LECTURE NOTES - MATH 215B (WINTER 2021)

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Contents

1. Jan 11, 2021: Definition of a smooth manifold via atlases, examples

In this course, we will study smooth manifolds using techniques you learned in calculus and ODE courses. We will be able think about the following sets (to begin with) as smooth manifolds soon:

- The Euclidean space $\mathbb{R}^n$ of dimension n for $n = 0, 1, 2, \ldots$.
- The unit sphere $\mathbb{S}^{n-1}$ inside $\mathbb{R}^n$, for $n = 1, 2, \ldots$.
- The possible (visual) states of a meteroid in otherwise empty three dimensional Euclidean space at an instant. Note that meteoroids can have different symmetry groups (a real life meteoroid will have none). You can assume that we have a real life meteoroid in the questions below.
- The solution set of the complex polynomial $x^2 - y^3 - 1$ inside $\mathbb{C}^2$.
- The set of all two dimensional linear subspaces of $\mathbb{R}^4$.

Question 1. What should be the dimensions of these? Explain what you mean by dimension. □

Question 2. Explain precisely how to equip each of these sets with a natural topology. □

Remark 1. Note that here even when we say X inside Y is a smooth manifold, we are only using Y to describe what X is. When we want to study X or do something in X we can forget about the rest of Y (only if we want to of course). □

Informally, a smooth manifold X of dimension n satisfies the following conditions:

(1) X is locally Euclidean: the points"sufficiently near" any $x_0 \in X$ are canonically determined by n coordinate functions (we refer to this as a coordinate system at x_0, the abstraction of "generalized coordinates" from mechanics if you are familiar with that).
(2) X might be globally complicated, in particular the coordinate systems from (1) do not necessarily extend to "large enough" neighborhoods. Note that this has the immediate consequence of non-uniqueness of coordinate systems at some points.
(3) For any function $f : X \to \mathbb{R}$ and $x_0 \in X$, there is a well-defined notion of f being k times differentiable at x_0 defined using the coordinate systems from (1). The non-uniqueness from (2) causes the resistance here.

Question 3. Make sure you understand what it means for a function $\mathbb{R}^n \to \mathbb{R}$ to be k times differentiable at a point. □

If X satisfies these conditions, we can imagine "doing calculus" in X. If we work inside the domain U of a particular coordinate system, then we are basically inside $\mathbb{R}^n$. Since we are not actually in $\mathbb{R}^n$ and we will have to change to a different coordinate system when we leave U (as in (2)), we need the consistency in (3) to make sure that what we did in U is also valid in the other charts.

Let us now move on to the formal definition of a smooth manifold. First, we recall the notion of smoothness. Let U be an open subset in an Euclidean space, then a function

$$f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$$

is called smooth if all iterated partial derivatives of f_i exist, for all $i = 1, \dots m$. If V is an open subset of $\mathbb{R}^m$, then $U \to V$ is smooth if $U \to V \subset \mathbb{R}^n$ is smooth.

The following definition guarantees conditions (1) and (2) above.

Definition 1. A topological manifold of dimension n is a second countable, Hausdorff topological space such that every point admits a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. □

Question 4. Recall the meanings of Hausdorff and second countable. These conditions are there to avoid certain pathologies and one could imagine removing them from the definition, though we will always assume them in this course. □

Question 5. Prove that the topological spaces from the beginning of the class are all topological manifolds. □

If X is a topological space, let us call an open subset $U \subset X$ and a homeomorphism $\phi : U \to \tilde{U}$, where $\tilde{U} \subset \mathbb{R}^n$ is an open set, a coordinate chart (or just chart). We can denote this chart by (U, ϕ). U is the domain of the chart, and the components of $\phi : U \to \mathbb{R}^n$ are the coordinates or coordinate functions. The final condition in the definition of a topological manifold says that every point in X is contained in the domain of a chart.

Let (U_1, ϕ_1) and (U_2, ϕ_2) be two charts in a topological space X. We automatically obtain a map

$$\phi_{12} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

called the transition map from the chart (U_1, ϕ_1) to (U_2, ϕ_2). Note that the transition map goes from an open subset of an Euclidean space to an open subset of an Euclidean space. Hence we know what it means for ϕ_{12} to be smooth.

Here is the key definition, which will take care of (3).

Definition 2. Let X be a topological manifold. A smooth atlas on X is a collection of charts

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{I}}$$

such that

- $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = X$.
- The transition map between any two charts in the collection is smooth.

$\square$

Remark 2. We declare that a map between two empty sets is smooth. This is true by definition but if it makes you confused do not spend time with it. $\square$

Question 6. Equip the topological manifolds from the beginning with smooth atlases. $\square$

Let us now spell out how exactly we deal with (3).

Definition 3. Let X be a topological manifold with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{I}}$. A function $f : X \to \mathbb{R}$ is k-times differentiable at $x_0 \in X$ if for some chart (U, ϕ) at x_0 in the smooth atlas, the map

$$f \circ \phi^{-1} : \tilde{U} \to \mathbb{R}$$

is k-times differentiable. $\square$

Question 7. Prove that this definition makes sense. $\square$

Question 8. Look up the definition of a maximal smooth atlas from Lee. $\square$

Here is our final definition.

Definition 4. A topological manifold equipped with a maximal smooth atlas is called a smooth manifold. $\square$

2. Jan 13, 2021: Smooth maps, smooth manifolds by gluing, tangent bundle

We extend the notion of smoothness of maps between open sets of the Euclidean space to smooth manifolds.

Definition 5. Let X and Y be smooth manifolds, and $f : X \to Y$ be a continuous map. We say that f is a smooth map if for every chart (U, ϕ) in X and (V, ψ) in Y such that $f(U) \subset V$, the map $\psi \circ f \circ \phi^{-1} : \tilde{U} \to \tilde{V}$ is smooth. $\square$

Question 9. What should be the definition of a differentiable map between smooth manifolds? $\square$

Question 10. Consider the unit circle $\mathbb{S}^1$ as a topological space. Equip it with two smooth atlases (i) with two charts whose domains are connected, and (ii) with three charts whose domains are connected and such that no two cover $\mathbb{S}^1$. To define the

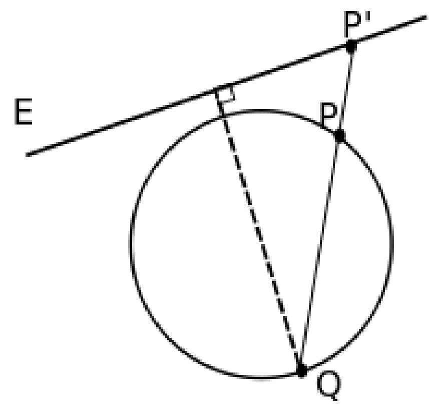

FIGURE 1. An example of stereographic projection, which sends P to P'. The only condition to be able to define it is that E should not contain Q.

coordinate maps use stereographic projections[1] to straight lines in the plane (see Figure 1).

Prove that if we did not require the smooth atlases to be maximal in the definition of smooth manifolds, a map $\mathbb{S}^1 \to \mathbb{S}^1$ that is not even continuous could be a smooth map using the previous definition. Here the source is equipped with smooth structure (i) and the target with (ii). This would be problematic.

Finally, prove that the maximal smooth atlas of (i) and (ii) are the same, and therefore that a non-continuous map cannot be smooth. □

Remark 3. Throughout this course, we will rarely work with maps between smooth manifolds that are not smooth. The situation is similar to continuous maps and topological spaces, but less strictly so. For most of our purposes it would be enough to require a certain finite number of iterated derivatives to exist, but we would need to say how many for every statement. We are just being super generous with our differentiability condition to focus on the issues that are more central to differential topology. The important point here is that there is a sufficient supply of smooth maps, which we will more explicitly discuss next week. One consequence of this generosity is that we will rarely have any function or map that is given by an explicit formula, instead we will use that smooth functions/maps that satisfy some conditions exist. □

We will soon introduce a number of important types of smooth maps but here is a definition that is conceptually very important.

Definition 6. Let X and Y be smooth manifolds, and $f : X \to Y$ be a bijective smooth map. If the inverse map f is also smooth, then we call f a diffeomorphism. We also say that X and Y are diffeomorphic. □

To the eyes of differential topology diffeomorphic smooth manifolds are the same, they are just obtained by giving different names to the elements of the set so to

[1]I mean this in the sense explained in the first paragraph of ”Generalizations” in Wikipedia page for Stereographic projection

speak. Yet, the choice of a diffeomorphism is still a choice, it is good practice to mention the diffeomorphism that is witnessing the "sameness" of diffeomorphic smooth manifolds.

Question 11. Let X be a topological manifold. Let S_1 and S_2 are two maximal smooth atlases[2] on X. Prove that the identity map $X \to X$ is a diffeomorphism if and only if $S_1 = S_2$. □

Question 12. Consider the real line $\mathbb{R}$ as a topological space. Equip it with (i) its "standard" smooth structure (ii) the smooth structure that admits a chart (U, ϕ) with $U = \mathbb{R}$ and $\phi(x) = x^3$. Prove that (i) and (ii) are not the same smooth structure, but they are diffeomorphic. □

We have been talking about the differentiability of maps between smooth manifolds but not taking any actual derivatives. Note that the partial derivatives of a function as we learned it in calculus courses depend on the coordinates that we were given and we do not have such preferred coordinates in a general smooth manifold. We have to develop the notion of the tangent bundle to get a head start. Then, we will define the differential of a smooth map.

First, we need to deal with the case of open subsets of Euclidean spaces (extending to general smooth manifolds is not going to be difficult). If you remember your multivariable calculus well, this is at best a reformulation of what you already know.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets, and $f : U \to V$ a smooth map. Then the Jacobian matrix at a point p is the following matrix

$$Jac_p(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}.$$

We define the tangent bundle of an open set $U \subset \mathbb{R}^n$ as

$$TU := U \times \mathbb{R}^n,$$

which is an open subset of $\mathbb{R}^{2n}$. It is very important to be able to visualize points of TU as a point p in U and a vector at p effectively.

We define the differential

$$df : TU \to TV$$

of $f : U \to V$ (as above) by the formula

$$df(p, v) = (f(p), Jac_p(f)v).$$

Question 13. Prove that if we have open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^k$ and smooth maps $f : U \to V$ and $g : V \to W$, we have the following reinterpretation of the chain rule

$$d(g \circ f) = dg \circ df.$$

You can use the multivariable calculus chain rule without proof. □

[2] we will also call this a smooth structure sometimes for brevity

Question 14. Let $v_0 = (0,1) \in T(-1,1)$. Let $\gamma : (-1,1) \to \mathbb{R}^n$ be a smooth map: a smooth trajectory of a point moving in the Euclidean space. Prove that $d\gamma(v_0)$ is the velocity vector at time 0 of γ as defined in multivariable calculus. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Prove that the velocity vector at 0 of the trajectory $f \circ \gamma$ is the image of the velocity vector at 0 of γ under the differential df. Visualize all of this. □

Before moving to the general case, let us precisely explain where the name "tangent bundle" is coming from. Even though we do not strictly know what this means yet, imagine a smooth manifold embedded in some Euclidean space $X \subset \mathbb{R}^n$, e.g. $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ works. One can think of the tangent bundle of the open subset $U \subset \mathbb{R}^n$ as the set of all possible velocity vectors of trajectories in U. This is what the tangent bundle of any smooth manifold is supposed to be. In particular, if we look at trajectories constrained to lie in $X \subset \mathbb{R}^n$, then we notice that the possible velocity vectors are precisely the vectors at points of X which are tangent to X. In other words, the tangent bundle is the union of all tangent (linear) spaces to X inside $\mathbb{R}^n$, e.g. tangent planes for $\mathbb{S}^2 \subset \mathbb{R}^3$. We will also use the phrase tangent space at a point of our smooth manifold in what follows.

Question 15. Taking this as a definition momentarily, concretely describe $T\mathbb{S}^1$ as a familiar shape. □

Question 16. Explain in words how you would define the differential of a smooth map between two embedded smooth manifolds using velocity vectors. □

This describes the tangent bundle perfectly for smooth manifolds embedded in an Euclidean space, but we do not have this data for a smooth manifold (it exists by itself). What we need is a "gluing description". We know what the tangent bundle of the domain of a chart should be. We will just take all of these tangent bundles and identify them with each other using the transition maps.

Gluing is in general a useful way of constructing smooth manifolds, so let discuss it in general.

Proposition 1. *Let X_α is a collection of smooth manifolds indexed by a countable*[3] *set $\alpha \in \mathcal{I}$. We are also given open subsets $X_{\alpha\beta} \subset X_\alpha$ for any $\alpha \neq \beta \in \mathcal{I}$, and diffeomorphisms (gluing maps)*

$$\varphi_{\alpha\beta} : X_{\alpha\beta} \to X_{\beta\alpha}.$$

These satisfy the following axioms:

(1) *For every $\alpha \neq \beta \in \mathcal{I}$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\alpha} = id$.*
(2) *For every pairwise distinct $\alpha, \beta, \gamma \in \mathcal{I}$,*

$$\varphi_{\alpha\beta}(X_{\alpha\beta} \cap X_{\alpha\gamma}) \subset X_{\beta\alpha} \cap X_{\beta\gamma}.$$

(3) *For every pairwise distinct $\alpha, \beta, \gamma \in \mathcal{I}$, the cocycle condition holds: on $X_{\alpha\beta} \cap X_{\alpha\gamma}$, we have*

$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}.$$

Under these assumptions, we can define the set

$$X := \coprod_{\alpha \in \mathcal{I}} X_\alpha / \sim,$$

[3]this can be weakened

where for $\alpha \neq \beta \in \mathcal{I}$, $a \in X_\alpha$ and $b \in X_\beta$, $a \sim b$ if $\phi_{\alpha\beta}(a) = b$. Taking the disjoint union topology on $\coprod_{\alpha \in \mathcal{I}} X_\alpha$, we can equip X with the quotient topology. Note that we have open topological embeddings $X_\alpha \to X$.

Finally, if X is in addition Hausdorff, then X has a canonical smooth structure such that the induced smooth structure on X_α is the given one.

Proof. The conditions (1)-(3) ensure that $\sim$ is an equivalence relation. Disjoint union and quotient topologies are discussed in standard topology textbooks. Finally, clearly each chart in X_α produces a chart in X. The fact that the gluing maps are diffeomorphisms imply that charts coming from different X_α's in X are compatible. $\square$

Remark 4. Note that under condition (1), condition (2) is equivalent to: for every every pairwise distinct $\alpha, \beta, \gamma \in \mathcal{I}$,

$$\varphi_{\alpha\beta}(X_{\alpha\beta} \cap X_{\alpha\gamma}) = X_{\beta\alpha} \cap X_{\beta\gamma}.$$

$\square$

Question 17. Read the "Smooth manifold construction lemma" from Lee. Compare the two statements. $\square$

Question 18. Given an example of a gluing of two real lines where the resulting topological space is not Hausdorff. $\square$

We can use a compatible smooth atlas (not necessarily maximal) on a smooth manifold to reconstruct it by gluing. Namely, the images of the domains of the charts under the coordinate maps are the X_α's and the transition maps are used as the gluing maps. Note what we are gluing here are open subsets of Euclidean space, which are in particular smooth manifolds.

Question 19. Make sure you understand this. $\square$

Definition 7. Let X be a smooth manifold, which is obtained by gluing open subsets $U_\alpha \subset \mathbb{R}^n$ by the gluing maps $\phi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$.

We define the tangent bundle TX as a smooth manifold by gluing the open subsets $TU_\alpha \subset \mathbb{R}^{2n}$ by the gluing maps $d\phi_{\alpha\beta} : TU_{\alpha\beta} \to TU_{\beta\alpha}$. $\square$

Question 20. Check that the construction satisfies the conditions of gluing. $\square$

Question 21. Construct the natural surjective smooth map $TM \to M$. The fibers of this map are the tangent spaces. Show that the tangent spaces have a canonical real vector space structure. $\square$

Remark 5. We will call $TM \to M$ a vector bundle within a couple of weeks. $\square$

3. Jan 15, 2021: Brief answers to selected questions

Answer to Question 2:

- $\mathbb{R}^n$ is a metric space with
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$
- We use the subspace topology.

- Let us call this set S. Recall that a map $\mathbb{R}^3 \to \mathbb{R}^3$ is called an isometry if it preserves distances. Isometries act transitively on S. Hence, as a set we have
$$S \simeq Iso(\mathbb{R}^3)/Stab(s),$$
where $s \in S$ is an arbitrary state and $Stab(s)$ is the subgroup of isometries that preserve s. $Stab(s)$ is the same as the symmetry group of the meteroid. It therefore suffices to topologize $Iso(\mathbb{R}^3)$.

  This can be done in many different ways. Perhaps the most abstract way is to note that $Iso(\mathbb{R}^3)$ is a subset of the set of continuous maps from $\mathbb{R}^3$ to itself, which has the compact open topology. Let us outline a much more useful method which is based on actually understanding the isometries.

  It is elementary to show that isometries send straight lines to straight lines, and with a little bit more work that they are affine transformations (composition of a linear map and a translation). Hence, as sets
$$Iso(\mathbb{R}^3) \simeq O(3) \times \mathbb{R}^3,$$
where $O(3)$ is the set of linear isometries. Of course, $O(3) \subset \mathbb{R}^9$ by representing linear maps as matrices. This results in the same topology as before.

  It is possible to further analyze $O(3)$. First, note that $O(3)$ is the trivial double cover of $SO(3)$, the linear isometries that preserve orientation. Euler's remarkable theorem tells us that each such isometry is given by rotation along an axis. Note that the set of axes is by definition the projective plane $\mathbb{RP}^2$, and if we consider oriented axes we obtain S^2. Therefore, we have a surjective map
$$S^2 \times (0, 2\pi) \to SO(3) - \{id\}.$$
All points have exactly two points in their preimage. Notice that by taking a two point compactification of $S^2 \times (0, 2\pi)$, we obtain S^3 - think about scanning the unit sphere $\mathbb{S}^3$ using parallel hyperplanes in $\mathbb{R}^4$, start with lower dimensions. It is easy to see that this map can be extended to a continuous map
$$S^3 \to SO(3),$$
where the two new points are both sent to the identity. Similar level of difficulty is there to show that the map is a double covering map, which identifies the anti-podal points. This shows that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$!

  There are more involved proofs of this fact using quaternions $\mathbb{H}$. Namely, one constructs a group homomorphism $S^3 \to SO(\mathbb{H}_p)$, where S^3 is the subgroup of unit quaternions, and $\mathbb{H}_p$ is the pure quaternions. You can read this in Lemma 8.2.1 (click).

  If you are still following, this is also a good time to understand the belt trick. This answer (click) is pretty clear.
- We use the subspace topology.
- One way is to construct a transitive action of a matrix group as in the third bullet point. Another way is to construct what is called the Plucker embedding. We call the set $Gr(2, 4)$. We will construct an injective map
$$Gr(2, 4) \to \mathbb{RP}^5.$$

Take any pair of vectors that form a basis of the plane in question. Write the vectors as the columns of a 4×2 matrix using the standard basis of $\mathbb{R}^4$. For any pair $i < j$ of elements in $\{1, 2, 3, 4\}$, define a number by taking the determinant of the minor with i and jth rows. This way we obtain 6 numbers, which we use to define the map. It is a good exercise to check that this map is well-defined and that it is an injection.

Here is a challenge question. Write down a quadratic polynomial in the homogeneous coordinates of $\mathbb{RP}^5$ whose vanishing locus is the image of $Gr(2, 4)$.

This becomes more clear if you think of $\mathbb{RP}^5$ as the projectivization of $\Lambda^2(\mathbb{R}^4)$, and the Plucker embedding as sending a basis v_1, v_2 to $[v_1 \wedge v_2]$. We will cover the linear algebra of these anti-symmetric tensors when we start talking about differential forms.

Answer to Question 5:

- Trivial.
- Example 1.2 from Lee.
- This follows because $\mathbb{S}^3$ is a topological manifold.
- Let us call this set $E \subset \mathbb{C}^2$. Consider the projection to the x-coordinate

$$E \to \mathbb{C}_x.$$

Clearly, away from $x = \pm 1$, this is a $3 : 1$ covering map. Therefore, we found a chart at all points of E other than $(\pm 1, 0)$. If we use the y-projection we cover those points as well.

E is a Riemann surface, more particularly an elliptic curve. It is homeomorphic to a punctured torus. Consider the projection to the y-coordinate as above. This is what is called a branched cover. It has three critical values at the cube roots of -1. Now above any line segment connecting two of these three points (and not intersecting any critical value in the interior), there is loop of E. Use this to try to convince yourself that E is indeed a punctured torus. You can also do this Riemann style, by taking two copies of the plane, making branch cuts and regluing.

- Let $P \subset \mathbb{R}^4$ be a plane. Consider $T(P) \subset Gr(2, 4)$, which is the set of all planes that are transverse to P. I claim that $T(P)$ is the domain of a chart, in fact we will prove that it is homeomorphic to $\mathbb{R}^4$ in a very explicit way.

It suffices to show this for $P = \{x_3 = x_4 = 0\}$ as the action of $GL(4, \mathbb{R})$ on $Gr(2, 4)$ is clearly by homeomorphisms (potentially by definition) and because this action preserves the transversality between planes.

It is elementary to show that any plane in $T(P)$ is canonically given by two equations of the form $x_1 = f_1(x_3, x_4)$ and $x_2 = f_2(x_3, x_4)$, where f_i are linear functions. This finishes the proof as such equations are equivalent to the choice of 4 real numbers as coefficients. To make this precise define the map $\mathbb{R}^4 \to Gr(2, 4)$, which is clearly injective and continuous with image $T(P)$, and check that the inverse map $T(P) \to \mathbb{R}^4$ is also continuous.

Answer to Question 6:

- Trivial.
- Example 1.20 from Lee.
- Use the smooth structure of $\mathbb{S}^3$.

- We prove that the charts we constructed in the previous answer give a smooth atlas. This boils down to showing the following. For any $(x_0, y_0) \in E$ with $x_0^2 \neq 1$ and $y_0^3 \neq 1$, we can find a small enough neighborhood U such that both x and y projections are coordinate maps for a chart. Then, we obtain a map between the two projections U_x and U_y, and it suffices to check that this map is a diffeomorphism. In one direction the map is given by $x \mapsto y(x)$, where $y(x)$ is a continuous solution of the equation
$$y(x)^3 = 1 - x^2,$$
near x_0 with $y(x_0) = y_0$. Inverse function theorem (or explicit computation) shows the desired result. The other direction is similar.
- Probably the shortest way to solve this is to find the quadratic equation from the answer to Question 2, and apply a similar strategy to the previous item. One can also show that the charts we constructed in Question 5 give a smooth atlas using some linear algebra. I will not write this down. See Lee Example 1.24 for details (in the general case of k-planes in $\mathbb{R}^n$).

Answer to Question 10: You could actually get away without doing any computation except Lee's Example 1.20, but it's also good if you got some practice with stereographic projections.

Answer to Question 15: It is diffeomorphic to $S^1 \times \mathbb{R}$. Construct an explicit map that better be a diffeomorphism.

Answer to Question 18: Take two copies of $\mathbb{R}$ and glue the two along the open sets $\mathbb{R} - \{0\}$ in both copies using the identity map. In the resulting topological space we have the images of the two origins. Show that these two points do not have disjoint open neighborhoods.

Answer to Question 21: Here do not forget to prove that the vector space structure is independent of the chosen chart, which follows from the obvious linearity of Jacobians.

4. Jan 18, 2021: Differential of a smooth map, submanifolds, immersions, Lie groups and closed subgroup theorem

With the definition of tangent bundle under our belts, we can now define the differential of a smooth map. First we give a definition that will help us talk about coordinate charts.

Definition 8. Let X be a smooth manifold. A smooth map $\varphi : U \to X$ is called a parametrization if U is an open subset of an Euclidean space, φ has open image and is a diffeomorphism onto its image. □

Question 22. Show that a smooth map $\varphi : U \to X$ is a parametrization if and only if it is injective and $(\varphi(U), \varphi^{-1})$ is a coordinate chart. □

Proposition 2. *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then, there is a canonical smooth map*
$$df : TX \to TY$$
with the following properties:

(1) *Let* $\phi : U \to \tilde{U}$ *be any chart of* X *and* $\psi : V \to \tilde{V}$ *be any one of* Y *such that* $f(U) \subset V$*. Note that by construction of the tangent bundle we have parametrizations* $T\tilde{U} \to TX$ *and* $T\tilde{V} \to TY$*. Finally, let* $\tilde{f} = \psi \circ f \circ \phi^{-1} : \tilde{U} \to \tilde{V}$ *and* $d\tilde{f}$ *be its differential as defined in the previous lecture. Then the following diagram commutes:*

$$\begin{tikzcd} T\tilde{U} \arrow[r, "d\tilde{f}"] \arrow[d] & T\tilde{V} \arrow[d] \\ TX \arrow[r, "df"'] & TY \end{tikzcd}$$

This property uniquely determines df*.*

(2) *The following diagram is commutative:*

$$\begin{tikzcd} TX \arrow[r, "df"] \arrow[d] & TY \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Moreover, the induced map $T_xX \to T_{f(x)}Y$ *between tangent spaces is linear for all* $x \in X$*.*

(3) *If* Z *another smooth manifold and* $g : Y \to Z$ *another smooth map, then we have the generalized chain rule*

$$d(g \circ f) = dg \circ df.$$

Proof. Let us prove (1). First of all, since the images of all parametrizations of the form $T\tilde{U} \to TX$ cover TX by the construction of TX[4], there can be at most one map $TX \to TY$ that satisfies this property. The given commutative diagram determines what *df* should do on $TU = im(T\tilde{U} \to TX)$.

What we need to show is that the maps defined in the subsets of the form TU by this requirement are compatible to each other. To formulate this take a second set of data: $\Phi : W \to \tilde{W}$ a chart of X and $\Psi : P \to \tilde{P}$ be one of Y such that $f(W) \subset P$. Then, *df* is defined in two different ways on $TU \cap TW = T(U \cap W)$, and we want to prove that in fact these definitions agree with each other. First of all the diagrams

$$\begin{tikzcd} T\phi^{-1}(U \cap W) \arrow[rr] \arrow[dr] & & T\Phi^{-1}(U \cap W) \arrow[dl] \\ & TX & \end{tikzcd}$$

$$\begin{tikzcd} T\psi^{-1}(V \cap P) \arrow[rr] \arrow[dr] & & T\Psi^{-1}(V \cap P) \arrow[dl] \\ & TY & \end{tikzcd}$$

[4]Here we are also using the fact that domains of charts in X whose image under f is contained in the domain of a chart in Y cover X. Hopefully, you are familiar with this by now.

commute by construction of the tangent bundle. Of course, here the vertical maps are the differentials of transition maps. Finally, note that

$$\begin{tikzcd} T\phi^{-1}(U \cap W) \arrow[r, "d\tilde{f}"] \arrow[d] & T\psi^{-1}(V \cap P) \arrow[d] \\ T\Phi^{-1}(U \cap W) \arrow[r] & T\Psi^{-1}(V \cap P), \end{tikzcd}$$

also commutes, where the bottom horizontal map is the differential of the analogue of $\tilde{f}$ and the vertical maps are the horizontal maps of the preceding two diagrams. This last commutativity is because of the chain rule and the commutativity of the diagram:

$$\begin{tikzcd} \phi^{-1}(U \cap W) \arrow[r, "\tilde{f}"] \arrow[d] & \psi^{-1}(V \cap P) \arrow[d] \\ \Phi^{-1}(U \cap W) \arrow[r] & \Psi^{-1}(V \cap P), \end{tikzcd}$$

which is essentially by definition. □

Question 23. Prove parts (2) and (3) of the proposition. □

All we did here is to glue the differentials of the induced maps on the domains of charts. You should not be put-off by the abstractness of the proof.

