Chapter 5

Functions on Metric Spaces and Continuity

When we studied real-valued functions of a real variable in calculus, the techniques and theory built on properties of continuity, differentiability, and integrability. All of these concepts are defined using the precise idea of a limit. In this chapter, we want to look at functions on metric spaces. In particular, we want to see how mapping metric spaces to metric spaces relates to properties of subsets of the metric spaces.

5.1 Limits of Functions

Recall the definitions of limit and continuity of real-valued functions of a real variable.

Definition 5.1.1 *Suppose that f is a real-valued function of a real variable, $p \in \mathbb{R}$, and there is an interval I containing p which, except possibly for p is in the domain of f. Then* ***the limit of f as x approaches p*** *is L if and only if*

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon))\,(\delta > 0 \wedge (\forall x)\,(0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon)))\,.$$

In this case, we write $\lim_{x \to p} f(x) = L$ which is read as "the limit of f of x as x approaches p is equal to L."

Definition 5.1.2 *Suppose that f is a real-valued function of a real variable and $p \in \operatorname{dom}(f)$. Then f is* ***continuous at*** *p if and only if $\lim_{x \to p} f(x) = f(p)$.*

These are more or less the way limit of a function and continuity of a function at a point were defined at the time of your first encounter with them. With our new terminology, we can relax some of what goes into the definition of limit. Instead of going for an interval (with possibly a point missing), we can specify that the point p be a limit point of the domain of f and then insert that we are only looking at the real numbers that are both in the domain of the function and in the open interval. This leads us to the following variation.

Definition 5.1.3 *Suppose that f is a real-valued function of a real variable,* $\operatorname{dom}(f) = A$, *and $p \in A'$ (i.e., p is a limit point of the domain of f). Then* ***the limit of*** f ***as*** x ***approaches*** p *is L if and only if*

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon) > 0)\, \left[(\forall x)\,(x \in A \wedge 0 < |x-p| < \delta \Rightarrow |f(x) - L| < \varepsilon)\right])$$

Example 5.1.4 *Use the definition to prove that* $\lim\limits_{x \to 3} \left(2x^2 + 4x + 1\right) = 31$.

Before we offer a proof, we'll illustrate some "expanded "scratch work that leads to the information needed in order to offer a proof. We want to show that, corresponding to each $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < |x-3| < \delta \Rightarrow \left|\left(2x^2+4x+1\right) - 31\right| < \varepsilon$. The easiest way to do this is to come up with a δ that is a function of ε. Note that

$$\left|\left(2x^2+4x+1\right) - 31\right| = \left|2x^2+4x-30\right| = 2\,|x-3|\,|x+5|\,.$$

The $|x-3|$ is good news because it is ours to make as small as we choose. But if we restrict $|x-3|$ there is a corresponding restriction on $|x+5|$; to take care of this part we will put a cap on δ which will lead to simpler expressions. Suppose that we place a 1^{st} restriction on δ of requiring that $\delta \le 1$. If $\delta \le 1$, then $0 < |x-3| < \delta \le 1 \Rightarrow |x+5| = |(x-3)+8| \le |x-3| + 8 < 9$. Now

$$\left|\left(2x^2+4x+1\right) - 31\right| = 2\,|x-3|\,|x+5| < 2 \cdot \delta \cdot 9 \le \varepsilon$$

whenever $\delta \le \dfrac{\varepsilon}{18}$. To get both bounds to be in effect we will take $\delta = \max\left\{1, \dfrac{\varepsilon}{18}\right\}$. This concludes that "expanded "scratch work.

Proof. *For $\varepsilon > 0$, let $\delta = \max\left\{1, \dfrac{\varepsilon}{18}\right\}$. Then*

$$0 < |x-3| < \delta \le 1 \Rightarrow |x+5| = |(x-3)+8| \le |x-3| + 8 < 9$$

and

$$\left|\left(2x^2+4x+1\right)-31\right| = 2\left|x-3\right|\left|x+5\right| < 2\cdot\delta\cdot 9 \le 18\cdot\frac{\varepsilon}{18} = \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that, for every* $\varepsilon > 0$*, there exists a* $\delta = \min\left\{1, \dfrac{\varepsilon}{18}\right\} > 0$*, such that* $0 < |x-3| < \delta \Rightarrow \left|\left(2x^2+4x+1\right)-31\right| < \varepsilon$*; i.e.,* $\lim_{x\to 3}\left(2x^2+4x+1\right) = 31$. ■

Excursion 5.1.5 *Use the definition to prove that* $\lim_{x\to 1}\left(x^2+5x\right) = 6$.

Space for scratch work.

A Proof.

*******For this one, the δ that you define will depend on the nature of the first restriction that you placed on δ in order to obtain a nice upper bound on $|x+6|$; if you chose $\delta \le 1$ as your first restriction, then $\delta = \min\left\{1, \dfrac{\varepsilon}{8}\right\}$ would have been what worked in the proof that was offered.*******

You want to be careful not to blindly take $\delta \le 1$ as the first restriction. For example, if you are looking at the greatest integer function as $x \to \dfrac{1}{2}$, you would need to make sure that δ never exceeded $\dfrac{1}{2}$ in order to stay away from the nearest "jumps"; if you have a rational function for which $\dfrac{1}{2}$ is a zero of the denominator and you are looking at the limit as $x \to \dfrac{1}{4}$, then you couldn't let δ be as great at $\dfrac{1}{4}$

so you might try taking $\delta \le \frac{1}{6}$ as a first restriction. Our next example takes such a consideration into account.

Example 5.1.6 *Use the definition to prove that* $\lim_{x\to -1} \frac{x^2+3}{2x+1} = -4$.

Space for scratch work.

Proof. *For* $\varepsilon > 0$, *let* $\delta = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{25}\right\}$. *From* $0 < |x+1| < \delta \le \frac{1}{4}$, *we have that*

$$|x+7| = |(x+1)+6| < |x+1| + 6 < \frac{1}{4} + 6 = \frac{25}{4}$$

and

$$|2x+1| = 2\left|x + \frac{1}{2}\right| = 2\left|(x+1) - \frac{1}{2}\right| \ge 2\left||x+1| - \frac{1}{2}\right| > 2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Furthermore,

$$\left|\left(\frac{x^2+3}{2x+1}\right) - (-4)\right| = \left|\frac{x^2+8x+7}{2x+1}\right| = \frac{|x+1|\,|x+7|}{|2x+1|} < \frac{\delta\cdot\left(\frac{25}{4}\right)}{\frac{1}{2}} = \frac{25\cdot\delta}{2} \le \frac{25}{2}\cdot\frac{2\varepsilon}{25} = \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, for every* $\varepsilon > 0$ *there exists a* $\delta = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{25}\right\} > 0$ *such that* $0 < |x+1| < \delta$ *implies that* $\left|\left(\frac{x^2+3}{2x+1}\right) - (-4)\right| < \varepsilon$*; that is,* $\lim_{x\to -1} \frac{x^2+3}{2x+1} = -4$. ■

In Euclidean $\mathbb{R}$ space, the metric is realized as the absolute value of the difference. Letting d denote this metric allows us to restate the definition of $\lim_{x \to p} f(x) = L$ as

$$(\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon) > 0) \\ \left[(\forall x)(x \in A \wedge 0 < d(x,p) < \delta \Rightarrow d(f(x), L) < \varepsilon)\right]).$$

Of course, at this point we haven't gained much; this form doesn't look particularly better than the one with which we started. On the other hand, it gets us nearer to where we want to go which is to the limit of a function that is from one metric space to another–neither of which is $\mathbb{R}^1$. As a first step, let's look at the definition when the function is from an arbitrary metric space into $\mathbb{R}^1$. Again we let d denote the Euclidean $1-$metric.

Definition 5.1.7 *Suppose that A is a subset of a metric space (S, d_S) and that f is a function with domain A and range contained in $\mathbb{R}^1$; i.e., $f : A \to \mathbb{R}^1$. then "f tends to L as x tends to p* ***through points of*** *A" if and only if*

(i) *p is a limit point of A, and*

(ii) $(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)((\forall x)(x \in A \wedge 0 < d_S(x,p) < \delta$
$\Rightarrow d(f(x), L) < \varepsilon)).$

In this case, we write $f(x) \to L$ as $x \to p$ for $x \in A$, or $f(x) \to L$ as $x \underset{A}{\to} p$, or

$$\lim_{\substack{x \to p \\ x \in A}} f(x) = L.$$

Example 5.1.8 *Let $f : \mathbb{C} \to \mathbb{R}$ be given by $f(z) = \mathrm{Re}(z)$. Prove that*

$$\lim_{\substack{z \to 3+i \\ z \in \mathbb{C}}} f(z) = 3.$$

Space for scratch work.

For this one, we will make use of the fact that for any complex number ζ, $|\mathrm{Re}\,(\zeta)| \leq |\zeta|$.

Proof. *For $\varepsilon > 0$, let $\delta = \varepsilon$. Then $0 < |z - (3 + i)| < \delta = \varepsilon$ implies that*

$$|f(z) - 3| = |\mathrm{Re}\,(z) - 3| = |\mathrm{Re}\,(z - (3 + i))| \leq |z - (3 + i)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim\limits_{\substack{z \to 3+i \\ z \in \mathbb{C}}} f(z) = 3$. ■

Remark 5.1.9 *Notice that, in the definition of $\lim\limits_{\substack{x \to p \\ x \in A}} f(x) = L$, there is neither a requirement that f be defined at p nor an expectation that p be an element of A. Also, while it isn't indicated, the $\delta > 0$ that is sought may be dependent on p.*

Finally we want to make the transition to functions from one arbitrary metric space to another.

Definition 5.1.10 *Suppose that A is a subset of a metric space (S, d_S) and that f is a function with domain A and range contained in a metric space (X, d_X); i.e., $f : A \to X$. Then "f tends to L as x tends to p* ***through points of*** *A" if and only if*

(i) *p is a limit point of A, and*

(ii) $(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\,((\forall x)\,(x \in A \wedge 0 < d_S(x, p) < \delta$

$$\Rightarrow d_X(f(x), L) < \varepsilon)).$$

In this case, we write $f(x) \to L$ as $x \to p$ for $x \in A$, or $f(x) \to L$ as $x \underset{A}{\to} p$, or

$$\lim_{\substack{x \to p \\ x \in A}} f(x) = L.$$

Example 5.1.11 *For $p \in \mathbb{R}^1$, let $f(p) = (2p+1, p^2)$. Then $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$. Use the definition of limit to from that $\lim_{p \to 1} f(p) = (3, 1)$ with respect to the Euclidean metrics on each space.*

Space for scratch work.

Proof. *For $\varepsilon > 0$, let $\delta = \min\left\{1, \frac{\varepsilon}{\sqrt{13}}\right\}$. Then $0 < d_{\mathbb{R}}(p, 1) = |p - 1| < \delta \le 1$ implies that*

$$|p+1| = |(p-1)+2| \le |p-1| + 2 < 3$$

and

$$\sqrt{4 + (p+1)^2} < \sqrt{4+9} = \sqrt{13}.$$

Hence, for $0 < d_{\mathbb{R}}(p, 1) = |p-1| < \delta$,

$$\begin{aligned} d_{\mathbb{R}^2}(f(p), (3,1)) &= \sqrt{((2p+1)-3)^2 + \left(p^2-1\right)^2} \\ &= |p-1| \sqrt{4 + (p+1)^2} < \delta \cdot \sqrt{13} \le \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{p \to 1} (2p+1, p^2) = (3, 1)$. ■

Remark 5.1.12 *With few exceptions our limit theorems for functions of real-valued functions of a real variable that involved basic combinations of functions have direct, straightforward analogs to functions on an arbitrary metric spaces. Things can get more difficult when we try for generalizations of results that involved comparing function values. For the next couple of excursions, you are just being asked to practice translating results from one setting to our new one.*

Excursion 5.1.13 *Let A be a subset of a metric space S and suppose that $f : A \to \mathbb{R}^1$ is given. If*

$$f \to L \text{ as } p \to p_0 \text{ in } A \text{ and } f \to M \text{ as } p \to p_0 \text{ in } A$$

prove that $L = M$. After reading the following proof for the case of real-valued functions of a real variable, use the space provided to write a proof for the new setting.

