

# LECTURE NOTES - MATH 58J (SPRING 2022)

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1. FEB 24, 2022: INTRODUCTION, 0th AND 1st COHOMOLOGY OF OPEN SUBSETS OF EUCLIDEAN SPACE (AD-HOC DEFINITIONS)

Consider the  $n$  dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\},$$

where  $[n] := \{1, \dots, n\}$ . Abusing notation  $x_i$ 's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$  has a metric given by

$$d(\vec{x}, \vec{y}) = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}$$

This induces a topology on  $\mathbb{R}^n$ . Let  $U \subset \mathbb{R}^n$  be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on  $U$  and see how these can be used to analyze the topology of  $U$ .

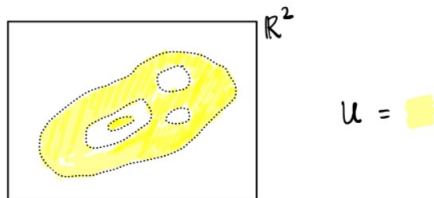


FIGURE 1. An example of an open subset of Euclidean space

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of  $U$  (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.
- Up to many technical details, you can intuitively think about a degree  $k$  cohomology class  $\beta$  on  $U$  as a way of associating a real number  $\beta(Z)$  to each compact, boundariless, not necessarily

connected<sup>1</sup>, oriented submanifold<sup>2</sup>  $Z$  of dimension  $k$  such that the following condition  $(\star)$  holds.

If  $Z$  is the oriented boundary of a  $(k + 1)$ -dimensional submanifold with boundary, then  $\beta(Z) = 0$ .

Let's refer to such  $Z$  as “ $k$ -cycles” - in quotation marks because we will use this word with a different meaning later

- The main operation that one does with a differential  $k$ -form is to integrate them along  $k$ -dimensional oriented submanifolds and we use this to associate real numbers to “ $k$ -cycles”.
- Property  $(\star)$  will only hold if the differential form is closed.

**1.1. Cohomology of  $U \subset \mathbb{R}^n$ .** Let  $\pi_0(U)$  be the set of all connected components of  $U$ .

*Definition 1.*  $H^0(U, \mathbb{R})$  is defined as the vector space of all maps from  $\pi_0(U)$  to  $\mathbb{R}$ .  $\square$

Let  $b \in U$  and  $\pi_1(U, b)$  be the fundamental group of  $U$  with base point  $b$ . Recall that

$$\pi_1(U, b) := \frac{\{(S^1, *) \rightarrow (U, b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where  $S^1 = \frac{[0,1]}{0 \sim 1}$  and  $* = [0] \in S^1$ .

Here are some properties

- $\pi_1(U, b)$  is a group.
- Choosing a continuous path  $\gamma : [0, 1] \rightarrow U$  from  $b$  to  $b'$  gives rise to a group isomorphism  $f_\gamma : \pi_1(U, b) \rightarrow \pi_1(U, b')$ .

*Definition 2.* Assuming that  $U$  is connected we define  $H^1(U, \mathbb{R})_b$  as the vector space of group homomorphisms

$$\pi_1(U, b) \rightarrow \mathbb{R}.$$

$\square$

*Exercise 1.* Prove that for any  $b, b' \in U$ , as long as  $U$  is connected, there is a canonical isomorphism  $H^1(U, \mathbb{R})_b \rightarrow H^1(U, \mathbb{R})_{b'}$ .  $\square$

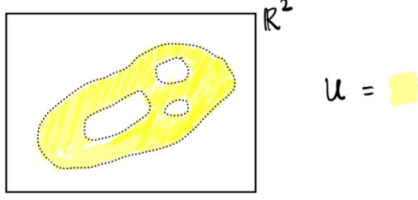
As a consequence of this exercise we can write  $H^1(U, \mathbb{R})$  without any ambiguity.

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<sup>1</sup>I write the last two only to stress the point

<sup>2</sup>submanifold here means a subset that locally looks like a  $k$  dimensional Euclidean space

*Example 1.* Let  $U$  be defined as below.



Then,  $\dim(H^1(U, \mathbb{R})) = 3$ . □

2. FEB 24, 2022: DEGREE 0 AND 1 DIFFERENTIAL FORMS ON OPEN SUBSETS OF EUCLIDEAN SPACE, A SPECIAL CASE OF DERHAM THEOREM

- A differential 0-form on  $U$  is a smooth<sup>3</sup> function  $U \rightarrow \mathbb{R}$ .
- A differential 0-form  $f$  is called closed if  $\frac{\partial f}{\partial x_i} \equiv 0, \forall i \in [n]$

**Proposition 1.** *There is a canonical linear isomorphism*

$$H_{dR}^0(U) := \{\text{closed differential 0-form on } U\} \simeq H^0(U; \mathbb{R})$$

*Remark 1.* In general  $H_{dR}^k := \frac{\{\text{closed differential k-form on } U\}}{\{\text{exact differential k-form on } U\}}$  □

- A differential 1-form on  $U$  is an expression  $f_1 dx_1 + \dots + f_n dx_n$  where  $f_i : U \rightarrow \mathbb{R}$  are smooth functions
- A differential 1-form  $\alpha = \sum_{i=1}^n f_i dx_i$  is called exact, if for some smooth  $V : U \rightarrow \mathbb{R}$ ,

$$f_i = \frac{\partial V}{\partial x_i}, \forall i \in [n].$$

In this case we write  $\alpha = dV$ .

- A differential 1-form is closed if for all  $i \neq j \in [n]$ ,

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0.$$

**Lemma 1.** *If  $\alpha = \sum_{i=1}^n f_i dx_i$  is exact, then it is closed.*

*Proof.* Since it is exact,  $\exists V : U \rightarrow \mathbb{R}$ , such that  $f_i = \frac{\partial V}{\partial x_i}$ , so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

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<sup>3</sup>note that this is a condition much stronger than differentiable, it means that all iterated partial derivatives exist. please read the wikipedia page if you are not familiar.