Question 24. Use the inverse function theorem to prove the following. Let X be a smooth manifold, U an open subset of an Euclidean space, and $\varphi : U \to X$ is a smooth map such that $d\phi_u : T_uU \to T_{\varphi(u)}X$ is a linear isomorphism for some $u \in U$. Then u has a neighborhood $W \subset U$ such that $\varphi |_W : W \to X$ is a parametrization. □

In differential topology we will also want to study certain well-behaved subsets of a smooth manifold, called smooth submanifolds. Just like the local model for a smooth manifold was an Euclidean space, the local model for a submanifold inside a manifold is a linear subspace of an Euclidean space.

Remark 6. If $Z \subset X$ is a submanifold, then Z is naturally inherits the structure of a smooth manifold, and it is sometimes helpful to call X the ambient space or ambient manifold (especially we are going to be working inside X for a while). □

Definition 9. Let X^n be a smooth manifold, and $Z \subset X$ be a subset. We call Z a k-dimensional submanifold if for every $z \in Z$, there is a coordinate chart (U, ϕ) at z inside X such that $\phi(U \cap Z)$ is the intersection of $\tilde{U}$ with a k-dimensional linear subspace of $\mathbb{R}^n$. □

Question 25. Prove that Z is a topological manifold with the subspace topology. Equip it with a smooth structure such that $Z \subset X$ is a smooth map. □

Here is an important definition:

Definition 10. Let Z and X be smooth manifolds, and $f : Z \to X$ be a smooth map. We call f an immersion if its differential is injective at every point of Z, i.e. $df_z : T_zZ \to T_{f(z)}X$ is injective for every $z \in Z$.

Let us also call charts of X as in this definition charts that are adapted to Z. □

Question 26. Prove that the inclusion map of a smooth submanifold $Z \subset X$ is an immersion. $\square$

On the other hand, it is not true that the image of an immersion is always a submanifold. We can bridge the gap as follows.

Proposition 3. *Let Z and X be smooth manifolds, and $f : Z \to X$ be an injective immersion. Then, $f(Z)$ is a submanifold if and only if $Z \to f(Z)$ is a homeomorphism, where we use the subspace topology on $f(Z)$.*

The hard part of this result is to obtain adapted charts in X from the charts of Z. This is a standard consequence of the inverse function theorem. We will come back to it later.

Definition 11. Let Z and X be smooth manifolds, and $f : Z \to X$ be an injective immersion, which is also a topological embedding (as in the previous proposition), then we call f a smooth embedding. Note that $f(Z)$ is a smooth submanifold under this assumption. $\square$

Remark 7. It might help to recall that by definition an injective continuous map that is also closed or open is a topological embedding. Confusingly, the converse is not true, just take any subset of a topological space that is neither open nor closed. $\square$

Question 27. Give an example of a non-injective immersion. Then, give an example of an injective immersion whose image is not a submanifold. $\square$

Recall that a continuous map between topological spaces is called proper if preimages of compact subsets are compact. Any continuous map with compact source is proper. Here is a less trivial lemma from topology.

Lemma 1. *Let X and Y be topological spaces. Assume that Y is Hausdorff and locally compact. Then, a proper continuous map $X \to Y$ is closed.*

A slick proof of this lemma can be found in the chosen answer here. Note that topological manifolds are locally compact (and Hausdorff of course).

We use this notion to give a more useful criterion.

Proposition 4. *Let Z and X be smooth manifolds, and $f : Z \to X$ be an injective immersion. Then, $f(Z)$ is a submanifold which is also closed as a subset of X if and only if f is proper.*

Question 28. Prove this proposition using Proposition 3 and Lemma 1. You will also use the basic fact (with cute proof) that if K is compact and Z is closed in topological space X, then $K \cap Z$ is compact inside Z with subspace topology. $\square$

Question 29. Give an example of submanifold that is not closed as a subset. $\square$

Let us also squeeze in an important definition.

Definition 12. Let G be a smooth manifold which is equipped with a group structure such that the multiplication

$$G \times G \to G$$

and the inverse $G \to G$ are smooth maps. We call such G a Lie group. $\square$

Question 30. Define what it should mean for a Lie group to smoothly act on a smooth manifold. Prove that $GL(4,\mathbb{R})$ is a Lie group and its the action on $Gr(2,4)$ is a smooth action. $\square$

Here is an interesting result called the closed-subgroup theorem. We need to know about the exponential map to prove it. I am pointing it out because it is of a similar nature to our discussion today.

Proposition 5. *Let G be a Lie group and $H \subset G$ be a subgroup, which is also a closed subset. Then, H is a submanifold, and with the induced smooth structure it becomes a Lie group itself.*

Question 31. Give an example of a subgroup of the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is not closed. Prove that it is not a submanifold, but is the injective image of an immersion. Note that above you had found an example of this already (hopefully not this one). What are the closed subgroups of the torus? $\square$

Question 32. Give an alternative description of the smooth structure on $SO(3)$ using the closed-subgroup theorem. Deduce that $SO(3)$ is a Lie group. $\square$

Next time, we will prove the following theorem of Whitney.

Theorem 1. *Any smooth manifold can be smoothly embedded inside $\mathbb{R}^N$ for sufficiently large N.*

5. Jan 20, 2021: Consequences of second countability, cut-off functions, partitions of unity, existence of Riemannian metrics, weak Whitney embedding theorem

In this lecture, we introduce the notion of a partitions of unity that is important in the theory of smooth manifolds. We then use related ideas to prove the existence of a Riemannian metric on any smooth manifold, and also a weak version of the Whitney embedding theorem.

A collection of subsets of a topological space X is called locally finite if for every $x \in X$, there is an open neighbhorhood of x which intersects only finitely many members of the collection.

Proposition 6. *Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ cover X. Then we can find another collection of open subsets $\{V_\alpha\}_{\alpha\in\mathcal{J}}$ with the following properties:*

1. *$\mathcal{J}$ is countable.*
2. *For every $\alpha \in \mathcal{J}$, there exists a $\beta \in \mathcal{I}$ such that $V_\alpha \subset U_\beta$.*
3. *For every $\alpha \in \mathcal{J}$, V_α is the domain of a coordinate chart (V_α, ϕ_α) with the image $\tilde{V}_\alpha$ being the open ball of radius 3 centered at the origin $B_3(0)$.*
4. *For every $\alpha \in \mathcal{J}$, define $W_\alpha := \phi_\alpha^{-1}(B_1(0))$. Then, the collection of open sets $\{W_\alpha\}_{\alpha\in\mathcal{J}}$ cover X.*
5. *$\{V_\alpha\}_{\alpha\in\mathcal{J}}$, which automatically covers X, is locally finite.*

Proof. I will omit the proof. Please see Lee's Proposition 2.24 if you are interested. $\square$

Definition 13. Let us call a cover $\{V_\alpha\}_{\alpha\in\mathcal{J}}$ with the properties (1), (3), (4), (5) a regular cover. $\square$

Question 33. In case you have not done so already, prove that any open subset of an Euclidean space is second countable. Find a more convenient necessary and sufficient condition for a Hausdorff, locally Euclidean topological space to be second countable. □

Question 34. Find a Hausdorff, locally Euclidean topological space for which Proposition 6 does not hold. Prove that a Hausdorff, locally Euclidean topological space which also has the property of satisfying Proposition 6 is second countable. □

Here is the second important input for partitions of unity. Recall that the support $supp(f)$ of a function f is the closure of the subset of points at which f does not vanish.

Lemma 2. *There exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with the following properties*

(1) $supp(f) \subset B_2(0)$.
(2) $f|_{B_1(0)} = 1$.
(3) $0 \leq f(x) \leq 1$ *for all* $x \in \mathbb{R}^n$.

It is customary to call such functions bump functions.

Proof. The key is that we can construct a smooth function $g : \mathbb{R} \to \mathbb{R}$, which vanishes on $\mathbb{R}_{\leq 0}$, but is positive and increasing on $\mathbb{R}_{>0}$. Here is an example

$$g(x) = \begin{matrix} 0, & \text{for } x \leq 0 \\ e^{-1/x}, & \text{for } x > 0 \end{matrix}$$

The smoothness is an easy consequence of the smoothness and decay of the exponential function e^{-x}. □

Question 35. Construct a bump function. For $n > 1$, you might find it convenient to construct one that only depends on the distance from the origin. □

Remark 8. A special case of the Whitney extension theorem says that for any closed subset $C \subset \mathbb{R}^n$, there exists a smooth function $\mathbb{R}^n \to \mathbb{R}$ that vanishes precisely on C. This becomes useful sometimes. Note that C can be wild, like the Cantor set. □

Now, we are in a position to construct partitions of unity.

Definition 14. Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ cover X. We call a collection of smooth functions

$$\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{I}}$$

a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ if the following properties are satisfied.

(1) For every $\alpha \in \mathcal{I}$, $supp(f_\alpha) \subset U_\alpha$.
(2) $\{supp(f_\alpha)\}_{\alpha \in \mathcal{I}}$ is locally finite.
(3) For every $\alpha \in \mathcal{I}$ and $x \in X$, $f_\alpha(x) \geq 0$.
(4) $\sum_{\alpha \in \mathcal{I}} f_\alpha = 1$.

□

Note that the sum in (4) makes sense because of (2). The name comes from (4), where one should think of the RHS as the unit of the ring of functions on X. If you have a partition of unity, you can write any function q on X as a sum of functions that are all supported on domains of coordinate charts:

$$\sum_{\alpha \in \mathcal{I}} f_\alpha q = 1 \cdot q.$$

More often though, you use partitions of unity to patch together locally defined "things" to a global one. We will see an example soon.

Proposition 7. *Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ cover X. Then, there exists a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{I}}$.*

Proof. First assume that $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is regular. Let ρ be a bump function, satisfying the properties in Lemma 2. Then define ρ_α via extending $\rho \circ \psi_\alpha : U_\alpha \to \mathbb{R}$ by zero. $\{\rho_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{I}}$ satisfies all the properties but the last one. The last stroke is to define

$$f_\alpha := \frac{\rho_\alpha}{\sum_{\alpha \in \mathcal{I}} \rho_\alpha}.$$

In the general case, first use Proposition 6 to obtain $\{V_\beta\}_{\beta \in \mathcal{J}}$. Using the previous paragraph we find a partition of unity subordinate to $\{V_\beta\}_{\beta \in \mathcal{J}}$, called $\{g_\beta : X \to \mathbb{R}\}_{\beta \in \mathcal{J}}$. The only thing to fix is that this collection is not indexed by $\mathcal{I}$.

We can choose a map $a : \mathcal{J} \to \mathcal{I}$ which satisfies the property that $V_\beta \subset U_{a(\beta)}$ for every $\beta \in \mathcal{J}$. We are using axiom of choice here. Finally, define

$$f_\alpha = \sum_{\beta \in \mathcal{J}, a(\beta) = \alpha} g_\beta.$$

□

We now give a typical application of partitions of unity.

Definition 15. Let X be a smooth manifold. A Riemannian metric on X is a smoothly varying positive definite symmetric bilinear form on $T_x X$ for every $x \in X$. □

Question 36. Give a rigorous definition of smoothly varying. □

We all know about the standard Riemannian metric on $\mathbb{R}^n$ defined using the inner product of vectors. Let us call this the flat metric.

We will also need the following simple fact from linear algebra. Let V be a real vector space, and $g_1(\cdot,\cdot), \ldots, g_k(\cdot,\cdot)$ positive definite symmetric bilinear forms. Choose any k-tuple of non-negative real numbers $a_1, \ldots, a_k$ at least one of which is positive. Then

$$a_1 g_1(\cdot,\cdot) + \ldots + a_k g_k(\cdot,\cdot)$$

is also a positive definite symmetric bilinear form on V.

Proposition 8. *Every smooth manifold admits a Riemannian metric.*

Proof. Let X be our manifold. Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be a regular cover with partition of unity $\{\rho_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{I}}$.

Note that using the flat metric on Euclidean space we obtain a Riemannian metric g_α on V_α. We define

$$g_x(\cdot,\cdot) := \sum_{\alpha \in \mathcal{I}} \rho_\alpha(x) g_{\alpha,x}(\cdot,\cdot),$$

for every $x \in X$. □

We move on to the Whitney embedding theorem for compact smooth manifolds. The non-compact case is not that much harder. If you are curious, it is in Lee.

Theorem 2. *Let X^n be a compact smooth manifold. Then, X can be smoothly embedded inside $\mathbb{R}^N$ for sufficiently large N.*

Proof. Using Proposition 6 and compactness, we find a regular cover $\{U_i\}_{i\in\mathcal{I}}$ with $I = \{1, \dots, k\}$ for some positive integer k. Let us call the coordinate maps $\phi_i : U_i \to \mathbb{R}^n$.

Let us take a collection of functions $\{\rho_i : X \to \mathbb{R}\}_{i\in\mathcal{I}}$ satisfying conditions (1), (2) and (3) of Definition 14 and define $\tilde{\phi}_i : X \to \mathbb{R}^n$ via extending $\rho_i\phi_i$[5] by zero.

We define a map $X \to \mathbb{R}^{kn+k}$ via

$$x \mapsto (\rho_1(x), \dots, \rho_k(x), \tilde{\phi}_1(x), \dots, \tilde{\phi}_k(x)).$$

$\square$

Question 37. Check that the final map is a smooth embedding. $\square$

Remark 9. Notice that a smooth submanifold of an Euclidean space can be equipped with a Riemannian metric by restricting the flat metric to the tangent spaces. It turns out that any Riemannian metric on a smooth manifold X can be obtained by embedding it into an Euclidean space and this restriction procedure. This is a much more difficult theorem called Nash embedding theorem. $\square$

Here is another nice property that an open cover of a smooth manifold can have, which will be useful in studying the topology of smooth manifolds.

Definition 16. Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ cover X. We call $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ good if it is locally finite and for any finite subset $J \in \mathcal{I}$,

$$\bigcap_{i\in J} U_i$$

is either empty or diffeomorphic to an open ball. $\square$

Question 38. Prove that the open unit ball in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. $\square$

The proof of the following theorem is discussed here. See the accepted answer which I think you might be able to follow (though it is not a problem if you cannot).

Theorem 3. *Every smooth manifold admits a good open cover.*

Finally, let us mention another result that can be proved with similar techniques. Proof is in Lee Proposition 2.28.

Theorem 4. *Every smooth manifold admits a proper smooth map to the real line.*

Such a function is called exhausting. By looking at the preimages of intervals $[-m, m]$ with $m \to \infty$, we obtain a what is called an exhaustion by compact subsets. You can also take the square of any exhausting function to obtain one which only takes non-negative values. This usually makes things a bit more clear conceptually as a non-compact smooth manifold can have any number of "ends" and the two sides of the real line creates an artificial division between them.

[5]Here we are using the scalar multiplication action on $\mathbb{R}^n$

6. Jan 22, 2021: Brief answers to selected questions

Answer to Question 24: This follows immediately from the most standard version of inverse function theorem once you pass to a smaller open neighborhood of u whose image is contained in the domain of a chart in X. You will have to go to an even smaller open neighborhood after using the inverse function theorem of course.

Answer to Question 25: Construct the charts of a smooth atlas by restricting the domains of adapted coordinates charts to their intersection with Z and the targets to the corresponding linear subspaces. It is trivial to see that the transition maps are smooth (do spell it out fully though).

Answer to Question 31: If you take an irrational slope line passing through the origin in $\mathbb{R} \times \mathbb{R}$ and project it to the torus, then you get a subset that by construction is the image of an immersion $\gamma : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, which is also a group homomorphism. The image is clearly a subgroup. The irrational slope assumption directly implies that γ is injective (look upstairs in the plane to what non-injectivity would mean).

In order to show density, consider all the circles $C_a := \{a\} \times \mathbb{R}/\mathbb{Z}$. It suffices to prove that the intersection of the image of γ with C_a is dense in C_a for all $a \in \mathbb{R}/\mathbb{Z}$. Irrational slope implies that these intersections are all of the form

$$\{b + qn \bmod 1 \mid n \in \mathbb{Z}\} \subset \mathbb{R}/\mathbb{Z},$$

where $b, q \in \mathbb{R}$ and with q irrational. It suffices to prove the density for $b = 0$. In that case, what we have is a subgroup of the circle and we are reduced to proving that 0 is an accumulation point, which is equivalent to proving the existence of two elements of $\{n \in \mathbb{Z} \mid qn \bmod 1\} \subset \mathbb{R}/\mathbb{Z}$ that are arbitrarily close to each other. I will leave this final elementary step to you.

The closed subgroups are given by projections of the following subgroups in the plane

- 0-dim: $\{nv_1 + mv_2 \mid n, m \in \mathbb{Z}\}$, where v_1, v_2 are rational vectors.
- 1-dim: $\{nv_1 + mv_2 \mid n \in \mathbb{Z}, a \in \mathbb{R}\}$, where v_1, v_2 are rational vectors.
- 2-dim: The whole torus.

Visualize how these can look like.

Answer to Question 36: It is easy to see that in a chart a Riemannian metric is given by $\frac{n(n+1)}{2}$ functions which are the entries of an $n \times n$ symmetric matrix. The requirement is that in all charts these functions are smooth functions. You should verify that it suffices to check this smoothness in a cover by charts.

Answer to Question 37: If you are not able to do this let me know. Our extra requirement that ρ_i is equal to 1 on $\phi_i^{-1}(B_1(0))$ (and that $\phi_i^{-1}(B_1(0))$ cover X) leads to a proof that does not require any computation.

Below are some additional questions.

Question 39. Prove that

$$\{(x, y) \mid |x| = |y|, y \geq 0\} \subset \mathbb{R}^2$$

is not a smooth submanifold. Prove that there does exist a smooth proper injective map $\mathbb{R} \to \mathbb{R}^2$ with image equal to this set. $\square$

Question 40. Prove that we can take $N = 2n + 1$ in the Whitney embedding theorem for compact smooth manifolds. Do this by starting with the embedding constructed above and projecting to carefully chosen hypersurfaces (until this is not possible anymore). More precisely, if you are given a smooth embedding $M \to \mathbb{R}^N$, and a hyperplane $H \subset \mathbb{R}^N$ that is perpendicular to vector v, then what are the conditions on v so that the the composition

$$M \to \mathbb{R}^N \to H,$$

where the last map is the projection map, is also a smooth embedding? Do some dimension counting and argue that there should be an H satisfying these properties as long as $N > 2n + 1$. □

Remark 10. One can actually get the number to $N = 2n$ by using the Whitney trick (look up!). It is also possible to prove that the real projective plane does not embed into $\mathbb{R}^3$, so this is the optimal general result. On the other hand $\mathbb{RP}^2$ does immerse in to $\mathbb{R}^3$. The image of this embedding is called the Boy surface (look up!). □

Question 41. Revisit the proof of the inverse function theorem. The most important point is to figure out how the contraction mapping principle is used in the proof. You might want to start by understanding how Newton's method (iteration) works for finding square roots (e.g. see the recursion here). □

Next week, we start with the constant rank theorem.

7. Jan 25, 2021: Constant rank theorem, submersions, fiber bundles, fundamental results on ODE's

We start with the ultimate application of inverse functions theorem to smooth manifolds.

Theorem 5 (Constant rank theorem)**.** *Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be a smooth map such that the rank of $df_x : T_xU \to T_{f(x)}\mathbb{R}^n$ is constant over all $x \in U$. Then, for any $x_0 \in U$, there exists coordinate charts (V, ϕ) at x_0 and (W, ψ) at $y_0 := f(x_0)$ such that $f(V) \subset W$ and the induced map $\tilde{V} \to \tilde{W}$ is the restriction of the map $\mathbb{R}^n \to \mathbb{R}^m$:*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0 \dots, 0).$$

Proof. Let us denote the rank by r. By pre- and post-composing f with affine linear isomorphisms we can assume (without loss of generality - why?) that $Jac_{x_0}(f)$ is the matrix with $r \times r$ identity matrix as its principal minor of order r and all the other entries zero.

Let us define a map $\phi : U \to \mathbb{R}^n$ by

$$x = (x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_r(x), x_{r+1}, \dots x_n).$$

It follows that $Jac_{x_0}(\phi)$ is the identity. Hence, we can find a ball neighborhood $U_1 \subset \mathbb{R}^n$ of $(y_{0,1}, \dots, y_{0,r}, x_{0,r+1}, \dots, x_{0,n})$ with an inverse map $\phi^{-1} : U_1 \to U$, which is a parametrization.

Consider the map $g := f \circ \phi^{-1} : U_1 \to \mathbb{R}$:

$$\begin{tikzcd} U \arrow[rr, "f"] & & \mathbb{R}^m \\ & U_1 \arrow[ul, "\phi^{-1}"] \arrow[ur, "g"'] & \end{tikzcd}$$

By construction, for every $x \in \phi^{-1}(U_1)$, we have

$$g((f_1(x), \dots, f_r(x), x_{r+1}, \dots x_n)) = (f_1(x), \dots, f_r(x), f_{r+1}(x), \dots f_m(x)).$$

This implies that g is a map of the form

$$g((z_1, \dots z_n)) = (z_1, \dots, z_r, g_{r+1}(z), \dots, g_m(z)).$$

In particular,

$$Jac_z(g) = \begin{bmatrix} Id_r & 0 \\ A(z) & B(z) \end{bmatrix},$$

for some family of matrices $A(z)$ and $B(z)$. Since ϕ^{-1} is a diffeomorphism onto its image, the rank of $Jac_z(g)$ is also constant at r. This implies that

$$B(z) = 0, \text{ for all } z \in U_1.$$

This means that the functions $g_{r+1}, \dots, g_n : U_1 \to \mathbb{R}$ in reality are independent of the last $n - r$ coordinates, i.e. g is of the form

$$g((z_1, \dots z_n)) = (z_1, \dots, z_r, g_{r+1}(z_1, \dots, z_r), \dots, g_m(z_1, \dots, z_r)).$$

As a final step, define $\psi : \mathbb{R}^m \to \mathbb{R}^m$ by

$$(y_1, \dots, y_n) \mapsto (y_1, \dots, y_r, y_{r+1} - g_{r+1}(y_1, \dots, y_r), \dots y_m - g_m(y_1, \dots, y_r)).$$

Using the inverse function theorem once again, ψ is a diffeomorphism. Hence, we have

$$\begin{tikzcd} U \arrow[r, "f"] & \mathbb{R}^m \arrow[d, "\psi"] \\ U_1 \arrow[u, "\phi^{-1}"] \arrow[ur, "g"'] \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

The lower horizontal map by construction is of the form desired in the statement. $\square$

Most important special cases of this theorem are when the differential has full rank:

- $n = m = r$ (local diffeomorphism)
- $r = n < m$ (immersion)
- $n > m = r$ (submersion)

Remark 11. Note that the it is an open and dense condition on $n \times m$ matrices to be of full rank. $\square$

Question 42. Write down the constant rank theorem for smooth maps between smooth manifolds and prove it using the local version above. $\square$

Question 43. Give the proof of Proposition 3. $\square$

Definition 17. Let $f : X \to Y$ be a smooth map between smooth manifolds. For $x \in X$, if df_x is surjective, x is called a regular point, if not it is called a critical point. If $y \in Y$ is the image of a critical point, it is called a critical value, otherwise y is called a regular value.

If all $x \in X$ are regular points, we call f a submersion. □

Question 44. Prove the regular value theorem using the constant rank theorem: if $f : X \to Y$ is a smooth map between smooth manifolds and y_0 is a regular value, then $f^{-1}(y_0) \subset X$ is a smooth submanifold. □

A special class of submersions are fiber bundles. Here is the definition:

Definition 18. Let $p : E \to B$ be a smooth map between smooth manifolds, and F a smooth manifold. Then, p is called a fiber bundle with fiber F if every point $b \in B$ has a neighborhood $U \subset B$ such that there exists a fiber preserving diffeomorphism $p^{-1}(U) \simeq U \times F$. Here fiber preserving means that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{proj.}"] \\ & U & \end{tikzcd}$$

commutes. □

Question 45. Prove that $TM \to M$ is a fiber bundle with fiber $\mathbb{R}^n$. In fact, it is a vector bundle, a bundle of vector spaces. You should be able to guess the definition (if you don't know already) and check this as well. □

Question 46. Prove that fiber bundles are submersions. □

It turns out that the difference between a submersion and a fiber bundle is not that huge.

Theorem 6 (Ehresmann)**.** *Let $p : E \to B$ be a proper surjective submersion. Then p is a fiber bundle.*

The proof of this theorem is very nice and underlying it is a geometric concept called an Ehresmann connection. This gadget allows you to lift tangent vectors from the base to the total space. We will come back to this later in the course.

We finish by reviewing the fundamental results of ODE theory.

If $U \subset \mathbb{R}^n$ is open, a (smooth) vector field on U is a smooth map $V : U \to \mathbb{R}^n$, equivalently a section of $TU \to U$. Colloquially, a vector field is a choice of smoothly varying collection of vectors at every point of U.

We are interested in the particle trajectories in U whose velocity at any time is equal to the vector specified by V at the point it's at. Finding such trajectories is by definition solving the following ODE for $\gamma : I \to U$, where I is a real interval:

$$\gamma'(t) = V(\gamma(t)), \tag{1}$$

for all $t \in I$.

Remark 12. A special class of vector fields are linear ones, i.e. linear maps $V : \mathbb{R}^n \to \mathbb{R}^n$. You have studied such ODE's in detail and it might be a good time to remember what was going on there (you should have the full picture for $n = 2$). If you had studied phase portraits of more complicated ODE's by doing local analysis near the constant solution etc., you might want to connect what we will be doing in this class with that as well. □

Theorem 7.
- *(existence) For every $x_0 \in U$, there exists an $\epsilon > 0$ such that the ODE* (1) *has a solution $\gamma : (-\epsilon, \epsilon) \to U$ satisfying the initial condition $\gamma(0) = x_0$.*
- *(uniqueness) For any $\epsilon > 0$, the initial value problem as in the previous bullet point has at most one solution*[6]*.*
- *(smooth dependence on initial data) For every $x_0 \in U$, there exists an $\epsilon > 0$ and a neighborhood $U_{x_0} \subset U$ of x_0 such that the solution of the IVP with initial condition $\gamma(0) = x$ on the interval $(-\epsilon, \epsilon)$ exists for all $x \in U_{x_0}$. Moreover, the induced map*

$$(-\epsilon, \epsilon) \times U_{x_0} \to U$$

is a smooth map.

Question 47. Prove the following rectification theorem. Assuming that $V(x_0) \neq 0$, one can find a coordinate system $y_1, \ldots, y_n$ at x_0 such that $V = \frac{\partial}{\partial y_1}$ in the domain of this coordinate system. □

Remark 13. We call the points where the vector field vanishes its singularities. Finding a normal form near singularities is much more difficult. For example, it is not true that every smooth vector field is equal to a linear one in a different coordinate system if the coordinate change is required to be smooth. This is possible using a continuous change of coordinates if the singularity is hyperbolic due to Hartman-Grobman theorem, which is a non-trivial result. □

The main change of perspective in differential topology from a standard ODE course will be to study the solutions of an ODE with all possible initial conditions at the same time (rather than solving a single initial value problem). This leads to the notion of the flow of a vector field. The well-definedness and good behaviour of flows rely heavily on the smooth dependence on initial data property, which may not have been at the forefront thus far in your thinking of the basic theory of ODE's.

