Chapter 9

Some Special Functions

Up to this point we have focused on the general properties that are associated with uniform convergence of sequences and series of functions. In this chapter, most of our attention will focus on series that are formed from sequences of functions that are polynomials having one and only one zero of increasing order. In a sense, these are series of functions that are "about as good as it gets." It would be even better if we were doing this discussion in the "Complex World"; however, we will restrict ourselves mostly to power series in the reals.

9.1 Power Series Over the Reals

In this section, we turn to series that are generated by sequences of functions $\left\{c_k (x-\alpha)^k\right\}_{k=0}^{\infty}$.

Definition 9.1.1 *A **power series in** $\mathbb{R}$ **about the point** $\alpha \in \mathbb{R}$ is a series in the form*

$$c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$$

where α and c_n, for $n \in \mathbb{J} \cup \{0\}$, are real constants.

Remark 9.1.2 *When we discuss power series, we are still interested in the different types of convergence that were discussed in the last chapter; namely, pointwise, uniform and absolute. In this context, for example, the power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ is said to be **pointwise convergent** on a set $S \subset \mathbb{R}$ if and only if, for each $x_0 \in S$, the series $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$ converges. If $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$*

*is **divergent**, then the power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ is said to diverge at the point x_0.*

When a given power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ is known to be pointwise convergent on a set $S \subset \mathbb{R}$, we define a function $f : S \longrightarrow \mathbb{R}$ by $f(x) = c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ whose range consists of the pointwise limits that are obtained from substituting the elements of S into the given power series.

We've already seen an example of a power series about which we know the convergence properties. The geometric series $1 + \sum_{n=1}^{\infty} x^n$ is a power series about the point 0 with coefficients $\{c_n\}_{n=0}^{\infty}$ satisfying $c_n = 1$ for all n. From the Convergence Properties of the Geometric Series and our work in the last chapter, we know that

- the series $\sum_{n=0}^{\infty} x^n$ is pointwise convergent to $\dfrac{1}{1-x}$ in $U = \{x \in \mathbb{R} : |x| < 1\}$,
- the series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent in any compact subset of U, and
- the series $\sum_{n=0}^{\infty} x^n$ is not uniformly convergent in U.

We will see shortly that this list of properties is precisely the one that is associated with any power series on its segment (usually known as interval) of convergence. The next result, which follows directly from the Necessary Condition for Convergence, leads us to a characterization of the nature of the sets that serve as domains for convergence of power series.

Lemma 9.1.3 *If the series $\sum_{n=0}^{\infty} c_n (x-\alpha)^n$ converges for $x_1 \neq \alpha$, then the series converges absolutely for each x such that $|x-\alpha| < |x_1-\alpha|$. Furthermore, there is a number M such that*

$$\left|c_n (x-\alpha)^n\right| \leq M \left(\frac{|x-\alpha|}{|x_1-\alpha|}\right)^n \text{ for } |x-\alpha| \leq |x_1-\alpha| \text{ and for all } n. \tag{9.1}$$

Proof. Suppose $\sum_{n=0}^{\infty} c_n (x-\alpha)^n$ converges at $x_1 \neq \alpha$. We know that a necessary condition for convergence is that the "nth terms" go to zero as n goes to infinity. Consequently, $\lim_{n\to\infty} c_n (x_1-\alpha)^n = 0$ and, corresponding to $\varepsilon = 1$, there exists a positive integer K such that

$$n > K \Rightarrow \left|c_n (x_1-\alpha)^n - 0\right| < 1.$$

Let $M = \max\left\{1, \max_{0\leq j\leq K} c_j\,(x_1-\alpha)^j\right\}$. Then

$$\left|c_n\,(x_1-\alpha)^n\right| \leq M \text{ for all } \ n \in \mathbb{J}\cup\{0\}\,.$$

For any fixed $x \in \mathbb{R}$ satisfying $|x-\alpha| \leq |x_1-\alpha|$, it follows that

$$\begin{aligned}\left|c_n\,(x-\alpha)^n\right| &= |c_n|\,|x-\alpha|^n = |c_n|\,|x_1-\alpha|^n \left|\frac{x-\alpha}{x_1-\alpha}\right|^n \\ &\leq M\left|\frac{x-\alpha}{x_1-\alpha}\right|^n \ \text{ for all } n\in\mathbb{J}\cup\{0\}\end{aligned}$$

as claimed in equation (9.1). Finally, for fixed $x \in \mathbb{R}$ satisfying $|x-\alpha| < |x_1-\alpha|$, the Comparison Test yields the absolute convergence of $\sum_{n=0}^{\infty} c_n\,(x-\alpha)^n$. ■

The next theorem justifies that we have uniform convergence on compact subsets of a segment of convergence.

Theorem 9.1.4 *Suppose that the series $\sum_{n=0}^{\infty} c_n\,(x-\alpha)^n$ converges for $x_1 \neq \alpha$. Then the power series converges uniformly on $I = \{x\in\mathbb{R} : \alpha-h \leq x \leq \alpha+h\}$ for each nonnegative h such that $h < |x_1-\alpha|$. Furthermore, there is a real number M such that*

$$\left|c_n\,(x-\alpha)^n\right| \leq M\left(\frac{h}{|x_1-\alpha|}\right)^n \ \textit{for } \ |x-\alpha| \leq h < |x_1-\alpha| \ \textit{and for all } n.$$

Proof. The existence of M such that $\left|c_n\,(x-\alpha)^n\right| \leq M\left(\frac{|x-\alpha|}{|x_1-\alpha|}\right)^n$ was just shown in our proof of Lemma 9.1.3. For $|x-\alpha| \leq h < |x_1-\alpha|$, we have that

$$\frac{|x-\alpha|}{|x_1-\alpha|} \leq \frac{h}{|x_1-\alpha|} < 1.$$

The uniform convergence now follows from the Weierstrass M-Test with $M_n = \left(\frac{h}{|x_1-\alpha|}\right)^n$. ■

Theorem 9.1.5 *For the power series $c_0 + \sum_{n=1}^{\infty} c_n\,(x-\alpha)^n$, either*

(i) the series converges only for $x = \alpha$; or

(ii) the series converges for all values of $x \in \mathbb{R}$; or

(iii) *there is a positive real number R such that the series converges absolutely for each x satisfying $|x-\alpha| < R$, converges uniformly in $\{x \in \mathbb{R} : |x-\alpha| \leq R_0\}$ for any positive $R_0 < R$, and diverges for $x \in \mathbb{R}$ such that $|x-\alpha| > R$.*

Proof. To see (i) and (ii), note that the power series $\sum_{n=1}^{\infty} n^n (x-\alpha)^n$ diverges for each $x \neq \alpha$, while $\sum_{n=0}^{\infty} \dfrac{(x-\alpha)^n}{n!}$ is convergent for each $x \in \mathbb{R}$. Now, for (iii), suppose that there is a real number $x_1 \neq \alpha$ for which the series converges and a real number x_2 for which it diverges. By Theorem 9.1.3, it follows that $|x_1-\alpha| \leq |x_2-\alpha|$. Let

$$S = \left\{\rho \in \mathbb{R} : \sum_{n=0}^{\infty} \left|c_n (x-\alpha)^n\right| \text{ converges for } |x-\alpha| < \rho\right\}$$

and define

$$R = \sup\ S.$$

Suppose that x^* is such that $|x^*-\alpha| < R$. Then there exists a $\rho \in S$ such that $|x^*-\alpha| < \rho < R$. From the definition of S, we conclude that $\sum_{n=0}^{\infty} \left|c_n (x^*-\alpha)^n\right|$ converges. Since x^* was arbitrary, the given series is absolutely convergent for each x in $\{x \in \mathbb{R} : |x-\alpha| < R\}$. The uniform convergence in $\{x \in \mathbb{R} : |x-\alpha| \leq R_0\}$ for any positive $R_0 < R$ was justified in Theorem 9.1.4.

Next, suppose that $\hat{x} \in \mathbb{R}$ is such that $\left|\hat{x}-\alpha\right| = \hat{\rho} > R$. From Lemma 9.1.3, convergence of $\sum_{n=0}^{\infty} \left|c_n \left(\hat{x}-\alpha\right)^n\right|$ would yield absolute convergence of the given series for all x satisfying $|x-\alpha| < \hat{\rho}$ and place $\hat{\rho}$ in S which would contradict the definition of R. We conclude that for all $x \in \mathbb{R}$, $|x-\alpha| > R$ implies that $\sum_{n=0}^{\infty} \left|c_n (x-\alpha)^n\right|$ as well as $\sum_{n=0}^{\infty} c_n (x-\alpha)^n$ diverge. ■

The nth Root Test provides us with a formula for finding the radius of convergence, R, that is described in Theorem 9.1.5.

Lemma 9.1.6 *For the power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and*

$$R = \begin{cases} +\infty & , \textit{if} \quad \rho = 0 \\ \dfrac{1}{\rho} & , \textit{if} \quad 0 < \rho < \infty \\ 0 & , \textit{if} \quad \rho = +\infty \end{cases} . \tag{9.2}$$

Then $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ converges absolutely for each $x \in (\alpha - R, \alpha + R)$, converges uniformly in $\{x \in \mathbb{R} : |x-\alpha| \le R_0\}$ for any positive $R_0 < R$, and diverges for $x \in \mathbb{R}$ such that $|x-\alpha| > R$. The number R is called the radius of convergence for the given power series and the segment $(\alpha - R, \alpha + R)$ is called the "interval of convergence."

Proof. For any fixed x_0, we have that

$$\limsup_{n\to\infty} \sqrt[n]{\left|c_n (x_0-\alpha)^n\right|} = \limsup_{n\to\infty} \left(|x_0-\alpha| \sqrt[n]{|c_n|}\right) = |x_0-\alpha| \rho.$$

From the Root Test, the series $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$ converges absolutely whenever $|x_0-\alpha| \rho < 1$ and diverges when $|x_0-\alpha| \rho > 1$. We conclude that the radius of convergence justified in Theorem 9.1.5 is given by equation (9.2). ■

Example 9.1.7 *Consider $\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} (x-2)^n$. Because $\limsup_{n\to\infty} \sqrt[n]{\frac{2\left(2^n\right)}{3^n}} = \lim_{n\to\infty} \left(\frac{2}{3}\right) \sqrt[n]{2} = \frac{2}{3}$, from Lemma 9.1.6, it follows that the given power series has radius of convergence $\frac{3}{2}$. On the other hand, some basic algebraic manipulations yield more information. Namely,*

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} (x-2)^n = -2 \sum_{n=0}^{\infty} \left[\frac{(-2)}{3} (x-2)\right]^n = -2 \frac{1}{1 - \left[\frac{(-2)}{3} (x-2)\right]}$$

as long as $\left|\frac{(-2)}{3} (x-2)\right| < 1$, from the Geometric Series Expansion Theorem. Therefore, for each $x \in \mathbb{R}$ such that $|x-2| < \frac{3}{2}$, we have that

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} (x-2)^n = \frac{6}{1-2x}.$$

Another useful means of finding the radius of convergence of a power series follows from the Ratio Test when the limit of the exists.

