LECTURE NOTES - MATH 58J (SPRING 2022)

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Contents

1. Feb 24, 2022: Introduction, $0th$ and $1st$ cohomology of open subsets of Euclidean space (ad-hoc definitions)

Consider the n dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, 1 \in [n]\},$$

where $[n] := \{1, \ldots, n\}$. Abusing notation x_i's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$ has a metric given by

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

This induces a topology on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on U and see how these can be used to analyze the topology of U.

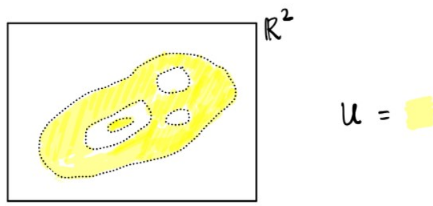

Figure 1. An example of an open subset of Euclidean space

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of U (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.
- Up to many technical details, you can intuitively think about a degree k cohomology class β on U as a way of associating a real number $\beta(Z)$ to each compact, boundariless, not necessarily

connected[1], oriented submanifold[2] Z of dimension k such that the following condition $(\star)$ holds.

> If Z is the oriented boundary of a $(k+1)-$dimensional submanifold with boundary, then $\beta(Z) = 0$.

Let's refer to such Z as "$k-$cycles" - in quotation marks because we will use this word with a different meaning later

- The main operation that one does with a differential $k-$form is to integrate them along $k-$dimensional oriented submanifolds and we use this to associate real numbers to "$k-$cycles".
- Property $(\star)$ will only hold if the differential form is closed.

1.1. **Cohomology of $U \subset \mathbb{R}^n$.**

Let $\pi_0(U)$ be the set of all connected components of U.

Definition 1. $H^0(U, \mathbb{R})$ is defined as the vector space of all maps from $\pi_0(U)$ to $\mathbb{R}$. □

Let $b \in U$ and $\pi_1(U, b)$ be the fundamental group of U with base point b. Recall that

$$\pi_1(U, b) \coloneqq \frac{\{(S^1, *) \to (U, b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where $S^1 = \frac{[0,1]}{0 \sim 1}$ and $* = [0] \in S^1$.
Here are some properties

- $\pi_1(U, b)$ is a group.
- Choosing a continuous path $\gamma : [0, 1] \to U$ from b to b' gives rise to a group isomorphism $f_\gamma : \pi_1(U, b) \to \pi_1(U, b')$.

Definition 2. Assuming that U is connected we define $H^1(U, \mathbb{R})_b$ as the vector space of group homomorphisms

$$\pi_1(U, b) \to \mathbb{R}.$$

□

Exercise 1. Prove that for any $b, b' \in U$, as long as U is connected, there is a canonical isomorphism $H^1(U, \mathbb{R})_b \to H^1(U, \mathbb{R})_{b'}$. □

As a consequence of this exercise we can write $H^1(U, \mathbb{R})$ without any ambiguity.

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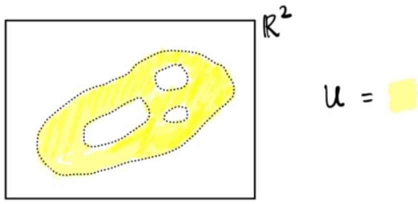
[1]I write the last two only to stress the point

[2]submanifold here means a subset that locally looks like a k dimensional Euclidean space

Example 1. Let U be defined as below.



Then, $\dim(H^1(U,\mathbb{R})) = 3$. $\square$

2. Feb 24, 2022: Degree 0 and 1 differential forms on open subsets of Euclidean space, a special case of deRham theorem

- A differential 0-form on U is a smooth[3] function $U \to \mathbb{R}$.
- A differential 0-form f is called closed if $\frac{\partial f}{\partial x_i} \equiv 0, \forall i \in [n]$

Proposition 1. *There is a canonical linear isomorphism*

$$H^0_{dR}(U) := \{\textit{closed differential 0-form on } U\} \simeq H^0(U;\mathbb{R})$$

Remark 1. In general $H^k_{dR} := \dfrac{\{\text{closed differential k-form on U}\}}{\{\text{exact differential k-form on U}\}}$ $\square$

- A differential 1-form on U is an expression $f_1dx_1 + ... + f_ndx_n$ where $f_i : U \to \mathbb{R}$ are smooth functions
- A differential 1-form $\alpha = \sum_{i=1}^{n} f_idx_i$ is called exact, if for some smooth $V : U \to \mathbb{R}$,
$$f_i = \frac{\partial V}{\partial x_i}, \forall i \in [n].$$
In this case we write $\alpha = dV$.
- A differential 1-form is closed if for all $i \neq j \in [n]$,
$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0.$$

Lemma 1. *If $\alpha = \sum_{i=1}^{n} f_idx_i$ is exact, then it is closed.*

Proof. Since it is exact, $\exists V : U \to \mathbb{R}$, such that $f_i = \frac{\partial V}{\partial x_i}$, so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

[3]note that this is a condition much stronger than differentiable, it means that all iterated partial derivatives exist. please read the wikipedia page if you are not familiar.

□

Exercise 2. For $n=2$ and $n=3$ explain what it means for the differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ to be closed in terms of the vector field $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ using terms from your calculus classes. Recall Green's and Stokes theorems. □

Theorem 1. *Assuming that U is connected, there exists a linear isomorphism,*

$$H^1_{dR} := \frac{\{\textit{closed differential 1-forms on } U\}}{\{\textit{exact differential 1-forms on } U\}} \simeq H^1(U;\mathbb{R}) \tag{1}$$

Proof sketch. First we want to define a linear map

$$\int : \{\text{closed 1-forms}\} \to \{\pi_1(U,b) \to \mathbb{R} \quad \text{group homomorphisms}\}$$

Recall: $X \subset \mathbb{R}^n$ arbitrary subset. A map $g : X \to \mathbb{R}^m$ is called smooth if it extends to a smooth map $N(X) \to \mathbb{R}^m$ where $N(X)$ is an open neighborhood of X.

Fact:

- Any class in $\pi_1(U,b)$ can be represented by a smooth map $(S^1,*) \to (U,b)$.
- Any two smooth maps $S^1 \to U$ that are homotopic continuously are homotopic smoothly.

Recall: Given $\alpha = \sum_{i=1}^{n} f_i dx_i$ and a smooth path $\gamma : [0,1] \to U$, we can define the line integral $\int_\gamma \alpha := \int_0^1 F \cdot \gamma' dt$, where $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$.

- The map $\int$ is independent of the parametrization of γ, meaning, if $\phi : [0,1] \to [0,1]$ is a smooth bijective map with $\phi' \neq 0$, then $\int_\gamma \alpha = \int_{\gamma\circ\phi} \alpha$. So line integral only depends on the image of γ.