8. Jan 27, 2021: Flow of a vector field, escape lemma, completeness of a vector field

Let's add the following to our vocabulary. If $f : X \to Y$ a smooth map and $v \in T_xX$ for some $x \in X$, the vector

$$f_*v := df_xv \in T_{f(x)}Y$$

is called the push-forward of v.

Definition 19. Let X be a smooth manifold. A smooth section of $TX \to X$ is called a (smooth) vector field. □

Question 48. Can you always push forward a vector field by a smooth map? Why? How about a diffeomorphism? □

[6] note that for large ϵ it may have no solutions!

Let us go through what it means to write a vector field on X in local coordinates. This is mostly about notation. Note that if we take a chart (U, ϕ) on X, we obtain a vector field on U by restriction, and one on $\tilde{U} = \phi(U) \subset \mathbb{R}^n$ by construction of TX.

Let us call $x_1, \ldots, x_n$ the coordinate functions[7] on the Euclidean space that $\tilde{U}$ resides. Then, it is customary to denote the vector field on $\tilde{U}$ equal to $(1, 0, 0 \ldots, 0)$ everywhere by $\frac{\partial}{\partial x_1}$, $(0, 1, 0, \ldots, 0)$ everywhere by $\frac{\partial}{\partial x_2}$ and so on. We can think about these constant vector fields also as living on U.

Notice that any vector field on $\tilde{U}$ can be written uniquely as

$$v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n},$$

for functions $v_i : \tilde{U} \to \mathbb{R}$, $i = 1, \ldots, n$. In particular, any vector field on X can be uniquely written in this form in the chart (U, ϕ).

Given a vector field $V : X \to TX$, we can write down the following differential equation for smooth maps $\gamma : I \to X$:

$$\gamma'(t) = V(\gamma(t)), \tag{1}$$

for every $t \in I$, where I is an interval inside the real line with coordinate t, where $\gamma'(t) := d\gamma_t \left(\frac{\partial}{\partial t}\right)$. In coordinate charts this equation is the same as the one we considered in the last class.

If we write V in a coordinate chart (U, ϕ) as above

$$V(x) = v_1(x) \frac{\partial}{\partial x_1} + \ldots + v_n(x) \frac{\partial}{\partial x_n}, \text{ for all } x \in \tilde{U},$$

and denote the components of γ by γ_i in the same chart, the equation (1) is equivalent to

$$\gamma_i'(t) = v_i(\gamma(t)), \text{ for } i = 1, \ldots n$$

which might be a more familiar form of an ODE ("a system of ODE's").

Solutions of the equation (1) are called integral curves. We know that for any $x_0 \in X$, there is an $\epsilon > 0$ and an integral curve $\gamma : (-\epsilon, \epsilon) \to X$ satisfying $\gamma(0) = x_0$. Note that an interval can sometimes be extended to larger intervals in time. If it cannot be extended we will call the integral curve maximal.

Question 49. Explain why the domain of a maximal integral curve should be an open interval. Then using the same idea prove the following lemma. □

Lemma 3 (Escape lemma)**.** *Let X be a smooth manifold and $V : X \to TX$ be a vector field. Assume that the domain of definition of a maximal integral curve γ is not the entire real line. Then prove that the image of γ is not contained in a compact subset of X.*

We finally come to the fundamental theorem of flows.

[7]Great confusion is caused by denoting the coordinate functions and the coordinates of an arbitrary point in $\mathbb{R}^n$ with the same symbols. This corresponds to the following: we usually denote the value of the coordinate function x_i at point x by x_i. In the real line with coordinate function x we sometimes make it even more confusing and denote the point which takes value x under the coordinate function x by just x. All three objects would ideally get their own symbol.

Theorem 8. *Let X be a smooth manifold and $V : X \to TX$ be a vector field. Then there exists a unique subset $\mathcal{U} \subset \mathbb{R} \times X$ containing $\{0\} \times X$ and continuous map $\Phi : \mathcal{U} \to X$ such that $\Phi(0, x) = x$ for all $x \in X$ with the following properties:*

(1) *$\mathcal{U} \subset \mathbb{R} \times X$ is open.*
(2) *$\Phi : \mathcal{U} \to X$ is smooth.*
(3) *For any $(t, x) \in \mathcal{U}$,*
$$d\Phi_{(t,x)}\left(\frac{\partial}{\partial t}\right) = V(\Phi(t, x))).$$
(4) *For any $x \in M$, $I_x := \mathcal{U} \cap (\mathbb{R} \times \{x\}) \subset \mathbb{R}$ is connected and the integral curve of V given by*
$$\Phi(\cdot, x) : I_x \to X$$
cannot be extended to a larger interval (i.e. it is maximal).

Question 50. Make sure you really understand what is meant by the vector field $\frac{\partial}{\partial t}$ in $\mathbb{R} \times X$. □

I would suggest taking this as a black box for the time. This is not because the proof is hard (see Theorem 17.9 in Lee.) As expected, the proof relies on the existence, uniqueness and the smooth dependence on initial data properties discussed in the previous lecture. Your priority should be to understand the statement. Rigorous proof can wait, but its basic inputs should also be clear.

Definition 20. The map $\Phi : \mathcal{U} \to X$ from Theorem 8 is called the flow of the vector field V. □

Question 51. Explicitly describe the flow (including its domain) of the vector fields $\frac{\partial}{\partial x_1}$, $x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ and $-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$ on the open unit disk in the plane with coordinates x_1 and x_2. □

Remark 14. The picture in your mind should be clear: the points in the manifold are all flowing (backwards and forwards in time) in the directions (and with speeds) dictated by the vector field. The only tricky point is that the integral curves might stop existing after some time. This last point is a common occurance, not just some theoretical what if - just think about $\frac{\partial}{\partial x}$ on an open interval finite in either side in the real line. If you want to come up with examples that look less clear, then use that any connected open interval is diffeoemorphic to the real line. □

Question 52. Find a diffeomorphism $[1, \infty) \to [0, 1)$ which sends the vector field $V(x) = x^2\frac{\partial}{\partial x}$ to a constant one. Explain the "blowing-up" of the unique solution of the IVP
$$x' = x^2 \text{ with } x(0) = 1$$
in this light. □

Definition 21. Let X be a smooth manifold and $V : X \to TX$ be a vector field. We call V complete, if the domain of definition of all maximal integral curves are the entire real line. This is equivalent to saying that the flow of V is defined on the entire $\mathbb{R} \times X$. □

Question 53. Prove that compactly supported vector fields are complete using the Escape lemma. What does this say about vector fields on compact smooth manifolds? □

The following actually is used in the proof of the fundamental theorem of flows, so logically it is not entirely accurate to state it here, but conceptually it makes full sense.

Proposition 9. *Let X be a smooth manifold and $V : X \to TX$ be a complete vector field. Then prove that the flow $\mathbb{R} \times X \to X$ of V defines a Lie group action of $\mathbb{R}$ with its additive group structure on X.*

This is a consequence of the uniqueness property of ODE's and equation (1) being autonomous, i.e. it does not matter at what time a particle starts out at a point, its trajectory looks the same. More succintly, if $\gamma(t)$ is an integral curve, so is $\gamma(t-\Delta)$ for any Δ, and it is the unique one with the initial condition $\tilde{\gamma}(\Delta) = \gamma(0)$.

Remark 15. Of course, there is a statement for non-complete vector fields but it is a bit confusing to state, so I omitted it. □

Question 54. Let X be a smooth manifold, $V : X \to TX$ be a complete vector field with flow $\Phi : \mathbb{R} \times X \to X$, and define $\Phi_t := \Phi(t, \cdot) : X \to X$. Prove that for every time t, Φ_t is a diffeomorphism. Moreover, show that Φ_t preserves V. □

9. Jan 29, 2021: Brief answers to selected questions

Answer to Question 43: First of all, let me point out that the only if part of this statement is utterly useless. Regardless, let us prove it. Assume that $f(Z)$ is a submanifold. It suffices to prove that $Z \to f(Z)$ is an open map. Let $V \subset Z$ be open. Let (U, ϕ) be any chart in X adapted to $f(Z)$ such that $\tilde{U}$ is $B_1(0)$. Domains of such charts cover $f(Z)$, therefore, it suffices to show that $U \cap f(V)$ is open in $f(Z)$. We will show that any $x \in U \cap f(V)$ has an open neighborhood W in X such that $W \cap Z \subset U \cap f(V)$.

Take the $z \in Z$ such that $f(z) = x$. Since $f^{-1}(U)$ is open, we can find a neighborhood N of z such that $f(N) \subset U$. Consider the composition $N \to U \to B_1(0)$ and note that by definition the image lies inside a linear subspace L in $B_1(0)$. Moreover, this map is an immersion and therefore if we restrict the target to L, i.e. consider the smooth map $N \to L \cap B_1(0)$, we see that the differential is an isomorphism everywhere on N. Therefore the image of $N \to L \cap B_1(0)$ is an open subset of $L \cap B_1(0)$ by the inverse function theorem. Take any open subset of $B_1(0)$ whose intersection with L is the open subset of $L \cap B_1(0)$ that we just produced, and define its preimage under ϕ to be the desired W.

The main point of this proposition was of course the if direction. Here we need the constant rank theorem. Let $z \in Z$ and we need to produce an adapted to $f(Z)$ chart in X at $f(z)$. The constant rank theorem gives us: (V, ϕ) at z and (W, ψ) at $f(z)$ such that $f(V) \subset W$ and the induced map $\tilde{V} \to \tilde{W}$ is the restriction of the map $\mathbb{R}^n \to \mathbb{R}^m$:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0 \ldots, 0).$$

The chart (W, ψ) almost does job but what we don't yet have is that W might be intersecting Z at points that are not in the image of V. This is where we need to use the topological embedding assumption. We will show that $\psi(f(z))$ has a neighborhood N in $\tilde{W}$ such that $N \cap \phi(f(Z) \cap W)$ is contained in the image of $\tilde{V}$.

Assume otherwise, that there is a sequence of points in $\phi(f(Z) \cap W)$ converging to $\psi(f(z))$ none of which are contained in the image of $\tilde{V}$. Move these points back to X. They all lie in the image of f by choice. Now consider the continuous map

$f^(Z) \to Z$. These points on $f(Z)$ converge to $f(z)$ by construction, hence their images need to converge to z by continuity. Therefore, some of them have to be contained in V, which is a contradiction. Hence, there is indeed such an N.

To conclude, we easily check that the chart $(\psi^{-1}(N), \psi|_{\psi^{-1}(N)}$ is an adapted chart by the choice of N.

Let's also do Proposition 4 very quickly. If $f(Z)$ is a closed submanifold, we need to show that if $K \subset X$ is compact, then $f^{-1}(K)$ is compact. This follows because $K \cap f(Z)$ is compact in $f(Z)$ by the closedness of $f(Z)$ and $Z \to f(Z)$ is a homeomorphism by the submanifoldness of $f(Z)$ using Proposition 3.

Conversely, using the Lemma we immediately obtain that $f : Z \to X$ is a closed map. We then use that a bijective continuous and closed map is a topological embedding and Proposition 3 to conclude that $f(Z)$ is a submanifold.

Answer to Question 44: For this one the constant rank theorem immediately gives you an adapted to $f^{-1}(y_0)$ chart in X at every point x of $f^{-1}(y_0)$. We find (V, ϕ) at x and (W, ψ) at y_0 such that $f(V) \subset W$ and the induced map $\tilde{V} \to \tilde{W}$ is the restriction of the map $\mathbb{R}^n \to \mathbb{R}^m$:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m),$$

where $n \geq m$. Also assume that $\tilde{W}$ is a ball. The image of $V \cap f^{-1}(y_0)$ under ϕ is contained in the affine linear subspace of the form $x_1 = c_1, \ldots x_m = c_m$, where c_i are constants.

Answer to Question 47: Choose a hyperplane H passing through x_0 in the Euclidean space that is transverse to $V(x_0)$. Let ϵ and U_{x_0} be as in the smooth dependence on initial data property. Define the map

$$(-\epsilon, \epsilon) \times (H \cap U_{x_0}) \to U,$$

by restricting the smooth map from again smooth dependence on initial data property.

The Jacobian at x_0 sends H to H by identiy and $\frac{\partial}{\partial t}$ to $V(x_0)$. Therefore, it is a linear isomorphism. We now use Question 24 to get the coordinates $y_1, \ldots, y_n$ with y_1 being the t-coordinate. The conclusion is immediate as $\frac{\partial}{\partial t}$ is sent to $V(x)$ for all points in the domain of the parametrization by construction.

Answer to Question 49: Assume the contrary, that the image is contained in a compact set K but also that the domain of definition cannot be extended say in the positive direction past $a \in \mathbb{R}$. Take a sequence of times $t_1, t_2, \ldots$ which monotonically converge to a from the left. By compactness $\gamma(t_1), \gamma(t_2), \ldots$ has a convergent subsequence in K. We pass to that subsequence of times and keep the notation the same. Let the accumulation point in K be p. We know that p has a neighborhood U in X such that all integral curves starting inside U can be defined for some time $\epsilon > 0$. Let t_N be so that $a - t_N < \epsilon$ and $\gamma(t_N) \in U$. Since the integral curve passing through $\gamma(t_N)$ can be defined for at least ϵ time in the forward direction, we find a contradiction using the uniqueness property of ODE's.

Answer to Question 52: The map $x \mapsto \frac{x-1}{x}$ does the job.

Answer to Question 54: We need to show that $(d\Phi_t)_p V(p) = V(\Phi_t(p))$ for all $p \in X$. Take an integral curve γ passing through p at time 0. Then, the group property of the flow shows that $\Phi_t \circ \gamma$ is the integral curve passing through $\Phi_t(p)$ at time 0. The velocity at time 0 of $\Phi_t \circ \gamma$ can be computed by being an integral curve but also using the chain rule. This proves the desired equality.

10. Feb 1, 2021: Gradient vector field, directional derivative, height normalized gradient vector field, fundamental theorem of Morse theory

Today we will start talking about Morse theory but we first have to understand the gradient of a real valued function (or just function) on a smooth manifold. The first thing to stress is that it depends on a Riemannian metric as well. The usual gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ is going to be a special case where we use the flat metric, so keep that in mind as we go.

Let g be a Riemannian metric on a smooth manifold X. In a coordinate chart (U, ϕ) with coordinates $x_1, \ldots, x_n$, we can define the smooth functions

$$g_{ij}(x) := g\left(\frac{\partial}{\partial x_i}(x), \frac{\partial}{\partial x_j}(x)\right).$$

In other words, in this chart, the bilinear form g at x is given by the matrix $(g_{ij}(x))_{i,j\in[n]}$ with respect to basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Let us define the inverse matrix of $(g_{ij}(x))_{i,j\in[n]}$ by $(g^{ij}(x))_{i,j\in[n]}$. This means that

$$\sum_{k=1}^{n} g_{ik}(x) g^{kj}(x) := \delta_{ij},$$

where the RHS is the Kronecker delta symbol.

Remark 16. These type of tensor computations in coordinates really get much more tractable with the Einstein conventions. It will not help us in this course so I will not introduce it but if I used Einstein convention I would not have made the mistake that I did (at least I would like to think so). □

Let $f : X \to \mathbb{R}$ be a smooth function. We define in the chart (U, ϕ):

$$grad_g f(x) := \sum_{j=1}^{n} \sum_{i=1}^{n} g^{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1}$$

We can do this for every chart and we need to prove that they are compatible with each other in the sense that they are sent to each other by the transition maps.

Question 55. First make sure you understand that the vector fields $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$ have no reason to be compatible across the charts. Then, check by direct computation that Equation (1) gives a well-defined vector field on X. □

The direct computation you just made shows that the vector field that you obtain on $\tilde{U}$ using Equation (1) is independent of coordinates. Namely, once the Riemannian metric is fixed no matter what coordinates you use on $\tilde{U}$, Equation (1) results in the same vector field. There is of course a better reason why this mysterious looking expression is coordinate independent. This better reason comes in the form of a coordinate free description.

Definition 22. Let X be a manifold and $x \in X$. A covector at x is a linear map $T_xX \to \mathbb{R}$, which is by definition an element of the linear dual $T_x^*X := (T_xX)^\vee$. A smoothly varying collection of covectors at every point of X is called a covector field. □

Remark 17. Soon we will define the cotangent bundle $T^*X \to X$ which is a vector bundle whose fiber over every $x \in X$ is T_x^*X. □

We will now also explain the concept of directional derivative of a function with respect to a tangent vector on a smooth manifold. This takes in a tangent vector, a function and returns a real number, which measures the change in the function in the direction of the vector. It does not require any extra data. Note that despite its name it depends on the "magnitude" of the vector as well.

Definition 23. Let $f : X \to \mathbb{R}$ and $v \in T_xX$. Let V be any vector field that is defined in a neighborhood U of x, which satisfies $V(x) = v$. Let $\gamma : (-\epsilon, \epsilon) \to U$ be an integral curve of V such that $\gamma(0) = x$. We define

$$v \cdot f := \frac{d}{dt} f(\gamma(t)) \mid_{t=0} .$$

Note that the RHS is just a fancy way of writing the derivative of the function $f \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}$ at 0. □

This way of defining the directional derivative is the most intuitive, but not the most practical. In particular, we need to show that $v \cdot f$ is independent of how the extension V is chosen. Note that we have the differential $df : TX \to T\mathbb{R}$, and we can find the canonical real number a (momentary notation) such that

$$df_x v = a \frac{\partial}{\partial y},$$

where we called y the coordinate on $\mathbb{R}$.

Question 56. Using the chain rule, prove that $v \cdot f = a$. □

This in particular shows the independence on the choice of V. It is also clear that $v \cdot f$ only depends on the restriction of f to an arbitrary neighborhood of x.

Proposition 10. *Using the notation as in the proposition,*

(1) *The directional derivative operation is linear in the vector variable*

$$(cv + w) \cdot f = c(v \cdot f) + w \cdot f,$$

for any $c \in \mathbb{R}$ and $w \in T_xX$.

(2) *The directional derivative operation satisfies the Leibniz rule:*

$$v \cdot (fg) = f(x)(v \cdot g) + g(x)(v \cdot f).$$

Proof. (1) follows from the linearity of df. For (2), if $v = 0$ we are done, otherwise choose some coordinates $x_1, \dots, x_n$ near x such that $v = \frac{\partial}{\partial x_1}$. Note that it easy to find such coordinates by choosing an arbitrary one and then applying a linear isomorphism. We can use $V = \frac{\partial}{\partial x_1}$ in the definition to check the equation.

Notice that $v \cdot f$ is nothing but the partial derivative of f at x with respect to the x_1 coordinate. This explains our arbitrary looking notation for constant vector fields. The Leibniz rule follows from the Leibniz rule for partial derivatives from calculus. □

Definition 24. Let $f : X \to \mathbb{R}$, then we define a covector field, which is called df by a slight abuse of notation as follows: for any tangent vector v, let $df(v) := v \cdot f$. □

Question 57. Check that df is indeed a covector field. □

Remark 18. The abuse is slight because df is nothing but $df : TX \to T\mathbb{R}$ thought of as a covector field in the only way that is possible. The way I presented the material might have hidden this a little but I hope you will unravel notation and see this. I wanted to keep the directional derivative viewpoint in focus. □

Now given a Riemannian metric g, the gradient is defined using the following equation of covector fields

$$g(grad_g f, \cdot) = df.$$

Remark 19. This definition works because (and only because) g is non-degenerate. A good way to express the non-degeneracy of a bilinear form β on a finite dimensional vector space V is to say that the map $\beta^{\#} : V \to V^{\vee}$ given by

$$w \mapsto \beta(w, \cdot)$$

is an isomorphism. Note that

$$w \mapsto \beta(\cdot, w)$$

being an isomorphism is an equivalent condition, even if β is not symmetric, antisymmetric etc. □

Question 58. Use the notation in the remark above. Fix a basis $e_1, \dots e_n$ of V, and let $e^1, \dots e^n$ denote the dual basis of $V^{\vee}$. Let $\beta_{ij} := \beta(e_i, e_j)$. Prove that the matrix of $\beta^{\#} : V \to V^{\vee}$ in the bases fixed is exactly $(\beta_{ij})_{i,j\in[n]}$. □

Question 59. Compute $grad_g f$ in charts and prove that it agrees with our previous definition. For notation, it might be helpful to define the local covector fields dx_i, which satisfy $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$, where the RHS is the Kronecker delta symbol. □

Question 60. Suitably interpret the following statement and prove it: $grad_g f$ is perpendicular to the level sets of f. □

We can also prove that f is non-decreasing in the direction of $grad_g f$. We have

$$grad_g f(x) \cdot f = df(grad_g f(x)) = g(grad_g f(x), grad_g f(x)) =: |grad_g f(x)|^2 \geq 0.$$

Clearly the singular points of $grad_g f$ are precisely the critical points of f, and if we are at a regular point, f is strictly increasing in the direction of the gradient vector field. Note that the gradient is not necessarily the direction that f increases the fastest anymore.

Let us also introduce the notion of a height normalized gradient vector field

$$V_g f := \frac{grad_g f}{|grad_g f|^2},$$

which of course is only defined on $M - crit(f)$. The geometric interpretation of $V_g f$ is much more clean: it sends level sets to level sets as long as it is defined. The advantage of the actual gradient vector field is that it is defined everywhere.

Lemma 4. *Let f contain no critical values in the interval $[a, b] \subset \mathbb{R}$. Take a point on $f^{-1}(a)$ and assume that its forward flow under $V_g f$ exists for time $b-a$. It ends up at a point of $f^{-1}(b)$.*

Proof. Let $\gamma : [0, b-a] \to X$ be the integral curve in question. We compute the derivative of the function $f \circ \gamma$. This is of course just the directional derivative of f in the direction of $V_g f$. We already computed the directional derivative of f in the direction of the actual gradient vector field and found $|grad_g f(x)|^2$. By the linearity of directional derivative, we find that the derivative of $\gamma \circ f$ is equal to 1! This proves the result. □

Proposition 11. *Let f contain no critical values in the interval $[a, b] \subset \mathbb{R}$ and assume that $f^{-1}([a, b])$ is compact. Then, the flow of $V_g f$ takes $f^{-1}(a)$ to $f^{-1}(b)$ diffeomorphically. In fact, we can naturally construct a diffeomorphism $f^{-1}(a) \times [a, b] \to f^{-1}([a, b])$ by following the flow.*

Question 61*.* Prove this proposition. □

11. Feb 3, 2021: Height functions as examples, stable/unstable sets, linearization of a vector field at a zero, non-degeneracy of a critical point, stable manifold theorem for gradient vector fields of Morse functions, change of level sets as we pass through critical points

We start by developing some intuition using actual height functions on Euclidean space. Consider a submanifold $Z \subset \mathbb{R}^n$ and let $h : Z \to \mathbb{R}$ be the projection map to one of the coordinate axes, say the first one. It is customary to imagine this coordinate as in the up-down direction, so h gives the heights of the points in Z. The critical points of h are precisely the points where the tangent spaces are horizontal.

Recall that for such Z, we automatically obtain a Riemannian metric by restricting the flat metric. At the critical points of h, we know that the gradient vanishes and the height normalized gradient is not defined. At regular points, we can describe the height normalized gradient as the canonical tangent vector that is perpendicular to the level set that it belongs to and whose first coordinate is 1. The gradient is the vector in the same direction with its lenght the inverse of the length of the height normalized gradient.

Question 62*.* Take a donut and put it on the table as you normally would (1). Then, hold it in up-right position with only one point of the donut touching the table (2). Finally, always having only one point touching the table, slant it a little bit (3). Each of these three positions desribe a height function on the torus that is the surface of the donut. Find the critical points of the height function for all three positions. Using the flat metric explain as much as you can about what happens under the gradient and height normalized gradient flows - what are different behaviors of the points on the torus, of level sets, sublevel sets...? We will do part of this analysis systematically below, so the goal is to just get a sense. □

Let us give some standard definitions for general vector fields. Let V be a vector field on X, and assume that $V(x) = 0$ for some x. Such points can be called singularities, equilibrium points or just zeros. If V is the gradient vector field of a function (for some metric), they are also the critical points of the function.

We call the set of points that converge to x in the forward direction the stable set of x and the ones converge to x in the backwards direction the unstable set. Precisely,

$$SS_x^V := \{y \in X \mid \text{the maximal integral curve } \gamma \text{ which is at } y \text{ at time } 0 \text{ exists for all future times and satisfies } \gamma(t) \to x \text{ as } t \to \infty\}$$

$$US_x^V := \{y \in X \mid \text{the maximal integral curve } \gamma \text{ which is at } y \text{ at time } 0 \text{ exists for all past times and satisfies } \gamma(t) \to x \text{ as } t \to -\infty\}$$

Remark 20. Note that if you move slightly from x to a point in the stable set you end up coming almost back to x with the flow after sufficient waiting. This explains the name stable set. The name unstable is less clear to me but if you remember which one is called stable, you will remember the other. Note that if all the nearby points to x belong to the stable set, then we recover the notion of a stable equilibrium from the ODE courses. □

Remark 21. We would like the stable set and unstable set to be submanifolds. They are not in general and the situation is quite complicated. If x is a hyperbolic equilibrium point, they are images of injective immersions but not necessarily submanifolds. Hyperbolicity condition comes up in the Hartman-Grobman theorem as well. It is a simple condition, it means that the linearization of the vector field at x does not have any eigenvalues with vanishing real part (naming is unfortunate). It turns out that for gradient vector fields of Morse functions stable/unstable sets of equilibrium points indeed are submanifolds! □

Question 63. Find a linear vector field in $2d$ with a non-hyperbolic isolated equilibrium point. □

Question 64. Name a dynamical phenomenon that you see in arbitrary flows but never in gradient flows. Bonus points if you can name two. □

Recall that the linearization of a vector field $V : U \to \mathbb{R}^n$ at a singular point $x \in U$, where U is an open subset of $\mathbb{R}^n$ is given by the vector field

$$dV_x : T_xU \to T_{V(x)}\mathbb{R}^n,$$

noting that both the domain and target of this map is equal to $\mathbb{R}^n$.

Definition 25. We call a singular point/equilibrium point/zero of a vector field non-degenerate if its linearization at any chart has only an isolated zero. □

We will soon see a coordinate invariant description of the linearization of a vector field at a singularity. This of course means that, suitably interpreted, the linearization on charts do not really depend on the coordinates, but we do not need this fact for the definition. It is easier to see that the non-degeneracy is independent of the choice of coordinates.