Proof. We want to prove that, if $f \to L$ as $x \to a$ and $f \to M$ as $x \to a$, then $L = M$. For $L \neq M$, let $\epsilon = \frac{1}{2} \cdot |L - M|$. By the definition of limit, there exists positive numbers δ_1 and δ_2 such that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon$ and $0 < |x - a| < \delta_2$ implies $|f(x) - M| < \epsilon$. Choose $x_0 \in \mathbb{R}$ such that $0 < |x_0 - a| < \min\{\delta_1, \delta_2\}$. Then $|L - M| \leq |L - f(x_0)| + |M - f(x_0)| < 2\epsilon$ which contradicts the trichotomy law. ■

Excursion 5.1.14 *Let f and g be real-valued functions with domain A, a subset of a metric space (S, d). If $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$ and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = M$, then $\lim\limits_{\substack{p \to p_0 \\ p \in A}} (f + g)(p) = L + M$.*

After reading the following proof for the case of real-valued functions of a real variable, use the space provided to write a proof for the new setting.

Proof. We want to show that, if $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} (f + g)(x) = L + M$. Let $\epsilon > 0$ be given. Then there exists positive numbers δ_1 and δ_2 such that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon/2$ and $0 < |x - a| < \delta_2$ implies $|g(x) - M| < \epsilon/2$. For $\delta = \min\{\delta_1, \delta_2\}$, $0 < |x - a| < \delta$ implies that

$|(f+g)(x)-(L+M)| \le |f(x)-L|+|g(x)-M| < \epsilon$. ■

Theorem 4.1.17 gave us a characterization of limit points in terms of limits of sequences. This leads nicely to a characterization of limits of functions in terms of behavior on convergent sequences.

Theorem 5.1.15 (Sequences Characterization for Limits of Functions) *Suppose that (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $f : E \longrightarrow Y$ and p is a limit point of E. Then $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$ if and only if*

$$(\forall \{p_n\}) \left[\left(\{p_n\} \subset E \wedge (\forall n)\, (p_n \neq p) \wedge \lim_{n \to \infty} p_n = p \right) \Rightarrow \lim_{n \to \infty} f(p_n) = q \right].$$

Excursion 5.1.16 *Fill in what is missing in order to complete the following proof of the theorem.*

Proof. Let X, Y, E, f, and p be as described in the introduction to Theorem 5.1.15. Suppose that $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$. Since p is a limit point of E, by Theorem $\underset{(1)}{\underline{\hspace{2cm}}}$, there exists a sequence $\{p_n\}$ of elements in E such that $p_n \neq p$ for all $n \in \mathbb{J}$, and $\underset{(2)}{\underline{\hspace{6cm}}}$. For $\varepsilon > 0$, because $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$, there exists $\delta > 0$ such that $0 < d_X(x, p) < \delta$ and $x \in E$ implies that $\underset{(3)}{\underline{\hspace{3cm}}}$. From $\lim_{n \to \infty} p_n = p$ and $p_n \neq p$, we also know that there exists a positive integer M such that $n > M$ implies that $\underset{(4)}{\underline{\hspace{6cm}}}$. Thus, it follows that

$d_Y(f(p_n), q) < \varepsilon$ for all $n > M$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} f(p_n) = q$. Finally, because $\{p_n\} \subset E$ was arbitrary, we have that

$$(\forall \{p_n\}) \left[\left(\underset{(5)}{\underline{\hspace{6cm}}} \right) \Rightarrow \lim_{n\to\infty} f(p_n) = q \right].$$

We will give a proof by contrapositive of the converse. Suppose that $\lim_{\substack{x\to p \\ x\in E}} f(x) \neq q$. Then there exists a positive real number ε such that corresponding to each positive real number δ there is a point $x_\delta \in E$ for which $0 < d_X(x_\delta, p) < \delta$ and $d_Y(f(x_\delta), q) \geq \varepsilon$. In particular, for each $n \in \mathbb{J}$, corresponding to $\frac{1}{n}$ there is a point $p_n \in E$ such that $\underset{(6)}{\underline{\hspace{2cm}}}$ and $d_Y(f(p_n), q) \geq \varepsilon$. Hence, $\lim_{n\to\infty} f(p_n) \neq q$. Thus, there exists a sequence $\{p_n\} \subset E$ such that $\lim_{n\to\infty} p_n = p$ and $\underset{(7)}{\underline{\hspace{2cm}}}$; i.e.,

$$(\exists \{p_n\}) \left[\left(\{p_n\} \subset E \wedge (\forall n)(p_n \neq p) \wedge \lim_{n\to\infty} p_n = p \right) \wedge \lim_{n\to\infty} f(p_n) \neq q \right]$$

which is equivalent to

$$\neg (\forall \{p_n\}) \left[\left(\{p_n\} \subset E \wedge (\forall n)(p_n \neq p) \wedge \lim_{n\to\infty} p_n = p \right) \Rightarrow \lim_{n\to\infty} f(p_n) = q \right].$$

Therefore, we have shown that $\lim_{\substack{x\to p \\ x\in E}} f(x) \neq q$ implies that

$$\neg (\forall \{p_n\}) \left[\left(\{p_n\} \subset E \wedge (\forall n)(p_n \neq p) \wedge \lim_{n\to\infty} p_n = p \right) \Rightarrow \lim_{n\to\infty} f(p_n) = q \right].$$

Since the $\underset{(8)}{\underline{\hspace{6cm}}}$ is logically equivalent to the converse, this concludes the proof. ■

Acceptable responses are: (1) 4.1.17,(2) $\lim_{n\to\infty} p_n = p$, (3) $d_Y(f(x), q) < \varepsilon$, (4) $0 < d_X(p_n, p) < \delta$, (5) $\{p_n\} \subset E \wedge (\forall n)(p_n \neq p) \wedge \lim_{n\to\infty} p_n = p$, (6) $0 < d_X(p_n, p) < \frac{1}{n}$, (7) $\lim_{n\to\infty} f(p_n) \neq q$, (8) contrapositive.

The following result is an immediate consequence of the theorem and Lemma 4.1.7.

Corollary 5.1.17 *Limits of functions on metric spaces are unique.*

Remark 5.1.18 *In view of Theorem 5.1.15, functions from metric spaces into subsets of the complex numbers will satisfy the "limits of combinations" properties of sequences of complex numbers that were given in Theorem 4.3.2. For completeness, we state it as a separate theorem.*

Theorem 5.1.19 *Suppose that (X, d_X) is a metric space, $E \subset X$, p is a limit point of E, $f : E \longrightarrow \mathbb{C}$, $g : E \longrightarrow \mathbb{C}$, $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = A$, and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = B$. Then*

(a) $\lim\limits_{\substack{x \to p \\ x \in E}} (f + g)(x) = A + B$

(b) $\lim\limits_{\substack{x \to p \\ x \in E}} (fg)(x) = AB$

(c) $\lim\limits_{\substack{x \to p \\ x \in E}} \dfrac{f}{g}(x) = \dfrac{A}{B}$ *whenever* $B \neq 0$.

While these statements are an immediate consequence of Theorem 4.3.2 and Theorem 5.1.15 completing the following excursions can help you to learn the approaches to proof. Each proof offered is independent of Theorems 4.3.2 and Theorem 5.1.15.

Excursion 5.1.20 *Fill in what is missing to complete a proof of Theorem 5.1.19(a).*

Proof. Suppose $\varepsilon > 0$ is given. Because $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = A$, there exists a positive real δ_1 such that $x \in E$ and $0 < d_X(x, p) < \delta_1$ implies that $|f(x) - A| < \dfrac{\varepsilon}{2}$. Since $\underset{(1)}{\underline{\hspace{3cm}}}$, there exists a positive real number δ_2 such that $x \in E$ and $0 < d_X(x, p) < \delta_2$ implies that $|g(x) - B| < \dfrac{\varepsilon}{2}$. Let $\delta = \underset{(2)}{\underline{\hspace{3cm}}}$. It follows from the triangular inequality that, if $x \in E$ and $0 < d_X(x, p) < \delta$, then

$$\begin{aligned} |(f + g)(x) - (A + B)| &= \left| (f(x) - A) + \left(\underset{(3)}{\underline{\hspace{3cm}}} \right) \right| \\ &\leq \underset{(4)}{\underline{\hspace{9cm}}} < \underset{(5)}{\underline{\hspace{0.7cm}}}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{\substack{x\to p\\ x\in E}} (f+g)(x) = A + B$ as claimed.

■

Acceptable responses are: (1) $\lim_{\substack{x\to p\\ x\in E}} g(x) = B$, (2) $\min\{\delta_1, \delta_2\}$, (3) $(g(x) - B)$, (4) $|f(x) - A| + |g(x) - B|$, (5) ε

Excursion 5.1.21 *Fill in what is missing to complete a proof of Theorem 5.1.19(b).*

Proof. Because $\lim_{\substack{x\to p\\ x\in E}} f(x) = A$, there exist a positive real number δ_1 such that $x \in E$ and $0 < d_X(x, p) < \delta_1$ implies that $|f(x) - A| < 1$; i.e., $|f(x)| - |A| < 1$. Hence, $|f(x)| < 1 + |A|$ for all $x \in E$ such that $0 < d_X(x, p) < \delta_1$.

Suppose that $\varepsilon > 0$ is given. If $B = 0$, then $\lim_{\substack{x\to p\\ x\in E}} g(x) = 0$ yields the existence of a positive real number δ_2 such that $x \in E$ and $0 < d_X(x, p) < \delta_2$ implies that

$$|g(x)| < \frac{\varepsilon}{1 + |A|}.$$

Then for $\delta^* = \underset{(1)}{\underline{\hspace{3cm}}}$, we have that

$$|(fg)(x)| = |f(x)|\,|g(x)| < (1 + |A|) \cdot \underset{(2)}{\underline{\hspace{3cm}}}.$$

Hence, $\lim_{\substack{x\to p\\ x\in E}} (fg)(x) = AB = 0$. Next we suppose that $B \neq 0$. Then there exists a positive real numbers δ_3 and δ_4 for which $|f(x) - A| < \dfrac{\varepsilon}{2|B|}$ and $|g(x) - B| < \dfrac{\varepsilon}{2(1 + |A|)}$ whenever $0 < d_X(x, p) < \delta_3$ and $0 < d_X(x, p) < \delta_4$, respectively, for $x \in E$. Now let $\delta = \min\{\delta_1, \delta_3, \delta_4\}$. It follows that if $x \in E$ and $0 < d_X(x, p) <$

δ, then

$$
\begin{aligned}
|(fg)(x) - AB| &= \left| f(x)\,g(x) - \underset{(3)}{\underline{\qquad\quad}} + \underset{(4)}{\underline{\qquad\quad}} - AB \right| \\
&\leq |f(x)|\,|g(x) - B| + \underset{(5)}{\underline{\qquad\qquad\quad}} \\
&< (1 + |A|)\,|g(x) - B| + |B| \left| \underset{(6)}{\underline{\qquad\qquad\qquad}} \right| \\
&< \underset{(7)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}} = \underset{(8)}{\underline{\qquad\qquad\quad}}.
\end{aligned}
$$

Again, since $\varepsilon > 0$ was arbitrary, we conclude that $\lim\limits_{\substack{x \to p \\ x \in E}} (fg)(x) = AB$ as needed. ■

Acceptable responses include: (1) $\min\{\delta_1, \delta_2\}$, (2) $\varepsilon(1 + |A|)^{-1}$, (3) $f(x)\,B$, (4) $f(x)\,B$, (5) $|B|\,|f(x) - A|$, (6) $|f(x) - A|$, (7) $(1 + |A|)\dfrac{\varepsilon}{2(1 + |A|)} + |B|\dfrac{\varepsilon}{2|B|}$ (8) ε.