Back to the map $\int$: we send a given $\alpha \in \{\text{closed 1-forms}\}$ to the map $\int_\cdot \alpha : \pi_1(U,b) \to \mathbb{R}$ defined by

$$a \mapsto \int_{\bar\gamma} \alpha,$$

where $\bar\gamma$ is an arbitrary smooth representative of a. In the exercise below you will show that this is well defined. Assuming that for now, it is easy to see that $\int_\cdot \alpha$ is a group homomorphism[4], so an element of $H^1(U;\mathbb{R})$, and the resulting $\int$ is a linear map.

[4] I forgot to say this in class, so please check it for yourself!

Exercise 3. Show that the map $\int$ is well defined by proving if $\alpha = \sum_{i=1}^{n} f_i dx_i$ is closed and $\gamma, \gamma' : S^1 \to U$ are smooth maps that are smoothly homotopic, then $\int_\gamma \alpha = \int_\gamma' \alpha$. (**Hint:** Start by analyzing $n = 2, 3$ and where the smooth homotopy $S^1 \times [0,1] \to U$ is injective, then reduce the statement to Green's theorem. Additionally, you may want to check proof of Stokes theorem.)

□

FIGURE 2. An example of an injective smooth homotopy's image for $n = 2$.

Goals:

(1) Prove that $\int$ sends an exact differential 1-form to zero. We obtain a linear map $\widetilde{\int} : H^1_{dR}(U, \mathbb{R}) \to H^1(U, \mathbb{R})$.
(2) Prove that $\widetilde{\int}$ is injective. ("Construct a potential")
(3) Prove that $\widetilde{\int}$ is surjective.

We start with 1). Let us integrate $\alpha = df$ along a smooth loop γ.

$$\int_\gamma \alpha = \int_0^1 \nabla f \cdot \gamma'(t) dt \overset{\text{FTC}}{=} f(b) - f(b) = 0.$$

Denote the resulting map by

$$\widetilde{\int} : \frac{\text{Closed differential 1-forms}}{\text{Exact differential 1-forms}} \to H^1(U, \mathbb{R}).$$

For 2), we need to show that if $\int_\gamma \alpha = 0$ for all $\gamma : (S^1, \star) \to (U, b)$, then $\alpha = df$ for some $f : U \to \mathbb{R}$.

Exercise 4. Do this! This is the same task as constructing a potential (recall work integrals, conservative fields etc.) for the corresponding vector field. □

Let us make a simplifying assumption on U to not deal with orthogonal difficulties in 3). Assume there exists $\gamma_1, \gamma_2, .., \gamma_n \in \pi_1(U, b)$ that freely generates the abelianization of $\pi_1(U, b)$. This implies that giving a group homomorphism $\pi_1(U, b) \to \mathbb{R}$ is equivalent to assigning real numbers to each of $\gamma_1, \gamma_2, .., \gamma_n$ (arbitrarily).

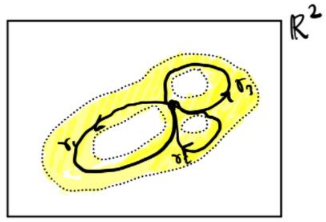


FIGURE 3

Now we need to create a differential 1-form that integrates to any $a_1, a_2, .., a_n \in \mathbb{R}$ along $\gamma_1, \gamma_2, .., \gamma_n$. This is still quite difficult. That will follow from DeRham Theorem, which will be the highlight of our course.

Remark 2. Once we give the general definition of singular cohomology, I will assign a homework exercise which shows that it agrees with what we defined today in degrees 0 and 1. The analogous statement will be automatic for DeRham cohomology. □

Exercise 5. Finish the proof of surjectivity in the case

$$U = \mathbb{R}^2 - \text{finitely many points}.$$

(**Hint:**Start with $\mathbb{R}^2 - (0, 0)$ and use the closed differential 1-form $\alpha = -\frac{ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$.) □

3. March 03, 2022: Manifolds

Riemann was looking for a class of spaces which exist by themselves, (for example, they don't have to be embedded in an Euclidean space $\mathbb{R}^N$) with the following properties[5]. Let X be such a space:

- X admits local coordinates. This means that the points x sufficiently near any $x_0 \in X$ are determined uniquely by the values of a set of real valued coordinates $x_1, x_2, \cdots, x_n$:

  $$x = (x_1, \ldots, x_n).$$

  This is sometimes called a generalized coordinate system in physics. There could be many such generalized coordinate systems near a given point. It is important that often generalized coordinates do not extend to the entirety of X.

[5]particularly vague phrases are underlined

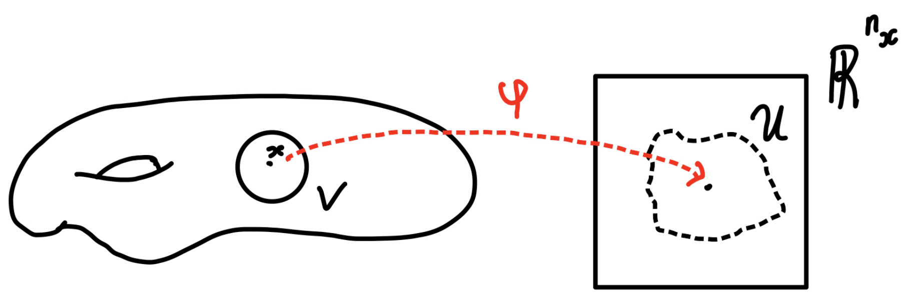


FIGURE 4

- One can use techniques of calculus. This means, in particular, that there should be a large class of $C^1/C^2/\cdots/$ smooth functions $X \to \mathbb{R}$. If two generalized coordinate systems are related to each other by a non-differentiable transformation, then a function $X \to \mathbb{R}$ that is differentiable with respect to one may not be differentiable with respect to the other.

Definition 3. A topological space X is called a topological manifold if for every $x \in X$, there exists a nonnegative integer $n_x \geq 0$, an open subset $U \subset \mathbb{R}^{n_x}$, an open neighborhood $V \subset X$ of x and a homeomorphism $\phi : V \to U$ (See Figure 4). □

Remark 3. Note that being a topological manifold is a property. □

Definition 4. Let X be a topological space. Let us call $U \subset \mathbb{R}^n$, $V \subset X$ open and $\phi : V \to U$ homeomorphism a coordinate chart in X. V is called the domain of the chart and the functions $x_1, \cdots, x_n$ obtained by $x_i : V \to U \xrightarrow{pr_i} \mathbb{R}$ the coordinates of the chart. □

Fact (A consequence of Invariance of Domain)
If an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$, then $m = n$.

Exercise 6. Using the fact above, prove that n_x in the definition is uniquely determined. Also, prove that $X \to \mathbb{Z}_{\geq 0}, x \mapsto n_x$ is constant on connected components of X. □

Definition 5. If $n_x = n$ for all $x \in X$, then we say that X is $n-$dimensional. We write this briefly by X^n. □

From now on, when we say X is a topological manifold, we assume that there is such an $n \geq 0$.