Definition 26. We call $f : M \to \mathbb{R}$ a Morse function if all of the singularities of its gradient vector field are non-degenerate. This is independent of the choice of the Riemannian metric. □

Question 65. Go back to the height functions on the torus above and decide which of the critical points are non-degenerate in all three cases. Conclude that only two positions give Morse functions. □

Definition 27. Consider a non-degenerate critical point x of f, the type of x is a pair of integers (s, u) defined as follows. Consider the Hessian of f at x in any coordinate chart. This is a symmetric matrix and non-degeneracy implies that it has no 0 eigenvalues. Therefore it is diagonalizable and all of its n eigenvalues are either positive or negative real numbers. We define s to be the number of negative eigenvalues, and u is the number of positive ones (both counted with multiplicities). Note that $s + u = dim(X)$. □

Again, we will give a better definition of this notion soon. It will then be clear that in fact the tangent space at a critical point is split into one s dimensional and one u dimensional transverse subspaces

$$T_xX = T_x^sX \oplus T_x^uX.$$

We will not prove the following difficult theorem. The proof can be found in Banyaga-Hurtubise "Lectures on Morse homology".

Theorem 9. *Let X be a compact smooth manifold with a Riemannian metric g, f a Morse function on X and x a critical point of f. Then, the stable set of $grad_g f$ at x is the image of a smooth embedding $T_x^sX \to X$ sending 0 to x and tangent to T_x^sX. The analogous statement for the unstable set is also true.*

It also follows that the non-empty intersections of $SS_x - \{x\}$ with level sets are $s-1$ dimensional spheres. One can also show that $SS_x - \{x\}$ intersects transversely with all the level sets. We will talk about transversality next time. Here and in what follows an omitted superscript in the notation of the stable/unstable sets mean that we are talking about the gradient vector field that is being considered at the time.

Question 66. Go back to the two Morse height functions on the torus and analyze the stable/unstable manifolds of each critical point. □

Proposition 12. *Let X be a compact smooth manifold with a Riemannian metric g and f a Morse function on X. Assume that f contains a single critical value in the interval $[a, b] \subset \mathbb{R}$ at $c \in (a, b)$ with (for simplicity) a single critical point $x \in f^{-1}(c)$. Then, the flow of $V_g f$ defines a continuous map from $f^{-1}(a)$ to $f^{-1}(c)$ as follows:*

- *We really just follow the flow in the complement of $f^{-1}(a) \cap SS_x$. This part of the map is a diffeomorphism onto $f^{-1}(c) - \{x\}$.*
- *$f^{-1}(a) \cap SS_x$, which is an embedded $s - 1$ dimensional sphere in $f^{-1}(a)$ if $s \geq 1$ and empty otherwise, is mapped to x entirely.*

There is an analogous statement for going from b to c with the backwards flow.

So colloquially, as we go from a to c the only interesting thing that happens is that a sphere contracts to the critical point, and as we go from c to b another sphere grows from the critical point. To really analyze this in sufficient detail, we would need to study handle attachments and consider what happens to the sublevel sets $f \leq a$ as we pass through critical points. I will not do this as we have many other things to cover. It can be done as in here `http://math.uchicago.edu/~may/REU2019/REUPapers/Bohm.pdf`, which seemed like a quite clear exposition to me.

One statement that would not be difficult to make is that up to homotopy equivalence, $f^{-1}(c)$ is obtained from attaching an s dimensional cell onto what is called the attaching sphere $f^{-1}(a) \cap SS_x$.

Question 67. Go back to the up-right Morse height function on the torus and analyze the changes in level sets. Start from $-\infty$ and go to ∞. □

Remark 22. Two dimensional handle attachments are easy but I have to warn you that they can be quite complicated in higher dimensions. Attaching spheres are simple as manifolds, but they can be embedded in very complicated ways. □

Next week, we will discuss transversality of submanifolds and clean up some of the mess from above. We will introduce the notion of pair (f, g) being Morse-Smale. We will define a chain complex generated by the critical points of a Morse-Smale pair which computes the singular homology of the manifold. To end our discussion, we will prove the Morse inequalities.

12. Feb 5, 2021: Brief answers to selected questions

Answer to Question 55: Below I used half-baked Einstein summation conventions. A^{jk} denotes the inverse matrix of A_{jk} and the sums are over the repeated indices.

Let $\Phi(x) = y$ be a change of coordinates (i.e. a transition function). Let us denote the Jacobian matrix by J_{ij}. Then we have that the vector field $\sum a_i \frac{\partial}{\partial x_i}$ in y coordinates (can think of it as the push-forward vector field) is

$$\sum J_{ij}(\phi^{-1}(y))a_j(\phi^{-1}(y))\frac{\partial}{\partial y_i}.$$

Also note that in y coordinates the components of the (inverse matrix of the) Riemannian metric are

$$g^{ij}(y) = \sum J_{ki}(\phi^{-1}(y))g^{kl}(\phi^{-1}(y))J_{jl}(\phi^{-1}(y))$$

and for the partial derivatives we have

$$\frac{\partial f}{\partial x_i}(\phi^{-1}(y)) = \sum J_{ij}(\phi^{-1}(y))\frac{\partial(f \circ \Phi^{-1})}{\partial y_j}(y),$$

or in the other direction

$$\sum J^{ji}(\phi^{-1}(y))\frac{\partial f}{\partial x_i}(\phi^{-1}(y)) = \frac{\partial(f \circ \Phi^{-1})}{\partial y_j}(y),$$

We need to therefore analyze

$$\sum_j J_{ij}(\phi^{-1}(y))g^{jk}(\phi^{-1}(y))\frac{\partial f}{\partial x_k}(\phi^{-1}(y))\frac{\partial}{\partial y_i}.$$

Inserting an identity (and omitting the arguments)

$$\sum J_{ij}g^{mj}J_{ml}J^{ln}\frac{\partial f}{\partial x_n}\frac{\partial}{\partial y_i}.$$

A slight rearrangement and using the above equations about how the metric tensor and partial derivative functions transform:

$$\sum J_{ml}g^{mj}J_{ij}J^{ln}\frac{\partial f}{\partial x_n}\frac{\partial}{\partial y_i} = \sum g^{li}(y)\frac{\partial(f \circ \Phi^{-1})}{\partial y_l}(y)\frac{\partial}{\partial y_i},$$

which is the desired expression.

Answer to Question 59: By testing with the vectors $\frac{\partial}{\partial x_i}$, one easily gets

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

Using the previous question's simple linear algebra we get the desired expression. Note that we are using the inverse of $\beta^{\#}$ in that question's notation when going from a covector to vector (from df to $grad_g f$).

Answer to Question 61: Strictly speaking, we did not develop the notion of a manifold with boundary so you cannot really answer this question. On the other hand, you can define the map

$$f^{-1}(a) \times [a, b] \to f^{-1}([a, b])$$

using the flow and check that it is an embedding (injective immersion, which is a topological embedding). Definitions should (and will) work out so that this is a diffeomorphism onto its image.

Answer to Question 63: The vector field generating rotation of the plane is an example.

Answer to Question 64: Periodic orbits, homoclinic orbits.

13. Feb 8, 2021: Transversality, Sard's theorem, parametric transversality theorem, vector bundles, transition functions, constructions of vector bundles, cotangent bundle, cleaning up the definitions of linearization and non-degeneracy from earlier

Today, we will start with two fundamental notions: transversality of submanifolds and constructions of vector bundles from old ones, in particular of cotangent bundle from the tangent bundle. Then we will clear up the definition of the linearization of a vector field and non-degeneracy.

Let's start with transversality. We call two linear subspaces W, W' of a vector space V transversal if

$$W + W' = V.$$

Note that this can only happen if

$$dim W + dim W' \geq dim V,$$

and in that case transversality is an open dense condition in the space of pairs of subspaces of the given dimensions.

Two submanifolds Z and Z' of a smooth manifold X in turn will be called transverse if at each intersection point they locally look like two transversely intersecting linear subspaces as above. As a special case: if

$$dim Z + dim Z' < dim X,$$

then this means that Z and Z' should not intersect. Now without any assumptions on dimensions, we expect Z and Z' to intersect transversely generically, but at this point this is more a heuristic than a mathematical statement.

It becomes useful give a definition of transversality that is slightly more general.

Definition 28. Let Z, Z' and X smooth manifolds; and $f : Z \to X$ and $f' : Z' \to X$ be smooth maps. We say that f and f' are transversal to each other if for every $z \in Z$ and $z' \in Z'$ such that $f(z) = f'(z') = x$,

$$im(df_z) + im(df'_{z'}) = T_x X.$$

If f and f' are transversal we write $f \pitchfork f'$. Finally if Z and Z' are submanifolds, we call them transverse if their inclusion maps are transverse and if this is the case write $Z \pitchfork Z'$. □

Question 68. Explain what it should mean for a smooth map $f : Z \to X$ to be transverse to a submanifold $Z' \subset X$. This is denoted by $f \pitchfork Z'$. □

Proposition 13. *If $Z \pitchfork Z'$ are submanifolds of X, then every $x \in Z \cap Z'$ admits a chart (U, ϕ) such that $\phi(U \cap Z)$ and $\phi(U \cap Z')$ are intersections of two transverse linear subpaces with $\tilde{U}$.*

Question 69. Prove this proposition. Start with a chart adapted to Z. Then, working inside the corresponding open subset of the Euclidean space, consider the projection of Z' to the linear subspace given by its tangent space at x. Use the inverse function theorem to get an inverse map. Extend this to a parametrization using transversality to finish (you will need to use inverse function theorem again). □

Proposition 14. *Let $f : Z \to X$ be transverse to a submanifold $Z' \subset X$. Then $f^{-1}(Z') \subset Z$ is a submanifold.*

Question 70. Prove this proposition. Start with an adapted chart and find a way to use the submersion theorem. □

Remark 23. We have the following generalization. Let Z, Z' and X smooth manifolds; and $f : Z \to X$ and $f' : Z' \to X$ be smooth maps such that $f \pitchfork f'$. Define the fiber product of $Z \times_X Z' \subset Z \times Z'$ as

$$\{(z, z') \mid f(z) = f'(z')\} \subset Z \times Z'.$$

It can be shown that $Z \times_X Z'$ is a smooth submanifold of $Z \times Z'$. You will need a "diagonal trick" to relate this to the proposition above.

Note that we obtain a diagram of smooth manifolds and smooth maps (how?):

$$\begin{tikzcd}
Z \times_X Z' \arrow[r] \arrow[d] & Z' \arrow[d] \\
Z \arrow[r] & X,
\end{tikzcd}$$

which is a pull-back diagram. □

A very important aspect of transversality of submanifolds is to learn how to make sense of and take advantage of our accurate heuristic that it generically holds. The main theorem here is Sard's theorem.

Theorem 10. *If $f : X \to Y$ is a smooth map, then $critv(f) \subset Y$ is of measure zero.*

Question 71. Prove that f is transverse to the submanifold $\{y\} \subset Y$ iff y is a regular value. □

Remark 24. This has a generalization to maps between Banach manifolds (replace Euclidean spaces with Banach spaces) called the Sard-Smale theorem. □

The proof of Sard's theorem is difficult but nice and elementary. I will take it as a blackbox here. Here is how to make Sard's theorem useful. It is called the parametric transversality theorem.

Theorem 11. *Let S, X, Y, Z be smooth manifolds, $F : S \times X \to Y$ and $g : Z \to Y$ be smooth maps. Assume that $F \pitchfork g$, then for $s \in S$, $F(s, \cdot) : X \to Y$ is transverse to g if and only if the s is a regular value of the projection map*

$$(S \times X) \times_Y Z \to S.$$

Question 72. Explain how Question 71 is a special case of this theorem. □

Note that Sard's theorem applies directly to the latter condition! Hence, we get that the desired transversality holds for all but a measure zero set of parameters $s \in S$. In most applications g is just the inclusion of a submanifold. The proof of parametric transversality theorem is easy but I would not worry about it for now. It is much more important to understand the statement.

Let us now shift gears and talk about vector bundles. First recall the definition.

Definition 29. Let $p : E \to B$ be a smooth map between smooth manifolds. Assume that each fiber of p is equipped with the structure of a real vector space. Then, p is called a vector bundle if there is a finite dimensional real vector space V such that every point $b \in B$ has a neighborhood $U \subset B$ such that there exists a fiber preserving fiberwise linear diffeomorphism $p^{-1}(U) \simeq U \times V$. Here fiber preserving means that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & & U \times V \arrow[dl, "\text{proj.}"] \\ & U & \end{tikzcd}$$

commutes.

Such a map $p^{-1}(U) \simeq U \times F$ is called a local trivialization. The dimension of V is called the rank of the vector bundle. □

Question 73. In case you were wondering: prove that if V is a smooth manifold with a smooth finite dimensional real vector space structure, then V is diffeomorphic by a linear map to an Euclidean space. When we say V is a real vector space in this lecture, this is what we mean. □

We can think of vector bundles as obtained by gluing local trivializations using transition functions. This means the following. Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open cover of B and V be a vector space. Assume that we are given smooth maps

$$t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V),$$

for every $\alpha, \beta \in \mathcal{I}$ satisfying ....

Question 74. Fill in the blank. This is similar to the smooth manifold construction lemma above but it is simpler. Why is it simpler? □

Then, we can construct a vector bundle over B via the formula

$$\bigcup_{\alpha\in\mathcal{I}} U_\alpha \times V/\sim,$$

where $(x, v) \sim (y, w)$ if $x = y$ in B and $w = t_{\alpha\beta}(x)v$.

Question 75. Go back to our construction of the tangent bundle and observe that it was actually constructed as a vector bundle. Show that every vector bundle has such a gluing description. □

Now let $E \to B$ be a vector bundle defined with a cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ and transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$. Let W be any other vector space and assume that we have a group homomorphism

$$\Psi : GL(V) \to GL(W).$$

Then we obtain a new vector bundle using the transition functions

$$\Psi \circ t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(W).$$

An example of such a Φ is obtained taking transpose and inverse

$$GL(V) \to GL(V^\vee) \to GL(V^\vee).$$

Note that the composition is indeed a group homomorphism, as each map is an anti-homomorphism. The resulting bundle is called the dual vector bundle $E^\vee \to B$.

Question 76. Prove that the fiber over $b \in B$ in $E^\vee \to B$ is canonically isomorphic to the dual vector space of the fiber over b in $E \to B$, i.e.

$$E^\vee_b = (E_b)^\vee.$$

□

Finally we have our cotangent bundle

$$T^*X := (TX)^\vee.$$

Covector fields are sections of the cotangent bundle.

Note that there are other similar ways to construct new vector bundles out of old ones. Here is another example.

Question 77. If $E \to B$ and $E' \to B$ are vector bundles, then construct the Whitney sum vector bundle $E\oplus E' \to B$ whose fibers are canonically isomorphic to the direct sum of the fibers:

$$(E \oplus E')_b = E_b \oplus E'_b.$$

□

The total space of every vector bundle $p : E \to B$ contains a canonical submanifold Z_E called the zero section. This is the image of the map that sends every point in B to the 0 element of the fiber above it. We can also talk about the zeros of an arbitrary section $s : B \to E$, which are $b \in B$ such that $s(b)$ lies in the zero section.

At an arbitrary point $e \in E$, we have the vertical subspace

$$ker(dp_e) \subset T_eE.$$

Question 78. If V is real vector space, then for every $v \in V$, we have a canonical isomorphism $T_vV = V$. Prove this and use it to show that if $p(e) = b$, then the vertical subspaces is canonically isomorphic to the vector space E_b. □

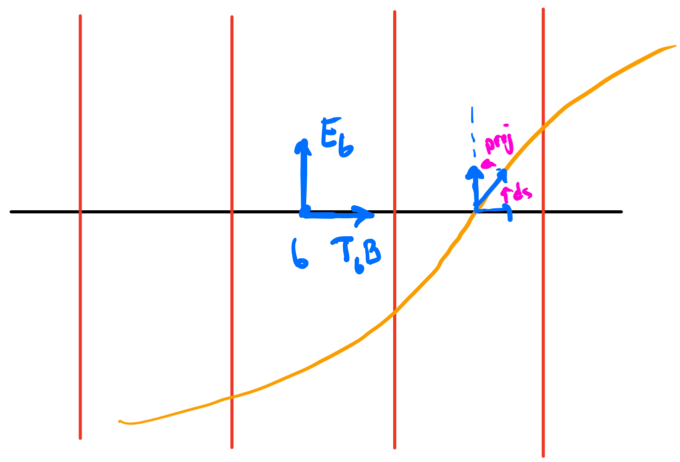

Figure 2. A local picture of how the tangent space splits at a point on the zero section. Also note how at the zero of a section we obtain a map $T_bB \to E_b$.

If $e \in Z_E$, it is the image of the zero section map $z : B \to E$ for some point $b \in B$, and therefore, in addition we have the horizontal subspace

$$im(dz_b) \subset T_eE.$$

Question 79. Prove that the horizontal and vertical subspaces at a point in the zero section are of complementary dimension and they intersect at one point (zero). Hence, they split the tangent space. In particular, we have a canonical projection map to the vertical subspace. □

Definition 30. Let V be a vector field on X and assume $V(x) = 0$ for $x \in X$. Note that the vertical subspace at $(x, 0)$ is canonically isomorphic to T_xX. Therefore, we have a map

$$T_xX \to T_{(x,0)}(TX) \to T_xX.$$

This composition is the linearization of V at x. □

Question 80. Check that this recovers the previous definition. □

Definition 31. Let us call a section of a vector bundle $E \to B$ non-degenerate if it is transverse to the zero section as a map. □

Question 81. Prove that this recovers the previous notion of non-degeneracy for a vector field. □

Definition 32. Let us call $f : X \to \mathbb{R}$ Morse, if *df* is a non-degenerate section of the cotangent bundle. □

Question 82. Prove that this also agrees with our previous definition. You might want to start by proving that a Riemannian metric g on X defines a vector bundle isomorphism and in particular a diffeomorphism

$$g^{\#} : TX \to T^*X.$$

We had already used the # symbol for turning a bilinear form on a vector space into a linear map from the vector space to its dual. That is what you should be doing fiberwise. Note that we can compute the differential of this map on the zero section easily. Use the splittings of the tangent space of the total space at the points of the zero section to express the result. □

14. Feb 10, 2021: Existence of Morse functions, Morse-Smale functions, moduli space of gradient flow lines and its compactification, Morse cohomology, examples

We know that a function $f : X \to \mathbb{R}$ is Morse if the image of the covector field $df : X \to T^*X$ is transverse to the zero section. We expect this to be generically true and the way to actually realize our expectation is as follows.

We will construct a family of smooth functions on X:

$$F : S \times X \to \mathbb{R},$$

where S is a smooth manifold so that the corresponding family of covector fields

$$S \times X \to T^*X$$

is transverse to the zero-section. Then, we will get using the parametric transversality theorem and Sard's theorem that for all but measure zero $s \in S$ we have that F_s is a Morse function.

Question 83. Construct this "large" family of functions for compact X. Take a regular cover of X and use cut-offs of coordinate functions. □

Let X be a compact smooth manifold, f a Morse function and g a Riemannian metric. Let x be a critical point of f. We have the linearization of the gradient vector field at x: $L_x : T_xX \to T_xX$. We know that L_x is an isomorphism. We can also prove that L_x is self-adjoint with respect to g_x:

$$g(L_x\cdot, \cdot) = g(\cdot, L_x\cdot).$$

Question 84. Prove this by recalling the definitions. We have the map $T_xX \to T_x^*X$ obtained from x being a zero of the section $df : X \to T^*X$. This map can be thought of as a bilinear map

$$H_x : T_xX \times T_xX \to \mathbb{R}.$$

If you compute this bilinear map in coordinates you will see that it is given by the Hessian of f at x in those coordinates. Hence H_x is a symmetric bilinar form. Finish by noticing that by definition $g(L_x\cdot, \cdot) = H_x(\cdot, \cdot)$, and symmetricity of g. □

So now the spectral theorem tells us that L_x admits an eigenbasis with positive and negative real eigenvalues. This leads to the promised decomposition:

$$T_xX = T_x^sX \oplus T_x^uX,$$

where T_x^sX is the span of negative eigenvectors and T_x^uX the positive ones. We can also easily see that the type (s_x, u_x) of x gives the dimensions of T_x^sX and T_x^uX. Let us recall the stable submanifold theorem in light of all this.

Theorem 12. *Let X be a compact smooth manifold with a Riemannian metric g, f a Morse function on X and x a critical point of f. Then, the stable set of $grad_g f$ at x is the image of a smooth embedding $T^s_x X \to X$ sending 0 to x and so that the induced map $T_0(T^s_x X) = T^s_x X \to T_x X$ is the defining inclusion. The analogous statement for the unstable set is also true.*

Let us start calling such a pair (f, g) a Morse pair.

Definition 33. We call a Morse pair (f, g) on a smooth manifold X Morse-Smale, if all unstable manifolds intersect transversely with all stable manifolds. □

Remark 25. There is a sense in which Morse-Smale pairs are generic as well but I will not further explain this. In fact it turns out that given a Morse function, it is generic for a Riemannian metric to give a Morse-Smale pair. □

Question 85. Show that the height function of the tilted torus gives an example of a Morse-Smale pair, whereas the up-right one fails to satisfy this property. □

Proposition 15. *For a Morse-Smale pair (f, g) on compact X^n, and two critical points $p \neq q$, we have that the intersection US_p and SS_q is empty if $u_p + s_q \leq n$. If $u_p + s_q > n$, then the intersection is a submanifold of dimension*

$$(u_p + s_q) - n = s_q - s_p \geq 1,$$

which could be empty.

In the latter case, moreover, the flow defines an action of $\mathbb{R}$ on $US_p \cap SS_q$. We define

$$M(p, q) := US_p \cap SS_q/\mathbb{R}$$

and equip it with a smooth structure as in the next paragraph, assuming $US_p \cap SS_q$ is not empty.

Take a regular value $c \in (f(p), f(q))$. The intersection of $US_p \cap SS_q$ with $f^{-1}(c)$ is canonically identified with $M(p, q)$. On the other hand, this intersection is transverse, and hence it is a submanifold of $f^{-1}(c)$ and has a canonical manifold structure. Finally, the smooth structure on $M(p, q)$ does not depend on the choice of c inside $M(p, q)$.

$M(p, q)$ is called the moduli space of gradient flow lines from p to q.

Question 86. Understand and prove this. □

Question 87. For the tilted torus compute the moduli space of flow lines for all possible pairs of critical points. □

Proposition 16. *For a Morse-Smale pair (f, g) on compact X, and two critical points $p \neq q$ such that $s_q - s_p = 1$, $M(p, q)$ is compact. This means that it is just a finite number of points.*

Proof. Recall that $M(p, q)$ is diffeomorphic to the intersection of $US_p \cap SS_q$ with $f^{-1}(c)$ for any regular value $c \in (f(p), f(q))$, which is a zero dimensional submanifold of $f^{-1}(c)$. This implies the claim because $f^{-1}(c)$ is compact. □

Remark 26. Note that a positive dimensional submanifold of a compact smooth manifold is not necesarily compact (or equivalently closed). □

Definition 34. For a Morse-Smale pair (f, g) on compact X, we define the Morse cochain complex over $\mathbb{Z}/2$ as follows. We take a vector space $CM^*(f, g)$ with a basis whose elements are in one to one correspondance with the critical points of f. The grading is given by

$$p \mapsto s_p.$$

The differential is defined by the following formula

$$dp = \sum_{s_q - s_p = 1} \#_2 M(p,q)q.$$

□

That $d^2 = 0$ follows from the next proposition and classification of 1 dimensional manifolds with boundary.

Proposition 17. *For a Morse-Smale pair (f,g) on compact X, and two critical points $p \neq q$ such that $s_q - s_p = 2$, $M(p,q)$ can be compactified to a 1 dimensional manifold with boundary. The added points correspond in a one-to-one fashion to once broken gradient flow lines from p to q.*

The proof is tricky and interesting, but hopefully you have some basic picture of what is going on. We will not have time to cover it.

Question 88. Expand the terms in d^2p and use this proposition to prove that it is zero. □

Question 89. For the tilted torus compute the compactified moduli space of flow lines from the minimum to maximum. There should be 8 boundary points corresponding to the 8 broken flow lines. The interior should be a disjoint union of four open intervals. Make sure you can see all of this on a clearly drawn picture. □

Theorem 13. *The homology of $CM^*(f,g)$ is isomorphic to singular cohomology of X with $\mathbb{Z}/2$ coefficients.*

Remark 27. As you can imagine this is also a difficult theorem. It is most transparent to relate Morse cohomology with cellular homology, but there are many approaches. Historically, Morse theory was already around in the 50's but the Morse cohomology viewpoint emerged only in the beginning of 80's with the work of Witten. This was through the study of certain simple supersymmetric quantum field theories, and was quite indirect. It was Floer who developed the theory in the form I explained. He went on to use this framework to invent Floer theory. This theory is extremely influential in symplectic geometry and gauge theory. □

Question 90. Check the theorem for the tilted torus. Try out higher genus surfaces as well. □

Question 91. Construct two Morse-Smale pairs on S^2, one where there are 2 critical points and another where there are 4. Go through everything you analyzed for the tilted torus in both cases. □

Corollary 1. *A Morse function on a compact smooth manifold has at least the dimension of $H^*(M,\mathbb{Z}/2)$ many critical points.*

Question 92. Prove this using the theorem. □

Remark 28. If you are interested, you can try to define the Morse cohomology over $\mathbb{Z}$. You will need to count gradient flow lines with signs, i.e. you need to orient the moduli spaces of gradient flow lines. Of course arbitrarily doing this will not work, you need to do it coherently. One way is to arbitrarily orient unstable manifolds (which automatically orients stable manifolds) and use that data to orient $M(p,q)$'s. The result ends up being isomorphic to singular cohomology with $\mathbb{Z}$ coefficients. □

15. Feb 12, 2021: Brief answers to selected questions

Answer to Question 73: Choose a basis $v_1, \dots, v_n$ of V. We can then construct canonically a smooth map, $\psi : \mathbb{R}^n \to V$, by

$$(a_1, \dots, a_n) \mapsto a_1 v_1 \dots a_n v_n.$$

Make sure you understand why this map is smooth. ψ is also clearly a linear isomorphism. This is not enough to conclude, we do not know if ψ is a diffeomorphism yet! We will first show that the map $d\psi_0$ is injective. Note that this implies that it is an isomorphism, because $dimV \leq n$ using the surjectivity of ψ and Sard's theorem.

Using the chain rule this is equivalent to the map $V \to T_0 V$ defined by

$$v \mapsto \gamma_v'(0),$$

where $\gamma_v : \mathbb{R} \to V$, $\gamma_v(t) = t \cdot v$, being injective.

Assume that $\gamma_v'(0) = 0$. Then, we claim that $\gamma_v'(t_0) = 0$ for all $t_0 \in \mathbb{R}$. Note that $(t_0 + t)v = t_0 v + tv$. This means that the composition of γ_v with $add_{t_0 v}$ is $\gamma_v(t_0 + \cdot)$. The velocity of the latter at 0 is $\gamma_v'(t_0)$. Therefore, by the chain rule we have

$$\gamma_v'(t) = (dadd_{tv_0})_0(\gamma_v'(0)),$$

which proves the claim. Hence, we get that $\gamma_v(1) = 0$, which implies that $v = 0$.

It is elementary to finish from here. We need to show that in fact $d\psi_a$ is injective for all $a \in \mathbb{R}^n$. This follows from the chain rule, linearity of ψ and that adding a vector is a diffeomorphism for both $\mathbb{R}^n$ and V.

Answer to Question 76: For every U_α containing b, we obtain such an identification automatically as for E we glue $V \times U_\alpha$ and for $E^\vee$ we glue $V^\vee \times U_\alpha$. What you have to do is to realize that relationship of the transition functions makes sure that these identifications are compatible.

