- Games popular in Rome & Greece
- No science: # system did not avail

Hindus & Arabs: arithmetic system.
~~second half of~~ 1500 on.

16th century : Cardano Italian mathem
publishes book calcul prob
in games.

17 Cent Fermat & Pascal

18 Cen Bernoulli : repeat coin tosses
Leibnitz
Bayes
Legendre

19 Cent Laplace : probability as quantitative
field

> Prob is common sense reduced
> to calculation
> — Laplace

Computing
Common sense = observed behavior

Probabilistic Model

A math description of uncertainty

1. Every probabilistic model involve an underlying process called the experiment.

2. Experiment produces outcomes or primitive events

3. Sample space = all outcomes = Ω

4. A subset of the space = event.

Key point: outcomes must be distinct and mutually exclusive

So that when the experiment is carried out there is a unique outcome.

+

5. Probability law / function $P : \Omega \to [0,1]$.

likelihood of the outcome

Axioms

Nonnegativity — 1) $P(A) \geq 0$

Additivity — 2) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

$P(A_1 \cup A_2 \ldots \cup \ldots) = P(A_1) + P(A_2) \ldots$

$A_i \cap A_j = \emptyset$

Normalization — 3) $P(\Omega) = 1$.

P.1

Probabilistic Model

A mathematical description of an uncertainty.

A probability model consists of

- A sample space Ω: set of all possible outcome
- the probability law:
  assigns to a set $A \subseteq \Omega$ (an event)
  a nonnegative number $P(A)$
  called the probability of A.

$P(A)$ encodes our belief ~~that~~ of the "likelihood" of A happening

Recap

Conditional Probability

Given Ω, P,

suppose we know that outcome is within event B.

we want to quantify the likelihood of outcome belong to some other event A.

$\Rightarrow$ We want a new probability law / function which gives us the conditional probability of A given B

write $P(A \mid B)$.

Example: $P(\text{the outcome is } 3 \mid \text{the outcome is odd})$
$= 1/3$

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

$P(A \cap B)$ → both events happen at the same time

$P(B)$ → prob. of the conditioned event happening

think of this as normalization.

conditional probabilities are a probability law

see p. 19.

on the new universe B.

1. ~~P(Ω~~ $P(A \mid B) \geq 0$

~~3~~2. $P(\Omega \mid B) = \dfrac{P(\Omega \cap B)}{P(B)} = 1.$ normalized satisfied.

Normalization

2. $P(A_1 \cup A_2 \mid B) =$

$= P((A_1 \cup A_2) \cap B) \;/\; P(B)$ $(A_1 \cap B) \cap (A_2 \cap B)$ are disjoint.

$= \dfrac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$

$= P(A_1 \mid B) + P(A_2 \mid B)$

Thus books'... * All properties remain valid:

$$\boxed{\begin{aligned} &P(A \cup C) \leq P(A) + P(C) \\ &P(A \cup C \mid B) \leq P(A \mid B) + P(C \mid B) \end{aligned}}$$

P. 5

Toss a 3 coin (fair)

$A = \{\text{more heads than tails}\}$

$B = \{\text{1st toss is heads}\}$

$\Omega = \{HHH, HHT, \cancel{HTH}, \cancel{HT}$
HTH, HTT
THH, THT
$TTH, TTT\}$

$P(B) = \frac{4}{8} = 1/2$

$P(A \cap B) = P(\{\underline{H}HH, \underline{H}HT, \underline{H}TH\})$

$= 3/8$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{1/2} = 3/4$$

→ all equally likely

$$\cancel{\frac{|P(A \cap B)|}{|\Omega|}} \quad \frac{|A \cap B|}{|B|} = \frac{3}{4}$$

P.5

INDEPENDENCE

$P(A|B) = P(A)$

occurrence of B has no bearing on A

~~P(come~~

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Leftrightarrow \boxed{P(A \cap B) = P(A).P(B)}$$

