Chapter 8

Sequences and Series of Functions

Given a set A, a sequence of elements of A is a function $F : \mathbb{J} \to A$; rather than using the notation $F(n)$ for the elements that have been selected from A, since the domain is always the natural numbers, we use the notational convention $a_n = F(n)$ and denote sequences in any of the following forms:

$$\{a_n\}_{n=1}^{\infty}, \qquad \{a_n\}_{n\in\mathbb{J}}, \qquad or \qquad a_1, a_2, a_3, a_4, ...$$

Given any sequence $\{c_k\}_{k=1}^{\infty}$ of elements of a set A, we have an associated sequence of nth partial sums

$$\{s_n\}_{n=1}^{\infty} \text{ where } s_n = \sum_{k=1}^{n} c_k;$$

the symbol $\sum_{k=1}^{\infty} c_k$ is called a series (or infinite series). Because the function $g(x) = x - 1$ is a one-to-one correspondence from $\mathbb{J}$ into $\mathbb{J} \cup \{0\}$, i.e., $g : \mathbb{J} \overset{1-1}{\twoheadrightarrow} \mathbb{J} \cup \{0\}$, a sequence could have been defined as a function on $\mathbb{J} \cup \{0\}$. In our discussion of series, the symbolic descriptions of the sequences of nth partial sums usually will be generated from a sequence for which the first subscript is 0. The notation always makes the indexing clear, when such specificity is needed.

Thus far, our discussion has focused on sequences and series of complex (and real) numbers; i.e., we have taken $A = \mathbb{C}$ and $A = \mathbb{R}$. In this chapter, we take A to be the set of complex (and real) functions on $\mathbb{C}$ (and $\mathbb{R}$).

8.1 Pointwise and Uniform Convergence

The first important thing to note is that we will have different types of convergence to consider because we have "more variables." The first relates back to numerical sequences and series. We start with an example for which the work was done in Chapter 4.

Example 8.1.1 *For each $n \in \mathbb{J}$, let $f_n(z) = z^n$ where $z \in \mathbb{C}$. We can use results obtained earlier to draw some conclusions about the convergence of $\{f_n(z)\}_{n=1}^{\infty}$. In Lemma 4.4.2, we showed that,* ***for any fixed complex number*** *z_0 such that $|z_0| < 1$, $\lim_{n\to\infty} z_0^n = 0$. In particular, we showed that for z_0, $0 < |z_0| < 1$, if $\varepsilon > 0$, then taking*

$$M = M(\varepsilon, z_0) = \begin{cases} 1 & , \quad \textit{for } \varepsilon \geq 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln|z_0|} \right\rceil & , \quad \textit{for } \varepsilon < 1 \end{cases}.$$

yields that $\left|z_0^n - 0\right| < \varepsilon$ for all $n > M$. When $z_0 = 0$, we have the constant sequence. In offering this version of the statement of what we showed, I made a "not so subtle" change in format; namely, I wrote the former $M(\varepsilon)$ and $M(\varepsilon, z_0)$. The change was to stress that our discussion was tied to the fixed point. In terms of our sequence $\{f_n(z)\}_{n=1}^{\infty}$, we can say that for each fixed point $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$, $\{f_n(z_0)\}_{n=1}^{\infty}$ is convergent to 0. This gets us to some new terminology: For this example, if $f(z) = 0$ for all $z \in \mathbb{C}$, then we say that $\{f_n(z_0)\}_{n=1}^{\infty}$ is ***pointwise convergent to f on Ω****.*

It is very important to keep in mind that our argument for convergence at each fixed point made clear and definite use of the fact that we had a point for which a known modulus was used in finding an $M(\varepsilon, z_0)$. It is natural to ask if the pointwise dependence was necessary. We will see that the answer depends on the nature of the sequence. For the sequence given in Example 8.1.1, the best that we will be able to claim over the set Ω is pointwise convergence. The associated sequence of nth partial sums for the functions in the previous example give us an example of a sequence of functions for which the pointwise limit is not a constant.

Example 8.1.2 *For $a \neq 0$ and each $k \in \mathbb{J} \cup \{0\}$, let $f_k(z) = az^k$ where $z \in \mathbb{C}$. In Chapter 4, our proof of the Convergence Properties of the Geometric Series*

Theorem showed that the associated sequence of nth partial sums $\{s_n(z)\}_{n=0}^{\infty}$ was given by

$$s_n(z) = \sum_{k=0}^{n} f_k(z) = \sum_{k=0}^{n} az^k = \frac{a\left(1 - z^{n+1}\right)}{1 - z}.$$

In view of Example 8.1.1, we see that for each fixed $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$, $\{s_n(z_0)\}_{n=1}^{\infty}$ is convergent to $\dfrac{a}{1 - z_0}$. Thus, $\{s_n(z)\}_{n=0}^{\infty}$ is ***pointwise convergent on Ω****. In terminology that is soon to be introduced, we more commonly say that "the series $\sum_{k=0}^{\infty} az^k$ is pointwise convergent on Ω."*

Our long term goal is to have an alternative way of looking at functions. In particular, we want a view that would give promise of transmission of nice properties, like continuity and differentiability. The following examples show that pointwise convergence proves to be insufficient.

Example 8.1.3 *For each $n \in \mathbb{J}$, let $f_n(z) = \dfrac{n^2 z}{1 + n^2 z}$ where $z \in \mathbb{C}$. For each fixed z we can use our properties of limits to find the pointwise limit of the sequences of functions. If $z = 0$, then $\{f_n(0)\}_{n=1}^{\infty}$ converges to 0 as a constant sequence of zeroes. If z is a fixed nonzero complex number, then*

$$\lim_{n\to\infty} \frac{n^2 z}{1 + n^2 z} = \lim_{n\to\infty} \frac{z}{\frac{1}{n^2} + z} = \frac{z}{z} = 1.$$

Therefore, $f_n \longrightarrow f$ *where* $f(z) = \begin{cases} 1 & , \textit{for} \quad z \in \mathbb{C} - \{0\} \\ \\ 0 & , \textit{for} \quad z = 0 \end{cases}$.

Remark 8.1.4 *From Theorem 4.4.3(c) or Theorem 3.20(d) of our text, we know that $p > 0$ and $\alpha \in \mathbb{R}$, implies that $\displaystyle\lim_{n\to\infty} \frac{n^\alpha}{(1 + p)^n} = 0$. Letting $\zeta = \dfrac{1}{1 + p}$ for $p > 0$ leads to the observation that*

$$\lim_{n\to\infty} n^\alpha \zeta^n = 0 \tag{8.1}$$

whenever $0 \le \zeta < 1$ and for any $\alpha \in R$. This is the form of the statement that is used by the author of our text in Example 7.6 where a sequence of functions for which the integral of the pointwise limit differs from the limit of the integrals of the functions in the sequence is given.

Example 8.1.5 (7.6 in our text) *Consider the sequence* $\{f_n\}_{n=1}^{\infty}$ *of real-valued functions on the interval* $I = [0,1]$ *that is given by* $f_n(x) = nx\left(1-x^2\right)^n$ *for* $n \in \mathbb{J}$. *For fixed* $x \in I - \{0\}$, *taking* $\alpha = 1$ *and* $\zeta = \left(1-x^2\right)$ *in (8.1) yields that* $n\left(1-x^2\right)^n \longrightarrow 0$ *as* $n \to \infty$. *Hence,* $f_n \underset{I-\{0\}}{\longrightarrow} 0$. *Because* $f_n(0) = 0$ *for all* $n \in \mathbb{J}$, *we see that for each* $x \in I$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx\left(1-x^2\right)^n = 0.$$

In contrast to having the Riemann integral of the limit function over I *being* 0, *we have that*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2}.$$

Note that, since α *in Equation (8.1) can be any real number, the sequence of real functions* $g_n(x) = n^2x\left(1-x^2\right)^n$ *for* $n \in \mathbb{J}$ *converges pointwise to* 0 *on* I *with*

$$\int_0^1 g_n(x)\,dx = \frac{n^2}{2n+2} \to \infty \text{ as } n \to \infty.$$

This motivates the search for a stronger sense of convergence; namely, uniform convergence of a sequence (and, in turn, of a series) of functions. Remember that our application of the term “uniform” to continuity required much nicer behavior of the function than continuity at points. We will make the analogous shift in going from pointwise convergence to uniform convergence.

Definition 8.1.6 *A sequence of complex functions* $\{f_n\}_{n=1}^{\infty}$ ***converges pointwise to a function*** f *on a subset* Ω *of* $\mathbb{C}$, *written* $f_n \longrightarrow f$ *or* $f_n \underset{z\in\Omega}{\longrightarrow} f$, *if and only if the sequence* $\{f_n(z_0)\}_{n=1}^{\infty} \longrightarrow f(z_0)$ *for each* $z_0 \in \Omega$; *i.e., for each* $z_0 \in \Omega$

$$(\forall \varepsilon > 0)\,(\exists M = M(\varepsilon, z_0) \in \mathbb{J})\,(n > M(\epsilon, z_0) \Rightarrow |f_n(z_0) - f(z_0)| < \varepsilon).$$

Definition 8.1.7 *A sequence of complex functions* $\{f_n\}$ ***converges uniformly*** *to* f *on a subset* Ω *of* $\mathbb{C}$, *written* $f_n \rightrightarrows f$, *if and only if*

$$(\forall \varepsilon > 0)\,(\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall n)\,(\forall z)\,(n > M \wedge z \in \Omega \Rightarrow |f_n(z) - f(z)| < \varepsilon)).$$

Remark 8.1.8 *Uniform convergence implies pointwise convergence. Given a sequence of functions, the only candidate for the uniform limit is the pointwise limit.*

Example 8.1.9 *The sequence considered in Example 8.1.1 exhibits the stronger sense of convergence if we restrict ourselves to compact subsets of* $\Omega = \{z \in \mathbb{C} : |z| < 1\}$. *For each* $n \in \mathbb{J}$, *let* $f_n(z) = z^n$ *where* $z \in \mathbb{C}$. *Then* $\{f_n(z)\}_{n=1}^{\infty}$ *is uniformly convergent to the constant function* $f(z) = 0$ *on any compact subset of* Ω.

Suppose $K \subset \Omega$ *is compact. From the Heine-Borel Theorem, we know that* K *is closed and bounded. Hence, there exists a positive real number* r *such that* $r < 1$ *and* $(\forall z)(z \in K \Rightarrow |z| \leq r)$. *Let* $\Omega_r = \{z \in \mathbb{C} : |z| \leq r\}$. *For* $\varepsilon > 0$, *let*

$$M = M(\varepsilon) = \begin{cases} 1 & , \quad \text{for } \varepsilon \geq 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln r} \right\rceil & , \quad \text{for } \varepsilon < 1 \end{cases}.$$

Then $n > M \Rightarrow n > \dfrac{\ln(\varepsilon)}{\ln r} \Rightarrow n \ln r < \ln(\varepsilon)$ *because* $0 < r < 1$. *Consequently,* $r^n < \varepsilon$ *and it follows that*

$$|f_n(z) - 0| = \left|z^n\right| = |z|^n \leq r^n < \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that* $f_n \underset{\Omega_r}{\rightrightarrows} f$. *Because* $K \subset \Omega_r$, $f_n \underset{K}{\rightrightarrows} f$ *as claimed.*

Excursion 8.1.10 *When we restrict ourselves to consideration of uniformly convergent sequences of real-valued functions on* $\mathbb{R}$, *the definition links up nicely to a graphical representation. Namely, suppose that* $f_n \underset{[a,b]}{\rightrightarrows} f$. *Then corresponding to any* $\varepsilon > 0$, *there exists a positive integer* M *such that* $n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon$ *for all* $x \in [a, b]$. *Because we have real-valued functions on the interval, the inequality translates to*

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \text{ for all } x \in [a, b]. \tag{8.2}$$

Label the following figure to illustrate what is described in (8.2) and illustrate the implication for any of the functions f_n *when* $n > M$.