Answer to Question 83: First of all, I actually could not make my hint work in the non-compact case without more machinery. Assuming compactness (you can do too), we start with a regular cover and consider functions ϕ_i, $i = 1, \dots, N$, which are cutoffs of squares of all coordinate functions shifted by 10 in all charts. Consider the family of functions

$$\mathbb{R}^N \times X \to \mathbb{R},$$

which is given by $(a, x) \mapsto \sum a_i \phi_i(x)$. We need to show that the induced family of covector fields

$$\Phi : \mathbb{R}^N \times X \to T^*X,$$

is transverse to the zero section. For this take a point (a, x) that maps to the zero section. We know that x is contained in the inner (radius 1) part U of a chart. There are n of the functions ϕ_i, which are equal to $(x_1 - 10)^2, \dots (x_n - 10)^2$ on U

where $x_1, \dots, x_n$ are the coordinate functions. Say these are the first n functions. Now, we have that

$$(a + (b, 0, \dots, 0), x) \mapsto (x, \sum_{i=1}^{n} b_i(2x_i - 20)dx_i).$$

Note that $2x_i - 20 \neq 0$ and hence using our freedom to choose b we hit all the covectors in T_x^*X. It is also easy to see directly that one can choose a curve of b's at 0 for time 0 whose image under Φ has velocity vector any given vertical vector in T^*X. This shows that all the vertical vectors are in the image of $d\Phi_{(a,x)}$, and finishes the proof.

To get the non-compact case people use the Whitney embedding theorem and consider their large family as the family of functions which are of the form $d(y, \cdot)^2$, where y is an arbitrary point in the Euclidean space. You can look at Theorem 6.6 of Milnor's famous book for the proof.

16. Feb 17, 2021: Tangent vectors as derivations at a point, vector fields as derivations on the algebra of smooth functions, Lie bracket, Lie algebra of a Lie group

Recall that we defined the directional derivative operation above. This took in a smooth function and a tangent vector and produced a real number which measures the change of the function in the direction of the vector. If you write in coordinates, this really is the directional derivative from calculus but we gave a coordinate free definition:

$$v \cdot f = df(v).$$

We also know that directional derivative satisfies the Leibniz rule and is $\mathbb{R}$-linear in both variables. There is a converse to this. Let us denote the $\mathbb{R}$-algebra of smooth functions on a smooth manifold X by

$$C^\infty(X, \mathbb{R}).$$

Lemma 5. *Let $L : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ be an $\mathbb{R}$-linear map, which satisfies the Leibniz rule at $x \in X$: for any $\phi, \psi \in C^\infty(X, \mathbb{R})$,*

$$L(\phi\psi) = \phi(x)L(\psi) + L(\phi)\psi(x).$$

Then, there exists a unique $v \in T_xX$ such that for any $\phi \in C^\infty(X, \mathbb{R})$

$$L(\phi) = v \cdot \phi.$$

Question 93. Show that L vanishes on constant functions. □

Proof. First, note that if ϕ vanishes in a neighborhood U of x, then $L(\phi) = 0$. To see this take a smooth function ρ which is 1 on $X - U$ and is zero in a smaller neighborhood of x. We have $\phi = \rho\phi$, which proves the claim using the Leibniz rule. By linearity, we get that if ϕ and ψ are the same in a neighborhood of x, then L sends them to the same number.

Let us prove the lemma when $X = \mathbb{R}^n$ and $x = 0$. The key to this is the following weak Taylor expansion property. For any $\phi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, there exists real numbers $a, b_1, \dots, b_n$ and $f_1, \dots, f_n \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f_i(0) = 0$ for all $i = 1, \dots, n$ and

$$f = a + \sum b_i x_i + \sum x_i f_i,$$

where $x_1, \dots, x_n$ are the coordinate functions. Using the Leibniz rule and $\mathbb{R}$-linearity, we get that L is canonically determined by what it does on linear functions $\sum b_i x_i$. Clearly, there exists a vector $v \in T_0\mathbb{R}^n$ such that $v \cdot \phi = L(\phi)$ on linear functions, but we proved that then this has to be the case for all smooth functions. □

Question 94. Finish the proof. □

Definition 35. Let A be an $\mathbb{R}$-algebra. An $\mathbb{R}$-linear map $D : A \to A$ is called a derivation if it satisfies

$$D(ab) = aD(b) + D(a)b$$

for all $a, b \in A$. Let us denote their set by $Der(A)$. Note that $Der(A)$ is naturally an A-module. □

Lemma 6. *If $f, g \in Der(A)$, then the commutator $f \circ g - g \circ f$ is also a derivation.*

Question 95. Do it! □

Let us now introduce the notation that if $E \to B$ is a vector bundle, we denote its set of smooth sections by $\Gamma(E)$. For example $\Gamma(TX)$ is the set of vector fields on X, whereas $\Gamma(T^*X)$ is the one of covector fields. $\Gamma(E)$ is naturally a $C^\infty(B, \mathbb{R})$-module.

Question 96. Let $E \to B$ and $E' \to B$ two vector bundles. Assume that we are given a $C^\infty(B, \mathbb{R})$-module map $T : \Gamma(E) \to \Gamma(E')$. Prove that T is obtained from a vector bundle map $E \to E'$ (i.e. a smooth, fiber-preserving and fiberwise linear map). Hopefully the converse is clear. This is Proposition 5.16 from Lee. □

Remark 29. It is customary among non-geometers to work only with the sections of a vector bundle and never talk about the vector bundle itself. Here you think of sections as a collection of local vector valued functions, which transform according to some rules (i.e. the transition functions, which in case of vector bundles related to the tangent bundle can be expressed in terms of changes of coordinates - this expression transforms as xxx, Einstein conventions etc.). I think it is a shame and the only reason to do this could be that the mental effort to conceptualize a non-trivial bundle is non-trivial. This is similar to the insistence of some physicists to never talk about the flow of a vector field but only about individual solutions of the corresponding ODE. Neither of these geometric notions (flows and global bundles) will help if all you want is to compute something, but they definitely help in thinking about what you are doing when you are doing the computation. Laziness turns into a defense mechanism that causes people to think mathematicians are just being fancy. □

Here is the upshot of the discussion so far. There is an isomorphism of $C^\infty(X, \mathbb{R})$-modules

$$\Gamma(TX) \to Der(C^\infty(X, \mathbb{R})).$$

Question 97. Make sure you can parse this and prove it using the results above. □

Hence, smooth vector fields are precisely the derivations on the algebra of smooth functions. You can think of such derivations as homogeneous first order differential operators acting on real valued functions. This is of course an entirely different viewpoint on vector fields (also very useful).

Let us finish by noting that for free we obtain an $\mathbb{R}$-bilinear operation

$$[\cdot,\cdot] : \Gamma(TX) \times \Gamma(TX) \to \Gamma(TX)$$

called the Lie bracket of vector fields. We will explore this operation and its geometric meaning next time.

Question 98. Let x, y be the coordinate functions on $\mathbb{R}^2$. Show that the Lie bracket of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is the zero vector field. Find two vector fields on $\mathbb{R}^2$ with a non-vanishing Lie bracket. □

Question 99. Read about the Lie algebra of a Lie group from Lee, pg. 93. Describe the Lie algebra of $SO(3)$. □

17. Feb 19, 2021: Brief answers to selected questions

Answer to Question 89: Here is a way to see this, which is in fact a general method (for Morse-Smale pairs with one local minimum and one local maximum say, but the latter assumption can be removed). Let us call the minimum p, maximum q and the two saddle points a and b, where $h(a) < h(b)$. Recall that we know how to analyze the change in level sets as we go from $-\infty$ to ∞. The main thing that happens is that as we approach a critical value, the stable sphere (I mean the intersection of the stable manifold with the level set, I will keep using this terminology) contracts to the critical point and as we pass that critical value the unstable sphere grows from the critical point.

We now want to modify this strategy to give a "scanning" description of the compactified moduli space of flow lines. Let us first introduce the following set

$$\bar{M}(c) : \{(x, \gamma) \mid h(x) = c, \gamma \text{ is a broken flow line from } p \text{ to } x\}.$$

Note that for $c = h(p) + \epsilon$, we have that

$$\bar{M}(c) = f^{-1}(c)$$

and $c = h(q) - \epsilon$, we have that

$$\bar{M}(c) = \bar{M}(p, q).$$

Our goal is to understand how $\bar{M}(c)$ changes as c goes between these values. Again, nothing really happens unless we cross a critical value.

In fact, this analysis might be the best way to equip $\bar{M}(p, q)$ with the structure of a manifold with corners in general. The framework that we would need here involves the blow-up procedure that takes a manifold with corners and a well behaved type of sub-manifold; and outputs a new manifold with corners, where the points on the submanifold are replaced with the sphere of its normal directions - we don't do anything in the complement of the submanifold, and replace the submanifold with its normal sphere bundle.

Assuming this made some sense (ok, if it didn't, but keep going, the specific case should make sense) what happens in general is that as you pass a critical value you replace $\bar{M}(c)$ with its blow-up along the proper transform of the stable sphere of the critical point. Note that after the first critical value, $\bar{M}(c)$ is not equal to a level set anymore, therefore we cannot talk about "the stable sphere" in it, but only a proper transform, which has to do with what happens to submanifolds after a blow-up.

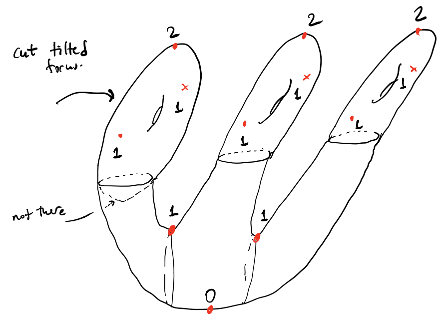

FIGURE 3. The numbers are the indices (i.e. the dimension of the stable manifold).

In our specific case all we need is to blow-up a 1 dimensional manifold with boundary along an interior point. This just means that you cut the manifold, the point is replaced with the two new boundary points. Because of the simplicity of the situation we also do not need to consider proper transforms. So in this specific case, we start with a circle as $\bar{M}(c)$. When, we pass $h(a)$, the stable sphere (this is just 2 points) of a in the circle is blown-up, each becoming two points, and the circle is now two closed intervals. Then, as we pass $h(b)$ the stable sphere of b (which is a subset of the interior of the two intervals, which naturally inside the corresponding level set) is also blown-up. This sphere is embedded so that each component contains one point, and therefore we get our union of four closed intervals.

One thing that is quite interesting to remark is that $\bar{M}(p.q)$ can be equipped with a natural topology that is not too easy to explain (need to make sense of in what sense unbroken gradient flow lines converge to broken ones). The procedure above also gives a topology. I am actually unsure if whether they are the same topology is analyzed anywhere. People generally try to equip $\bar{M}(p.q)$ with extra structure by trying to realize them as submanifolds of products of level sets as we touched upon in class. This turns out to be more complicated than one imagines at first.

Answer to Question 90: I gave an example of a Morse-Smale pair in Figure 3 using the height function and the restriction of the flat metric. This is combination of the 4 critical point Morse function on the sphere as explained below and the tilted torus, so you should understand those that first.

In Figure 4, I compute the Morse cohomology for this example.

Answer to Question 91 Note that the 2 critical point example with just one minimum and one maximum has the following property. The moduli space of gradient flow lines from min to max is a circle. In Figure 5 there is a Morse-Smale pair, as usual using height function and the restriction of the flat metric.

Answer to Question 96: This is an important result and the proof is in Lee. Please make sure you understand it.

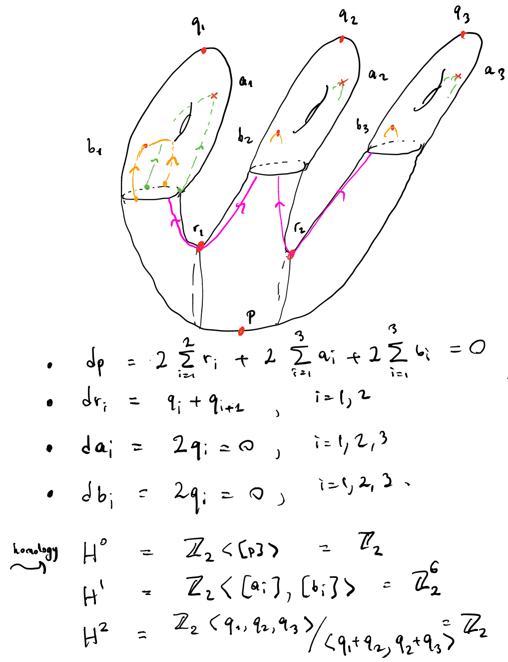

FIGURE 4. I drew the orange and green gradient flow lines only in the left-most torus out of laziness. Note that we need to make sure that the purple, orange and green points are all distinct on each of the three horizontal circles, which you should imagine all lie in a level set - this can be arranged. You should be able to fill in the rest of the relevant information with no problem.

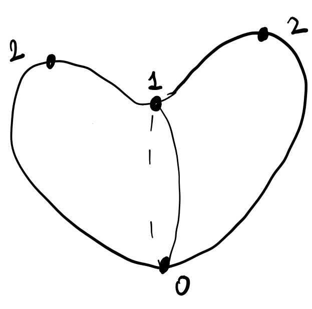

FIGURE 5. The numbers are the indices (i.e. the dimension of the stable manifold).

18. Feb 22, 2021: Lie bracket in coordinates, Lie bracket as Lie derivative, Lie bracket as an obstruction to commutativity of flows

Let X be a manifold, $V \in \Gamma(TX)$ and U be the domain of a coordinate chart with coordinates $x_1, \dots, x_n$. We write V in these coordinates

$$V(x) = v_1(x)\frac{\partial}{\partial x_1} + \dots + v_n(x)\frac{\partial}{\partial x_n}.$$

If we have a function $f : X \to \mathbb{R}$. on U:

$$V \cdot f(x) = v_1(x)\frac{\partial f}{\partial x_1}(x) + \dots + v_n(x)\frac{\partial f}{\partial x_n}(x).$$

Now assume that we have another vector field W with coordinate expression

$$W(x) = w_1(x)\frac{\partial}{\partial x_1} + \dots + w_n(x)\frac{\partial}{\partial x_n}.$$

As we discussed above, we can then talk about the Lie bracket vector field $[V, W]$, which corresponds to the commutator of V and W as operators. More precisely, for every $f \in C^\infty(X, \mathbb{R})$, we have:

$$[V, W] \cdot f = V \cdot (W \cdot f) - W \cdot (V \cdot f).$$

Note that it really seems like $[V.W]$ acts as a second order operator (i.e. involves second partial derivatives), but from our discussion above we know that it is given by a vector field and this is not the case. Let's compute and see what happens:

$$[V, W] \cdot f = \sum v_j \frac{\partial}{\partial x_j}\left(w_i \frac{\partial f}{\partial x_i}\right) - \sum w_j \frac{\partial}{\partial x_j}\left(v_i \frac{\partial f}{\partial x_i}\right).$$

Now use the Leibniz rule and notice the terms with second partial derivatives cancel and we end up with

$$\sum \left(v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i},$$

or in other terms

$$[V, W] = \sum \left(v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}.$$

Let us move on to the geometric meaning of the Lie bracket vector field. The first thing to say is that $[V, W]$ measures the change of W under the flow of V.

Question 100. Assume that we are in a chart with coordinates $x_1, \dots, x_n$, and that $V = \frac{\partial}{\partial x_1}$. Write $W(x) = w_1(x)\frac{\partial}{\partial x_1} + \dots + w_n(x)\frac{\partial}{\partial x_n}$, and using our local formula, note that

$$[V, W] = \frac{\partial w_1}{\partial x_1}\frac{\partial}{\partial x_1} + \dots + \frac{\partial w_n}{\partial x_1}\frac{\partial}{\partial x_n},$$

Hopefully, you the relationship with the claim just made before the question. □

The way to formalize this is to define the Lie derivative of W along the flow of V, which is also a vector field. Let $\Phi_V : \mathcal{U} \to M$ denote the flow of V. We define for every $p \in X$,

$$\mathcal{L}_V W(p) := \lim_{t \to 0} \frac{\Phi(-t, \cdot)_* W(\Phi(t, p)) - W(p)}{t}.$$

Proposition 18. *The vector fields $[V, W]$ and $\mathcal{L}_V W$ are the same.*

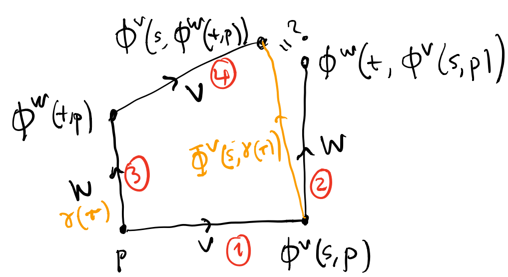

FIGURE 6. Used in the proof of Proposition 19

Proof. We have actually almost done this in Question 100. If at a point $p \in X$, $V(p) \neq 0$, then we use the rectification theorem and be done. If $V(p) = 0$, then we have two cases. If $V = 0$ in a neighborhood of p, then the result is trivial. Otherwise, we take a sequence of points converging to p which are not zeros of V and use that both vector fields are continuous maps $X \to TX$. $\square$

Here is a more striking proposition, which says that if the Lie bracket of two vector fields vanish, then their flows commute!

Proposition 19. *If* $[V, W] = 0$*, then*

$$\Phi^W(t, \Phi^V(s, p)) = \Phi^V(s, \Phi^W(t, p)),$$

for $t, s \in \mathbb{R}$ *and* $p \in X$ *as long as both sides of the equation make sense.*

Proof. Let us assume that s and t are positive without loss of generality. You can follow along from Figure 6. Let us define Path 1 to be the integral curve $[0, s] \to X$ from p to $\Phi^V(s, p)$, and Path 2 to be the integral curve $[0, t] \to X$ from $\Phi^V(s, p)$ to $\Phi^W(t, \Phi^V(s, p))$. Note that we can first follow Path 1 and then Path 2 to obtain a continuous map $[0, t + s] \to X$, call this Path 12. We can also define Path 3, Path 4 and Path 34 in the same fashion for the RHS of the equation. We need to prove that the end points of Path 12 and Path 34 are the same.

Because of what we proved in the previous proposition

$$\Phi^V(t, \cdot)_* W(p) = W(\Phi^V(t, p))$$

as long as $(t, p) \in \mathcal{U}$. It follows from the chain rule that if $\gamma(\tau)$, for $\tau \in [0, t]$, is Path 3, then $\Phi^V(s, \gamma(\tau))$ is the integral curve $[0, t] \to X$ of W starting at $\Phi^V(s, p)$. Therefore it has to be equal to Path 2. This proves that the endpoint of Path 12 is $\Phi^V(s, \gamma(t)) = \Phi^V(s, \Phi^W(t, p))$ as desired. $\square$

I want to warn of you the statement that you will hear a lot "Lie bracket measures the non-commutativity of the flows of the vector fields". This is true but you have to know what you mean.

For $p \in X$ and small enough $\epsilon > 0$, we can construct a smooth path $\gamma : (-\epsilon, \epsilon) \to X$:

$$\gamma(\tau) = \Phi^W(-\tau, \Phi^V(-\tau, \Phi^W(\tau, \Phi^V(\tau, p)))).$$

This requires a small bit of thinking for not complete vector fields. Of course, if we have commuting vector fields, γ would be a constant path. We would like to say that the vector $\gamma'(0)$ at p is $[V, W](p)$. This turns out to be wrong.

Let us introduce the smooth map $\beta : (-\epsilon, \epsilon)^2 \to X$ for possibly smaller $\epsilon > 0$, where

$$\beta(\tau_1, \tau_2) = \Phi^W(-\tau_2, \Phi^V(-\tau_1, \Phi^W(\tau_2, \Phi^V(\tau_1, p)))).$$

Note that $\gamma(\tau) = \beta(\tau, \tau)$.

Remark 30. It requires an argument to show that such an ϵ exists and also that the map is smooth. □

Question 101. Prove that for all $\tau_1, \tau_2 \in (-\epsilon, \epsilon)$,

$$\beta_*\left(\frac{\partial}{\partial \tau_1}(\tau_1, 0)\right) = \beta_*\left(\frac{\partial}{\partial \tau_2}(0, \tau_2)\right) = 0.$$

Then note that γ is the composition of β with the diagonal inclusion map $\tau \mapsto (\tau, \tau)$ and use the chain rule to show that $\gamma'(0) = 0$. □

It turns out that we need to take a second derivative to get to $[V, W](p)$. Here is how to make sense of this. We can consider the pullback vector bundle

$$\gamma^* TX \to (-\epsilon, \epsilon).$$

We have a section of this bundle given by

$$\tau \to \gamma'(\tau)$$

with a zero at $\tau = 0$. Hence, we can "linearize" and get a well defined map $T_0(-\epsilon, \epsilon) \to T_pX$. We define $\gamma''(0)$ to be the image of $\frac{\partial}{\partial \tau}$ under this map. Note that we can compute $\gamma''(0)$ using the Calculus way inside a coordinate chart.

Proposition 20. $[V, W](p) = \frac{\gamma''(0)}{2}$

Question 102. Show that we can make sense of second partial derivatives of β:

$$\frac{\partial^2 \beta}{\partial \tau_1^2}(0,0), \frac{\partial^2 \beta}{\partial \tau_2^2}(0,0), \frac{\partial^2 \beta}{\partial \tau_1 \partial \tau_2}(0,0), \frac{\partial^2 \beta}{\partial \tau_2 \partial \tau_1}(0,0)$$

as tangent vectors at p. Note that if we do choose coordinates in X, they can be computed as in Calculus. Prove that the first two are zero. Then, prove that the last two are the same and finally relate them to $\gamma''(0)$ using the diagonal map. □

Question 103. Prove the proposition. □

Next time we will prove a simultaneous rectification theorem for vector fields whose Lie brackets vanish pairwise. We will then discuss subbundles, distributions, foliations and Frobenius theorem.

19. Feb 24, 2021: Simultaneously rectifying vector fields with commuting flows, distributions, foliations, Frobenius theorem

Let us start with a generalization of the rectification theorem to multiple vector fields.

Proposition 21. *Let X be a manifold and $V_1, \dots V_k$ be vector fields such that*

$$[V_i, V_j] = 0, \text{ for every } i, j = 1, \dots, k$$

and $V_1(p), \dots, V_k(p)$ linearly independent at some point $p \in X$. Then, we can find coordinates $x_1, \dots, x_n$ around p such that

$$V_i = \frac{\partial}{\partial x_i}, \text{ for every } i = 1, \dots, k$$

in the domain of these coordinates.

Proof. First of all, we can find a smooth embedding $\psi : B \to X$ where $B \subset \mathbb{R}^{n-k}$ is an open ball containing the origin such that $\psi(0) = p$ and image of $T_0 B$ under $d\psi_0$ is transverse to the span of $V_1(p), \dots, V_k(p)$.

We would like to define a map

$$\Psi : B \times (-\epsilon, \epsilon)^k \to X$$

with the formula

$$\Psi(b, t_1, \dots, t_k) = \Phi^{V_k}(t_k, \Phi^{V_{k-1}}(t_{k-1}, \dots (\Phi^{V_2}(t_2, \Phi^{V_1}(t_1, \psi(b))) \dots),$$

where Φ denotes the flow maps as usual. With some work one can prove that for small enough radius B and small enough ϵ not just Ψ but also the similar potential maps obtained going in all possible 2^k different orders along the vector fields are all defined. I will omit the proof of this.

Note that because of the Lie bracket condition, all of these 2^k maps are the same as the flows of any two vector fields commute. It suffices to prove the following two statements by the inverse function theorem.

(1) $d\Psi_0$ is an isomorphism.
(2) $d\Psi_{(b,t)} \frac{\partial}{\partial x_i} = V_i(\Psi(b, t))$, for every $i = 1, \dots, k$.

Here I denoted the coordinate functions on $(-\epsilon, \epsilon)^k$ by $x_1, \dots x_k$ and the ones on B by $x_{k+1}, \dots x_n$.

(1) follows from (2) by construction, so let us prove (2). Notice that for $i = k$, (2) is actually automatic: if we keep b and $t_1, \dots, t_{k-1}$ constant but vary t_k in $(-\epsilon, \epsilon)$, we simply move along the integral curve of V_k which is at the point $\Phi^{V_{k-1}}(t_{k-1}, \dots (\Phi^{V_2}(t_2, \Phi^{V_1}(t_1, \psi(b))) \dots)$ when $t_k = 0$.

But actually, by reordering the flows, we can make the result for all the other i's equally easy. Just rewrite Ψ as

$$\Psi(b, t_1, \dots, t_k) = \Phi^{V_i}(t_i, \dots \psi(b) \dots).$$

□

We now discuss what is essentially a restatement of this result called Frobenius theorem. I want to start with simple situation.

Assume that we are inside $\mathbb{R}^3$ and we are given smoothly varying planes (a two dimensional linear subspace) inside $T_p\mathbb{R}^3$ at every point $p \in \mathbb{R}^3$ (a plane field). The question we are interested in is trying to find two dimensional submanifolds in $\mathbb{R}^3$ whose tangent spaces at every point is given by the plane field. Let us momentarily call these tangent submanifolds.

I warn you that I called this a submanifold here for simplicity but globally things can get complicated leading to injective immersions. Let's gloss over this point until we start being precise.

Remark 31. If instead we consider a line field, finding tangent submanifolds is essentially the same as finding integral curves of a vector field. In particular, by the uniqueness and existence theorems through each point there is a unique tangent submanifold. If you consider all possible tangent submanifolds, then every point lies in exactly one of them and moreover, locally, we can find coordinates near every point such that the tangent submanifolds look like straight lines, by the rectification theorem. The situation is more complicated for plane fields and their higher dimensional generalizations. □

Remark 32. We also keep saying things like integral curve, integrating a vector field etc. We will also use the word integrable soon, which sounds like some kind of integration is possible. There is nothing too deep here. None of these operations are given by actual integrals unless we are in very special situations. For example, for a time dependent vector field on $\mathbb{R}$, which is spatially constant at all times, finding integral curves of such a system is equivalent to computing an anti-derivative. Sometimes one can also do separation of variables and so on. Otherwise, what is meant is really just that what we are doing is vaguely resembling integration, which is true.

In old texts, sometimes you see the phrase that some differential equation or more generally a problem is solvable by quadratures. This means that the solutions (numbers, functions...) can be expressed using actual integration operations. Note that solving differential equations in general is much harder than integration, so when this is possible it is good news. $\square$

Back to plane fields in $\mathbb{R}^3$. Here is a picture of one:

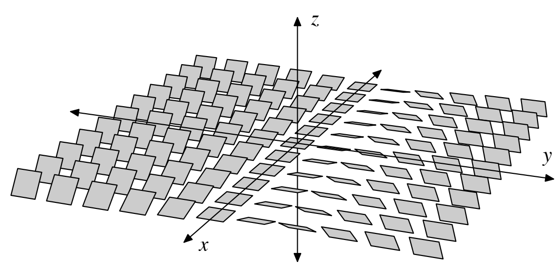

FIGURE 7. A plane field. It is extended to the rest of space by being invariant under vertical translations. They are horizontal along the x-axis, but they get rotated around the y direction positively if we move to the right.

The striking fact is that you cannot even locally (but not just at one point of course) find tangent submanifolds near any point for this plane field. It is strongly not "integrable". So how do we understand whether a plane field is "integrable" or not? It helps to generalize the situation and give some precise definitions.