If independent

If this then independent

Definition of independence works even when

$P(B) = \emptyset$ ~~(undefined)~~ then the def. undefined

$\boxed{\text{Disjoint events are never independent}}$

$P(A) \cap P(B) \Rightarrow 0$

$P(A \cap B) = \emptyset$

~~information gained~~

Example

Two rolls of dice

A

$A_i = \{ \text{1st roll is } i \}$

$B_j = \{ 2^{nd} \text{ roll is } j \}$

$$P(A_i \cap B_j) = P(\text{roll is } (i,j)) = \frac{1}{36}$$

$$P(A_i) = 6/36 = 1/6$$

$$P(B_j) = 6/36 = 1/6$$

$$P(A_i \cap B_j) = \frac{1}{36} \Rightarrow \underline{\text{independent}}$$

$\boxed{\text{no information gained}}$

B $A = \{ \text{1st roll is } 5 \}$ $\quad B = \{ \text{sum is } 7 \}$

$$P(A \cap B) = P\{(5,2)\} = 1/36$$

$$P(A) = 1/6$$

Thus independent!!

$$P(B) = P\{\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}\}$$

$\underline{6/36 = 1/6}$

p. 5-1

(C) Say a (B) sum is 10

$P(B) = P(\{(4,6), (5,5), (6,4)\})$

$3/36 = 1/12$

not independent !

(D) $A = \{\text{Min is } 2\}$

$B = \{\text{Max is } 2\}$

$\cancel{P(A) = P(\{2,}$

$P(A \cap B) = P(\text{roll is } (2,2)) = 1/36$

$P(A) =$ all − all 1's.

Show Exercise

No, they are not independent

because min is 2 $\Rightarrow$ max must

more likely be greater than 2

P.S.2

RANDOM VARIABLES

* A random variable is a real-valued function of the outcome of an experiment

$$X : \Omega \to \mathbb{R}$$

* A function of a random variable is another random variable

Example $\Omega =$ Roll dice $= \{1, - 6\}$

$$X = \begin{cases} 0 & \text{if odd} \\ 1 & \text{otherwise} \end{cases}$$

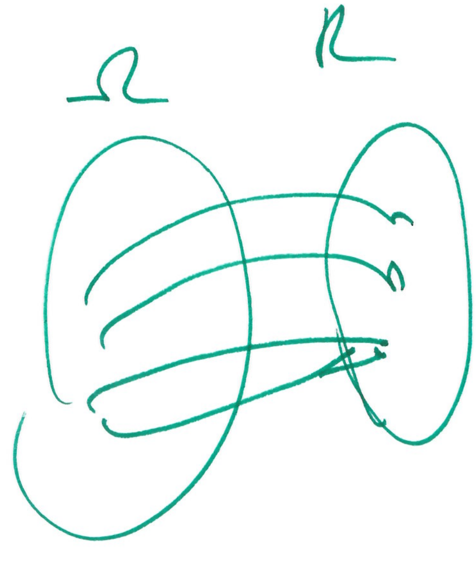
INDICATOR RANDOM VARIABLE

Ex

Range is 0 or 1.

Example Ω : two dice rolls.

$X =$ sum of the rolls.



Probability mass function (PMF)

