Chapter 6

Differentiation: Our First View

We are now ready to reflect on a particular application of limits of functions; namely, the derivative of a function. This view will focus on the derivative of real-valued functions on subsets of $\mathbb{R}^1$. Looking at derivatives of functions in $\mathbb{R}^k$ requires a different enough perspective to necessitate separate treatment; this is done with Chapter 9 of our text. Except for the last section, our discussion is restricted to aspects of differential calculus of one variable. You should have seen most of the results in your first exposure to calculus–MAT21A on this campus. However, some of the results proved in this chapter were only stated when you first saw them and some of the results are more general than the earlier versions that you might have seen. The good news is that the presentation here isn't dependent on previous exposure to the topic; on the other hand, reflecting back on prior work that you did with the derivative can enhance your understanding and foster a deeper level of appreciation.

6.1 The Derivative

Definition 6.1.1 *A real-valued function f on a subset Ω of $\mathbb{R}$ is* ***differentiable at a point*** $\zeta \in \Omega$ *if and only if f is defined in an open interval containing ζ and*

$$\lim_{w \to \zeta} \frac{f(w) - f(\zeta)}{w - \zeta} \tag{6.1}$$

exists. The value of the limit is denoted by $f'(\zeta)$. The function is said to be differentiable on Ω if and only if it is differentiable at each $\zeta \in \Omega$.

Remark 6.1.2 *For a function f and a fixed point ζ, the expression*

$$\phi(w) = \frac{f(w) - f(\zeta)}{w - \zeta}$$

is one form of what is often referred to as a "difference quotient". Sometimes it is written as

$$\frac{\Delta f}{\Delta w}$$

where the Greek letter Δ is being offered as a reminder that difference starts with a "d". It is the latter form that motivates use of the notation $\dfrac{df}{dw}$ for the first derivative of f as a function of w. Other commonly used notations are D_w and D_1; these only become useful alternatives when we explore functions over several real variables.

There is an alternative form of (6.1) that is often more useful in terms of computation and formatting of proofs. Namely, if we let $w = \zeta + h$, (6.1) can be written as

$$\lim_{h\to 0} \frac{f(\zeta + h) - f(\zeta)}{h}. \tag{6.2}$$

Remark 6.1.3 *With the form given in (6.2), the difference quotient can be abbreviated as $\dfrac{\Delta f}{h}$.*

Definition 6.1.4 *A real-valued function f on a subset Ω of $\mathbb{R}$ is **right-hand differentiable at a point** $\zeta \in \Omega$ if and only if f is defined in a half open interval in the form $[\zeta, \zeta + \delta)$ for some $\delta > 0$ and the one-sided derivative from the right, denoted by $D^+ f(\zeta)$,*

$$\lim_{h\to 0^+} \frac{f(\zeta + h) - f(\zeta)}{h}$$

*exists; the function f is **left-hand differentiable at a point** $\zeta \in \Omega$ if and only if f is defined in a half open interval in the form $(\zeta - \delta, \zeta]$ for some $\delta > 0$ and the one-sided derivative from the left, denoted by $D^- f(\zeta)$,*

$$\lim_{h\to 0^-} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists.

Definition 6.1.5 *A real-valued function* f *is* ***differentiable on a closed interval*** $[a, b]$ *if and only if* f *is differentiable in* (a, b)*, right-hand differentiable at* $x = a$ *and left-hand differentiable at* $x = b$*.*

Example 6.1.6 *Use the definition to prove that* $f(x) = \dfrac{x+2}{x-1}$ *is differentiable at* $x = 2$*.*

Note that f *is defined in the open interval* $(1, 3)$ *which contains* $w = 2$*. Furthermore,*

$$\lim_{w\to 2}\frac{f(w) - f(2)}{w-2} = \lim_{w\to 2}\frac{\left(\dfrac{w+2}{w-1}\right) - \dfrac{4}{1}}{w-2} = \lim_{w\to 2}\frac{-3(w-2)}{w-2} = \lim_{w\to 2}(-3) = -3.$$

Hence, f *is differentiable at* $w = 2$ *and* $f'(2) = -3$*.*

Example 6.1.7 *Use the definition to prove that* $g(x) = |x-2|$ *is not differentiable at* $x = 2$*.*

Since $\operatorname{dom}(g) = \mathbb{R}$*, the function* g *is defined in any open interval that contains* $x = 2$*. Hence,* g *is differentiable at* $x = 2$ *if and only if*

$$\lim_{h\to 0}\frac{g(2+h) - g(2)}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

exists. Let $\phi(h) = \dfrac{|h|}{h}$ *for* $h \neq 0$*. Note that*

$$\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = -1.$$

Thus, $\phi(0+) \neq \phi(0-)$ *from which we conclude that* $\lim_{h\to 0}\phi(h)$ *does not exist. Therefore,* g *is not differentiable at* $x = 2$*.*

Remark 6.1.8 *Because the function* g *given in Example 6.1.7 is left-hand differentiable at* $x = 2$ *and right-hand differentiable at* $x = 2$*, we have that* g *is differentiable in each of* $(-\infty, 2]$ *and* $[2, \infty)$*.*

Example 6.1.9 *Discuss the differentiability of each of the following at* $x = 0$*.*

1. $G(x) = \begin{cases} x\sin\dfrac{1}{x} & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0 \end{cases}$

2. $F(x) = \begin{cases} x^3 \sin \dfrac{1}{x} & \text{, for } \quad x \neq 0 \\ 0 & \text{, for } \quad x = 0 \end{cases}$

First of all, notice that, though the directions did not specify appeal to the definition, making use of the definition is the only viable option because of the way the function is defined. Discussing the differentiability of functions that are defined "in pieces" requires consideration of the pieces. On segments where the functions are realized as simple algebraic combinations of nice functions, the functions can be declared as differentiable based on noting the appropriate nice properties. If the function is defined one way at a point and a different way to the left and/or right, then appeal to the difference quotient is mandated.

For (1), we note that G is defined for all reals, consequently, it is defined in every interval that contains 0. Thus, G is differentiable at 0 if and only if

$$\lim_{h\to 0} \frac{G(0+h) - G(0)}{h} = \lim_{h\to 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h\to 0} \left(\sin \frac{1}{h} \right)$$

exists. For $h \neq 0$, let $\phi(h) = \sin \frac{1}{h}$. For each $n \in \mathbb{J}$, let $p_n = \dfrac{2}{\pi(2n-1)}$. Now, $\{p_n\}_{n=1}^{\infty}$ converges to 0 as n approaches infinity; but $\{\phi(p_n)\}_{n=1}^{\infty} = \left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$ diverges. From the Sequences Characterization for Limits of Functions (Theorem 5.1.15), we conclude that $\lim_{h\to 0} \phi(h)$ does not exist. Therefore, G is not differentiable at $x = 0$.

The function F given in (2) is also defined in every interval that contains 0. Hence, F is differentiable at 0 if and only if

$$\lim_{h\to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h\to 0} \frac{h^3 \sin \frac{1}{h} - 0}{h} = \lim_{h\to 0} \left(h^2 \sin \frac{1}{h} \right)$$

exists. Now we know that, for $h \neq 0$, $\left| \sin \frac{1}{h} \right| \leq 1$ and $\lim_{h\to 0} h^2 = 0$; it follows from a simple modification of what was proved in Exercise #6 of Problem Set D that $\lim_{h\to 0} \left(h^2 \sin \frac{1}{h} \right) = 0$. Therefore, F is differentiable at $x = 0$ and $F'(0) = 0$.

Excursion 6.1.10 *In the space provided, sketch graphs of G and F on two different representations of the Cartesian coordinate system in intervals containing* 0*.*

For the sketch of G using the curves $y = x$ and $y = -x$ as guides to stay within should have helped give a nice sense for the appearance of the graph; the guiding (or bounding) curves for F are $y = x^3$ and $y = -x^3$.

Remark 6.1.11 *The two problems done in the last example illustrate what is sometimes referred to as a smoothing effect. In our text, it is shown that*

$$K(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ \\ 0 & , \textit{for} \quad x = 0 \end{cases}$$

is also differentiable at $x = 0$. The function

$$L(x) = \begin{cases} \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ \\ 0 & , \textit{for} \quad x = 0 \end{cases}$$

is not continuous at $x = 0$ with the discontinuity being of the second kind. The "niceness" of the function is improving with the increase in exponent of the "smoothing function" x^n.

In the space provided, sketch graphs of K and L on two different representations of the Cartesian coordinate system in intervals containing 0*.*

The function L is not continuous at $x = 0$ while G is continuous at $x = 0$ but not differentiable there. Now we know that K and F are both differentiable at $x = 0$; in fact, it can be shown that F can be defined to be differentiable at $x = 0$ while at most continuity at $x = 0$ can be gained for the derivative of K at $x = 0$. Our first theorem in this section will justify the claim that being differentiable is a stronger condition than being continuous; this offers one sense in which we claim that F is a nicer function in intervals containing 0 than K is there.

Excursion 6.1.12 *Fill in what is missing in order to complete the following proof that the function $f(x) = \sqrt{x}$ is differentiable in $\mathbb{R}^+ = (0, \infty)$.*

Proof. *Let $f(x) = \sqrt{x}$ and suppose that $a \in \mathbb{R}^+$. Then f is* $\underset{(1)}{\underline{\hspace{3cm}}}$ *in the segment $\left(\frac{a}{2}, 2a\right)$ that contains $x = a$. Hence, f is differentiable at $x = a$ if and only if*

$$\lim_{h \to 0} \underset{(2)}{\boxed{\hspace{3cm}}} = \lim_{h \to 0} \underset{(3)}{\boxed{\hspace{3cm}}}$$

exists. Now

$$\lim_{h\to 0}\underbrace{\boxed{}}_{(3)} = \lim_{h\to 0}\left[\frac{\left(\sqrt{a+h}-\sqrt{a}\right)\left(\sqrt{a+h}+\sqrt{a}\right)}{h\left(\sqrt{a+h}+\sqrt{a}\right)}\right]$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{(4)}$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{(5)}$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{(6)}.$$

Consequently, f is differentiable at $x = a$ and $f'(a) = \underbrace{\qquad}_{(7)}$. Since $a \in \mathbb{R}^+$ was arbitrary, we conclude that

$$(\forall x)\left[\left(x \in \mathbb{R}^+ \wedge f(x) = \sqrt{x}\right) \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}\right].$$

■

Acceptable responses are: (1) defined, (2) $\left[\left(f(a+h) - f(a)\right)\left(h^{-1}\right)\right]$, (3) $\left[\left(\sqrt{a+h}-\sqrt{a}\right)\left(h^{-1}\right)\right]$, (4) $\lim_{h\to 0}\left[\frac{(a+h)-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)}\right]$, (5) $\lim_{h\to 0}\left(\sqrt{a+h}+\sqrt{a}\right)^{-1}$, (6) $\left(2\sqrt{a}\right)^{-1}$, and (7) $\frac{1}{2\sqrt{a}}$.

The next result tells us that differentiability of a function at a point is a stronger condition than continuity at the point.