Lemma 9.1.8 *Let α be a real constant and suppose that, for the sequence of nonzero real constants $\{c_n\}_{n=0}^{\infty}$, $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$ for $0 \le L \le \infty$.*

(i) If $L = 0$*, then* $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ *is absolutely convergent for all* $x \in \mathbb{R}$ *and uniformly convergent on compact subsets of* $\mathbb{R}$*;*

(ii) If $0 < L < \infty$*, then* $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ *is absolutely convergent* $\left(\alpha - \frac{1}{L}, \alpha + \frac{1}{L}\right)$*, uniformly convergent in any compact subset of* $\left(\alpha - \frac{1}{L}, \alpha + \frac{1}{L}\right)$*, and divergent for any* $x \in R$ *such that* $|x - \alpha| > \frac{1}{L}$*;*

(iii) If $L = \infty$*, then* $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ *is convergent only for* $x = \alpha$*.*

The proof is left as an exercise.

Remark 9.1.9 *In view of Lemma 9.1.8, whenever the sequence of nonzero real constants* $\{c_n\}_{n=0}^{\infty}$ *satisfies* $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$ *for* $0 \le L \le \infty$ *an alternative formula for the radius of convergence* R *of* $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ *is given by*

$$R = \begin{cases} +\infty & \text{, if } \quad L = 0 \\ \dfrac{1}{L} & \text{, if } \quad 0 < L < \infty \\ 0 & \text{, if } \quad L = +\infty \end{cases} . \tag{9.3}$$

Example 9.1.10 *Consider* $\sum_{n=1}^{\infty} \frac{(-1)^n \, 2 \cdot 4 \cdots (2n)}{1 \cdot 4 \cdot 7 \cdots (3n-2)} (x+2)^n$.

Let $c_n = \frac{(-1)^n \, 2 \cdot 4 \cdots (2n)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$*. Then*

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(-1)^n \, 2 \cdot 4 \cdots (2n) \cdot 2\,(n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2) \cdot (3\,(n+1)-2)} \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(-1)^n \, 2 \cdot 4 \cdots (2n)}\right| = \frac{2\,(n+1)}{3n+1} \longrightarrow \frac{2}{3}$$

as $n \to \infty$*. Consequently, from Lemma 9.1.8, the radius of convergence of the given power series is* $\frac{3}{2}$*. Therefore, the "interval of convergence" is* $\left(-\frac{8}{3}, -\frac{4}{3}\right)$*.*

The simple manipulations illustrated in Example 9.1.7 can also be used to derive power series expansions for rational functions.

Example 9.1.11 *Find a power series about the point $\alpha = 1$ that sums pointwise to $\dfrac{8x-5}{(1+4x)(3-2x)}$ and find its interval of convergence.*

Note that

$$\frac{8x-5}{(1+4x)(3-2x)} = \frac{1}{3-2x} + \frac{-2}{1+4x},$$

$$\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty} [2(x-1)]^n = \sum_{n=0}^{\infty} 2^n (x-1)^n \ \textit{for} \ |x-1| < \frac{1}{2}$$

and

$$\begin{aligned}\frac{-2}{1+4x} &= \frac{-2}{5} \frac{1}{1-\left[\left(\frac{-4}{5}\right)(x-1)\right]} \\ &= \frac{-2}{5}\sum_{n=0}^{\infty}\left[\left(\frac{-4}{5}\right)(x-1)\right]^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{5^{n+1}}(x-1)^n \ \textit{for} \ |x-1| < \frac{5}{4}.\end{aligned}$$

We have pointwise and absolute convergence of both sums for $|x-1| < \min\left\{\frac{1}{2}, \frac{5}{4}\right\}$. It follows that

$$\frac{8x-5}{(1+4x)(3-2x)} = \sum_{n=0}^{\infty}\left[(-1)^{n+1}\frac{2^{2n+1}}{5^{n+1}} + 2^n\right](x-1)^n \ \textit{for} \ |x-1| < \frac{1}{2}.$$

The nth partial sums of a power series are polynomials and polynomials are among the nicest functions that we know. The nature of the convergence of power series allows for transmission of the nice properties of polynomials to the limit functions.

Lemma 9.1.12 *Suppose that the series $f(x) = \sum_{n=0}^{\infty} c_n (x-\alpha)^n$ converges in $\{x \in \mathbb{R} : |x-\alpha| < R\}$ with $R > 0$. Then f is continuous and differentiable in $(\alpha - R, \alpha + R)$, f' is continuous in $(\alpha - R, \alpha + R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-\alpha)^{n-1} \ \textit{for} \ \alpha - R < x < \alpha + R.$$

Space for comments and scratch work.

Proof. For any x_1 such that $|x_1 - \alpha| < R$, there exists an $h \in \mathbb{R}$ with $0 < h < R$ such that $|x_1 - \alpha| < h$. Let $I = \{x : |x - \alpha| \le h\}$. Then, by Theorem 9.1.4, $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ is uniformly convergent on I. From Theorem 8.3.3, f is continuous on I as the continuous limit of the polynomials $\sum_{j=0}^{n} c_j (x - \alpha)^j$. Consequently, f is continuous at x_1. Since the x_1 was arbitrary, we conclude that f is continuous in $|x - \alpha| < R$.

Note that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is a power series whose limit, when it is convergent, is the limit of $\left\{s_n'\right\}$ where $s_n(x) = \sum_{j=0}^{n} c_j (x - \alpha)^j$. Thus, the second part of the theorem will follow from showing that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ converges at least where f is defined; i.e., in $|x - \alpha| < R$. Let $x_0 \in \{x \in \mathbb{R} : 0 < |x - \alpha| < R\}$. Then there exists an x^* with $|x_0 - \alpha| < |x^* - \alpha| < R$. In the proof of Lemma **??**, it was shown that there exists an $M > 0$ such that $\left|c_n (x^* - \alpha)^n\right| \le M$ for $n \in \mathbb{J} \cup \{0\}$. Hence,

$$\left|nc_n (x_0 - \alpha)^{n-1}\right| = \frac{n}{|x^* - \alpha|} \cdot |c_n| \left|x^* - \alpha\right|^n \left|\frac{x_0 - \alpha}{x^* - \alpha}\right|^{n-1} \le \frac{M}{|x^* - \alpha|} \cdot nr^{n-1}$$

for $r = \left|\dfrac{x_0 - \alpha}{x^* - \alpha}\right| < 1$. From the ratio test, the series $\sum_{n=1}^{\infty} nr^{n-1}$ converges. Thus, $\sum_{n=1}^{\infty} \dfrac{M}{|x^* - \alpha|} \cdot nr^{n-1}$ is convergent and we conclude that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is convergent at x_0. Since x_0 was arbitrary we conclude that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is convergent in $|x - \alpha| < R$. Applying the Theorems 9.1.4 and 8.3.3 as before leads to the desired conclusion for f'. ∎

Theorem 9.1.13 (Differentiation and Integration of Power Series) *Suppose f is given by $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ for $x \in (\alpha - R, \alpha + R)$ with $R > 0$.*

(a) The function f possesses derivatives of all orders. For each positive integer m, the mth derivative is given by

$f^{(m)}(x) = \sum_{n=m}^{\infty} \binom{n}{m} c_n (x - \alpha)^{n-m}$ for $|x - \alpha| < R$

where $\binom{n}{m} = n(n-1)(n-2)\cdots(n-m+1)$.

(b) *For each x with $|x-\alpha|<R$, define the function F by $F(x)=\int_\alpha^x f(t)\,dt$. Then F is also given by $\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-\alpha)^{n+1}$ which is obtained by term-by-term integration of the given series for f.*

(c) *The constants c_n are given by $c_n=\dfrac{f^{(n)}(\alpha)}{n!}$.*

Excursion 9.1.14 *Use the space that is provided to complete the following proof of the Theorem.*

Proof. Since (b) follows directly from Theorem 8.3.3 and (c) follows from substituting $x=\alpha$ in the formula from (a), we need only indicate some of the details for the proof of (a).

Let

$$S=\left\{m\in\mathbb{N}: f^{(m)}(x)=\sum_{n=m}^{\infty}\binom{n}{m}c_n(x-\alpha)^{n-m}\ \ for\ \ |x-\alpha|<R\right\}$$

where $\binom{n}{m}=n(n-1)(n-2)\cdots(n-m+1)$. By Lemma 9.1.12, we know that $1\in S$. Now suppose that $k\in S$ for some k; i.e.,

$$f^{(k)}(x)=\sum_{n=k}^{\infty}n(n-1)(n-2)\cdots(n-k+1)\,c_n(x-\alpha)^{n-k}\ \text{ for }\ |x-\alpha|<R.$$

■

Remark 9.1.15 *Though we have restricted ourselves to power series in $\mathbb{R}$, note that none of what we have used relied on any properties of $\mathbb{R}$ that are not possessed* *by $\mathbb{C}$. With that in mind, we state the following theorem and note that the proofs are the same as the ones given above. However, the region of convergence is a disk rather than an interval.*

Theorem 9.1.16 *For the complex power series $c_0 + \sum_{n=1}^{\infty} c_n (z-\alpha)^n$ where α and c_n, for $n \in \mathbb{J} \cup \{0\}$, are complex constants, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and*

$$R = \begin{cases} +\infty & , \text{if} \quad \rho = 0 \\ \dfrac{1}{\rho} & , \text{if} \quad 0 < \rho < \infty \\ 0 & , \text{if} \quad \rho = +\infty \end{cases} .$$

Then the series

(i) converges only for $z = \alpha$ when $R = 0$;

(ii) converges for all values of $z \in \mathbb{C}$ when $R = +\infty$; and

(iii) converges absolutely for each $z \in N_R(\alpha)$, converges uniformly in

$$\{x \in \mathbb{R} : |x-\alpha| \le R_0\} = \overline{N_{R_0}(\alpha)}$$

for any positive $R_0 < R$, and diverges for $z \in \mathbb{C}$ such that $|z-\alpha| > R$ whenever $0 < R < \infty$. In this case, R is called the radius of convergence for the series and $N_R(\alpha) = \{z \in \mathbb{C} : |z-\alpha| < R\}$ is the corresponding disk of convergence.

Both Lemma 9.1.12 and Theorem 9.1.13 hold for the complex series in their disks of convergence.