Definition 36. Let $\pi : E \to B$ be a vector bundle. $S \subset E$ is called a subbundle if

(1) $S \cap E_b$ is a subspace of E_b for every $b \in B$.
(2) $\pi|_S : S \to B$ is a vector bundle.

$\square$

Remark 33. In fact, assuming (1), the much weaker condition that S is a submanifold of E implies (2). $\square$

A subbundle of the tangent bundle $TX \to X$ gets a special name: a distribution. This is the generalization of the line and plane fields that we discussed above.

Remark 34. The name has nothing to do with distributions from analysis. It is unfortunate. $\square$

Now let us define what it means for a distribution to be integrable.

Definition 37. Let X be a manifold. We call a collection of pairwise disjoint subsets of X indexed by $\mathcal{F}$ (called leaves):

$$\{L_\nu\}_{\nu\in\mathcal{F}}$$

a foliation if

- $\bigcup_{\nu\in\mathcal{F}} L_\nu = X$.
- At every $p \in X$ there exists a coordinate chart (U, ϕ) such that the intersection of each leaf with U is sent by ϕ to $\mathbb{R}^k \times \{(a_1, \dots a_{n-k})\} \cap \tilde{U}$ for some $(n-k)$-tuple of real numbers.

□

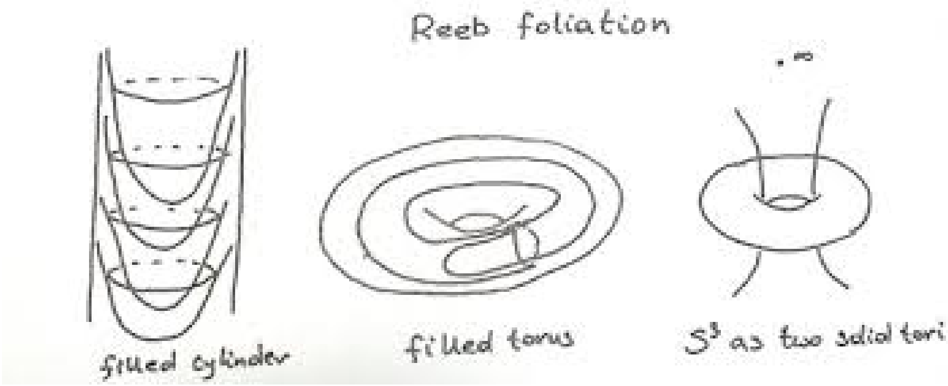

FIGURE 8. A foliation of S^3 or $\mathbb{R}^3$ can be obtained by looking at these pictures.

Note that a foliation induces a distribution by taking the tangent subspaces to leaves. If a distribution is the tangent distribution to a foliation it is called integrable. Finally, the Frobenious theorem.

Theorem 14. *Let $\mathcal{D} \subset TX$ be a distribution. It is integrable if and only if for any two vector fields V and W whose vectors belong to $\mathcal{D}$ at every point, the vectors of $[V, W]$ also belong to $\mathcal{D}$.*

Question 104. Prove the easy direction of this theorem. □

Remark 35. One might imagine less idealized notions of integrability of distributions than being tangent to a foliation but we will leave it at this. □

Question 105. Let D be a distribution on a manifold M.

(1) Show that we can always extend a tangent vector v at p belonging to D to a vector field V defined in an open neighborhood of p, all of whose members belong to D.
(2) Let W be a vector field defined near p, not necessarily in D. Take v and extend it to V as in the previous part. Take the image of $[V, W](p)$ under the natural projection to T_pM/D_p. Prove that the result does not depend

on the extension of v to V. Therefore for every $W \in \Gamma(TM)$, we obtain a map $D \to TM/D$, which is linear on fibers.

(3) Now use the previous part to obtain a map $D \times_M D \to TM/D$ which is bilinear on fibers. Prove that the condition in the Frobenius integrability theorem is equivalent to the vanishing of this bilinear pairing. This is a much better condition because it requires showing that a tensorial object vanishes pointwise.

$\square$

20. Feb 26, 2021: Brief answers to selected questions

Answer to Question 99: You have shown before the $SO(3)$:

$$\{S \in GL(n, \mathbb{R}) \mid A^t A = I\}$$

was a sub Lie group of $GL(n, \mathbb{R})$ using the closed subgroup theorem. Below I will use e for identity elements.

It is a general fact that if you have a Lie subgroup $H \subset G$, then we have an inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Of course, as vector spaces this inclusion is nothing but the inclusion of tangent spaces at the identity.

Let me briefly explain how you might see the fact that $T_e H \subset T_e G$ as in the previous paragraph is respects the Lie algebra structures. Any left invariant vector fields V, W on H canonically extends to left invariant vector fields $V^\#, W^\#$ on G (compatibly with the inclusion of tangent spaces at identity). All we need show here is that the Lie bracket $[V^\#, W^\#]$ is tangent to $H \subset G$, and that the restriction of $[V^\#, W^\#]$ to H is the same as $[V, W]$. Perhaps the easiest way is to consider the foliation of G by the orbits of the left action of H on G. The leaves of this foliation is indexed by the set G/H. Now, notice that in fact $V^\#, W^\#$ are tangent to this entire foliation and use the argument for the easy direction of Frobenius theorem.

Going back to $SO(3)$, it will then suffice to describe the Lie algebra of $GL(3, \mathbb{R})$ and then understand how $\mathfrak{so}(3)$ sits inside $T_e GL(3, \mathbb{R})$.

First of all, note that $GL(n, \mathbb{R})$ is an open subset of the n^2 dimentional vector space of $n \times n$ real matrices $Mat(n, \mathbb{R})$. Therefore,

$$T_e GL(n, \mathbb{R}) = Mat(n, \mathbb{R}).$$

How do we figure out the induced Lie algebra structure on $Mat(n, \mathbb{R})$? Let $A, B \in Mat(n, \mathbb{R})$, we need to find out what is $[A, B]$. We know that this is a left invariant vector field so it suffices to find its value at e. The corresponding left invariant vector fields $GL(n, \mathbb{R}) \to Mat(n, \mathbb{R})$ are given by $T \mapsto TA$ and $T \mapsto TB$, where we are are using matrix multiplication of course.

What are the flows of these vector fields? You can check that they are given by matrix exponentials! More precisely, the flow of $T \mapsto TA$ is

$$(t, T) \mapsto T \exp(tA),$$

where

$$exp(tA) = 1 + tA + \frac{1}{2} t^2 A^2 + \dots$$

Therefore, all we are left with computing

$$\frac{d^2}{dt^2} |_0 \exp(tA) \exp(tB) \exp(-tA) \exp(-tB).$$

Direct computation gives $2(AB - BA)$, dividing by 2 as in the formula we get that

$$[A, B] = AB - BA,$$

as could be expected. There are other ways to figure this out, we could use the Lie derivative apprach, or directly use the definition involving derivations. I like this one the best.

Now we need to compute the subspace $T_eSO(3) \subset Mat(3, \mathbb{R})$. Here is how you do this. These are the matrices A such that $I + \delta A$ is an element of $SO(3)$ up to an error of $O(\delta^2)$. Writing this down

$$(I + \delta A)^t(I + \delta A) \sim 1 + \delta(A^t + A) + O(\delta^2).$$

Therefore A satisfies $A^t + A = 0$, in other words the subspace we are looking for is the subspace of skew-symmetric matrices. This computation can be phrased as finding the best approximating linear subspace to $SO(3) \subset Mat(3, \mathbb{R})$ at e.

Hence the Lie algebra of $SO(3)$ is 3×3 skew-symmetric matrices with Lie bracket given by the commutator. A more concrete way to describe this is to say that it is isomorphic to $\mathbb{R}^3$ with cross product (prove this).

Answer to Question 20: We are going to compute $\frac{\partial^2 \beta}{\partial \tau_1 \partial \tau_2}(0, 0)$. Let us work locally, meaning $X = \mathbb{R}^n$. This means that we can think of this mixed partial as just the value at $(0, 0)$ of the τ_1 partial derivative of the vector valued function

$$(-\epsilon, \epsilon)^2 \to \mathbb{R}^n$$

given by $(\tau_1, \tau_2) \mapsto \frac{\partial \beta}{\partial \tau_2}(\tau_1, \tau_2)$. Of course similarly, we are thinking of the latter map as the τ_2 partial derivative of the vector valued function β.

We need to understand the map

$$\tau_1 \mapsto \frac{\partial \beta}{\partial \tau_2}(\tau_1, 0).$$

For fixed τ_1, the value of this map is the velocity vector of the curve

$$\Phi^W(-t, \Phi^V(-\tau_1, \Phi^W(t, \Phi^V(\tau_1, p))))$$

at $t = 0$.

By the same diagonal argument, this velocity vector is the sum of the velocity vectors at $t = 0$ of

$$\Phi^W(-0, \Phi^V(-\tau_1, \Phi^W(t, \Phi^V(\tau_1, p)))) = \Phi^V(-\tau_1, \Phi^W(t, \Phi^V(\tau_1, p)))$$

and

$$\Phi^W(-t, \Phi^V(-\tau_1, \Phi^W(0, \Phi^V(\tau_1, (0, 0))))) = \Phi^W(-t, (0, 0)).$$

To deal with the first one, we use a general lemma:

Lemma 7. *X manifold, W vector field $\phi : X \to X$ diffeomorphism, $p \in M$. Then, the velocity vector at $t = 0$ of*

$$\phi^{-1}(\Phi^W(t, \phi(p))$$

is equal to $\phi_*^{-1}W(\phi(p))$.

Proof. $\Phi^W(t, \phi(p))$ is by definition the integral curve of W passing through $\phi(p)$ at time 0. Then use the chain rule. $\square$

Applying the lemma, we find that the first velocity vector we were trying to find is

$$\Phi^V(-t,\cdot)_* W(\Phi^V(t,p)),$$

whereas the second one is obviously $-W(p)$.

Now recall that we are trying to find out the t-derivative of the sum of these two vector valued functions of t (using t in place of the original τ_1). The second term vanishes but the first term is exactly what we want. Looking back at the definition of the Lie derivative $\mathcal{L}_V W$ you will notice that the second term is exactly equal to $\mathcal{L}_V W(p)$, finishing the proof.

If you are getting confused about the coordinate independent definitions of partials, mixed partials etc. at $(0,0)$, just define them using coordinates without knowing that they do not depend on the coordinates. The relationship with the Lie bracket that we just proved shows a posteriori that they are coordinate independent.

Answer to Question 105: I just want to restate the construction of the vector bundle map $D \to TM/D$ obtained from $W \in \Gamma(TM)$ slightly differently, which should make everything clear.

First, note that we have a map $\Gamma(D) \to \Gamma(TM)$ given by

$$V \mapsto [W, V].$$

Let's check whether this map comes from a map of vector bundles. We need to check whether the map commutes with multiplication by a function $f \in C^\infty(M,\mathbb{R})$, i.e. we need to understand $[W, fV]$. It turns out that there is another Leibniz rule here. Thinking of vector fields as derivations for a moment:

$$[W, fV]\cdot g = W\cdot(fV\cdot g) - fV\cdot(W\cdot g).$$

Using the Leibniz rule for directional derivatives for the first term, we end up with

$$f[W,V]\cdot g + (W\cdot f)V\cdot g,$$

therefore

$$[W, fV] = f[W,V] + (W\cdot f)V.$$

Of course the second term ruins the commutation we were looking for.

But... after we kill the parts of the vectors lying in D by post-composing

$$\Gamma(D) \to \Gamma(TM) \to \Gamma(TM/D),$$

we do not have that term anymore and therefore we get the $C^\infty(M,\mathbb{R})$-linearity we need. This is a way to describe the vector bundle map $D \to TM/D$. It is easy to check that it gives the same result as what is explained in the question. I leave the rest to you.

21. Mar 1, 2021: Exterior algebra of a vector space, wedge product, exterior algebra of the dual of a finite dimensional vector space as alternating multilinear maps

Today we start our discussion differential forms on smooth manifolds. Unfortunately, we have to start with some abstract linear algebra.

Let V be a vector space over a field $\mathbb{F}$. Then, we define the underlying vector space of its tensor algebra as follows

$$T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n.}$$

Here $V^{\otimes 0} := \mathbb{F}$. We equip $T(V)$ with a grading so that $V^{\otimes n}$ is the set of elements in grade n, for every $n \in \mathbb{Z}_{\geq 0}$. There are no negatively graded elements.

Remark 36. If you don't know how to take tensor product of two vector spaces, you will still be able to understand today's lecture but I will assume that you will learn as soon as you can after this class. It is probably easiest to start by thinking about how to construct a basis of the tensor product. $\square$

Question 106. Explain the natural structure of $T^*(V)$ as a graded associative $\mathbb{F}$-algebra. When is this algebra commutative? $\square$

$T^*(V)$ has a two-sided graded ideal I, which is generated as an ideal by all the elements of the form $v \otimes v$, $v \in V$. Note that I being graded means $I = \bigoplus_{n=0}^{\infty} I^n$, where $I^n := I \cap V^{\otimes n}$.

Question 107. Find an ideal of $T^*(\mathbb{F})$ that is not graded. $\square$

Question 108. Prove that $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ is in I if and only if $v_1, \ldots, v_n$ is linearly dependent in V. $\square$

Definition 38. We define the exterior algebra of V as

$$\Lambda^*(V) := T^*(V)/I.$$

The induced product is denoted by $\cdot \wedge \cdot$ and called the wedge product. $\square$

Question 109. Assume that V is finite dimensional. Compute the dimension of $\Lambda^n(V)$, for every $n \in \mathbb{Z}$. $\square$

Question 110. Prove that for homogenous elements α, β of $\Lambda^*(V)$:

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha.$$

This is called super-commutativity for graded algebras. You might want to prove the special case

$$[v_1 \otimes v_2] = -[v_2 \otimes v_1]$$

in $\Lambda^*(V)$ for $v_1, v_2 \in V$. $\square$

All this is fairly abstract. Now let W be a finite dimensional vector space over a field of characteristic zero , and let $V := W^\vee = Hom_{\mathbb{F}}(W, \mathbb{F})$. We will give a more concrete model for $\Lambda^*(V)$ in this case. Recall that we call a multilinear map $W \times \ldots \times W \to \mathbb{R}$ alternating if exchanging two inputs negate the result.

Lemma 8. *$\Lambda^n(V)$ is canonically isomorphic to the vector space of alternating multilinear maps*

$$\underbrace{W \times \ldots \times W}_{n} \to \mathbb{R}.$$

The isomorphism sends $[v_1 \otimes \ldots \otimes v_n]$ to the map

$$(w_1, \ldots, w_n) \mapsto \sum_{\sigma \in \Sigma_n} sign(\sigma) v_1(w_{\sigma(1)}) \ldots v_n(w_{\sigma(n)}),$$

where Σ_n is the set of permutations on $(1, \ldots, n)$, i.e. the set of all bijections $\{1, \ldots, n\} \to \{1, \ldots, n\}$.

Question 111. Prove this lemma. I added a sketch argument below. □

Remark 37. Read this about what goes wrong in infinite dimensions. It should be clear when you work out the proof why we needed the assumption on characteristic. □

Hence, we have that $Alt^*(W) := \bigoplus_{n=0}^{\infty} Alt^n(W)$ is isomorphic as a graded vector space to $\Lambda^*(V)$. In particular, we see that there is an algebra structure on $Alt^*(W)$, the product of which we again denote by the wedge sign and call wedge product. It turns out that we can explicitly write this down.

Lemma 9. *Let $\alpha \in Alt^k(W)$ and $\beta \in Alt^l(W)$, then $\alpha \wedge \beta \in Alt^{k+l}(W)$ is the following alternating bilinear map*

$$\alpha \wedge \beta(w_1, \ldots, w_{k+l}) = \sum_{\sigma \in Sh(k,l)} sign(\sigma)\alpha(w_{\sigma(1)}, \ldots, w_{\sigma(k)})\beta(w_{\sigma(k+1)}, \ldots, w_{\sigma(k+l)}),$$

where $Sh(k, l)$ are the bijections $\sigma : \{1, \ldots, k\} \sqcup \{1, \ldots, l\} \to \{1, \ldots, k + l\}$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(1) < \ldots < \sigma(l)$.

Question 112. First, fully expand the sum for $k = l = 2$. Then, prove the statement. Finally, check super-commutativity of the wedge product directly from the formula. □

Remark 38. Elements of $Sh(k, l)$ are called shuffles, by analogy with playing card shuffles. □

For our purposes, it would suffice to only talk about alternating multilinear maps. The down side of that is that the wedge product would look very mysterious. It's good to have multiple viewpoints anyways.

Question 113. Let $e_1, \ldots, e_n$ be basis of W and let $e_1^\vee, \ldots, e_n^\vee$ be the dual basis of V. Prove that

$$e_{j(1)}^\vee \wedge \ldots \wedge e_{j(k)}^\vee,$$

for all injective order preserving maps $j : \{1, \ldots, k\} \to \{1, \ldots, n\}$ give a basis of $\Lambda^k(W^\vee)$, for all $k \geq 0$. □

Question 114. Let $e_1, \ldots, e_n$ be basis of W and let $e_1^\vee, \ldots, e_n^\vee$ be the dual basis of V. Compute

$$e_1^\vee \wedge \ldots \wedge e_n^\vee(w_1, \ldots, w_n),$$

for arbitrary elements $w_1, \ldots, w_n \in W$ by writing them as a linear combination of the basis elements. Assuming that $W = \mathbb{R}^n$ and $e_1, \ldots, e_n$ is the standard basis interpret your result as a geometric quantity related to $w_1, \ldots, w_n$ as vectors in the Euclidean space. □

Remark 39. The main point of differential forms is that you should be able to integrate them over submanifolds (with no extra data except an orientation). This computation gives you a hint of how that is going to work if you think about the change of variables formula. We will come back to this point next time. □

From now on we think of $\Lambda^*(W^\vee)$ for a finite dimensional vector space W as $Alt^*(W)$ unless otherwise specified. Now we define the interior product and pull-back operations on alternating multilinear maps.

Let $\alpha \in \Lambda^k(W^\vee)$ and $w \in W$, we define the interior product $\iota_w \alpha$ of α with w as follows:

$$\iota_w \alpha(\cdot, \dots, \cdot) := \alpha(w, \cdot, \dots, \cdot).$$

Question 115. Prove that interior product is an anti-derivation:

$$\iota_w(\alpha \wedge \beta) = \iota_w \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \iota_w \beta.$$

□

Finally, if you have a linear map $f : W \to U$, you can define

$$f^* : \Lambda^*(U^\vee) \to \Lambda^*(W^\vee),$$

by the formula

$$f^*\alpha(w_1, \dots, w_n) = \alpha(f(w_1), \dots, f(w_n)).$$

This is called the pull-back operation.

Question 116. Prove that f^* is a homomorphism of graded algebras. □

Question 117. Given a vector bundle $E \to B$, construct the vector bundle $\Lambda^n E^\vee \to B$ with fiber over b canonically identified with $\Lambda^n E_b^\vee$. You can do this in two different ways now: (1) transition maps (2) Whitney embedding theorem. □

22. Mar 3, 2021: Differential forms, exterior product, pull-back of differential forms, change of variables formula, integration of differential forms

We define the following vector bundles, for $k = 0, 1, \dots, dim(M)$:

$$\Lambda^k T^* X \to X.$$

Also, let $\Lambda^* T^* X$ be the Whitney sum $\bigoplus_{k=0}^{dim(X)} \Lambda^k T^* X$. The fibers $(\Lambda^k T^* X)_p$ are canonically identified with $\Lambda^k T_p^* X$, and the fibers $(\Lambda^* T^* X)_p$ with $\Lambda^*(T_p^* X)$.

Sections of $\Lambda^k T^* X \to X$ are smoothly varying k-linear alternating maps

$$\underbrace{T_p^* X \times \dots \times T_p^* X}_{k} \to \mathbb{R},$$

for all $p \in X$. We call these differential k-forms.

Question 118. What are differential 0 and 1-forms? □

Sections of $\Lambda^* T^* X$ are finite sums of differential k-forms for $k = 0, 1, \dots, dim(X)$. They are called differential forms. Let us also introduce some notations.

The $C^\infty(X, \mathbb{R})$-module of differential k-forms are denoted by

$$\Omega^k(X) := \Gamma(\Lambda^k T^* X)$$

. Similarly, we define

$$\Omega^*(X) := \Gamma(\Lambda^* T^* X).$$

Question 119. Show that $\Omega^*(X) = \bigoplus_{k=0}^{dim(X)} \Omega^k(X)$. Therefore the elements of $\Omega^*(M)$ are precisely differential forms. □

Question 120. Prove that $\Omega^*(X)$ is a graded super-commutative algebra over $C^\infty(X,\mathbb{R})$, where we use the wedge product pointwise to define the product structure. $\square$

The product structure on $\Omega^*(X)$ is also called wedge product, or sometimes exterior product.

We can also define the pull-back of differential forms. Given $\alpha \in \Omega^k(Y)$ and smooth map $f : X \to Y$, we can define

$$(f^*\alpha)_p(v_1,\dots,v_k) := \alpha_{f(p)}(f_*v_1,\dots,f_*v_k)$$

for all $p \in X$ and $v_1,\dots,v_k \in T_pX$.

Question 121. Check that $f^*\alpha$ is indeed a differential k-form. $\square$

Question 122. With the pull-back operation, we can extend the assignment $X \to \Omega^*(X)$ into a contravariant functor from the category of smooth manifolds with smooth maps to the category of graded super-commutative $\mathbb{R}$-algebras with algebra homomorphisms. If you don't know what this means, learn it. Then prove the statement. $\square$

Question 123. Does f induce a smooth map $T^*Y \to T^*X$? $\square$

Let us now switch gears and talk about differential forms concretely (i.e. using coordinates) on an open subset $U \subset \mathbb{R}^n$.

Calling the coordinates $x_1,\dots,x_n$, we have the vector fields $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ and the covector fields $dx_1,\dots,dx_n$. For every point $p \in U$, the former gives a basis of T_pU and the latter gives the dual basis of T_p^*U. Note that in our new notation $dx_1,\dots,dx_n \in \Omega^1(U)$.

Question 124. Prove that

$$dx_{j(1)} \wedge \ldots \wedge dx_{j(k)},$$

for all injective order preserving maps $j : \{1,\dots,k\} \to \{1,\dots,n\}$ give a $C^\infty(U,\mathbb{R})$-basis of $\Omega^k(U)$, for all $k \geq 0$. $\square$

More concretely, every differential k-form on U can be written uniquely as

$$\sum_j f_j dx_{j(1)} \wedge \ldots \wedge dx_{j(k)},$$

where $f_j : U \to \mathbb{R}$ are smooth functions.

Question 125. Compute the exterior product of differential forms on U in coordinates. $\square$

Let's now compute how the coordinate expressions change if we change coordinates. Let $y_1,\dots,y_n$ be a different set of coordinates in some open subset $V \subset U$. We can think of what we are about to do in two different ways:

(1) Think of $y_1,\dots,y_n$ as functions in the coordinates $x_1,\dots,x_n$. We can talk about the covector fields $dy_1,\dots,dy_n$ in $x_1,\dots,x_n$ coordinates and note that

$$dy_j = \sum_i \frac{\partial y_j}{\partial x_i} dx_i.$$

That $y_1, \ldots, y_n$ are a set of coordinates mean that $dy_1, \ldots, dy_n$ are linearly independent at all points of V. Therefore, any differential k-form on V can also be written as
$$\sum_j g_j dy_{j(1)} \wedge \ldots \wedge dy_{j(k)}.$$
We want to compute g_j.

(2) Think of $y_1, \ldots, y_n$ as defining a map $V \to \mathbb{R}^n$. That $y_1, \ldots, y_n$ are a set of coordinates mean that this map has open image and is a diffeomorphism onto its image $y : V \to V'$. The standard coordinates on V' can be rightly denoted by $y_1, \ldots, y_n$ (but this is not important, call them something else if you wish). We can then pull-back our differential k-form on V to V' by the inverse of y. We are then trying to compute the coefficient functions of this pull-back differential form in standard basis of V'. These coordinate functions are the "same as" g_j from (1).

These two are absolutely the same in content. The difference is that in (1) we think of $y_1, \ldots, y_n$ as functions on some open subset of the same ambient space where $x_1, \ldots, x_n$ are standard coordinates, whereas in (2) we think of them as the standard coordinates in some other Euclidean space, which is explicitly identified with the original Euclidean space along some open subsets.

I will go with (1) for this computation. It turns out to be cleaner to write $\{f_j\}$ in terms of $\{g_j\}$ and the reason is visible from viewpoint (2). Differential forms naturally pull-back, and writing $\{f_j\}$ in terms of $\{g_j\}$ is equivalent to pulling back by the map defined with the map $y : V \to V'$.

Here we go,
$$\sum_j g_j dy_{j(1)} \wedge \ldots \wedge dy_{j(k)} = \sum_j g_j \left(\sum_i \frac{\partial y_{j(1)}}{\partial x_i} dx_i \right) \wedge \ldots \wedge \left(\sum_i \frac{\partial y_{j(k)}}{\partial x_i} dx_i \right).$$
Well, then you expand and write it in the form $\sum_j f_j dx_{j(1)} \wedge \ldots \wedge dx_{j(k)}$. Make sure to not forget what the sum over j is indexed by. There is no point of actually doing it in this generality.

Note that each g_j is a function on V and hence can be written in coordinates $x_1, \ldots, x_n$ or $y_1, \ldots, y_n$ equally well. On the other hand, if you work with viewpoint (2), when you write down the corresponding pull-back, strictly speaking you need to define new functions on V' by pre-composing g_j with the map y.

Question 126. Write down the form $dx \wedge dy$ on $\mathbb{R}^2$ in polar coordinates on $\mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}$. □

Question 127. Write down $\{g_j\}$ in terms of $\{f_j\}$ by introducing the inverse map of y or equivalently by introducing the functions that express $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_n$. □

Let us now specialize to the case of differential n-forms in $\mathbb{R}^n$. The formula above reads:
$$g dy_1 \wedge \ldots \wedge dy_n = g \left(\sum_i \frac{\partial y_1}{\partial x_i} dx_i \right) \wedge \ldots \wedge \left(\sum_i \frac{\partial y_n}{\partial x_i} dx_i \right).$$
You compute and realize that
$$g dy_1 \wedge \ldots \wedge dy_n = g \cdot det \left(\frac{\partial y_j}{\partial x_i} \right) dx_1 \wedge \ldots \wedge dx_n.$$

Question 128. Deduce this from Question 114. □

This formula is the key to the integration of differential forms along manifolds. Recall the change of variables formula (assuming that integrals exist):

$$\int_V g dy_1 \dots dy_n = \int_V g \cdot |det\left(\frac{\partial y_j}{\partial x_i}\right)| dx_1 \dots dx_n.$$

In calculus exercises, this also involves writing the function g and domain V in terms of $y_1, \dots, y_n$ on LHS and in terms of $x_1, \dots, x_n$ on RHS.

In calculus $dy_1 \dots dy_n$ does not mean much - it just reminds the person who is taking the integral what coordinates we have been using. Now we will interpret $gdy_1 \dots dy_n$ as the differential n-form $gdy_1 \wedge \dots \wedge dy_n$ and $g \cdot |det\left(\frac{\partial y_j}{\partial x_i}\right)| dx_1 \dots dx_n$ as $g \cdot |det\left(\frac{\partial y_j}{\partial x_i}\right)| dx_1 \wedge \dots \wedge dx_n$.