Excursion 5.1.22 *Fill in what is missing to complete a proof of Theorem 5.1.19(c).*

Proof. In view of Theorem 5.1.19(b), it will suffice to prove that, under the given hypotheses, $\lim\limits_{\substack{x \to p \\ x \in E}} \dfrac{1}{g(x)} = \dfrac{1}{B}$. First, we will show that, for $B \neq 0$, the modulus of g is bounded away from zero. Since $|B| > 0$ and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = B$, there exists a positive real number $\delta_1 > 0$ such that $x \in E$ and $\underset{(1)}{\underline{\qquad\qquad\quad}}$ implies that $|g(x) - B| < \dfrac{|B|}{2}$. It follows from the (other) $\underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$ that, if $x \in E$ and $0 < d_X(x, p) < \delta_1$, then

$$
\begin{aligned}
|g(x)| = |(g(x) - B) + B| &\geq \left| |g(x) - B| \quad \underline{\qquad\qquad\quad} \right| \\
&> \frac{|B|}{2}.
\end{aligned}
$$

Suppose that $\varepsilon > 0$ is given. Then $\dfrac{|B|^2 \varepsilon}{2} > 0$ and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = B$ yields the existence of a positive real number δ_2 such that $|g(x) - B| < \dfrac{|B|^2 \varepsilon}{2}$ whenever $x \in E$ and $0 < d_X(x, p) < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in E$ and $0 < d_X(x, p) < \delta$ we have that

$$\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \frac{|g(x) - B|}{|B|\,|g(x)|} < \underset{(3)}{\underline{\qquad\qquad\qquad}} = \underset{(4)}{\underline{\qquad\quad}}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\underset{(5)}{\underline{\qquad\qquad\qquad}}$.

Finally, letting $h(x) = \dfrac{1}{g(x)}$, by Theorem $\underset{(6)}{\underline{\qquad\quad}}$,

$$\lim_{\substack{x \to p \\ x \in E}} \frac{f}{g}(x) = \lim_{\substack{x \to p \\ x \in E}} f(x) \underset{(7)}{\underline{\qquad\quad}} = \underset{(8)}{\underline{\qquad\quad}}.$$

■

Acceptable responses are: (1) $0 < d_X(x, p) < \delta_1$, (2) triangular inequality (3) $\dfrac{\varepsilon |B|^2}{2|B|\left(\dfrac{|B|}{2}\right)}$, (4) ε (5) $\lim\limits_{\substack{x \to p \\ x \in E}} \dfrac{1}{g(x)} = \dfrac{1}{B}$, (6) 5.1.19(b), (7) $h(x)$, (8) $A \cdot \dfrac{1}{B}$.

From Lemma 4.3.1, it follows that the limit of the sum and the limit of the product parts of Theorem 5.1.19 carry over to the sum and inner product of functions from metric spaces to Euclidean $k - space$.

Theorem 5.1.23 *Suppose that X is a metric space, $E \subset X$, p is a limit point of E, $\mathbf{f} : E \longrightarrow \mathbb{R}^k$, $\mathbf{g} : E \longrightarrow \mathbb{R}^k$, $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = \mathbf{A}$, and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = \mathbf{B}$. Then*

(a) $\lim\limits_{\substack{x \to p \\ x \in E}} (\mathbf{f} + \mathbf{g})(x) = \mathbf{A} + \mathbf{B}$ *and*

(b) $\lim\limits_{\substack{x \to p \\ x \in E}} (\mathbf{f} \bullet \mathbf{g})(x) = \mathbf{A} \bullet \mathbf{B}$

In the set-up of Theorem 5.1.23, note that $\mathbf{f}+\mathbf{g} : E \longrightarrow \mathbb{R}^k$ while $\mathbf{f}\bullet\mathbf{g} : E \longrightarrow \mathbb{R}$.

5.2 Continuous Functions on Metric Spaces

Recall that in the case of real-valued functions of a real variable getting from the general idea of a functions having limits to being continuous simply added the property that the values approached are actually the values that are achieved. There is nothing about that transition that was tied to the properties of the reals. Consequently, the definition of continuous functions on arbitrary metric spaces should come as no surprise. On the other hand, an extra adjustment is needed to allow for the fact that we can consider functions defined at isolated points of subsets of metric spaces.

Definition 5.2.1 *Suppose that (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $f : E \longrightarrow Y$ and $p \in E$. Then f is* ***continuous at*** *p if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall x)\,(x \in E \wedge d_X(x, p) < \delta) \Rightarrow d_Y(f(x), f(p)) < \varepsilon\right].$$

Theorem 5.2.2 *Suppose that (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $f : E \longrightarrow Y$ and $p \in E$ and p is a limit point of E. Then f is continuous at p if and only if $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = f(p)$*

Definition 5.2.3 *Suppose that (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$ and $f : E \longrightarrow Y$. Then f is* ***continuous on*** *E if and only if f is continuous at each $p \in E$.*

Remark 5.2.4 *The property that was added in order to get the characterization that is given in Theorem 5.2.2 was the need for the point to be a limit point. The definition of continuity at a point is satisfied for isolated points of E because each isolated point p has the property that there is a neighborhood of p, $N_{\delta^*}(p)$, for which $E \cap N_{\delta^*}(p) = \{p\}$; since $p \in \mathrm{dom}(f)$ and $d_X(p, p) = d_Y(f(p), f(p)) = 0$, we automatically have that $(\forall x)\,(x \in E \wedge d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$ for any $\varepsilon > 0$ and any positive real number δ such that $\delta < \delta^*$.*

Remark 5.2.5 *It follows immediately from our limit theorems concerning the algebraic manipulations of functions for which the limits exist, the all real-valued polynomials in k real variables are continuous in $\mathbb{R}^k$.*

Remark 5.2.6 *Because $f(1) = (3, 1)$ for the $f(p) = \left(\left(2p + 1, p^2\right)\right) : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ that was given in Example 5.1.11, our work for the example allows us to claim that f is continuous at $p = 1$.*

Theorem 5.1.15 is not practical for use to show that a specific function is continuous; it is a useful tool for proving some general results about continuous functions on metric spaces and can be a nice way to show that a given function is not continuous.

Example 5.2.7 *Prove that the function* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f((x, y)) = \begin{cases} \dfrac{xy}{x^3 + y^3} & , \text{ for } (x, y) \neq (0, 0) \\ 0 & , \text{ for } x = y = 0 \end{cases}$ *is not continuous at* $(0, 0)$.

Let $p_n = \left(\dfrac{1}{n}, \dfrac{1}{n}\right)$*. Then* $\{p_n\}_{n=1}^{\infty}$ *converges to* $(0, 0)$*, but*

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} \frac{\left(\dfrac{1}{n}\right)\left(\dfrac{1}{n}\right)}{\left(\dfrac{1}{n}\right)^3 + \left(\dfrac{1}{n}\right)^3} = \lim_{n\to\infty} \frac{n}{2} = +\infty \neq 0.$$

Hence, by the Sequences Characterization for Limits of Functions, we conclude that the given f *is not continuous at* $(0, 0)$.

Example 5.2.8 *Use the definition to prove that* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f((x, y)) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & , \text{ for } (x, y) \neq (0, 0) \\ 0 & , \text{ for } x = y = 0 \end{cases}$ *is continuous at* $(0, 0)$.

We need to show that $\displaystyle\lim_{(x,y)\to(0,0)} f((x, y)) = 0$*. Because the function is defined in two parts, it is necessary to appeal to the definition. For* $\varepsilon > 0$*, let* $\delta = \varepsilon$*. Then*

$$0 < d_{\mathbb{R}\times\mathbb{R}}((x, y), (0, 0)) = \sqrt{x^2 + y^2} < \delta = \varepsilon$$

implies that

$$|f((x, y)) - 0| = \left|\frac{x^2 y}{x^2 + y^2}\right| \leq \frac{\left(x^2 + y^2\right)|y|}{x^2 + y^2} = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} < \varepsilon.$$

Because $\varepsilon > 0$ *was arbitrary, we conclude that* $\displaystyle\lim_{(x,y)\to(0,0)} f((x, y)) = 0 = f((0, 0))$*. Hence,* f *is continuous at* $(0, 0)$.

It follows from the definition and Theorem 5.1.19 that continuity is transmitted to sums, products, and quotients when the ranges of our functions are subsets of the complex field. For completeness, the general result is stated in the following theorem.

Theorem 5.2.9 *If f and g are complex valued functions that are continuous on a metric space X, then $f+g$ and fg are continuous on X. Furthermore, $\dfrac{f}{g}$ is continuous on $X - \{p \in X : g(p) = 0\}$.*

From Lemma 4.3.1, it follows immediately that functions from arbitrary metric spaces to Euclidean k-space are continuous if and only if they are continuous by coordinate. Furthermore, Theorem 5.1.23 tells us that continuity is transmitted to sums and inner products.

Theorem 5.2.10 *(a) Let $f_1, f_2, \ldots, f_k$ be real valued functions on a metric space X, and $\mathbf{F} : X \longrightarrow \mathbb{R}^k$ be defined by $\mathbf{F}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$. Then $\mathbf{F}$ is continuous if and only if f_j is continuous for each j, $1 \le j \le k$.*

(b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous functions from a metric space X into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f} \bullet \mathbf{g}$ are continuous on X.

The other combination of functions that we wish to examine on arbitrary metric spaces is that of composition. If X, Y, and Z are metric spaces, $E \subset X$, $f : E \longrightarrow Y$, and $g : f(E) \longrightarrow Z$, then the composition of f and g, denoted by $g \circ f$, is defined by $g(f(x))$ for each $x \in E$. The following theorem tells us that continuity is transmitted through composition.

Theorem 5.2.11 *Suppose that X, Y, and Z are metric spaces, $E \subset X$, $f : E \longrightarrow Y$, and $g : f(E) \longrightarrow Z$. If f is continuous at $p \in E$ and g is continuous at $f(p)$, then the composition $g \circ f$ is continuous at $p \in E$.*

Space for scratch work.

