LECTURE NOTES - MATH 58J (SPRING 2022)

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1. Feb 24, 2022: Introduction, cohomology of $U \subset \mathbb{R}^n$

Consider the n dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, 1 \in [n]\},$$

where $[n] := \{1, \ldots, n\}$. Abusing notation x_i's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$ has a metric given by

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

This induces a topology on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on U and see how these can be used to analyze the topology of U.

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of U (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different
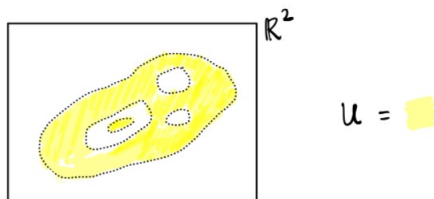

FIGURE 1. An example of an open subset of Euclidean space

ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.

- Up to many technical details, you can intuitively think about a degree k cohomology class β on U as a way of associating a real number $\beta(Z)$ to each compact, boundariless, not necessarily connected[1], oriented submanifold[2] Z of dimension k such that the following condition $(\star)$ holds.

> If Z is the oriented boundary of a $(k+1)-$dimensional submanifold with boundary, then $\beta(Z) = 0$.

Let's refer to such Z as "$k-$cycles" - in quotation marks because we will use this word with a different meaning later

- The main operation that one does with a differential $k-$form is to integrate them along $k-$dimensional oriented submanifolds and we use this to associate real numbers to "$k-$cycles".
- Property $(\star)$ will only hold if the differential form is closed.

1.1. **Cohomology of $U \subset \mathbb{R}^n$.** Let $\pi_0(U)$ be the set of all connected components of U.

Definition 1. $H^0(U, \mathbb{R})$ is defined as the vector space of all maps from $\pi_0(U)$ to $\mathbb{R}$. $\qquad\square$

Let $b \in U$ and $\pi_1(U, b)$ be the fundamental group of U with base point b. Recall that

$$\pi_1(U, b) := \frac{\{(S^1, *) \to (U, b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where $S^1 = \frac{[0,1]}{0 \sim 1}$ and $* = [0] \in S^1$.
Here are some properties

- $\pi_1(U, b)$ is a group.
- Choosing a continuous path $\gamma : [0, 1] \to U$ from b to b' gives rise to a group isomorphism $f_\gamma : \pi_1(U, b) \to \pi_1(U, b')$.

Definition 2. Assuming that U is connected we define $H^1(U, \mathbb{R})_b$ as the vector space of group homomorphisms

$$\pi_1(U, b) \to \mathbb{R}.$$

$\qquad\square$

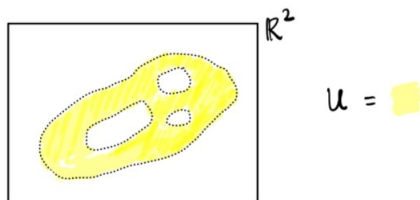*Exercise* 1. Prove that for any $b, b' \in U$, as long as U is connected, there is a canonical isomorphism $H^1(U, \mathbb{R})_b \to H^1(U, \mathbb{R})_{b'}$. $\qquad\square$

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[1] I write the last two only to stress the point

[2] submanifold here means a subset that locally looks like a k dimensional Euclidean space

As a consequence of this exercise we can write $H^1(U, \mathbb{R})$ without any ambiguity.

Example 1. Let U be defined as below.

Then, $\dim(H^1(U, \mathbb{R})) = 3$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2. Feb 24, 2022: Differential Forms on $U \subset \mathbb{R}^n$

- A differential 0-form on U is a smooth[3] function $U \to \mathbb{R}$.
- A differential 0-form f is called closed if $\frac{\partial f}{\partial x_i} \equiv 0, \forall i \in [n]$

Proposition 1. *There is a canonical linear isomorphism*

$$H^0_{dR}(U) := \{\text{closed differential 0-form on } U\} \simeq H^0(U; \mathbb{R})$$

Remark 1. In general $H^k_{dR} := \dfrac{\{\text{closed differential k-form on U}\}}{\{\text{exact differential k-form on U}\}}$ $\qquad\square$