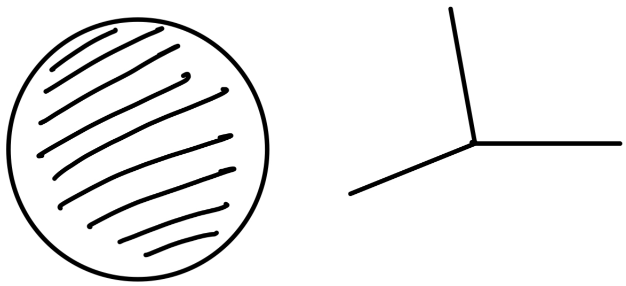


FIGURE 5. Non-Examples of Manifolds

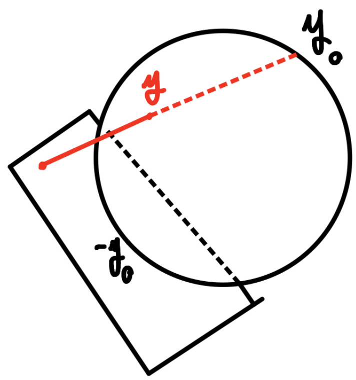


FIGURE 6. Stereographic Projection

Example 2. Topological Manifolds

- $\mathbb{R}^n$
- $S^n : \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

$\square$

To see that S^n is a topological manifold, we can use the stereographic projection. Let us consider a point $y_0 \in S^n$, and let H_0 be the hyperplane that is tangent to the point opposite to $y_0 \in S^n$ (we call this point $-y_0$). For every, $y \in S^n \setminus \{y_0\}$ the straight line l_y passing through y and y_0 intersects H_0 at precisely one point.

$$S_{y_0} : S^n \setminus \{y_0\} \longrightarrow H_0 \simeq \mathbb{R}^n$$
$$y \longmapsto l_y \cap H_0$$

Note that H_0 $\cong$Parallel hyperplane passing through the origin$\cong$ $\mathbb{R}^n$.

FIGURE 7. $n = 0$ case

The second homeomorphism can be obtained by choosing a basis.

Proposition 2. *S_{y_0} is a homeomorphism.*

Proof. (sketch)
$n = 0$ case is given in the figure. In this case the map is identity.

Exercise 7. Do the $n = 1$ case. □

We can deduce the $n > 1$ case by using rotational symmetry around the line that contains the diameter (passing through y_0 that is perpendicular to H_0). The stereographic projection in dimension n is given by spinning around l_y the stereographic projection in dimension $n-1$. □

Remark 4. Stereographic projection

- Preserves angles.
- It preserves circles ($n = 2$).
- But, it distorts distances.

□

Exercise 8. Prove that $\{x^2 - y^3 = 1\} \subset \mathbb{C}^2$ is a topological manifold. (**Hint:** Use projections to x and y.) □

Definition 6. Let X be a topological space, and $\phi_1 : V_1 \to U_1$ and $\phi_2 : V_2 \to U_2$ be coordinate charts. Then, the map $\phi_2 \circ \phi_1^{-1} : \phi_1(V_1 \cap V_2) \to \phi_2(V_1 \cap V_2)$ is called the transition map form the chart ϕ_1 to the chart ϕ_2. Note that transition maps are automatically homeomorphisms. □

Definition 7. A smooth atlas on a topological space X is a collection of charts $\{\phi_i : V_i \to U_i\}_{i \in I}$ such that
1) $\bigcup_{i \in I} V_i = X$
2) The transition map from any chart in the collection to any other in the collection is smooth. □

Remark 5. Atlas means a book of maps, i.e. images of charts $\phi : V \to U \subset \mathbb{R}^2$ on the manifold that is the surface of the earth. It is likely that some of these maps are drawn using stereographic projection. □

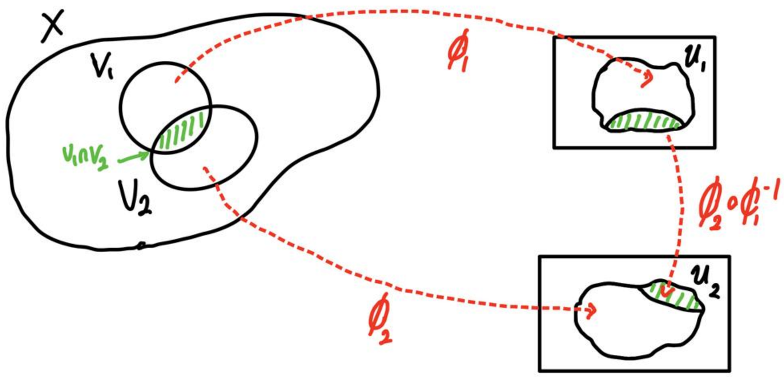


FIGURE 8. Transition map between charts

Definition 8. A smooth manifold is a topological space equipped with a maximal[6] smooth atlas. □

Exercise 9. Using latitude and longitude, define a chart on S^2 with domain $S^2 \backslash 0^{th}-$meridian and prove that it has smooth transition maps to all stereographic projections. You can assume that stereographic projection charts form a smooth atlas. □

4. March 07, 2022: Definition of singular homology

Convention: Unless otherwise stated, all of our vector spaces and chain complexes (to be defined) are over $\mathbb{R}$.

Definition 9. A graded vector space V_* is a collection of vector spaces $\{V_i\}_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$. V_* is non-negatively graded if $V_i = 0$ for $i < 0$.[7] □

Definition 10. A chain complex (C_*, ∂_*) is a graded vector space C_* with a collection of linear maps $\partial_n : C_n \to C_{n-1}$, $n \in \mathbb{Z}$, such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We call ∂_n's boundary maps. □

$$\ldots \leftarrow C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \ldots$$

[6]we did not have time to define this in the lecture and we will come back to it. it roughly means that if any chart has smooth transition maps to all the charts in the atlas, then it is contained in the atlas

[7]V=0 stands for the trivial vector space with 0 as the only element, $V = \{0\}$.

Definition 11. The homology of a chain complex (C_*, ∂_*) is a graded vector space $H_*(C_*, \partial_*)$ defined by

$$H_n\left((C_*, \partial_*)\right) := \frac{\mathrm{Ker}(\partial_n : C_n \to C_{n-1})}{\mathrm{Im}(\partial_{n+1} : C_{n+1} \to C_n)}$$

□

It immediately follows from $\partial_i \circ \partial_{i+1} = 0$ that $\mathrm{Im}(\partial_{i+1}) \subset \mathrm{Ker}(\partial_i)$.
There is a slight variant of the last two definitions.

Definition 12. C^* graded vector space with $d_n : C^n \to C^{n+1}$ coboundary maps such that $d_n \circ d_{n-1} = 0$. (C^*, d_*) is called a co-chain complex.

$$H^n\left((C^*, d_*)\right) := \frac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n-1})}$$

is called cohomology. □

$$\ldots \to C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_{-1}} C^2 \to \ldots$$

Now we move on to define the singular chain complex $C_*(X; \mathbb{R})$ of a topological manifold X.