Proof. Suppose that (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces, $E \subset X$, $f : E \longrightarrow Y$, $g : f(E) \longrightarrow Z$, f is continuous at $p \in E$, and g is continuous at $f(p)$. Let $\varepsilon > 0$ be given. Since g is continuous at $f(p)$, there exists a positive real number δ_1 such that $d_Z(g(y), g(f(p))) < \varepsilon$ for any $y \in f(E)$ such that $d_Y(y, f(p)) < \delta_1$. From f being continuous at $p \in E$ and δ_1 being a positive real number, we deduce the existence of another positive real number δ such that $x \in E$ and $d_X(x, p) < \delta$ implies that $d_Y(f(x), f(p)) < \delta_1$. Substituting $f(x)$ for y, we have that $x \in E$ and $d_X(x, p) < \delta$ implies that $d_Y(f(x), f(p)) < \delta_1$ which further implies that $d_Z(g(f(x)), g(f(p))) < \varepsilon$. That is, $d_Z((g \circ f)(x), (g \circ f)(p)) < \varepsilon$ for any $x \in E$ for which $d_X(x, p) < \delta$. Therefore, $g \circ f$ is continuous at p. ■

Remark 5.2.12 *The "with respect to a set" distinction can be an important one to note. For example, the function* $f(x) = \begin{cases} 1 & , \text{ for } x \text{ rational} \\ 0 & , \text{ for } x \text{ irrational} \end{cases}$ *is continuous with respect to the rationals and it is continuous with respect to the irrationals. However, it is not continuous on* $\mathbb{R}^1$.

5.2.1 A Characterization of Continuity

Because continuity is defined in terms of proximity, it can be helpful to rewrite the definition in terms of neighborhoods. Recall that, for (X, d_X), $p \in X$, and $\delta > 0$,

$$N_\delta(p) = \{x \in X : d_X(x, p) < \delta\}.$$

For a metric space (Y, d_Y), $f : X \longrightarrow Y$ and $\varepsilon > 0$,

$$N_\varepsilon(f(p)) = \{y \in Y : d_Y(y, f(p)) < \varepsilon\}.$$

Hence, for metric spaces (X, d_X) and (Y, d_Y), $E \subset X$, $f : E \longrightarrow Y$ and $p \in E$, f is continuous at p if and only if

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)\left[f(N_\delta(p) \cap E) \subset N_\varepsilon(f(p))\right].$$

Because neighborhoods are used to define open sets, the neighborhood formulation for the definition of continuity of a function points us in the direction of the following theorem.

Theorem 5.2.13 (Open Set Characterization of Continuous Functions) *Let f be a mapping on a metric space (X, d_X) into a metric space (Y, d_Y). Then f is continuous on X if and only if for every open set V in Y, the set $f^{-1}(V)$ is open in X.*

Space for scratch work.

Excursion 5.2.14 *Fill in what is missing in order to complete the following proof of the theorem.*

Proof. Let f be a mapping from a metric space (X, d_X) into a metric space (Y, d_Y).

Suppose that f is continuous on X, V is an open set in Y, and $p_0 \in f^{-1}(V)$. Since V is open and $f(p_0) \in V$, we can choose $\varepsilon > 0$ such that $N_\varepsilon(f(p_0)) \subset V$ from which it follows that

$$\underbrace{\qquad\qquad}_{(1)} \subset f^{-1}(V).$$

Because f is continuous at $p_0 \in X$, corresponding to $\varepsilon > 0$, there exists a $\delta > 0$, such that $f(N_\delta(p_0)) \subset N_\varepsilon(f(p_0))$ which implies that

$$N_\delta(p_0) \subset \underbrace{\qquad\qquad}_{(2)}.$$

From the transitivity of subset, we concluded that $N_\delta(p_0) \subset f^{-1}(V)$. Hence, p_0 is an interior point of $f^{-1}(V)$. Since p_0 was arbitrary, we conclude that each $p \in f^{-1}(V)$ is an interior point. Therefore, $f^{-1}(V)$ is open.

To prove the converse, suppose that the inverse image of every open set in Y is open in X. Let p be an element in X and $\varepsilon > 0$ be given. Now the neighborhood $N_\varepsilon(f(p))$ is open in Y. Consequently, $\underbrace{\qquad\qquad}_{(3)}$ is open in X.

Since p is an element of $\underset{(3)}{\underline{\hspace{4cm}}}$, there exists a positive real number δ such that $N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p)))$; i.e., $\underset{(4)}{\underline{\hspace{4cm}}} \subset N_\varepsilon(f(p))$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{x\to p} f(x) = \underset{(5)}{\underline{\hspace{2cm}}}$. Finally, because p was an arbitrary point in X, it follows that f $\underset{(6)}{\underline{\hspace{8cm}}}$ as needed. ■

Acceptable responses are: (1) $f^{-1}(N_\varepsilon(f(p_0))$, (2) $f^{-1}(N_\varepsilon(f(p_0)))$, (3) $f^{-1}(N_\varepsilon(f(p)))$, (4) $f(N_\delta(p))$, (5) $f(p)$, (6) is continuous on X.

Excursion 5.2.15 *Suppose that f is a mapping on a metric space (X, d_X) into a metric space (Y, d_Y) and $E \subset X$. Prove that $f^{-1}\left[E^c\right] = \left(f^{-1}[E]\right)^c$.*

The following corollary follows immediately from the Open Set Characterization for Continuity, Excursion 5.2.15, and the fact that a set is closed if and only if its complement is open. Use the space provided after the statement to convince yourself of the truth of the given statement.

Corollary 5.2.16 *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

Remark 5.2.17 *We have stated results in terms of open sets in the full metric space. We could also discuss functions restricted to subsets of metric spaces and then the characterization would be in terms of relative openness. Recall that given two sets X and Y and $f : X \longrightarrow Y$, the corresponding set induced functions satisfy the following properties for $C_j \subset X$ and $D_j \subset Y$, $j = 1, 2$:*

- $f^{-1}[D_1 \cap D_2] = f^{-1}[D_1] \cap f^{-1}[D_2]$,
- $f^{-1}[D_1 \cup D_2] = f^{-1}[D_1] \cup f^{-1}[D_2]$,
- $f[C_1 \cap C_2] \subset f[C_1] \cap f[C_2]$, *and*
- $f[C_1 \cup C_2] = f[C_1] \cup f[C_2]$

Because subsets being open to subsets of metric spaces in characterized by their realization as intersections with open subsets of the parent metric spaces, our neighborhoods characterization tells us that we loose nothing by looking at restrictions of given functions to the subsets that we wish to consider rather that stating things in terms of relative openness.

5.2.2 Continuity and Compactness

Theorem 5.2.18 *If f is a continuous function from a compact metric space X to a metric space Y, then $f(X)$ is compact.*

Excursion 5.2.19 *Fill in what is missing to complete the following proof of Theorem 5.2.18.*

Space for scratch work.

Proof. Suppose that f is a continuous function from a compact metric space X to a metric space Y and $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for $f(X)$. Then G_α is open in Y for each $\alpha \in \Delta$ and $\underset{(1)}{\underline{\hspace{3cm}}}$.

From the Open Set Characterization of $\underset{(2)}{\underline{\hspace{3cm}}}$ Functions, $f^{-1}(G_\alpha)$ is $\underset{(3)}{\underline{\hspace{3cm}}}$ for each $\alpha \in \Delta$.

Since $f : X \twoheadrightarrow f(X)$ and $f(X) \subset \bigcup_{\alpha\in\Delta} G_\alpha$, we have that

$$X = f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{\alpha\in\Delta} G_\alpha\right) = \underset{(4)}{\underline{\hspace{3cm}}}.$$

Hence, $\mathcal{F} = \left\{ f^{-1}(G_\alpha) : \alpha \in \Delta \right\}$ is an $\underset{(5)}{\underline{\hspace{5cm}}}$ for X. Since X is $\underset{(6)}{\underline{\hspace{2.5cm}}}$, there is a finite subcollection of $\mathcal{F}$, $\left\{ f^{-1}\left(G_{\alpha_j}\right) : j = 1, 2, ..., n \right\}$, that covers X; i.e.,

$$X \subset \bigcup_{j=1}^{n} f^{-1}\left(G_{\alpha_j}\right).$$

It follows that

$$f(X) \subset f\left(\bigcup_{j=1}^{n} f^{-1}\left(G_{\alpha_j}\right)\right) = \bigcup_{j=1}^{n} \underset{(7)}{\underline{\hspace{1.5cm}}} = \bigcup_{j=1}^{n} G_{\alpha_j}.$$

Therefore, $\left\{ G_{\alpha_j} : j = 1, 2, ..., n \right\}$ is a finite subcollection of $\mathcal{G}$ that covers $f(X)$. Since $\mathcal{G}$ was arbitrary, every $\underset{(8)}{\underline{\hspace{7.5cm}}}$; i.e., $f(X)$ is $\underset{(9)}{\underline{\hspace{2.5cm}}}$. ■

Remark 5.2.20 *Just to stress the point, in view of our definition of relative compactness the result just stated is also telling us that the continuous image of any compact subset of a metric space is a compact subset in the image.*

Definition 5.2.21 *For a set E, a function $\mathbf{f} : E \longrightarrow \mathbb{R}^k$ is said to be **bounded** if and only if*

$$(\exists M)\left(M \in \mathbb{R} \wedge (\forall x)\left(x \in E \Rightarrow |\mathbf{f}(x)| \leq M\right)\right).$$

When we add compactness to domain in the metric space, we get some nice analogs.

Theorem 5.2.22 (Boundedness Theorem) *Let A be a compact subset of a metric space (S, d) and suppose that $\mathbf{f} : A \longrightarrow \mathbb{R}^k$ is continuous. Then $f(A)$ is closed and bounded. In particular, f is bounded.*

Excursion 5.2.23 *Fill in the blanks to complete the following proof of the Boundedness Theorem.*

Proof. *By the* $\underset{(1)}{\underline{\hspace{6cm}}}$, *we know that compactness in* $\mathbb{R}^k$ *for any* $k \in \mathbb{J}$ *is equivalent to being closed and bounded. Hence, from Theorem 5.2.18, if* $f : A \longrightarrow \mathbb{R}^k$ *where* A *is a compact metric space, then* $f(A)$ *is compact. But* $f(A) \subset \underset{(2)}{\underline{\hspace{1.5cm}}}$ *and compact yields that* $f(A)$ *is* $\underset{(3)}{\underline{\hspace{6cm}}}$. *In particular,* $f(A)$ *is bounded as claimed in the Boundedness Theorem.* ■

Expected responses are: (1) Heine-Borel Theorem, (2) $\mathbb{R}^k$, and (3) closed and bounded.

Theorem 5.2.24 (Extreme Value Theorem) *Suppose that* f *is a continuous function from a compact subset* A *of a metric space* S *into* $\mathbb{R}^1$,

$$M = \sup_{p \in A} f(p) \qquad \textit{and} \qquad m = \inf_{p \in A} f(p).$$

Then there exist points u *and* v *in* A *such that* $f(u) = M$ *and* $f(v) = m$.

Proof. From Theorem 5.2.18 and the Heine-Borel Theorem, $f(A) \subset \mathbb{R}$ and f continuous implies that $f(A)$ is closed and bounded. The Least Upper and Greatest Lower Bound Properties for the reals yields the existence of finite real numbers M and m such that $M = \sup_{p \in A} f(p)$ and $m = \inf_{p \in A} f(p)$. Since $f(A)$ is closed, by Theorem 3.3.26, $M \in f(A)$ and $m \in f(A)$. Hence, there exists u and v in A such that $f(u) = M$ and $f(v) = m$; i.e., $f(u) = \sup_{p \in A} f(p)$ and $f(v) = \inf_{p \in A} f(p)$. ■

Theorem 5.2.25 *Suppose that* f *is a continuous one-to-one mapping of a compact metric space* X *onto a metric space* Y. *Then the inverse mapping* f^{-1} *which is defined by* $f^{-1}(f(x)) = x$ *for all* $x \in X$ *is a continuous mapping that is a one-to-one correspondence from* Y *to* X.