- A differential 1-form on U is an expression $f_1 dx_1 + ... + f_n dx_n$ where $f_i : U \to \mathbb{R}$ are smooth functions
- A differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ is called exact, if for some smooth $V : U \to \mathbb{R}$,

$$f_i = \frac{\partial V}{\partial x_i}, \forall i \in [n].$$

  In this case we write $\alpha = dV$.
- A differential 1-form is closed if for all $i \neq j \in [n]$,

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0.$$

Lemma 1. *If $\alpha = \sum_{i=1}^{n} f_i dx_i$ is exact, then it is closed.*

Proof. Since it is exact, $\exists V : U \to \mathbb{R}$, such that $f_i = \frac{\partial V}{\partial x_i}$, so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

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[3]note that this is a condition much stronger than differentiable, it means that all iterated partial derivatives exist. please read the wikipedia page if you are not familiar.

$\square$

Exercise 2. For $n = 2$ and $n = 3$ explain what it means for the differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ to be closed in terms of the vector field $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ using terms from your calculus classes. Recall Green's and Stokes theorems. $\square$

Theorem 1. *Assuming that U is connected, there exists a linear isomorphism,*

$$(1) \qquad H^1_{dR} := \frac{\{closed\ differential\ 1\text{-}forms\ on\ U\}}{\{exact\ differential\ 1\text{-}forms\ on\ U\}} \simeq H^1(U; \mathbb{R})$$

Proof sketch. First we want to define a linear map

$$\int : \{\text{closed 1-forms}\} \to \{\pi_1(U, b) \to \mathbb{R} \quad \text{group homomorphisms}\}$$

Recall: $X \subset \mathbb{R}^n$ arbitrary subset. A map $g : X \to \mathbb{R}^m$ is called smooth if it extends to a smooth map $N(X) \to \mathbb{R}^m$ where $N(X)$ is an open neighborhood of X.

Fact:

- Any class in $\pi_1(U, b)$ can be represented by a smooth map $(S^1, *) \to (U, b)$.
- Any two smooth maps $S^1 \to U$ that are homotopic continuously are homotopic smoothly.

Recall: Given $\alpha = \sum_{i=1}^{n} f_i dx_i$ and a smooth path $\gamma : [0, 1] \to U$, we can define the line integral $\int_\gamma \alpha := \int_0^1 F \cdot \gamma' dt$, where $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$.

- The map $\int$ is independent of the parametrization of γ, meaning, if $\phi : [0, 1] \to [0, 1]$ is a smooth bijective map with $\phi' \neq 0$, then $\int_\gamma \alpha = \int_{\gamma \circ \phi} \alpha$. So line integral only depends on the image of γ.

Back to the map $\int$: we send a given $\alpha \in \{\text{closed 1-forms}\}$ to the map $\int \alpha : \pi_1(U, b) \to \mathbb{R}$ defined by

$$a \mapsto \int_{\bar\gamma} \alpha,$$

where $\bar\gamma$ is an arbitrary smooth representative of a. In the exercise below you will show that this is well defined. Assuming that for now, it is easy to see that $\int \alpha$ is a group homomorphism[4], so an element of $H^1(U; \mathbb{R})$, and the resulting $\int$ is a linear map.

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[4]I forgot to say this in class, so please check it for yourself!

Exercise 3. Show that the map $\int$ is well defined by proving if $\alpha = \sum_{i=1}^{n} f_i dx_i$ is closed and $\gamma, \gamma' : S^1 \to U$ are smooth maps that are smoothly homotopic, then $\int_\gamma \alpha = \int_{\gamma'} \alpha$. (**Hint:** Start by analyzing $n = 2, 3$ and where the smooth homotopy $S^1 \times [0, 1] \to U$ is injective, then reduce the statement to Green's theorem. Additionally, you may want to check proof of Stokes theorem.)

$\square$

FIGURE 2. An example of an injective smooth homotopy's image for $n = 2$.