Definition 13. The n-dimensional simplex Δ^n for $n \geq 0$ is defined as

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_i \geq 0, \quad \forall i = 0, \ldots, n \\ x_0 + \ldots + x_n = 1 \end{array} \right\}$$

□

FIGURE 9. 0-dimensional simplex.

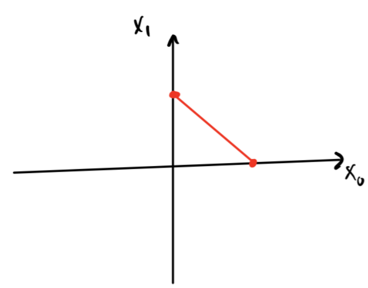


FIGURE 10. 1-dimensional simplex.

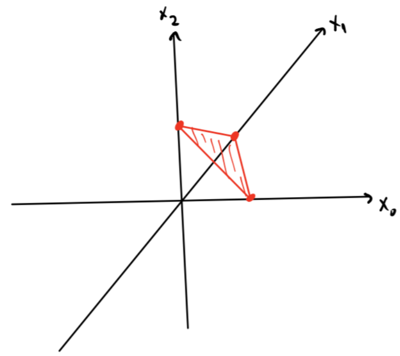


FIGURE 11. 2-dimensional simplex.

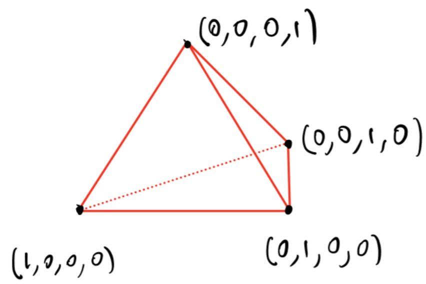


FIGURE 12. 3-dimensional simplex.

For each subset $S \subset \{0, 1, \ldots, n\}$, we can define a subset (a face) by

$$F_S := \left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_i = 0, \quad \text{if } \ i \in \{0, \ldots, n\} \setminus S \\ (x_0, \ldots, x_n) \in \Delta^n \end{array} \right\}$$

As an example, $F_{\{i\}}$, $i = 0, \ldots, n$ correspond to vertices.

Exercise 10.
- Prove that the dimension of F_S is $|S| - 1$. Explain what you mean by dimension.
- Prove that $F_{S_1} \cap F_{S_2} = F_{S_1 \cap S_2}$.

□

Definition 14. For each $n \geq 1$ and $0 \leq i \leq n$ we define the face map $f_{i,n} : \Delta^{n-1} \to \Delta^n$ with $(x_0, ..., x_{n-1}) \mapsto (y_0, ..., y_n)$ where

$$y_j = \begin{cases} x_j \ , & j < i \\ 0 \ , & j = i \\ x_{j-1} \ , & j > i \end{cases}$$

□

- This simply adds a zero to the $(i+1)$th slot.
- The image of $f_{i,n}$ is $F_{\{0,\ldots,n\}\setminus\{i\}}$

We need one last notion before we define the singular chain complex.

Definition 15. Given any set A, we define the vector space generated by A as the vector space of all finite formal linear combinations of the elements of A.

$$\left\{ \sum_{a\in A} c_a \cdot a \Big| c_a \in \mathbb{R} \text{ and } c_a \neq 0 \text{ for finitely many elements} \right\}.$$

□

Exercise 11. Construct a natural linear map from the vector space generated by A to the vector space of all maps $A \to \mathbb{R}$. Prove that this map is an isomorphism if and only if A is finite. Bonus: analyze when these two vector spaces are isomorphic - by an arbitrary map. □

Consider the subspace topology on simplices.

4.1. **Singular homology of a topological space.** Let X be a topological space. For $n \geq 0$, $C_n(X;\mathbb{R})$ is defined to be the vector space generated by the set of all continuous maps $\Delta^n \to X$. The elements of $C_n(X;\mathbb{R})$ are called singular chains of degree n. We set $C_n(X;\mathbb{R}) = 0$ for all $n < 0$. So $C_*(X;\mathbb{R})$ is a non-negatively graded vector space. Now we will equip it with boundary maps and turn it into a chain complex. Let $n \geq 1$. For any continuous $g : \Delta^n \to X$ we define

$$\partial_n g = \sum_{i=0}^{n} (-1)^i g \circ f_{i,n} \in C_{n-1}(X;\mathbb{R})$$

where $g \circ f_{i,n} : \Delta^{n-1} \xrightarrow{f_{i,n}} \Delta^n \xrightarrow{g} X$. We then extend to all singular chains so that map is linear and we get

$$\partial_n : C_n(X;\mathbb{R}) \to C_{n-1}(X;\mathbb{R}) \text{ for } n \geq 1$$

and $\partial_n = 0$ for $n < 1$.

Example 3. As an example, consider the following figure □

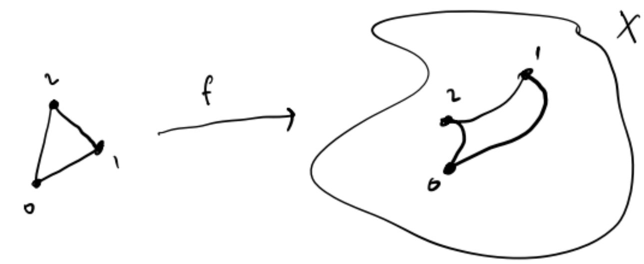


FIGURE 13. Example

Now we look at the boundary maps $\partial_2 f$ of f.

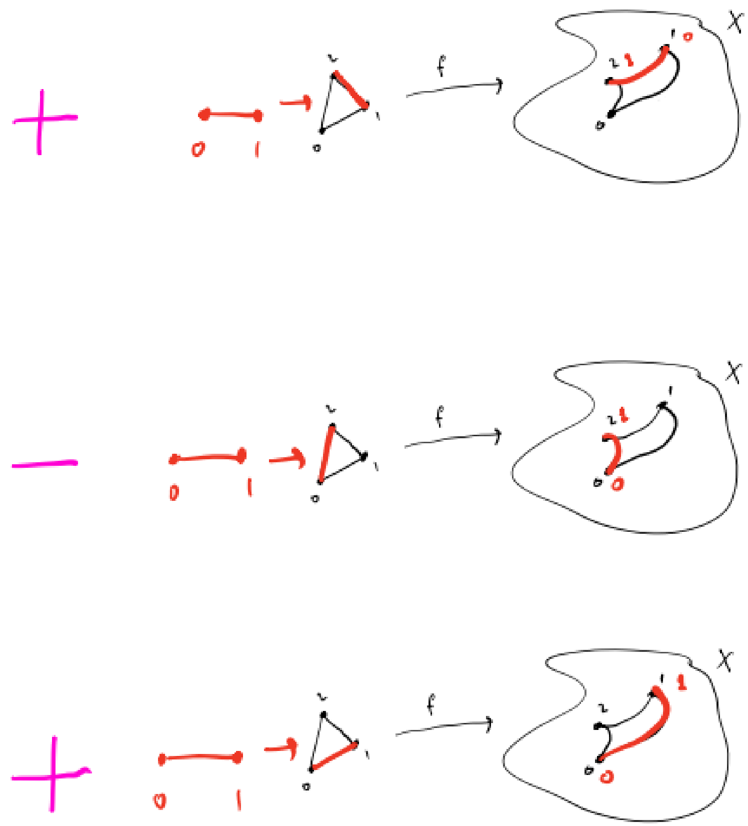


FIGURE 14. Boundary maps of the example above.