Proof. *Suppose that* f *is a continuous one-to-one mapping of a compact metric space* X *onto a metric space* Y. *Because* f *is one-to-one, the inverse* f^{-1} *is a function from* $\operatorname{rng}(f) = Y$ *in* X. *From the Open Set Characterization of Continuous Functions, we know that* f^{-1} *is continuous in* Y *if* $f(U)$ *is open in* Y *for every* U *that is open in* X. *Suppose that* $U \subset X$ *is open. Then, by Theorem 3.3.37,* U^c *is compact as a closed subset of the compact metric space* X. *In view of Theorem*

5.2.18, $f(U^c)$ is compact. Since every compact subset of a metric space is closed (Theorem 3.3.35), we conclude that $f(U^c)$ is closed. Because f is one-to-one, $f(U^c) = f(X) - f(U)$; then f onto yields that $f(U^c) = Y - f(U) = f(U)^c$. Therefore, $f(U)^c$ is closed which is equivalent to $f(U)$ being open. Since U was arbitrary, for every U open in X, we have that $f(U)$ is open in Y. Hence, f^{-1} is continuous in Y. ■

5.2.3 Continuity and Connectedness

Theorem 5.2.26 *Suppose that f is a continuous mapping for a metric space X into a metric space Y and $E \subset X$. If E is a connected subset of X, then $f(E)$ is connected in Y.*

Excursion 5.2.27 *Fill in what is missing in order to complete the following proof of Theorem 5.2.26.*

Space for scratch work.

Proof. Suppose that f is a continuous mapping from a metric space X into a metric space Y and $E \subset X$ is such that $f(E)$ is not connected. Then we can let $f(E) = A \cup B$ where A and B are nonempty $\underset{(1)}{\underline{\hspace{3cm}}}$ subsets of Y; i.e., $A \neq \emptyset$, $B \neq \emptyset$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Consider $G \underset{def}{=} E \cap f^{-1}(A)$ and $H \underset{def}{=} E \cap f^{-1}(B)$. Then neither G nor H is empty and

$$\begin{aligned} G \cup H &= \left(E \cap f^{-1}(A)\right) \cup \underset{(2)}{\underline{\hspace{3cm}}} \\ &= E \cap \left(\underset{(3)}{\underline{\hspace{3cm}}}\right) \\ &= E \cap f^{-1}(A \cup B) = \underset{(4)}{\underline{\hspace{3cm}}}. \end{aligned}$$

Because $A \subset \overline{A}$, $f^{-1}(A) \subset f^{-1}\left(\overline{A}\right)$. Since $G \subset f^{-1}(A)$, the transitivity of containment yields that $\underset{(5)}{\underline{\hspace{3cm}}}$. From the Corollary to the Open Set Characterization for Continuous Functions, $f^{-1}\left(\overline{A}\right)$ is $\underset{(6)}{\underline{\hspace{3cm}}}$.

It follows that $\overline{G} \subset f^{-1}\left(\overline{A}\right)$. From $\overline{G} \subset f^{-1}\left(\overline{A}\right)$ and $H \subset f^{-1}(B)$, we have that

$$\overline{G} \cap H \subset f^{-1}\left(\overline{A}\right) \cap f^{-1}(B) = f^{-1}\left(\underset{(7)}{\underline{\hspace{6em}}}\right) = f^{-1}\left(\underset{(8)}{\underline{\hspace{1.5em}}}\right) = \underset{(9)}{\underline{\hspace{1.5em}}}.$$

The same argument yields that $G \cap \overline{H} = \emptyset$.
From $E = G \cup H$, $G \neq \emptyset$, $H \neq \emptyset$ and $\overline{G} \cap H = G \cap \overline{H} = \emptyset$, we conclude that E is $\underset{(10)}{\underline{\hspace{6em}}}$. Hence, for f a continuous mapping from a metric space X into a metric space Y and $E \subset X$, if $f(E)$ is not connected, then E is not connected. According to the contrapositive, we conclude that, if $\underset{(11)}{\underline{\hspace{6em}}}$, then $\underset{(12)}{\underline{\hspace{6em}}}$, as needed. ■

*******Acceptable responses are: (1) separated (2) $E \cap f^{-1}(B)$, (3) $f^{-1}(A) \cup f^{-1}(B)$, (4) E, (5) $G \subset f^{-1}\left(\overline{A}\right)$, (6) closed, (7) $\overline{A} \cap B$, (8) $\emptyset$, (9) $\emptyset$, (10) not connected, (11) E is connected, and (12) $f(E)$ is connected.*******

Theorem 5.2.28 *Suppose that f is a real-valued function on a metric space (X, d). If f is continuous on S, a nonempty connected subset of X, then the range of $f|_S$, denoted by $R(f|_S)$, is either an interval or a point.*

Theorem 5.2.29 (The Intermediate Value Theorem) *Let f be a continuous real-valued function on an interval $[a, b]$. If $f(a) < f(b)$ and if $c \in (f(a), f(b))$, then there exist a point $x \in (a, b)$ such that $f(x) = c$.*

Proof. Let $E = f([a, b])$. Because $[a, b]$ is an interval, from Theorem 3.3.60, we know that $[a, b]$ is connected. By Theorem 5.2.26, E is also connected as the continuous image of a connected set. Since $f(a)$ and $f(b)$ are in E, from Theorem 3.3.60, it follows that if c is a real number satisfying $f(a) < c < f(b)$, then c is in E. Hence, there exists x in $[a, b]$ such that $f(x) = c$. Since $f(a)$ is not equal to c and $f(b)$ is not equal to c, we conclude that x is in (a, b). Therefore, there exists x in (a, b) such that $f(x) = c$, as claimed. ■

5.3 Uniform Continuity

Our definition of continuity works from continuity at a point. Consequently, point dependency is tied to our $\delta - \varepsilon$ proofs of limits. For example, if we carried out a $\delta - \varepsilon$ proof that $f(x) = \dfrac{2x+1}{x-1}$ is continuous at $x = 2$, corresponding to $\varepsilon > 0$, taking $\delta = \min\left\{\dfrac{1}{2}, \dfrac{\varepsilon}{6}\right\}$ will work nicely to show that $\lim\limits_{x\to 2} \dfrac{2x+1}{x-1} = 5 = f(2)$; however, it would not work for showing continuity at $x = \dfrac{3}{2}$. On the other hand, corresponding to $\varepsilon > 0$, taking $\delta = \min\left\{\dfrac{1}{4}, \dfrac{\varepsilon}{24}\right\}$ will work nicely to show that $\lim\limits_{x\to \frac{3}{2}} \dfrac{2x+1}{x-1} = 8 = f\left(\dfrac{3}{2}\right)$. The point dependence of the work is just buried in the focus on the local behavior. The next concept demands a "niceness" that is global.

Definition 5.3.1 *Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \longrightarrow Y$ is* ***uniformly continuous*** *on X if and only if*

$$(\forall \varepsilon > 0)\, (\exists \delta > 0) \left[(\forall p)\, (\forall q)\, (p, q \in X \wedge d_X(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon)\right].$$

Example 5.3.2 *The function $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on* $[1, 3]$.

For $\varepsilon > 0$ let $\delta = \dfrac{\varepsilon}{6}$. For $x_1, x_2 \in [1, 3]$, the triangular inequality yields that

$$|x_1 + x_2| \le |x_1| + |x_2| \le 6.$$

Hence, $x_1, x_2 \in [1, 3]$ and $|x_1 - x_2| < \delta$ implies that

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2|\,|x_1 + x_2| < \delta \cdot 6 = \varepsilon.$$

Since $\varepsilon > 0$ and $x_1, x_2 \in [1, 3]$ were arbitrary, we conclude that f is uniformly continuous on $[1, 3]$.

Example 5.3.3 *The function $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is not uniformly continuous on* $\mathbb{R}$.

We want to show that there exists a positive real number ε such that corresponding to every positive real number δ we have (at least two points) $x_1 = x_1(\delta)$ and $x_2 = x_2(\delta)$ for which $|x_1(\delta) - x_2(\delta)| < \delta$ and $|f(x_1) - f(x_2)| \geq \varepsilon$. This statement is an exact translation of the negation of the definition. For the given function, we want to exploit the fact the as x increases x^2 increase at a rapid (not really uniform) rate.

Take $\varepsilon = 1$. For any positive real number δ, let $x_1 = x_1(\delta) = \dfrac{\delta}{2} + \dfrac{1}{\delta}$ and $x_2 = x_2(\delta) = \dfrac{1}{\delta}$. Then x_1 and x_2 are real numbers such that

$$|x_1 - x_2| = \left| \left(\frac{\delta}{2} + \frac{1}{\delta} \right) - \left(\frac{1}{\delta} \right) \right| = \frac{\delta}{2} < \delta$$

while

$$|f(x_1) - f(x_2)| = \left| x_1^2 - x_2^2 \right| = |x_1 - x_2| \, |x_1 + x_2| = \left(\frac{\delta}{2} \right) \left(\frac{\delta}{2} + \frac{2}{\delta} \right) = \frac{\delta^2}{4} + 1 \geq 1 = \varepsilon.$$

Hence, f is not uniformly continuous on $\mathbb{R}$.

Example 5.3.4 *For $p \in \mathbb{R}^1$, let $f(p) = (2p + 1, \ p^2)$. Then $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ is uniformly continuous on the closed interval $[0, 2]$.*

For $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{2\sqrt{5}}$. If $p_1 \in [0, 2]$ and $p_2 \in [0, 2]$, then

$$4 + (p_1 + p_2)^2 = 4 + |p_1 + p_2|^2 \leq 4 + (|p_1| + |p_2|)^2 \leq 4 + (2 + 2)^2 = 20$$

and

$$\begin{aligned} d_{\mathbb{R}^2}(f(p_1), f(p_2)) &= \sqrt{((2p_1 + 1) - (2p_2 + 1))^2 + \left(p_1^2 - p_2^2\right)^2} \\ &= |p_1 - p_2| \sqrt{4 + (p_1 + p_2)^2} < \delta \cdot \sqrt{20} = \frac{\varepsilon}{2\sqrt{5}} \cdot \sqrt{20} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ and $p_1, p_2 \in [1, 2]$ were arbitrary, we conclude that f is uniformly continuous on $[0, 2]$.

Theorem 5.3.5 (Uniform Continuity Theorem) *If f is a continuous mapping from a compact metric space X to a metric space Y, then f is uniformly continuous on X.*

Excursion 5.3.6 *Fill in what is missing in order to complete the proof of the Uniform Continuity Theorem.*

Proof. Suppose f is a continuous mapping from a compact metric space (X, d_X) to a metric space (Y, d_Y) and that $\varepsilon > 0$ is given. Since f is continuous, for each $p \in X$, there exists a positive real number δ_p such that $q \in X \wedge d_X(q, p) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. Let $\mathcal{G} = \left\{ N_{\frac{1}{2}\delta_p}(p) : p \in X \right\}$. Since neighborhoods are open sets, we conclude that $\mathcal{G}$ is an $\underset{(1)}{\underline{\hspace{6cm}}}$. Since X is compact there exists a finite number of elements of $\mathcal{G}$ that covers X, say $p_1, p_2, \ldots, p_n$. Hence,

$$X \subset \bigcup_{j=1}^{n} N_{\frac{1}{2}\delta_{p_j}}(p_j).$$

Let $\delta \underset{def}{=} \frac{1}{2} \min_{1 \le j \le n} \{\delta_{p_j}\}$. Then, $\delta > 0$ and the minimum of a finite number of positive real numbers.