Remark 9.1.17 *Theorem 9.1.13 tells us that every function that is representable as a power series in some segment $(\alpha - R, \alpha + R)$ for $R > 0$ has continuous derivatives of all orders there and has the form $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$. It is natural to ask if the converse is true? The answer to this question is no. Consider the function*

$$g(x) = \begin{cases} \exp\left(-1/x^2\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} .$$

It follows from l'Hôpital's Rule that g is infinitely differentiable at $x = 0$ with $g^{(n)}(0) = 0$ for all $n \in \mathbb{J} \cup \{0\}$. Since the function is clearly not identically equal to zero in any segment about 0, we can't write g in the "desired form." This prompts us to take a different approach. Namely, we restrict ourselves to a class of functions that have the desired properties.

Definition 9.1.18 *A function that has continuous derivatives of all orders in the neighborhood of a point is said to be* ***infinitely differentiable at the point****.*

Definition 9.1.19 *Let f be a real-valued function on a segment I. The function f is said to be* ***analytic at the point*** *α if it is infinitely differentiable at $\alpha \in I$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$ is valid in a segment $(\alpha - R, \alpha + R)$ for some $R > 0$. The function f is called* ***analytic on a set*** *if and only if it is analytic at each point of the set.*

Remark 9.1.20 *The example mentioned above tells us that infinitely differentiable at a point is not enough to give analyticity there.*

9.2 Some General Convergence Properties

There is a good reason why our discussion has said nothing about what happens at the points of closure of the segments of convergence. This is because there is no one conclusion that can be drawn. For example, each of the power series $\sum_{n=0}^{\infty} x^n$, $\sum_{n=0}^{\infty} \frac{x^n}{n}$, and $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ has the same "interval of convergence" $(-1, 1)$; however, the first is divergent at each of the endpoints, the second one is convergent at -1 and divergent at 1, and the last is convergent at both endpoints. The fine point to keep in mind is that the series when discussed from this viewpoint has nothing to do with the functions that the series represent if we stay in $(-1, 1)$. On the other hand, if a power series that represents a function in its segment is known to converge at an endpoint, we can say something about the relationship of that limit in relation to the given function. The precise set-up is given in the following result.

Theorem 9.2.1 *If $\sum_{n=0}^{\infty} c_n$ converges and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$, then $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} c_n$.*

Excursion 9.2.2 *Fill in what is missing in order to complete the following proof of Theorem 9.2.1.*

Proof. Let $s_n = \sum_{k=0}^{n} c_k$ and $s_{-1} = 0$. It follows that

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = \left((1-x) \sum_{n=0}^{m-1} s_n x^n \right) + s_m x^m.$$

Since $|x| < 1$ and $\lim_{m\to\infty} s_m = \sum_{n=0}^{\infty} c_n$, we have that $\lim_{m\to\infty} s_m x^m = 0$ and we conclude that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n. \tag{9.4}$$

Let $s = \sum_{n=0}^{\infty} c_n$. For each $x \in (-1, 1)$, we know that $(1-x) \sum_{n=0}^{\infty} x^n = 1$. Thus,

$$s = (1-x) \sum_{n=0}^{\infty} s x^n. \tag{9.5}$$

Suppose that $\varepsilon > 0$ is given. Because $\lim_{n\to\infty} s_n = s$ there exists a positive integer M such that $\underset{(1)}{\underline{\hspace{3cm}}}$ implies that $|s_n - s| < \frac{\varepsilon}{2}$. Let

$$K = \max \left\{ \frac{1}{2}, \max_{0 \le j \le M} \left| s - s_j \right| \right\}$$

and

$$\delta = \begin{cases} \dfrac{1}{4} & , \text{if} \quad \varepsilon \ge 2KM \\ \\ \dfrac{\varepsilon}{2KM} & , \text{if} \quad \varepsilon < 2KM \end{cases}.$$

Note that, if $2KM \le \varepsilon$, then $\dfrac{KM}{4} = \dfrac{2KM}{8} \le \dfrac{\varepsilon}{8} < \dfrac{\varepsilon}{2}$. For $1 - \delta < x < 1$, it follows that

$$(1-x) \sum_{n=0}^{M} |s_n - s| \, |x|^n \le (1-x) \underset{(2)}{\underline{\hspace{1cm}}} \sum_{n=0}^{M} |x|^n < (1-x) \underset{(2)}{\underline{\hspace{1cm}}} M < \frac{\varepsilon}{2}. \tag{9.6}$$

Use equations (9.4) and (9.5), to show that, if $1-\delta < x < 1$, then

$$|f(x) - s| < \varepsilon.$$

(3)

■

Acceptable responses are: (1) $n > M$, (2) K, (3) Hopefully, you noted that $|f(x)-s|$ is bounded above by the sum of $(1-x)\sum_{n=0}^{M}|s_n - s||x|^n$ and $(1-x)\sum_{n=M+1}^{\infty}|s_n - s||x|^n$. The first summation is bounded above by $\frac{\varepsilon}{2}$ as shown in equation (9.6) while the latter summation is bounded above by $\frac{\varepsilon}{2}\left((1-x)\sum_{n=M+1}^{\infty}|x|^n\right)$; with $x > 0$ this yields that $(1-x)\sum_{n=M+1}^{\infty}|x|^n = (1-x)\sum_{n=M+1}^{\infty}x^n < (1-x)\sum_{n=0}^{\infty}|x|^n = 1$.

An application of Theorem 9.2.1 leads to a different proof of the following result concerning the Cauchy product of convergent numerical series.

Corollary 9.2.3 *If $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$ are convergent to A, B, and C, respectively, and $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, then $C = AB$.*

Proof. For $0 \le x \le 1$, let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,\ g(x) = \sum_{n=0}^{\infty} b_n x^n, \text{ and } h(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Because each series converges absolutely for $|x| < 1$, for each fixed $x \in [0, 1)$ we have that

$$f(x)\,g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n = h(x).$$

From Theorem 9.2.1,

$$\lim_{x\to 1^-} f(x) = A,\ \lim_{x\to 1^-} g(x) = B, \text{ and } \lim_{x\to 1^-} h(x) = C.$$

The result follows from the properties of limits. ■

One nice argument justifying that a power series is analytic at each point in its interval of convergence involves rearrangement of the power series. We will make use of the Binomial Theorem and the following result that justifies the needed rearrangement.

Lemma 9.2.4 *Given the double sequence* $\{a_{ij}\}_{i,j\in J}$ *suppose that* $\sum_{j=1}^{\infty}|a_{ij}| = b_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converges. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. Let $E = \{x_n : n \in \mathbb{J} \cup \{0\}\}$ be a denumerable set such that $\lim_{n\to\infty} x_n = x_0$ and, for each $i, n \in \mathbb{J}$ let

$$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \text{ and } f_i(x_n) = \sum_{j=1}^{n} a_{ij}.$$

Furthermore, for each $x \in E$, define the function g on E by

$$g(x) = \sum_{i=1}^{\infty} f_i(x).$$

From the hypotheses, for each $i \in \mathbb{J}$, $\lim_{n\to\infty} f_i(x_n) = f_i(x_0)$. Furthermore, the definition of E ensures that for any sequence $\{w_k\}_{k=1}^{\infty} \subset E$ such that $\lim_{k\to\infty} w_k = x_0$, $\lim_{k\to\infty} f_i(w_k) = f_i(x_0)$. Consequently, from the Limits of Sequences Characterization for Continuity Theorem, for each $i \in \mathbb{J}$, f_i is continuous at x_0. Because $(\forall x)(\forall i)(i \in \mathbb{J} \wedge x \in E \Rightarrow |f_i(x)| \le b_i)$ and $\sum_{i=1}^{\infty} b_i$ converges, $\sum_{i=1}^{\infty} f_i(x)$ is uniformly convergent in E. From the Uniform Limit of Continuous Functions Theorem (8.3.3), g is continuous at x_0. Therefore,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n\to\infty} g(x_n).$$

Now

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=\lim_{n\to\infty}\sum_{i=1}^{\infty}f_i\left(x_n\right)=\lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{n}a_{ij}$$
$$=\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{\infty}a_{ij}=\sum_{ji=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}.$$

■

Theorem 9.2.5 *Suppose that* $f(x)=\sum_{n=0}^{\infty}c_nx^n$ *converges in* $|x|<R$. *For* $a\in(-R,R)$, f *can be expanded in a power series about the point* $x=a$ *which converges in* $\{x\in\mathbb{R}:|x-a|<R-|a|\}$ *and* $f(x)=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(a)}{n!}(x-a)^n$.

In the following proof, extra space is provided in order to allow more room for scratch work to check some of the claims.

Proof. For $f(x)=\sum_{n=0}^{\infty}c_nx^n$ in $|x|<R$, let $a\in(-R,R)$. Then $f(x)=\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}c_n\left[(x-a)+a\right]^n$and, from the Binomial Theorem,

$$f(x)=\sum_{n=0}^{\infty}c_n\sum_{j=0}^{n}\binom{n}{j}a^j(x-a)^{n-j}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}c_n\binom{n}{j}a^j(x-a)^{n-j}.$$

We can think of this form of summation as a “summing by rows.” In this context, the first row would could be written as $c_0(x-a)^0$, while the second row could be written as $c_1\left[\binom{1}{0}a^0(x-a)^1+\binom{1}{1}a^1(x-a)^0\right]$. In general, the $(\ell+1)\,st$ row is given by

$$c_\ell\left[\sum_{j=0}^{\ell}\binom{\ell}{j}a^j(x-a)^{\ell-j}\right]$$
$$=c_\ell\left[\binom{\ell}{0}a^0(x-a)^\ell+\binom{\ell}{1}a^1(x-a)^{\ell-1}+\cdots+\binom{\ell}{\ell}a^\ell(x-a)^0\right].$$

In the space provided write 4-5 of the rows aligned in such a way as to help you envision what would happen if we decided to arrange the summation "by columns."

If $w_{nk} \underset{def}{=} \begin{cases} c_n \binom{n}{k} a^k (x-a)^{n-k} & \text{, if} \quad k \le n \\ 0 & \text{, if} \quad k > n \end{cases}$, then it follows that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n [(x-a)+a]^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} w_{nk}.$$

In view of Lemma 9.2.4, $\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} w_{nk} \right) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} w_{nk} \right)$ whenever

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} |c_n| \binom{n}{j} |a|^j |x-a|^{n-j} = \sum_{n=0}^{\infty} |c_n| \left(|x-a| + |a| \right)^n < \infty;$$

i.e., at least when $(|x-a| + |a|) < R$. Viewing the rearrangement as "summing by columns," yields that first column as $(x-a)^0 \left[c_0 a^0 + c_1 a^1 + \cdots + \binom{n}{n} c_n a^n + \cdots \right]$ and the second column as $(x-a)^1 \left[\binom{1}{0} c_1 a^0 + \binom{2}{1} c_2 a^1 + \cdots + \binom{n}{n-1} c_n a^{n-1} + \cdots \right]$. In general, we have that the $(k+1)st$ column if given by

$$(x-a)^k \left[c_k a^0 + \binom{k+1}{1} c_{k+1} a^1 + \cdots + \binom{n}{n-k} c_n a^{n-k} + \cdots . \right]$$

Use the space that is provided to convince yourself concerning the form of the general term.

