UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2015 FINAL EXAMINATION

MAT240H5F -Algebra I

J. Thind

Duration: 3 hours

Aids: None

NAME (PRINT): ____

Last / Surname First / Given Name

STUDENT #: ______________________ SIGNATURE: ______________________

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*Please note, you **CANNOT** petition to **re-write** an examination once the exam has begun.*

INSTRUCTIONS

(1) There are three parts to this examination:
  PART I (30 marks): Ten (10) multiple choice questions. Each question is worth 3 marks.
  PART II (20 marks): Four (4) short answer questions. Each question is worth 5 marks.
  PART III (50 marks): Five (5) written questions. Each question is worth 10 marks.

(2) This examination has 13 different pages including this page. Make sure your copy of the examination has 13 different pages and **sign** at the top of this page. You can use page 9, 13, and its back for rough work.

Part I	Part II	Part III Q#1	Part III Q#2	Part III Q#3	Part III Q#4	Part III Q#5	**TOTAL**
/30	/20	/10	/10	/10	/10	/10	/100

PART I: True or False (30 marks) Each question is worth 3 points. Give a brief explanation of your answer, which may be simply a counterexample if the statement is false.

In this section of the test V, W denote finite dimensional vector spaces over a field $\mathbb{F}$, unless otherwise indicated.

(1) Let $n \geq 2$. Every element in $\mathbb{Z}_n$ has an additive inverse.

TRUE FALSE

Explanation:

(2) The equation $z^{10} - 17z^8 + (1-3i)z^2 = z^5 - 3iz^4 + z + 1$ has no solutions over $\mathbb{C}$.

TRUE FALSE

Explanation:

(3) The set $\{A \in Mat_{n\times n}(\mathbb{F}) \mid tr(A + A^t) = 0\}$ is a subspace of $Mat_{n\times n}(\mathbb{F})$.

TRUE FALSE

Explanation:

(4) If $\dim V = \dim W$ and $T : V \to W$ is a one-to-one linear transformation, then T must be onto.

TRUE FALSE

Explanation:

(5) Suppose that $T : V \to W$ is a linear transformation, and $w \neq 0_W$ is a non-zero vector in W. The set $T^{-1}(w) = \{v \in V \mid T(v) = w\}$ is a subspace of V.

TRUE FALSE

Explanation:

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(6) Every square matrix can be expressed as a product of elementary matrices.

TRUE FALSE

Explanation:

(7) Let A, B be square matrices. If A is similar to B, then $c_A(x) = c_B(x)$. (A is similar to B if $QAQ^{-1} = B$ for some invertible matrix Q.)

TRUE FALSE

Explanation:

(8) Let A, B be square matrices. If $c_A(x) = c_B(x)$, then A and B are similar.

TRUE FALSE

Explanation:

(9) All square matrices can be diagonalized over $\mathbb{C}$.

TRUE FALSE

Explanation:

(10) If $T : V \to V$ is a linear transformation, and p is a polynomial such that $p(T) = 0$, then $p(x)$ must divide $c_A(x)$.

TRUE FALSE

Explanation:

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PART II: Short Answer (20 marks): Please show any relevant work. However, only your final answers will be graded. Part marks may be available when warranted.

(1) Find all solutions to the equation $z^4 = -\frac{1}{4}$. (Express your answers in the form $z = a + bi$.)

So the solutions are $z =$__________________________.

(2) Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be given by $T(p(x)) = p(x) + x^2p''(x)$. Let $\beta = \{1, x, x^2, x^3\}$ be the standard basis for $P_3(\mathbb{R})$. Find $[T]_\beta$, the matrix for T using β-coordinates.

So the matrix for T is given by $[T]=$__________________________.

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(3) What is the dimension of $N(A)$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$?

(4) Determine if the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$. If so, find a basis β for which $[A]_\beta$ is diagonal. If not, explain why not.

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PART III: Long Answer (50 marks) Please answer the following questions. Your responses require full justification and proof. Part marks are available, when warranted.

(1) Let V be a finite dimensional vector space, and $\beta = \{v_1, v_2, \dots v_n\}$ a basis for V. Prove that the expression $v = \sum_{i=1}^{n} c_i v_i$, for a vector $v \in V$, is unique.

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(2) Let V, W denote finite dimensional vector spaces over $\mathbb{F}$ of dimension n, m respectively. Let $\mathcal{L}(V, W)$ denote the set of linear transformations from V to W, and $Mat_{m\times n}$ denote the set of $m \times n$ matrices.

(a) Determine $\dim(\mathcal{L}(V, W))$ by finding a basis for $\mathcal{L}(V, W)$. (You must show that your basis, is in fact a basis.)

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(b) Use your basis in (a) to construct an isomorphism $\Phi : \mathcal{L}(V, W) \to Mat_{m\times n}$. (You must show that your map Φ is linear, one-to-one, and onto.)

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(3) Recall that a relation $\sim$ on a set S, is an equivalence relation if the following conditions hold:

(a) For every $x \in S$, $x \sim x$.

(b) If $x \sim y$, then $y \sim x$.

(c) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Define $\sim$ on $Mat_{n \times n}$ as follows: $A \sim B$ if A is similar to B (i.e. $B = QAQ^{-1}$ for some invertible matrix Q). Prove that this relation is an equivalence relation.

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(4) Let $T : V \to V$ be a linear transformation, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues, with eigenvectors $v_1, v_2, \ldots v_k$. (Note: v_i has eigenvalue λ_i.) Prove that $\{v_1, v_2, \ldots v_k\}$ is linearly independent.

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(5) Let V be a finite dimensional vector space over $\mathbb{F}$. Let $T : V \to V$ be a linear transformation.
Recall: We say a subspace W is T-invariant, if $T(W) \subseteq W$.

(a) Let λ be an eigenvalue for T. Prove that the corresponding eigenspace E_λ is T-invariant.

(b) Prove that if $\mathbb{F} = \mathbb{C}$ and $\dim V \geq 2$, then every linear transformation has at least one non-trivial invariant subspace.

(c) Is Part (b) necessarily true if we replace $\mathbb{C}$ with $\mathbb{R}$? If so, give a proof. If not, give a counterexample.

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