Theorem 6.1.13 *If a function is differentiable at $\zeta \in \mathbb{R}$, then it is continuous there.*

Excursion 6.1.14 *Make use of the following observations and your understanding of properties of limits of functions to prove Theorem 6.1.13*

Some observations to ponder:

- *The function f being differentiable at ζ assures the existence of a $\delta > 0$ such that f is defined in the segment $(\zeta - \delta, \zeta + \delta)$;*

- *Given a function G defined in a segment (a, b), we know that G is continuous at any point $p \in (a, b)$ if and only if $\lim_{x \to p} G(x) = G(p)$ which is equivalent to having $\lim_{x \to p} \left[G(x) - G(p)\right] = 0$.*

Space for scratch work.

Proof.

$\blacksquare$

Once you think of the possibility of writing $\left[G(x) - G(p)\right]$ as $\left[(G(x) - G(p))(x - p)^{-1}\right](x - p)$ for $x \neq p$ the limit of the product theorem does the rest of the work.

Remark 6.1.15 *We have already seen two examples of functions that are continuous at a point without being differentiable at the point; namely, $g(x) = |x - 2|$ at $x = 2$ and, for $x = 0$,*

$$G(x) = \begin{cases} x \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ 0 & , \textit{for} \quad x = 0 \end{cases}.$$

To see that G is continuous at $x = 0$*, note that* $\left|\sin \frac{1}{x}\right| \leq 1$ *for* $x \neq 0$ *and* $\lim_{x\to 0} x = 0$ *implies that* $\lim_{x\to 0}\left((x)\left(\sin \frac{1}{x}\right)\right) = 0$*. Alternatively, for* $\varepsilon > 0$*, let* $\delta(\varepsilon) = \varepsilon$*; then* $0 < |x - 0| < \delta$ *implies that*

$$\left|x \sin \frac{1}{x} - 0\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x| < \delta = \varepsilon.$$

Hence, $\lim_{x\to 0}\left(x \sin \frac{1}{x}\right) = 0 = G(0)$*. Either example is sufficient to justify that the converse of Theorem 6.1.13 is not true.*

Because the derivative is defined as the limit of the difference quotient, it should come as no surprise that we have a set of properties involving the derivatives of functions that follow directly and simply from the definition and application of our limit theorems. The set of basic properties is all that is needed in order to make a transition from finding derivatives using the definition to finding derivatives using simple algebraic manipulations.

Theorem 6.1.16 (Properties of Derivatives) *(a) If c is a constant function, then* $c'(x) = 0$*.*

(b) If f *is differentiable at* ζ *and* k *is a constant, then* $h(x) = kf(x)$ *is differentiable at* ζ *and* $h'(\zeta) = kf'(\zeta)$*.*

(c) If f *and* g *are differentiable at* ζ*, then* $F(x) = (f + g)(x)$ *is differentiable at* ζ *and* $F'(\zeta) = f'(\zeta) + g'(\zeta)$*.*

(d) If u *and* υ *are differentiable at* ζ*, then* $G(x) = (u\upsilon)(x)$ *is differentiable at* ζ *and*

$$G'(\zeta) = u(\zeta)\upsilon'(\zeta) + \upsilon(\zeta)u'(\zeta).$$

(e) If f *is differentiable at* ζ *and* $f(\zeta) \neq 0$*, then* $H(x) = [f(x)]^{-1}$ *is differentiable at* ζ *and* $H'(\zeta) = -\frac{f'(\zeta)}{[f(\zeta)]^2}$*.*

(f) If $p(x) = x^n$ *for* n *an integer,* p *is differentiable wherever it is defined and*

$$p'(x) = nx^{n-1}.$$

The proofs of (a) and (b) are about as easy as it gets while the straightforward proofs of (c) and (f) are left as exercises. Completing the next two excursions will provide proofs for (d) and (e).

Excursion 6.1.17 *Fill is what is missing in order to complete the following proof that, if u and υ are differentiable at ζ, then $G(x) = (u\upsilon)(x)$ is differentiable at ζ and*

$$G'(\zeta) = u(\zeta)\upsilon'(\zeta) + \upsilon(\zeta)u'(\zeta).$$

Proof. *Suppose u, υ, and G are as described in the hypothesis. Because u and υ are differentiable at ζ, they are defined in a segment containing ζ. Hence, $G(x) = u(x)\upsilon(x)$ is defined in a segment containing ζ. Hence, G is differentiable at ζ if and only if* $\lim\limits_{h\to 0}\underbrace{\boxed{}}_{(1)}$ *exists. Note that*

$$\begin{aligned}\lim_{h\to 0}\underbrace{\boxed{}}_{(1)} &= \lim_{h\to 0}\underbrace{\boxed{}}_{(2)}\\ &= \lim_{h\to 0}\frac{\upsilon(\zeta+h)\left[u(\zeta+h)-u(\zeta)\right]+u(\zeta)\left[\upsilon(\zeta+h)-\upsilon(\zeta)\right]}{h}\\ &= \lim_{h\to 0}\left[\upsilon(\zeta+h)\left(\frac{u(\zeta+h)-u(\zeta)}{h}\right)+u(\zeta)\left(\frac{\upsilon(\zeta+h)-\upsilon(\zeta)}{h}\right)\right].\end{aligned}$$

Since υ is differentiable at ζ it is continuous there; thus, $\lim\limits_{h\to 0}\upsilon(\zeta+h) = \underbrace{\qquad\qquad}_{(3)}$.

Now the differentiability of u and υ with the limit of the product and limit of the sum theorems yield that

$$\lim_{h\to 0}\underbrace{\boxed{}}_{(1)} = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{(4)}.$$

Therefore, G is differentiable at ζ. ■

Acceptable responses are: (1) $\left[(G(\zeta+h)-G(\zeta))h^{-1}\right]$, (2) $\left[(u(\zeta+h)\upsilon(\zeta+h)-u(\zeta)\upsilon(\zeta))h^{-1}\right]$, (3) $\upsilon(\zeta)$, and (4) $\upsilon(\zeta)u'(\zeta)+u(\zeta)\upsilon'(\zeta)$.

Excursion 6.1.18 *Fill is what is missing in order to complete the following proof that, if f is differentiable at ζ and $f(\zeta) \neq 0$, then $H(x) = [f(x)]^{-1}$ is differentiable at ζ and $H'(\zeta) = -\dfrac{f'(\zeta)}{[f(\zeta)]^2}$.*

Proof. *Suppose that the function f is differentiable at ζ and $f(\zeta) \neq 0$. From Theorem 6.1.13, f is $\underset{(1)}{\underline{\hspace{3cm}}}$ at ζ. Hence. $\lim_{x\to\zeta} f(x) = \underset{(2)}{\underline{\hspace{1.5cm}}}$. Since $\varepsilon = \dfrac{|f(\zeta)|}{2} > 0$, it follows that there exists $\delta > 0$ such that $\underset{(3)}{\underline{\hspace{6cm}}}$ implies that $|f(x) - f(\zeta)| < \dfrac{|f(\zeta)|}{2}$. The (other) triangular inequality, yields that, for $\underset{(3)}{\underline{\hspace{6cm}}}$, $|f(\zeta)| - |f(x)| < \dfrac{|f(\zeta)|}{2}$ from which we conclude that $|f(x)| > \dfrac{|f(\zeta)|}{2}$ in the segment $\underset{(4)}{\underline{\hspace{3cm}}}$. Therefore, the function $H(x) \underset{def}{=} [f(x)]^{-1}$ is defined in a segment that contains ζ and it is differentiable at ζ if and only if $\lim_{h\to 0} \dfrac{H(\zeta+h) - H(\zeta)}{h}$ exists. Now simple algebraic manipulations yield that*

$$\lim_{h\to 0} \frac{H(\zeta+h) - H(\zeta)}{h} = \lim_{h\to 0} \left[\left(\frac{f(\zeta+h) - f(\zeta)}{h} \right) \left(\frac{-1}{f(\zeta+h)\, f(\zeta)} \right) \right].$$

From the $\underset{(5)}{\underline{\hspace{3cm}}}$ of f at ζ, it follows that $\lim_{h\to 0} f(\zeta+h) = \underset{(6)}{\underline{\hspace{3cm}}}$. In view of the differentiability of f and the limit of the product theorem, we have that

$$\lim_{h\to 0} \frac{H(\zeta+h) - H(\zeta)}{h} = \underset{(7)}{\underline{\hspace{3cm}}}.$$

■

Acceptable responses are: (1) continuous, (2) $f(\zeta)$, (3) $|x - \zeta| < \delta$, (4) $(\zeta - \delta, \zeta + \delta)$, (5) continuity, (6) $f(\zeta)$, and (7) $-\left(f'(\zeta)\right)[f(\zeta)]^{-2}$.

The next result offers a different way to think of the difference quotient.

Theorem 6.1.19 (Fundamental Lemma of Differentiation) *Suppose that f is differentiable at x_0. Then there exists a function η defined on an open interval containing 0 for which $\eta(0) = 0$ and*

$$f(x_0 + h) - f(x_0) = [f'(x_0) + \eta(h)] \cdot h \tag{6.3}$$

and η is continuous at 0.

Before looking at the proof take a few moments to reflect on what you can say about

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)$$

for $|h| > 0$.

Proof. Suppose that $\delta > 0$ is such that f is defined in $|x - x_0| < \delta$ and let

$$\eta(h) = \begin{cases} \dfrac{1}{h}\left[f(x_0 + h) - f(x_0)\right] - f'(x_0) & \text{, if } 0 < |h| < \delta \\ 0 & \text{, if } h = 0 \end{cases}.$$

Because f is differentiable at x_0, it follows from the limit of the sum theorem that $\lim_{h \to 0} \eta(h) = 0$. Since $\eta(0) = 0$, we conclude that η is continuous at 0. Finally, solving $\eta = \dfrac{1}{h}\left[f(x_0 + h) - f(x_0)\right] - f'(x_0)$ for $f(x_0 + h) - f(x_0)$ yields (6.3). ■

Remark 6.1.20 *If f is differentiable at x_0, then*

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

for h very small; i.e., the function near to x_0 is approximated by a linear function whose slope is $f'(x_0)$.

Next, we will use the Fundamental Lemma of Differentiation to obtain the derivative of the composition of differentiable functions.

Theorem 6.1.21 (Chain Rule) *Suppose that g and u are functions on $\mathbb{R}$ and that $f(x) = g(u(x))$. If u is differentiable at x_0 and g is differentiable at $u(x_0)$, then f is differentiable at x_0 and*

$$f'(x_0) = g'(u(x_0)) \cdot u'(x_0).$$

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Before reviewing the offered proof, look at the following and think about what prompted the indicated rearrangement; What should be put in the boxes to enable us to relate to the given information?