Goals:
   (1) Prove that $\int$ sends an exact differential 1-form to zero. We obtain a linear map $\widetilde{\int} : H^1_{dR}(U, \mathbb{R}) \to H^1(U, \mathbb{R})$.
   (2) Prove that $\widetilde{\int}$ is injective. ("Construct a potential")
   (3) Prove that $\widetilde{\int}$ is surjective.
We start with 1). Let us integrate $\alpha = df$ along a smooth loop γ.

$$\int_\gamma \alpha = \int_0^1 \nabla f \cdot \gamma'(t) dt \overset{\text{FTC}}{=} f(b) - f(b) = 0.$$

Denote the resulting map by

$$\widetilde{\int} : \frac{\text{Closed differential 1-forms}}{\text{Exact differential 1-forms}} \to H^1(U, \mathbb{R}).$$

For 2), we need to show that if $\int_\gamma \alpha = 0 \quad$ for all $\quad \gamma : (S^1, \star) \to (U, b)$, then $\alpha = df$ for some $f : U \to \mathbb{R}$.

Exercise 4. Do this! This is the same task as constructing a potential (recall work integrals, conservative fields etc.) for the corresponding vector field. $\square$

Let us make a simplifying assumption on U to not deal with orthogonal difficulties in 3). Assume there exists $\gamma_1, \gamma_2, .., \gamma_n \in \pi_1(U, b)$ that

freely generates the abelianization of $\pi_1(U, b)$. This implies that giving a group homomorphism $\pi_1(U, b) \to \mathbb{R}$ is equivalent to assigning real numbers to each of $\gamma_1, \gamma_2, .., \gamma_n$ (arbitrarily).
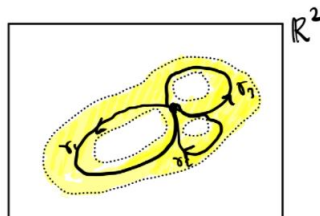

FIGURE 3

Now we need to create a differential 1-form that integrates to any $a_1, a_2, .., a_n \in \mathbb{R}$ along $\gamma_1, \gamma_2, .., \gamma_n$. This is still quite difficult. That will follow from DeRham Theorem, which will be the highlight of our course.

Remark 2. Once we give the general definition of singular cohomology, I will assign a homework exercise which shows that it agrees with what we defined today in degrees 0 and 1. The analogous statement will be automatic for DeRham cohomology. □

Exercise 5. Finish the proof of surjectivity in the case

$$U = \mathbb{R}^2 - \text{finitely many points.}$$

(**Hint:**Start with $\mathbb{R}^2 - (0, 0)$ and use the closed differential 1-form $\alpha = -\frac{ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$.) □

3. MARCH 03, 2022

3.1. **Manifolds.** Riemann was looking for a class of spaces which exist by themselves, (for example, they don't have to be embedded in an Euclidean space $\mathbb{R}^N$) with the following properties[5]. Let X be such a space:

- X admits local coordinates. This means that the points x <u>sufficiently near</u> any $x_0 \in X$ are determined uniquely by the <u>values of a set of real valued coordinates</u> $x_1, x_2, \cdots, x_n$:

$$x = (x_1, \ldots, x_n).$$
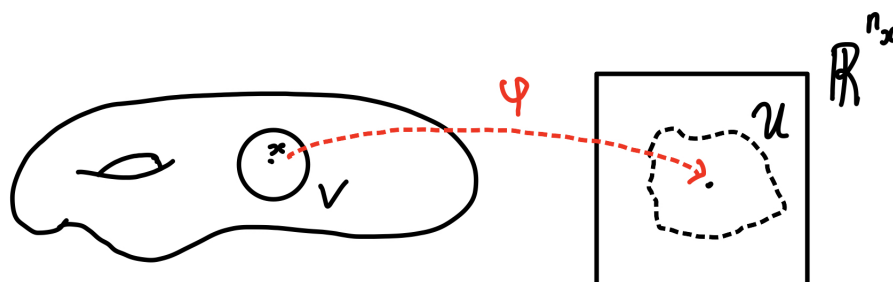
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[5]particularly vague phrases are underlined

FIGURE 4

This is sometimes called a generalized coordinate system in physics. There could be many such generalized coordinate systems near a given point. It is important that often generalized coordinates do not extend to the entirety of X.

- One can use techniques of calculus. This means, in particular, that there should be a large class of $C^1/C^2/\cdots/$ smooth functions $X \to \mathbb{R}$. If two generalized coordinate systems are related to each other by a non-differentiable transformation, then a function $X \to \mathbb{R}$ that is differentiable with respect to one may not be differentiable with respect to the other.