Proposition 3. $\partial_{n-1} \circ \partial_n = 0$

Proof. For $n < 2$ it is obvious. For $n \geq 2$ it suffices to show that for $g : \Delta^n \to X$ we should have $\partial_{n-1}(\partial_n(g)) = 0$.

$$\partial_{n-1}(\partial_n(g)) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i g \circ f_{i,n}\right) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(g \circ f_{i,n})$$
$$= \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{n-1}(-1)^j g \circ f_{i,n} \circ f_{j,n-1}\right) = \sum_{i,j}(-1)^{i+j} g \circ f_{i,n} \circ f_{j,n-1}$$

Where $f_{i,n} \circ f_{j,n-1} : \Delta^{n-2} \to \Delta^n$...

□

Exercise 12. Finish the proof of this proposition. If you were not able to follow in class, first do it for $n = 2$ using the pictures above - no need to write this, just to get yourself oriented. □

Definition 16. We define the singular homology of X as the homology of its singular chain complex

$$H_n(X;\mathbb{R}) \coloneqq H_n(C_*(X;\mathbb{R}), \partial_*)\ .$$

□

5. March 10, 2022: Constructing singular cycles, homology of a point, star shaped open subsets of Euclidean space

Definition 17. Let us call the elements of

$$Z_n(X;\mathbb{R}) \coloneqq ker(\partial_n : C_n(X;\mathbb{R}) \to C_{n-1}(X;\mathbb{R}))$$

the singular n-cycles and the elements of

$$B_n(X;\mathbb{R}) \coloneqq im(\partial_{n+1} : C_{n+1}(X;\mathbb{R}) \to C_n(X;\mathbb{R}))$$

the singular n-boundaries. □

In this class, often we will omit the adjective singular from these phrases for brevity. Then, by definition

$$\begin{aligned} H_n(X;\mathbb{R}) &= \frac{Z_n(X;\mathbb{R})}{B_n(X;\mathbb{R})} \\ &= \frac{n\text{-cycles}}{n\text{-boundaries}} \end{aligned}$$

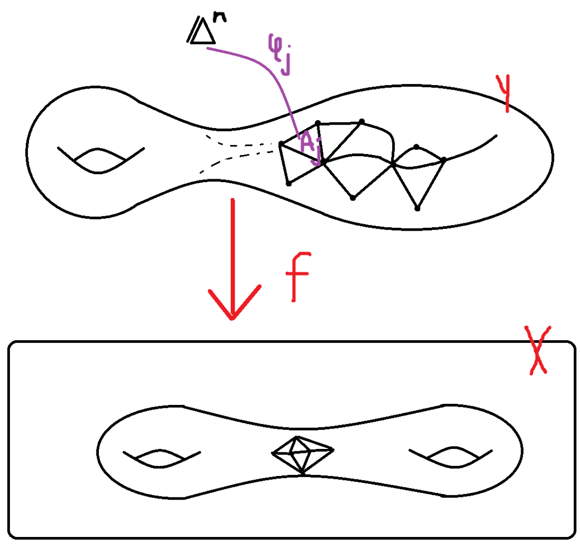
5.1. **Singular n-cycles from geometric n-cycles - slightly informal discussion.** Let X be a topological manifold. Let us define a geometric n-cycle to be the image of a continuous map $f : Y^n \to X$ where Y is a compact oriented (We will define precisely for smooth manifolds later) topological manifold.

I want to briefly explain how a geometric n-cycle gives rise to an n-cycle on X. Oriented compact submanifolds are examples of geometric n-cycles.

Under some mild conditions (for example if it is Hausdorff and admits a smooth structure), Y admits a triangulation. This, in particular, means we can find

$$Y = \bigcup_{j=1}^{N} A_j \quad \text{with homeomorphisms} \quad \phi_j : \Delta^n \to A_j$$

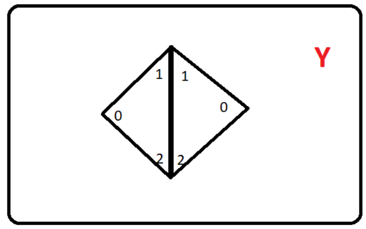
such that the intersections $A_i \cap A_j$ for $i \neq j$ are either empty or equal to the image of both $\phi_i \circ \mathrm{face}_{k,n}$ $(: \Delta^{n-1} \to \Delta^n \to Y)$ and $\phi_j \circ \mathrm{face}_{l,n}$ for some k and l.



The idea then is to add up all $f \circ \phi_j : \Delta^n \to X$ and get an n-cycle seeing how the boundaries seem to cancel each other.

The issue is that we do not actually know, it depends on whether the signs work out.

We could also add $\pm f \circ \phi_j$ of course. Note that we can modify ϕ_j also by homeomorphism, $\Delta^n \to \Delta^n$ which permute the coordinates of



NO CANCELLATION

$\mathbb{R}^{n+1}$. If the result is $\widetilde{\phi}_j$ and the permutation π^{-1}

$$\begin{aligned} \partial_n(f \circ \widetilde{\phi}_j) &= \sum_{i=0}^{n} (-1)^i f \circ \widetilde{\phi}_j \circ \mathrm{face}_{i,n} \\ \pi(l) = i \qquad &= \sum_{l=0}^{n} (-1)^{\pi(l)} f \circ \widetilde{\phi}_j \circ \mathrm{face}_{\pi(l),n} \\ &= \sum_{l=0}^{n} (-1)^{\pi(l)-l} (-1)^l f \circ \phi_j \circ \mathrm{face}_{l,n} \end{aligned}$$

The interesting result is that whether one can modify ϕ_i's using there so that $\partial_n(\sum_{i=1}^{N} \pm \widetilde{\phi}_i) = 0$ is a condition that depends only on Y and is called orientability.

If this is true, then we obtain at least two n-cycles in Y, we can multiply everything by -1. Actually orienting Y, we would pick out one of them.

Exercise 13. Consider $S^1 \subset \mathbb{R}^2$. Construct a nonzero 1-cycle in $\mathbb{R}^2$ corresponding to this geometric 1-cycle. Prove that it is actually a 1-boundary directly. □

5.2. **Some computations.** Let X be a point. What is $H_*(X;\mathbb{R})$?