We want to consider

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\\
&= \lim_{h\to 0}\frac{g(u(x_0+h))-g(u(x_0))}{h}\\
&= \lim_{h\to 0}\left(\frac{g(u(x_0+h))-g(u(x_0))}{\boxed{}}\cdot\frac{\boxed{}}{h}\right)
\end{aligned}$$

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Proof. Let $\Delta f = f(x_0+h) - f(x_0)$, $\Delta u = u(x_0+h) - u(x_0)$ and $u_0 = u(x_0)$. Then

$$\Delta f = g(u(x_0+h)) - g(u(x_0)) = g(u_0 + \Delta u) - g(u_0).$$

Because u is continuous at x_0, we know that $\lim_{h\to 0}\Delta u = 0$. By the Fundamental Lemma of Differentiation, there exists a function η, with $\eta(0) = 0$, that is continuous at 0 and is such that $\Delta f = [g'(u_0) + \eta(\Delta u)]\Delta u$. Hence,

$$\lim_{h\to 0}\frac{\Delta f}{h} = \lim_{h\to 0}\left([g'(u_0) + \eta(\Delta u)]\frac{\Delta u}{h}\right) = g'(u_0)\,u'(x_0)$$

from the limit of the sum and limit of the product theorems. ■

6.1.1 Formulas for Differentiation

As a consequence of the results in this section, we can justify the differentiation of all polynomials and rational functions. From Excursion 6.1.12, we know that the formula given in the Properties of Derivatives Theorem (f) is valid for $n = \frac{1}{2}$. In fact, it is valid for all nonzero real numbers. Prior to the Chain Rule, the only way to find the derivative of $f(x) = \left(x^3 + \left(3x^2 - 7\right)^{12}\right)^8$, other than appeal to the definition, was to expand the expression and apply the Properties of Derivatives Theorem, parts (a), (b), (c) and (f); in view of the Chain Rule and the Properties of Derivatives Theorem, we have

$$f'(x) = 8\left(x^3 + \left(3x^2 - 7\right)^{12}\right)^7 \left[3x^2 + 72x\left(3x^2 - 7\right)^{11}\right].$$

What we don't have yet is the derivatives of functions that are not realized as algebraic combinations of polynomials; most notably this includes the trigonometric functions, the inverse trig functions, α^x for any fixed positive real number α, and the logarithm functions.

For any $x \in \mathbb{R}$, we know that

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{\sin(h)\cos(x) + \cos(h)\sin x - \sin x}{h} \\ &= \lim_{h\to 0}\left[(\cos x)\left(\frac{\sin(h)}{h}\right) + (\sin x)\left(\frac{\cos(h) - 1}{h}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} &= \lim_{h\to 0} \frac{\cos(h)\cos(x) - \sin(h)\sin x - \cos x}{h} \\ &= \lim_{h\to 0}\left[(\cos x)\left(\frac{\cos(h) - 1}{h}\right) - (\sin x)\left(\frac{\sin(h)}{h}\right)\right].\end{aligned}$$

Consequently, in view of the limit of the sum and limit of the product theorems, finding the derivatives of the sine and cosine functions depends on the existence of $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$ and $\lim_{h\to 0}\left(\frac{\cos(h) - 1}{h}\right)$. Using elementary geometry and trigonometry, it can be shown that the values of these limits are 1 and 0, respectively. An outline for the proofs of these two limits, which is a review of what is shown in an elementary calculus course, is given as an exercise. The formulas for the derivatives of the other trigonometric functions follow as simple applications of the Properties of Derivatives.

Recall that $e = \lim\limits_{\zeta\to 0}(1+\zeta)^{1/\zeta}$ and $y = \ln x \Leftrightarrow x = e^y$. With these in addition to basic properties of logarithms, for x a positive real,

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} &= \lim_{h\to 0}\left[\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right]\\ &= \lim_{h\to 0}\left[\ln\left(1+\frac{h}{x}\right)^{1/h}\right].\end{aligned}$$

Keeping in mind that x is a constant, it follows that

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} &= \lim_{h\to 0}\left[\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}\right]\\ &= \frac{1}{x}\lim_{h\to 0}\left[\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]\right]\end{aligned}$$

Because $\left[\left(1+\frac{h}{x}\right)^{x/h}\right] \longrightarrow e$ as $h \longrightarrow 0$ and $\ln(e) = 1$, the same argument that was used for the proof of Theorem 5.2.11 allows us to conclude that

$$\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \frac{1}{x}.$$

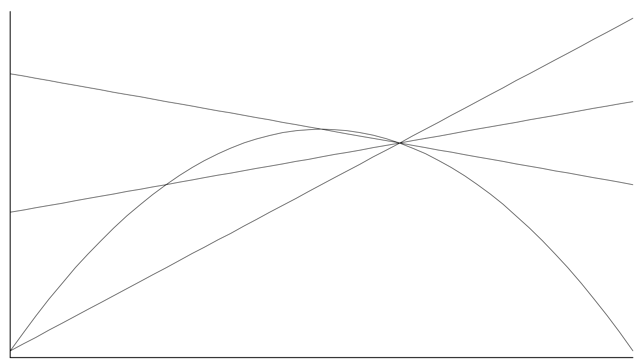
Formulas for the derivatives of the inverse trigonometric functions and α^x, for any fixed positive real number α, will follow from the theorem on the derivative of the inverses of a function that is proved at the end of this chapter.

6.1.2 Revisiting A Geometric Interpretation for the Derivative

Completing the following figure should serve as a nice reminder of one of the common interpretations and applications of the derivative of a function at the point.

- On the x-axis, label the x-coordinate of the common point of intersection of the curve, $f(x)$, and the three indicated lines as c.

- Corresponding to each line–ℓ_1, ℓ_2, and ℓ_3, on the x-axis label the x-coordinate of the common point of intersection of the curve, $f(x)$, with the line as $c+h_1$, $c+h_2$, and $c+h_3$ in ascending order. Note that h_1, h_2 and h_3 are negative in the set-up that is shown. Each of the lines ℓ_1, ℓ_2, and ℓ_3 are called **secant lines**.
- Find the slopes m_1, m_2, and m_3, respectively, of the three lines.



Excursion 6.1.22 *Using terminology associated with the derivative, give a brief description that applies to each of the slopes m_j for $j = 1, 2, 3$.*

Excursion 6.1.23 *Give a concise well-written description of the geometric interpretation for the derivative of f at $x = c$, if it exists.*

6.2 The Derivative and Function Behavior

The difference quotient is the ratio of the change in function values to the change in arguments. Consequently, it should come as no surprise that the derivative provides information related to monotonicity of functions.

In the following, continuity on an interval $I = [a, b]$ is equivalent to having continuity on (a, b), right-hand continuity at $x = a$ and left-hand continuity at $x = b$. For right-hand continuity at $x = a$; $f(a+) = f(a)$, while left-hand continuity at $x = b$ requires that $f(b-) = f(b)$.

Definition 6.2.1 *A real valued function f on a metric space (X, d_X) has a **local maximum** at a point $p \in X$ if and only if*

$$(\exists \delta > 0)\left[(\forall q)\left(q \in N_\delta(p) \Rightarrow f(q) \leq f(p)\right)\right];$$

*the function has a **local minimum** at a point $p \in X$ if and only if*

$$(\exists \delta > 0)\left[(\forall q)\left(q \in N_\delta(p) \Rightarrow f(p) \leq f(q)\right)\right].$$

Definition 6.2.2 *A real valued function f on a metric space (X, d_X) has a **(global) maximum** at a point $p \in X$ if and only if*

$$\left[(\forall x)\left(x \in X \Rightarrow f(x) \leq f(p)\right)\right];$$

*the function has a **(global) minimum** at a point $p \in X$ if and only if*

$$\left[(\forall x)\left(q \in X \Rightarrow f(p) \leq f(x)\right)\right].$$

Theorem 6.2.3 (Interior Extrema Theorem) *Suppose that f is a function that is defined on an interval $I = [a, b]$. If f has a local maximum or local minimum at a point $x_0 \in (a, b)$ and f is differentiable at x_0, then $f'(x_0) = 0$.*

Space for scratch work or motivational picture.

Proof. Suppose that the function f is defined in the interval $I = [a, b]$, has a local maximum at $x_0 \in (a, b)$, and is differentiable at x_0. Because f has a local maximum at x_0, there exists a positive real number δ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $(\forall t)\left[t \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(t) \leq f(x_0)\right]$. Thus, for $t \in (x_0 - \delta, x_0)$,

$$\frac{f(t) - f(x_0)}{t - x_0} \geq 0 \tag{6.4}$$

while $t \in (x_0, x_0 + \delta)$ implies that

$$\frac{f(t) - f(x_0)}{t - x_0} \leq 0. \tag{6.5}$$

Because f is differential at x_0, $\lim_{t \to x_0} \dfrac{f(t) - f(x_0)}{t - x_0}$ exists and is equal to $f'(x_0)$. From (6.4) and (6.5), we know that $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$, respectively. The Trichotomy Law yields that $f'(x_0) = 0$. ■

The Generalized Mean-Value Theorem that follows the next two results contains Rolle's Theorem and the Mean-Value Theorem as special cases. We offer the results in this order because it is easier to appreciate the generalized result after reflecting upon the geometric perspective that is offered by the two lemmas.

Lemma 6.2.4 (Rolle's Theorem) *Suppose that f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$. If $f(a) = f(b)$, then there is a number $x_0 \in I^\circ$ such that $f'(x_0) = 0$.*

Space for scratch work or building intuition via a typical picture.

Proof. If f is constant, we are done. Thus, we assume that f is not constant in the interval (a, b). Since f is continuous on I, by the Extreme Value Theorem, there exists points ζ_0 and ζ_1 in I such that

$$f(\zeta_0) \leq f(x) \leq f(\zeta_1) \text{ for all } x \in I.$$

Because f is not constant, at least one of $\{x \in I : f(x) > f(a)\}$ and

$\{x \in I : f(x) < f(a)\}$ is nonempty. If $\{x \in I : f(x) > f(a)\} = (a, b)$, then $f(\zeta_0) = f(a) = f(b)$ and, by the Interior Extrema Theorem, $\zeta_1 \in (a, b)$ is such that $f'(\zeta_1) = 0$. If $\{x \in I : f(x) < f(a)\} = (a, b)$, then $f(\zeta_1) = f(a) = f(b)$, $\zeta_0 \in (a, b)$, and the Interior Extrema Theorem implies that $f'(\zeta_0) = 0$. Finally, if $\{x \in I : f(x) > f(a)\} \neq (a, b)$ and $\{x \in I : f(x) < f(a)\} \neq (a, b)$, then both ζ_0 and ζ_1 are in (a, b) and $f'(\zeta_0) = f'(\zeta_1) = 0$. ■

Lemma 6.2.5 (Mean-Value Theorem) *Suppose that f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$. Then* *there exists a number $\xi \in I^\circ$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} .$$

Excursion 6.2.6 *Use the space provided to complete the proof of the Mean-Value Theorem.*

Proof. *Consider the function F defined by*

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

as a candidate for application of Rolle's Theorem.

■

Theorem 6.2.7 (Generalized Mean-Value Theorem) *Suppose that f and F are functions that are continuous on the interval $I = [a, b]$ and differentiable on the segment I°. If $F'(x) \neq 0$ on I°, then*

(a) $F(b) - F(a) \neq 0$, *and*

(b) $(\exists \xi)\left(\xi \in I^\circ \wedge \dfrac{f(b) - f(a)}{F(b) - F(a)} = \dfrac{f'(\xi)}{F'(\xi)}\right)$.

Excursion 6.2.8 *Fill in the indicated steps in order to complete the proof of the Generalized Mean-Value Theorem.*

Proof. To complete a proof of (a), apply the Mean-Value Theorem to F.

For (b), for $x \in I$, define the function by

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)} \cdot [F(x) - F(a)].$$

It follows directly that $\varphi(a) = \varphi(b) = 0$.

■

Theorem 6.2.9 (Monotonicity Test) *Suppose that a function f is differentiable in the segment (a, b).*

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing in (a, b).