□

*Exercise 2.* For  $n = 2$  and  $n = 3$  explain what it means for the differential 1-form  $\alpha = \sum_{i=1}^n f_i dx_i$  to be closed in terms of the vector field

$F = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$  using terms from your calculus classes. Recall Green's and Stokes theorems. □

**Theorem 1.** *Assuming that  $U$  is connected, there exists a linear isomorphism,*

$$(1) \quad H_{dR}^1 := \frac{\{\text{closed differential 1-forms on } U\}}{\{\text{exact differential 1-forms on } U\}} \simeq H^1(U; \mathbb{R})$$

*Proof sketch.* First we want to define a linear map

$$\int : \{\text{closed 1-forms}\} \rightarrow \{\pi_1(U, b) \rightarrow \mathbb{R} \mid \text{group homomorphisms}\}$$

**Recall:**  $X \subset \mathbb{R}^n$  arbitrary subset. A map  $g : X \rightarrow \mathbb{R}^m$  is called smooth if it extends to a smooth map  $N(X) \rightarrow \mathbb{R}^m$  where  $N(X)$  is an open neighborhood of  $X$ .

**Fact:**

- Any class in  $\pi_1(U, b)$  can be represented by a smooth map  $(S^1, *) \rightarrow (U, b)$ .
- Any two smooth maps  $S^1 \rightarrow U$  that are homotopic continuously are homotopic smoothly.

**Recall:** Given  $\alpha = \sum_{i=1}^n f_i dx_i$  and a smooth path  $\gamma : [0, 1] \rightarrow U$ , we

can define the line integral  $\int_{\gamma} \alpha := \int_0^1 F \cdot \gamma' dt$ , where  $F = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ .

- The map  $\int$  is independent of the parametrization of  $\gamma$ , meaning, if  $\phi : [0, 1] \rightarrow [0, 1]$  is a smooth bijective map with  $\phi' \neq 0$ , then  $\int_{\gamma} \alpha = \int_{\gamma \circ \phi} \alpha$ . So line integral only depends on the image of  $\gamma$ .

Back to the map  $\int$ : we send a given  $\alpha \in \{\text{closed 1-forms}\}$  to the map  $\int \alpha : \pi_1(U, b) \rightarrow \mathbb{R}$  defined by

$$a \mapsto \int_{\bar{\gamma}} \alpha,$$

where  $\bar{\gamma}$  is an arbitrary smooth representative of  $a$ . In the exercise below you will show that this is well defined. Assuming that for now, it is easy to see that  $\int \alpha$  is a group homomorphism<sup>4</sup>, so an element of  $H^1(U; \mathbb{R})$ , and the resulting  $\int$  is a linear map.

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<sup>4</sup>I forgot to say this in class, so please check it for yourself!

*Exercise 3.* Show that the map  $\int$  is well defined by proving if  $\alpha = \sum_{i=1}^n f_i dx_i$  is closed and  $\gamma, \gamma' : S^1 \rightarrow U$  are smooth maps that are smoothly homotopic, then  $\int_{\gamma} \alpha = \int_{\gamma'} \alpha$ . (**Hint:** Start by analyzing  $n = 2, 3$  and where the smooth homotopy  $S^1 \times [0, 1] \rightarrow U$  is injective, then reduce the statement to Green's theorem. Additionally, you may want to check proof of Stokes theorem.)

□



FIGURE 2. An example of an injective smooth homotopy's image for  $n = 2$ .

Goals:

- (1) Prove that  $\int$  sends an exact differential 1-form to zero. We obtain a linear map  $\tilde{\int} : H_{dR}^1(U, \mathbb{R}) \rightarrow H^1(U, \mathbb{R})$ .
- (2) Prove that  $\tilde{\int}$  is injective. ("Construct a potential")
- (3) Prove that  $\tilde{\int}$  is surjective.

We start with 1). Let us integrate  $\alpha = df$  along a smooth loop  $\gamma$ .

$$\int_{\gamma} \alpha = \int_0^1 \nabla f \cdot \gamma'(t) dt \stackrel{\text{FTC}}{=} f(b) - f(b) = 0.$$

Denote the resulting map by

$$\tilde{\int} : \frac{\text{Closed differential 1-forms}}{\text{Exact differential 1-forms}} \rightarrow H^1(U, \mathbb{R}).$$

For 2), we need to show that if  $\int_{\gamma} \alpha = 0$  for all  $\gamma : (S^1, \star) \rightarrow (U, b)$ , then  $\alpha = df$  for some  $f : U \rightarrow \mathbb{R}$ .

*Exercise 4.* Do this! This is the same task as constructing a potential (recall work integrals, conservative fields etc.) for the corresponding vector field. □

Let us make a simplifying assumption on  $U$  to not deal with orthogonal difficulties in 3). Assume there exists  $\gamma_1, \gamma_2, \dots, \gamma_n \in \pi_1(U, b)$  that

freely generates the abelianization of  $\pi_1(U, b)$ . This implies that giving a group homomorphism  $\pi_1(U, b) \rightarrow \mathbb{R}$  is equivalent to assigning real numbers to each of  $\gamma_1, \gamma_2, \dots, \gamma_n$  (arbitrarily).

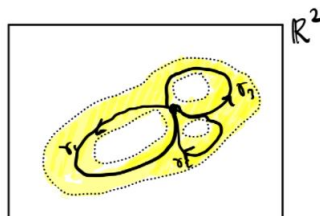


FIGURE 3

Now we need to create a differential 1-form that integrates to any  $a_1, a_2, \dots, a_n \in \mathbb{R}$  along  $\gamma_1, \gamma_2, \dots, \gamma_n$ . This is still quite difficult. That will follow from DeRham Theorem, which will be the highlight of our course.