Remark 8.1.11 *The negation of the definition offers us one way to prove that a sequence of functions is not uniformly continuous. Given a sequence of functions $\{f_n\}$ that are defined on a subset Ω of $\mathbb{C}$, the convergence of $\{f_n\}$ to a function f on Ω is not uniform if and only if*

$$(\exists\varepsilon > 0)\,(\forall M)\,[M \in \mathbb{J} \Rightarrow \\ (\exists n)\left(\exists z_{M_n}\right)\left(n > M \wedge z_{M_n} \in \Omega \wedge \left|f_n\left(z_{M_n}\right) - f\left(z_{M_n}\right)\right| \geq \varepsilon\right)].$$

Example 8.1.12 *Use the definition to show that the sequence $\left\{\frac{1}{nz}\right\}_{n=1}^{\infty}$ is pointwise convergent, but not uniformly convergent, to the function $f(x) = 0$ on $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$.*

Suppose that z_0 is a fixed element of Ω. For $\varepsilon > 0$, let $M = M(\varepsilon, z_0) = \left\lceil \frac{1}{|z_0|\,\varepsilon} \right\rceil$. Then $n > M \Rightarrow n > \frac{1}{|z_0|\,\varepsilon} \Rightarrow \frac{1}{n\,|z_0|} < \varepsilon$ because $|z_0| > 0$. Hence,

$$\left|\frac{1}{nz_0} - 0\right| = \frac{1}{n\,|z_0|} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\left\{\frac{1}{nz_0}\right\}_{n=1}^{\infty}$ is convergent to 0 for each $z_0 \in \Omega$. Therefore, $\left\{\frac{1}{nz}\right\}_{n=1}^{\infty}$ is pointwise convergent on Ω.

On the other hand, let $\varepsilon = \frac{1}{2}$ and for each $n \in \mathbb{J}$, set $z_n = \frac{1}{n+1}$. Then

$z_n \in \Omega$ and

$$\left|\frac{1}{nz_n} - 0\right| = \left|\frac{1}{n\left(\dfrac{1}{n+1}\right)}\right| = 1 + \frac{1}{n} \geq \varepsilon.$$

Hence, $\left\{\dfrac{1}{nz}\right\}_{n=1}^{\infty}$ is not uniformly convergent on Ω.

Example 8.1.13 *Prove that the sequence $\left\{\dfrac{1}{1+nz}\right\}_{n=1}^{\infty}$ converges uniformly for $|z| \geq 2$ and does not converge uniformly in $\Omega^* = \{z \in \mathbb{C} : |z| \leq 2\} - \left\{-\dfrac{1}{n} : n \in \mathbb{J}\right\}$.*

Let $\Omega = \{z \in \mathbb{C} : |z| \geq 2\}$ and, for each $n \in \mathbb{J}$, let $f_n(z) = \dfrac{1}{1+nz}$. From the limit properties of sequences, $\{f_n(z)\}_{n=1}^{\infty}$ is pointwise convergent on $\mathbb{C}$ to

$$f(z) = \begin{cases} 0 & , \text{for} \quad z \in \mathbb{C} - \{0\} \\ 1 & , \text{for} \quad z = 0 \end{cases}.$$

Thus, the pointwise limit of $\{f_n(z)\}_{n=1}^{\infty}$ on Ω is the constant function 0. For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \dfrac{1}{2}\left(\dfrac{1}{\varepsilon} + 1\right)\right\rceil$. Then $n > M \Rightarrow n > \dfrac{1}{2}\left(\dfrac{1}{\varepsilon} + 1\right) \Rightarrow \dfrac{1}{2n-1} < \varepsilon$ because $n > 1$. Furthermore, $|z| \geq 2 \Rightarrow n|z| \geq 2n \Rightarrow n|z| - 1 \geq 2n - 1 > 0$. Hence, $|z| \geq 2 \wedge n > M \Rightarrow$

$$|f_n(z) - 0| = \left|\frac{1}{1+nz}\right| \leq \frac{1}{|n|z|-1|} \leq \frac{1}{n|z|-1} \leq \frac{1}{2n-1} < \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that $f_n \underset{\Omega}{\rightrightarrows} 0$.

On the other hand, let $\varepsilon = \dfrac{1}{2}$ and, corresponding to each $n \in \mathbb{J}$, set $z_n = \dfrac{1}{n}$. Then $z_n \in \Omega^*$ and

$$|f_n(z_n) - 0| = \left|\frac{1}{1+n\left(\dfrac{1}{n}\right)}\right| = \frac{1}{2} \geq \varepsilon.$$

Hence, $\{f_n(z)\}_{n=1}^{\infty}$ *is not uniformly convergent in* Ω^*.

Excursion 8.1.14 *Use the definition to prove that the sequence* $\{z^n\}$ *is not uniformly convergent in* $|z| < 1$.

Hopefully, you thought to make use of the choices $\zeta_n = \left(1 - \dfrac{1}{n}\right)$ that could be related back to e^{-1}.

Using the definition to show that a sequence of functions is not uniformly convergent, usually, involves exploitation of "bad points." For Examples 8.1.12 and 8.1.13, the exploitable point was $x = 0$ while, for Example 8.1.14, it was $x = 1$.

Because a series of functions is realized as the sequence of nth partial sums of a sequence of functions, the definitions of pointwise and uniform convergence of series simply make statements concerning the nth partial sums. On the other hand, we add the notion of absolute convergence to our list.

Definition 8.1.15 *Corresponding to the sequence* $\{c_k(z)\}_{k=0}^{\infty}$ *of complex-valued functions on a set* $\Omega \subset \mathbb{C}$, *let*

$$S_n(z) = \sum_{k=0}^{n} c_k(z)$$

denote the sequence of nth *partial sums. Then*

(a) *the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***pointwise convergent on*** Ω ***to*** S *if and only if, for each* $z_0 \in \Omega$, $\{S_n(z_0)\}_{n=0}^{\infty}$ *converges to* $S(z_0)$; *and*

(b) *the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***uniformly convergent on*** Ω ***to*** S *if and only if* $S_n \underset{\Omega}{\rightrightarrows} S$.

Definition 8.1.16 *Corresponding to the sequence* $\{c_k(z)\}_{k=0}^{\infty}$ *of complex-valued functions on a set* $\Omega \subset \mathbb{C}$*, the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***absolutely convergent on*** Ω *if and only if* $\sum_{k=0}^{\infty} |c_k(z)|$ *is convergent for each* $z \in \Omega$*.*

Excursion 8.1.17 *For* $a \neq 0$ *and* $k \in \mathbb{J} \cup \{0\}$*, let* $c_k(z) = az^k$*. In Example 8.1.2, we saw that*

$$\sum_{k=0}^{\infty} c_k(z) = \sum_{k=0}^{\infty} az^k$$

is pointwise convergent for each $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$ *to* $a(1 - z_0)^{-1}$*. Show that*

(i) $\sum_{k=0}^{\infty} c_k(z)$ *is absolutely convergent for each* $z_0 \in \Omega$*;*

(ii) $\sum_{k=0}^{\infty} c_k(z)$ *is uniformly convergent on every compact subset* K *of* Ω*;*

(iii) $\sum_{k=0}^{\infty} c_k(z)$ *is not uniformly convergent on* Ω*.*

For part (i), hopefully you noticed that the formula derived for the proof of the Convergence Properties of the Geometric Series applied to the real series that results from replacing az^k with $|a|\,|z|^k$. Since $\sum_{k=0}^{n} |a|\,|z|^k = \dfrac{|a|\left(1 - |z|^{n+1}\right)}{1 - |z|}$, we conclude that $\left\{\sum_{k=0}^{n} |a| \, |z|^k\right\}_{n=0}^{\infty} \longrightarrow \dfrac{|a|}{1-|z|}$ for each $z \in C$ such that $|z| < 1$; i.e., $\sum_{k=0}^{\infty} c_k(z) = \sum_{k=0}^{\infty} az^k$ is absolutely convergent for each $z \in \Omega$. To show part (ii) it is helpful to make use of the fact that if K is a compact subset of Ω then there exists a positive real number r such that $r < 1$ and $K \subset \Omega_r = \{z \in \mathbb{C} : |z| \leq r\}$. The uniform convergence of $\sum_{k=0}^{\infty} c_k(z)$ on Ω_r then yields uniform convergence on K. For $S_n(z) = \sum_{k=0}^{n} c_k(z) = \sum_{k=0}^{n} az^k = \dfrac{a\left(1-z^{n+1}\right)}{1-z}$ and $S(z) = \dfrac{a}{1-z}$, you should have noted that $|S_n(z) - S(z)| \leq \dfrac{|a| \, r^{n+1}}{1-r}$ for all $z \in \Omega_r$ which leads to $M(\varepsilon) = \max\left\{1, \left\lceil \dfrac{\ln\left(\varepsilon(1-r)|a|^{-1}\right)}{\ln r} - 1 \right\rceil\right\}$ as one possibility for justifying the uniform convergence. Finally, with (iii), corresponding to each $n \in \mathbb{J}$, let $z_n = \left(1 - \dfrac{1}{n+1}\right)$; then $z_n \in \Omega$ for each n and $S_n(z_n) - S(z_n) = (n+1)|a| \left(1 - \dfrac{1}{n+1}\right)^{n+1}$ can be used to justify that we do not have uniform convergence.

8.1.1 Sequences of Complex-Valued Functions on Metric Spaces

In much of our discussion thus far and in numerous results to follow, it should become apparent that the properties claimed are dependent on the properties of the codomain for the sequence of functions. Indeed our original statement of the definitions of pointwise and uniform convergence require bounded the distance between images of points from the domain while not requiring any “nice behavior relating the points of the domain to each other.” To help you keep this in mind, we state the definitions again for sequences of functions on an arbitrary metric space.

Definition 8.1.18 *A sequence of complex functions* $\{f_n\}_{n=1}^{\infty}$ ***converges pointwise to a function*** f *on a subset* Ω *of a metric space* (X, d)*, written* $f_n \longrightarrow f$ *or* $f_n \underset{w \in \Omega}{\longrightarrow} f$*, if and only if the sequence* $\{f_n(w_0)\}_{n=1}^{\infty} \longrightarrow f(w_0)$ *for each* $w_0 \in \Omega$*; i.e., for each* $w_0 \in \Omega$

$$(\forall \varepsilon > 0)\left(\exists M = M(\varepsilon, w_0) \in \mathbb{J}\right)\left(n > M(\epsilon, w_0) \Rightarrow |f_n(w_0) - f(w_0)| < \varepsilon\right).$$

Definition 8.1.19 *A sequence of complex functions* $\{f_n\}$ ***converges uniformly*** *to* f *on a subset* Ω *of a metric space* (X, d)*, written* $f_n \rightrightarrows f$ *on* Ω *or* $f_n \underset{\Omega}{\rightrightarrows} f$*, if and only if*

$$(\forall \varepsilon > 0)\,(\exists M = M(\varepsilon))\,[M \in \mathbb{J} \wedge (\forall n)\,(\forall w)\,(n > M \wedge w \in \Omega \\ \Rightarrow |f_n(w) - f(w)| < \varepsilon)].$$

8.2 Conditions for Uniform Convergence

We would like some other criteria that can allow us to make decisions concerning the uniform convergence of given sequences and series of functions. In addition, if can be helpful to have a condition for uniform convergence that does not require knowledge of the limit function.