value

$P_X(x)$ = $P(\{X = x\})$ = $P(X=x)$

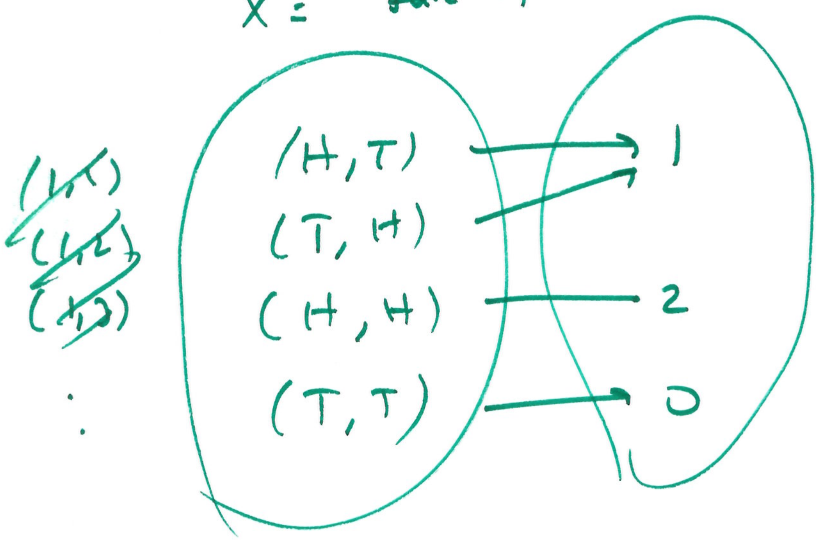
random variable

event

All outcomes that give rise to X being x.

Example Two rolls of ~~die~~ coin

X = ~~sum of the two rolls~~ # heads



$$P_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \text{ or } 2 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\text{Note} \quad \sum_x P(X=x) = 1.}$$

* X

How to calculate

For each outcome that give rise to the event $\underline{\underline{X = x}}$

Add the probability.

<u>Example</u> Bernoulli R.V.

$$X = \begin{cases} 1 & \text{if head} \\ 0 & \text{if tail} \end{cases}$$

<u>Example</u> Binomial R.V

Toss n coins.

X = # of heads in an n-toss sequence

binomial with parameter n & p

probability of head

<u>Ex</u> Geometric R.V

Coin, head $= p$. Num X of toss needed to

see <u>a head</u>. $P_X(k) = (1-p)^{k-1} \cdot p$

Example ~~X=~~ Two indep coin tosses

w/ 3/4 prob of head.

X = # of heads. (Binomial dist w/ $n=2$ $p=3/4$)

$P\{X=0) = \cancel{1/4} \; 1/4 \cdot 1/4 = 1/16$

$P(X=1) = 2(1/4 \cdot 3/4) = 3/8$

$P(X=2) = 3/4 \cdot 3/4 = 9/16$

$E[X] = 0 \cdot \frac{1}{16} + 1 \cdot 3/8 + 2 \cdot 9/16 = 3/2$

1.5 heads makes sense more than less.

Expected value Rule for Functn of Random Var

$X =$ R.V.

$g(x) =$ a function of X. (anoth random variabl)

$$E(g(x)) = \sum_x g(x) P(X=x)$$

easy

Corollary

$$Y = aX + b$$

$$E[Y] = a E[X] + b$$

Exmpl

n coin tosses

~~E[X~~

$X = \#$ of heads.

Direct $\sum \cancel{X} x \cdot P(X=x)$

$\cancel{x \in} \sum x \cdot \binom{n}{x} \left(\frac{1}{2}\right)^x \left(1-\frac{1}{2}\right)^{n-x}$

$X = X_1 + \cdots + X_n$

$E[X] = \underbrace{E[X_1]}_{0.5} + \cdots + \underbrace{E(X_n)}_{0.5}$

Diff

$E[X.Y] \stackrel{?}{=} E[X].E[Y]$

~~Head~~

~~Ω~~ 0

Coin Toss

$X =$ Head $= 1$ tail $= 0$

$Y =$ Head $= 5$ Tail $= 10$

Ω :	H	T	E
X :	1	0	$\Rightarrow 0.5$
Y :	5	10	$\Rightarrow 7.5$
$X.Y =$	5	0	~~2.5~~ 2.5

$0.5 \times 7.5 = 3.75$

$2.5 < 3.75$

No But if independent

$E[XY] = E[X].E[Y]$

independent

$X \Rightarrow$

$Y = \#$ of heads.