Assuming that $det\left(\frac{\partial y_j}{\partial x_i}\right) > 0$, these two differential n-forms are the same! The integral of this differential form over V (defined using the symbolic trickery in the previous paragraph) is independent of coordinates, as long as we restrict ourselves to coordinates with the same orientation.

Question 129. Explain what it should mean for a diffeomorphism between open subsets of $\mathbb{R}^n$ to be orientation preserving. □

Because of this orientation issue, we can only integrate an n-form on an oriented n-dimensional manifold, which we define now.

Definition 39. A smooth manifold is called orientable if it admits a subatlas all of whose transition maps are orientation preserving. Let us call such a subatlas positive. An orientation on a smooth manifold is the choice of a maximal positive subatlas. A smooth manifold with a specified orientation is called oriented. When we talk about an oriented smooth manifold, the only allowed smooth charts are the ones that belong to the maximal positive subatlas. □

Question 130. Show that a connected orientable smooth manifold has precisely two orientations. □

Remark 40. There are plenty of non-orientable smooth manifolds. The simplest example is $\mathbb{RP}^2 := Gr(1,3)$, the real projective plane. □

We are now almost ready to define integration of a differential n-form α on an oriented smooth manifold X of dimension n. It turns out that it takes some extra care if one wants to integrate non-compactly supported differential forms. This is to be expected perhaps as we then get into issues about the convergence of the integral (see here for some discussion.) We will be content with integrating compactly supported forms.

Choose a regular cover $\{U_i\}$ of X and a sub-ordinate partitions of unity $\rho_i : X \to \mathbb{R}$. We first temporarily define

$$\int_X \rho_i \alpha := \int_{\tilde{U}_i} (\phi^{-1})^* \rho_i \alpha,$$

where the RHS is defined using coordinates in $\tilde{U}_i$ as above. Note that we still have not used the change of variables formula, we just made a definition by the rewriting of the symbol $gdx_1 \dots dx_n$ as $gdx_1 \wedge \dots \wedge dx_n$.

Now we make the big definition:

$$\int_X \alpha := \sum_i \int_X \rho_i \alpha.$$

Question 131. For this definition to be "big" it better not depend on the choices of the regular cover and the partitions of unity. This is where the change of variables formula is used. Carefully prove this. We will talk about it on Friday. □

Remark 41. An alternative route to defining $\int_X \alpha$ would be to do the following.

- Assume that the support of α is contained in some coordinate chart. Then prove that the integral can be defined unambiguously using the change of variables formula.
- For general α, take any partitions of unity $\sum \rho_i = 1$ such that the support of each ρ_i is contained in some coordinate chart and define $\int_X \alpha$ as the sum of $\int_X \rho_i \alpha$ as defined in the previous step.
- We only need to prove independence on the choice of the coordinate chart. This is done by taking the "product" partitions of unity.

□

Question 132. Assuming we have a k-dimensional submanifold $Z \subset X$ and a compactly supported differential k-form β on X, define the integral of β over Z. □

Remark 42. An important fact about integration on manifolds is the degree formula. Let $\phi : M \to N$ be a proper smooth map of oriented smooth manifolds of dimension n. Assume that N is connected. Then, there exists an integer $deg(\phi)$ such that for every compactly supported $\eta \in \Omega^n(N)$

$$\int_M \phi^* \eta = deg(\phi) \int_N \eta.$$

For proof, and topological interpretation/consequences check out the book by Guillemin-Haine. □

23. Mar 5, 2021: Brief answers to selected questions

Answer to Question 111: Let's first understand the maps in this diagram

$$\begin{tikzcd}
 & & Alt^k(W) \arrow[d, "1"] \\
T^k(W^\vee) \arrow[r, shift left, "4"] \arrow[d, "2"'] & T^k(W)^\vee \arrow[l, shift left, "5"] \arrow[r, "3"] & Mult^k(W) \\
\Lambda^k(W^\vee) & &
\end{tikzcd}$$

- The map 1 is just the inclusion map of alternating k-linear maps into k-linear maps.
- The map 2 is the quotient by I^k map.

- The map 3 is an isomorphism that comes from the fact that giving a k-linear map
$$\underbrace{W \times \ldots \times W}_{k} \to \mathbb{R}$$
is the same thing as giving a linear map
$$\underbrace{W \otimes \ldots \otimes W}_{k} \to \mathbb{R}.$$
- The map 4 is the map that sends a pure tensor $v_1 \otimes \ldots \otimes v_k$ to the linear map that is defined on pure tensors by
$$w_1 \otimes \ldots \otimes w_k \to v_1(w_1) \ldots v_k(w_k).$$
Map 4 can be shown to be an isomorphism for W finite dimensional by showing that is injective and that the source and target have the same dimension.
- The map 5 is the inverse of 4.

Unfortunately, this diagram does not define a map from $\Lambda^k(W^\vee)$ to $Alt^k(W)$ as the map 4 does not factor through $\Lambda^k(W^\vee)$ or the map $3 \circ 4$ does not lie inside $Alt^k(W)$.

So we construct an alternative diagram that fixes these problems:

$$\begin{tikzcd}
 & & Alt^k(W) \arrow[d, "1"] \\
T^k(W^\vee) \arrow[r, shift left, "4'"] \arrow[d, "2"'] & T^k(W)^\vee \arrow[l, shift left, "5'"] \arrow[r, "3"] & Mult^k(W) \\
\Lambda^k(W^\vee) \arrow[ur, "6"'] & &
\end{tikzcd}$$

- The map $4'$ is the map that sends a pure tensor $v_1 \otimes \ldots \otimes v_k$ to the linear map that is defined on pure tensors by
$$w_1 \otimes \ldots \otimes w_k \to \sum_{\sigma \in \Sigma_k} sign(\sigma) v_1(w_{\sigma(1)}) \ldots v_k(w_{\sigma(k)}).$$
It is also helpful to write $4'$ as the post-composition of 4 with the map $T^k(W^\vee) \to T^k(W^\vee)$:
$$w_1 \otimes \ldots \otimes w_k \to \sum_{\sigma \in \Sigma_k} sign(\sigma) w_\sigma(1) \otimes \ldots \otimes w_\sigma(k).$$
- The map $5'$ is the map defined by multiplying 5 with $\frac{1}{k!}$.
- Now notice that I^k is sent to 0 under $4'$. Therefore, the map $4'$ factors through the quotient and hence we get the map 6.
- $4'$ and $5'$ are not two-sided inverses anymore.

Also notice that $3 \circ 4'$ now lands in the image of 1. Hence, we obtain our map $\Lambda^k(W^\vee) \to Alt^k(W)$, which is the map described in the statement of the lemma.

We can also define a map in the other direction $Alt^k(W) \to \Lambda^k(W^\vee)$ as the composition $2 \circ 5' \circ 3^{-1} \circ 1$.

Showing that these two maps are inverses of each other boils down to showing that on the image of $3^{-1} \circ 1$, $6 \circ 2 \circ 5'$ is the identity map. The image of $3^{-1} \circ 1$ is easily seen to be the linear maps that kill elements of $I^k \subset T^k(W)$. I leave it to you check that such elements are closed under applying $4' \circ 5'$ and in fact the map is the identity map. By construction $4' = 2 \circ 6$, which finishes the proof.

Answer to Question 112: We need to show that the diagram

$$\begin{tikzcd}
\Lambda^k(W^\vee) \times \Lambda^l(W^\vee) \arrow[r] \arrow[d] & Alt^k(W) \times Alt^l(W) \arrow[d] \\
\Lambda^{k+l}(W^\vee) \arrow[r] & Alt^{k+l}(W)
\end{tikzcd}$$

commutes.

We take $([\alpha],[\beta]) \in \Lambda^k(W^\vee) \times \Lambda^l(W^\vee)$ and consider the two ways of going to the lower right corner. To see that the results are the same, we test on $(w_1, \ldots, w_{k+l})$.

Going through the lower left corner, we get the result

$$\sum_{\sigma \in \Sigma_{k+l}} sign(\sigma) 4(\alpha \otimes \beta)(w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(k+l)}).$$

Here 4 is the map defined above.

Going through the upper right corner, we get the result

$$sum_{(\sigma_1', \sigma_2') \in \Sigma_k \times \Sigma_l} \sum_{\sigma \in Shk,l} s(\sigma) s(\sigma_1') 4(\alpha)(w_{\sigma(\sigma_1'(1))} \otimes \ldots \otimes w_{\sigma(\sigma_1'(k))})$$
$$s(\sigma_2') 4(\beta)(w_{\sigma(\sigma_2'(1))} \otimes \ldots \otimes w_{\sigma_2(\sigma'(l))}).$$

These two sums are the same, which boils down to fact that any permutation of a deck of cards can be obtained by first splitting the deck into two groups of originally adjacent cards, then permuting each group separately and then finally shuffling. The signs also work out essentially because if you compose two permutations the signs multiply, and the implication that sign is a local invariant in the sense that it is computed by the mod 2 number of transpositions needed to define the permutation.

Answer to Question 130: I will only show that there can be at most two different orientations. I am assuming you are familiar with orienting finite dimensional vector spaces. Define an orientation of TX to be a continuously varying choice of orientations on each tangent fiber. An orientation of a manifold in the sense that we defined it canonically defines an orientation of TX. Hopefully, you see this.

Now, I claim that if two positive smooth atlases on X give rise to the same orientation on TX then they have to be positively compatible. Take a chart from each atlas. Then consider the transition map. The fact that these two charts give rise to the same orientations of the tangent spaces in the intersection of their domains is equivalent to the positivity of the determinants of the Jacobians.

Finally, note that if M is connected, then orienting one tangent space automatically orients all tangent spaces by the continuity requirement. Hence, there can be at most two orientations of TX, which finishes the proof.

24. Mar 8. 2021: Exterior derivative, first pass at deRham theorem, Lie derivative of differential forms, Cartan formula

Today, we will discuss some more operations on differential forms. We start with exterior derivative of a differential form, which takes in a k-form and returns a $k+1$ form:

$$d : \Omega^*(X) \to \Omega^{*+1}(X).$$

For $k = 0$, we have already defined this operation, which took $f \in \Omega^0(X) = C^\infty(X, \mathbb{R})$ and returned $df \in \Omega^1(X) = \Gamma(T^*X)$. We want d to satisfy the following properties:

(1) It is $\mathbb{R}$-linear.
(2) It agrees with our existing definition on smooth functions.
(3) $d^2 = 0$, i.e. $(\Omega^*(X), d)$ is a chain complex.
(4) d satisfies the graded Leibniz rule,
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|x|}\alpha \wedge d\beta,$$
i.e. $(\Omega^*(X), d, \wedge)$ is a commutative differential graded algebra (cdga).
(5) If $\phi : X \to Y$ is smooth, then $\phi^* : \Omega^*(Y) \to \Omega^*(X)$ is a chain map:
$$d \circ \phi^* = \phi^* \circ d.$$

Definition 40. We call $(\Omega^*(X)\, d, \wedge)$ the deRham cdga of X, and denote the whole structure by $\Omega^*_{dR}(X)$. Let us also define

$$H^*_{dR}(X) := H^*(\Omega^*_{dR}(X)),$$

which is a super-commutative algebra. □

Remark 43. Hence, in fact we have a functor from smooth manifolds with smooth maps to cdga's with cdga homomorphisms (algebra homomorphism+chain map). □

Let us take a moment to state our end goal.

Theorem 15. *Let $C^*(X, \mathbb{R})$ be the singular cochain complex of X with real coefficients. Then, there exists a quasi-isomorphism*

$$\int : \Omega^*_{dR}(X) \to C^*(X, \mathbb{R})$$

obtained by integration of forms along smooth chains. The induced isomorphism

$$H^*_{dR}(X) \to H^*(X, \mathbb{R})$$

respects the algebra structures. Moreover, if we have a smooth map $X \to Y$, then the following diagram commutes:

$$\begin{tikzcd} H^*_{dR}(Y) \arrow[r] \arrow[d] & H^*(Y, \mathbb{R}) \arrow[d] \\ H^*_{dR}(X) \arrow[r] & H^*(X, \mathbb{R}). \end{tikzcd}$$

Remark 44. Note that $\int$ is not an algebra map (only after passing to homologies). This failure can be measured but I do not remember how to do this for example. The cup product is in fact not super-commutative and the fact that the deRham algebra is super-commutative at the chain level is an important plus for it.

Recall that Morse cochain complex with integer coefficients (requires a discussion of signed counting as we had briefly mentioned) is quasi-isomorphic to $C^*(X, \mathbb{Z})$, so it recovers integral cohomology, which deRham theory does not. On the other hand the product structure on Morse cohomology (which we did not discuss) is non-trivial to even define and at the chain level it leads to a complicated structure called an A_∞-algebra.

Witten's work that lead to Morse cohomology does something extremely fascinating. He shows that Morse cohomology arises as a certain limit of a deformed (using a smooth function $h : X \to \mathbb{R}$) deRham complex:

$$d_t := e^{-th} d e^{th}, \text{ as } t \to \infty.$$

This way he actually discovers what Morse cohomology should be. Making this precise is no small feat.

Along with cellular homology, now you know four major ways of accessing the cohomology of a smooth manifold. There is also the sheaf theoretic viewpoint, which is generally called Cech theory. We will briefly mention this next week. As far as I know, this are all the major ways - there are many variants etc. of course. It's a fruitful exercise to try to relate these directly to each other. As an example, the theorem above is the deRham to singular comparison theorem. □

We of course have a long way to go for this theorem and we will have to be brief for most of the steps. First, we have to actually define the exterior derivative.

By looking at the properties that it is supposed to satisfy you can work out what d should do in coordinates:

$$d(f dx_I) = df \wedge dx_I,$$

for $I \subset [n]$. Here we introduce the notation

$$dx_I := dx_{i(1)} \wedge \ldots \wedge dx_{i(k)},$$

where $i : [k] \to [n]$ is the unique order preserving injective map with image I.

Question 133. Let $U \subset \mathbb{R}^n$ be an open subset and call the coordinates $x_1, \ldots, x_n$. Prove that there exists a unique $d : \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying the properties (1)-(4) above. □

Question 134. For a diffeomorphism $U \to V$ between open subsets of an Euclidean space, prove that (5) follows from (1)-(4). Use this to define the exterior derivative on an arbitrary manifold. □

Let us now also define the Lie derivative of a differential form α along a vector field V. This is an $\mathbb{R}$-bilinear operation. If α is a k-form the result is a differential k-form. The idea for the definition is the same as the Lie derivative of a vector field along a vector field.

Let $\Phi_V : \mathcal{U} \to M$ denote the flow of V. We define for every $p \in X$,

$$\mathcal{L}_V \alpha(p) := \lim_{t \to 0} \frac{(\Phi(t, \cdot)^* \alpha)(p) - \alpha(p)}{t}.$$

Question 135. What is the Lie derivative of a function along a vector field? □

Question 136. Prove that $\mathcal{L}_V$ is a derivation on differential forms, i.e. if $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^l(X)$, then

$$\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_V(\beta).$$

□

Question 137. Prove that the Lie derivative operation also satisfies the following Leibniz rule. If V, W vector fields, and α a covector field, then

$$V \cdot \alpha(W) = (\mathcal{L}_V \alpha)(W) + \alpha(\mathcal{L}_V W).$$

Generalize the result to differential k-forms. □

Finally, also note that given a vector field V and differential k-form α, we can define the interior product $\iota_V \alpha$ pointwise:

$$\iota_V \alpha(p) := \iota_{V(p)} \alpha(p).$$

The following important result is generally called Cartan's magic formula.

Theorem 16. *If V is a vector field and α is a differential form:*

$$\mathcal{L}_V \alpha = \iota_V d\alpha + d(\iota_V \alpha).$$

Proof. The formula holds at the zeros of V. Both sides are $\mathbb{R}$-linear in α. Moreover, note that both $\mathcal{L}_V \cdot$ and $\iota_V d \cdot + d(\iota_V \cdot)$ are derivations on $\Omega^*(X)$ - former was mentioned above and the latter is because the anti-commutator of two anti-derivations is a derivation. Therefore, it suffices to check the formula for α being a smooth function or one of the 1-forms $dx_1, \ldots, dx_n$ in some coordinate neighborhood of every point where V does not vanish. Choosing coordinates that rectify V this becomes trivial. □

Question 138. Please come back to this question after next class. I state it here to make its connection with Cartan's formula clear. It is an integral version of Cartan's formula.

Let X be a compact oriented k-manifold with boundary embedded inside a manifold M. Take a vector field v on M, and a time dependent k-form ω_t. Prove:

$$\frac{d}{dt}\Big|_{t=0} \int_{X(t)} \omega_t = \int_{\partial X} \iota_v \omega_0 + \int_X \iota_v d\omega_0 + \int_X \frac{d}{dt}\Big|_{t=0} \omega_t,$$

where $X(t)$ is the image of X under the time-t flow of v.

Prove the special cases where ω is time independent, and (i) ω is closed or (ii) ∂X is empty or (iii) $M = \mathbb{R}^2, X = \{0\} \times [0,1]$ without using Cartan's formula. □

Question 139. Let α be a 1-form, V, W vector fields. Prove the curvature formula:

$$d\alpha(V, W) = V \cdot \beta(W) - W \cdot \beta(V) - \alpha([V, W]).$$

This formula generalizes to k-forms as well! If α be a k-form, $V_1, \ldots, V_{k+1}$ vector fields, then

$$\begin{aligned} d\alpha(V_1, \ldots, V_{k+1}) = &\sum_{i=1}^{k+1} V_i \cdot \alpha(\ldots, \widehat{V_i}, \ldots) + \\ &\sum_{1 \le i < j \le k+1} (-1)^{i+j} \alpha([V_i, V_j], \ldots, \widehat{V_i}, \ldots, \widehat{V_j}, \ldots), \end{aligned}$$

where the hat denotes the omitted vector fields from the ordered list $V_1, \ldots, V_{k+1}$. □

Question 140. Refresh your memory about chain homotopies between two chain maps. Next time, we will start by showing that smoothly homotopic smooth maps induce chain homotopic maps on the deRham algebra. □

25. Mar 10, 2021: Manifolds with boundary, Stokes theorem, closed and exact differential forms, homotopy formula for differential forms, Poincare lemma

We start by introducing manifolds with boundary. The local models for manifolds with boundary are open subsets of

$$\mathbb{H}^n := \{(x_1, \dots, x_n) \mid x_1 \geq 0\} \subset \mathbb{R}^n.$$

Recall that we call a function from a subset of an Euclidean space smooth if it can be extended to a smooth map in an open neighborhood.

Lemma 10. *Let U and V be open subsets of $\mathbb{H}^n$, if $\phi : U \to V$ is a smooth bijection with a smooth inverse, then it sends $U \cap (\{0\} \times \mathbb{R}^{n-1})$ diffeomorphically to $V \cap (\{0\} \times \mathbb{R}^{n-1})$.*

Proof. It suffices to show $\phi(U \cap (\{0\} \times \mathbb{R}^{n-1})) \subset V \cap (\{0\} \times \mathbb{R}^{n-1})$. This follows from the inverse function theorem. □

Remark 45. If ϕ in the statement is a homeomorphism, then it still sends $U \cap (\{0\} \times \mathbb{R}^{n-1})$ to $V \cap (\{0\} \times \mathbb{R}^{n-1})$. This is a more difficult result, called invariance of domain. □

For $U \subset \mathbb{H}^n$ open, let us define:

$$\partial U := U \cap (\{0\} \times \mathbb{R}^{n-1}).$$

Question 141. Define a smooth manifold with boundary using atlases. Define the boundary of a smooth manifold with boundary and prove that it is canonically a smooth manifold. □

Most of the theory we developed thus far can be extended to smooth manifolds with boundary without too much trouble. We can define tangent bundle, cotangent bundle, differential forms, integration of differential forms... When talking about orientations, don't forget the equivalent formulation as a continuously varying choice of orientations on each tangent space.

Remark 46. Sometimes it becomes useful to define manifolds with corners as well. You can probably guess the local models, but generally further assumptions are made on the naive definition for various reasons. There is not a uniform definition in the literature. □

Here is why it was worth discussing manifolds with boundary all of a sudden.

Theorem 17 (Stokes' theorem)**.** *Let M be an oriented manifold with boundary of dimension n and α a compactly supported differential $(n-1)$-form. Then ∂M can be oriented so that*

$$\int_M d\alpha = \int_{\partial M} \iota_{\partial M}^* \alpha.$$

Question 142. Fully prove this for a cube $M = [0,1]^n \subset \mathbb{R}^n$. Yes, this has corners but it's ok - you can make sense of it, you are grown ups. Then, explain very briefly how you can get to the general Stokes' theorem from this special case. □

Remark 47. Something that we will not have time to cover in this class are global neighborhood theorems: tubular neighborhood theorem for submanifolds, collar neighborhood theorem for the boundary of a manifold with boundary. Please read about these. □

Stokes' theorem tells us that integrals of certain differential forms over compact boundariless submanifolds are independent of continuous movements of the submanifold. These are the ones that satisfy $d\alpha = 0$. We call these closed. To explain what we mean let us recall an important definition, which is the rigorous and more general way of saying "continuous movements".

Remark 48. It is much more intuitive to think about integration over submanifolds but in fact we can integrate over any smooth map from a manifold and this flexibility is useful. □

Definition 41. Two smooth maps $f_0, f_1 : X \to Y$ are called (smoothly) homotopic if there is a smooth map

$$F : [0,1] \times X \to Y$$

such that $F|_{\{0\}\times X} = f_0$ and $F|_{\{1\}\times X} = f_1$ under the standard identification of X with $\{0\} \times X$ and $\{1\} \times X$. □

Proposition 22. *Let X be a k-dimensional oriented closed (boundariless and compact) manifold and let $f_0, f_1 : X \to Y$ be homotopic smooth maps. If α is a closed k-form on Y, then*

$$\int_X f_0^*\alpha = \int_X f_1^*\alpha.$$

Question 143. Prove this! □

For example if the image of $f : X \to Y$ is contained in the domain of a coordinate chart whose image in $\mathbb{R}^n$ is a ball, then the integral of any closed form over it vanishes! There is a class of closed forms where this vanishing happens without any condition. These are the exact forms, the ones that can be expressed as the exterior derivative of another differential form:

$$\alpha = d\beta.$$

Make sure you understand why from the Stokes formula.

The point is that not every closed form has to be exact. Even though for "small" local closed submanifolds closed forms integrate to zero, for the ones that are in a certain sense complicated, that do not fit in the domain of a chart, they do not necessarily do. But even so, these integrals are not sensitive to continuous movements. deRham theory is the systematization of this idea.

Recall that last time we had defined the deRham complex $(\Omega^*(X), d)$. Note that with terminology from today

$$H^i_{dR}(X) = \frac{\text{closed i-forms}}{\text{exact i-forms}}.$$

Let us now switch gears and explain the homotopy formula for pullback. We know that homotopic continuous maps induce chain homotopic maps on the singular cochain complex. We will show that the same is true in for the deRham complex.

Let $F : \mathbb{R} \times M \to N$ be smooth, and define $\iota_t : M \to \mathbb{R} \times M$ as the inclusion to t-level and $f_t : M \to N$ as $F \circ \iota_t$. Let us also define $tr_t : \mathbb{R} \times M \to \mathbb{R} \times M$ be the map that increases $\mathbb{R}$ by t. Now let us compute:

$$\begin{aligned}
\frac{d}{dt}|_{t=t_0} f_t^*\omega &= \frac{d}{dt}|_{t=t_0} \iota_0^* tr_t^* F^*\omega \\
&= \iota_0^* \frac{d}{dt}|_{t=0} tr_{t_0}^* tr_t^* F^*\omega \\
&= \iota_0^* tr_{t_0}^* \mathcal{L}_{\frac{\partial}{\partial t}} F^*\omega \\
&= \iota_{t_0}^* (\iota_{\frac{\partial}{\partial t}} dF^*\omega + d(\iota_{\frac{\partial}{\partial t}} F^*\omega)) \\
&= (\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*) d\omega + d(\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*)\omega
\end{aligned}$$

This is called the infinitesimal homotopy formula. We get the full homotopy formula by integrating.

Proposition 23. *Let $F : [0,1] \times M \to N$ be smooth, and define $\iota_t : M \to [0,1] \times M$ as the inclusion to t-level and $f_t : M \to N$ as $F \circ \iota_t$. There exists an $\mathbb{R}$-linear map*

$$h : \Omega^*(N) \to \Omega^{*-1}(M)$$

such that

$$f_1^*\omega - f_0^*\omega = hd\omega + dh\omega.$$

An explicit formula for h is given in the proof.

Proof. We can extend F to a smooth map $F : \mathbb{R} \times M \to N$. Integrating the infinitesimal homotopy formula (the extension does not appear at all):

$$\begin{aligned}
\int_0^1 \left(\frac{d}{dt}|_{t=t_0} f_t^*\omega\right) dt_0 &= \int_0^1 \left((\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*) d\omega\right) dt_0 + \int_0^1 \left(d(\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*)\omega\right) dt_0 \\
&= \int_0^1 \left((\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*) d\omega\right) dt_0 + d\int_0^1 \left(\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*\omega\right) dt_0.
\end{aligned}$$

We therefore define

$$h(\alpha) := \int_0^1 \left(\iota_{t_0}^* \iota_{\frac{\partial}{\partial t}} F^*\alpha\right) dt_0.$$

The desired relationship follows since by the fundamental theorem of calculus:

$$\int_0^1 \left(\frac{d}{dt}|_{t=t_0} f_t^*\omega\right) dt_0 = f_1^*\omega - f_0^*\omega.$$

□

Corollary 2. *Homotopic smooth maps induce the same map on deRham cohomology.*

Corollary 3 (Poincare lemma)**.** *Let $U \subset \mathbb{R}^n$ be star-shaped, which means that for some point $p \in U$, which we can assume without loss of generality to be the origin, and for every $c \leq 1$,*

$$cU \subset U.$$

Then $H_{dR}^*(U) = \mathbb{R}[0]$. This means that the homology is trivial in all non-zero degrees and is one dimensional in the zeroth degree.

Question 144. Deduce this from the homotopy formula. This is the last question you are responsible for in the exam. □

On Monday, we finish the discussion of differential forms with a sketch proof of the deRham theorem.

26. Mar 12, 2021: Brief answers to selected questions

Answer to Question 137: For any diffeomorphism $\phi : X \to X$, we have

$$\alpha(W)(\phi(p)) = \phi^* \alpha(\phi_*^{-1} V)(p).$$

If we denote the flow of V by ϕ_t, then we have:

$$V \cdot \alpha(W) = \frac{d}{dt}|_{t=0} \phi_t^*(\alpha(W)) = \frac{d}{dt}|_{t=0} \phi_t^* \alpha((\phi_{-t})_* V).$$

Now, we apply the diagonal trick to the very right:

$$(\frac{d}{dt}|_{t=0}\phi_t^*\alpha)(V) + \frac{d}{dt}|_{t=0}\alpha((\phi_{-t})_*V) = (\frac{d}{dt}|_{t=0}\phi_t^*\alpha)(V) + \alpha(\frac{d}{dt}|_{t=0}(\phi_{-t})_*V).$$

Finally use the definitions of the Lie derivative of a vector field and of a differential form.

