LECTURE NOTES - MATH 58J (SPRING 2022)

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1. Feb 24, 2022: Introduction, cohomology of $U \subset \mathbb{R}^n$

Consider the n dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, 1 \in [n]\},$$

where $[n] := \{1, \ldots, n\}$. Abusing notation x_i's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$ has a metric given by

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

This induces a topology on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on U and see how these can be used to analyze the topology of U.

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of U (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different

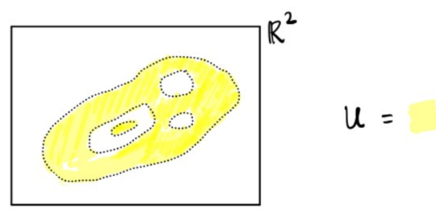


Figure 1. An example of an open subset of Euclidean space

ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.

- Up to many technical details, you can intuitively think about a degree k cohomology class β on U as a way of associating a real number $\beta(Z)$ to each compact, boundariless[1], oriented submanifold[2] Z of dimension k such that the following condition $(\star)$ holds.

If Z is the oriented boundary of a $(k+1)-$dimensional submanifold with boundary, then $\beta(Z) = 0$.

  Let's refer to such Z as "$k-$cycles" - in quotation marks because we will use this word with a different meaning later
- The main operation that one does with a differential $k-$form is to integrate them along $k-$dimensional oriented submanifolds and we use this to associate real numbers to "$k-$cycles".
- Property $(\star)$ will only hold if the differential form is closed.

1.1. **Cohomology of $U \subset \mathbb{R}^n$.** Let $\pi_0(U)$ be the set of all connected components of U.

Definition 1. $H^0(U, \mathbb{R})$ is defined as the vector space of all maps from $\pi_0(U)$ to $\mathbb{R}$. □

Let $b \in U$ and $\pi_1(U, b)$ be the fundamental group of U with base point b. Recall that

$$\pi_1(U, b) := \frac{\{(S^1, *) \to (U, b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where $S^1 = \frac{[0,1]}{0 \sim 1}$ and $* = [0] \in S^1$.
Here are some properties

- $\pi_1(U, b)$ is a group.
- Choosing a continuous path $\gamma : [0, 1] \to U$ from b to b' gives rise to a group isomorphism $f_\gamma : \pi_1(U, b) \to \pi_1(U, b')$.

Definition 2. Assuming that U is connected we define $H^1(U, \mathbb{R})_b$ as the vector space of group homomorphisms

$$\pi_1(U, b) \to \mathbb{R}.$$

□

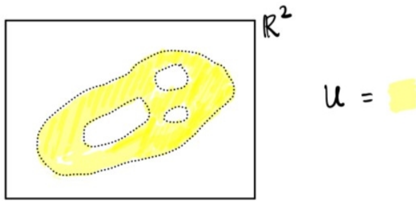
Exercise 1. Prove that for any $b, b' \in U$, as long as U is connected, there is a canonical isomorphism $H^1(U, \mathbb{R})_b \to H^1(U, \mathbb{R})_{b'}$. □

[1] I write this only to stress the point

[2] submanifold here means a subset that locally looks like a k dimensional Euclidean space

As a consequence of this exercise we can write $H^1(U, \mathbb{R})$ without any ambiguity.

Example 1. Let U be defined as below.



Then, $\dim(H^1(U, \mathbb{R})) = 3$. $\square$