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant in (a, b).

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing in (a, b).

Excursion 6.2.10 *Fill in what is missing in order to complete the following proof of the Monotonicity Test.*

Proof. *Suppose that f is differentiable in the segment (a, b) and $x_1, x_2 \in (a, b)$ are such that $x_1 < x_2$. Then f is continuous in $[x_1, x_2]$ and* $\underset{(1)}{\rule{5cm}{0.4pt}}$

in (x_1, x_2). *From the* $\underset{(2)}{\underline{\hspace{6cm}}}$, *there exists* $\xi \in (x_1, x_2)$ *such that*

$$f'(\xi) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

If $f'(x) \geq 0$ *for all* $x \in (a, b)$, *then* $f'(\xi) \geq 0$. *Since* $x_1 - x_2 < 0$, *it follows that* $\underset{(3)}{\underline{\hspace{6cm}}}$; *i.e.,* $f(x_1) \leq f(x_2)$. *Since* x_1 *and* x_2 *were arbitrary, we have that*

$$(\forall x_1)(\forall x_2)\left[\left(x_1, x_2 \in (a, b) \wedge \underset{(4)}{\underline{\hspace{3cm}}}\right) \Rightarrow f(x_1) \leq f(x_2)\right].$$

Hence, f *is* $\underset{(5)}{\underline{\hspace{6cm}}}$ *in* (a, b).

If $f'(x) = 0$ *for all* $x \in (a, b)$, *then*

$$\underset{(6)}{\underline{\hspace{9cm}}}.$$

Finally, if $f'(x) \leq 0$ *for all* $x \in (a, b)$,

$$\underset{(7)}{\underline{\hspace{9cm}}} \quad \blacksquare$$

Acceptable responses are: (1) differentiable, (2) Mean-Value Theorem, (3) $f(x_1) - f(x_2) \leq 0$, (4) $x_1 < x_2$, (5) monotonically increasing, (6) $f(x_1) - f(x_2) = 0$; i.e., $f(x_1) = f(x_2)$. Since x_1 and x_2 were arbitrary, we have that f is constant throughout (a, b)., (7) then $f'(\xi) \leq 0$ and $x_1 - x_2 < 0$ implies that $f(x_1) - f(x_2) \geq 0$; i.e., $f(x_1) \geq f(x_2)$. Because x_1 and x_2 were arbitrary we conclude that f is monotonically decreasing in (a, b).

Example 6.2.11 *Discuss the monotonicity of* $f(x) = 2x^3 + 3x^2 - 36x + 7$.

For $x \in \mathbb{R}$, $f'(x) = 6x^2 + 6x - 36 = 6(x + 3)(x - 2)$. *Since* f' *is positive in* $(-\infty, -3)$ *and* $(2, \infty)$, f *is monotonically increasing there, while* f' *negative in* $(-3, 2)$ *yields that* f *is monotonically decreasing in that segment.*

Remark 6.2.12 *Actually, in each of open intervals* $(-\infty, -3)$, $(2, \infty)$, *and* $(-3, 2)$ *that were found in Example 6.2.11, we have strict monotonicity; i.e., for* $x_1, x_2 \in (-\infty, -3)$ *or* $x_1, x_2 \in (2, \infty)$, $x_1 < x_2$ *implies that* $f(x_1) < f(x_2)$, *while* $x_1, x_2 \in (-3, 2)$ *and* $x_1 < x_2$ *yields that* $f(x_1) > f(x_2)$.

6.2.1 Continuity (or Discontinuity) of Derivatives

Given a real-valued function f that is differentiable on a subset Ω of $\mathbb{R}$, the derivative $F = f'$ is a function with domain Ω. We have already seen that F need not be continuous. It is natural to ask if there are any nice properties that can be associated with the derivative. The next theorem tells us that the derivative of a real function that is differentiable on an interval satisfies the intermediate value property there.

Theorem 6.2.13 *Suppose that* f *is a real valued function that is differentiable on* $[a, b]$ *and* $f'(a) < f'(b)$. *Then for any* $\lambda \in \mathbb{R}$ *such that* $f'(a) < \lambda < f'(b)$, *there exists a point* $x \in (a, b)$ *such the* $f'(x) = \lambda$.

Proof. *Suppose that* f *is a real valued function that is differentiable on* $[a, b]$ *and* $\lambda \in \mathbb{R}$ *is such that* $f'(a) < \lambda < f'(b)$. *Let* $G(t) = f(t) - \lambda t$. *From the Properties of Derivatives,* G *is differentiable on* $[a, b]$. *By Theorem 6.1.13,* G *is continuous on* $[a, b]$ *from which the Extreme Value Theorem yields that* G *has a minimum at some* $x \in [a, b]$. *Since* $G'(a) = f'(t) - \lambda < 0$ *and* $G'(b) = f'(t) - \lambda > 0$, *there exists a* $t_1 \in (a, b)$ *and* $t_2 \in (a, b)$ *such that* $G(t_1) < G(a)$ *and* $G(t_2) < G(b)$. *It follows that neither* $(a, G(a))$ *nor* $(b, G(b))$ *is a minimum of* G *in* $[a, b]$. *Thus,* $a < x < b$. *In view of the Interior Extrema Theorem, we have that* $G'(x) = 0$ *which is equivalent to* $f'(x) = \lambda$ ■

Remark 6.2.14 *With the obvious algebraic modifications, it can be shown that the same result holds if the real valued function that is differentiable on* $[a, b]$ *satisfies* $f'(a) > f'(b)$.

Corollary 6.2.15 *If* f *is a real valued function that is differentiable on* $[a, b]$, *then* f' *cannot have any simple(first kind) discontinuities on* $[a, b]$.

Remark 6.2.16 *The corollary tells us that any discontinuities of real valued functions that are differentiable on an interval will have only discontinuities of the second kind.*

6.3 The Derivative and Finding Limits

The next result allows us to make use of derivatives to obtain some limits: It can be used to find limits in the situations for which we have been using the Limit of Almost Equal Functions and to find some limits that we have not had an easy means of finding.

Theorem 6.3.1 (L'Hôpital's Rule I) *Suppose that f and F are functions such that f' and F' exist on a segment $I = (a, b)$ and $F' \neq 0$ on I.*

(a) If $f(a+) = F(a+) = 0$ and $\left(\dfrac{f'}{F'}\right)(a+) = L$, then $\left(\dfrac{f}{F}\right)(a+) = L$.

(b) If $f(a+) = F(a+) = \infty$ and $\left(\dfrac{f'}{F'}\right)(a+) = L$, then $\left(\dfrac{f}{F}\right)(a+) = L$.

Excursion 6.3.2 *Fill in what is missing in order to compete the following proof of part (a).*

Proof. *Suppose that f and F are differentiable on a segment $I = (a, b)$, $F' \neq 0$ on I, and $f(a+) = F(a+) = 0$. Setting $f(a) = f(a+)$ and $F(a) = F(a+)$ extends f and F to functions that are $\underset{(1)}{\underline{\hspace{4cm}}}$ in $[a, b)$. With this, $F(a) = 0$ and $F'(x) \neq 0$ in I yields that $F(x)$ $\underset{(2)}{\underline{\hspace{2cm}}}$.*

Suppose that $\varepsilon > 0$ is given. Since $\left(\dfrac{f'}{F'}\right)(a+) = L$, there exists $\delta > 0$ such that $a < w < a + \delta$ implies that

$$\underset{(2)}{\underline{\hspace{8cm}}}$$

From the Generalized Mean-Value Theorem and the fact that $F(a) = f(a) = 0$, it follows that

$$\left|\frac{f(x)}{\underbrace{\hspace{1.5cm}}_{(3)}} - L\right| = \left|\frac{\overbrace{\hspace{1.5cm}}^{(4)}}{F(x) - F(a)} - L\right| = \left|\frac{\hspace{2cm}}{\underbrace{\hspace{2cm}}_{(5)}} - L\right|$$

for some ξ satisfying $a < \xi < a+\delta$. Hence, $\left|\frac{f(x)}{F(x)} - L\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\underset{(6)}{\underline{\hspace{10em}}}$. ∎

Acceptable responses are: (1) continuous, (2) $\neq 0$, (3) $\left|\frac{f'(w)}{F'(w)} - L\right| < \varepsilon$, (4) $F(x)$, (5) $f(x) - f(a)$, (6) $\frac{f'(\xi)}{F'(\xi)}$, and (7) $\lim_{x\to a^+} \frac{f(x)}{F(x)} = L$.

Proof. Proof of (b). Suppose that f and F are functions such that f' and F' exist on an open interval $I = \{x : a < x < b\}$, $F' \neq 0$ on I, $f(a+) = F(a+) = \infty$ and $\left(\frac{f'}{F'}\right)(a+) = L$. Then f and F are continuous on I and there exists $h > 0$ such that $F' \neq 0$ in $I_h = \{x : a < x < a+h\}$. For $\epsilon > 0$, there exists a δ with $0 < \delta < h$ such that

$$\left|\frac{f'(\xi)}{F'(\xi)} - L\right| < \frac{\epsilon}{2} \text{ for all } \xi \text{ in } I_\delta = \{x : a < x < a+\delta\}.$$

Let x and c be such that $x < c$ and $x, c \in I_\delta$. By the Generalized Mean-Value Theorem, there exists a ξ in I_δ such that $\frac{f(x)-f(c)}{F(x)-F(c)} = \frac{f'(\xi)}{F'(\xi)}$. Hence,

$$\left|\frac{f(x)-f(c)}{F(x)-F(c)} - L\right| < \frac{\epsilon}{2}.$$

In particular, for $\epsilon < 1$, we have that

$$\left|\frac{f(x)-f(c)}{F(x)-F(c)}\right| = \left|\frac{f(x)-f(c)}{F(x)-F(c)} - L + L\right| < |L| + \frac{1}{2}.$$

With a certain amount of playing around we claim that

$$\begin{aligned}\left|\frac{f(x)}{F(x)} - \frac{f(x)-f(c)}{F(x)-F(c)}\right| &= \left|\frac{f(c)}{F(x)} - \frac{F(c)}{f(x)} \cdot \frac{f(x)-f(c)}{F(x)-F(c)}\right| \\ &\leq \left|\frac{f(c)}{F(x)}\right| - \left|\frac{F(c)}{f(x)}\right| \left(|L| + \frac{1}{2}\right).\end{aligned}$$

For c fixed, $\frac{f(c)}{F(x)} \to 0$ and $\frac{F(c)}{f(x)} \to 0$ as $x \to a^+$. Hence, there exists δ_1, $0 < \delta_1 < \delta$, such that

$$\left|\frac{f(c)}{F(x)}\right| < \frac{\epsilon}{4} \text{ and } \left|\frac{F(c)}{f(x)}\right| < \frac{1}{4(|L|+1/2)}\ .$$

Combining the inequalities leads to

$$\left|\frac{f(x)}{F(x)} - L\right| \leq \left|\frac{f(x)}{F(x)} - \frac{f(x) - f(c)}{F(x) - F(c)}\right| + \left|\frac{f(x) - f(c)}{F(x) - F(c)} - L\right| < \epsilon$$

whenever $a < x < a + \delta_1$. Since $\epsilon > 0$ was arbitrary, we conclude that

$$\left(\frac{f}{F}\right)(a+) = \lim_{x \to a^+} \frac{f(x)}{F(x)} = L.$$

■

Remark 6.3.3 *The two statements given in L'Hôpital's Rule are illustrative of the set of such results. For example, the* $x \to a^+$ *can be replaced with* $x \to b^-$*,* $x \to +\infty$*,* $x \to \infty$*, and* $x \to -\infty$*, with some appropriate modifications in the statements. The following statement is the one that is given as Theorem 5.13 in our text.*

Theorem 6.3.4 (L'Hôpital's Rule II) *Suppose* f *and* g *are real and differentiable in* (a, b)*, where* $-\infty \leq a < b \leq \infty$*,* $g'(x) \neq 0$ *for all* $x \in (a, b)$*, and* $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$*. If* $\lim_{x \to a} f(x) = 0 \wedge \lim_{x \to a} g(x) = 0$ *or* $\lim_{x \to a} g(x) = +\infty$*, then* $\lim_{x \to a} \frac{f(x)}{g(x)} = A$*.*

Excursion 6.3.5 *Use an appropriate form of L'Hôpital's Rule to find*

1. $\lim_{x \to 3} \dfrac{x^2 - 5x + 6 - 7\sin(x - 3)}{2x - 6}$.