Definition 8.2.1 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *of complex-valued functions satisfies the* ***Cauchy Criterion for Convergence*** *on* $\Omega \subset \mathbb{C}$ *if and only if*

$$(\forall \varepsilon > 0)\,(\exists M \in \mathbb{J})\,[(\forall n)\,(\forall m)\,(\forall z)\,(n > M \wedge m > M \wedge z \in \Omega \\ \Rightarrow |f_n(z) - f_m(z)| < \varepsilon)].$$

Remark 8.2.2 *Alternatively, when a sequence satisfies the Cauchy Criterion for Convergence on a subset* $\Omega \subset \mathbb{C}$ *it may be described as being* ***uniformly Cauchy on*** Ω *or simply as being Cauchy.*

In Chapter 4, we saw that in $\mathbb{R}^n$ being convergent was equivalent to being Cauchy convergent. The same relationship carries over to uniform convergence of functions.

Theorem 8.2.3 *Let* $\{f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions on a set* $\Omega \subset \mathbb{C}$*. Then* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly on* Ω *if and only if* $\{f_n\}_{n=1}^{\infty}$ *satisfies the Cauchy Criterion for Convergence on* Ω*.*

Space for scratch work.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of complex-valued functions on a set $\Omega \subset \mathbb{C}$ that converges uniformly on Ω to the function f and let $\varepsilon > 0$ be given. Then there exists $M \in \mathbb{J}$ such that $n > M$ implies that $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$ for all $z \in \Omega$. Taking any other $m > M$ also yields that $|f_m(z) - f(z)| < \frac{\varepsilon}{2}$ for all $z \in \Omega$. Hence, for $m > M \wedge n > M$,

$$\begin{aligned} |f_m(z) - f_n(z)| &= |(f_m(z) - f(z)) - (f_n(z) - f(z))| \\ &\leq |f_m(z) - f(z)| + |f_n(z) - f(z)| < \varepsilon \end{aligned}$$

for all $z \in \Omega$. Therefore, $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on Ω.

Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions on a set $\Omega \subset \mathbb{C}$ satisfies the Cauchy Criterion for Convergence on Ω and let $\varepsilon > 0$ be given. For $z \in \Omega$, $\{f_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$; because $\mathbb{C}$ is complete, it follows that $\{f_n(z)\}_{n=1}^{\infty}$ is convergent to some $\zeta_z \in \mathbb{C}$. Since $z \in \Omega$ was arbitrary, we can define a function $f : \Omega \longrightarrow \mathbb{C}$ by $(\forall z)(z \in \Omega \Rightarrow f(z) = \zeta_z)$. Then, f is the pointwise limit of $\{f_n\}_{n=1}^{\infty}$. Because $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy, there exists an $M \in \mathbb{J}$ such that $m > M$ and $n > M$ implies that

$$|f_n(z) - f_m(z)| < \frac{\varepsilon}{2} \text{ for all } z \in \Omega.$$

Suppose that $n > M$ is fixed and $z \in \Omega$. Since $\lim_{m \to \infty} f_m(\xi) = f(\xi)$ for each $\xi \in \Omega$, there exists a positive integer $M^* > M$ such that $m > M^*$ implies that $|f_m(z) - f(z)| < \frac{\varepsilon}{2}$. In particular, we have that $|f_{M^*+1}(z) - f(z)| < \frac{\varepsilon}{2}$. Therefore,

$$\begin{aligned} |f_n(z) - f(z)| &= |f_n(z) - f_{M^*+1}(z) + f_{M^*+1}(z) - f(z)| \\ &\leq |f_n(z) - f_{M^*+1}(z)| + |f_{M^*+1}(z) - f(z)| < \varepsilon. \end{aligned}$$

But $n > M$ and $z \in \Omega$ were both arbitrary. Consequently,

$$(\forall n)\,(\forall z)\,(n > M \wedge z \in \Omega \Rightarrow |f_n(z) - f(z)| < \varepsilon)\,.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f_n \underset{\Omega}{\rightrightarrows} f$. ■

Remark 8.2.4 *Note that in the proof just given, the positive integer M^* was dependent on the point z and the ε; i.e., $M^* = M^*(\varepsilon, z)$. However, the final inequality obtained via the intermediate travel through information from M^*, $|f_n(z) - f(z)| < \varepsilon$, was independent of the point z. What was illustrated in the proof was a process that could be used repeatedly for each $z \in \Omega$.*

Remark 8.2.5 *In the proof of both parts of Theorem 8.2.3, our conclusions relied on properties of the codomain for the sequence of functions. Namely, we used the metric on $\mathbb{C}$ and the fact that $\mathbb{C}$ was complete. Consequently, we could allow Ω to be any metric space and claim the same conclusion. The following corollary formalizes that claim.*

Corollary 8.2.6 *Let $\{f_n\}_{n=1}^{\infty}$ denote a sequence of complex-valued functions defined on a subset Ω of a metric space (X, d). Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on Ω if and only if $\{f_n\}_{n=1}^{\infty}$ satisfies the Cauchy Criterion for Convergence on Ω.*

Theorem 8.2.7 *Let $\{f_n\}_{n=1}^{\infty}$ denote a sequence of complex-valued functions on a set $\Omega \subset \mathbb{C}$ that is pointwise convergent on Ω to the function f; i.e.,*

$$\lim_{n\to\infty} f_n(z) = f(z)\,;$$

and, for each $n \in \mathbb{J}$, let $M_n = \sup_{z\in\Omega} |f_n(z) - f(z)|$. Then $f_n \underset{\Omega}{\rightrightarrows} f$ if and only if $\lim_{n\to\infty} M_n = 0$.

Use this space to fill in a proof for Theorem 8.2.7.

Theorem 8.2.8 (Weierstrass M-Test) *For each $n \in \mathbb{J}$, let $u_n(w)$ be a complex-valued function that is defined on a subset Ω of a metric space (X, d). Suppose that there exists a sequence of real constants $\{M_n\}_{n=1}^{\infty}$ such that $|u_n(w)| \leq M_n$ for all $w \in \Omega$ and for each $n \in \mathbb{J}$. If the series $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(w)$ and $\sum_{n=1}^{\infty} |u_n(w)|$ converge uniformly on Ω.*

Excursion 8.2.9 *Fill in what is missing in order to complete the following proof of the Weierstrass M-Test.*

Proof. Suppose that $\{u_n(w)\}_{n=1}^{\infty}$, Ω, and $\{M_n\}_{n=1}^{\infty}$ are as described in the hypotheses. For each $n \in J$, let

$$S_n(w) = \sum_{k=1}^{n} u_k(w) \text{ and } T_n(w) = \sum_{k=1}^{n} |u_k(w)|$$

and suppose that $\varepsilon > 0$ is given. Since $\sum_{n=1}^{\infty} M_n$ converges and $\{M_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\left\{\sum_{k=1}^{n} M_k\right\}_{n=1}^{\infty}$ is a convergent sequence of real numbers. In view of the completeness of the reals, we have that $\left\{\sum_{k=1}^{n} M_k\right\}_{n=1}^{\infty}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$. Hence, there exists a positive integer K such that $n > K$ implies that

$$\sum_{k=n+1}^{n+p} M_k < \varepsilon \text{ for each } p \in \mathbb{J}.$$

Since $|u_k(w)| \leq M_k$ for all $w \in \Omega$ and for each $k \in \mathbb{J}$, we have that

$$\left|T_{n+p}(w) - T_n(w)\right| = \left|\sum_{k=n+1}^{n+p} |u_k(w)|\right| = \sum_{k=n+1}^{n+p} |u_k(w)| \text{ for all } w \in \Omega.$$

Therefore, $\{T_n\}_{n=1}^{\infty}$ is $\underset{(2)}{\underline{\hspace{3cm}}}$ in Ω. It follows from the $\underset{(3)}{\underline{\hspace{4.5cm}}}$ that

$$\underset{(4)}{\underline{\hspace{6cm}}} = \underset{(5)}{\underline{\hspace{3cm}}} \leq \sum_{k=n+1}^{n+p} |u_k(w)| \leq \sum_{k=n+1}^{n+p} M_k < \varepsilon$$

for all $w \in \Omega$. Hence, $\{S_n\}_{n=1}^{\infty}$ is uniformly Cauchy in Ω. From Corollary 8.2.6, we conclude that $\underset{(6)}{\underline{\hspace{6cm}}}$. ∎

Acceptable responses include: (1) Cauchy, (2) uniformly Cauchy, (3) triangular inequality, (4) $\left|S_{n+p}(w) - S_n(w)\right|$, (5) $\left|\sum_{k=n+1}^{n+p} u_k(w)\right|$, and (6) $\sum_{n=1}^{\infty} u_n(w)$ and $\sum_{n=1}^{\infty} |u_n(w)|$ converge uniformly on Ω.

Excursion 8.2.10 *Construct an example to show that the converse of the Weierstrass M-Test need not hold.*

8.3 Property Transmission and Uniform Convergence

We have already seen that pointwise convergence was not sufficient to transmit the property of continuity of each function in a sequence to the limit function. In this section, we will see that uniform convergence overcomes that drawback and allows for the transmission of other properties.

Theorem 8.3.1 *Let $\{f_n\}_{n=1}^{\infty}$ denote a sequence of complex-valued functions defined on a subset Ω of a metric space (X, d) such that $f_n \underset{\Omega}{\rightrightarrows} f$. For w a limit point of Ω and each $n \in \mathbb{J}$, suppose that*

$$\lim_{\substack{t \to w \\ t \in \Omega}} f_n(t) = A_n.$$

Then $\{A_n\}_{n=1}^{\infty}$ converges and $\lim_{t \to w} f(t) = \lim_{n \to \infty} A_n$.

Excursion 8.3.2 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. *Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions defined on a subset Ω of a metric space (X, d) is such that $f_n \underset{\Omega}{\rightrightarrows} f$, w is a limit point of Ω and, for each $n \in \mathbb{J}$, $\lim_{t \to w} f_n(t) = A_n$. Let $\varepsilon > 0$ be given. Since $f_n \underset{\Omega}{\rightrightarrows} f$,* *by Corollary 8.2.6,* $\{f_n\}_{n=1}^{\infty}$ *is* $\underset{(1)}{\underline{\hspace{6em}}}$ *on* Ω*. Hence, there exists a positive integer* M *such that* $\underset{(2)}{\underline{\hspace{6em}}}$ *implies that*

$$|f_n(t) - f_m(t)| < \frac{\varepsilon}{3} \text{ for all } \underset{(3)}{\underline{\hspace{6em}}}.$$

Fix m *and* n *such that* $m > M$ *and* $n > M$. *Since* $\lim\limits_{t\to w} f_k(t) = A_k$ *for each* $k \in \mathbb{J}$*, it follows that there exists a* $\delta > 0$ *such that* $0 < d(t, w) < \delta$ *implies that*

$$|f_m(t) - A_m| < \frac{\varepsilon}{3} \text{ and } \underset{(4)}{\underline{\hspace{12em}}}$$

From the triangular inequality,

$$|A_n - A_m| \leq |A_n - f_n(t)| + \left|\underset{(5)}{\underline{\hspace{6em}}}\right| + |f_m(t) - A_m| < \varepsilon.$$

Since m *and* n *were arbitrary, for each* $\varepsilon > 0$ *there exists a positive integer* M *such that* $(\forall m)(\forall n)(n > M \wedge m > M \Rightarrow |A_n - A_m| < \varepsilon)$*; i.e.,* $\{A_n\}_{n=1}^{\infty} \subset \mathbb{C}$ *is Cauchy. From the completeness of the complex numbers, if follows that* $\{A_n\}_{n=1}^{\infty}$ *is convergent to some complex number; let* $\lim\limits_{n\to\infty} A_n = A$.