*Remark 2.* Once we give the general definition of singular cohomology, I will assign a homework exercise which shows that it agrees with what we defined today in degrees 0 and 1. The analogous statement will be automatic for DeRham cohomology.  $\square$

*Exercise 5.* Finish the proof of surjectivity in the case

$$U = \mathbb{R}^2 - \text{finitely many points.}$$

(**Hint:** Start with  $\mathbb{R}^2 - (0, 0)$  and use the closed differential 1-form  $\alpha = -\frac{ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$ .)  $\square$

### 3. MARCH 03, 2022: MANIFOLDS, CHARTS, SMOOTH ATLASES

Riemann was looking for a class of spaces which exist by themselves, (for example, they don't have to be embedded in an Euclidean space  $\mathbb{R}^N$ ) with the following properties<sup>5</sup>. Let  $X$  be such a space:

- $X$  admits local coordinates. This means that the points  $x$  sufficiently near any  $x_0 \in X$  are determined uniquely by the values of a set of real valued coordinates  $x_1, x_2, \dots, x_n$ :

$$x = (x_1, \dots, x_n).$$

This is sometimes called a generalized coordinate system in physics. There could be many such generalized coordinate systems near a given point. It is important that often generalized coordinates do not extend to the entirety of  $X$ .

<sup>5</sup>particularly vague phrases are underlined

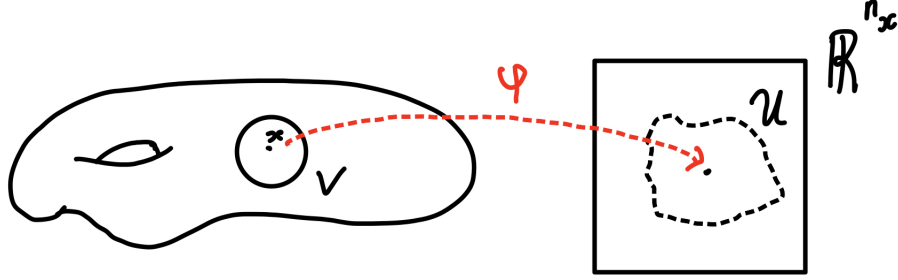


FIGURE 4

- One can use techniques of calculus. This means, in particular, that there should be a large class of  $C^1/C^2/\dots$  smooth functions  $X \rightarrow \mathbb{R}$ . If two generalized coordinate systems are related to each other by a non-differentiable transformation, then a function  $X \rightarrow \mathbb{R}$  that is differentiable with respect to one may not be differentiable with respect to the other.

*Definition 3.* A topological space  $X$  is called a topological manifold if for every  $x \in X$ , there exists a nonnegative integer  $n_x \geq 0$ , an open subset  $U \subset \mathbb{R}^{n_x}$ , an open neighborhood  $V \subset X$  of  $x$  and a homeomorphism  $\phi : V \rightarrow U$  (See Figure 4).  $\square$

*Remark 3.* Note that being a topological manifold is a property. Usually in this definition one also assumes that  $X$  is Hausdorff and, less often but still quite often, second countable. We will focus on the core part of the definition today. Later on we will start assuming these two properties when they are needed.  $\square$

*Definition 4.* Let  $X$  be a topological space. Let us call  $U \subset \mathbb{R}^n$ ,  $V \subset X$  open and  $\phi : V \rightarrow U$  homeomorphism a coordinate chart in  $X$ .  $V$  is called the domain of the chart and the functions  $x_1, \dots, x_n$  obtained by  $x_i : V \rightarrow \mathbb{R} \xrightarrow{pr_i}$  the coordinates of the chart.  $\square$

**Fact** (A consequence of Invariance of Domain)

If an open subset of  $\mathbb{R}^n$  is homeomorphic to an open subset of  $\mathbb{R}^m$ , then  $m = n$ .

*Exercise 6.* Using the fact above, prove that  $n_x$  in the definition is uniquely determined. Also, prove that  $X \rightarrow \mathbb{Z}_{\geq 0}, x \mapsto n_x$  is constant on connected components of  $X$ .  $\square$

*Definition 5.* If  $n_x = n$  for all  $x \in X$ , then we say that  $X$  is  $n$ -dimensional. We write this briefly by  $X^n$ .  $\square$



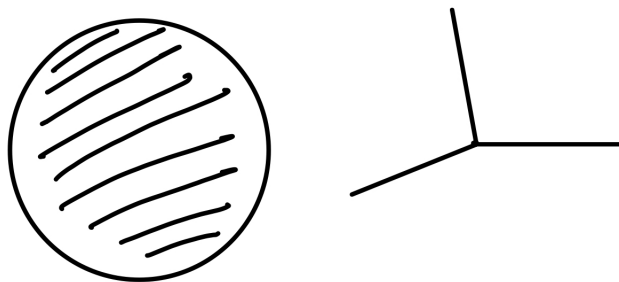


FIGURE 5. Non-Examples of Manifolds

From now on, when we say  $X$  is a topological manifold, we assume that there is such an  $n \geq 0$ .

*Example 2.* Topological Manifolds

- $\mathbb{R}^n$
- $S^n : \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ .

□

To see that  $S^n$  is a topological manifold, we can use the stereographic projection. Let us consider a point  $y_0 \in S^n$ , and let  $H_0$  be the hyperplane that is tangent to the point opposite to  $y_0 \in S^n$  (we call this point  $-y_0$ ). For every,  $y \in S^n \setminus \{y_0\}$  the straight line  $l_y$  passing through  $y$  and  $y_0$  intersects  $H_0$  at precisely one point.

$$S_{y_0} : S^n \setminus \{y_0\} \longrightarrow H_0 \simeq \mathbb{R}^n$$

$$y \longmapsto l_y \cap H_0$$

Note that  $H_0 \cong$  Parallel hyperplane passing through the origin  $\cong \mathbb{R}^n$ . The second homeomorphism can be obtained by choosing a basis.