2. $\lim_{w\to\infty}\left(1+\dfrac{1}{w-1}\right)^{w}$

Hopefully, you got -3 and e, respectively.

6.4 Inverse Functions

Recall that for a relation, S, on $\mathbb{R}$, the inverse relation of S, denoted by S^{-1}, is the set of all ordered pairs (y, x) such that $(x, y) \in S$. While a function is a relation that is single-valued, its inverse need not be single-valued. Consequently, we cannot automatically apply the tools of differential calculus to inverses of functions. What follows if some criteria that enables us to talk about "inverse functions." The first result tells us that where a function is increasing, it has an inverse that is a function.

Remark 6.4.1 *If u and v are monotonic functions with the same monotonicity, then their composition (if defined) is increasing. If u and v are monotonic functions with the opposite monotonicity, then their composition (if defined) is decreasing.*

Theorem 6.4.2 (Inverse Function Theorem) *Suppose that f is a continuous function that is strictly monotone on an interval I with $f(I) = J$. Then*

(a) J is an interval;

(b) the inverse relation g of f is a function with domain J that is continuous and strictly monotone on J; and

(c) we have $g(f(x)) = x$ for $x \in I$ and $f(g(y)) = y$ for $y \in J$.

Proof. Because the continuous image of a connected set is connected and f is strictly monotone, J is an interval. Without loss of generality, we take f to be decreasing in the interval I. Then $f(x_1) \neq f(x_2)$ implies that $x_1 \neq x_2$ and we conclude that, for each w_0 in J, there exists one and only one $\zeta_0 \in I$ such that $w_0 = f(\zeta_0)$. Hence, the inverse of f is a function and the formulas given in (c) hold. It follows from the remark above and (c) that g is strictly decreasing.

To see that g is continuous on J, let w_0 be an interior point of J and suppose that $g(w_0) = x_0$; i.e., $f(x_0) = w_0$. Choose points w_1 and w_2 in J such that $w_1 < w_0 < w_2$. Then there exist points x_1 and x_2 in I, such that $x_1 < x_0 < x_2$, $f(x_1) = w_2$ and $f(x_2) = w_1$. Hence, x_0 is an interior point of I. Now, without loss of generality, take $\epsilon > 0$ small enough that the interval $(x_0 - \epsilon, x_0 + \epsilon)$ is contained in I and define $w_1^* = f(x_0 + \epsilon)$ and $w_2^* = f(x_0 - \epsilon)$ so $w_1^* < w_2^*$. Since g is decreasing,

$$x_0 + \epsilon = g(w_1^*) \geq g(w) \geq g(w_2^*) = x_0 - \epsilon \text{ for } \quad w \text{ such that } \quad w_1^* \leq w \leq w_2^*.$$

Hence,

$$g(w_0) + \epsilon \geq g(w) \geq g(w_0) - \epsilon \text{ for } \quad w \text{ such that } \quad w_1^* \leq w \leq w_2^*.$$

Now taking δ to be the minimum of $w_2^* - w_0$ and $w_0 - w_1^*$ leads to

$$|g(w) - g(w_0)| < \epsilon \text{ whenever } |w - w_0| < \delta.$$

■

Remark 6.4.3 *While we have stated the Inverse Function Theorem in terms of intervals, please note that the term intervals can be replaced by segments (a, b) where a can be $-\infty$ and/or b can be ∞.*

In view of the Inverse Function Theorem, when we have strictly monotone continuous functions, it is natural to think about differentiating their inverses. For a proof of the general result concerning the derivatives of inverse functions, we will make use with the following partial converse of the Chain Rule.

Lemma 6.4.4 *Suppose the real valued functions F, G, and u are such that $F(x) = G(u(x))$, u is continuous at $x_0 \in \mathbb{R}$, $F'(x_0)$ exists, and $G'(u(x_0))$ exists and differs from zero. Then $u'(x_0)$ is defined and $F'(x_0) = G'(u(x_0))\, u'(x_0)$.*

Excursion 6.4.5 *Fill in what is missing to complete the following proof of the Lemma.*

Proof. *Let* $\Delta F = F(x_0+h) - F(x_0)$, $\Delta u = u(x_0+h) - u(x_0)$ *and* $u_0 = u(x_0)$. *Then*

$$\Delta F = \underset{(1)}{\underline{\hspace{6cm}}} = G(u_0 + \Delta u) - G(u_0).$$

Since u *is continuous at* x_0, *we know that* $\lim_{h \to 0} \Delta u = 0$. *By the Fundamental Lemma of Differentiation, there exists a function* η, *with* $\underset{(2)}{\underline{\hspace{3cm}}}$, *that is continuous at* 0 *and is such that* $\Delta F = \underset{(3)}{\underline{\hspace{6cm}}}$. *Hence,*

$$\frac{\Delta u}{h} = \frac{\frac{\Delta F}{h}}{[G'(u_0) + \eta(\Delta u)]}.$$

From $\lim_{h \to 0} \Delta u = 0$, *it follows that* $\eta(\Delta u) \longrightarrow 0$ *as* $h \longrightarrow 0$. *Because* $G'(u_0)$ *exists and is nonzero,*

$$u'(x_0) = \lim_{h \to 0} \frac{u(x_0+h) - u(x_0)}{h} = \lim_{h \to 0} \frac{\frac{\Delta F}{h}}{[G'(u_0) + \eta(\Delta u)]} = \frac{F'(x_0)}{G'(u_0)}.$$

Therefore, $u'(x_0)$ *exists and* $\underset{(4)}{\underline{\hspace{6cm}}}$. ■

Acceptable responses are: (1) $G(u(x_0+h)) - G(u(x_0))$, (2) $\eta(0) = 0$, (3) $[G'(u_0) + \eta(\Delta u)]\Delta u$, and (4) $F'(x_0) = G'(u_0)\, u'(x_0)$.

Theorem 6.4.6 (Inverse Differentiation Theorem) *Suppose that* f *satisfies the hypotheses of the Inverse Function Theorem. If* x_0 *is a point of* J *such that* $f'(g(x_0))$ *is defined and is different from zero, then* $g'(x_0)$ *exists and*

$$g'(x_0) = \frac{1}{f'(g(x_0))}. \tag{6.6}$$

Proof. From the Inverse Function Theorem, $f(g(x)) = x$. Taking $u = g$ and $G = f$ in Lemma 6.4.4 yields that $g'(x_0)$ exists and $f'(g(x))\, g'(x) = 1$. Since $f'(g(x_0)) \neq 0$, it follows that $g'(x_0) = \frac{1}{f'(g(x_0))}$ as needed. ■

Corollary 6.4.7 *For a fixed nonnegative real number α, let $g(x) = \alpha^x$. Then* $\operatorname{dom}(g) = \mathbb{R}$ *and, for all $x \in \mathbb{R}$, $g'(x) = \alpha^x \ln \alpha$.*

Proof. *We know that $g(x) = \alpha^x$ is the inverse of $f(x) = \log_\alpha x$ where f is a strictly increasing function with domain $(0, \infty)$ and range $(-\infty, \infty)$. Because $A = \log_\alpha B \Leftrightarrow \alpha^A = B \Leftrightarrow A \ln \alpha = \ln B$, it follows that*

$$\log_\alpha B = \frac{\ln B}{\ln \alpha}.$$

Hence

$$f'(x) = \left(\log_\alpha x\right)' = \left(\frac{\ln x}{\ln \alpha}\right)' = \frac{1}{x \ln \alpha}.$$

From the Inverse Differentiation Theorem, we have that $g'(x) = \dfrac{1}{f'(g(x))} = g(x) \ln \alpha = \alpha^x \ln \alpha$. ■

Remark 6.4.8 *Taking $\alpha = e$ in the Corollary yields that $(e^x)' = e^x$.*

In practice, finding particular inverses is usually carried out by working directly with the functions given rather than by making a sequence of substitutions.

Example 6.4.9 *Derive a formula, in terms of x, for the derivative of $y = \arctan x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.*

We know that the inverse of $u = \tan v$ is a relation that is not a function; consequently we need to restrict ourselves to a subset of the domain. Because u is strictly increasing and continuous in the segment $I = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$; the corresponding segment is $(-\infty, \infty)$. We denote the inverse that corresponds to this segment by $y = f(x) = \arctan x$. From $y = \arctan x$ if and only if $x = \tan y$, it follows directly that $\left(\sec^2 y\right) \dfrac{dy}{dx} = 1$ or $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$. On the other hand, $\tan^2 y + 1 = \sec^2 y$ with $x = \tan y$ implies that $\sec^2 y = x^2 + 1$. Therefore, $\dfrac{dy}{dx} = f'(x) = \dfrac{1}{x^2 + 1}$.

Excursion 6.4.10 *Use $f(x) = \dfrac{x}{1 - x}$ to verify the Inverse Differentiation Theorem on the segment $(2, 4)$; i.e., show that the theorem applies, find the inverse g and* *its derivative by the usual algebraic manipulations, and then verify the derivative satisfies equation (6.6)*

*******Hopefully, you thought to use the Monotonicity Test; that f is strictly increasing in $I = (2,4)$ follows immediately upon noting that $f'(x) = (1-x)^{-2} > 0$ in I. The corresponding segment $J = \left(-2, -\frac{4}{3}\right)$ is the domain for the inverse g that we seek. The usual algebraic manipulations for finding inverses leads us to solving $x = y(1-y)^{-1}$ for y. Then application of the quotient rule should have led to $g'(x) = (1+x)^{-2}$. Finally, to verify agreement with what is claimed with equation (6.6), substitute g into $f'(x) = (1-x)^{-2}$ and simplify.*******

6.5 Derivatives of Higher Order

If f is a differentiable function on a set Ω then corresponding to each $x \in \Omega$, there is a uniquely determined $f'(x)$. Consequently, f' is also a function on Ω. We have already seen that f' need not be continuous on Ω. However, if f' is differentiable on a set $\Lambda \subset \Omega$, then its derivative is a function on Λ which can also be considered for differentiability. When they exist, the subsequent derivatives are called higher order derivatives. This process can be continued indefinitely; on the other hand, we could arrive at a function that is not differentiable or, in the case of polynomials, we'll eventually obtain a higher order derivative that is zero everywhere. Note that we can speak of higher order derivatives only after we have isolated the set on which the previous derivative exists.