Definition 3. A topological space X is called a topological manifold if for every $x \in X$, there exists a nonnegative integer $n_x \geq 0$, an open subset $U \subset \mathbb{R}^{n_x}$, an open neighborhood $V \subset X$ of x and a homeomorphism $\phi : V \to U$ (See Figure 4). $\square$

Remark 3. Note that being a topological manifold is a property. $\square$

Definition 4. Let X be a topological space. Let us call $U \subset \mathbb{R}^n, V \subset X$ open and $\phi : V \to U$ homeomorphism a coordinate chart in X. V is called the domain of the chart and the functions $x_1, \cdots, x_n$ obtained by $x_i : V \to U \xrightarrow{pr_i} \mathbb{R}$ the coordinates of the chart. $\square$

Fact (A consequence of Invariance of Domain)
If an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$, then $m = n$.

Exercise 6. Using the fact above, prove that n_x in the definition is uniquely determined. Also, prove that $X \to \mathbb{Z}_{\geq 0}, x \mapsto n_x$ is constant on connected components of X. $\square$

Definition 5. If $n_x = n$ for all $x \in X$, then we say that X is $n-$dimensional. We write this briefly by X^n. $\square$
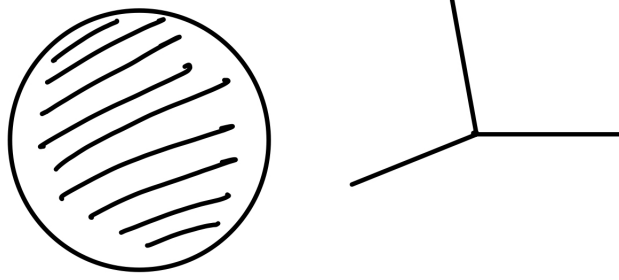
FIGURE 5. Non-Examples of Manifolds

From now on, when we say X is a topological manifold, we assume that there is such an $n \geq 0$.

Example 2. Topological Manifolds
- $\mathbb{R}^n$
- $S^n : \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

$\square$

To see that S^n is a topological manifold, we can use the stereographic projection. Let us consider a point $y_0 \in S^n$, and let H_0 be the hyperplane that is tangent to the point opposite to $y_0 \in S^n$ (we call this point $-y_0$). For every, $y \in S^n \setminus \{y_0\}$ the straight line l_y passing through y and y_0 intersects H_0 at precisely one point.

$$S_{y_0} : S^n \setminus \{y_0\} \longrightarrow H_0 \simeq \mathbb{R}^n$$

$$y \longmapsto l_y \cap H_0$$

Note that $H_0 \cong$ Parallel hyperplane passing through the origin $\cong \mathbb{R}^n$. The second homeomorphism can be obtained by choosing a basis.

Proposition 2. *S_{y_0} is a homeomorphism.*

Proof. (sketch)
$n = 0$ case is given in the figure. In this case the map is identity.

Exercise 7. Do the $n = 1$ case. $\square$

We can deduce the $n > 1$ case by using rotational symmetry around the line that contains the diameter (passing through y_0 that is perpendicular to H_0). The stereographic projection in dimension n is given by spinning around l_y the stereographic projection in dimension $n-1$. $\square$
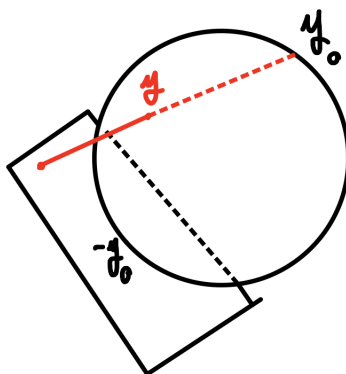
FIGURE 6. Stereographic Projection

FIGURE 7. $n = 0$ case

Remark 4. Stereographic projection
- Preserves angles.
- It preserves circles ($n = 2$).
- But, it distorts distances.