**Proposition 2.**  $S_{y_0}$  is a homeomorphism.

*Proof.* (sketch)

$n = 0$  case is given in the figure. In this case the map is identity.

*Exercise 7.* Do the  $n = 1$  case. □

We can deduce the  $n > 1$  case by using rotational symmetry around the line that contains the diameter (passing through  $y_0$  that is perpendicular to  $H_0$ ). The stereographic projection in dimension  $n$  is given by spinning around  $l_y$  the stereographic projection in dimension  $n - 1$ . □

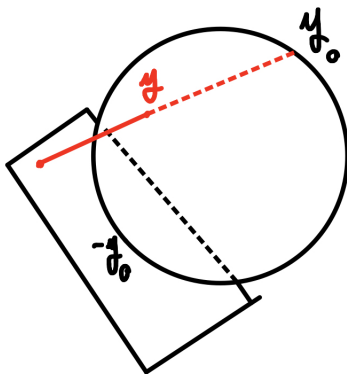


FIGURE 6. Stereographic Projection

FIGURE 7.  $n = 0$  case

*Remark 4.* Stereographic projection

- Preserves angles.
- It preserves circles ( $n = 2$ ).
- But, it distorts distances.

□

*Exercise 8.* Prove that  $\{x^2 - y^3 = 1\} \subset \mathbb{C}^2$  is a topological manifold.  
**(Hint:** Use projections to  $x$  and  $y$ .) □

*Definition 6.* Let  $X$  be a topological space, and  $\phi_1 : V_1 \rightarrow U_1$  and  $\phi_2 : V_2 \rightarrow U_2$  be coordinate charts. Then, the map  $\phi_2 \circ \phi_1^{-1} : \phi_1(V_1 \cap V_2) \rightarrow \phi_2(V_1 \cap V_2)$  is called the transition map from the chart  $\phi_1$  to the chart  $\phi_2$ . Note that transition maps are automatically homeomorphisms. □

*Definition 7.* A smooth atlas on a topological space  $X$  is a collection of charts  $\{\phi_i : V_i \rightarrow U_i\}_{i \in I}$  such that

- 1)  $\bigcup_{i \in I} V_i = X$
- 2) The transition map from any chart in the collection to any other in the collection is smooth. □

*Remark 5.* Atlas means a book of maps, i.e. images of charts  $\phi : V \rightarrow U \subset \mathbb{R}^2$  on the manifold that is the surface of the earth. It is likely that some of these maps are drawn using stereographic projection. □

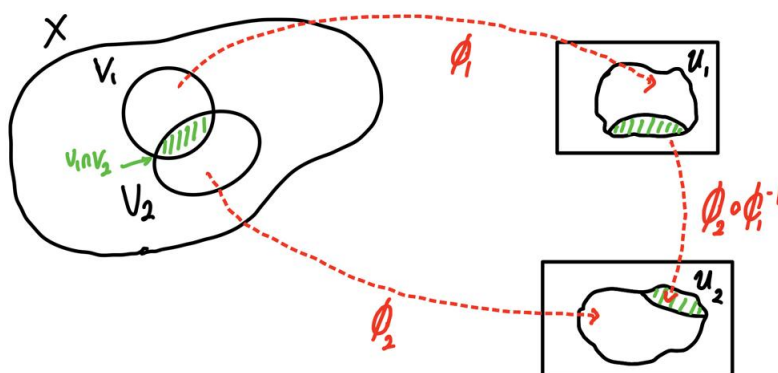


FIGURE 8. Transition map between charts

*Exercise 9.* Using latitude and longitude, define a chart on  $S^2$  with domain  $S^2 \setminus 0^{th}$ –meridian and prove that it has smooth transition maps to all stereographic projections. You can assume that stereographic projection charts form a smooth atlas.  $\square$

#### 4. MARCH 07, 2022: DEFINITION OF SINGULAR HOMOLOGY

*Convention:* Unless otherwise stated, all of our vector spaces and chain complexes (to be defined) are over  $\mathbb{R}$ .

*Definition 8.* A graded vector space  $V_*$  is a collection of vector spaces  $\{V_i\}_{i \in \mathbb{Z}}$  indexed by  $\mathbb{Z}$ .  $V_*$  is non-negatively graded if  $V_i = 0$  for  $i < 0$ .<sup>6</sup>  $\square$

*Definition 9.* A chain complex  $(C_*, \partial_*)$  is a graded vector space  $C_*$  with a collection of linear maps  $\partial_n : C_n \rightarrow C_{n-1}$ ,  $n \in \mathbb{Z}$ , such that  $\partial_n \circ \partial_{n+1} = 0$  for all  $n \in \mathbb{Z}$ . We call  $\partial_n$ 's boundary maps.  $\square$

$$\dots \leftarrow C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \dots$$

*Definition 10.* The homology of a chain complex  $(C_*, \partial_*)$  is a graded vector space  $H_*(C_*, \partial_*)$  defined by

$$H_n((C_*, \partial_*)) := \frac{\text{Ker}(\partial_n : C_n \rightarrow C_{n-1})}{\text{Im}(\partial_{n+1} : C_{n+1} \rightarrow C_n)}$$

$\square$

It immediately follows from  $\partial_i \circ \partial_{i+1} = 0$  that  $\text{Im}(\partial_{i+1}) \subset \text{Ker}(\partial_i)$ . There is a slight variant of the last two definitions.

<sup>6</sup> $V=0$  stands for the trivial vector space with 0 as the only element,  $V = \{0\}$ .