Definition 6.5.1 *If f is differentiable on a set Ω and f' is differentiable on a set $\Omega_1 \subset \Omega$, then the derivative of f' is denoted by f'' or $\frac{d^2 f}{dx^2}$ and is called the* ***second***

derivative of *f; if the second derivative of f is differentiable on a set $\Omega_2 \subset \Omega_1$, then the derivative of f'', denoted by f''' or $f^{(3)}$ or $\dfrac{d^3 f}{dx^3}$, is called the* ***third derivative of*** *f. Continuing in this manner, when it exists, $f^{(n)}$ denotes the n^{th} derivative of f and is given by $\left(f^{(n-1)}\right)'$.*

Remark 6.5.2 *The statement "$f^{(k)}$ exists at a point x_0" asserts that $f^{(k-1)}(t)$ is defined in a segment containing x_0 (or in a half-open interval having x_0 as the included endpoint in cases of one-sided differentiability) and differentiable at x_0. If $k > 2$, then the same two claims are true for $f^{(k-2)}$. In general, "$f^{(k)}$ exists at a point x_0" implies that each of $f^{(j)}$, for $j = 1, 2, ..., k-1$, is defined in a segment containing x_0 and is differentiable at x_0.*

Example 6.5.3 *Given $f(x) = \dfrac{3}{(5+2x)^2}$ in $\mathbb{R} - \left\{-\dfrac{5}{2}\right\}$, find a general formula for $f^{(n)}$.*

From $f(x) = 3(5+2x)^{-2}$, it follows that $f'(x) = 3\cdot(-2)(5+2x)^{-3}(2)$, $f''(x) = 3\cdot(-2)(-3)(5+2x)^{-4}\left(2^2\right)$, $f^{(3)}(x) = 3\cdot(-2)(-3)(-4)(5+2x)^{-5}\left(2^3\right)$, and $f^{(4)}(x) = 3\cdot(-2)(-3)(-4)(-5)(5+2x)^{-6}\left(2^4\right)$. Basic pattern recognition suggests that

$$f^{(n)}(x) = (-1)^n \cdot 3 \cdot 2^n \cdot (n+1)!\,(5+2x)^{-(n+2)}. \tag{6.7}$$

Remark 6.5.4 *Equation (6.7) was not proved to be the case. While it can be proved by Mathematical Induction, the set-up of the situation is direct enough that claiming the formula from a sufficient number of carefully illustrated cases is sufficient for our purposes.*

Theorem 6.5.5 (Taylor's Approximating Polynomials) *Suppose f is a real function on $[a,b]$ such that there exists $n \in \mathbb{J}$ for which $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists for every $t \in (a,b)$. For $\gamma \in [a,b]$, let*

$$P_{n-1}(\gamma; t) = \sum_{k=0}^{(n-1)} \frac{f^{(k)}(\gamma)}{k!}(t-\gamma)^k.$$

Then, for α and β distinct points in $[a,b]$, there exists a point x between α and β such that

$$f(\beta) = P_{n-1}(\alpha;\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{6.8}$$

Excursion 6.5.6 *Fill in what is missing to complete the following proof of Taylor's Approximating Polynomials Theorem.*

Proof. *Since* $P_{n-1}(\alpha;\beta)$, $(\beta-\alpha)^n$ *and* $f(\beta)$ *are fixed, we have that*

$$f(\beta) = P_{n-1}(\alpha;\beta) + M(\beta-\alpha)^n$$

for some $M \in \mathbb{R}$. *Let*

$$g(t) \underset{def}{=} f(t) - P_{n-1}(\alpha;t) - M(t-\alpha)^n .$$

Then g *is a real function on* $[a,b]$ *for which* $\underset{(1)}{\underline{\qquad\qquad\qquad}}$ *is continuous and* $g^{(n)}$ *exists in* (a,b) *because* $\underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$. *From the Properties of Derivatives, for* $t \in (a,b)$, *we have that*

$$g'(t) = f'(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-1)!}(t-\alpha)^{k-1} - nM(t-\alpha)^{n-1},$$

and

$$g''(t) = \underset{(3)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

In general, for j *such that* $1 \le j \le (n-1)$ *and* $t \in (a,b)$, *it follows that*

$$g^{(j)}(t) = f^{(j)}(t) - \sum_{k=j}^{n-1} \frac{f^{(k)}(\alpha)}{(k-j)!}(t-\alpha)^{k-j} - \frac{n!}{(n-j)!}M(t-\alpha)^{n-j}.$$

Finally,

$$g^{(n)}(t) = \underset{(4)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}. \tag{6.9}$$

Direct substitution yields that $g(\alpha) = 0$. *Furthermore, for each* j, $1 \le j \le (n-1)$, $t = \alpha$ *implies that* $\sum_{k=j}^{n-1} \dfrac{f^{(k)}(\alpha)}{(k-j)!}(t-\alpha)^{k-j} = f^{(k)}(\alpha)$; *consequently,*

$$g(\alpha) = g^{(j)}(\alpha) = 0 \textit{ for each } j, 1 \le j \le (n-1).$$

In view of the choice of M, we have that $g(\beta) = 0$. Because g is differentiable in (a, b), continuous in $[a, b]$, and $g(\alpha) = g(\beta) = 0$, by $\underset{(5)}{\underline{\hspace{6cm}}}$, *there exists x_1 between α and β such that $g'(x_1) = 0$. Assuming that $n > 1$, from Rolle's Theorem, g' differentiable in (a, b) and* $\underset{(6)}{\underline{\hspace{3cm}}}$ *in $[a, b]$ with $g'(\alpha) = g'(x_1) = 0$ for $\alpha, x_1 \in (a, b)$ yields the existence of x_2 between α and x_1 such that* $\underset{(7)}{\underline{\hspace{3cm}}}$. *If $n > 2$, Rolle's Theorem can be applied to g'' to obtain x_3 between* $\underset{(8)}{\underline{\hspace{3cm}}}$ *such that $g^{(3)}(x_3) = 0$. We can repeat this process through $g^{(n)}$, the last higher order derivative that we are assured exists. After n steps, we have that there is an x_n between α and x_{n-1} such that $g^{(n)}(x_n) = 0$. Substituting x_n into equation (6.9) yields that*

$$0 = g^{(n)}(x_n) = \underset{(9)}{\underline{\hspace{6cm}}}.$$

Hence, there exists a real number x $(= x_n)$ that is between α and β such that $f^{(n)}(x) = n!M$; i.e., $\dfrac{f^{(n)}(x)}{n!} = M$. The definition of M yields equation (6.8). ■

*******Acceptable responses are: (1) $g^{(n-1)}$, (2) g is the sum of functions having those properties, (3) $f''(t) - \sum_{k=2}^{n-1} \dfrac{f^{(k)}(\alpha)}{(k-2)!}(t-\alpha)^{k-2} - n(n-1)M(t-\alpha)^{n-2}$, (4) $f^{(n)}(t) - n!M$, (5) Rolle's Theorem or the Mean-Value Theorem, (6) continuous, (7) $g''(x_2) = 0$, (8) α and x_2, and (9) $f^{(n)}(x_n) - n!M$.*******

Remark 6.5.7 *For $n = 1$, Taylor's Approximating Polynomials Theorem is the Mean-Value Theorem. In the general case, the error from using $P_{n-1}(\alpha; \beta)$ instead of $f(\beta)$ is $\dfrac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$ for some x between α and β; consequently, we have an approximation of this error whenever we have bounds on $\left|f^{(n)}(x)\right|$.*

Example 6.5.8 *Let $f(x) = (1-x)^{-1}$ in $\left[-\dfrac{3}{4}, \dfrac{7}{8}\right]$. Then, for each $n \in \mathbb{J}$, $f^{(n)}(x) = n!(1-x)^{-(n+1)}$ is continuous in $\left[-\dfrac{3}{4}, \dfrac{7}{8}\right]$. Consequently, the hypotheses for Tay-*

lor's Approximating Polynomial Theorem are met for each $n \in \mathbb{J}$. For $n = 2$,

$$P_{n-1}(\gamma\,;t) = P_1(\gamma\,;t) = \frac{1}{1-\gamma} + \frac{1}{(1-\gamma)^2}(t-\gamma).$$

If $\alpha = \frac{1}{4}$ and $\beta = -\frac{1}{2}$, the Theorem claims the existence of $x \in \left(-\frac{1}{2}, \frac{1}{4}\right)$ such that

$$f\left(-\frac{1}{2}\right) = P_1\left(\frac{1}{4}; -\frac{1}{2}\right) + \frac{f^{(2)}(x)}{2!}\left(-\frac{1}{2} - \frac{1}{4}\right)^2.$$

Since

$$P_1\left(\frac{1}{4}; -\frac{1}{2}\right) = \frac{1}{1-\frac{1}{4}} + \frac{1}{\left(1-\frac{1}{4}\right)^2}\left(-\frac{1}{2} - \frac{1}{4}\right) = 0$$

we wish to find $x \in \left(-\frac{1}{2}, \frac{1}{4}\right)$ such that $\frac{2}{3} = 0 + \frac{1}{(1-x)^3}\left(\frac{9}{16}\right)$; the only real solution to the last equation is $x_0 = 1 - \frac{3}{2\sqrt[3]{4}}$ which is approximately equal to $.055$. *Because x_0 is between $\alpha = \frac{1}{4}$ and $\beta = -\frac{1}{2}$, this verifies the Theorem for the specified choices.*

6.6 Differentiation of Vector-Valued Functions

In the case of limits and continuity we have already justified that for functions from $\mathbb{R}$ into $\mathbb{R}^k$, properties are ascribed if and only if the property applies to each coordinate. Consequently, it will come as no surprise that the same "by co-ordinate property assignment" carries over to differentiability.

Definition 6.6.1 *A **vector-valued function** $\mathbf{f}$ from a subset Ω of $\mathbb{R}$ into $\mathbb{R}^k$ is **differentiable at a point** $\zeta \in \Omega$ if and only if $\mathbf{f}$ is defined in a segment containing ζ and there exists an element of $\mathbb{R}^k$, denoted by $\mathbf{f}'(\zeta)$, such that*

$$\lim_{t\to\zeta}\left|\frac{\mathbf{f}(t) - \mathbf{f}(\zeta)}{t-\zeta} - \mathbf{f}'(\zeta)\right| = 0$$

where $|\cdot|$ denotes the Euclidean k-metric.

Lemma 6.6.2 *Suppose that* $f_1, f_2, \ldots, f_k$ *are real functions on a subset* $\mathbf{\Omega}$ *of* $\mathbb{R}$ *and* $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ *for* $x \in \mathbf{\Omega}$. *Then* $\mathbf{f}$ *is differentiable at* $\zeta \in \mathbf{\Omega}$ *with derivative* $\mathbf{f}'(\zeta)$ *if and only if each of the functions* $f_1, f_2, \ldots, f_k$ *is differentiable at* ζ *and* $\mathbf{f}'(\zeta) = \left(f_1'(\zeta), f_2'(\zeta), \ldots, f_k'(\zeta)\right)$.