$\square$

Exercise 8. Prove that $\{x^2 - y^3 = 1\} \subset \mathbb{C}^2$ is a topological manifold. (**Hint:** Use projections to x and y.) $\square$

Definition 6. Let X be a topological space, and $\phi_1 : V_1 \to U_1$ and $\phi_2 : V_2 \to U_2$ be coordinate charts. Then, the map $\phi_2 \circ \phi_1^{-1} : \phi_1(V_1 \cap V_2) \to \phi_2(V_1 \cap V_2)$ is called the transition map form the chart ϕ_1 to the chart ϕ_2. Note that transition maps are automatically homeomorphisms. $\square$

Definition 7. A smooth atlas on a topological space X is a collection of charts $\{\phi_i : V_i \to U_i\}_{i \in I}$ such that
1) $\bigcup_{i \in I} V_i = X$
2) The transition map from any chart in the collection to any other in the collection is smooth. $\square$

Remark 5. Atlas means a book of maps, i.e. images of charts $\phi : V \to U \subset \mathbb{R}^2$ on the manifold that is the surface of the earth. It is likely that some of these maps are drawn using stereographic projection. $\square$
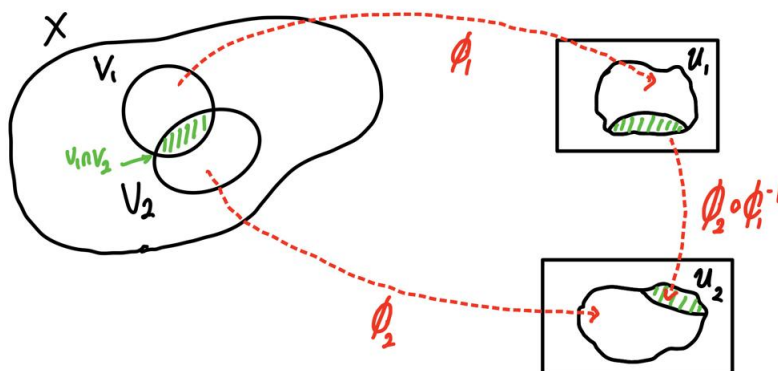
FIGURE 8. Transition map between charts

Definition 8. A smooth manifold is a topological space equipped with a maximal[6] smooth atlas. □

Exercise 9. Using latitude and longitude, define a chart on S^2 with domain $S^2 \setminus 0^{th}-$meridian and prove that it has smooth transition maps to all stereographic projections. You can assume that stereographic projection charts form a smooth atlas. □

4. MARCH 07, 2022

Convention: Unless otherwise stated, all of our vector spaces and chain complexes (to be defined) are over $\mathbb{R}$.

Definition 9. A graded vector space V_* is a collection of vector spaces $\{V_i\}_{i \in \mathbb{Z}}$ indexed by $\mathbb{Z}$. V_* is non-negatively graded if $V_i = 0$ for $i < 0$.[7] □

Definition 10. A chain complex (C_*, ∂_*) is a graded vector space C_* with a collection of linear maps $\partial_n : C_n \to C_{n-1}$, $n \in \mathbb{Z}$, such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We call ∂_n's boundary maps. □

$$\ldots \leftarrow C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \ldots$$

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[6]we did not have time to define this in the lecture and we will come back to it. it roughly means that if any chart has smooth transition maps to all the charts in the atlas, then it is contained in the atlas

[7]V=0 stands for the trivial vector space with 0 as the only element, $V = \{0\}$.

Definition 11. The homology of a chain complex (C_*, ∂_*) is a graded vector space $H_*(C_*, \partial_*)$ defined by

$$H_n\left((C_*, \partial_*)\right) := \frac{\mathrm{Ker}(\partial_n : C_n \to C_{n-1})}{\mathrm{Im}(\partial_{n+1} : C_{n+1} \to C_n)}$$

$\square$

It immediately follows from $\partial_i \circ \partial_{i+1} = 0$ that $\mathrm{Im}(\partial_{i+1}) \subset \mathrm{Ker}(\partial_i)$.
There is a slight variant of the last two definitions.

Definition 12. C^* graded vector space with $d_n : C^n \to C^{n+1}$ coboundary maps such that $d_n \circ d_{n-1} = 0$. (C^*, d_*) is called a co-chain complex.

$$H^n\left((C^*, d_*)\right) := \frac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n-1})}$$

is called cohomology.

$\square$

$$\ldots \to C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_{-1}} C^2 \to \ldots$$

Now we move on to define the singular chain complex $C_*(X; \mathbb{R})$ of a topological manifold X.