Hence, for any $x \in \mathbb{R}$ such that $|x - a| < R - |a|$, we have that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (x-a)^k \left(\sum_{n=k}^{\infty} \binom{n}{n-k} c_n a^{n-k} \right) \\ &= \sum_{k=0}^{\infty} (x-a)^k \left(\sum_{n=k}^{\infty} \frac{n!}{k!\,(n-k)!} c_n a^{n-k} \right) \\ &= \sum_{k=0}^{\infty} (x-a)^k \frac{1}{k!} \left(\sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\, a^{n-k} c_n \right) \\ &= \sum_{k=0}^{\infty} (x-a)^k \frac{f^{(k)}(a)}{k!} \end{aligned}$$

as needed. ■

Theorem 9.2.6 (Identity Theorem) *Suppose that the series* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *both converge in the segment* $S = (-R, R)$. *If*

$$E = \left\{ x \in S : \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \right\}$$

has a limit point in S, *then* $(\forall n)\,(n \in \mathbb{J} \cup \{0\} \Rightarrow a_n = b_n)$ *and* $E = S$.

Excursion 9.2.7 *Fill in what is missing in order to complete the following proof of the Identity Theorem.*

Proof. Suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ both converge in the segment $S = (-R, R)$ and that

$$E = \left\{ x \in S : \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \right\}$$

has a limit point in S. For each $n \in \mathbb{J} \cup \{0\}$, let $c_n = a_n - b_n$. Then $f(x) \underset{def}{=} \sum_{n=0}^{\infty} c_n x^n = 0$ for each $x \in E$. Let

$$A = \left\{ x \in S : x \in E' \right\} \text{ and } B = S - A = \{ x \in S : x \notin A \}$$

where E' denotes the set of limit points of E. Note that S is a connected set such that $S = A \cup B$ and $A \cap B = \emptyset$. First we will justify that B is open. If B is empty, then we are done. If B is not empty and not open, then there exists a $w \in B$ such that $\neg (\exists N_\delta (w)) (N_\delta (w) \subset B)$.

(1)

Next we will show that A is open. Suppose that $x_0 \in A$. Because $x_0 \in S$, by Theorem 9.2.5,

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n \text{ for } \underset{(2)}{\underline{\hspace{3cm}}}.$$

Suppose that $T = \left\{ j \in \mathbb{J} \cup \{0\} : d_j \neq 0 \right\} \neq \emptyset$. By the $\underset{(3)}{\underline{\hspace{3cm}}}$, T has a least element, say k. It follows that we can write $f(x) = (x - x_0)^k g(x)$ where $g(x) = \sum_{m=0}^{\infty} d_{k+m} (x - x_0)^n$ for $\underset{(2)}{\underline{\hspace{3cm}}}$. Because g is continuous at x_0, we know that $\lim_{x \to x_0} g(x) = \underset{(4)}{\underline{\hspace{1.5cm}}} = \underset{(5)}{\underline{\hspace{1.5cm}}} \neq 0$. Now we will make use of the fact that $\dfrac{|g(x_0)|}{2} > 0$ to show that there exists $\delta > 0$ such that $g(x) \neq 0$ for $|x - x_0| < \delta$.

(6)

Hence, $g(x) \neq 0$ for $|x - x_0| < \delta$ from which it follows that

$$f(x) = (x - x_0)^k g(x) \neq 0$$

in $\underset{(7)}{\underline{\hspace{3cm}}}$. But this contradicts the claim that x_0 is a limit point of zeroes of f. Therefore, $\underset{(8)}{\underline{\hspace{3cm}}}$ and we conclude that $\underset{(9)}{\underline{\hspace{6cm}}}$. Thus, $f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n = 0$ for all x in a neighborhood $N(x_0)$ of x_0. Hence, $N(x_0) \subset A$. Since x_0 was arbitrary, we conclude that

$$(\forall w)\left(w \in A \Rightarrow \underset{(10)}{\underline{\hspace{6cm}}}\right); \text{ i.e.,}$$

$\underset{(11)}{\underline{\hspace{3cm}}}$.

Because S is a connected set for which A and B are open sets such that $S = A \cup B$, $A \neq \emptyset$, and $A \cap B = \emptyset$, we conclude that $\underset{(12)}{\underline{\hspace{3cm}}}$. ■

Acceptable responses are: (1) Your argument should have generated a sequence of elements of E that converges to w. This necessitated an intermediate step because at each step you could only claim to have a point that was in E'. For example, if $N_\delta(w)$ is not contained in B, then there exists a $v \in S$ such that $v \notin B$ which places v in E'. While this does not place v in E, it does insure that any neighborhood of v contains an element of E. Let u_1 be an element of E such that $u_1 \neq w$ and $|u_1 - w| < \delta$. The process can be continued to generate a sequence of elements of E, $\{u_n\}_{n=1}^{\infty}$, that converges to w. This would place w in $A \cap B$ which contradicts the choice of B. (2) $|x - x_0| < R - |x_0|$, (3) Well-Ordering Principle, (4) $g(x_0)$, (5) d_k, (6) We've seen this one a few times before. Corresponding to $\varepsilon = \dfrac{|g(x_0)|}{2}$, there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$. The (other) triangular inequality, then yields that $|g(x_0)| - |g(x)| < \dfrac{|g(x_0)|}{2}$ which implies that $|g(x)| > \dfrac{|g(x_0)|}{2}$ whenever $|x - x_0| < \delta$. (7) $0 < |x - x_0| < \delta$, (8) $T = \emptyset$, (9) $(\forall n)(n \in \mathbb{J} \cup \{0\} \Rightarrow d_n = 0)$, (10) $(\exists N(w))(N(w) \subset A)$, (11) A is open, (12) B is empty.

9.3 Designer Series

With this section, we focus attention on one specific power series expansion that satisfies some special function behavior. Thus far we have been using the definition of e that is developed in most elementary calculus courses, namely, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. There are alternative approaches that lead us to e. In this section, we will obtain e as the value of power series at a point. In Chapter 3 of Rudin, e was defined as $\sum_{n=0}^{\infty}\frac{1}{n!}$ and it was shown that $\sum_{n=0}^{\infty}\frac{1}{n!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We get to this point from work on a specially chosen power series. The series leads to a definition for the function e^x and $\ln x$ as well as a "from series perspective" view of trigonometric functions.

For each $n \in \mathbb{J}$, if $c_n = (n!)^{-1}$, then $\limsup_{n\to\infty}\left(|c_{n+1}|\,|c_n|^{-1}\right) = 0$. Hence, the Ratio Test yields that $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for each $z \in \mathbb{C}$. Consequently, we can let

$$E(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} \text{ for } z \in \mathbb{C}. \tag{9.7}$$

Complete the following exercises in order to obtain some general properties of $E(z)$. If you get stuck, note that the following is a working excursion version of a subset of what is done on pages 178-180 of our text.

From the absolute convergence of the power series given in (9.7), for any fixed $z, w \in \mathbb{C}$, the Cauchy product, as defined in Chapter 4, of $E(z)$ and $E(w)$ can be written as

$$E(z)\,E(w) = \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{w^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^k w^{n-k}}{k!\,(n-k)!}.$$

From $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, it follows that

$$\begin{aligned} E(z)\,E(w) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\frac{n!}{k!\,(n-k)!}z^k w^{n-k} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}z^k w^{n-k}\right) \\ &= \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}. \end{aligned}$$

Therefore,

$$E(z)\,E(w) = E(z+w)\,. \tag{9.8}$$

Suppose there exists a $\zeta \in \mathbb{C}$ such that $E(\zeta) = 0$. Taking $z = \zeta$ and $w = -\zeta$ in (9.8) yields that

$$E(\zeta)\,E(-\zeta) = E(0) = 1 \tag{9.9}$$

which would contradict our second Property of the Additive Identity of a Field (Proposition 1.1.4) from which we have to have that $E(\zeta)\,E(w) = 0$ for all $w \in \mathbb{C}$. Consequently $(\forall z)\,(z \in \mathbb{C} \Rightarrow E(z) \neq 0)$.

1. For x real, use basic bounding arguments and field properties to justify each of the following.

   (a) $(\forall x)\,(x \in \mathbb{R} \Rightarrow E(x) > 0)$

   (b) $\lim\limits_{x \to -\infty} E(x) = 0$

   (c) $(\forall x)\,(\forall y)\,[(x, y \in \mathbb{R} \wedge 0 < x < y) \Rightarrow (E(x) < E(y) \wedge E(-y) < E(-x))]$

What you have just shown justifies that $E(x)$ over the reals is a strictly increasing function that is positive for each $x \in \mathbb{R}$.

2. Use the definition of the derivative to prove that

$$(\forall z)\left(z \in \mathbb{C} \Rightarrow E'(z) = E(z)\right).$$

Note that when x is real, $E'(x) = E(x)$ and $(\forall x)(x \in \mathbb{R} \Rightarrow E(x) > 0)$ with the Monotonicity Test yields an alternative justification that E is increasing in $\mathbb{R}$.

A straight induction argument allows us to claim from (9.8) that

$$(\forall n)\left[n \in \mathbb{J} \Rightarrow E\left(\sum_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} E\left(z_j\right)\right]. \tag{9.10}$$

3. Complete the justification that

$$E(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

For each $n \in \mathbb{J}$, let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \qquad and \qquad t_n = \left(1 + \frac{1}{n}\right)^n$$

(a) Use the Binomial Theorem to justify that,

$$t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).$$

(b) Use part (a) to justify that $\limsup_{n\to\infty} t_n \leq E(1)$.

(c) For $n > m > 2$, justify that

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots$$
$$+ \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

(d) Use the inequality you obtain by keeping m fixed and letting $n \to \infty$ in the equation from part (c) to obtain a lower bound on $\liminf_{n\to\infty} t_n$ and an upper bound on s_m for each m.

(e) Finish the argument.

4. Use properties of E to justify each of the following claims.

(a) $(\forall n)\,(n \in \mathbb{J} \Rightarrow E\,(n) = e^n)$.

(b) $(\forall u)\,(u \in \mathbb{Q} \wedge u > 0 \Rightarrow E\,(u) = e^u)$

Using field properties and the density of the rationals can get us to a justification that $E\,(x) = e^x$ for x real.

5. Show that, for $x > 0$, $e^x > \dfrac{x^{n+1}}{(n+1)!}$ and use the inequality to justify that $\lim\limits_{x \to +\infty} x^n e^{-x} = 0$ for each $n \in \mathbb{J}$.