Suppose that $p, q \in X$ are such that $d(p, q) < \delta$. Because $p \in X$ and $X \subset \bigcup_{j=1}^{n} N_{\frac{1}{2}\delta_{p_j}}(p_j)$, there exists a positive integer k, $1 \le k \le n$, such that $\underset{(2)}{\underline{\hspace{6cm}}}$. Hence, $d(p, p_k) < \frac{1}{2}\delta_{p_k}$. From the triangular inequality

$$d_X(q, p_k) \le d_X(q, p) + d_X(p, p_k) < \delta + \underset{(3)}{\underline{\hspace{1cm}}} \le \delta_{p_k}.$$

Another application of the triangular inequality and the choices that were made for δ_p yield that

$$d_Y(f(p), f(q)) \le \underset{(4)}{\underline{\hspace{6cm}}} < \underset{(5)}{\underline{\hspace{3cm}}} = \varepsilon$$

■

***Acceptable fill ins are: (1) open cover for X (2) $p \in N_{\frac{1}{2}\delta_k}(p_k)$, (3) $\frac{1}{2}\delta_{p_k}$, (4) $d_Y(f(p), f(p_k)) + d_Y(f(p_k), f(q))$, $\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. ***

5.4 Discontinuities and Monotonic Functions

Given two metric spaces (X, d_X) and (Y, d_Y) and a function f from a subset A of X into Y. If $p \in X$ and f is not continuous at p, then we can conclude that f is not defined at p ($p \notin A = \operatorname{dom}(f)$), $\lim\limits_{\substack{x \to p \\ x \in A}} f(x)$ does not exist, and/or $p \in A$ and $\lim\limits_{\substack{x \to p \\ x \in A}} f(x)$ exists but $f(p) \neq \lim\limits_{\substack{x \to p \\ x \in A}} f(x)$; a point for which any of the three conditions occurs is called a point of discontinuity. In a general discussion of continuity of given functions, there is no need to discuss behavior at points that are not in the domain of the function; consequently, our consideration of points of discontinuity is restricted to behavior at points that are in a specified or implied domain. Furthermore, our discussion will be restricted to points of discontinuity for real-valued functions of a real-variable. This allows us to talk about one-sided limits, behavior on both sides of discontinuities and growth behavior.

Definition 5.4.1 *A function f is* ***discontinuous*** *at a point $c \in \operatorname{dom}(f)$ or* ***has a discontinuity at*** *c if and only if either $\lim\limits_{x \to c} f(x)$ doesn't exist or $\lim\limits_{x \to c} f(x)$ exists and is different from $f(c)$.*

Example 5.4.2 *The domain of $f(x) = \dfrac{|x|}{x}$ is $\mathbb{R} - \{0\}$. Consequently, f has no points of discontinuity on its domain.*

Example 5.4.3 *For the function* $f(x) = \begin{cases} \dfrac{|x|}{x} & , \quad \text{for } x \in \mathbb{R} - \{0\} \\ 1 & , \quad \text{for } x = 0 \end{cases}$,

$\operatorname{dom}(f) = \mathbb{R}$ *and $x = 0$ is a point of discontinuity of f. To see that $\lim\limits_{x \to 0} f(x)$ does not exist, note that, for every positive real number δ,*

$$f\left(-\frac{\delta}{2}\right) = -1 \qquad \text{and} \qquad \left| f\left(-\frac{\delta}{2}\right) - f(0) \right| = 2.$$

Hence, if $\varepsilon = \dfrac{1}{2}$, then, for every positive real number δ, there exists $x \in \operatorname{dom}(f)$ such that $0 < |x| < \delta$ and $|f(x) - f(0)| \geq \varepsilon$. Therefore, f is not continuous at $x = 0$.

Example 5.4.4 *If* $g(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & , \quad \textit{for } x \in \mathbb{R} - \{0\} \\ 0 & , \quad \textit{for } x = 0 \end{cases}$*, then* g *has no discontinuities in* $\mathbb{R}$.

Excursion 5.4.5 *Graph the following function* f *and find*

$$A = \{x \in \operatorname{dom}(f) : f \text{ is continuous at } x\}$$

and $B = \{x \in \operatorname{dom}(f) : f \text{ is discontinuous at } x\}$.

$$f(x) = \begin{cases} \dfrac{4x+1}{1+x} & , \quad x \leq \dfrac{1}{2} \\ 2 & , \quad \dfrac{1}{2} < x \leq 1 \\ -2x+4 & , \quad 1 < x \leq 3 \\ \lfloor x \rfloor + 2 & , \quad 3 < x \leq 6 \\ \dfrac{14(x-10)}{x-14} & , \quad (6 < x < 14) \vee (14 < x) \end{cases}$$

Hopefully, your graph revealed that $A = \mathbb{R} - \{-1, 3, 4, 5, 6, 14\}$ and $B = \{3, 4, 5, 6\}$.

Definition 5.4.6 *Let f be a function that is defined on the segment (a, b). Then, for any point $x \in [a, b)$, the **right-hand limit** is denoted by $f(x+)$ and*

$$f(x+) = q \Leftrightarrow \left(\forall \{t_n\}_{n=1}^{\infty}\right)\left[\left(\{t_n\} \subset (x, b) \wedge \lim_{n\to\infty} t_n = x\right) \Rightarrow \lim_{n\to\infty} f(t_n) = q\right]$$

*and, for any $x \in (a, b]$, the **left-hand limit** is denoted by $f(x-)$ and*

$$f(x-) = q \Leftrightarrow \left(\forall \{t_n\}_{n=1}^{\infty}\right)\left[\left(\{t_n\} \subset (a, x) \wedge \lim_{n\to\infty} t_n = x\right) \Rightarrow \lim_{n\to\infty} f(t_n) = q\right].$$

Remark 5.4.7 *From the treatment of one-sided limits in frosh calculus courses, recall that $\lim_{t\to x^+} f(t) = q$ if and only if*

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall t)(t \in \operatorname{dom}(f) \wedge x < t < x + \delta \Rightarrow |f(t) - q| < \varepsilon)\right]$$

and $\lim_{t\to x^-} f(t) = q$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall t)(t \in \operatorname{dom}(f) \wedge x - \delta < t < x \Rightarrow |f(t) - q| < \varepsilon)\right].$$

The Sequences Characterization for Limits of Functions justifies that these definitions are equivalent to the definitions of $f(x+)$ and $f(x-)$, respectively.

Excursion 5.4.8 *Find $f(x+)$ and $f(x-)$ for every $x \in B$ where B is defined in Excursion 5.4.5.*

For this function, we have $f(3-) = -2$, $f(3+) = 5$; $f(4-) = 5$, $f(4+) = 6$; $f(5-) = 6$, $f(5+) = 7$; and $f(6-) = 7$, $f(6+) = 7$.

Remark 5.4.9 *For each point x where a function f is continuous, we must have $f(x+) = f(x-) = f(x)$.*

Definition 5.4.10 *Suppose the function f is defined on the segment (a, b) and discontinuous at $x \in (a, b)$. Then f has a* ***discontinuity of the first kind at*** *x or a* ***simple discontinuity at*** *x if and only if both $f(x+)$ and $f(x-)$ exist. Otherwise, the discontinuity is said to be a* ***discontinuity of the second kind****.*

Excursion 5.4.11 *Classify the discontinuities of the function f in Excursion 5.4.5.*

Remark 5.4.12 *The function* $F(x) = \begin{cases} f(x) & , \textit{for} \quad x \in R - \{-1, 14\} \\ 0 & , \textit{for} \quad x = -1 \vee x = 14 \end{cases}$ *where f is given in Excursion 5.4.5 has discontinuities of the second kind at $x = -1$ and $x = 14$.*

Excursion 5.4.13 *Discuss the continuity of each of the following.*

1. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2} & , \quad x < -2 \\ 2x - 1 & , \quad x \geq -2 \end{cases}$

2. $g(x) = \begin{cases} 2 & , \quad x \textit{ rational} \\ 1 & , \quad x \textit{ irrational} \end{cases}$

Your discussion of (1) combines considers some cases. For $-\infty < x < -2$, $\dfrac{x^2 - x - 6}{x+2}$ is continuous as the quotient of polynomials for which the denominator is not going to zero, while continuity of $2x - 1$ for $x > -2$ follows from the limit of the sum theorem or because $2x - 1$ is a polynomial; consequently, the only point in the domain of f that needs to be checked is $x = -2$. Since $f((-2)-) = \lim_{x \to -2^-} \dfrac{x^2 - x - 6}{x+2} = \lim_{x \to -2^-} (x-3) = -5$, $f((-2)+) = \lim_{x \to -2^-} (2x - 1) = -5$, and $f(-2) = -5$, it follows that f is also continuous at $x = -2$. That the function given in (2) is not continuous anywhere follows from the density of the rationals and the irrationals; each point of discontinuity is a "discontinuity of the second kind."

Definition 5.4.14 *Let f be a real-valued function on a segment (a, b). Then f is said to be **monotonically increasing** on (a, b) if and only if*

$$(\forall x_1)(\forall x_2)\left[x_1, x_2 \in (a, b) \wedge x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)\right]$$

*and f is said to be **monotonically decreasing** on (a, b) if and only if*

$$(\forall x_1)(\forall x_2)\left[x_1, x_2 \in (a, b) \wedge x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)\right]$$

*The **class of monotonic functions** is the set consisting of both the functions that are increasing and the functions that are decreasing.*

Excursion 5.4.15 *Classify the monotonicity of the function f that was defined in Excursion 5.4.5*

Based on the graph, we have that f is monotonically increasing in each of $(-\infty, -1)$, $(-1, 1)$, and $(3, 6)$; the function is monotonically decreasing in each of $\left(\dfrac{1}{2}, 3\right)$, $(6, 14)$, and $(14, \infty)$. The section $\left(\dfrac{1}{2}, 1\right)$ is included in both statements because the function is constant there. As an alternative, we could have claimed that f is both monotonically increasing and monotonically decreasing in each of $\left(\frac{1}{2}, 1\right)$, $(3, 4)$, $(4, 5)$, and $(5, 6)$ and distinguished the other segments according to the property of being **strictl**y monotonically increasing and **strictly** monotonically decreasing.

Now we will show that monotonic functions do not have discontinuities of the second kind.

Theorem 5.4.16 *Suppose that the real-valued function f is monotonically increasing on a segment (a, b). Then, for every $x \in (a, b)$ both $f(x-)$ and $f(x+)$ exist,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

and

$$(\forall x)(\forall y)(a < x < y < b \Rightarrow f(x+) \leq f(y-)).$$

Excursion 5.4.17 *Fill in what is missing in order to complete the following proof of the theorem.*

Space for scratch work.

Proof. Suppose that f is monotonically increasing on the segment (a, b) and $x \in (a, b)$. Then, for every $t \in (a, b)$ such that $a < t < x$, $\underset{(1)}{\underline{\qquad\qquad}}$. Hence, $B \underset{def}{=} \{f(t) : a < t < x\}$ is bounded above by $f(x)$. By the $\underset{(2)}{\underline{\qquad\qquad\qquad\qquad}}$, the set B has a least upper bound; let $u = \sup(B)$. Now we want to show that $u = f(x-)$.