We want to show that A *is also equal to* $\lim\limits_{\substack{t\to w\\ t\in\Omega}} f(t)$. *Again we suppose that* $\varepsilon > 0$ *is given. From* $f_n \underset{\Omega}{\rightrightarrows} f$ *there exists a positive integer* M_1 *such that* $n > M_1$ *implies that* $\left|\underset{(6)}{\underline{\hspace{6em}}}\right| < \frac{\varepsilon}{3}$ *for all* $t \in \Omega$*, while the convergence of* $\{A_n\}_{n=1}^{\infty}$ *yields a positive integer* M_2 *such that* $|A_n - A| < \frac{\varepsilon}{3}$ *whenever* $n > M_2$*. Fix* n *such that* $n > \max\{M_1, M_2\}$*. Then, for all* $t \in \Omega$*,*

$$|f(t) - f_n(t)| < \frac{\varepsilon}{3} \quad \text{and} \quad |A_n - A| < \frac{\varepsilon}{3}.$$

Since $\lim\limits_{\substack{t\to w\\ t\in\Omega}} f_n(t) = A_n$*, there exists a* $\delta > 0$ *such that*

$$|f_n(t) - A_n| < \frac{\varepsilon}{3} \text{ for all } t \in (N_\delta(w) - \{w\}) \cap \Omega.$$

From the triangular inequality, for all $t \in \Omega$ *such that* $0 < d(t, w) < \delta$,

$$|f(t) - A| \leq \underset{(7)}{\underline{\hspace{6cm}}} < \varepsilon.$$

Therefore, $\underset{(8)}{\underline{\hspace{2cm}}}$. ■

Acceptable responses are: (1) uniformly Cauchy, (2) $n > M \wedge m > M$, (3) $t \in \Omega$, (4) $|f_n(t) - A_n| < \dfrac{\varepsilon}{3}$, (5) $f_n(t) - f_m(t)$, (6) $f(t) - f_n(t)$, (7) $|f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, and (8) $\lim\limits_{\substack{t \to w \\ t \in \Omega}} f(t) = A$.

Theorem 8.3.3 (The Uniform Limit of Continuous Functions) *Let* $\{f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions that are continuous on a subset* Ω *of a metric space* (X, d). *If* $f_n \underset{\Omega}{\rightrightarrows} f$, *then* f *is continuous on* Ω.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of complex-valued functions that are continuous on a subset Ω of a metric space (X, d). Then for each $\zeta \in \Omega$, $\lim\limits_{t \to \zeta} f_n(t) = f_n(\zeta)$. Taking $A_n = f_n(\zeta)$ in Theorem 8.3.1 yields the claim. ■

Remark 8.3.4 *The contrapositive of Theorem 8.3.3 affords us a nice way of showing that we do not have uniform convergence of a given sequence of functions. Namely, if the limit of a sequence of complex-valued functions that are continuous on a subset* Ω *of a metric space is a function that is not continuous on* Ω, *we may immediately conclude that the convergence in not uniform. Be careful about the appropriate use of this: The limit function being continuous IS NOT ENOUGH to conclude that the convergence is uniform.*

The converse of Theorem 8.3.3 is false. For example, we know that $\left\{\dfrac{1}{nz}\right\}_{n=1}^{\infty}$ converges pointwise to the continuous function $f(z) = 0$ in $\mathbb{C} - \{0\}$ and the convergence is not uniform. The following result offers a list of criteria under which continuity of the limit of a sequence of real-valued continuous functions ensures that the convergence must be uniform.

Theorem 8.3.5 *Suppose that* Ω *is a compact subset of a metric space* (X, d) *and* $\{f_n\}_{n=1}^{\infty}$ *satisfies each of the following:*

(i) $\{f_n\}_{n=1}^{\infty}$ *is a sequence of real-valued functions that are continuous on* Ω*;*

(ii) $f_n \underset{\Omega}{\longrightarrow} f$ *and* f *is continuous on* Ω*; and*

(iii) $(\forall n)(\forall w)(n \in \mathbb{J} \wedge w \in \Omega \Rightarrow f_n(w) \geq f_{n+1}(w))$.

Then $f_n \underset{\Omega}{\rightrightarrows} f$.

Excursion 8.3.6 *Fill in what is missing in order to complete the following proof of Theorem 8.3.5.*

Proof. For $\{f_n\}_{n=1}^{\infty}$ satisfying the hypotheses, set $g_n = f_n - f$. Then, for each $n \in \mathbb{J}$, g_n is continuous on Ω and, for each $\zeta \in \Omega$, $\lim_{n\to\infty} g_n(\zeta) = \underset{(1)}{\underline{\qquad}}$. Since $f_n(w) \geq f_{n+1}(w)$ implies that $f_n(w) - f(w) \geq f_{n+1}(w) - f(w)$, we also have that $(\forall n)(\forall w)\left(n \in \mathbb{J} \wedge w \in \Omega \Rightarrow \underset{(2)}{\underline{\qquad\qquad\qquad}}\right)$.

To see that $g_n \underset{\Omega}{\rightrightarrows} 0$, suppose that $\varepsilon > 0$ is given. For each $n \in \mathbb{J}$, let

$$K_n = \{x \in \Omega : g_n(x) \geq \varepsilon\}.$$

Because Ω and $\mathbb{R}$ are metric spaces, g_n is continuous, and $\{w \in \mathbb{R} : w \geq \varepsilon\}$ is a closed subset of $\mathbb{R}$, by Corollary 5.2.16 to the Open Set Characterization of Continuous Functions, $\underset{(3)}{\underline{\qquad\qquad\qquad}}$. As a closed subset of a compact metric space, from Theorem 3.3.37, we conclude that K_n is $\underset{(4)}{\underline{\qquad\qquad\qquad}}$. If $x \in K_{n+1}$, then $g_{n+1}(x) \geq \varepsilon$ and $g_n(x) \geq g_{n+1}(x)$; it follows from the transitivity of $\geq$ that $\underset{(5)}{\underline{\qquad\qquad\qquad}}$. Hence, $x \in K_n$. Since x was arbitrary, $(\forall x)(x \in K_{n+1} \Rightarrow x \in K_n)$; i.e., $\underset{(6)}{\underline{\qquad\qquad\qquad}}$. Therefore, $\{K_n\}_{n=1}^{\infty}$ is a $\underset{(7)}{\underline{\qquad\qquad\qquad}}$ sequence of compact subsets of Ω. From Corollary 3.3.44 to Theorem 3.3.43, $((\forall n \in \mathbb{J})(K_n \neq \emptyset)) \Rightarrow \bigcap_{k\in\mathbb{J}} K_k \neq \emptyset$.

Suppose that $w \in \Omega$. Then $\lim_{n\to\infty} g_n(w) = 0$ and $\{g_n(x)\}$ decreasing yields the existence of a positive integer M such that $n > M$ implies that $0 \leq g_n(w) < \varepsilon$. In particular, $w \notin K_{M+1}$ from which it follows that $w \notin \bigcap_{n\in\mathbb{J}} K_n$. Because w was arbitrary, $(\forall w \in \Omega)\left(w \notin \bigcap_{n \in \mathbb{J}} K_n\right)$; i.e., $\bigcap_{n \in \mathbb{J}} K_n = \emptyset$. We conclude that there exists a positive integer P such that $K_P = \emptyset$. Hence, $K_n = \emptyset$ for all $\underset{(8)}{\underline{\hspace{2cm}}}$; that is, for all $n \geq P$, $\{x \in \Omega : g_n(x) \geq \varepsilon\} = \emptyset$. Therefore,

$$(\forall x)(\forall n)(x \in \Omega \wedge n > P \Rightarrow 0 \leq g_n(x) < \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we have that $g_n \underset{\Omega}{\rightrightarrows} 0$ which is equivalent to showing that $f_n \underset{\Omega}{\rightrightarrows} f$. ■

Acceptable responses are: (1) 0, (2) $g_n(w) \geq g_{n+1}(w)$, (3) K_n is closed, (4) compact, (5) $g_n(x) \geq \varepsilon$, (6) $K_{n+1} \subset K_n$, (7) nested, and (8) $n \geq P$.

Remark 8.3.7 *Since compactness was referred to several times in the proof of Theorem 8.3.5, it is natural to want to check that the compactness was really needed. The example offered by our author in order to illustrate the need is* $\left\{\dfrac{1}{1+nx}\right\}_{n=1}^{\infty}$ *in the segment* $(0, 1)$.

Our results concerning transmission of integrability and differentiability are for sequences of functions of real-valued functions on subsets of $\mathbb{R}$.

Theorem 8.3.8 (Integration of Uniformly Convergent Sequences) *Let α be a function that is (defined and) monotonically increasing on the interval $I = [a, b]$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions such that*

$$(\forall n)(n \in \mathbb{J} \Rightarrow f_n \in \Re(\alpha) \text{ on } I)$$

and $f_n \underset{[a,b]}{\rightrightarrows} f$. Then $f \in \Re(\alpha)$ on I and

$$\int_a^b f(x)\, d\alpha(x) = \lim_{n \to \infty} \int_a^b f_n(x)\, d\alpha(x)$$

Excursion 8.3.9 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. For each $n \in J$, let $\varepsilon_n = \sup_{x \in I} |f_n(x) - f(x)|$. Then

$$f_n(x) - \varepsilon_n \leq f(x) \leq \underset{(1)}{\underline{\hspace{6em}}} \text{ for } a \leq x \leq b$$

and if follows that

$$\int_a^b (f_n(x) - \varepsilon_n)\, d\alpha(x) \leq \underline{\int_a^b} f(x)\, d\alpha(x) \leq \overline{\int_a^b} f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x). \tag{8.3}$$

Properties of linear ordering yield that

$$0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x) - \underset{(2)}{\underline{\hspace{9em}}}. \tag{8.4}$$

Because the upper bound in equation (8.4) is equivalent to $\underset{(3)}{\underline{\hspace{12em}}}$,

we conclude that

$(\forall n \in \mathbb{J})\left(0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) \leq \underset{(4)}{\underline{\hspace{9em}}}\right)$. By Theorem 8.2.7, $\varepsilon_n \to 0$ as $n \to \infty$. Since $\overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x)$ is constant, we conclude that $\underset{(5)}{\underline{\hspace{12em}}}$. Hence $f \in \Re(\alpha)$.

Now, from equation 8.3, for each $n \in J$,

$$\int_a^b (f_n(x) - \varepsilon_n)\, d\alpha(x) \leq \int_a^b f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x).$$

(6) Finish the proof in the space provided.

■

Acceptable responses are:(1) $f_n(x)+\varepsilon_n$ (2) $\int_a^b (f_n(x)-\varepsilon_n)\,d\alpha(x)$, (3) $\int_a^b e\varepsilon_n d\alpha(x)$, (4) $2\varepsilon_n[\alpha(b)-\alpha(a)]$, (5) $\overline{\int_a^b} f(x)\,d\alpha(x)=\underline{\int_a^b} f(x)\,d\alpha(x)$, (6) Hopefully, you thought to repeat the process just illustrated. From the modified inequality it follows that $\left|\int_a^b f(x)\,d\alpha(x)-\int_a^b f_n(x)\,d\alpha(x)\right|\leq\varepsilon_n[\alpha(b)-\alpha(a)]$; then because $\varepsilon_n\to 0$ as $n\to\infty$, given any $\varepsilon>0$ there exists a positive integer M such that $n>M$ implies that $\varepsilon_n[\alpha(b)-\alpha(a)]<\varepsilon$.

Corollary 8.3.10 *If $f_n\in\Re(\alpha)$ on $[a,b]$, for each $n\in\mathbb{J}$, and $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a,b]$ to a function f, then $f\in\Re(\alpha)$ on $[a,b]$ and*

$$\int_a^b f(x)\,d\alpha(x)=\sum_{k=1}^{\infty}\int_a^b f_k(x)\,d\alpha(x).$$

Having only uniform convergence of a sequence of functions is insufficient to make claims concerning the sequence of derivatives. There are various results that offer some additional conditions under which differentiation is transmitted. If we restrict ourselves to sequences of real-valued functions that are continuous on an interval $[a,b]$ and Riemann integration, then we can use the Fundamental Theorems of Calculus to draw analogous conclusions. Namely, we have the following two results.