EXP & PROBABILITY

Markov Inequality

X: nonneg R.V X.

How much can X exceed its $E[X]$.

Since X is nonneg cannot exceed its expect frequently

$$\boxed{\Pr\left(X \geq \beta E[X]\right) \leq \frac{1}{\beta}}$$

if it did $E[X] > E[X]$ ✗

$\beta = \alpha / E[X]$

$$\boxed{\Pr\left(X \geq \alpha\right) \leq \frac{E[X]}{\alpha}}$$

Key points to remember

Review

1

$$E[X] = \sum_{x \in R(X)} (X=x) \cdot P(X=x)$$

Range of

$$E(X_1+X_2) = E(X_1)+E(X_2)$$

$$E(X \cdot Y) = E(X) \cdot E(Y) \quad \text{iff } X \,\&\, Y$$

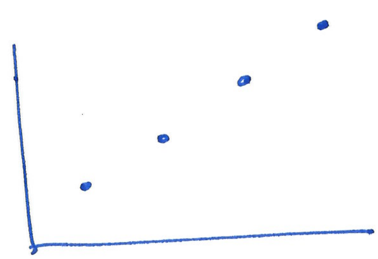
are independent

$\cancel{E(X) =}$

$$E(f(X)) \neq f(E(X)) \qquad E(f(X,Y)) = f(E(X,$$

Example $\quad f = \max$

$\cancel{E(f(X}$ $\qquad E(X^2) = (E[X])^2$



1, 2, 3, 4 $\quad E[X] = \frac{10}{4} = 2.5 \quad - (2.5)^2 = 6.25$

2 4 9 16 $\quad E(X) = \frac{31}{4} \simeq 8$

$E[X \cdot Y] = E[X] \cdot E[Y]$?

only if X & Y are independent

$E[\max(X,Y)] = ?$

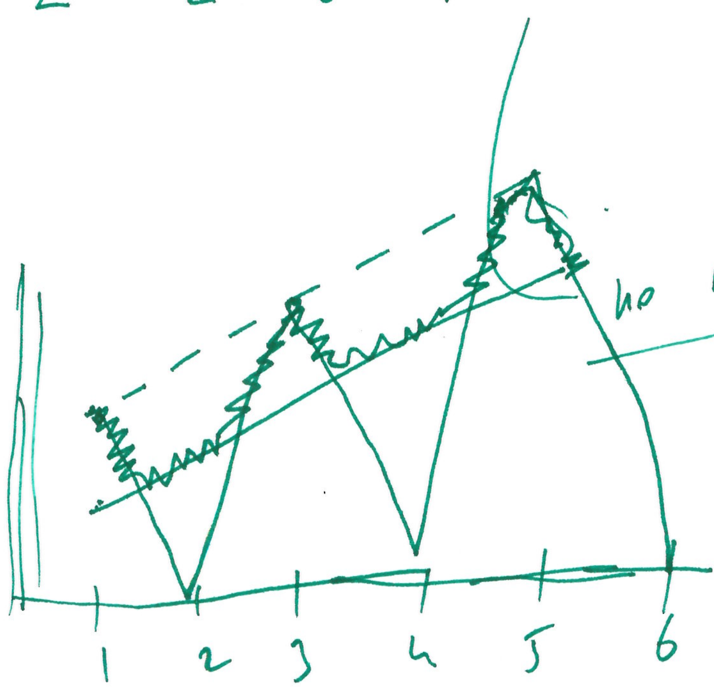
	1	2	3	4	5	6
X	1	2	3	4	5	6
Y	1	0	3	0	5	0
max(X,Y)	1	2	3	4	5	6

→ $\underline{\underline{3.5}}$

Y	1	2	3	4	5	6
2Y	2	0	6	0	10	0
	1/6	1/6	1/6	1/6	1/6	1/6
	2	2	6	4	10	6

— ? completely different

neither one nor other



no regularity

observe primitive events & map

5