*Definition 11.*  $C^*$  graded vector space with  $d_n : C^n \rightarrow C^{n+1}$  coboundary maps such that  $d_n \circ d_{n-1} = 0$ .  $(C^*, d_*)$  is called a co-chain complex.

$$H^n((C^*, d_*)) := \frac{\text{Ker}(d_n)}{\text{Im}(d_{n-1})}$$

is called cohomology. □

$$\dots \rightarrow C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \rightarrow \dots$$

Now we move on to define the singular chain complex  $C_*(X; \mathbb{R})$  of a topological manifold  $X$ .

*Definition 12.* The  $n$ -dimensional simplex  $\Delta^n$  for  $n \geq 0$  is defined as

$$\Delta^n := \left\{ (x_0, \dots, x_n) \mid \begin{array}{l} x_i \geq 0, \quad \forall i = 0, \dots, n \\ x_0 + \dots + x_n = 1 \end{array} \right\}$$

□

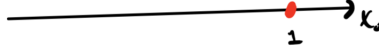


FIGURE 9. 0-dimensional simplex.

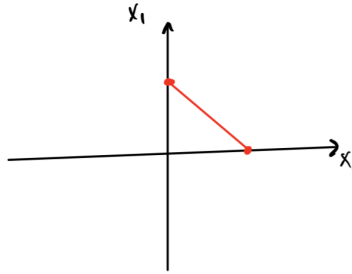


FIGURE 10. 1-dimensional simplex.

For each subset  $S \subset \{0, 1, \dots, n\}$ , we can define a subset (a face) by

$$F_S := \left\{ (x_0, \dots, x_n) \mid \begin{array}{l} x_i = 0, \quad \text{if } i \in \{0, \dots, n\} \setminus S \\ (x_0, \dots, x_n) \in \Delta^n \end{array} \right\}$$

As an example,  $F_{\{i\}}$ ,  $i = 0, \dots, n$  correspond to vertices.

*Exercise 10.* • Prove that the dimension of  $F_S$  is  $|S| - 1$ . Explain what you mean by dimension.

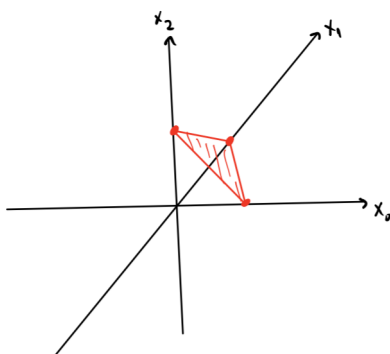


FIGURE 11. 2-dimensional simplex.

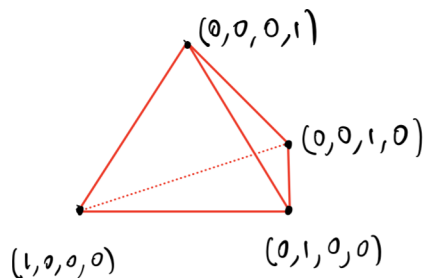


FIGURE 12. 3-dimensional simplex.

- Prove that  $F_{S_1} \cap F_{S_2} = F_{S_1 \cap S_2}$ .

□

*Definition 13.* For each  $n \geq 1$  and  $0 \leq i \leq n$  we define the face map  $f_{i,n} : \Delta^{n-1} \rightarrow \Delta^n$  with  $(x_0, \dots, x_{n-1}) \mapsto (y_0, \dots, y_n)$  where

$$y_j = \begin{cases} x_j & , \quad j < i \\ 0 & , \quad j = i \\ x_{j-1} & , \quad j > i \end{cases}$$

□

- This simply adds a zero to the  $(i+1)$ th slot.
- The image of  $f_{i,n}$  is  $F_{\{0, \dots, n\} \setminus \{i\}}$

We need one last notion before we define the singular chain complex.

*Definition 14.* Given any set  $A$ , we define the vector space generated by  $A$  as the vector space of all finite formal linear combinations of the

elements of  $A$ .

$$\left\{ \sum_{a \in A} c_a \cdot a \mid c_a \in \mathbb{R} \text{ and } c_a \neq 0 \text{ for finitely many elements} \right\}.$$

□

*Exercise 11.* Construct a natural linear map from the vector space generated by  $A$  to the vector space of all maps  $A \rightarrow \mathbb{R}$ . Prove that this map is an isomorphism if and only if  $A$  is finite. Bonus: analyze when these two vector spaces are isomorphic - by an arbitrary map. □

Consider the subspace topology on simplices.

**4.1. Singular homology of a topological space.** Let  $X$  be a topological space. For  $n \geq 0$ ,  $C_n(X; \mathbb{R})$  is defined to be the vector space generated by the set of all continuous maps  $\Delta^n \rightarrow X$ . The elements of  $C_n(X; \mathbb{R})$  are called singular chains of degree  $n$ . We set  $C_n(X; \mathbb{R}) = 0$  for all  $n < 0$ . So  $C_*(X; \mathbb{R})$  is a non-negatively graded vector space. Now we will equip it with boundary maps and turn it into a chain complex. Let  $n \geq 1$ . For any continuous  $g : \Delta^n \rightarrow X$  we define

$$\partial_n g = \sum_{i=0}^n (-1)^i g \circ f_{i,n} \in C_{n-1}(X; \mathbb{R})$$

where  $g \circ f_{i,n} : \Delta^{n-1} \xrightarrow{f_{i,n}} \Delta^n \xrightarrow{g} X$ . We then extend to all singular chains so that map is linear and we get

$$\partial_n : C_n(X; \mathbb{R}) \rightarrow C_{n-1}(X; \mathbb{R}) \text{ for } n \geq 1$$

and  $\partial_n = 0$  for  $n < 1$ .