For every $n \geq 0$, there exists exactly one continuous map $\Delta^n \xrightarrow{c_n} X$. Therefore,

$$C_n(X;\mathbb{R}) = \mathbb{R} \cdot c_n.$$

How about the boundary map?

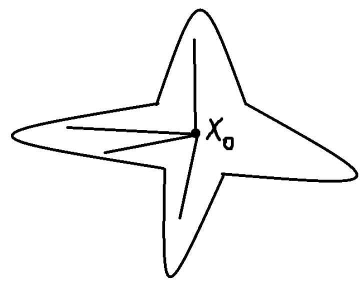
$$\begin{aligned} \partial_n c_n &= \sum_{i=0}^{n} (-1)^i c_{n-1} \\ &= \begin{cases} c_{n-1} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \end{aligned}$$

The singular chain complex looks like

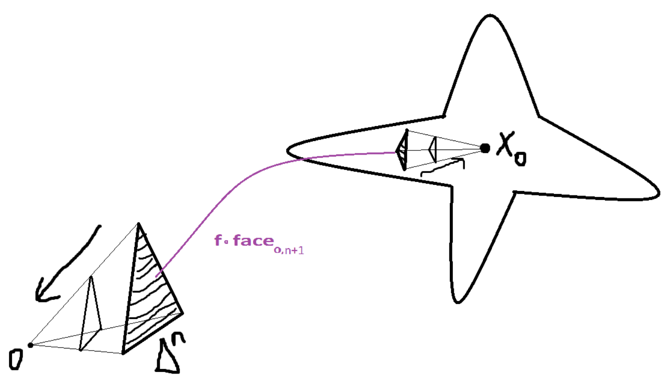
$$\leftarrow 0 \leftarrow 0 \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \ldots$$

$$\implies \qquad H_0(X;\mathbb{R}) = \mathbb{R} \text{ and } H_n(X;\mathbb{R}) = 0, \ \ n \neq 0.$$

Let us now consider $U \subset \mathbb{R}^n$ open and star-shaped, that is, there exists an $x_0 \in U$ such that the line segment x_0 and y lies inside U for all $y \in U$.



We claim that $H_*(U;\mathbb{R}) \cong H_*(\text{point};\mathbb{R})$. The idea is that for a given $f : \Delta^n \to U$, we can define a $P_f : \Delta^{n+1} \to U$ as: follows



Exercise 14. Write down an explicit formula for P_f in terms of f. Extending $f \mapsto P_f$ linearly, define a linear map $P : C_n(X;\mathbb{R}) \to C_{n+1}(X;\mathbb{R})$. For $n > 0$, $\sigma \in C_n(X;\mathbb{R})$, prove that

$$\partial_{n+1} P\sigma = -P(\partial_n \sigma) + \sigma.$$

For $n = 0$, $\sigma \in C_0(X;\mathbb{R})$, what is $\partial_1 P\sigma$? □

If $\sigma \in Z_n(U;\mathbb{R})$ and $n > 0$, then $\partial_{n+1} P\sigma = \sigma$. This implies $\sigma \in B_n(U;\mathbb{R})$ and hence $H_n(U;\mathbb{R}) = 0$ for $n \neq 0$.

Exercise 15. Let Y be a topological space. Prove that $H_0(Y;\mathbb{R})$ is isomorphic to the vector space generated by the set of connected components of Y. $\square$

6. March 14, 2022: Induced maps on homology, homeomorphism invariance, homotopy invariance of induced maps

Definition 18. Let $(C_\bullet, \partial_\bullet)$ and $(\tilde{C}_\bullet, \tilde{\partial}_\bullet)$ be chain complexes. A chain map is a collection of linear maps $C_n \xrightarrow{f_n} \tilde{C}_n$, $\forall n \in \mathbb{Z}$ such that each square in the diagram

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "\partial_n"'] \arrow[d, "f_n"] & \cdots \arrow[l] \\
\cdots & \tilde{C}_{n-1} \arrow[l] & \tilde{C}_n \arrow[l, "\tilde{\partial}_n"'] & \cdots \arrow[l]
\end{tikzcd}$$

is commutative, i.e.

$$f_{n-1} \circ \partial_n = \tilde{\partial}_n \circ f_n, \quad \forall n \in \mathbb{Z}.$$

$\square$

Given a chain map $f_\bullet : C_\bullet \to \tilde{C}_\bullet$ we canonically obtain a linear map of graded vector spaces

$$H(f) : H_*(C) \to H_*(\tilde{C}).$$

Remark 6. Let us make a notational clarification. For any chain complex (C, ∂) we define

$Z_n(C) := \ker(\partial_n) \cdot n$-cycles
$B_n(C) := \operatorname{im}(\partial_{n+1}) \cdot n$-boundaries
$H_n(C) = {Z_n(C)}/{B_n(C)}$

When $C_\bullet = C_*(X;\mathbb{R})$ then we may add the adjective "singular". $\square$

Back to constructing $H_n(C) \to H_n(\tilde{C})$.

(1) If $\sigma \in Z_n(C)$, then $f(\sigma) \in Z_n(\tilde{C})$:
$f(\partial\sigma) = \tilde{\partial} f(\sigma) \implies \tilde{\partial} f(\sigma) = 0$
(2) If $\sigma \in B_n(C)$, then $f(\sigma) \in B_n(\tilde{C})$:
$\sigma = \partial\gamma \implies f(\sigma) = f(\partial\gamma) = \tilde{\partial} f(\gamma)$

$\implies$ We obtain the map shown in dashes below.

$$\begin{tikzcd}
B_n(C) \arrow[r, hook] & Z_n(C) \arrow[r] \arrow[d] & Z_n(\tilde{C}) \arrow[r] & Z_n(\tilde{C})\big/B_n(\tilde{C}) \\
& Z_n(C)\big/B_n(C) \arrow[urr, dashed] & &
\end{tikzcd}$$

We can compose chain maps as in the following exercise.

Exercise 16. Prove that if $f : C_* \to \tilde{C}_*$ and $g : \tilde{C}_* \to \tilde{\tilde{C}}_*$ are chain maps, then $h : C_* \to \tilde{\tilde{C}}_*$ defined by $h_n \coloneqq g_n \circ f_n$ is a chain map. For such f, g, h prove that $H(h) = H(g) \circ H(f)$. $\square$

Definition 19. Let $\varphi : X \to Y$ be continuous. Then for every $\rho : \Delta^n \to X$, we obtain $\varphi \circ \rho : \Delta^n \to Y$. Linearly extending we obtain a map $(\varphi_*)_n : C_n(X; \mathbb{R}) \to C_n(Y; \mathbb{R})$. These form a chain map

$$\varphi_* : C_*(X; \mathbb{R}) \to C_*(Y; \mathbb{R}).$$

$\square$

Exercise 17. Prove that φ_* is indeed a chain map. $\square$

We also obtain

$$H\varphi_* : H_*(X; \mathbb{R}) \to H_*(Y; \mathbb{R}).$$

Exercise 18. For continuous maps $\tilde{\varphi} : X \to Y$ and $\varphi : Y \to Z$, prove that $\varphi_* \circ \tilde{\varphi}_* = (\varphi \circ \tilde{\varphi})_*$. $\square$

Corollary 1. $H\varphi_* \circ H\tilde{\varphi}_* = H(\varphi \circ \tilde{\varphi})_*$. $\square$

Since $(\mathrm{id})_*$ is the identity map, we immediately obtain that if $\varphi : X \to Y$ is a homeomorphism then $H\varphi_* : H_*(X; \mathbb{R}) \to H_*(Y; \mathbb{R})$ is a linear isomorphism.