Proof. For t and ζ in $\mathbb{R}$, we have that

$$\frac{\mathbf{f}(t) - \mathbf{f}(\zeta)}{t - \zeta} - \mathbf{f}'(\zeta) = \left(\frac{f_1(t) - f_1(\zeta)}{t - \zeta} - f_1'(\zeta), \ldots, \frac{f_k(t) - f_k(\zeta)}{t - \zeta} - f_k'(\zeta)\right).$$

Consequently, the result follows immediately from Lemma 4.3.1 and the Limit of Sequences Characterization for the Limits of Functions. ■

Lemma 6.6.3 *If* $\mathbf{f}$ *is a vector-valued function from* $\mathbf{\Omega} \subset \mathbb{R}$ *into* $\mathbb{R}^k$ *that is differentiable at a point* $\zeta \in \mathbf{\Omega}$, *then* $\mathbf{f}$ *is continuous at* ζ.

Proof. Suppose that $\mathbf{f}$ is a vector-valued function from $\mathbf{\Omega} \subset \mathbb{R}$ into $\mathbb{R}^k$ that is differentiable at a point $\zeta \in \mathbf{\Omega}$. Then f is defined in a segment I containing ζ and, for $t \in I$, we have that

$$\begin{aligned}\mathbf{f}(t) - \mathbf{f}(\zeta) &= \left(\frac{f_1(t) - f_1(\zeta)}{t - \zeta}(t - \zeta), \ldots, \frac{f_k(t) - f_k(\zeta)}{t - \zeta}(t - \zeta)\right)\\ &\longrightarrow \left(f_1'(\zeta) \cdot 0, f_2'(\zeta) \cdot 0, \ldots, f_k'(\zeta) \cdot 0\right) \text{ as } t \longrightarrow \zeta.\end{aligned}$$

Hence, for each $j \in \mathbb{J}$, $1 \le j \le k$, $\lim_{t \to \zeta} f_j(t) = f_j(\zeta)$; i.e., each f_j is continuous at ζ. From Theorem 5.2.10(a), it follows that $\mathbf{f}$ is continuous at ζ. ■

We note that an alternative approach to proving Lemma 6.6.3 simply uses Lemma 6.6.2. In particular, from Lemma 6.6.2, $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ differentiable at ζ implies that f_j is differentiable at ζ for each j, $1 \le j \le k$. By Theorem 6.1.13, f_j is continuous at ζ for each j, $1 \le j \le k$, from which Theorem 5.2.10(a) allows us to conclude that $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ is continuous at ζ.

Lemma 6.6.4 *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are vector-valued functions from* $\mathbf{\Omega} \subset \mathbb{R}$ *into* $\mathbb{R}^k$ *that are differentiable at a point* $\zeta \in \mathbf{\Omega}$, *then the sum and inner product are also differentiable at* ζ.

Proof. Suppose that $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ and $\mathbf{g}(x) = (g_1(x), g_2(x), \ldots, g_k(x))$ are vector-valued functions from $\mathbf{\Omega} \subset \mathbb{R}$ into $\mathbb{R}^k$ that are differentiable at a point $\zeta \in \mathbf{\Omega}$. Then

$$(\mathbf{f} + \mathbf{g})(x) = ((f_1 + g_1)(x), (f_2 + g_2)(x), \ldots, (f_k + g_k)(x))$$

and

$$(\mathbf{f} \bullet \mathbf{g})(x) = ((f_1 g_1)(x), (f_2 g_2)(x), ..., (f_k g_k)(x)).$$

From the Properties of Derivatives (c) and (d), for each $j \in \mathbb{J}$, $1 \leq j \leq k$, $\left(f_j + g_j\right)$ and $\left(f_j g_j\right)$ are differentiable at ζ with $\left(f_j + g_j\right)'(\zeta) = f_j'(\zeta) + g_j'(\zeta)$ and $\left(f_j g_j\right)'(\zeta) = f_j'(\zeta) g_j(\zeta) + f_j(\zeta) g_j'(\zeta)$. From Lemma 6.6.2, it follows that $(\mathbf{f} + \mathbf{g})$ is differentiable at ζ with

$$(\mathbf{f} + \mathbf{g})(\zeta) = \left(\mathbf{f}' + \mathbf{g}'\right)(\zeta) = \left(\left(f_1' + g_1'\right)(\zeta), \left(f_2' + g_2'\right)(\zeta), ..., \left(f_k' + g_k'\right)(\zeta)\right)$$

and $(\mathbf{f} \bullet \mathbf{g})$ is differentiable at ζ with

$$(\mathbf{f} \bullet \mathbf{g})'(\zeta) = \left(\mathbf{f}' \bullet \mathbf{g}\right)(\zeta) + \left(\mathbf{f} \bullet \mathbf{g}'\right)(\zeta).$$

■

The three lemmas might prompt an unwarranted leap to the conclusion that all of the properties that we have found for real-valued differentiable functions on subsets of $\mathbb{R}$ carry over to vector-valued functions on subsets of $\mathbb{R}$. A closer scrutiny reveals that we have not discussed any results for which the hypotheses or conclusions either made use of or relied on the linear ordering on $\mathbb{R}$. Since we loose the existence of a linear ordering when we go to $\mathbb{R}^2$, it shouldn't be a shock that the Mean-Value Theorem does not extend to the vector-valued functions from subsets of $\mathbb{R}$ to $\mathbb{R}^2$.

Example 6.6.5 *For $x \in \mathbb{R}$, let $\mathbf{f}(x) = (\cos x, \sin x)$. Show that there exists an interval $[a, b]$ such that $\mathbf{f}$ satisfies the hypotheses of the Mean-Value Theorem without yielding the conclusion.*

From Lemma 6.6.2 and Lemma 6.6.3, we have that $\mathbf{f}$ is differentiable in (a, b) and continuous in $[a, b]$ for any $a, b \in \mathbb{R}$ such that $a < b$. Since $\mathbf{f}(0) = \mathbf{f}(2\pi) = (1, 0)$, $\mathbf{f}(2\pi) - \mathbf{f}(0) = (0, 0)$. Because $\mathbf{f}'(x) = (-\sin x, \cos x)$, $\left|\mathbf{f}'(x)\right| = 1$ for each $x \in (0, 2\pi)$. In particular, $(\forall x \in (0, 2\pi))\left(\mathbf{f}'(x) \neq (0, 0)\right)$ from which we see that $(\forall x \in (0, 2\pi))\left(\mathbf{f}(2\pi) - \mathbf{f}(0) \neq (2\pi - 0)\mathbf{f}'(x)\right)$; i.e.,

$$\neg (\exists x)\left[x \in (0, 2\pi) \wedge \left(\mathbf{f}(2\pi) - \mathbf{f}(0) = (2\pi - 0)\mathbf{f}'(x)\right)\right].$$

Remark 6.6.6 *Example 5.18 in our text justifies that L'Hôpital's Rule is also not valid for functions from $\mathbb{R}$ into $\mathbb{C}$.*

When we justify that a result known for real-valued differentiable functions on subsets of $\mathbb{R}$ does not carry over to vector-valued functions on subsets of $\mathbb{R}$, it is natural to seek modifications of the original results in terms of properties that might carry over to the different situation. In the case of the Mean-Value Theorem, success in achieved with an inequality that follows directly from the theorem. From the Mean-Value Theorem, if f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$, then there exists a number $\xi \in I^\circ$ such that $f(b) - f(a) = f'(\xi)(b - a)$. Since $\xi \in I^\circ$, $\left|f'(\xi)\right| \le \sup_{x \in I^\circ} \left|f'(x)\right|$. This leads to the weaker statement that $|f(b) - f(a)| \le |b - a| \sup_{x \in I^\circ} \left|f'(x)\right|$. On the other hand, this statement has a natural candidate for generalization because the absolute value or Euclidean 1-metric can be replaced with the Euclidean k-metric. We end this section with a proof of a vector-valued adjustment of the Mean-Value Theorem.

Theorem 6.6.7 *Suppose that* $\mathbf{f}$ *is a continuous mapping of* $[a, b]$ *into* $\mathbb{R}^k$ *that is differentiable in* (a, b)*. Then there exists* $x \in (a, b)$ *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)\left|\mathbf{f}'(x)\right| \tag{6.10}$$

Proof. Suppose that $\mathbf{f} = (f_1, f_2)$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ that is differentiable in (a, b) and let $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$. Equation 6.10 certainly holds if $\mathbf{z} = (\mathbf{0}, \mathbf{0})$; consequently, we suppose that $\mathbf{z} \ne (\mathbf{0}, \mathbf{0})$. By Theorem 5.2.10(b) and Lemma 6.6.4, the real-valued function

$$\phi(t) = \mathbf{z} \bullet \mathbf{f}(t) \text{ for } t \in [a, b]$$

is continuous in $[a, b]$ and differentiable in (a, b). Applying the Mean-Value Theorem to ϕ, we have that there exists $x \in (a, b)$ such that

$$\phi(b) - \phi(a) = \phi'(x)(b - a). \tag{6.11}$$

Now,

$$\begin{aligned}
\phi(b) - \phi(a) &= \mathbf{z} \bullet \mathbf{f}(b) - \mathbf{z} \bullet \mathbf{f}(a) \\
&= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(b) - (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(a) \\
&= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet (\mathbf{f}(b) - \mathbf{f}(a)) \\
&= \mathbf{z} \bullet \mathbf{z} = |\mathbf{z}|^2.
\end{aligned}$$

For $z_1 = (f_1(b) - f_1(a))$ and $z_2 = (f_2(b) - f_2(a))$,

$$\begin{aligned} |\phi(x)| &= \left|\mathbf{z} \bullet \mathbf{f}'(x)\right| = \left|z_1 f_1'(x) + z_2 f_2'(x)\right| \\ &\le \sqrt{|z_1| + |z_2|}\sqrt{\left|f_1'(x)\right| + \left|f_2'(x)\right|} = |\mathbf{z}|\left|\mathbf{f}'(x)\right| \end{aligned}$$

by Schwarz's Inequality. Substituting into equation (6.11) yields

$$|\mathbf{z}|^2 = (b-a)\left|\mathbf{z} \bullet \mathbf{f}'(x)\right| \le (b-a)\,|\mathbf{z}|\left|\mathbf{f}'(x)\right|$$

which implies $|\mathbf{z}| \le (b-a)\left|\mathbf{f}'(x)\right|$ because $|\mathbf{z}| \ne 0$. ■

6.7 Problem Set F

1. Use the definition to determine whether or not the given function is differentiable at the specified point. When it is differentiable, give the value of the derivative.

   (a) $f(x) = x^3$; $x = 0$

   (b) $f(x) = \begin{cases} x^3 & \text{, for } 0 \le x \le 1 \\ \sqrt{x} & \text{, for } x > 1 \end{cases}$; $x = 1$

   (c) $f(x) = \begin{cases} \sqrt{x}\sin\dfrac{1}{x} & \text{, for } x \ne 0 \\ 0 & \text{, for } x = 0 \end{cases}$; $x = 0$

   (d) $f(x) = \dfrac{9}{2x^2+1}$; $x = 2$

2. Prove that, if f and g are differentiable at ζ, then $F(x) = (f+g)(x)$ is differentiable at ζ and $F'(\zeta) = f'(\zeta) + g'(\zeta)$.

