).

(b) Show that if z is a complex number, then $z = w - \bar{w}$ for some complex number w if and only if z is imaginary.

(c) For which complex numbers z is it true that $z = w\bar{w}$ for some complex number w? (Of course, you must prove your answer correct.)

(d) For which complex numbers z is it true that $z = w/\bar{w}$ for some complex number w?

Appendix D, ''**Exercise 2**'': Show that neither of the equations

$$z\,\bar{z} = -1, \qquad z + \bar{z} = i$$

has a solution in C. Why do these facts not contradict the Fundamental Theorem of Algebra?

Answers to True/False questions on Appendix A: (a) F. (b) T. (c) T. (d) F. (e) T. (f) T. (g) F.