*Example 3.* As an example, consider the following figure □

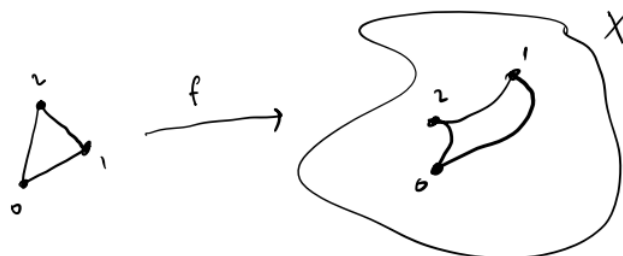


FIGURE 13. Example

Now we look at the boundary maps  $\partial_2 f$  of  $f$ .

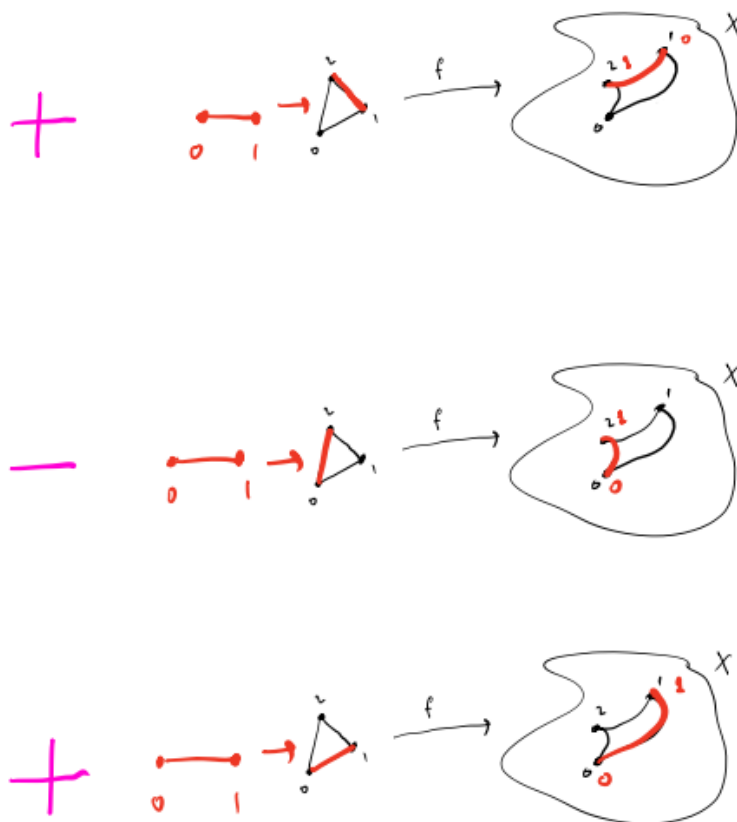


FIGURE 14. Boundary maps of the example above.

**Proposition 3.**  $\partial_{n-1} \circ \partial_n = 0$

*Proof.* For  $n < 2$  it is obvious. For  $n \geq 2$  it suffices to show that for  $g : \Delta^n \rightarrow X$  we should have  $\partial_{n-1}(\partial_n(g)) = 0$ .

$$\begin{aligned} \partial_{n-1}(\partial_n(g)) &= \partial_{n-1} \left( \sum_{i=0}^n (-1)^i g \circ f_{i,n} \right) = \sum_{i=0}^n (-1)^i \partial_{n-1}(g \circ f_{i,n}) \\ &= \sum_{i=0}^n (-1)^i \left( \sum_{j=0}^{n-1} (-1)^j g \circ f_{i,n} \circ f_{j,n-1} \right) = \sum_{i,j} (-1)^{i+j} g \circ f_{i,n} \circ f_{j,n-1} \end{aligned}$$

Where  $f_{i,n} \circ f_{j,n-1} : \Delta^{n-2} \rightarrow \Delta^n \dots$

□

*Exercise 12.* Finish the proof of this proposition. If you were not able to follow in class, first do it for  $n = 2$  using the pictures above - no need to write this, just to get yourself oriented. □

*Definition 15.* We define the singular homology of  $X$  as the homology of its singular chain complex

$$H_n(X; \mathbb{R}) := H_n(C_*(X; \mathbb{R}), \partial_*) .$$

□

## 5. MARCH 10, 2022: CONSTRUCTING SINGULAR CYCLES, HOMOLOGY OF A POINT, STAR SHAPED OPEN SUBSETS OF EUCLIDEAN SPACE

*Definition 16.* Let us call the elements of

$$Z_n(X; \mathbb{R}) := \ker(\partial_n : C_n(X; \mathbb{R}) \rightarrow C_{n-1}(X; \mathbb{R}))$$

the singular  $n$ -cycles and the elements of

$$B_n(X; \mathbb{R}) := \operatorname{im}(\partial_{n+1} : C_{n+1}(X; \mathbb{R}) \rightarrow C_n(X; \mathbb{R}))$$

the singular  $n$ -boundaries. □

In this class, often we will omit the adjective singular from these phrases for brevity. Then, by definition

$$\begin{aligned} H_n(X; \mathbb{R}) &= \frac{Z_n(X; \mathbb{R})}{B_n(X; \mathbb{R})} \\ &= \frac{n\text{-cycles}}{n\text{-boundaries}} \end{aligned}$$



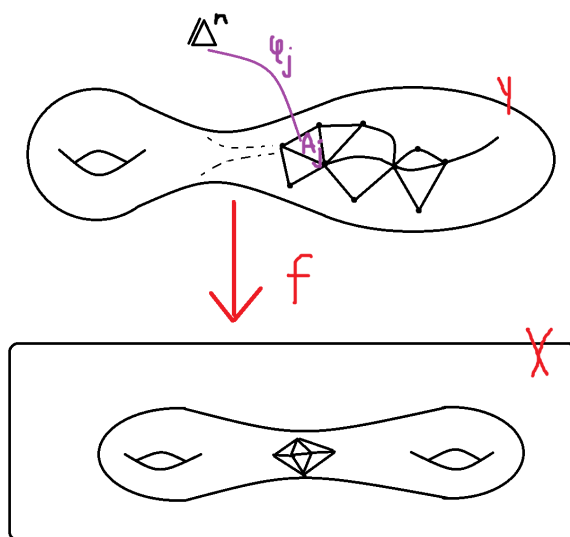
**5.1. Singular  $n$ -cycles from geometric  $n$ -cycles - slightly informal discussion.** Let  $X$  be a topological manifold. Let us define a geometric  $n$ -cycle to be the image of a continuous map  $f : Y^n \rightarrow X$  where  $Y$  is a compact oriented (We will define precisely for smooth manifolds later) topological manifold.