9.3.1 Another Visit With the Logarithm Function

Because the function $E\ \restriction_{\mathbb{R}}$ is strictly increasing and differentiable from $\mathbb{R}$ into $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, by the Inverse Function Theorem, $E\ \restriction_{\mathbb{R}}$ has an inverse function $L : \mathbb{R}^+ \to \mathbb{R}$, defined by $E\,(L\,(y)) = y$ that is strictly increasing and differentiable on $\mathbb{R}^+$. For $x \in \mathbb{R}$, we have that $L\,(E\,(x)) = x$, for x real and the Inverse Differentiation Theorem yields that

$$L'\,(y) = \frac{1}{y} \text{ for } y > 0 \tag{9.11}$$

where $y = E(x)$. Since $E(0) = 1$, $L(1) = 0$ and (9.11) implies that

$$L(y) = \int_1^y \frac{dx}{x}$$

which gets us back to the natural logarithm as it was defined in Chapter 7 of these notes. A discussion of some of the properties of the natural logarithm is offered on pages 180-182 of our text.

9.3.2 A Series Development of Two Trigonometric Functions

The development of the real exponential and logarithm functions followed from restricting consideration of the complex series $E(z)$ to $\mathbb{R}$. In this section, we consider $E(z)$ restricted the subset of $\mathbb{C}$ consisting of numbers that are purely imaginary. For $x \in \mathbb{R}$,

$$E(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n x^n}{n!}.$$

Since

$$i^n = \begin{cases} 1 & \text{, if } \; 4 \mid n \\ i & \text{, if } \; 4 \mid (n-1) \\ -1 & \text{, if } \; 4 \mid (n-2) \\ -i & \text{, if } \; 4 \mid (n-3) \end{cases} \quad \textit{and} \quad (-i)^n = \begin{cases} 1 & \text{, if } \; 4 \mid n \\ -i & \text{, if } \; 4 \mid (n-1) \\ -1 & \text{, if } \; 4 \mid (n-2) \\ i & \text{, if } \; 4 \mid (n-3) \end{cases},$$

it follows that each of

$$C(x) = \frac{1}{2}\left[E(ix) + E(-ix)\right] \quad \textit{and} \quad S(x) = \frac{1}{2i}\left[E(ix) - E(-ix)\right] \tag{9.12}$$

have real coefficients and are, thus, real valued functions. We also note that

$$E(ix) = C(x) + iS(x) \tag{9.13}$$

from which we conclude that $C(x)$ and $S(x)$ are the real and imaginary parts of $E(ix)$, for $x \in \mathbb{R}$.

Complete the following exercises in order to obtain some general properties of $C(x)$ and $S(x)$ for $x \in \mathbb{R}$. If you get stuck, note that the following is a working excursion version of a subset of what is done on pages 182-184 of our text. Once completed, the list of properties justify that $C(x)$ and $S(x)$ for $x \in \mathbb{R}$ correspond to the $\cos x$ and $\sin x$, respectively, though appeal to triangles or the normal geometric view is never made in the development.

1. Show that $|E(ix)| = 1$.

2. By inspection, we see that $C(0) = 1$ and $S(0) = 0$. Justify that $C'(x) = -S(x)$ and $S'(x) = C(x)$.

3. Prove that $(\exists x)\left(x \in \mathbb{R}^{+} \wedge C(x) = 0\right)$.

4. Justify that there exists a smallest positive real number x_0 such that $C(x_0) = 0$.

5. Define the symbol π by $\pi = 2x_0$ where x_0 is the number from #4 and justify each of the following claims.

(a) $S\left(\frac{\pi}{2}\right) = 1$

(b) $E\left(\frac{\pi i}{2}\right) = i$

(c) $E(\pi i) = -1$

(d) $E(2\pi i) = 1$

It follows immediately from equation (9.8) that E is periodic with period $2\pi i$; i.e.,

$$(\forall z)\,(z \in \mathbb{C} \Rightarrow E(z + 2\pi i) = E(z)).$$

Then the formulas given in equation (9.12) immediately yield that both C and S are periodic with period $2\pi i$.

Also shown in Theorem 8.7 of our text is that $(\forall t)\,(t \in (0, 2\pi) \Rightarrow E(it) \neq 1)$ and

$$(\forall z)\,[(z \in \mathbb{C} \wedge |z| = 1) \Rightarrow (\exists! t)\,(t \in [0, 2\pi) \wedge E(it) = z)].$$

The following space is provided for you to enter some helpful notes towards justifying each of these claims.

9.4 Series from Taylor's Theorem

The following theorem supplies us with a sufficient condition for a given function to be representable as a power series. The statement and proof should be strongly reminiscent of Taylor's Approximating Polynomials Theorem that we saw in Chapter 6.

Theorem 9.4.1 (Taylor's Theorem with Remainder) *For $a < b$, let $I = [a, b]$. Suppose that f and $f^{(j)}$ are in $\mathcal{C}(I)$ for $1 \leq j \leq n$ and that $f^{(n+1)}$ is defined for each $x \in \operatorname{Int}(I)$. Then, for each $x \in I$, there exists a ξ with $a < \xi < x$ such that*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$ is known as the Lagrange Form of the Remainder .

Excursion 9.4.2 *Fill in what is missing to complete the following proof.*

Proof. It suffices to prove the theorem for the case $x = b$. Since f and $f^{(j)}$ are in $\mathcal{C}(I)$ for $1 \leq j \leq n$, $R_n = f(b) - \sum_{j=0}^{n} \dfrac{f^{(j)}(a)}{j!}(b-a)^j$ is well defined. In order to find a different form of R_n, we introduce a function φ. For $x \in I$, let

$$\varphi(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(b-x)^j - \frac{(b-x)^{n+1}}{(b-a)^{n+1}} R_n.$$

From the hypotheses and the properties of continuous and $\underset{(1)}{\underline{\hspace{3cm}}}$ functions, we know that φ is $\underset{(2)}{\underline{\hspace{3cm}}}$ and differentiable for each $x \in I$. Furthermore,

$$\varphi(a) = \underset{(3)}{\underline{\hspace{9cm}}} = \underset{(4)}{\underline{\hspace{1cm}}}$$

and $\varphi(b) = 0$. By $\underset{(5)}{\underline{\hspace{4cm}}}$, there exists a $\xi \in I$ such that $\varphi'(\xi) = 0$. Now

$$\varphi'(x) = -\sum_{j=1}^{n}\left[\frac{-f^{(j)}(x)}{(j-1)!}(b-x)^{j-1}\right] - \underset{(6)}{\underline{\hspace{4cm}}} + \frac{(n+1)(b-x)^n}{(b-a)^{n+1}}R_n.$$

Because

$$\sum_{j=0}^{n}\frac{f^{(j+1)}(x)}{j!}(b-x)^j = f'(x) + \sum_{j=1}^{n}\frac{f^{(j+1)}(x)}{j!}(b-x)^j = f'(x) + \sum_{j=2}^{n+1}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1},$$

it follows that

$$\begin{aligned}\varphi'(x) - \tfrac{(n+1)(b-x)^n}{(b-a)^{n+1}}R_n &= \textstyle\sum_{j=2}^{n}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1} - \left(\textstyle\sum_{j=2}^{n+1}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1}\right)\\ &= \underset{(7)}{\underline{\hspace{4cm}}}.\end{aligned}$$

If $\varphi'(\xi) = 0$, then $\dfrac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n = \dfrac{(n+1)(b-\xi)^n}{(b-a)^{n+1}}R_n$. Therefore,

$\underset{(8)}{\underline{\hspace{4cm}}}$. ■

Acceptable responses are: (1) differentiable, (2) continuous, (3) $f(b) - \sum_{j=0}^{n}\frac{f^{(j)}(x)}{j!}(b-a)^j - R_n$, (4) 0, (5) Rolle's or the Mean-Value Theorem, (6) $\sum_{j=0}^{n}\dfrac{f^{(j+1)}(x)}{j!}(b-x)^j$, (7) $-\dfrac{f^{(n+1)}(x)}{n!}(b-x)^n$, (8) $R_n = \dfrac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}$.

Remark 9.4.3 *Notice that the inequality $a < b$ was only a convenience for framing the argument; i.e., if we have the conditions holding in a neighborhood of a point α we have the Taylor's Series expansion to the left of α and to the right of α. In this case, we refer to the expansion as a Taylor's Series with Lagrange Form of the Remainder about α.*

Corollary 9.4.4 *For $\alpha \in \mathbb{R}$ and $R > 0$, suppose that f and $f^{(j)}$ are in $\mathcal{C}((\alpha - R, \alpha + R))$ for $1 \le j \le n$ and that $f^{(n+1)}$ is defined for each $x \in (\alpha - R, \alpha + R)$. Then, for each $x \in (\alpha - R, \alpha + R)$, there exists a $\xi \in (\alpha - R, \alpha + R)$ such that*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(\alpha)}{j!} (x - \alpha)^j + R_n$$

where $R_n = \dfrac{f^{(n+1)}(\xi)(x - \alpha)^{n+1}}{(n+1)!}$.

9.4.1 Some Series To Know & Love

When all of the derivatives of a given function are continuous **in a neighborhood of a point** α, the Taylor series expansion about α simply takes the form $f(x) = \sum_{j=0}^{\infty} \dfrac{f^{(j)}(\alpha)}{j!}(x - \alpha)^j$ with its radius of convergence being determined by the behavior of the coefficients. Alternatively, we can justify the series expansion by proving that the remainder goes to 0 as $n \to \infty$. There are several series expansions that we should just know and/or be able to use.

Theorem 9.4.5

(a) For all real α and x, we have

$$e^x = e^\alpha \sum_{n=0}^{\infty} \frac{(x - \alpha)^n}{n!}. \tag{9.14}$$

(b) For all real α and x, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{\sin\left(\alpha + \frac{n\pi}{2}\right)}{n!} (x - \alpha)^n \tag{9.15}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{\cos\left(\alpha + \frac{n\pi}{2}\right)}{n!} (x-\alpha)^n . \tag{9.16}$$

(c) For $|x| < 1$, we have

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \tag{9.17}$$

and

$$\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}. \tag{9.18}$$

(d) *The Binomial Series Theorem.* For each $m \in \mathbb{R}^1$ and for $|x| < 1$, we have

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n. \tag{9.19}$$

We will offer proofs for (a), and the first parts of (b) and (c). A fairly complete sketch of a proof for the Binomial Series Theorem is given after discussion of a different form of Taylor's Theorem.