Theorem 8.3.11 *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions that are continuous on the interval $[a,b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$. For $c\in[a,b]$ and each $n\in\mathbb{J}$, let*

$$F_n(x)\underset{def}{=}\int_c^x f_n(t)\,dt.$$

Then f is continuous on $[a,b]$ and $F_n \underset{[a,b]}{\rightrightarrows} F$ where

$$F(x)=\int_c^x f(t)\,dt.$$

The proof is left as an exercise.

Theorem 8.3.12 *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is such that* $f_n \underset{[a,b]}{\longrightarrow} f$ *and, for each* $n \in \mathbb{J}$, f_n' *is continuous on an interval* $[a,b]$. *If* $f_n' \underset{[a,b]}{\rightrightarrows} g$ *for some function* g *that is defined on* $[a,b]$, *then* g *is continuous on* $[a,b]$ *and* $f'(x) = g(x)$ *for all* $x \in [a,b]$.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is such that $f_n \underset{[a,b]}{\longrightarrow} f$, f_n' is continuous on an interval $[a,b]$ for each $n \in \mathbb{J}$, and $f_n' \underset{[a,b]}{\rightrightarrows} g$ for some function g that is defined on $[a,b]$. From the Uniform Limit of Continuous Functions Theorem, g is continuous. Because each f_n' is continuous and $f_n' \underset{[a,b]}{\rightrightarrows} g$, by the second Fundamental Theorem of Calculus and Theorem 8.3.11, for $[c,x] \subset [a,b]$

$$\int_c^x g(t)\,dt = \lim_{n\to\infty}\int_c^x f_n'(t)\,dt = \lim_{n\to\infty}\left[f_n(x) - f_n(c)\right].$$

Now the pointwise convergence of $\{f_n\}$ yields that $\lim_{n\to\infty}\left[f_n(x) - f_n(c)\right] = f(x) - f(c)$. Hence, from the properties of derivative and the first Fundamental Theorem of Calculus, $g(x) = f'(x)$. ■

We close with the variation of 8.3.12 that is in our text; it is more general in that it does not require continuity of the derivatives and specifies convergence of the original sequence only at a point.

Theorem 8.3.13 *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of real-valued functions that are differentiable on an interval* $[a,b]$ *and that there exists a point* $x_0 \in [a,b]$ *such that* $\lim_{n\to\infty} f_n(x_0)$ *exists. If* $\left\{f_n'\right\}_{n=1}^{\infty}$ *converges uniformly on* $[a,b]$ *then* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly on* $[a,b]$ *to some function* f *and*

$$(\forall x)\left(x \in [a,b] \Rightarrow f'(x) = \lim_{n\to\infty} f_n'(x)\right).$$

Excursion 8.3.14 *Fill in what is missing in order to complete the following proof of Theorem 8.3.13.*

Proof. Suppose $\varepsilon > 0$ is given. Because $\{f_n(x_0)\}_{n=1}^{\infty}$ is convergent sequence of real numbers and $\mathbb{R}$ is complete, $\{f_n(x_0)\}_{n=1}^{\infty}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$. Hence, there exists a positive integer M_1 such that $n > M_1$ and $m > M_1$ implies that

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Because $\left\{f_n'\right\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$, by Theorem $\underset{(2)}{\underline{\qquad\qquad}}$, there exists a positive integer M_2 such that $n > M_2$ and $m > M_2$ implies that

$$\left|f_n'(\xi) - f_m'(\xi)\right| < \frac{\varepsilon}{2(b-a)} \text{ for } \underset{(3)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

For fixed m and n, let $F = f_n - f_m$. Since each f_k is differentiable on $[a,b]$, F is differentiable on (a,b) and continuous on $[a,b]$. From the $\underset{(4)}{\underline{\qquad\qquad\qquad}}$ Theorem, for any $[x,t] \subset (a,b)$, there exists a $\xi \in (x,t)$ such that $F(x) - F(t) = F'(\xi)(x-t)$. Consequently, if $m > M_2$ and $n > M_2$, for any $[x,t] \subset (a,b)$, there exists a $\xi \in (x,t)$, it follows that

$$\begin{aligned} \left|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))\right| &= \left|f_n'(\xi) - f_m'(\xi)\right| |x-t| \\ &< \frac{\varepsilon}{2(b-a)} |x-t| \le \underset{(5)}{\underline{\qquad\qquad}}. \end{aligned} \tag{8.5}$$

Let $M = \max\{M_1, M_2\}$. Then $m > M$ and $n > M$ implies that

$$\begin{aligned} &|f_n(w) - f_m(w)| \\ &\le \quad |(f_n(w) - f_m(w)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| < \varepsilon \end{aligned}$$

for any $w \in [a,b]$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to some function. Let f denote the limit function; i.e., $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [a,b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$.

Now we want to show that, for each $x \in [a,b]$, $f'(x) = \lim_{n\to\infty} f_n'(x)$; i.e., for fixed $x \in [a,b]$,

$$\lim_{n\to\infty}\lim_{t\to x}\frac{f_n(t) - f_n(x)}{t-x} = \lim_{t\to x}\frac{f(t) - f(x)}{t-x}$$

where the appropriate one-sided limit is assumed when $x = a$ or $x = b$. To this end, for fixed $x \in [a,b]$, let

$$\phi_n(t) \underset{def}{=} \frac{f_n(t) - f_n(x)}{t-x} \quad and \quad \phi(t) \underset{def}{=} \frac{f(t) - f(x)}{t-x}$$

for $t \in [a,b] - \{x\}$ and $n \in \mathbb{J}$. Then, $x \in (a,b)$ implies that $\lim_{t\to x}\phi_n(t) = f_n'(x)$, while $x = a$ and $x = b$ yield that $\lim_{t\to a^+}\phi_n(t) = f_n'(a)$ and $\lim_{t\to b^-}\phi_n(t) = f_n'(b)$, respectively. Suppose $\varepsilon > 0$ if given. If $m > M_2$, $n > M_2$, and $t \in [a,b] - \{x\}$, then

$$|\phi_n(t) - \phi_m(t)| = \underset{(6)}{\underline{\hspace{6cm}}} < \frac{\varepsilon}{2(b-a)}$$

from equation (8.5). Thus, $\{\phi_n\}_{n=1}^{\infty}$ is uniformly Cauchy and, by Theorem 8.2.3, uniformly convergent on $t \in [a,b] - \{x\}$. Since $f(t) = \lim_{n\to\infty} f_n(t)$ for $t \in [a,b]$, we have that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t).$$

Consequently, $\phi_n \rightrightarrows \phi$ on $[a,b] - \{x\}$. Finally, applying Theorem 8.3.1 to the sequence $\{\phi_n\}_{n=1}^{\infty}$, where $A_n = f_n'(x)$ yields that

$$f'(x) = \lim_{t\to x} \phi(t) = \underset{(7)}{\underline{\hspace{3cm}}}.$$

■

***Acceptable responses are: (1) Cauchy, (2) 8.2.3, (3) all $\xi \in [a,b]$, (4) Mean-Value, (5) $\frac{\varepsilon}{2}$,
(6) $\left|\frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x}\right| = \frac{|(f_n(t) - f_m(t)) - (f_n(x) - f_m(x))|}{|t-x|}$,
(7) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} f_n'(x)$.***

Rudin ends the section of our text that corresponds with these notes by constructing an example of a real-valued continuous function that is nowhere differentiable.

Theorem 8.3.15 *There exists a real-valued function that is continuous on $\mathbb{R}$ and nowhere differentiable on $\mathbb{R}$.*

Proof. First we define a function ϕ that is continuous on $\mathbb{R}$, periodic with period 2, and not differentiable at each integer. To do this, we define the function in a interval that is "2 wide" and extend the definition by reference to the original part. For $x \in [-1,1]$, suppose that $\phi(x) = |x|$ and, for all $x \in R$, let $\phi(x+2) = \phi(x)$.

In the space provided sketch a graph of ϕ.

The author shows that the function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi\left(4^n x\right)$$

satisfies the needed conditions. Use the space provided to fill in highlights of the justification.

■

8.4 Families of Functions

Since any sequence of functions is also a set of functions, it is natural to ask questions about sets of functions that are related by some commonly shared nice behavior. The general idea is to seek additional properties that will shared by such sets of functions. For example, if $\mathcal{F}$ is the set of all real-valued functions from $[0, 1]$ into $[0, 1]$ that are continuous, we have seen that an additional shared property is that $(\forall f)(f \in \mathcal{F} \Rightarrow (\exists t)(t \in [0, 1] \wedge f(t) = t))$. In the last section, we considered sets of functions from a metric space into $\mathbb{C}$ or $\mathbb{R}$ and examined some of the consequences of uniform convergence of sequences.

Another view of sets of functions is considering the functions as points in a metric space. Let $\mathcal{C}([a,b])$ denote the family of real-valued functions that are continuous on the interval $I=[a,b]$. For f and g in $\mathcal{C}([a,b])$, we have seen that

$$\rho_\infty(f,g)=\max_{a\le x\le b}|f(x)-g(x)|$$

and

$$\rho(f,g)=\int_a^b|f(x)-g(x)|\,dx.$$

are metrics on $\mathcal{C}([a,b])$. As a homework problem (Problem Set H, #14), you will show that $(\mathcal{C}([a,b]),\rho)$ is not a complete metric space. On the other hand, $(\mathcal{C}([a,b]),\rho_\infty)$ is complete. In fact, the latter generalizes to the set of complex-valued functions that are continuous and bounded on the same domain.

Definition 8.4.1 *For a metric space (X,d), let $\mathcal{C}(X)$ denote the set of all complex-valued functions that are continuous and bounded on the domain X and, corresponding to each $f\in\mathcal{C}(X)$ the* ***supremum norm*** *or* ***sup norm*** *is given by*

$$\|f\|=\|f\|_X=\sup_{x\in X}|f(x)|.$$

It follows directly that $\|f\|_X=0\Leftrightarrow f(x)=0$ for all $x\in X$ and

$$(\forall f)(\forall g)\left(f,g\in\mathcal{C}(X)\Rightarrow\|f+g\|_X\le\|f\|_X+\|g\|_X\right).$$

The details of our proof for the corresponding set-up for $\mathcal{C}([a,b])$ allow us to claim that $\rho_\infty(f,g)=\|f-g\|_X$ is a metric for $\mathcal{C}(X)$.

Lemma 8.4.2 *The convergence of sequences in $\mathcal{C}(x)$ with respect to ρ_∞ is equivalent to uniform convergence of sequences of continuous functions in subsets of X.*

Use the space below to justify the claim made in the lemma.

Hopefully, you remembered that the metric replaces the occurrence of the absolute value (or modulus) is the statement of convergence. The immediate translation is that for every $\varepsilon > 0$, there exists a positive integer M such that $n > M$ implies that $\rho_\infty(f_n, f) < \varepsilon$. Of course, you don't want to stop there; the statement $\rho_\infty(f_n, f) < \varepsilon$ translates to $\sup_{x\in X} |f_n(x) - f(x)| < \varepsilon$ which yields that $(\forall x)(x \in X \Rightarrow |f_n(x) - f(x)| < \varepsilon)$. This justifies that convergence of $\{f_n\}_{n=1}^{\infty}$ with respect to ρ_∞ implies that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Since the converse also follows immediately from the definitions, **we can conclude that convergence of sequences in $C(X)$ with respect to ρ_∞ is equivalent to uniform convergence**.