Let $\varepsilon > 0$ be given. Then $u = \sup(B)$ and $u - \varepsilon < u$ yields the existence of a $w \in B$ such that $\underset{(3)}{\underline{\qquad\qquad\qquad\qquad}}$. From the definition of B, w is the image of a point in $\underset{(4)}{\underline{\qquad\qquad}}$. Let $\delta > 0$

be such that $x - \delta \in (a, x)$ and $f(x - \delta) = w$. If $t \in (x - \delta, x)$, then

$$f(x-\delta) \le f(t) \qquad and \qquad \underbrace{\rule{4cm}{0.4pt}}_{(5)}.$$

Since $u - \varepsilon < w$ and $f(x) \le u$, the transitivity of less than or equal to yields that

$$\underbrace{\rule{2cm}{0.4pt}}_{(6)} < f(t) \qquad and \qquad f(t) \le \underbrace{\rule{2cm}{0.4pt}}_{(7)}.$$

Because t was arbitrary, we conclude that

$$(\forall t)\,(x - \delta < t < x \Rightarrow u - \varepsilon < f(t) \le u)\,.$$

Finally, it follows from $\varepsilon > 0$ being arbitrary that

$$(\forall \varepsilon > 0)\left(\underbrace{\rule{6cm}{0.4pt}}_{(8)}\right);\text{ i.e.,}$$

$f(x-) = \lim_{t \to x^-} f(t) = u$.

For every $t \in (a, b)$ such that $x < t < b$, we also have that $f(x) \le f(t)$ from which it follows that $C \underset{def}{=} \{f(t) : x < t < b\}$ is bounded $\underbrace{\rule{2cm}{0.4pt}}_{(9)}$ by $f(x)$. From the greatest lower bound property of the reals, C has a greatest lower bound that we will denote by υ.

Use the space provided to prove that $\upsilon = f(x+)$.

(10)

Next suppose that $x, y \in (a, b)$ are such that $x < y$.
Because $f(x+) = \lim_{t \to x^+} f(t) = \inf\{f(t) : x < t < b\}$, $(x, y) \subset (x, b)$ and f is monotonically increasing, it follows that

$$\underbrace{\qquad\qquad\qquad}_{(11)} = \inf\{f(t) : x < t < y\}.$$

From our earlier discussion, ■
$f(y-) = \lim_{t \to y^-} f(t) = \underbrace{\qquad\qquad\qquad\qquad\qquad}_{(12)}$.
Now, $(x, y) \subset (a, y)$ yields that

$$f(y-) = \sup\{f(t) : x < t < y\}.$$

Therefore, $\underbrace{\qquad\qquad\qquad\qquad\qquad}_{(13)}$ as claimed.

*******The expected responses are: (1) $f(t) \le f(x)$, (2) least upper bound property, (3) $u - \varepsilon < w < u$, (4) (a, x), (5) $f(t) \le f(x)$, (6) $u - \varepsilon$, (7) u, (8) $(\exists \delta > 0)\left[(\forall t)\,(x - \delta < t < x \Rightarrow u - \varepsilon < f(t) < u)\right]$, (9) below, (10) Let $\varepsilon > 0$ be given. Then $v = \inf C$ implies that there exists $w \in C$ such that $v < w < v + \varepsilon$. Since $w \in C$, w is the image of some point in (x, b). Let $\delta > 0$ be such that $x + \delta \in (x, b)$ and $f(x + \delta) = w$. Now suppose $t \in (x, x + \delta)$. Then $f(x) \le f(t)$ and $f(t) \le f(x + \delta) = w$. Since $v \le f(x)$ and $w < v + \varepsilon$, it follows that $v \le f(t)$ and $f(t) < v + \varepsilon$. Thus, $(\exists \delta > 0)\left[(\forall t)\,(x < t < x + \delta \Rightarrow v < f(t) \le v + \varepsilon)\right]$. Because $\varepsilon > 0$ was arbitrary, we conclude that $v = \lim_{t \to x^+} f(t) = f(x+)$., (11) $f(x+)$, (12) $\sup\{f(t) : a < t < y\}$ and (13) $f(x+) \le f(y-)$.*******

Corollary 5.4.18 *Monotonic functions have no discontinuities of the second kind.*

The nature of discontinuities of functions that are monotonic on segments allows us to identify points of discontinuity with rationals in such a way to give us a limit on the number of them.

Theorem 5.4.19 *If f is monotonic on the segment (a, b), then*

$$\{x \in (a, b) : f \text{ is discontinuous at } x\}$$

is at most countable.

Excursion 5.4.20 *Fill in what is missing in order to complete the following proof of Theorem 5.4.19.*

Proof. *Without loss of generality, we assume that f is a function that is monotonically increasing in the segment $(a, b) \underset{def}{=} I$. If f is continuous in I, then f has no points of discontinuity there and we are done. Suppose that f is not continuous on I and let $D = \{w \in I : f$ is not continuous at $w\}$.*

From our assumption $D \neq \emptyset$ and we can suppose that $\zeta \in D$. Then $\zeta \in \mathrm{dom}\,(f)$,

$$(\forall x)\,(x \in I \wedge x < \zeta \Rightarrow f(x) \leq f(\zeta))$$

and

$$(\forall x)\left(x \in I \wedge \zeta < x \Rightarrow \underset{(1)}{\underline{\hspace{6em}}}\right).$$

From Theorem 5.4.16, $f(\zeta-)$ and $f(\zeta+)$ exist; furthermore,

$$f(\zeta-) = \sup\{f(x) : x < \zeta\},\ f(\zeta+) = \underset{(2)}{\underline{\hspace{12em}}}.$$

and $f(\zeta-) \leq f(\zeta+)$. Since ζ is a discontinuity for f, $f(\zeta-) \underset{(3)}{\underline{\hspace{2em}}} f(\zeta+)$. From the Density of the Rationals, it follows that there exists a rational r_ζ such that $f(\zeta-) < r_\zeta < f(\zeta+)$. Let $I_{r_\zeta} = (f(\zeta-), f(\zeta+))$. If $D - \{\zeta\} = \emptyset$, then $|D| = 1$ and we are done. If $D - \{\zeta\} \neq \emptyset$ then we can choose another $\xi \in D$ such that $\xi \neq \zeta$. Without loss of generality suppose that $\xi \in D$ is such that $\zeta < \xi$. Since ζ was an arbitrary point in the discussion just completed, we know that there exists a rational r_ξ, $r_\xi \neq r_\zeta$, such that $\underset{(4)}{\underline{\hspace{12em}}}$ and we can let $I_{r_\xi} = (f(\xi-), f(\xi+))$. Since $\zeta < \xi$, it follows from Theorem 5.4.16 that $\underset{(5)}{\underline{\hspace{5em}}} \leq f(\xi-)$. Thus, $I_{r_\zeta} \cap I_{r_\xi} = \underset{(6)}{\underline{\hspace{1.5em}}}$.

Now, let $\mathcal{G} = \left\{I_{r_\gamma} : \gamma \in D\right\}$ and $H : D \to \mathcal{G}$ be defined by $H(w) = I_{r_w}$. Now we claim that H is a one-to-one correspondence. To see that H is $\underline{\hspace{12em}}$, suppose that $w_1, w_2 \in D$ and $H(w_1) = H(w_2)$.

Then

(7)

To see that H is onto, note that by definition $H(D) \subset \mathcal{G}$ and suppose that $A \in \mathcal{G}$. Then

(8)

Finally, $H : D \overset{1-1}{\twoheadrightarrow} \mathcal{G}$ yields that $D \sim \mathcal{G}$. Since $r_\gamma \in \mathbb{Q}$ for each $I_{r_\gamma} \in \mathcal{G}$, we have that $|\mathcal{G}| \leq |\mathbb{Q}| = \aleph_0$. Therefore, $|D| \leq \aleph_0$; i.e., D is __________________________.
(9)

■

***Acceptable responses are : (1) $f(\zeta) \leq f(x)$, (2) $\inf\{f(x) : \zeta < x\}$, (3) $<$, (4) $f(\xi-) < r_\xi < f(\xi+)$, (5) $f(\zeta+)$, (6) $\emptyset$, (7) $I_{r_{w_1}} = I_{r_{w_2}}$. From the Trichotomy Law, we know that one and only one of $w_1 < w_2$, $w_1 = w_2$, or $w_2 < w_1$ holds. Since either $w_1 < w_2$ or $w_2 < w_1$ implies that $I_{r_{w_1}} \cap I_{r_{w_2}} = \emptyset$, we conclude that $w_1 = w_2$. Since w_1 and w_2 were arbitrary,
$(\forall w_1)(\forall w_2)[H(w_1) = H(w_2) \Rightarrow w_1 = w_2]$; i.e., H is one-to-one., (8) there exists $r \in \mathbb{Q}$ such that $A = I_r$ and $r \in (f(\lambda-), f(\lambda+))$ for some $\lambda \in D$. It follows that $H(\lambda) = A$ or $A \in H(D)$. Since A was an arbitrary element of $\mathcal{G}$, we have that $(\forall A)[A \in \mathcal{G} \Rightarrow A \in H(D)]$; i.e., $\mathcal{G} \subset H(D)$. From $H(D) \subset \mathcal{G}$ and $\mathcal{G} \subset H(D)$, we conclude that $\mathcal{G} = H(D)$. Hence, H is onto., and (9) at most countable.***

Remark 5.4.21 *The level of detail given in Excursion 5.4.20 was more that was needed in order to offer a well presented argument. Upon establishing the ability to associate an interval I_{r_ζ} with each $\zeta \in D$ that is labelled with a rational and justifying that the set of such intervals is pairwise disjoint, you can simply assert that you have established a one-to-one correspondence with a subset of the rationals and the set of discontinuities from which it follows that the set of discontinuities is at most countable. I chose the higher level of detail–which is also acceptable–in* *order to make it clearer where material prerequisite for this course is a part of the foundation on which we are building. For a really concise presentation of a proof of Theorem 5.4.19, see pages 96-97 of our text.*

Remark 5.4.22 *On page 97 of our text, it is noted by the author that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of a segment (a, b), he constructs a function f that is monotonic on (a, b) with E as set of all discontinuities of f in (a, b). More consideration of the example is requested in our exercises.*

5.4.1 Limits of Functions in the Extended Real Number System

Recall the various forms of definitions for limits of real valued functions in relationship to infinity:
Suppose that f is a real valued function on $\mathbb{R}$, c is a real number, and L real number, then

- $$\lim_{x\to+\infty} f(x) = L \Leftrightarrow (\forall\varepsilon > 0)\,(\exists K > 0)\,(x > K \Rightarrow |f(x) - L| < \varepsilon)$$
  $$\Leftrightarrow (\forall\varepsilon > 0)\,(\exists K > 0)\,(x > K \Rightarrow f(x) \in N_\varepsilon(L))$$
- $$\lim_{x\to-\infty} f(x) = L \Leftrightarrow (\forall\varepsilon > 0)\,(\exists K > 0)\,(x < -K \Rightarrow |f(x) - L| < \varepsilon)$$
  $$\Leftrightarrow (\forall\varepsilon > 0)\,(\exists K > 0)\,(x < -K \Rightarrow f(x) \in N_\varepsilon(L))$$
- $$\lim_{x\to c} f(x) = +\infty \Leftrightarrow (\forall M \in \mathbb{R})\,(\exists\delta > 0)\,(0 < |x - c| < \delta \Rightarrow f(x) > M)$$
  $$\Leftrightarrow (\forall M \in \mathbb{R})\,(\exists\delta > 0)\,\left(x \in N_\delta^d(c) \Rightarrow f(x) > M\right)$$
  where $N_\delta^d(c)$ denotes the deleted neighborhood of c, $N_\delta(c) - \{c\}$.
- $$\lim_{x\to c} f(x) = -\infty \Leftrightarrow (\forall M \in \mathbb{R})\,(\exists\delta > 0)\,(0 < |x - c| < \delta \Rightarrow f(x) < M)$$
  $$\Leftrightarrow (\forall M \in \mathbb{R})\,(\exists\delta > 0)\,\left(x \in N_\delta^d(c) \Rightarrow f(x) < M\right)$$

Based of the four that are given, complete each of the following.