Proof. Let $f(x) = e^x$. Then f is continuously differentiable on all of $\mathbb{R}$ and $f^{(n)}(x) = e^x$ for each $n \in \mathbb{J}$. For $\alpha \in \mathbb{R}$, from Taylor's Theorem with Remainder, we have that

$$f(x) = e^x = e^\alpha \sum_{j=0}^{n} \frac{1}{j!} (x-\alpha)^j + R_n(\alpha, x) \text{ where } R_n = \frac{e^\xi (x-\alpha)^{n+1}}{(n+1)!}$$

where ξ is between α and x. Note that

$$0 \le \left| e^\alpha \sum_{n=0}^{\infty} \frac{(x-\alpha)^n}{n!} - f(x) \right| = |R_n| .$$

Furthermore, because $x > \alpha$ with $\alpha < \xi < x$ implies that $e^{\xi} < e^{x}$, while $x < a$ yields that $x < \xi < \alpha$ and $e^{\xi} < e^{\alpha}$,

$$|R_n| = \frac{e^{\xi}\,|x-\alpha|^{n+1}}{(n+1)!} \leq \begin{cases} e^{x}\dfrac{|x-\alpha|^{n+1}}{(n+1)!} & \text{, if} \quad x \geq \alpha \\[2ex] e^{\alpha}\dfrac{|x-\alpha|^{n+1}}{(n+1)!} & \text{, if} \quad x < \alpha \end{cases}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{k^n}{n!} = 0$ for any fixed $k \in \mathbb{R}$, we conclude that $R_n \to 0$ as $n \to \infty$. From the Ratio Test, $\displaystyle\sum_{n=0}^{\infty} \frac{(x-\alpha)^n}{n!}$ is convergent for all $x \in \mathbb{R}$. We conclude that the series given in (9.14) converges to f for each x and α.

The expansion claimed in (9.17) follows from the Integrability of Series because

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} \qquad \textit{and} \qquad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n \text{ for } |t| < 1.$$

■

There are many forms of the remainder for "Taylor expansions" that appear in the literature. Alternatives can offer different estimates for the error entailed when a Taylor polynomial is used to replace a function in some mathematical problem. The integral form is given with the following

Theorem 9.4.6 (Taylor's Theorem with Integral Form of the Remainder) *Suppose that f and its derivatives of order up to $n+1$ are continuous on a segment I containing α. Then, for each $x \in I$, $f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(\alpha)(x-\alpha)^j}{j!} + R_n(\alpha, x)$ where*

$$R_n(\alpha, x) = \int_{\alpha}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

Proof. Since f' is continuous on the interval I, we can integrate the derivative to obtain

$$f(x) = f(\alpha) + \int_{\alpha}^{x} f'(t)\,dt.$$

As an application of Integration-by-Parts, for fixed x, corresponding to $u = f'(t)$ and $dv = dt$, $du = f''(t)\,dt$ and we can choose $v = -(x-t)$. Then

$$\begin{aligned} f(x) &= f(\alpha) + \int_\alpha^x f'(t)\,dt = f(\alpha) - f'(t)(x-t)\,|_{t=\alpha}^{t=x} + \int_\alpha^x (x-t)\, f''(t)\,dt \\ &= f(\alpha) + f'(\alpha)(x-\alpha) + \int_\alpha^x (x-t)\, f''(t)\,dt. \end{aligned}$$

Next suppose that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(j)}(\alpha)(x-\alpha)^j}{j!} + \int_\alpha^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt$$

and $f^{(k+1)}$ is differentiable on I. Then Integration-by-Parts can be applied to $\int_\alpha^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt$; taking $u = f^{(k+1)}(t)$ and $dv = \dfrac{(x-t)^k}{k!}dt$ leads to $u = f^{(k+2)}(t)\,dt$ and $v = -\dfrac{(x-t)^{k+1}}{(k+1)!}$. Substitution and simplification justifies the claim. ■

As an application of Taylor's Theorem with Integral Form of Remainder, complete the following proof of the *The Binomial Series Theorem.*

Proof. For fixed $m \in \mathbb{R}^1$ and $x \in \mathbb{R}$ such that $|x| < 1$, from Taylor's Theorem with Integral Form of Remainder, we have

$$(1+x)^m = 1 + \sum_{n=1}^{k} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + R_k(0,x).$$

where

$$R_k(0,x) = \int_0^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

We want to show that

$$R_k(0,x) = \int_0^x m(m-1)\cdots(m-k)\frac{(x-t)^k}{k!}(1+t)^{m-k-1}\,dt \longrightarrow 0 \text{ as } k \to \infty$$

for all x such that $|x| < 1$. Having two expressions in the integrand that involve a power k suggests a rearrangement of the integrand; i.e.,

$$R_k(0,x) = \int_0^x \frac{m(m-1)\cdots(m-k)}{k!}\left(\frac{x-t}{1+t}\right)^k (1+t)^{m-1}\,dt.$$

We discuss the behavior of $(1+t)^{m-1}$, when t is between 0 and x, and $\int_0^x \left(\frac{x-t}{1+t}\right)^k dt$ separately.

On one hand, we have that

$$(1+t)^{m-1} \le 1 \text{ whenever } (m \ge 1 \wedge -1 < t \le 0) \vee (m \le 1 \wedge 1 > t \ge 0).$$

On the other hand, because t is between 0 and x, if $m \ge 1 \wedge x \ge 0$ or $m \le 1 \wedge x \le 0$, then

$$g(t) = (1+t)^{m-1} \text{ implies that } g'(t) = (m-1)(1+t)^{m-2}\begin{cases} >0 & \text{for} \quad m>1 \\ \\ <0 & \text{for} \quad m<1 \end{cases}.$$

Consequently, if $m \ge 1 \wedge x \ge 0$, then $0 < t < x$ and g increasing yields the $g(t) \le g(x)$; while $m \le 1 \wedge x \le 0$, $0 < t < x$ and g decreasing, implies that $g(x) \ge g(t)$. With this in mind, define $C_m(x)$, for $|x| < 1$ by

$$C_m(x) = \begin{cases} (1+x)^{m-1} & , \quad m \ge 1, x \ge 0 \text{ OR } m \le 1, x \le 0 \\ 1 & , \quad m \ge 1, x < 0 \text{ OR } m \le 1, x \ge 0 \end{cases}.$$

We have shown that

$$(1+t)^{m-1} = C_m(t) \le C_m(x), \text{ for } t \text{ between 0 and } x. \tag{9.20}$$

Next, we turn to $\int_0^x \left(\frac{x-t}{1+t}\right)^k dt$. Since we want to bound the behavior in terms of x or a constant, we want to get the x out of the limits of integration. The standard way to do this is to effect a change of variable. Let $t = xs$. Then $dt = xds$ and

$$\int_0^x \left(\frac{x-t}{1+t}\right)^k dt = \int_0^1 x^{k+1}\left(\frac{1-s}{1+xs}\right)^k ds.$$

Since $s(1+x) \geq 0$, we immediately conclude that $\left(\dfrac{1-s}{1+xs}\right)^k \leq 1$. Hence, it follows that

$$\left|\int_0^x \left(\frac{x-t}{1+t}\right)^k dt\right| \leq |x|^{k+1}. \tag{9.21}$$

From (9.20) and (9.21), if follows that

$$\begin{aligned} 0 &\leq |R_k(0,x)| \\ &\leq \int_0^1 \frac{m(m-1)\cdots(m-k)}{k!}|x|^{k+1} C_m(x)\,dt \\ &= \frac{|m(m-1)\cdots(m-k)|}{k!}|x|^{k+1} C_m(x). \end{aligned}$$

For $u_k(x) = \dfrac{|m(m-1)\cdots(m-k)|}{k!}|x|^{k+1} C_m(x)$ consider $\sum_{n=1}^{\infty} u_n(x)$. Because

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \left|\frac{m}{n+1} - 1\right| |x| \to |x| \text{ as } n \to \infty,$$

$\sum_{n=1}^{\infty} u_n(x)$ is convergent for $|x| < 1$. From the nth term test, it follows that $u_k(x) \to 0$ as $k \to \infty$ for all x such that $|x| < 1$. Finally, from the Squeeze Principle, we conclude that $R_k(0,x) \to 0$ as $k \to \infty$ for all x with $|x| < 1$. ■

9.4.2 Series From Other Series

There are some simple substitutions into power series that can facilitate the derivation of series expansions from some functions for which series expansions are "known." The proof of the following two examples are left as an exercise.

Theorem 9.4.7 *Suppose that $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ for $|u-b| < R$ with $R > 0$.*

(a) If $b = kc + d$ with $k \neq 0$, then $f(kx+d) = \sum_{n=0}^{\infty} c_n k^n (x-c)^n$ for $|x-c| < \dfrac{R}{|k|}$.

(b) *For every fixed positive integer* k, $f\left[(x-c)^k+b\right]=\sum_{n=0}^{\infty}c_n(x-c)^{kn}$ *for* $|x-c|<R^{1/k}$.

The proofs are left as an exercise.

We close this section with a set of examples.

Example 9.4.8 *Find the power series expansion for* $f(x)=\dfrac{1}{1-x^2}$ *about the point* $\alpha=\dfrac{1}{2}$*and give the radius of convergence.*

Note that

$$\begin{aligned}
f(x)&=\frac{1}{(1+x)(1-x)}=\frac{1}{2}\left[\frac{1}{(1-x)}+\frac{1}{(1+x)}\right]\\
&=\frac{1}{2}\left[\frac{1}{\left(\frac{1}{2}-\left(x-\frac{1}{2}\right)\right)}+\frac{1}{\left(\frac{3}{2}+\left(x-\frac{1}{2}\right)\right)}\right]\\
&=\frac{1}{\left(1-2\left(x-\frac{1}{2}\right)\right)}+\frac{1}{3}\frac{1}{\left(1+\frac{2}{3}\left(x-\frac{1}{2}\right)\right)}.
\end{aligned}$$

Since $\dfrac{1}{\left(1-2\left(x-\frac{1}{2}\right)\right)}=\displaystyle\sum_{n=0}^{\infty}2^n\left(x-\frac{1}{2}\right)^n$ *for* $\left|2\left(x-\frac{1}{2}\right)\right|<1$ *or* $\left|x-\frac{1}{2}\right|<\frac{1}{2}$*and* $\dfrac{1}{\left(1+\frac{2}{3}\left(x-\frac{1}{2}\right)\right)}=\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n\left(x-\frac{1}{2}\right)^n$ *for* $\left|\frac{2}{3}\left(x-\frac{1}{2}\right)\right|<1$ *or* $\left|x-\frac{1}{2}\right|<\frac{3}{2}$*. Because both series expansions are valid in* $\left|x-\frac{1}{2}\right|<\frac{1}{2}$*, it follows that*

$$f(x)=\sum_{n=0}^{\infty}\left(2^n+\frac{(-2)^n}{3^{n+1}}\right)\left(x-\frac{1}{2}\right)^n\ \text{for}\ \left|x-\frac{1}{2}\right|<\frac{1}{2}.$$

Example 9.4.9 *Find the power series expansion for $g(x) = \arcsin(x)$ about the point $\alpha = 0$.*

We know that, for $|x| < 1$, $\arcsin x = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$. From the Binomial Series Theorem, for $m = -\frac{1}{2}$, we have that
$(1+u)^{-1/2} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} u^n$ *for $|u| < 1$. Since $|u| < 1$ if and only if $\left|u^2\right| < 1$, it follows that*

$$\left(1-t^2\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} (-1)^n t^{2n} \text{ for } |t| < 1.$$

Note that

$$\underbrace{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}_{n\,terms}(-1)^n = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+(n-1)\right)$$
$$= \frac{1\cdot 3\cdots(2n-1)}{2^n}.$$

Consequently,

$$\left(1-t^2\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!} t^{2n} \text{ for } |t| < 1$$

with the convergence being uniform in each $|t| \leq h$ for any h such that $0 < h < 1$. Applying the Integration of Power Series Theorem (Theorem 9.1.13), it follows that

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{(2n+1)\,2^n n!} x^{2n+1}, \text{ for } |x| < 1$$

where $\arcsin 0 = 0$.