I want to briefly explain how a geometric  $n$ -cycle gives rise to an  $n$ -cycle on  $X$ . Oriented compact submanifolds are examples of geometric  $n$ -cycles.

Under some mild conditions (for example if it is Hausdorff and admits a smooth structure),  $Y$  admits a triangulation. This, in particular, means we can find

$$Y = \bigcup_{j=1}^N A_j \quad \text{with homeomorphisms} \quad \phi_j : \Delta^n \rightarrow A_j$$

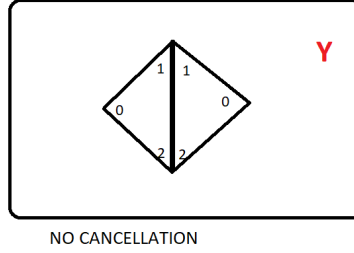
such that the intersections  $A_i \cap A_j$  for  $i \neq j$  are either empty or equal to the image of both  $\phi_i \circ \text{face}_{k,n} (: \Delta^{n-1} \rightarrow \Delta^n \rightarrow Y)$  and  $\phi_j \circ \text{face}_{l,n}$  for some  $k$  and  $l$ .



The idea then is to add up all  $f \circ \phi_j : \Delta^n \rightarrow X$  and get an  $n$ -cycle seeing how the boundaries seem to cancel each other.

The issue is that we do not actually know, it depends on whether the signs work out.

We could also add  $\pm f \circ \phi_j$  of course. Note that we can modify  $\phi_j$  also by homeomorphism,  $\Delta^n \rightarrow \Delta^n$  which permute the coordinates of



$\mathbb{R}^{n+1}$ . If the result is  $\tilde{\phi}_j$  and the permutation  $\pi^{-1}$

$$\begin{aligned}
 \partial_n(f \circ \tilde{\phi}_j) &= \sum_{i=0}^n (-1)^i f \circ \tilde{\phi}_j \circ \text{face}_{i,n} \\
 \pi(l) = i &= \sum_{l=0}^n (-1)^{\pi(l)} f \circ \tilde{\phi}_j \circ \text{face}_{\pi(l),n} \\
 &= \sum_{l=0}^n (-1)^{\pi(l)-l} (-1)^l f \circ \phi_j \circ \text{face}_{l,n}
 \end{aligned}$$

The interesting result is that whether one can modify  $\phi_i$ 's using there so that  $\partial_n(\sum_{i=1}^N \pm \tilde{\phi}_i) = 0$  is a condition that depends only on  $Y$  and is called orientability.

If this is true, then we obtain at least two  $n$ -cycles in  $Y$ , we can multiply everything by  $-1$ . Actually orienting  $Y$ , we would pick out one of them.

*Exercise 13.* Consider  $S^1 \subset \mathbb{R}^2$ . Construct a nonzero 1-cycle in  $\mathbb{R}^2$  corresponding to this geometric 1-cycle. Prove that it is actually a 1-boundary directly.  $\square$

**5.2. Some computations.** Let  $X$  be a point. What is  $H_*(X; \mathbb{R})$  ?

For every  $n \geq 0$ , there exists exactly one continuous map  $\Delta^n \xrightarrow{c_n} X$ . Therefore,

$$C_n(X; \mathbb{R}) = \mathbb{R} \cdot c_n.$$

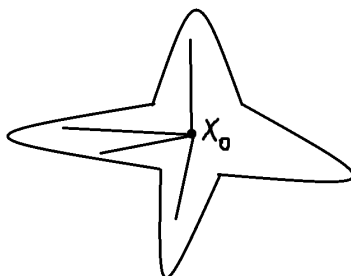
How about the boundary map?

$$\begin{aligned}
 \partial_n c_n &= \sum_{i=0}^n (-1)^i c_{n-1} \\
 &= \begin{cases} c_{n-1} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}
 \end{aligned}$$

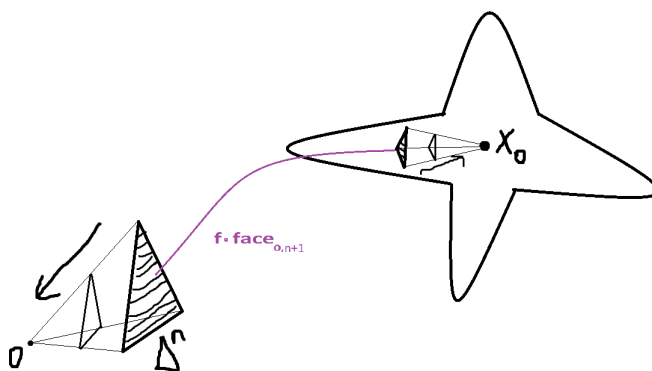
The singular chain complex looks like

$$\begin{aligned} & \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \dots \\ \implies & H_0(X; \mathbb{R}) = \mathbb{R} \text{ and } H_n(X; \mathbb{R}) = 0, \quad n \neq 0. \end{aligned}$$

Let us now consider  $U \subset \mathbb{R}^n$  open and star-shaped, that is, there exists an  $x_0 \in U$  such that the line segment  $x_0$  and  $y$  lies inside  $U$  for all  $y \in U$ .