Answer to Question 138: The tricky step here is to write

$$\int_{X(t)} \omega_t = \int_{\phi_t(X)} \omega_t = \int_X \phi_t^* \omega_t.$$

The last step is the general change of variables formula. The proof of this is easy, I leave it to you. Since we are just relabeling everything by a diffeomorphism it is also quite expected.

Also see Remark 42, which is called the degree formula, and which generalizes the change of variables formula. For a local diffeomorphism this is also not difficult, but in general it is slightly involved.

To finish the question use the diagonal trick, Cartan formula and Stokes theorem.

Remark 49. If you write things in coordinates, the diagonal trick that I used in the two answers above simply becomes the product rule for coefficient functions from Calculus. This was not the case when we used the diagonal trick before for the Lie bracket and non-commutativity of the flows stuff. □

Answer to Question 142: Actually it would suffice to do the local computation for a compactly supported n-form on $\mathbb{H}^n$. Cube is good exercise though, I will assume that you did this part. As long as you are careful about orientations, you should be able to do it.

So how to finish from this local computation? The intution is to divide your manifold with boundary into little cubes, use Stokes for each of the cubes, and finish using the cancellations for faces that are common to more than one cube. Dividing into cubes step ends up being unnecessarily complicated.

It is easier to do the following. Take a partitions of unity $\sum_i \rho_i$ where the support of each ρ_i is contained in the domain of a coordinate chart. Note that $\rho_i|\partial M$ is a partitions of unity on ∂M with the same property! Also note that

$$0 = d(\sum_i \rho_i) = \sum_i d\rho_i.$$

The following shows that it suffices to prove Stokes theorem for $\rho_i\alpha$:

$$\begin{aligned}\int_M d\alpha &= \sum_i \int_M \rho_i d\alpha \\ &= \sum_i \int_M \rho_i d\alpha + \sum_i \int_M d\rho_i \wedge \alpha \\ &= \sum_i \int_M d(\rho_i \alpha) \\ &= \sum_i \int_{\partial M} \iota_{\partial M}^*(\rho_i \alpha) \\ &= \sum_i \int_{\partial M} \rho_i \mid_{\partial M} \iota_{\partial M}^*(\alpha) \\ &= \int_{\partial M} \iota_{\partial M}^*(\alpha)\end{aligned}$$

But of course $\int_M \rho_i\alpha$ can be computed inside $\mathbb{H}^n$. Make sure you understand how your local computation covers this.

27. Mar 15, 2021: Proof of deRham theorem, further directions in deRham theory: Poincare duality, intersection theory

Let us briefly recall the construction of singular cohomology of a topological space X. The integral singular chain complex in degree $i \geq 0$

$$C_i(X, \mathbb{Z})$$

is the abelian group generated by all continuous maps from the ith simplex Δ^i to X.

The differential

$$\delta : C_*(X, \mathbb{Z}) \to C_{*-1}(X, \mathbb{Z})$$

is defined as an alternating sum of all the face maps. The singular cochain complex over a ring R

$$C^*(X, R)$$

is defined as the dual complex. Cup product turns $C^*(X, R)$ into a differential graded algebra over R.

In what follows, we will take X to be a smooth manifold, in which case only considering smooth maps $\Delta^i \to X$ gives rise to chain homotopy equivalent results. We do this without comment from now on and ignore some technicalities that arise. Please see the corresponding section in Lee for details.

Given a smooth map $f : \Delta^i \to X$ and a smooth differential i-form α on X, we obtain a real number by integration

$$\int_{\Delta^i} f^*\alpha.$$

Clearly this gives rise to an $\mathbb{R}$-linear map

$$\int : \Omega^*_{dR}(X) \to C^*(X, \mathbb{R}).$$

Remark 50. When the dimension of the manifold and the degree of the form no not match, the integral is formally defined to be zero for convenience. $\square$

Question 145. Use the Stokes' theorem to prove that this map is a chain map! □

Hence, we made our first step in the pursuit of deRham theorem.

Theorem 18. *Let $C^*(X, \mathbb{R})$ be the singular cochain complex of X with real coefficients. Then, there exists a quasi-isomorphism*

$$\int : \Omega^*_{dR}(X) \to C^*(X, \mathbb{R})$$

obtained by integration of forms along smooth chains. The induced isomorphism

$$H^*_{dR}(X) \to H^*(X, \mathbb{R})$$

respects the algebra structures. Moreover, if we have a smooth map $X \to Y$, then the following diagram commutes:

$$\begin{tikzcd} H^*_{dR}(Y) \arrow[r] \arrow[d] & H^*(Y, \mathbb{R}) \arrow[d] \\ H^*_{dR}(X) \arrow[r] & H^*(X, \mathbb{R}). \end{tikzcd}$$

Question 146. Use the general change of variables formula for integration of differential forms to obtain the last statement about naturality of the deRham isomorphism. □

We already know that if U is a smooth manifold diffeomorphic to $\mathbb{R}^n$, then $H^*_{dR}(U) = \mathbb{R}[0]$. We also know that the same result is true for $H^*(U, \mathbb{R})$. Consider the map

$$H^*(\int) : H^*_{dR}(U) \to H^*(U, \mathbb{R}).$$

Question 147. Chech that this is an isomorphism. □

Now recall that very early on in the quarter, we had mentioned the existence of good covers.

Definition 42. Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ cover X. We call $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ good if it is locally finite and for any finite subset $J \in \mathcal{I}$,

$$\bigcap_{i \in J} U_i$$

is either empty or diffeomorphic to $\mathbb{R}^n$ (or equivalently an open ball). □

We want derive the fact that $H^*(\int)$ is an isomorphism for all smooth manifolds from the local result for $\mathbb{R}^n$ and the existence of a good open cover. What we need is a "local-to-global" result.

Recall the Mayer-Vietoris exact sequence for singular cohomology:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
H^{n+1}(U \cup V, R) \arrow[r] & H^{n+1}(U, R) \oplus H^{n+1}(V, R) \arrow[r] & H^{n+1}(U \cap V, R) \arrow[ull] \\
H^{n}(U \cup V, R) \arrow[r] & H^{n}(U, R) \oplus H^{n}(V, R) \arrow[r] & H^{n}(U \cap V, R) \arrow[ull] \\
\cdots & \cdots & \cdots \arrow[ull]
\end{tikzcd}$$

The way to construct this long exact sequence is to show that the following is a short exact sequence of cochain complexes:

$$0 \to C^*(U \cup V, R) \to C^*(U, R) \oplus C^*(V, R) \to C^*(U \cap V, R) \to 0.$$

Proposition 24. *Let U, V be open subsets of X. Then, the following is a short exact sequence:*

$$0 \to \Omega^*_{dR}(U \cup V) \to \Omega^*_{dR}(U) \oplus \Omega^*_{dR}(V) \to \Omega^*_{dR}(U \cap V) \to 0.$$

Question 148. Prove this. It is actually quite easy. You will need a partitions of unity argument in one of the slots. □

Hence, we obtain the Mayer-Vietoris exact sequence for deRham cohomology:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
H^{n+1}_{dR}(U \cup V) \arrow[r] & H^{n+1}_{dR}(U) \oplus H^{n+1}_{dR}(V) \arrow[r] & H^{n+1_{dR}}(U \cap V) \arrow[ull] \\
H^{n}_{dR}(U \cup V) \arrow[r] & H^{n}_{dR}(U) \oplus H^{n}_{dR}(V) \arrow[r] & H^{n}_{dR}(U \cap V) \arrow[ull] \\
\cdots & \cdots & \cdots \arrow[ull]
\end{tikzcd}$$

Question 149. Concretely describe the connecting homomorphisms in both Mayer-Vietoris sequences for fun. □

The following proposition is immediate.

Proposition 25. *Let U, V be open subsets of X. Then the following diagram commutes, where the vertical maps are the integration map.*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_{dR}(U \cup V) \arrow[r] \arrow[d] & \Omega^*_{dR}(U) \oplus \Omega^*_{dR}(V) \arrow[r] \arrow[d] & \Omega^*_{dR}(U \cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^*(U \cup V, \mathbb{R}) \arrow[r] & C^*(U, \mathbb{R}) \oplus C^*(V, \mathbb{R}) \arrow[r] & C^*(U \cap V, \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Corollary 4. *If two out of the three vertical maps in this proposition are quasi-isomorphisms, then so is the third.*

Proof. Use the five lemma in the induced map of long exact sequences. □

Theorem 19. *$H^*(\int)$ is an isomorphism for any smooth manifold with a finite good open cover.*

Proof. Do induction on the number of elements in the cover. ☐

Question 150. Prove that if $H^*(\int)$ is an isomorphism for X_i, $i \in \mathcal{I}$, where $\mathcal{I}$ is countable, then it is an isomorphism for

$$\bigsqcup_{i\in\mathcal{I}} X_i$$

as well. ☐

Theorem 20. *$H^*(\int)$ is an isomorphism for any smooth manifold.*

Proof. Use an exhaustion to write the manifold as the union of two open subsets each of which is a disjoint union of open subsets with finite good covers. ☐

All that is left from the main theorem is the statement about product structures. We do not have time to actually do this. The right way to go about this is through understanding Poincare duality and Thom isomorphisms in deRham theory.

Remark 51. Developing the theory for non-compact smooth manifolds require the introduction of compactly supported deRham complex. ☐

Probably the most striking idea here is the following. For simplicity assume that X is a closed oriented manifold and Z is an oriented submanifold. Then, we can construct a differential form η_Z that is supported near Z with the following property:

$$\int_Z \iota_Z^*\alpha = \int_M \eta_Z \wedge \alpha,$$

for every differential form α on M.

Doing some work one can prove for example that for any two oriented submanifolds Z_1 and Z_2, the cup product

$$PD([Z_1]) \cup PD([Z_2]) \in H^*(X, \mathbb{R})$$

is equal to the the class of $\eta_{Z_1} \wedge \eta_{Z_2}$ under the deRham isomorphism.

Remark 52. Note that $PD([Z_1]) \cup PD([Z_2])$ can be computed as the Poincare dual of the class of the submanifold that is obtained by taking the transverse intersection of a generically perturbed Z_1 and Z_2. ☐

Assume now that the oriented submanifolds Z_1 and Z_2 are transverse and of complementary dimension. Then we can prove

$$Z_1 \cdot Z_2 = \int_{Z_1} \eta_{Z_2},$$

where the LHS is the signed count of intersection. One roughly thinks of η_Z as a Dirac delta "function" supported at Z.

Question 151. I had mentioned that one can also access the cohomology of a smooth manifold using sheaf theoretic techniques. Read and understand the notes here. Solve the exercise at the end. I think you will find this useful and fun. ☐

28. Mar 17, 2021: Diffeomorphism relatedness of vector fields, Vector fields depending on extra parameters, time dependent flows, Ehresmann connections, Proof of Ehresmann theorem

Today we will prove the following theorem. Assume that B is connected throughout.

Theorem 21 (Ehresmann)**.** *Let $\pi : E \to B$ be a proper surjective submersion. Then π is a fiber bundle.*

Before doing that I want to introduce two more general concepts regarding vector fields and flows.

Definition 43. Let $\varphi : X \to Y$ be a smooth map. Let V be a vector field on X and W be one on Y. We call V and W φ-related if for every $p \in X$,

$$\varphi_* V(p) = W(\varphi(p)).$$

□

Lemma 11. *In the notation of Definition 43, if $\gamma : (a,b) \to X$ is an integral curve of V, then $\varphi \circ \gamma : (a,b) \to Y$ is an integral curve of W.*

Proof. Use the chain rule. □

Even though we really will not use it today, let us also note the following.

Lemma 12. *In the notation of Definition 43, if $f : Y \to \mathbb{R}$ is a smooth function, then*

$$V \cdot (f \circ \varphi) = (W \cdot f) \circ \varphi.$$

In fact, φ-relatedness is equivalent to this condition.

Proof. For every $p \in X$,

$$V \cdot (f \circ \varphi)(p) = df_{\varphi(p)}(d\varphi_p V(p)) = df_{\varphi(p)}(W_{\varphi(p)}) = (W \cdot f)(\varphi(p)),$$

as desired. The converse follows from the assumed equality in the middle, and the fact if two vectors define the same derivation at a point, then they are the same vector. □

Question 152. Prove that if V_1 and V_2 are φ-related to W_1 and W_2 (resp.), then $[V_1, V_2]$ is φ-related to $[W_1, W_2]$. □

Remark 53. Note that in the same way, one can make sense of two differential operators being φ-related. Hence you can in fact prove the stronger statement that $V_1 \circ V_2$ and $W_1 \circ W_2$ are φ-related. □

Let us now move on to the second point that I want to make. Sometimes, one encounters vector fields on manifolds that depend on extra parameters. To be rigorous these are smooth maps $S \times X \to TX$, where S is a smooth manifold and fixing the parameter to any $s \in S$, we obtain a section $X \to TX$.

For simplicity assume that S is an open subset of $\mathbb{R}^N$ with coordinates $s_1, \dots, s_N$. On a coordinate chart in X with coordinates $x_1, \dots, x_n$, then this S-family of vector fields look like

$$\sum_i f_i(s_1, \dots, s_N, x_1, \dots, x_n) \frac{\partial}{\partial x_i}.$$

We want to of course talk about the flows of these vector fields, specifically we want make a statement that the flows of vector fields depend smoothly on parameters. This sort of thing can be a bit confusing but all you have to do is the following.

Consider the S-family of vector fields as a single vector field on $S \times X$ in the only possible way. Now for this vector field we have developed the theory of flows. All we need to do is to use the results that we proved there.

Question 153. Assume that X is closed so that there is no issue of completeness (just so that the result you get can be expressed easily). Construct the flow map

$$\mathbb{R} \times S \times X \to X$$

so that if we fix s, what we obtain is the flow of the vector field of the parameter s. Prove that this map is smooth using our results for the flow of a single vector field. □

This is a useful technique in general: if you have something that depends on extra parameters, you can just think of those parameters as extra degrees of freedom in your space and consider one static something.

An important special case of vector fields depending on parameters is time-dependent vector fields. This is the case where S is one-dimensional. Typically S also has a specified coordinate given to us, which we think of as time. Let us just take $S = \mathbb{R}$ for simplicity. Here, there is something more interesting we can consider than just looking at the flows of the vector fields for each value of the parameter. We can change the vector field as we are flowing in the sense that our trajectories (integral curves) are now tangent at time t to the vector field at time t (which we call V_t.)

Question 154. Assuming X is closed again, show that this defines a smooth map

$$\mathbb{R} \times X \to X.$$

Do this by defining the vector field

$$\frac{\partial}{\partial t} + V(t)$$

on $\mathbb{R} \times X$, and relating the flow of this vector field on $\mathbb{R} \times X$ to the time dependent flow. Notice that the map $\mathbb{R} \times X \to X$ is not an action of $\mathbb{R}$ for a time dependent flow. □

Remark 54. If you understand this method, you should be able to use it when completeness is not given. □

Ok, let's go back to Ehresmann theorem. Here is the key definition.

Definition 44. Let $\pi : E \to B$ be a submersion.

- The vertical subbundle V of π is the distribution on E given by the kernel of $d\pi$ at every point.
- An Ehresmann connection (or a horizontal subbundle/distribution) on π is a subbundle H of TE (i.e. a distribution) such that at every point $p \in E$,

$$H_p \oplus V_p = T_pE.$$

□

Proposition 26. *Let $\pi : E \to B$ be a submersion. We can always find an Ehresmann connection for π.*

Proof. Choose a Riemannian metric on E (we know that they exist). Then, define the horizontal subspaces by taking orthogonal complements to the vertical subspaces at every point. □

Lemma 13. *Let $\pi : E \to B$ be a submersion and H be an Ehresmann connection. Let v be a vector field on B. Then, there exists a unique vector field $v^{\#}$ on E whose vectors belong to H and so that v and $v^{\#}$ are π-related.*

Question 155. This is easy once you understand what is going on. After you prove this you will have made a big step towards understanding how Ehresmann connections work. □

In particular, we can try to "lift" the flow of a vector field on B to the flow of this canonical vector field on E. The only reason I say try to is because of potential incompleteness issues, leading us to the properness assumption for the Ehresmann theorem.

Question 156. Find a surjective submersion $\pi : E \to B$ with B closed (so every vector field on B is complete) and an Ehresmann connection on π such that the canonical lift $v^{\#}$ of some vector field v on B is incomplete.

Similarly, find a surjective submersion $\pi : E \to B$ that is not a fiber bundle. □

Question 157. A problem that I should have given you long ago is the following. For any connected smooth manifold and two points A and B there exists a self diffeomorphism sending A to B. The way to do this is to find a vector field whose flow takes A to B.

Now using the same strategy prove the following weaker version of Ehresmann theorem. Any two fibers of a proper surjective submersion $\pi : E \to B$ are diffeomorphic. □

Now we finish the proof fully.

Proof of Ehresmann's theorem. First of all, it suffices to prove that for $B = \mathbb{R}^n$, we can find a fiber preserving diffeomorphism (a trivialization)

$$\mathbb{R}^n \times E_0 \to E.$$

We make the crucial step of choosing an Ehresmann connection H.

For every $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, define the vector field

$$V_a := \sum_i a_i \frac{\partial}{\partial x_i}.$$

The time 1 flow of V_a takes the origin to a.

Using our discussion above we can canonically lift these vector fields to E using the Ehresmann connection: $V_a^{\#}$. Now consider the smooth map

$$\Phi : \mathbb{R} \times \mathbb{R}^n \times E \to E,$$

which is the flows of all vector fields $V_a^{\#}$ put together. The key point here is that indeed all the flows are defined for all times because of the properness assumption (and because we know that the flow of V_a is complete).

We claim that the restriction of Φ to $\{1\}\times\mathbb{R}^n\times E_0$ gives the desired trivialization:

$$\phi := \Phi|_{\{1\}\times\mathbb{R}^n\times E_0} : \mathbb{R}^n \times E_0 \to E.$$

By construction and π-relatedness, ϕ sends $\{a\} \times E_0$ diffeomorphically to E_a. It follows that ϕ is a bijection. To finish we need to check that ϕ has invertible differential at every point. This is another good exercise so I will leave it to you. $\square$

Question 158. Finish the proof using the splittings of the tangent space to vertical and horizontal subspaces. $\square$

This is a nice theorem but Ehresmann connections are useful way beyond this proof. The proof simply illustrates the principle. If you have a fiber bundle, the nice way to relate/connect different fibers in a coherent way is to choose a horizontal subbundle and use the canonical lifting of tangent vectors from the base. Note that if you only have a fiber bundle, there is no way of canonically identifying nearby fibers to each other - different trivializations give different identifications.

Even with an Ehresmann connection the identifications of fibers are not entirely canonical, they depend on a path that is chosen between the two points in the base. Diffeomorphisms obtained this way are called parallel transport maps. Next time I will start by giving a careful definition of this.

The infinitesimal measure of how the parallel transport maps depend on the choice of the path is called the curvature! Curvature is the tensorial quantity that tests the integrability of the horizontal distribution, which we talked about when we were discussing Frobenius integrability. I will also clarify all this.

Finally I will talk about linear Ehresmann connections on vector bundles, and equivariant Ehresmann connections on principle G-bundles. These are the cases that are used a lot. In these cases, it is possible to entirely hide the geometric viewpoint via Ehresmann connections, so my goal is to make you aware that it exists.

29. Mar 19, 2021: Parallel transport using an Ehresmann connection, curvature of an Ehresmann connection, connections on principal G-bundles and vector bundles

Let $\pi : E \to B$ be a fiber bundle and let H be a choice of a horizontal subbundle. For safety we assume that B is connected and the fiber is a closed manifold, but it should be clear that there are some other situations where we can make sense of what is below. We start with the definition of parallel transport maps.

Definition 45. Let $\gamma : [0, 1] \to B$ be a smooth path. The corresponding parallel transport diffeomorphism

$$P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

is defined as follows. Consider the pull-back (can be thought of as restriction if we have an embedded path) fiber bundle

$$\gamma^*\pi : \gamma^*E \to [0, 1]$$

with the induced horizontal subbundle. Take the canonical horizontal vector field on γ^*E which is $\gamma^*\pi$-related to $\frac{\partial}{\partial t}$. The parallel transport diffeomorphism is the time-1 map of this vector field. $\square$

Question 159. Define the induced horizontal subbundle in this definition. Generalize the construction to the pullback bundle along any smooth map $X \to B$. This is called the pull-back connection. □

Question 160. Show that P_γ only depends on the image of γ. □

Question 161. Assume that the parallel transport maps only depend on the endpoints of paths. Prove that π is a trivial bundle. □

Remark 55. Note that the converse is not true. Even if a fiber bundle is trivial, the parallel transport maps might be highly dependent on the path. If you choose a connection that is compatible with a trivialization (called the trivial connection if a trivialization is specified) then it only depends on the endpoints, but most connections are not compatible with any trivialization. □

There is a weaker notion of independence of paths: two smooth paths that are homotopic through smooth paths with fixed endpoints induce the same parallel transport diffeomorphisms.

Proposition 27. *Any two smoothly homotopic rel. boundary paths (what we just explained) induce the same parallel transport diffeomorphism if and only if H is an integrable distribution.*

Proof. Let's first assume the independence in paths as in the statement and construct a foliation tangent to H. We first construct local sections: for every $b \in B$, $e \in E_b$, and simply connected neighborhood U of b, there is a unique section of π over U sending b to e and so that the image is tangent to H.

For every $u \in U$, we take a smooth path connecting b to u that lies entirely inside U and send u to the parallel transport of e over this path. This defines a section $U \to E$. It's smoothness can be shown similarly to the way we showed smoothness in the proof of Ehresmann's theorem. It is clear from the way parallel transport is defined (via lifting vectors to horizontal vectors) that the image is tangent to H.

To prove uniqueness, notice that if we have such a local section, then it is a diffeomorphism onto its image and paths simply lift to their images under this diffeomorphism. This shows that if u is sent to e', then e' has to be the parallel transport of e for any path from b to u. Also note that the simply connectedness of U is not necessary for this argument.

We define the leaves by patching together the images of these local sections in the only possible way using both the uniqueness and existence part of our previous claim. Finally, by the same argument in the proof of Ehresmann's theorem, we can see that we have trivialization diffeomorphisms

$$E_b \times U \to E|_U,$$

which certify that we do indeed have a foliation if we choose a coordinate chart in E_b.

Converse is easier. If H is integrable, one easily sees that the leaves of the tangent foliation are covering spaces over B. We finish using the properties of path lifting for covering spaces. □

Hence, dependence on paths in the same homotopy class is equivalent to the non-integrability of H. Recall that we had constructed a skew-symmetric (check!) bundle map

$$D \otimes D \to TM/D$$

for any distribution D on a smooth manifold M, whose vanishing was equivalent to integrability of D. In the case at hand, this obstruction takes the form of a skew-symmetric bundle map

$$H \otimes H \to TE/H \simeq V$$

over E.

This map and its millions of reinterpretations are called the curvature of H. Let us briefly indicate one of these reinterpretations.

The data of an Ehresmann connection is equivalent to a bundle map $h : TE \to TE$ that is a projection ($p^2 = p$) at every tangent space whose kernel is the vertical subspace. It is also equivalent to a bundle map $v : TE \to V$ that is the identity on V.

Definition 46. Let $L \to M$ be a vector bundle. An L-valued differential k-form on M is a smoothly varying collection of alternating k-linear maps

$$\underbrace{T_p \times \ldots \times T_p}_{k} \to L_p$$

at every $p \in M$. □

Assuming that there is a skew-symmetric bilinear pairing

$$\Gamma(L) \times \Gamma(L) \to \Gamma(L),$$

we can define the wedge product of L-valued differential forms:

$$\alpha \wedge \beta(V_1, \ldots, V_{k+l}) = \sum_{\sigma \in Sh(k,l)} \operatorname{sign}(\sigma)[\alpha(V_{\sigma(1)}, \ldots, V_{\sigma(k)}), \beta(V_{\sigma(k+1)}, \ldots, V_{\sigma(k+l)})].$$

Assuming $\Gamma(L)$ is a Lie algebra, this makes $L-$valued differential forms into a Lie algebra. Note that $\Gamma(TE)$ and $\Gamma(V)$ are Lie algebras via the Lie bracket of vector fields and the integrability of V.

Thinking of h as a TE valued 1-form, we see immediately that curvature is the same data as the TE-valued 2-form

$$\frac{1}{2}[h, h].$$

We also use the projection $TE \to V$ to turn this into a V-valued 2-form without loss of data (why?).

Finally one can also very easily define the covariant derivative of sections of $\Gamma(E)$:

$$\nabla : TM \times \Gamma(E) \to V.$$

You take a section s and a tangent vector l, push-forward l via s and project to the vertical subspace:

$$\nabla_l s := v(s_* l).$$

You can develop the theory a little further in this generality and I think it is useful to do so, but we are losing sight of B (and the clock is ticking!). As our final point, let H and H' be two Ehresmann connections with V-valued one forms v and v'. Then, $v - v'$ has a nice property, it vanishes on vertical vectors. Hence, it can be considered as a very complicated kind of 1-form on B: every tangent vector at $b \in B$ is sent to a vector field on E_b linearly (lift the vector to every point above arbitrarily and project using $v - v'$). This is a vector field valued 1-form on B.

We now start assuming that $\pi : E \to B$ is a vector bundle. Of course we want to then consider Ehresmann connections that respect this extra structure. Namely, we want the parallel transport maps to be linear maps, not just diffeomorphisms. One can express this condition directly on the horizontal subbundle:

- H is preserved under scalar multiplication maps $E \to E$.
- Consider the addition map $E \times E \to E$. Under the differential of this map, the image of a pair horizontal vectors at two points on a fiber E_b that project to the same vector in T_bB has to map to a horizontal vector that also projects to the same vector in T_bB.

These are weird conditions to express which is why in the context of vector bundles it is not very common to talk about Ehressmann connections. I am not even going to attempt at stating the extra property of the map $h : TE \to TE$.

The vector fields on E_b that we discussed above for $v - v'$ become linear vector fields! Note that a linear vector field on E_b is the same thing as a linear map $E_b \to E_b$. Therefore, we say that the space of connections is a torsor over $Hom(E, E)$ valued 1-forms on B!

The covariant derivative perspective becomes much more useful here, because we can think of it as a map

$$\nabla : TM \times \Gamma(E) \to E$$

using the fact that $V = \pi^* E$.

Lemma 14. *∇ satisfies the Leibniz rule in the $\Gamma(E)$ variable and it is $\mathbb{R}$-linear in the TM-variable.*

Proof. This is a good exercise. □

We could go on for a lot longer, for example relate the curvature as defined above to the more familiar definitions. To do this one has to re-express our definition entirely in terms of vertical projection, which can then be related to covariant derivative and so on.

After pointing out the slight inconvenience of using linear Ehresmann connections, I should also mention that for principle G-bundles the relevant notion, an equivariant Ehresmann connection, becomes again extremely convenient. In this case something that really helps is that $V \to E$ is canonically isomorphic to a trivial bundle with fibers $\mathfrak{g}$, so v is a vector (and not vector bundle) valued 1-form, so we can talk about its exterior derivative in the standard way etc.

Question 162. Prove that the curvature in this case is given by

$$dv + \frac{1}{2}[v, v].$$

You will need to use the formula of Cartan that we called curvature formula. To begin note that up to a little detail $v + h = id$. □