Excursion 9.4.10 *Find the power series expansion about $\alpha = 0$ for $f(x) =$*

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ *and give the radius of convergence.*

Upon noting that $f(0) = 1$, $f'(0) = 0$, $f^{(2n)}(x) = f(x)$ and $f^{(2n-1)}(x) = f'(x)$, it follows that we can write f as $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

Example 9.4.11 *Suppose that we want the power series expansion for* $f(x) = \ln(\cos(x))$ *about the point* $\alpha = 0$. *Find the Taylor Remainder* R_3 *in both the Lagrange and Integral forms.*

Since the Lagrange form for R_3 *is given by* $\dfrac{f^{(4)}(\xi)}{4!}x^4$ *for* $0 < \xi < x$, *we have that*

$$R_3 = \frac{-\left(4\sec^2\xi\tan^2\xi + 2\sec^4\xi\right)x^4}{24} \text{ for } 0 < \xi < x.$$

In general, the integral form is given by $R_n(\alpha, x) = \displaystyle\int_\alpha^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$. *For this problem,* $\alpha = 0$ *and* $n = 3$, *which gives*

$$R_3(\alpha, x) = \int_\alpha^x \frac{-(x-t)^3}{6}\left(4\sec^2 t\tan^2 t + 2\sec^4 t\right)dt$$

Excursion 9.4.12 *Fill in what is missing in the following application of the geometric series expansion and the theorem on the differentiation of power series to find* $\sum_{n=1}^{\infty} \dfrac{3n-1}{4^n}$.

Because $\sum_{n=1}^{\infty} x^n = \dfrac{x}{1-x}$ *for* $|x| < 1$*, it follows that*

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \underset{(1)}{\underline{\hspace{6cm}}}.$$

From the theorem on differentiation of power series,

$$\sum_{n=1}^{\infty} n x^n = x \left(\frac{x}{1-x}\right)' = \underset{(2)}{\underline{\hspace{6cm}}}$$

in $|x| < 1$*. Hence,*

$$3\sum_{n=1}^{\infty} \frac{n}{4^n} = \underset{(3)}{\underline{\hspace{6cm}}}.$$

Combining the results yields that

$$\sum_{n=1}^{\infty} \frac{3n-1}{4^n} = \sum_{n=1}^{\infty} \left(3\frac{n}{4^n} - \frac{1}{4^n}\right) = \underset{(4)}{\underline{\hspace{4.5cm}}}.$$

*******Expected responses are: (1) $\dfrac{1}{3}$, (2) $x(1-x)^{-2}$, (3) $\dfrac{4}{3}$, and (4) 1.*******

9.5 Fourier Series

Our power series expansions are only useful in terms of representing functions that are nice enough to be continuously differentiable, infinitely often. We would like to be able to have series expansions that represent functions that are not so nicely behaved. In order to obtain series expansions of functions for which we may have only a finite number of derivatives at some points and/or discontinuities at other points, we have to abandon the power series form and seek other "generators." The set of generating functions that lead to what is known as Fourier series is $\{1\} \cup \{\cos nx : n \in \mathbb{J}\} \cup \{\sin nx : n \in \mathbb{J}\}$.

Definition 9.5.1 *A **trigonometric series** is defined to be a series that can be written in the form*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{9.22}$$

where $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of constants.*

Definition 9.5.2 *A **trigonometric polynomial** is a finite sum in the form*

$$\sum_{k=-N}^{N} c_k e^{ikx},\ x \in \mathbb{R} \tag{9.23}$$

where c_k, $k = -N, -N+1, ..., N-1, N$, is a finite sequence of constants.

Remark 9.5.3 *The trigonometric polynomial given in (9.23) is real if and only if $c_{-n} = \overline{c_n}$ for $n = 0, 1, ..., N$.*

Remark 9.5.4 *It follows from equation (9.12) that the Nth partial sum of the trigonometric series given in (9.22) can be written in the form given in (9.23). Consequently, a sum in the form $\frac{1}{2}a_0 + \sum_{k=0}^{N}(a_k \cos kx + b_k \sin kx)$ is also called a trigonometric polynomial. The form used is often a matter of convenience.*

The following "orthogonality relations" are sometimes proved in elementary calculus courses as applications of some methods of integration:

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} \pi & \text{, if} \quad m = n \\ 0 & \text{, if} \quad m \neq n \end{cases}$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for all } m, n \in \mathbb{J}.$$

We will make use of these relations in order to find useful expressions for the coefficients of trigonometric series that are associated with specific functions.

Theorem 9.5.5 *If f is a continuous function on $I = [-\pi, \pi]$ and the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to f on I, then*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt \textit{ for } n \in \mathbb{J} \cup \{0\} \tag{9.24}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt. \tag{9.25}$$

Proof. For each $k \in \mathbb{J}$, let $s_k(x) = \frac{1}{2}a_0 + \sum_{m=1}^{k}(a_m \cos mx + b_m \sin mx)$ and suppose that $\varepsilon > 0$ is given. Because $s_k \underset{I}{\rightrightarrows} f$ there exists a positive integer M such that $k > M$ implies that $|s_k(x) - f(x)| < \varepsilon$ for all $x \in I$. It follows that, for each fixed $n \in \mathbb{J}$,

$$|s_k(x)\cos nx - f(x)\cos nx| = |s_k(x) - f(x)|\,|\cos nx| \le |s_k(x) - f(x)| < \varepsilon$$

and

$$|s_k(x)\sin nx - f(x)\sin nx| = |s_k(x) - f(x)|\,|\sin nx| \le |s_k(x) - f(x)| < \varepsilon$$

for all $x \in I$ and all $k > M$. Therefore, $s_k(x)\cos nx \underset{I}{\rightrightarrows} f(x)\cos nx$ and $s_k(x)\sin nx \underset{I}{\rightrightarrows} f(x)\sin nx$ for each fixed n. Then for fixed $n \in \mathbb{J}$,

$$f(x)\cos nx = \frac{1}{2}a_0\cos nx + \sum_{m=1}^{\infty}(a_m \cos mx \cos nx + b_m \sin mx \cos nx)$$

and

$$f(x)\sin nx = \frac{1}{2}a_0\sin nx + \sum_{m=1}^{\infty}(a_m \cos mx \sin nx + b_m \sin mx \sin nx);$$

the uniform convergence allows for term-by-term integration over the interval $[-\pi, \pi]$ which, from the orthogonality relations yields that

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \pi a_n \quad \textit{and} \quad \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \pi b_n.$$

■

Definition 9.5.6 *If f is a continuous function on $I = [-\pi, \pi]$ and the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to f on I, then the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the ***Fourier series*** *for the function f and the numbers a_n and b_n are called the* ***Fourier coefficients*** *of f.*

Given any Riemann integrable function on an interval $[-\pi, \pi]$, we can use the formulas given by (9.24) and (9.25) to calculate Fourier coefficients that could be associated with the function. However, the Fourier series formed using those coefficients may not converge to f. Consequently, a major concern in the study of Fourier series is isolating or describing families of functions for which the associated Fourier series can be identified with the "generating functions"; i.e., we would like to find classes of functions for which each Fourier series generated by a function in the class converges to the generating function.

The discussion of Fourier series in our text highlights some of the convergence properties of Fourier series and the estimating properties of trigonometric polynomials. The following is a theorem that offers a condition under which we have pointwise convergence of the associated Fourier polynomials to the function. The proof can be found on pages 189-190 of our text.

Theorem 9.5.7 *For f a periodic function with period 2π that is Riemann integrable on $[-\pi, \pi]$, let*

$$s_N(f;x) = \sum_{m=-N}^{N} c_m e^{imx} \text{ where } c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{imt} dt.$$

If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that

$$|f(x+t) - f(x)| \leq M\,|t|$$

for all $t \in (-\delta, \delta)$, then $\lim_{N\to\infty} s_N(f;x) = f(x)$.

The following theorem that is offered on page 190 of our text can be thought of as a trigonometric polynomial analog to Taylor's Theorem with Remainder.

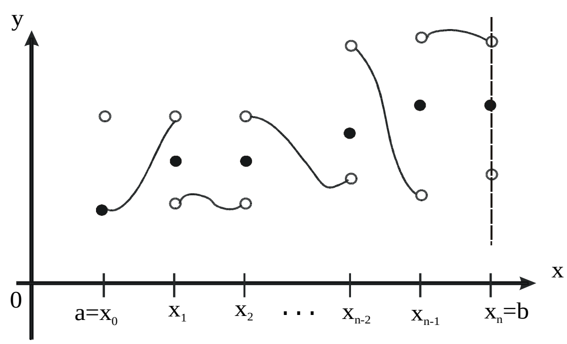
Theorem 9.5.8 *If f is a continuous function that is periodic with period 2π and $\varepsilon > 0$, then there exists a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$.*

For the remainder of this section, we will focus briefly on the process of finding Fourier series for a specific type of functions.

Definition 9.5.9 *A function f defined on an interval $I = [a, b]$ is* ***piecewise continuous*** *on I if and only if there exists a partition of I, $\{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ such that (i) f is continuous on each segment (x_{k-1}, x_k) and (ii) $f(a+)$, $f(b-)$ and, for each $k \in \{1, 2, \ldots, n-1\}$ both $f(x_k+)$ and $f(x_k-)$ exist.*

Definition 9.5.10 *If f is piecewise continuous on an interval I and $x_k \in I$ is a point of discontinuity, then $f(x_k+) - f(x_k-)$ is called the* ***jump at*** *x_k. A piecewise continuous function on an interval I is said to be* ***standardized*** *if the values at points of discontinuity are given by $f(x_k) = \frac{1}{2}\left[f(x_k+) + f(x_k-)\right]$.*

Note that two piecewise continuous functions that differ only at a finite number of points will generate the same associated Fourier coefficients. The following figure illustrates a standardized piecewise continuous function.