Definition 13. The n-dimensional simplex Δ^n for $n \geq 0$ is defined as

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_i \geq 0, \quad \forall i = 0, \ldots, n \\ x_0 + \ldots + x_n = 1 \end{array} \right\}$$
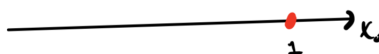
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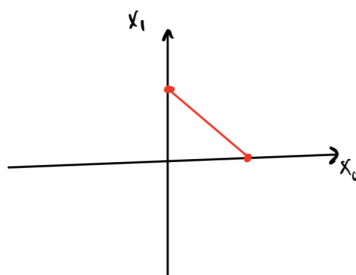

FIGURE 9. 0-dimensional simplex.
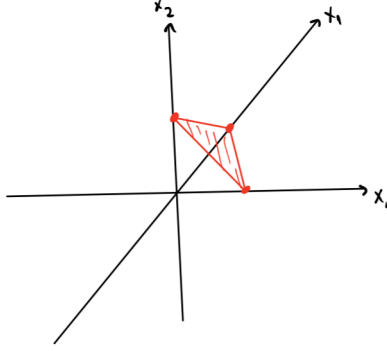

FIGURE 10. 1-dimensional simplex.
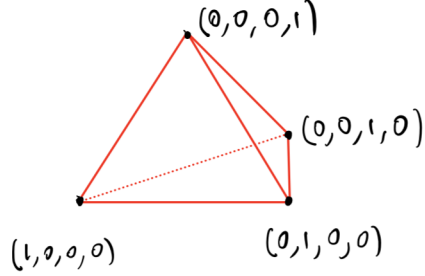
FIGURE 11. 2-dimensional simplex.

FIGURE 12. 3-dimensional simplex.

For each subset $S \subset \{0, 1, \ldots, n\}$, we can define a subset (a face) by

$$F_S := \left\{ (x_0, \ldots, x_n) \,\middle|\, \begin{array}{l} x_i = 0, \quad \text{if } i \in \{0, \ldots, n\} \setminus S \\ (x_0, \ldots, x_n) \in \Delta^n \end{array} \right\}$$

As an example, $F_{\{i\}}$, $i = 0, \ldots, n$ correspond to vertices.

Exercise 10.   • Prove that the dimension of F_S is $|S| - 1$. Explain what you mean by dimension.
   • Prove that $F_{S_1} \cap F_{S_2} = F_{S_1 \cap S_2}$.

$\square$

Definition 14. For each $n \geq 1$ and $0 \leq i \leq n$ we define the face map $f_{i,n} : \Delta^{n-1} \to \Delta^n$ with $(x_0, ..., x_{n-1}) \mapsto (y_0, ..., y_n)$ where

$$y_j = \left\{ \begin{array}{ll} x_j \;, & j < i \\ 0 \;, & j = i \\ x_{j-1} \;, & j > i \end{array} \right.$$

$\square$

- This simply adds a zero to the $(i+1)$th slot.
- The image of $f_{i,n}$ is $F_{\{0,\dots,n\}\setminus\{i\}}$

We need one last notion before we define the singular chain complex.

Definition 15. Given any set A, we define the vector space generated by A as the vector space of all finite formal linear combinations of the elements of A.

$$\left\{ \sum_{a \in A} c_a \cdot a \,\middle|\, c_a \in \mathbb{R} \ \text{ and } c_a \neq 0 \ \text{ for finitely many elements} \right\}.$$

$\square$

Exercise 11. Prove that the vector space of maps $A \to \mathbb{R}$ is isomorphic to the vector space generated by A if and only if A is finite. $\square$

Consider the subspace topology on simplices.

4.1. **Singular homology of a topological space.** Let X be a topological space. For $n \geq 0$, $C_n(X;\mathbb{R})$ is defined to be the vector space generated by the set of all continuous maps $\Delta^n \to X$. The elements of $C_n(X;\mathbb{R})$ are called singular chains of degree n. We set $C_n(X;\mathbb{R}) = 0$ for all $n < 0$. So $C_*(X;\mathbb{R})$ is a non-negatively graded vector space. Now we will equip it with boundary maps and turn it into a chain complex. Let $n \geq 1$. For any continuous $g : \Delta^n \to X$ we define

$$\partial_n g = \sum_{i=0}^{n} (-1)^i g \circ f_{i,n} \in C_{n-1}(X;\mathbb{R})$$

where $g \circ f_{i,n} : \Delta^{n-1} \xrightarrow{f_{i,n}} \Delta^n \xrightarrow{g} X$. We then extend to all singular chains so that map is linear and we get

$$\partial_n : C_n(X;\mathbb{R}) \to C_{n-1}(X;\mathbb{R}) \ \text{ for } n \geq 1$$

and $\partial_n = 0$ for $n < 1$.