Hence singular homology can distinguish non-homeomorphic topological spaces (it does not have to!) Actually singular homology is in general only sensitive to the homotopy equivalence class. Let's explain this.

Recall: $\bullet f, g : X \xrightarrow{cts} Y$ are called homotopic if there exists a continuous

$$F : X \times [0,1] \to Y \text{ s.t.}$$

$$F|_{\{0\}} = f \;\&\; F|_{\{1\}} = g.$$

• A continuous $f : X \to Y$ is called a homotopy equivalence if $\exists g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps (of Y and X).

Exercise 19. Let $U \subset \mathbb{R}^n$ be star shaped with respect to x_0. Prove the map pt. $\to U$ with image x_0 is a homotopy equivalence. $\square$

Theorem 2. *If f & $g : X \to Y$ are homotopic, then $Hf_* = Hg_*$.*

I don't want to spend time proving this homotopy invariance theorem, but it is quite important and the proof is not too difficult. If you want we can discuss during office hours. You are responsible from the statement, not the proof. The key idea in the proof is essentially the one that we used on proving $H_*(\text{star shaped}) \cong H_*(\text{pt.})$.

The corollary below can be proved using the same logic that showed that homeomorphisms induce isomorphisms on homology but this time using Theorem 2.

Corollary 2. *If $f : X \to Y$ is a homotopy equivalence, then $Hf : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R})$ is an isomorphism.* $\square$

Exercise 20. Prove this corollary assuming Theorem 2. $\square$

7. March 17, 2022: Mayer-Vietoris property, special properties of singular homology for manifolds, singular cohomology

Definition 20. An exact sequence is a sequence of vector spaces V_n, $n \in \mathbb{Z}$ and maps $V_n \xrightarrow{f_n} V_{n-1}$ such that $\forall n \in \mathbb{Z} \quad ker(f_n) = im(f_{n+1})$

$$\begin{tikzcd}
V_{n-1} \arrow[r, bend right=60, "\mathsf{ker}"'] & V_n \arrow[l, "f_n"'] & V_{n+1} \arrow[l, "f_{n+1}"'] \arrow[l, bend left=60, "\mathsf{Im}"] & V_{n+2} \arrow[l, "f_{n+2}"'] \\
 & = & &
\end{tikzcd}$$

$\square$

Remark 7. This is the same data as chain complex with 0-Homology. $\square$

Exercise 21. Let V_n be an exact sequence. Assume that $\sum \dim V_n < \infty$. Show that $\sum_{n \text{ even}} \dim V_n = \sum_{n \text{ odd}} \dim V_n$. $\square$

Theorem 3 (Mayer-Vietoris Theorem)**.** *Let X be a topological space and $U, V \subseteq X$ open subsets.*
There are canonical maps

$$H_{n+1}(U \cup V) \xrightarrow{c_{n+1}} H_n(U \cap V)$$

called connecting maps that makes the following graded vector space

$$\begin{tikzcd}
& \cdots \arrow[r, dashed] & H_{n+2}(U \cup V) \arrow[dll, dashed, "c_{n+2}"'] \\
H_{n+1}(U \cap V) \arrow[r] & H_{n+1}(U) \oplus H_{n+1}(V) \arrow[r] & H_{n+1}(U \cup V) \arrow[dll, "c_{n+1}"'] \\
H_n(U \cap V) \arrow[r, "i_n"] & H_n(U) \oplus H_n(V) \arrow[r, "j_n"] & H_n(U \cup V) \arrow[dll, dashed, "c_n"'] \\
H_{n-1}(U \cap V) \arrow[r, dashed] & \cdots &
\end{tikzcd}$$

an exact sequence, where i_n and j_n are the natural maps given by

$$\begin{aligned} i_n : H_n(U \cap V) &\to H_n(U) \oplus H_n(V) \\ a &\mapsto (i_*^{U\cap V \subset U} a, i_*^{U \cap V \subset V} a) \end{aligned}$$

and

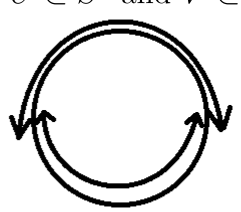
$$\begin{aligned} j_n : H_n(U) \oplus H_n(V) &\to H_n(U \cup V) \\ (a, b) &\mapsto i_*^{U \subset U \cup V} a - i_*^{V \subset U \cup V} b. \end{aligned}$$

We will discuss the proof of Mayer-Vietoris theorem later when we state it for DeRham Theory.

7.1. **Applications.**

Example 4. Let Y, Z be topological spaces. Consider their direct sum [8] $X = Y \sqcup Z$. Since $Y \cap Z = \emptyset$, by Mayer-Vietoris' Theorem, we have $H_n(X) \simeq H_n(Y) \oplus H_n(Z)$. This also follows from the fact that $C_n(X) = C_n(Y) \oplus C_n(Z)$. □

Example 5. Computing $H_*(S^1)$ Now for a more serious application. Consider the two intervals $U \subset S^1$ and $V \subset S^1$.



Both are homeomorphic to an open interval $(0, 1) \subset \mathbb{R}$, which is contractible. Hence $H_*(U) \cong H_*(V) \cong H_*((0, 1)) \cong H_*(pt)$. And $U \cap V$ is homeomorphic to a disjoint union of two open intervals. So $H_*(U \cap V) \cong H_*(\mathbb{R}) \oplus H_*(\mathbb{R}) \cong H_*(pt) \oplus H_*(pt)$. Since S^1 is connected, we have $H_0(S^1) = \mathbb{R}$. Plugging all this data into the Mayer-Vietoris sequence we get

[8]The disjoint union of Y and Z, equipped with the topology consisting of open sets of the form $U \cup V$ where $U \subset Y$ and $V \subset Z$ are open.