Definition 9.5.11 *A function f is* ***piecewise smooth*** *on an interval $I = [a, b]$ if and only if (i) f is piecewise continuous on I, and (ii) f' both exists and is piecewise continuous on the segments corresponding to where f is continuous. The function f is* ***smooth*** *on I if and only if f and f' are continuous on I.*

Definition 9.5.12 *Let f be a piecewise continuous function on $I = [-\pi, \pi]$. Then the* ***periodic extension*** *$\tilde{f}$ of f is defined by*

$$\tilde{f}(x) = \begin{cases} f(x) & , \text{if} \quad -\pi \le x < \pi \\ \dfrac{f(-\pi+) + f(\pi-)}{2} & , \text{if} \quad x = \pi \vee x = -\pi \\ \tilde{f}(x - 2\pi) & , \text{if} \quad x \in \mathbb{R} \end{cases},$$

where f is continuous and by $\tilde{f}(x) = \dfrac{f(x+) + f(x-)}{2}$ an each point of discontinuity of f in $(-\pi, \pi)$.

It can be shown that, if f is periodic with period 2π and piecewise smooth on $[-\pi, \pi]$, then the Fourier series of f converges for every real number x to the limit $\frac{f(x+)+f(x-)}{2}$. In particular, the series converges to the value of the given function f at every point of continuity and to the standardized value at each point of discontinuity.

Example 9.5.13 *Let $f(x) = x$ on $I = [-\pi, \pi]$. Then, for each $j \in \mathbb{Z}$, the periodic extension $\tilde{f}$ satisfies $\tilde{f}(j\pi) = 0$ and the graph in each segment of the form $(j\pi, (j+1)\pi)$ is identical to the graph in $(-\pi, \pi)$. Use the space provided to sketch a graph for f.*

The associated Fourier coefficients for f are given by (9.24) and (9.25) from Theorem 9.5.5. Because $t\cos nt$ is an odd function,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0 \text{ for } n \in \mathbb{J}\cup\{0\}.$$

According to the formula for integration-by parts, if $n \in \mathbb{J}$, then

$$\int t\sin nt\,dt = -\frac{t\cos nt}{n} + \frac{1}{n}\int \cos nt\,dt + C$$

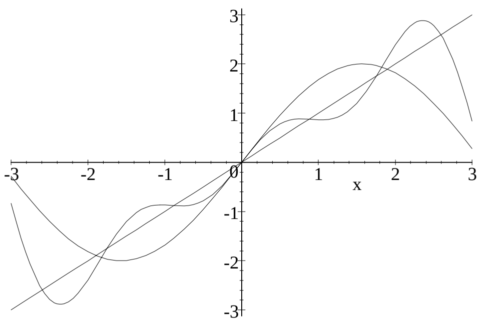
for any constant C. Hence, $\cos n\pi = (-1)^n$ for $n \in \mathbb{J}$ yields that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\sin nt\,dt = \begin{cases} \dfrac{2}{n} & , \text{if} \quad 2 \nmid n \\ -\dfrac{2}{n} & , \text{if} \quad 2 \mid n \end{cases}.$$

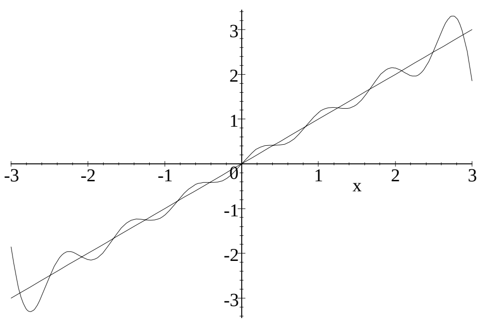
Thus, the Fourier series for f is given by

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

The following figure shows the graphs of f, $s_1(x) = 2\sin x$, *and* $s_3(x) = 2\sin x - \sin 2x + \dfrac{2}{3}\sin 3x$ *in* $(-3, 3)$.



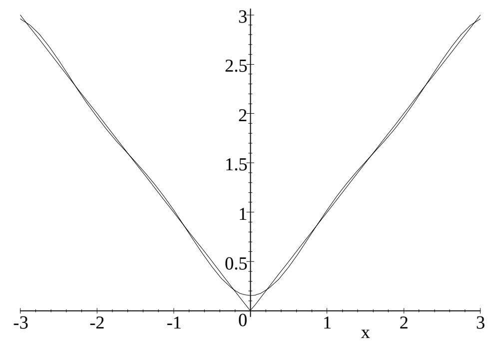
while the following shows the graphs of f *and* $s_7(x) = 2\sum_{n=1}^{7}(-1)^{n+1}\dfrac{\sin nx}{n}$ *in* $(-3, 3)$.



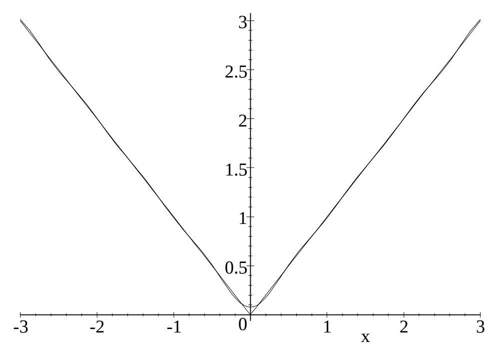
Example 9.5.14 *Find the Fourier series for* $f(x) = |x|$ *in* $-\pi \leq x \leq \pi$. *Note that, because* f *is an even function,* $f(t)\sin nt$ *is odd.*

Hopefully, you noticed that $b_n = 0$ for each $n \in \mathbb{J}$ and $a_n = 0$ for each even natural number n. Furthermore, $a_0 = \pi$ while, integration-by-parts yielded that $a_n = -4n^{-2}(\pi)^{-1}$ for n odd.

The following figure shows $f(x) = |x|$ and the corresponding Fourier polynomial $s_3(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\cos x + \dfrac{1}{9}\cos 3x\right]$ in $(-3, 3)$.



We close with a figure that shows $f(x) = |x|$ and the corresponding Fourier polynomial $s_7(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\sum_{n=1}^{7}\dfrac{1}{(2n-1)^2}\cos(2n-1)x$ in $(-3, 3)$. Note how the difference is almost invisible to the naked eye.



9.6 Problem Set I

1. Apply the Geometric Series Expansion Theorem to find the power series expansion of $f(x) = \dfrac{3}{4-5x}$ about $\alpha = 2$ and justify where the expansion is valid. Then verify that the coefficients obtained satisfy the equation given in part (c) of Theorem 9.1.13.

2. Let

$$g(x) = \begin{cases} \exp\left(-1/x^2\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

where $\exp w = e^w$.

   (a) Use the Principle of Mathematical Induction to prove that, for each $n \in \mathbb{J}$ and $x \in \mathbb{R} - \{0\}$, $g^{(n)}(x) = x^{-3n} P_n(x) \exp\left(-1/x^2\right)$ where $P_n(x)$ is a polynomial.

   (b) Use l'Hôpital's Rule to justify that, for each $n \in \mathbb{J} \cup \{0\}$, $g^{(n)}(0) = 0$.

3. Use the Ratio Test, as stated in these Companion Notes, to prove Lemma 9.1.8.

4. For each of the following use either the Root Test or the Ratio Test to find the "interval of convergence."

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(7x)^n}{n!}$

   (b) $\displaystyle\sum_{n=0}^{\infty} 3n(x-1)^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{\sqrt[n]{n}}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 (x-3)^n}{(2n)!}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln n)\, 3^n (x+1)^n}{5^n n\sqrt{n}}$

5. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\ln(n+1)\, 2^n (x+1)^n}{n+1}$ is convergent in $\left(-\frac{3}{2}, -\frac{1}{2}\right)$.

6. For each of the following, derive the power series expansion about the point α and indicate where it is valid. Remember to briefly justify your work.

   (a) $g(x) = \dfrac{3x-1}{3x+2}$; $\alpha = 1$

   (b) $h(x) = \ln x$; $\alpha = 2$

7. For each of the following, find the power series expansion about $\alpha = 0$.

   (a) $f(x) = \left(1 - x^2\right)^{-1/2}$

   (b) $f(x) = (1-x)^{-2}$

   (c) $f(x) = (1-x)^{-3}$

   (d) $f(x) = \arctan\left(x^2\right)$

8. Find the power series expansion for $h(x) = \ln\left(x + \sqrt{1+x^2}\right)$ about $\alpha = 0$ and its interval of convergence. (Hint: Consider h'.)

9. Prove that if $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ for $|u-b| < R$ with $R > 0$ and $b = kc + d$ with $k \neq 0$, then $f(kx+d) = \sum_{n=0}^{\infty} c_n k^n (x-c)^n$ for $|x-c| < \dfrac{R}{|k|}$.

10. Prove that if $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ for $|u-b| < R$ with $R > 0$, then $f\left[(x-c)^k + b\right] = \sum_{n=0}^{\infty} c_n (x-c)^{kn}$ in $|x-c| < R^{1/k}$ for any fixed positive integer k.

11. Find the power series expansions for each of the following about the specified point α.

    (a) $f(x) = (3x+5)^{-2}$; $\alpha = 1$

    (b) $g(x) = \sin x \cos x$; $\alpha = \dfrac{\pi}{4}$

    (c) $h(x) = \ln\left(\dfrac{x}{(1-x)^2}\right)$; $\alpha = 2$

12. Starting from the geometric series $\sum_{n=1}^{\infty} x^n = x(1-x)^{-1}$ for $|x| < 1$, derive closed form expressions for each of the following.

    (a) $\sum_{n=1}^{\infty} (n+1)\, x^n$

    (b) $\sum_{n=1}^{\infty} (n+1)\, x^{2n}$

    (c) $\sum_{n=1}^{\infty} (n+1)\, x^{n+2}$

    (d) $\sum_{n=1}^{\infty} \frac{n+1}{n+3} x^{n+3}$

13. Find each of the following, justifying your work carefully.

    (a) $\sum_{n=1}^{\infty} \frac{n^2+2n-1}{3^n}$

    (b) $\sum_{n=1}^{\infty} \frac{n\,(3^n - 2^n)}{6^n}$

14. Verify the orthogonality relations that were stated in the last section.

    (a) $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} \pi & \text{, if } \quad m = n \\ 0 & \text{, if } \quad m \neq n \end{cases}$.

    (b) $\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$ for all $m, n \in \mathbb{J}$.

15. For each of the following, verify that the given Fourier series is the one associated with the function f according to Theorem 9.5.5.

    (a) $f(x) = \begin{cases} 0 & \text{, if } \quad -\pi \le x < 0 \\ 1 & \text{, if } \quad 0 \le x \le \pi \end{cases}$; $\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$

(b) $f(x) = x^2$ for $x \in [-\pi, \pi]$; $\dfrac{\pi^2}{3} + 4\sum_{k=1}^{\infty} (-1)^k \dfrac{\cos(kx)}{k^2}$

(c) $f(x) = \sin^2 x$ for $x \in [-\pi, \pi]$; $\dfrac{1}{2} - \dfrac{\cos 2x}{2}$