3. Use the definition of the derivative to prove that $f(x) = x^n$ is differentiable on $\mathbb{R}$ for each $n \in \mathbb{J}$.

4. Let $f(x) = \begin{cases} x^2 & \text{, for } x \in \mathbb{Q} \\ 0 & \text{, for } x \notin \mathbb{Q} \end{cases}$.

   Is f differentiable at $x = 0$? Carefully justify your position.

5. If f is differentiable at ζ, prove that

$$\lim_{h\to 0}\frac{f(\zeta+\alpha h)-f(\zeta-\beta h)}{h}=(\alpha+\beta)\,f'(\zeta).$$

6. Discuss the differentiability of the following functions on $\mathbb{R}$.

   (a) $f(x)=|x|+|x+1|$

   (b) $f(x)=x\cdot|x|$

7. Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is differentiable at a point $c\in\mathbb{R}$. Given any two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n\neq b_n$ for each $n\in\mathbb{J}$ and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=c$, is it true that

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(c)?$$

   State your position and carefully justify it.

8. Use the Principle of Mathematical Induction to prove the Leibnitz Rule for the n^{th} derivative of a product:

$$(fg)^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)\,g^{(k)}(x)$$

   where $\binom{n}{k}=\dfrac{n!}{(n-k)!\,(k!)}$ and $f^{(0)}(x)=f(x)$.

9. Use derivative formulas to find $f'(x)$ for each of the following. Do only the obvious simplifications.

   (a) $f(x)=\dfrac{4x^6+3x-1}{\left(x^5+4x^2\left(5x^3-7x^4\right)^7\right)}$

   (b) $f(x)=\left(4x^2+\dfrac{1+2x}{\left(2+x^2\right)^3}\right)^3\left(4x^9-3x^2+10\right)^2$

   (c) $f(x)=\left(\dfrac{\left(2x^2+3x^5\right)^3+7}{14+\left(4+\sqrt{x^2+3}\right)^4}\right)^{15}$

(d) $f(x) = \left(\left(3x^5 + \frac{1}{x^5}\right)^{10} + \left(4\sqrt{7x^4+3} - 5x^2\right)^5\right)^{12}$

(e) $f(x) = \sqrt{3 + \sqrt{2 + \sqrt{1+x}}}$

10. Complete the following steps to prove that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \textit{and} \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0.$$

(a) Draw a figure that will serve as an aid towards completion of a proof that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

i. On a copy of a Cartesian coordinate system, draw a circle having radius 1 that is centered at the origin. Then pick an arbitrary point on the part of the circle that is in the first quadrant and label it P.

ii. Label the origin, the point $(1, 0)$, and the point where the line $x = P$ would intersect the x-axis,

$$O, B, \text{ and } A, \text{ respectively.}$$

iii. Suppose that the argument of the point P, in radian measure, is θ. Indicate the coordinates of the point P and show the line segment joining P to A in your diagram.

iv. If your completed diagram is correctly labelled, it should illustrate that

$$\frac{\sin\theta}{\theta} = \frac{\left|\overline{PA}\right|}{\text{length of } \overset{\frown}{PB}}$$

where $\left|\overline{PA}\right|$ denotes the length of the line segment joining points P and A and $\overset{\frown}{PB}$ denotes the arc of the unit circle from the point B to the point P.

v. Finally, the circle having radius $\left|\overline{OA}\right|$ and centered at the origin will pass through the point A and a point and a point on the ray $\overrightarrow{OP}$. Label the point of intersection with $\overrightarrow{OP}$ with the letter C and show the arc $\overset{\frown}{CA}$ on your diagram.

(b) Recall that, for a circle of radius r, the area of a sector subtended by θ radians is given by $\dfrac{\theta r^2}{2}$. Prove that

$$\frac{\theta\cos^2\theta}{2} < \frac{\cos\theta\sin\theta}{2} < \frac{\theta}{2}$$

for θ satisfying the set-up from part (a).

(c) Prove that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(d) Recall that $\sin^2\theta + \cos^2\theta = 1$. Prove that $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$.

11. The result of Problem 10 in conjunction with the discussion that was offered in the section on Formulas for Derivatives justifies the claim that, for any $x \in \mathbb{R}$, $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, where x is interpreted as radians. Use our Properties of Derivatives and trig identities to prove each of the following.

   (a) $(\tan x)' = \sec^2 x$
   (b) $(\sec x)' = \sec x \tan x$
   (c) $(\csc x)' = -\csc x \cot x$
   (d) $(\ln|\sec x + \tan x|)' = \sec x$
   (e) $(\ln|\csc x - \cot x|)' = \csc x$

12. Use derivative formulas to find $f'(x)$ for each of the following. Do only the obvious simplifications.

   (a) $f(x) = \sin^5\left(3x^4 + \cos^2\left(2x^2 + \sqrt{x^4+7}\right)\right)$
   (b) $f(x) = \dfrac{\tan^3\left(4x+3x^2\right)}{1+\cos^2\left(4x^5\right)}$
   (c) $f(x) = \left(1+\sec^3(3x)\right)^4\left(x^3 + \dfrac{3}{2x^2+1} - \tan x\right)^2$
   (d) $f(x) = \cos^3\left(x^4 - 4\sqrt{1+\sec^4 x}\right)^4$

13. Find each of the following. Use L'Hôpital's Rule when it applies.

    (a) $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan x}{x-(\pi/2)}$

    (b) $\lim_{x\to 0} \dfrac{\tan^5 x - \tan^3 x}{1-\cos x}$

    (c) $\lim_{x\to\infty} \dfrac{x^3}{e^{2x}}$

    (d) $\lim_{x\to\infty} \dfrac{4x^3+2x^2-x}{5x^3+3x^2+2x}$

    (e) $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$

    (f) $\lim_{x\to 2^+} (x-2)\ln(x-2)$

14. For $f(x) = x^3$ and $x_0 = 2$ in the Fundamental Lemma of Differentiation, show that $\eta(h) = 6h + h^2$.

15. For $f(x) = \dfrac{x+1}{2x+1}$ and $x_0 = 1$ in the Fundamental Lemma of Differentiation, find the corresponding $\eta(h)$.

16. Suppose that f, g, and h are three real-valued functions on $\mathbb{R}$ and c is a fixed real number such that $f(c) = g(c) = h(c)$ and $f'(c) = g'(c) = h'(c)$. If $\{A_1, A_2, A_3\}$ is a partition of $\mathbb{R}$, and

$$L(x) = \begin{cases} f(x) & \text{, for} \quad x \in A_1 \\ g(x) & \text{, for} \quad x \in A_2 \\ h(x) & \text{, for} \quad x \in A_3 \end{cases},$$

    prove that L is differentiable at $x = c$.

17. If the second derivative for a function f exists at $x_0 \in \mathbb{R}$, show that

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

18. For each of the following, find formulas for $f^{(n)}$ in terms of $n \in \mathbb{J}$.

(a) $f(x) = \dfrac{3}{(3x+2)^2}$

(b) $f(x) = \sin(2x)$

(c) $f(x) = \ln(4x+3)$

(d) $f(x) = e^{(5x+7)}$

19. For $f(x) = \begin{cases} e^{-x^{-2}} & \text{, for } x > 0 \\ 0 & \text{, for } x \le 0 \end{cases}$, show that $f^{(n)}(0)$ exists for each $n \in \mathbb{J}$ and is equal to 0.

20. Discuss the monotonicity of each of the following.

    (a) $f(x) = x^4 - 4x + 5$

    (b) $f(x) = 2x^3 + 3x + 5$

    (c) $f(x) = \dfrac{3x+1}{2x-1}$

    (d) $f(x) = x^3 e^{-x}$

    (e) $f(x) = (1+x)e^{-x}$

    (f) $f(x) = \dfrac{\ln x}{x^2}$

21. Suppose that f is a real-valued function on $\mathbb{R}$ for which both the first and second derivatives exist. Determine conditions on f' and f'' that will suffice to justify that the function is increasing at a decreasing rate, increasing at an increasing rate, decreasing at an increasing rate, and decreasing at a decreasing rate.

22. For a function f from a metric space X to a metric space Y, let F_f denote the inverse relation from Y to X. Prove that F_f is a function from rng f into X if and only if f is one-to-one.

23. For each of the following,

    - find the segments I_k, $k = 1, 2, \ldots$, where f is strictly increasing and strictly decreasing,

- find the corresponding segments $J_k = f(I_k)$ on which the corresponding inverses g_k of f are defined,
- graph f on one Cartesian coordinate system, and each of the corresponding inverses on a separate Cartesian coordinate system, and
- whenever possible, with a reasonable amount of algebraic manipulations, find each g_k.

(a) $f(x) = x^2 + 2x + 2$

(b) $f(x) = \dfrac{2x}{x+2}$

(c) $f(x) = \dfrac{x^2}{2} + 3x - 4$

(d) $f(x) = \sin x$ for $-\dfrac{3\pi}{2} \le x \le 2\pi$

(e) $f(x) = \dfrac{2x^3}{3} + x^2 - 4x + 1$

24. Suppose that f and g are strictly increasing in an interval I and that

$$(f - g)(x) > 0$$

for each $x \in I$. Let F and G denote the inverses of f and g, respectively, and J_1 and J_2 denote the respective domains for those inverses. Prove that $F(x) < G(x)$ for each $x \in J_1 \cap J_2$.

25. For each of the following, the Inverse Function Theorem applies on the indicated subset of $\mathbb{R}$. For each given f find the corresponding inverse g. Use the properties of derivatives to find f' and g'. Finally, the formulas for f' and g' to verify equation (6.6).

(a) $f(x) = x^3 + 3x$ for $(-\infty, \infty)$

(b) $f(x) = \dfrac{4x}{x^2+1}$ for $\left(\dfrac{1}{2}, \infty\right)$

(c) $f(x) = e^{4x}$ for $(-\infty, \infty)$

26. For $f\,(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{, for} \quad 0 < x \leq 1 \\ 0 & \text{, for} \quad x = 0 \end{cases}$, find the segments I_k, $k{-}1, 2, \ldots$, where f is strictly increasing and strictly decreasing and the corresponding segments J_k where the Inverse Function Theorem applies.

27. For each of the following, find the Taylor polynomials $P\,(t)$ as described in Taylor's Approximating Polynomials Theorem about the indicated point γ .

    (a) $f\,(x) = \dfrac{2}{5-2x}$; $\gamma\ = 1$
    (b) $f\,(x) = \sin x$; $\gamma\ = \dfrac{\pi}{4}$
    (c) $f\,(x) = e^{2x-1}$; $\gamma\ = 2$
    (d) $f\,(x) = \ln\,(4-x)$; $\gamma\ = 1$

28. For each of the following functions from $\mathbb{R}$ into $\mathbb{R}^3$, find $\mathbf{f}'$.

    (a) $\mathbf{f}\,(x) = \left(\dfrac{x^3 \sin x}{x^2+1}, x \tan\,(3x)\,, e^{2x} \cos\,(3x-4)\right)$
    (b) $\mathbf{f}\,(x) = \left(\ln\left(2x^2+3\right), \sec x, \sin^3(2x) \cos^4\left(2+3x^2\right)\right)$

29. For $\mathbf{f}\,(x) = \left(x^2+2x+2, 3x+2\right)$ in $[0, 2]$, verify equation (6.10).