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & 0 \oplus 0 \arrow[r] & H_2(S^1) \arrow[dll, "c_2"'] \\
0 \oplus 0 \arrow[r] & 0 \oplus 0 \arrow[r] & H_1(S^1) \arrow[dll, "c_1"'] \\
\mathbb{R} \oplus \mathbb{R} \arrow[r, "i_0"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "j_0"] & H_0(S^1) = \mathbb{R} \arrow[dll, "c_0"'] \\
0 \arrow[r, dashed] & \cdots &
\end{tikzcd}
$$

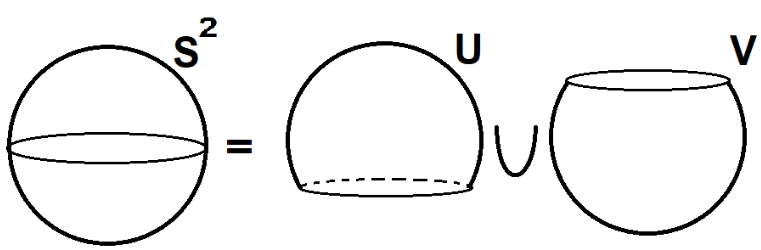
Simply by observing the diagram and counting dimensions we get $H_i(S^1) = 0$ for $i > 1$ and $H_1(S^1) \cong \mathbb{R}$. The only non-canonical isomorphism we have here is $H_1(S^1) \cong \mathbb{R}$. To understand this isomorphism better we have to inspect this sequence further.

Since $ker\, c_1 = 0$, we may identify $H_1(S^1) = im\, c_1 = ker\, i_0$. Let's introduce some notation to communicate better, denote by $W_1 \sqcup W_2 = U \cap V$ where W_i are the two disjoint intervals. Let $p_i : \Delta^0 \to W_i$ be any specific map. It's clear that each element of $H_0(U \cap V)$ is represented uniquely by a cycle of the form $ap_1 + bp_2$, where $a, b \in \mathbb{R}$. Also, we have $i_*^{U\cap V \subset U} p_1 = i_*^{U\cap V \subset U} p_2 \in H_0(U)$ and $i_*^{U\cap V \subset V} p_1 = i_*^{U\cap V \subset V} p_2 \in H_0(V)$, so $ker\, i_0 = \{ap_1 - ap_2 \in H_0(U \cap V)\}$. There are two natural bases for this space: $p_1 - p_2$ and $p_2 - p_1$. These two choices give us two choices of isomorphisms $H_1(S^1) \cong \mathbb{R}$ and correspond to the two choices of orientation we have on S^1

□

Exercise 22. Compute $H_*(S^n, \mathbb{R})$. □

Example 6. Sketch for $H_*(S^2)$
Consider the following open sets $U, V \subset S^2$.



Since $U, V \cong \mathbb{R}^2$, we have $H_*(U) \cong H_*(V) \cong H_*(\mathbb{R}^2) \cong H_*(pt)$. Notice that the circular belt $U \cap V$ can be retracted onto the equator of the sphere, which is homeomorphic to S^1. Since this is a homotopy equivalence, by Theorem 2 we have $H_*(U \cap V) \cong H_*(S^1)$. Plugging all we know into the Mayer-Vietoris sequence

$$\begin{array}{lll}
0 \to 0 \oplus 0 \to H_3(S^2) & \cong 0 \\
\to 0 \to 0 \oplus 0 \to H_2(S^2) & \cong \mathbb{R} \\
\to \mathbb{R} \to 0 \oplus 0 \to H_1(S^2) & \cong 0 \\
\to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H_0(S^2) & \cong \mathbb{R}
\end{array}$$

we get that $H_i(S^2) = 0$ for $i < 2$, $H_1(S^2) = 0$ and $H_2(S^2) \cong H_0(S^2) \cong \mathbb{R}$. □

7.2. **Singular Homology Of Manifolds.** From now on, we'll assume that our manifolds are Hausdorff. When we write manifold we will mean a topological manifold below.

Remark 8. We needed this condition for the existence of a triangulation as well. □

Later when we go back to smooth manifolds we'll add another condition, being second countable.

Exercise 23. Find a non-Hausdorff manifold which can be equipped with a smooth atlas. □

Theorem 4. *Let M be an n-dimensional manifold, then $H_i(M;\mathbb{R}) = 0$ for $i > n$.*

Hence the singular homology of M can only live in degrees $i = 0, 1, \dots, n$.

If we assume that M is also connected, then $H_0(M;\mathbb{R}) \cong \mathbb{R}$ This isomorphism is canonical, where we identify any map from a point to M with $1 \in \mathbb{R}$. It turns out that we know quite a bit about the top degree as well

Theorem 5. *Let M be an n-dimensional manifold. Then*

$$H_n(M;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \textit{if } M \textit{ is compact and orientable} \\ 0 & \textit{otherwise.} \end{cases}$$

Here the isomorphism with $\mathbb{R}$ depends on our choice of orientation.

Theorem 6 (Poincaré Duality)**.** *Let M be a compact, oriented[9] n-dimensional manifold. Then we have the canonical isomorphisms*

$$H_{n-k}(M;\mathbb{R}) \cong \big(H_k(M;\mathbb{R})\big)^\vee$$

for all $k \in \mathbb{Z}$.

[9]That is, we have chosen a specific orientation and so we are equipped with an isomorphism $H_n(M;\mathbb{R}) \cong \mathbb{R}$.

Exercise 24. Recall that the linear dual $V^\vee$ of a vector space V is the vector space of linear maps $V \to \mathbb{R}$. Prove the following: $V^\vee \cong V$ if and only if $\dim V < \infty$. The only if part is optional, similar to Exercise 11 □

Exercise 25. Let $f : V \to W$ be a linear map. Define $f^\vee : W^\vee \to V^\vee$ by $f^\vee \alpha = \alpha \circ f$. Show that $f^\vee$ is a linear map. Describe $ker f^\vee$ and $im f^\vee$ in terms of $ker f$ and $im f$.
Let $g : W \to P$ be another linear map, show that $(g \circ f)^\vee = f^\vee \circ g^\vee$. □

7.3. **Singular Cohomology.**

Definition 21. Let X be a topological space. The singular cochain complex $C^*(X; \mathbb{R})$ is defined by

$$C^n(X; \mathbb{R}) = \big(C_n(X; \mathbb{R})\big)^\vee$$

and the coboundary maps $\delta_n = \partial_{n+1}^\vee$ are given by

$$\begin{aligned} \delta_n : C^n(X; \mathbb{R}) &\to C^{n+1}(X; \mathbb{R}) \\ \alpha &\mapsto \alpha \circ \partial_{n+1}. \end{aligned}$$

The cohomology of this complex is called the singular cohomology of X. We denote it by

$$H^*(X; \mathbb{R}) = H^*(C^*(X; \mathbb{R})).$$

□

Exercise 26. Prove that the above defined graded vector space is indeed a cochain complex, *viz* $\delta_{n+1} \circ \delta_n = 0$. □

Exercise 27. Prove that $H^n(X; \mathbb{R}) \cong \big(H_n(X; \mathbb{R})\big)^\vee$. Hint: Start by constructing a map. □

Exercise 28. Deduce a Mayer-Vietoris sequence for singular cohomology from the Mayer-Vietoris sequence for singular homology. Be careful! □