Theorem 8.4.3 *For a metric space X, $(\mathcal{C}(X), \rho_\infty)$ is a complete metric space.*

Excursion 8.4.4 *Fill in what is missing in order to complete the following proof of Theorem 8.4.3.*

Proof. Since $(\mathcal{C}(X), \rho_\infty)$ is a metric space, from Theorem 4.2.9, we know that any convergent sequence in $\mathcal{C}(X)$ is Cauchy.

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$ and that $\varepsilon > 0$ is given. Then there exists a positive integer M such that $n > M$ and $m > M$ implies that $\underset{(1)}{\underline{\hspace{8em}}}$; i.e., for $n > M$ and $m > M$,

$$\sup_{\xi\in X} |f_n(\xi) - f_m(\xi)| < \varepsilon.$$

Hence, $(\forall x)\left(x \in X \Rightarrow \underset{(2)}{\underline{\hspace{5em}}}\right)$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\{f_n\}_{n=1}^{\infty}$ is $\underset{(3)}{\underline{\hspace{8em}}}$. As a $\underset{(3)}{\underline{\hspace{8em}}}$

sequence of complex-valued functions on a metric space X, by Theorem $\underset{(4)}{\underline{\hspace{2cm}}}$, $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent. Let $f : X \to \mathbb{C}$ denote the uniform limit. Because $f_n \underset{X}{\rightrightarrows} f$, for any $\varepsilon > 0$ there exists a positive integer M such that $n > M$ implies that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in X.$$

In particular, $\underset{(5)}{\underline{\hspace{3cm}}} = \sup_{\xi \in X} |f_n(\xi) - f(\xi)| \le \frac{\varepsilon}{2} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\rho_\infty(f_n, f) \to 0$ as $n \to \infty$. Hence, $\{f_n\}_{n=1}^{\infty}$ is convergent to f in $(\mathcal{C}(X), \rho_\infty)$.

Now we want to show that $f \in \mathcal{C}(X)$. As the uniform limit of continuous functions from a metric space X in $\mathbb{C}$, we know that f is $\underset{(6)}{\underline{\hspace{3cm}}}$. Because $f_n \underset{X}{\rightrightarrows} f$, corresponding to $\varepsilon = 1$ there exists a positive integer M such that $n > M$ implies that $|f_n(x) - f(x)| < 1$ for all $x \in X$. In particular, from the (other) triangular inequality, we have that

$$(\forall x)\,(x \in X \Rightarrow |f(x)| < |f_{M+1}(x)| + 1)\,. \tag{8.6}$$

Since $f_{M+1} \in \mathcal{C}(X)$, f_{M+1} is continuous and $\underset{(7)}{\underline{\hspace{3cm}}}$ on X. From equation (8.6), it follows that f is $\underset{(7)}{\underline{\hspace{3cm}}}$ on X. Because $f : X \to \mathbb{C}$ is continuous and bounded on X, $\underset{(8)}{\underline{\hspace{3cm}}}$. The sequence $\{f_n\}_{n=1}^{\infty}$ was an arbitrary Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$. Consequently, we conclude that every Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$ is convergent in $(\mathcal{C}(X), \rho_\infty)$. This concludes that proof that convergence in $(\mathcal{C}(X), \rho_\infty)$ is equivalent to being Cauchy in $(\mathcal{C}(X), \rho_\infty)$. ■

Acceptable responses are: (1) $\rho_\infty(f_n, f_m) < \varepsilon$, (2) $|f_n(x) - f_m(x)| < \varepsilon$, (3) uniformly Cauchy, (4) 8.2.3, (5) $\rho_\infty(f_n, f)$, (6) continuous, (7) bounded, and (8) $f \in \mathcal{C}(X)$.

Remark 8.4.5 *At first, one might suspect that completeness is an intrinsic property of a set. However, combining our prior discussion of the metric spaces* $(\mathbb{R}, d)$ *and* $(\mathbb{Q}, d)$ *where* d *denotes the Euclidean metric with our discussion of the two metrics* *on $\mathcal{C}([a,b])$ leads us to the conclusion that completeness depends on two things: the nature of the underlying set and the way in which distance is measured on the set.*

We have made a significant transition from concentration on sets whose elements are points on a plane or number line (or Euclidean n-space) to sets where the points are functions. Now that we have seen a setting that gives us the notion of completeness in this new setting, it is natural to ask about generalization or transfer of other general properties. What might characterizations of compactness look like? Do we have an analog for the Bolzano-Weierstrass Theorem? In this discussion, we will concentrate on conditions that allow us to draw conclusions concerning sequences of bounded functions and subsequences of convergent sequences. We will note right away that care must be taken.

Definition 8.4.6 *Let $\mathcal{F}$ denote a family of complex-valued functions defined on a metric space (Ω, d). Then*

(a) *$\mathcal{F}$ is said to be* ***uniformly bounded*** *on Ω if and only if*

$$(\exists M \in \mathbb{R})\,(\forall f)\,(\forall w)\,(f \in \mathcal{F} \wedge w \in \Omega \Rightarrow |f(w)| \leq M).$$

(b) *$\mathcal{F}$ is said to be* ***locally uniformly bounded*** *on Ω if and only if*

$$(\forall w)\,(w \in \Omega \Rightarrow (\exists N_w)\,(N_w \subset \Omega \wedge \mathcal{F} \text{ is uniformly bounded on } N_w)).$$

(c) *any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ is said to be* ***pointwise bounded*** *on Ω if and only if*

$(\forall w)\left(w \in \Omega \Rightarrow \{f_n(w)\}_{n=1}^{\infty} \text{ is bounded}\right)$; i.e., corresponding to each $w \in \Omega$, there exists a positive real number $M_w \underset{def}{=} \phi(w)$ such that

$$|f_n(w)| < M_w \text{ for all } n \in \mathbb{J}.$$

Example 8.4.7 *For $x \in \Omega = \mathbb{R} - \{0\}$, let $\mathcal{F} = \left\{ f_n(x) = \dfrac{x}{n^2 + x^2} : n \in \mathbb{J} \right\}$. Then, for $w \in \Omega$, taking $\phi(w) = \dfrac{2|w|}{1 + w^2}$ implies that $|f_n(w)| < \phi(w)$ for all $n \in \mathbb{J}$. Thus, $\mathcal{F}$ is pointwise bounded on Ω.*

Remark 8.4.8 *Uniform boundedness of a family implies that each member of the family is bounded but not conversely.*

Excursion 8.4.9 *Justify this point with a discussion of* $\mathcal{F} = \{f_n(z) = nz : n \in \mathbb{J}\}$ *on* $U_r = \{z \in \mathbb{C} : |z| < r\}$.

Hopefully, you observed that each member of $\mathcal{F}$ is bounded in U_r but no single bound works for all of the elements in $\mathcal{F}$.

Remark 8.4.10 *Uniform boundedness of a family implies local uniform boundedness but not conversely.*

Excursion 8.4.11 *To see this, show that* $\left\{\dfrac{1}{1-z^n} : n \in \mathbb{J}\right\}$ *is locally uniformly bounded in* $U = \{z : |z| < 1\}$ *but not uniformly bounded there.*

Neighborhoods that can justify local uniform boundedness vary; the key is to capitalize on the fact that you can start with an arbitrary fixed $z \in U$ and make use of its distance from the origin to define a neighborhood. For example, given $z_0 \in U$ with $|z_0| = r < 1$, let $N_{z_0} = N\left(z_0, \dfrac{1-r}{4}\right)$; now, $N_{z_0} \subset U$ and $\left|(1-z^n)^{-1}\right| \leq (1-|z|)^{-1}$ can be used to justify that, for each $n \in \mathbb{J}$, $\left|(1-z^n)^{-1}\right| < 4(1-r)/3$. The latter allows us to conclude that the given family is uniformly bounded on N_{z_0}. Since z_0 was arbitrary, we can claim local uniform boundedness in U. One way to justify the lack of uniform boundedness is to investigate the behavior of the functions in the family at the points $\sqrt[n]{1-n^{-1}}$.

The following theorem gives us a characterization for local uniform boundedness when the metric space is a subset of $\mathbb{R}$ or $\mathbb{C}$.

Theorem 8.4.12 *A family of complex valued functions $\mathcal{F}$ on a subset Ω of $\mathbb{C}$ is locally uniformly bounded in Ω if and only if $\mathcal{F}$ is uniformly bounded on every compact subset of Ω.*

Proof. ($\Leftarrow$) This is an immediate consequence of the observation that the closure of a neighborhood in $\mathbb{C}$ or $\mathbb{R}$ is compact.

($\Rightarrow$) Suppose $\mathcal{F}$ is locally uniformly bounded on a domain Ω and K is a compact subset of Ω. Then, for each $z \in K$ there exists a neighborhood of z, $N(z; \epsilon_z)$ and a positive real number, M_z, such that

$$|f(\zeta)| \leq M_z, \text{ for all } \zeta \in N(z; \epsilon_z).$$

Since $\{N(z; \epsilon_z) : z \in K\}$ covers K, we know that there exists a finite subcover, say $\left\{N\left(z_j; \epsilon_{z_j}\right) : j = 1, 2, \cdots, n\right\}$. Then, for $M = \max\left\{M_{z_j} : 1 \leq j \leq n\right\}$, $|f(z)| \leq M$, for all $z \in K$, and we conclude that $\mathcal{F}$ is uniformly bounded on K. ■

Remark 8.4.13 *Note that Theorem 8.4.12 made specific use of the Heine-Borel Theorem; i.e., the fact that we were in a space where compactness is equivalent to being closed and bounded.*

Remark 8.4.14 *In our text, an example is given to illustrate that a uniformly bounded sequence of real-valued continuous functions on a compact metric space need not yield a subsequence that converges (even) pointwise on the metric space. Because the verification of the claim appeals to a theorem given in Chapter 11 of the text, at this point we accept the example as a reminder to be cautious.*

Remark 8.4.15 *Again by way of example, the author of our text illustrates that it is not the case that every convergent sequence of functions contains a uniformly convergent subsequence. We offer it as our next excursion, providing space for you to justify the claims.*

Excursion 8.4.16 *Let* $\Omega = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$ *and*

$$\mathcal{F} = \left\{ f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} : n \in \mathbb{J} \right\}.$$

(a) *Show that $\mathcal{F}$ is uniformly bounded in $\boldsymbol{\Omega}$.*

(b) *Find the pointwise limit of $\{f_n\}_{n=1}^{\infty}$ for $x \in \boldsymbol{\Omega}$.*

(c) *Justify that no subsequence of $\{f_n\}_{n=1}^{\infty}$ can converge uniformly on $\boldsymbol{\Omega}$.*

*******For (a), observing that $x^2 + (1 - nx)^2 \geq x^2 > 0$ for $x \in (0, 1]$ and $f_n(0) = 0$ for each $n \in \mathbb{J}$ yields that $|f_n(x)| \leq 1$ for $x \in \boldsymbol{\Omega}$. In (b), since the only occurrence of n is in the denominator of each f_n, for each fixed $x \in \boldsymbol{\Omega}$, the corresponding sequence of real goes to 0 as $n \to \infty$. For (c), in view of the negation of the definition of uniform convergence of a sequence, the behavior of the sequence $\{f_n\}_{n=1}^{\infty}$ at the points $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ allows us to conclude that no subsequences of $\{f_n\}_{n=1}^{\infty}$ will converge uniformly on $\boldsymbol{\Omega}$.*******

Now we know that we don't have a "straight" analog for the Bolzano-Weierstrass Theorem when we are in the realm of families of functions in $\mathcal{C}(X)$. This poses the challenge of finding an additional property (or set of properties) that will yield such an analog. Towards that end, we introduce define a property that requires "local and global" uniform behavior over a family.