- $\lim_{x\to+\infty} f(x) = +\infty \Leftrightarrow$ ____

- $\lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow$ ________________________________

- $\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow$ ________________________________

- $\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow$ ________________________________

Hopefully, the neighborhood formulation and the pattern of the various statements suggests that we could pull things together if we had comparable descriptions for neighborhoods of $+\infty$ and $-\infty$.

Definition 5.4.23 *For any positive real number K,*

$$N_K(\infty) = \{x \in \mathbb{R} \cup \{+\infty, -\infty\} : x > K\}$$

and $N_K(-\infty) = \{x \in \mathbb{R} \cup \{+\infty, -\infty\} : x < K\}$ are neighborhoods of $+\infty$ and $-\infty$, respectively.

With this notation we can consolidate the above definitions.

Definition 5.4.24 *Let f be a real valued function defined on $\mathbb{R}$. Then for A and c in the extended real number system, $\lim_{x \to c} f(x) = A$ if and only if for every neighborhood of A, $N(A)$ there exists a deleted neighborhood of c, $N^{*d}(c)$, such that $x \in N^{*d}(c)$ implies that $f(x) \in N(A)$. When specification is needed this will be referred to a the* ***limit of a function in the extended real number system****.*

Hopefully, the motivation that led us to this definition is enough to justify the claim that this definition agrees with the definition of $\lim_{x \to c} f(x) = A$ when c and A are real. Because the definition is the natural generalization and our proofs for the properties of limits of function built on information concerning neighborhoods, we note that we can establish some of the results with only minor modification in the proofs that have gone before. We will simply state analogs.

Theorem 5.4.25 *Let f be a real-valued function that is defined on a set $E \subset \mathbb{R}$ and suppose that $\lim_{t\to c} f(t) = A$ and $\lim_{t\to c} f(t) = C$ for c, A, and C in the extended real number system. Then $A = C$.*

Theorem 5.4.26 *Let f and g be real-valued functions that are defined on a set $E \subset \mathbb{R}$ and suppose that $\lim_{t\to c} f(t) = A$ and $\lim_{t\to c} g(t) = B$ for c, A, and B in the extended real number system. Then*

1. $\lim_{t\to c} (f+g)(t) = A + B$,
2. $\lim_{t\to c} (fg)(t) = AB$, *and*
3. $\lim_{t\to c} \left(\frac{f}{g}\right)(t) = \frac{A}{B}$

whenever the right hand side of the equation is defined.

Remark 5.4.27 *Theorem 5.4.26 is not applicable when the algebraic manipulations lead to the need to consider any of the expressions $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, or $\frac{A}{0}$ because none of these symbols are defined.*

The theorems in this section have no impact on the process that you use in order to find limits of real functions as x goes to infinity. At this point in the coverage of material, given a specific function, we find the limit as x goes to infinity by using simple algebraic manipulations that allow us to apply our theorems for algebraic combinations of functions having finite limits. We close this chapter with two examples that are intended as memory refreshers.

Example 5.4.28 *Find* $\lim_{x\to\infty} \frac{\left(x^2 - 3x^3 + 5\right) + i\left(x^3 + x\sin x\right)}{4x^3 - 7}$.

Since the given function is the quotient of two functions that go to infinity as x goes to infinity, we factor in order to transform the given in to the quotient of functions that will have finite limits. In particular, we want to make use of the fact that, for any $p \in J$, $\lim_{x\to\infty} \frac{1}{x^p} = 0$. From

$$\lim_{x\to\infty} \frac{\left(x^2 - 3x^3 + 5\right) + i\left(x^3 + x\sin x\right)}{4x^3 - 7} = \lim_{x\to\infty} \frac{\left(\frac{1}{x} - 3 + \frac{5}{x^3}\right) + i\left(1 + \frac{\sin x}{x^2}\right)}{\left(4 - \frac{7}{x^3}\right)},$$

The limit of the quotient and limit of the sum theorem yields that

$$\lim_{x\to\infty}\frac{\left(x^2-3x^3+5\right)+i\left(x^3+x\sin x\right)}{4x^3-7}=\frac{(0-3+0)+i\,(1+0)}{4-0}=\frac{-3+i}{4}.$$

Example 5.4.29 *Find* $\lim_{x\to-\infty}\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)$.

In its current form, it looks like the function is tending to $\infty-\infty$ which is undefined. In this case, we will try "unrationalizing" the expression in order to get a quotient at will allow some elementary algebraic manipulations. Note that

$$\begin{aligned}
&\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\\
&= \frac{\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\
&= \frac{\left(2x^2+x+2\right)-\left(2x^2-x-1\right)}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\
&= \frac{2x+3}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}.
\end{aligned}$$

Furthermore, for $x<0$, $\sqrt{x^2}=|x|=-x$. Hence,

$$\begin{aligned}
&\lim_{x\to-\infty}\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\\
&= \lim_{x\to-\infty}\frac{2x+3}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\
&= \lim_{x\to-\infty}\frac{2x+3}{\sqrt{x^2}\left(\sqrt{2+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{2-\frac{1}{x}-\frac{1}{x^2}}\right)}\\
&= \lim_{x\to-\infty}(-1)\frac{2+\frac{3}{x}}{\left(\sqrt{2+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{2-\frac{1}{x}-\frac{1}{x^2}}\right)}\\
&= (-1)\left(\frac{2}{\sqrt{2}+\sqrt{2}}\right)=\frac{-1}{\sqrt{2}}=\frac{-\sqrt{2}}{2}.
\end{aligned}$$

5.5 Problem Set E

1. For each of the following real-valued functions of a real variable give a well-written $\delta - \varepsilon$ proof of the claim.

   (a) $\lim_{x \to 2} \left(3x^2 - 2x + 1\right) = 9$

   (b) $\lim_{x \to -1} 8x^2 = 8$

   (c) $\lim_{x \to 16} \sqrt{x} = 4$

   (d) $\lim_{x \to 1} \dfrac{3}{x - 2} = -3$

   (e) $\lim_{x \to 3} \dfrac{x + 4}{2x - 5} = 7.$

2. For each of the following real-valued functions of a real variable find the implicit domain and range.

   (a) $f(x) = \dfrac{\sin x}{x^2 - 1}$

   (b) $f(x) = \sqrt{2x + 1}$

   (c) $f(x) = \dfrac{x}{x^2 + 5x + 6}$

3. Let $f(x) = \begin{cases} \dfrac{-3}{x+3} & , \quad x < 0 \\ \dfrac{|x-2|}{x-2} & , \quad 0 \leq x < 2 \wedge x > 2 \\ 1 & , \quad x = 2 \end{cases}$

   (a) Sketch a graph for f.

   (b) Determine where the function f is continuous.

4. Let $f(x) = \begin{cases} \left|x^2 - 5x - 6\right| & \text{, for } \left|x - \frac{7}{2}\right| \geq \frac{5}{2} \\ \sqrt{36 - 6x} & \text{, for } \left|x - \frac{7}{2}\right| < \frac{5}{2} \end{cases}$ and $g(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{, for } x \neq -1 \\ 3 & \text{, for } x = -1 \end{cases}$.

   (a) Discuss the continuity of f at $x = 1$.

   (b) Discuss the continuity of $(fg)(x) = f(x)g(x)$ at $x = -1$.

5. For $f : \mathbb{C} \longrightarrow \mathbb{R}$ given by $f(z) = |z|$ give a $\delta - \varepsilon$ proof that $\lim_{z \to (1+i)} f(z) = \sqrt{2}$.

6. When it exists, find

   (a) $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}, \sqrt{3x^2 + 2} \right)$

   (b) $\lim_{x \to 1} \left(\frac{x - 1}{x^2 + 3x - 4}, \sqrt{x^4 + 5}, \frac{|x - 1|}{x - 1} \right)$

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose that $\lim_{x \to a} f(x) = L > 0$. Prove that

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}.$$

8. Using only appropriate definitions and elementary bounding processes, prove that if g is a real-valued function on $\mathbb{R}$ such that $\lim_{x \to a} g(x) = M \neq 0$, then $\lim_{x \to a} \frac{1}{[g(x)]^2} = \frac{1}{M^2}$.

9. Suppose that A is a subset of a metric space (S, d), $f : A \longrightarrow \mathbb{R}^1$, and $g : A \longrightarrow \mathbb{R}^1$. Prove each of the following.

   (a) If c is a real number and $f(p) = c$ for all $p \in A$, then, for any limit point p_0 of A, we have that $\lim_{\substack{p \to p_0 \\ p \in A}} f(p) = c$.

(b) If $f(p) = g(p)$ for all $p \in A - \{p_0\}$ were p_0 is a limit point of A and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$, then $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = L$.

(c) If $f(p) \leq g(p)$ for all $p \in A$, $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$ and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = M$, then $L \leq M$.

10. For each of the following functions on $\mathbb{R}^2$, determine whether or not the given function is continuous at $(0,0)$. Use $\delta - \varepsilon$ proofs to justify continuity or show lack of continuity by justifying that the needed limit does not exist.

(a) $f((x,y)) = \begin{cases} \dfrac{xy^2}{\left(x^2+y^2\right)^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } x = y = 0 \end{cases}$

(b) $f((x,y)) = \begin{cases} \dfrac{x^3y^3}{\left(x^2+y^2\right)^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } x = y = 0 \end{cases}$

(c) $f((x,y)) = \begin{cases} \dfrac{x^2y^4}{\left(x^2+y^4\right)^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } x = y = 0 \end{cases}$

11. Discuss the uniform continuity of each of the following on the indicated set.

(a) $f(x) = \dfrac{x^2+1}{2x+3}$ in the interval $[4,9]$.

(b) $f(x) = x^3$ in $[1, \infty)$.

12. For $a < b$, let $\mathcal{C}([a,b])$ denote the set of all real valued functions that are continuous on the interval $[a,b]$. Prove that $d(f,g) = \max\limits_{a \leq x \leq b} |f(x) - g(x)|$ is a metric on $\mathcal{C}([a,b])$.

13. Correctly formulate the monotonically decreasing analog for Theorem 5.4.16 and prove it.

14. Suppose that f is monotonically increasing on a segment $I = (a, b)$ and that $(\exists M)\left[M \in \mathbb{R} \wedge (\forall x)\,(x \in I \Rightarrow f(x) \le M)\right]$. Prove that there exists a real number C such that $C \le M$ and $f(b-) = C$.

15. A function f defined on an interval $I = [a, b]$ is called strictly increasing on I if and only if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ for $x_1, x_2 \in I$. Furthermore, a function f is said to have the intermediate value properly in I if and only if for each c between $f(a)$ and $f(b)$ there is an $x_0 \in I$ such that $f(x_0) = c$. Prove that a function f that is strictly increasing and has the intermediate value property on an interval $I = [a, b]$ is continuous on (a, b).

16. Give an example of a real-valued function f that is continuous and bounded on $[0, \infty)$ while not satisfying the Extreme Value Theorem.

17. Suppose that f is uniformly continuous on the intervals I_1 and I_2. Prove that f is uniformly continuous on $S = I_1 \cup I_2$.

18. Suppose that a real-valued function f is continuous on I° where $I = [a, b]$. If $f(a+)$ and $f(b-)$ exist, show that the function

$$f_0(x) = \begin{cases} f(a+) & \text{, for } \quad x = a \\ f(x) & \text{, for } \quad a < x < b \\ f(b-) & \text{, for } \quad x = b \end{cases}$$

is uniformly continuous on I.

19. If a real valued function f is uniformly continuous on the half open interval $(0, 1]$, is it true that f is bounded there. Carefully justify the position taken.