1 Definitions

Chapter 5

5.1

A linear operator T on a finite dimensional vector space V is called *diagonalizable* if there exists an ordered basis β of V such that $[T]_\beta$ is diagonal. A square matrix A is called diagonalizable if L_A is diagonalizable.

Given a linear operator T on a vector space V, an *eigenvector* is a non-zero vector x such that $T(x) = \lambda x$, for some $\lambda \in F$. This λ is called the *eigenvalue* corresponding to eigenvector x.

For a square matrix A, the *characteristic polynomial* of A is the polynomial $f(t) = \det(A - tI)$. The characteristic polynomial of a linear operator T is the polynomial $f(t) = \det([T]_\beta - tI)$, for any basis β of V.

5.2

A polynomail $f(t)$ *splits* over a field F if it can be expressed as a product of linear terms. That is, $f(t) = \prod_i (\lambda_i - t)$ for $\lambda_i \in F$.

The *algebraic multiplicity* of a root λ of a polynomial $f(t)$ is the greatest m such that $(t - \lambda)^m$ divides $f(t)$.

The *eigenspace* of an eigenvalue λ for a linear operator $T : V \to V$ is the subspace $E_\lambda = \{x : T(x) = \lambda x\}$.

The *geometric multiplicity* of an eigenvalue λ is the dimension of E_λ.

Given a vector space V and an arbitrary family of subspaces $(W_\alpha)_{\alpha \in A}$, the *sum* of subspaces, denoted by $\sum_{\alpha \in A} W_\alpha$ is the set of vectors $\left\{\sum_{\alpha \in A} x_\alpha : x_\alpha \in W_\alpha\right\}$.

The sum $\sum_{\alpha \in A} W_\alpha$ is called *direct* if it decomposes the vector space V uniquely. Explicitly, for an arbitrary $x \in V$, if $x = \sum_{\alpha \in A} \zeta_\alpha w_\alpha$, then the scalars ζ_α are unique.

5.3

For a linear operator T on a vector space V, a subsapce W is called *T-invariant* if $T(W) = W$. In other words, $x \in W \Rightarrow T(x) \in W$.

For a vector $x \in V$, The *T-cyclic subspace generated by v* is the subspace given by $\mathrm{Span}\{x, T(x), T^2(x), \ldots\}$.

For a linear operator T on a vector space V and some T-invariant subspace W, define $\bar{T} : V/W \to V/W$ by $x + W \mapsto T(x) + W$ for $x \in V$.

Chapter 6

6.1

Given a vector space V over field F, we define an *inner product* to be a function $\langle x, y \rangle : V \times V \mapsto F$ field which is

1. Linear in the first component,
2. Symmetric under complex conjugation,
3. Positive definite.

Note that conjugate linearity in the second component follows immediately from these properties.

If $F = \mathbb{R}$ or $F = \mathbb{C}$ and the $V = F^n$, we call the inner product $\langle x, y \rangle = \sum_i x_i \overline{y_i}$ the *standard inner product* on F^n.

For a square matrix A, the *conjugate transpose* of A, denoted A^*, is the matrix given by $A^*_{ij} = \overline{A_{ji}}$. Note that if $F = \mathbb{R}$, $A^* = A^t$.

A vector space V over F, endowed with a specific inner product is called an *inner product space.* Naturally, if $F = \mathbb{R}$, it is a *real inner product space* and if $F = \mathbb{C}$ it is a *complex inner product space.*

The *length* of a vector v in an inner product space V, denoted by $||x||$, is given by $||x|| = \sqrt{\langle x, x \rangle}$.

Two vectors in an inner product space V are called *orthogonal* if $\langle x, y \rangle = 0$. A subset $S \subset V$ is called orthogonal if $\langle x, y \rangle = 0$ for all distinct $x, y \in S$. A *unit vector* is a vector with length one. An *orthonormal* subset S is an orthogonal set of unit vectors. Equivalently, S is orthonormal if $\langle x, y \rangle = \delta_{xy}$.

6.2

An *orthonormal basis* is a basis of a inner product space V which is orthonormal.

Given a subset S of an inner product space V, we obtain a natural subspace called the *orthogonal complement* of S, denoted by $S^\perp$, which is the set $S^\perp = \{x \in V : \langle x, y \rangle = 0, \ \forall y \in S\}$. If V is finite dimensionl, and W a subspace, the sum $W + W^\perp$ is direct.

For a vector x in an inner product space We defined the *orthogonal projection* of x onto W by $x \mapsto y$, where $x = y + z$ for $y \in W^\perp$ and $z \in W$ Moreover, $y = x_{W^\perp}$ is the unique vector in $W^\perp$ such that $x - y \in W$.