Definition 8.4.17 *A family $\mathcal{F}$ of complex-valued functions defined on a metric space (Ω, d) is **equicontinuous** on Ω if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall f)\,(\forall u)\,(\forall v)\,(f \in \mathcal{F} \wedge u \in \Omega \wedge v \in \Omega \wedge d\,(u, v) < \delta \\ \Rightarrow |f\,(u) - f\,(v)| < \varepsilon).$$

Remark 8.4.18 *If $\mathcal{F}$ is equicontinuous on Ω, then each $f \in \mathcal{F}$ is clearly uniformly continuous in Ω.*

Excursion 8.4.19 *On the other hand, for $U_R = \{z : |z| \leq R\}$, show that each function in $\mathcal{F} = \{nz : n \in \mathbb{J}\}$ is uniformly continuous on U_R though $\mathcal{F}$ is not equicontinuous on U_R.*

Excursion 8.4.20 *Use the Mean-Value Theorem to justify that*

$$\left\{ f_n\,(x) = n \sin\frac{x}{n} : n \in \mathbb{J} \right\}$$

is equicontinuous in $\Omega = [0, \infty)$

The next result is particularly useful when we can designate a denumerable subset of the domains on which our functions are defined. When the domain is an open connected subset of $\mathbb{R}$ or $\mathbb{C}$, then the rationals or points with the real and imaginary parts as rational work very nicely. In each case the denumerable subset is dense in the set under consideration.

Lemma 8.4.21 *If $\{f_n\}_{n=1}^{\infty}$ is a pointwise bounded sequence of complex-valued functions on a denumerable set E, then $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\left\{f_{n_k}\right\}_{k=1}^{\infty}$ that converges pointwise on E.*

Excursion 8.4.22 *Finish the following proof.*

Proof. Let $\{f_n\}$ be sequence of complex-valued functions that is pointwise bounded on a denumerable set E. Then the set E can be realized as a sequence$\{w_k\}$ of distinct points. This is a natural setting for application of the Cantor diagonalization process that we saw earlier in the proof of the denumerability of the rationals. From the Bolzano–Weierstrass Theorem, $\{f_n(w_1)\}$ bounded implies that there exists a convergent subsequence $\left\{f_{n,1}(w_1)\right\}$. The process can be applied to $\left\{f_{n,1}(w_2)\right\}$ to obtain a subsequence $\left\{f_{n,2}(w_2)\right\}$ that is convergent.

$$\begin{array}{cccc} f_{1,1} & f_{2,1} & f_{3,1} & \cdots \\ f_{1,2} & f_{2,2} & f_{3,2} & \cdots \\ & \cdot\ \cdot\ \cdot & & \\ & \cdot\ \cdot\ \cdot & & \end{array}$$

In general, $\left\{f_{n,j}\right\}_{n=1}^{\infty}$ is such that $\left\{f_{n,j}\left(w_j\right)\right\}_{n=1}^{\infty}$ is convergent and $\left\{f_{n,j}\right\}_{n=1}^{\infty}$ is a subsequence of each of $\left\{f_{n,k}\right\}_{n=1}^{\infty}$ for $k = 1, 2, ..., j-1$. Now consider $\left\{f_{n,n}\right\}_{n=1}^{\infty}$

■

*** For $x \in E$, there exists an $M \in \mathbb{J}$ such that $x = w_M$. Then $\left\{f_{n,n}\right\}_{n=M+1}^{\infty}$ is a subsequence of $\left\{f_{n,M}\right\}_{n=M+1}^{\infty}$ from which it follows that $\left\{f_{n,n}(x)\right\}$ is convergent at x. ***

The next result tells us that if we restrict ourselves to domains K that are compact metric spaces that any uniformly convergent sequence in $\mathcal{C}(K)$ is also an equicontinuous family.

Theorem 8.4.23 *Suppose that (K, d) is a compact metric space and the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is such that $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly on K, then $\mathcal{F} = \{f_n : n \in \mathbb{J}\}$ is equicontinuous on K.*

Proof. Suppose that (K, d) is a compact metric space, the sequence of functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}(K)$ converges uniformly on K and $\varepsilon > 0$ is given. By Theorem 8.2.3, $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on K. Thus, there exists a positive integer M such that $n \geq M$ implies that $\|f_n - f_m\|_K < \dfrac{\varepsilon}{3}$. In particular,

$$\|f_n - f_M\|_K < \frac{\varepsilon}{3} \text{ for all } n > M.$$

Because each f_n is continuous on a compact set, from the Uniform Continuity Theorem, for each $n \in \mathbb{J}$, f_n is uniformly continuous on K. Hence, for each $j \in \{1, 2, ..., M\}$, there exists a $\delta_j > 0$ such that $x, y \in K$ and $d(x, y) < \delta_j$ implies that $\left|f_j(x) - f_j(y)\right| < \dfrac{\varepsilon}{3}$. Let $\delta = \min\limits_{1 \leq j \leq M} \delta_j$. Then

$$(\forall j)\,(\forall x)\,(\forall y)\left[\begin{array}{c} (j \in \{1, 2, ..., M\} \wedge x, y \in K \wedge d(x, y) < \delta) \Rightarrow \\ \left|f_j(x) - f_j(y)\right| < \dfrac{\varepsilon}{3} \end{array}\right]. \tag{8.7}$$

For $n > M$ and $x, y \in K$ such that $d(x, y) < \delta$, we also have that

$$\begin{aligned} |f_n(x) - f_n(y)| \leq {} & |f_n(x) - f_M(x)| + \\ & |f_M(x) - f_M(y)| + |f_M(y) - f_n(y)| < \varepsilon. \end{aligned} \tag{8.8}$$

From (8.7) and (8.8) and the fact that $\varepsilon > 0$ was arbitrary, we conclude that

$$\begin{aligned} (\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall f_n)\,(\forall u)\,(\forall v)\,(& f_n \in \mathcal{F} \wedge u, v \in K \wedge d(u, v) < \delta \\ & \Rightarrow |f(u) - f(v)| < \varepsilon); \text{ i.e.,} \end{aligned}$$

$\mathcal{F}$ is equicontinuous on K. ■

We are now ready to offer conditions on a subfamily of $\mathcal{C}(K)$ that will give us an analog to the Bolzano-Weierstrass Theorem.

Theorem 8.4.24 *Suppose that (K, d) is a compact metric space and the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is such that $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$. If $\{f_n : n \in \mathbb{J}\}$ is pointwise bounded and equicontinuous on K, then*

(a) $\{f_n : n \in \mathbb{J}\}$ is uniformly bounded on K and

(b) $\{f_n\}_{n=1}^{\infty}$ *contains a subsequence that is uniformly convergent on* K.

Excursion 8.4.25 *Fill in what is missing in order to complete the following proof of Theorem 8.4.24.*

Proof. Suppose that (K, d) is a compact metric space, the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is such that $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$, the family $\{f_n : n \in \mathbb{J}\}$ is pointwise bounded and equicontinuous on K.

Proof of part (a):

Let $\varepsilon > 0$ be given. Since $\{f_n : n \in \mathbb{J}\}$ is equicontinuous on K, there exists a $\delta > 0$ such that

$$(\forall n)\,(\forall x)\,(\forall y)\left[(n \in \mathbb{J} \wedge x, y \in K \wedge d(x, y) < \delta) \Rightarrow |f_n(x) - f_n(y)| < \varepsilon\right]. \tag{8.9}$$

Because $\{N_\delta(u) : u \in K\}$ forms an $\underset{(1)}{\underline{\hspace{3cm}}}$ for K and K is compact, there exists a finite number of points, say $p_1, p_2, \ldots, p_k$, such that $K \subset \underset{(2)}{\underline{\hspace{3cm}}}$. On the other hand, $\{f_n : n \in \mathbb{J}\}$ is pointwise bounded; consequently, for each p_j, $j \in \{1, 2, \ldots, k\}$, there exists a positive real number M_j such that

$$(\forall n)\left(n \in \mathbb{J} \Rightarrow \left|f_n\left(p_j\right)\right| < M_j\right).$$

For $M = \underset{(3)}{\underline{\hspace{3cm}}}$, it follows that

$$(\forall n)\,(\forall j)\left((n \in \mathbb{J} \wedge j \in \{1, 2, \ldots, k\}) \Rightarrow \left|f_n\left(p_j\right)\right| < M\right). \tag{8.10}$$

Suppose that $x \in K$. Since $K \subset \bigcup_{j=1}^{k} N_\delta\left(p_j\right)$ there exists an $m \in \{1, 2, \ldots, k\}$ such that $\underset{(4)}{\underline{\hspace{3cm}}}$. Hence, $d(x, p_m) < \delta$ and, from (8.9), we conclude that $\underset{(5)}{\underline{\hspace{3cm}}}$ for all $n \in \mathbb{J}$. But then $|f_n(x)| - |f_n(p_m)| \leq |f_n(x) - f_n(p_m)|$ yields that $|f_n(x)| < |f_n(p_m)| + \varepsilon$ for $\underset{(6)}{\underline{\hspace{3cm}}}$. From (8.10), we conclude that $|f_n(x)| < M + \varepsilon$ for all $n \in \mathbb{J}$. Since x was arbitrary, it follows that

$$(\forall n)\,(\forall x)\left[(n \in \mathbb{J} \wedge x \in K) \Rightarrow |f_n(x)| < M + \varepsilon\right]; \text{ i.e.,}$$

$\{f_n : n \in \mathbb{J}\}$ is $\underset{(7)}{\underline{\qquad\qquad\qquad}}$.

Almost a proof of part (b):

If K were finite, we would be done. For K infinite, let E be a denumerable subset of K that is dense in K. (The reason for the "Almost" in the title of this part of the proof is that we did not do the Exercise #25 on page 45 for homework. If $K \subset \mathbb{R}$ or $K \subset \mathbb{C}$, then the density of the rationals leads immediately to a set E that satisfies the desired property; in the general case of an arbitrary metric space, Exercise #25 on page 45 indicates how we can use open coverings with rational radii to obtain such a set.) Because $\{f_n : n \in \mathbb{J}\}$ is $\underset{(8)}{\underline{\qquad\qquad\qquad}}$ on E, by Lemma 8.4.21, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$, say $\{g_j\}_{j=1}^{\infty}$, that is convergent for each $x \in E$.

Suppose that $\varepsilon > 0$ is given. Since $\{f_n : n \in \mathbb{J}\}$ is equicontinuous on K, there exists a $\delta > 0$ such that

$$(\forall n)(\forall x)(\forall y)\left[(n \in \mathbb{J} \wedge x, y \in K \wedge d(x, y) < \delta) \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}\right].$$

Because E is dense in K, $\{N_\delta(u) : u \in E\}$ forms an open cover for K. Because K is compact, we conclude that there exists a finite number of elements of E, say $w_1, w_2, ..., w_q$, such that

$$K \subset \bigcup_{j=1}^{q} N_\delta(w_j). \tag{8.11}$$

Since $\{w_1, w_2, ..., w_q\} \subset E$ and $\{g_j(x)\}_{j=1}^{\infty}$ is a convergent sequence of complex numbers for each $x \in E$, the completeness of $\mathbb{C}$, yields Cauchy convergence of $\{g_j(w_s)\}_{j=1}^{\infty}$ for each w_s, $s \in \{1, 2, ..., q\}$. Hence, for each $s \in \{1, 2, ..., q\}$, there exists a positive integer M_s such that $n > M_s$ and $m > M_s$ implies that

$$|g_n(w_s) - g_m(w_s)| < \frac{\varepsilon}{3}.$$

Suppose that $x \in K$. From (8.11), there exists an $s \in \{1, 2, ..., q\}$ such that $\underset{(9)}{\underline{\qquad\qquad\qquad}}$. Then $d(x, w_s) < \delta$ implies that

$$|f_n(x) - f_n(w_s)| < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{J}$. Let $M = \max\{M_s : s \in \{1, 2, ..., q\}\}$. It follows that, for $n > M$ and $m > M$,

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_n(w_s)| + \left|\underset{(10)}{\underline{\qquad\qquad}}\right| + |g_m(w_s) - g_m(x)| < \varepsilon.$$

Since $\varepsilon > 0$ and $x \in K$ were arbitrary, we conclude that

$$(\forall \varepsilon > 0)\,(\exists M \in \mathbb{J})\left[n > M \wedge m > M \Rightarrow (\forall x)\,(x \in K \Rightarrow |g_n(x) - g_m(x)| < \varepsilon)\right];$$

i.e., $\{g_j\}_{j=1}^{\infty}$ is $\underset{(11)}{\underline{\qquad\qquad}}$. By Theorem 8.4.23, $\{g_j\}_{j=1}^{\infty}$ is uniformly convergent on K as needed. ■

Acceptable responses are: (1) open cover, (2) $\bigcup_{j=1}^{k} N_\delta(p_j)$, (3) $\max\{M_j : j = 1, 2, ..., k\}$, (4) $N_\delta(p_m)$, (5) $|f_n(x) - f_n(p_m)| < \varepsilon$, (6) all $n \in \mathbb{J}$, (7) uniformly bounded on K, (8) pointwise bounded on K, (9) $x \in N_\delta(w_s)$, (10) $|g_n(w_s) - g_m(w_s)|$, (11) uniformly Cauchy on K.

