University of Toronto
Faculty of Arts and Science
Final Examinations, December 12, 2015
MAT240HF – Algebra I
Instructor: Stephen S. Kudla
Duration – 3 hours
No aids allowed or needed

Please write clearly and show all of your work.
The point value of each problem is indicated.

1. (*50 points*) Let $T : V \longrightarrow W$ be a linear transformation.
(i) Give the definition of the null space $N(T)$ and the range $R(T)$ of T.
(ii) Prove that $R(T)$ is a subspace of W.
(iii) Assume that $\dim V = n < \infty$. State and prove the conservation of dimension theorem.

2. (*40 points*) (i) For what values of a does the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_2 + a\,x_3 &= 2 \\ a\,x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

have more than one solution? Explain your answer.
(ii) Viewing the solutions as vectors in $\mathbb{R}^3$, what does the set of solutions look like for such an a?

3. (*60 points*) Let $P_2(F)$ be the vector space of polynomials in x of degree at most 2 with coefficients in a field F. Define a function $S : P_2(F) \longrightarrow P_2(F)$ by

$$S(f)(x) = x^2 f''(x) - f'(x) + f(x).$$

(i) Find the matrix A for S with respect to the standard basis $\beta = \{1, x, x^2\}$ for $P_2(F)$.
(ii) Show that A is invertible and find A^{-1}.
(iii) Using the results of parts (i) and (ii), solve the equation

$$x^2 f''(x) - f'(x) + f(x) = a_0 + a_1 x + a_2 x^2.$$

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MAT 244H1F
Ordinary Differential Equations
Duration: 3 hours
Instructors: A. Izosimov, B. Khesin, R. Rotman

Total marks: 50. No Aids Allowed.

1. (4 pts) Solve the initial value problem
$$2y' + (\cos t)y = -3\cos t, \quad y(0) = -4\,.$$

2. (5 pts) Find the general solution of the homogeneous equation
$$xy' = \sqrt{x^2 - y^2} + y\,, \qquad x > 0.$$

3. (7 pts) Assume that
$$N(x,y)y' + x^2 + y^2 \sin x = 0 \tag{1}$$
is an exact equation.
   (a) (3 pts) Determine the general form of the function $N(x,y)$.
   (b) (3 pts) Find the general solution to equation (1) given that $N(0,y) = -2y + 5$ for all values of y.
   (c) (1 pt) Solve the initial value problem $y(0) = -3$.

4. (6 pts) Find the general solution of the equation
$$t^2y'' - 4ty' + 6y = t^4e^t, \quad t > 0\,.$$

5. (a) (2 pts) Find the general solution of the equation $y''' + 4y'' + 4y' = 0\,.$
   (b) (2 pts) Find the value of the Wronskian $W(3)$ of solutions to the equation in 5(a), if it is known that $W(0) = 5$.
   (c) (3 pts) Find the general solution of the equation $y''' + 4y'' + 4y' = 2t^3 - 4t\,.$

6. (7 pts) Find the general solution of the system
$$\begin{cases} x' = x - y + 3\,, \\ y' = 5x - y + 7\,. \end{cases}$$

Please continue to the next page.

7. (6 pts) Find the general solution of the system

$$\begin{cases} x' = x + y - 2z\,, \\ y' = -x + 2y + z\,, \\ z' = y - z \end{cases}$$

if it is known that one of the eigenvalues of the corresponding matrix is 1.

8. (8 pts) Consider the nonlinear system

$$\begin{cases} x' = (x - 2y - 1)(y - 2)\,, \\ y' = -x + y^2 - 2\,. \end{cases}$$

(a) (2 pts) Describe the locations of all critical points.

(b) (3 pts) Classify their types (i.e. specify whether they are nodes, saddles, etc.) and stability.

(c) (2 pts) Sketch the phase portraits near the critical points.

(d) (1 pt) Sketch the phase portrait of the whole system.