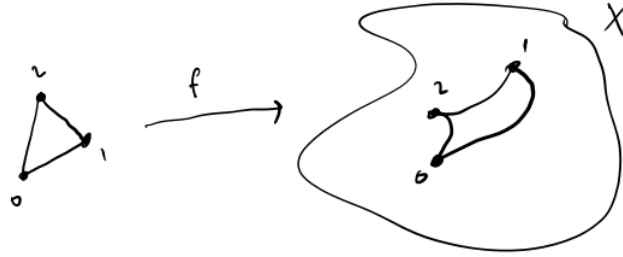*Example* 3. As an example, consider the following figure □

FIGURE 13. Example

Now we look at the boundary maps $\partial_2 f$ of f.
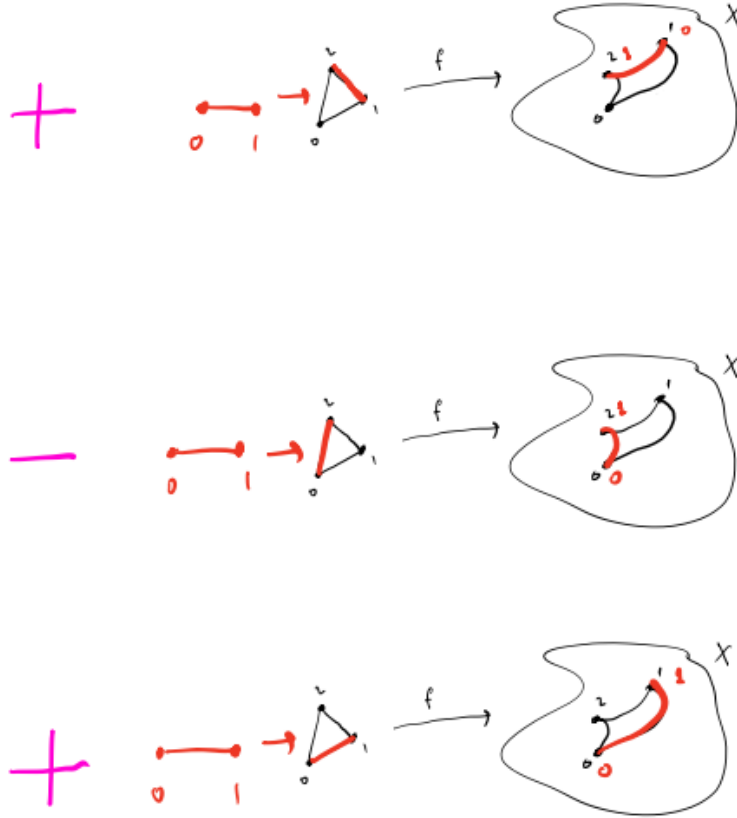

FIGURE 14. Boundary maps of the example above.

Proposition 3. $\partial_{n-1} \circ \partial_n = 0$

Proof. For $n < 2$ it is obvious. For $n \geq 2$ it suffices to show that for $g : \Delta^n \to X$ we should have $\partial_{n-1}(\partial_n(g)) = 0$.

$$\partial_{n-1}(\partial_n(g)) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i g \circ f_{i,n} \right) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(g \circ f_{i,n})$$

$$= \sum_{i=0}^{n}(-1)^i \left(\sum_{j=0}^{n-1}(-1)^j g \circ f_{i,n} \circ f_{j,n-1} \right) = \sum_{i,j}(-1)^{i+j} g \circ f_{i,n} \circ f_{j,n-1}$$

Where $f_{i,n} \circ f_{j,n-1} : \Delta^{n-2} \to \Delta^n$.

$\square$

Exercise 12. Prove this proposition. If you were not able to follow in class, first check this for $n = 2$ using the pictures above - no need to write this just to get yourself oriented. $\square$

Definition 16. We define the singular homology of X as the homology of its singular chain complex

$$H_n(X; \mathbb{R}) := H_n(C_*(X; \mathbb{R}), \partial_*) .$$

$\square$