6.3

For a finite dimensional inner product space V, the *adjoint* of a linear operator $T : V \to V$ is the unique linear operator $T^* : V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$.

6.4

For a finite dimensional inner product space V, a linear operator T is called *normal* if it commutes with its adjoint. That is, $TT^* = T^*T$. A matrix A is normal if $AA^* = A^*A$.

With V as before, a linear operator T is called *self-adjoint* if it is its own adjoint: $T = T^*$. A matrix A is called self-adjoint if $A = A*$. *Hermitian* is a synonym for self-adjoint.

A square matrix A with $F = \mathbb{R}$ is called *Gramian* if there exists a real matrix B such that $A = B^t B$.

With V as before, a linear operator T is called *positive definite* if T is self-adjoint and $\langle T(x), x \rangle > 0$ for all $x \neq 0$. A linear operator T is called *semi-positive definite* if T is self-adjoint and $\langle T(x), x \rangle \geq 0$ for all $x \neq 0$. The definitions for matrices are analagous in the obvious way.

2 Theorems

Chapter 5

5.1

Theorem 5.2 A scalar λ is an eigenvalue of a square matrix A if and only if $\det(A - \lambda I) = 0$.

5.2

Theorem 5.5 For a linear operator T over n-dimensional V with distinct eigenvalues λ_i, for v_i an eigenvector corresponding to λ_i, the set $\{v_i\}$ is linearly independent. Consequently, if T has n distinct eigenvalues, it is diagonalizable.

Theorem 5.9 A linear operator T over V with a splitting characteristic polynomial is diagonalizable if and only if its geometric and algebraic multiplicities are equal. Furthermore, the union of the bases of the eigenspaces form a basis for V.
Theorem 5.11 A linear operator T on a finite dimensionl vector space V is diagonalizabl if and only if its eigenspaces form a direct decomposition.

5.4

Theorem 5.22 If T is a linear operator over a k-dimensional vector space V, and W is the cyclic subspace generate by a non-zero vector x, then the set $\{x, T(x), \ldots T^{k-1}(x)\}$ is a basis for W and the scalars ζ_i in the linear comibnation $\sum_i \zeta_i T^i(x) = -T^k(x)$ give the characteristic polynomial of T_w by $f(t) = (-1)^k \sum_i a_i t^i$
Cayley-Hamilton Theorem A linear operator on a finite dimensional vector space satisfies its characteristic polynomial. The same holds for square matrices.

Chapter 6

6.1

Theorem 6.1 (e) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.
Cauchy-Schwarz Inequality $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$ with equality when $x = \lambda y$.
Triangle Inequality $||x + y|| \leq ||x|| + ||y||$ with equality when x and y are orthogonal.

6.2

Theorem 6.3 If V is an inner product space and $S = \{x_i : i = 1, \ldots, k\}$ is an orthogonal subset such that $v_i \neq 0$, then, if $y \in \operatorname{Span} S$, then

$$y = \sum_{i=1}^{k} \frac{\langle y, x_i\rangle}{\langle x_i, x_i\rangle} v_i$$

Gram-Schmidt Orthogonalization Process With V and S as above, if we define $S' = \{v_i : 1, \ldots, k\}$ by $v_1 = x_1$, and otherwise by

$$v_i = x_i - \sum_{j=1}^{k-1} \frac{\langle x_i, v_j\rangle}{\langle v_j, v_j\rangle} v_j$$

Then S' is orthogonal and $\operatorname{Span} S = \operatorname{Span} S'$.
Theorem 6.6 If W is a finite dimensional subspace of an inner product space V, then $x = V$ can uniquly be expressed as a sum of vectors from W and $W^{\perp}$. If V is finite dimensional,

$$W \oplus W^{\perp} = V$$

6.3

Theorem 6.8 Every linear functional $g : V \to F$ is some inner product $g(x) = \langle x, y\rangle$ for a fixed $y \in V$.
Theorem 6.9 If $\dim V < \infty$, Given $T : V \to V$, there exists a linear function $T^* : V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.
Theorem 6.10 If β is an orthonormal basis of V, then $[T^*]_\beta = ([T]_\beta)^*$.

6.4

Lemma If T, a linear operator on a finite-dimensional inner product space, has an eigenvector, then so does T^*.
Theorem 6.14 (Schur) With T as above, if the characteristic polynomial of T splits then there is an orthonormal basis β of V such that $[T]_\beta$ is upper triangular.
Spectral Theorem(s) If T is a linear operator on a finite-dimensional vector space V over field F, if $F = \mathbb{R}$ and T is self adjoint OR if $f = \mathbb{C}$ and T is normal, then V has an orthonormal basis of eigenvectors. Furthermore, the converse is also true, in both cases.