Since we now know that for families of functions it is not the case that every convergent sequence of functions contains a uniformly convergent subsequence, families that do have that property warrant a special label.

Definition 8.4.26 *A family $\mathcal{F}$ of complex-valued functions defined on a metric space Ω is said to be **normal** in Ω if and only if every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of Ω.*

Remark 8.4.27 *In view of Theorem 8.4.24, any family that is pointwise bounded and equicontinuous on a compact metric space K is normal in K.*

Our last definition takes care of the situation when the limits of the sequences from a family are in the family.

Definition 8.4.28 *A normal family of complex-valued functions $\mathcal{F}$ is said to be **compact** if and only if the uniform limits of all sequences converging in $\mathcal{F}$ are also members of $\mathcal{F}$.*

8.5 The Stone-Weierstrass Theorem

In view of our information concerning the transmission of nice properties of functions in sequences (and series), we would like to have results that enable us to realize a given function as the uniform limit of a sequence of nice functions. The last result that we will state in this chapter relates a given function to a sequence of polynomials. Since polynomials are continuously differentiable functions the theorem is particularly good news. We are offering the statement of the theorem without discussing the proof. Space is provided for you to insert a synopsis or comments concerning the proof that is offered by the author of our text on pages 159-160.

Theorem 8.5.1 *If $f \in \mathcal{C}([a,b])$ for $a < b$, then there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty} \subset \mathcal{C}([a,b])$ such that $\lim_{n\to\infty} P_n(x) = f(x)$ where the convergence is uniform of $[a,b]$. If f is a real-valued function then the polynomials can be taken as real.*

Space for Comments.

8.6 Problem Set H

1. Use properties of limits to find the pointwise limits for the following sequences of complex-valued functions on $\mathbb{C}$.

   (a) $\left\{\dfrac{nz}{1+nz^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{nz^2}{z+3n}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{z^n}{1+z^n}\right\}_{n=1}^{\infty}$

(d) $\left\{\dfrac{n^2 z}{1+n^3 z^2}\right\}_{n=1}^{\infty}$

(e) $\left\{\dfrac{1+n^2 z}{1-n^2 z}+\dfrac{n}{1+2n}\right\}_{n=1}^{\infty}$

(f) $\left\{ze^{-n|z|}\right\}_{n=1}^{\infty}$

2. For each $n \in \mathbb{J}$, let $f_n(x) = \dfrac{nx}{e^{nx}}$. Use the definition to prove that $\{f_n\}_{n=1}^{\infty}$ is pointwise convergent on $[0,\infty)$, uniformly convergent on $[\alpha,\infty)$ for any fixed positive real number α, and not uniformly convergent on $(0,\infty)$.

3. For each of the following sequences of real-valued functions on $\mathbb{R}$, use the definition to show that $\{f_n(x)\}_{n=1}^{\infty}$ converges pointwise to the specified $f(x)$ on the given set I; then determine whether or not the convergence is uniform. Use the definition or its negation to justify your conclusions concerning uniform convergence.

(a) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{2x}{1+nx}\right\}$; $f(x) = 0$; $I = [0,1]$

(b) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{\cos nx}{\sqrt{n}}\right\}$; $f(x) = 0$; $I = [0,1]$

(c) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{n^3 x}{1+n^4 x}\right\}$; $f(x) = 0$; $I = [0,1]$

(d) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{n^3 x}{1+n^4 x^2}\right\}$; $f(x) = 0$; $I = [a,\infty)$ where a is a positive fixed real number

(e) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{1-x^n}{1-x}\right\}$; $f(x) = \dfrac{1}{1-x}$; $I = \left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

(f) $\{f_n(x)\}_{n=1}^{\infty} = \left\{nxe^{-nx^2}\right\}$; $f(x) = 0$; $I = [0,1]$

4. For the sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $\mathbb{R}$ given by $f_n(x) = \dfrac{(n+1)(n+2)x^n}{1-x}$ for $n \in \mathbb{J}$ and $f(x) = 0$ for $x \in I = [0,1]$, show that $f_n(x) \longrightarrow f(x)$ as $n \to \infty$ for each $x \in I$. Is is true that

$$\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx \text{ as } n \to \infty?$$

5. Suppose that the sequences of functions $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ converge uniformly to f and g, respectively, on a set A in a metric space (S,d). Prove that the sequence $\{f_n + g_n\}_{n=1}^{\infty}$ converges uniformly to $f+g$.

6. Determine all the values of h such that $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{\left(1+nx^2\right)\sqrt{n}}$ is uniformly convergent in $I = \{x \in \mathbb{R} : |x| < h\}$. (Hint: Justify that each $f_n(x) = \dfrac{x^2}{\left(1+nx^2\right)}$ is increasing as a function x and make use that the obtain an upper bound on the summand.)

7. Prove that, if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly for all $x \in \mathbb{R}$.

8. Suppose that $\displaystyle\sum_{n=1}^{\infty} n\,|b_n|$ is convergent and let $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$ for $x \in \mathbb{R}$. Show that

$$f'(x) = \sum_{n=1}^{\infty} nb_n \cos nx$$

and that both $\displaystyle\sum_{n=1}^{\infty} b_n \sin nx$ and $\displaystyle\sum_{n=1}^{\infty} nb_n \cos nx$ converge uniformly for all $x \in \mathbb{R}$.

9. Prove that if a sequence of complex-valued functions on $\mathbb{C}$ converges uniformly on a set A and on a set B, then it converges uniformly on $A \cup B$.

10. Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions on $\mathbb{C}$ is uniformly convergent on a set Ω to a function f that is bounded on Ω, then

there exists a positive real number K and a positive integer M such that $(\forall n)(\forall x)(n > M \wedge x \in \Omega \Rightarrow |f_n(x)| < K)$.

11. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions each of which is continuous on an interval $I = [a,b]$. If $\{f_n\}_{n=1}^{\infty}$ is uniformly continuous on I, prove that there exists a positive real number K such that

    $(\forall n)(\forall x)(n \in J \wedge x \in I \Rightarrow |f_n(x)| < K)$.

12. Without appeal to Theorem 8.3.8; i.e., using basic properties of integrals, prove Theorem 8.3.11: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions that are continuous on the interval $[a,b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$. For $c \in [a,b]$ and each $n \in \mathbb{J}$, let $F_n(x) \underset{def}{=} \int_c^x f_n(t)\,dt$.Then f is continuous on $[a,b]$ and $F_n \underset{[a,b]}{\rightrightarrows} F$ where $F(x) = \int_c^x f(t)\,dt$.

13. Compare the values of the integrals of the nth partials sums over the interval $[0,1]$ with the integral of their their limit in the case where $\sum_{k=1}^{\infty} f_k(x)$ is such that

$$f_1(x) = \begin{cases} x+1 & , \quad -1 \le x \le 0 \\ -x+1 & , \quad 0 < x \le 1 \end{cases},$$

and, for each $n = 2, 3, 4, \ldots$,

$$S_n(x) = \begin{cases} 0 & , \quad -1 \le x < \dfrac{-1}{n} \\ n^2x+n & , \quad \dfrac{-1}{n} \le x \le 0 \\ -n^2x+n & , \quad 0 < x \le \dfrac{1}{n} \\ 0 & , \quad \dfrac{1}{n} < x \le 1 \end{cases}.$$

Does your comparison allow you to conclude anything concerning the uniform convergence of the given series n $[0,1]$? Briefly justify your response.

14. For each $n \in \mathbb{J}$, let $f_n(x) = \begin{cases} 0 & , \text{if } \ -1 \leq x \leq -\dfrac{1}{n} \\ \dfrac{nx+1}{2} & , \text{if } \ -\dfrac{1}{n} < x < \dfrac{1}{n} \\ 0 & , \text{if } \ \dfrac{1}{n} \leq x \leq 1 \end{cases}$.

 Then $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}([-1,1])$ where $\mathcal{C}([-1,1])$ is the set of real-valued functions that are continuous on $[-1,1]$. Make use of $\{f_n\}_{n=1}^{\infty}$ to justify that the metric space $(\mathcal{C}([-1,1]), \rho)$ is not complete, where

$$\rho(f,g) = \int_{-1}^{1} |f(x) - g(x)|\, dx.$$

15. For each of the following families $\mathcal{F}$ of real-valued functions on the specified sets $\mathbf{\Omega}$, determine whether of not $\mathcal{F}$ is pointwise bounded, locally uniformly bounded, and/or uniformly bounded on $\mathbf{\Omega}$. Justify your conclusions.

 (a) $\mathcal{F} = \left\{1 - \dfrac{1}{nx} : n \in \mathbb{J}\right\}, \mathbf{\Omega} = (0,1]$

 (b) $\mathcal{F} = \left\{\dfrac{\sin nx}{\sqrt{n}} : n \in \mathbb{J}\right\}, \mathbf{\Omega} = [0,1]$

 (c) $\mathcal{F} = \left\{\dfrac{nx}{1+n^2x^2} : n \in \mathbb{J}\right\}, \mathbf{\Omega} = \mathbb{R}$

 (d) $\mathcal{F} = \left\{\dfrac{x^{2n}}{1+x^{2n}} : n \in \mathbb{J}\right\}, \mathbf{\Omega} = \mathbb{R}$

 (e) $\mathcal{F} = \left\{n^2x^n(1-x) : n \in \mathbb{J}\right\}, \mathbf{\Omega} = [0,1)$

16. Suppose that $\mathcal{F}$ is a family of real-valued functions on $\mathbb{R}$ that are differentiable on the interval $[a,b]$ and $\mathcal{F}' = \left\{f' : f \in \mathcal{F}\right\}$ is uniformly bounded on $[a,b]$. Prove that $\mathcal{F}$ is equicontinuous on (a,b).

17. Is $\mathcal{F} = \left\{nxe^{-nx^2} : n \in \mathbb{J} \wedge x \in \mathbb{R}\right\}$ uniformly bounded on $[0,\infty)$? State your position clearly and carefully justify it.

18. Is $\mathcal{G} = \left\{n \cos \dfrac{x}{2n} : n \in \mathbb{J} \wedge x \in \mathbb{R}\right\}$ equicontinuous on $\mathbb{R}$? State your position clearly and carefully justify it.

19. Is $\left\{ \sum_{k=1}^{n} \frac{k^2 x \sin kx}{1 + k^4 x} \right\}_{n=1}^{\infty}$ uniformly convergent on $[0, \infty)$? State your position clearly and carefully justify it.