We claim that  $H_*(U; \mathbb{R}) \cong H_*(\text{point}; \mathbb{R})$ . The idea is that for a given  $f : \Delta^n \rightarrow U$ , we can define a  $P_f : \Delta^{n+1} \rightarrow U$  as follows



*Exercise 14.* Write down an explicit formula for  $P_f$  in terms of  $f$ . Extending  $f \mapsto P_f$  linearly, define a linear map  $P : C_n(X; \mathbb{R}) \rightarrow C_{n+1}(X; \mathbb{R})$ . For  $n > 0$ ,  $\sigma \in C_n(X; \mathbb{R})$ , prove that

$$\partial_{n+1} P\sigma = -P(\partial_n \sigma) + \sigma.$$

For  $n = 0$ ,  $\sigma \in C_0(X; \mathbb{R})$ , what is  $\partial_1 P\sigma$  ? □

If  $\sigma \in Z_n(U; \mathbb{R})$  and  $n > 0$ , then  $\partial_{n+1} P\sigma = \sigma$ . This implies  $\sigma \in B_n(U; \mathbb{R})$  and hence  $H_n(U; \mathbb{R}) = 0$  for  $n \neq 0$ .

*Exercise 15.* Let  $Y$  be a topological space. Prove that  $H_0(Y; \mathbb{R})$  is isomorphic to the vector space generated by the set of connected components of  $Y$ .  $\square$

6. MARCH 14, 2022: INDUCED MAPS ON HOMOLOGY,  
HOMEOMORPHISM INVARIANCE, HOMOTOPY INVARIANCE OF  
INDUCED MAPS

*Definition 17.* Let  $(C_\bullet, \partial_\bullet)$  and  $(\tilde{C}_\bullet, \tilde{\partial}_\bullet)$  be chain complexes. A chain map is a collection of linear maps  $C_n \xrightarrow{f_n} \tilde{C}_n$ ,  $\forall n \in \mathbb{Z}$  such that each square in the diagram

$$\begin{array}{ccccccc} \cdots & \longleftarrow & C_{n-1} & \xleftarrow{\partial_n} & C_n & \longleftarrow & \cdots \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \\ \cdots & \longleftarrow & \tilde{C}_{n-1} & \xleftarrow{\tilde{\partial}_n} & \tilde{C}_n & \longleftarrow & \cdots \end{array}$$

is commutative, i.e.

$$f_{n-1} \circ \partial_n = \tilde{\partial}_n \circ f_n, \quad \forall n \in \mathbb{Z}.$$

$\square$

Given a chain map  $f_\bullet : C_\bullet \rightarrow \tilde{C}_\bullet$  we canonically obtain a linear map of graded vector spaces

$$H(f) : H_*(C) \rightarrow H_*(\tilde{C}).$$

*Remark 6.* Let us make a notational clarification. For any chain complex  $(C, \partial)$  we define

$$Z_n(C) := \ker(\partial_n) \cdot n\text{-cycles}$$

$$B_n(C) := \text{im}(\partial_{n+1}) \cdot n\text{-boundaries}$$

$$H_n(C) = Z_n(C) / B_n(C)$$

When  $C_\bullet = C_*(X; \mathbb{R})$  then we may add the adjective "singular".  $\square$

*Definition 18.* If two  $n$ -cycles in a chain complex differ by an  $n$ -boundary (that is they define the same class in homology), we say that these two cycles are homologous.  $\square$

Back to constructing  $H_n(C) \rightarrow H_n(\tilde{C})$ .

- (1) If  $\sigma \in Z_n(C)$ , then  $f(\sigma) \in Z_n(\tilde{C})$  :  
 $f(\partial\sigma) = \tilde{\partial}f(\sigma) \implies \tilde{\partial}f(\sigma) = 0$
- (2) If  $\sigma \in B_n(C)$ , then  $f(\sigma) \in B_n(\tilde{C})$  :  
 $\sigma = \partial\gamma \implies f(\sigma) = f(\partial\gamma) = \tilde{\partial}f(\gamma)$

$\implies$  We obtain the map shown in dashes below.

$$\begin{array}{ccccccc}
 B_n(C) & \hookrightarrow & Z_n(C) & \longrightarrow & Z_n(\tilde{C}) & \longrightarrow & Z_n(\tilde{C})/B_n(\tilde{C}) \\
 & & \downarrow & & & \nearrow & \\
 & & Z_n(C)/B_n(C) & & & & 
 \end{array}$$

We can compose chain maps as in the following exercise.

*Exercise 16.* Prove that if  $f : C_* \rightarrow \tilde{C}_*$  and  $g : \tilde{C}_* \rightarrow \tilde{\tilde{C}}_*$  are chain maps, then  $h : C_* \rightarrow \tilde{\tilde{C}}_*$  defined by  $h_n := g_n \circ f_n$  is a chain map. For such  $f, g, h$  prove that  $H(h) = H(g) \circ H(f)$ .  $\square$

*Definition 19.* Let  $\varphi : X \rightarrow Y$  be continuous. Then for every continuous map  $\rho : \Delta^n \rightarrow X$ , we obtain the continuous map  $\varphi \circ \rho : \Delta^n \rightarrow Y$ . Linearly extending we obtain a map  $(\varphi_*)_n : C_n(X; \mathbb{R}) \rightarrow C_n(Y; \mathbb{R})$ . These form a chain map

$$\varphi_* : C_*(X; \mathbb{R}) \rightarrow C_*(Y; \mathbb{R}).$$

$\square$

*Exercise 17.* Prove that  $\varphi_*$  is indeed a chain map.  $\square$

We also obtain

$$H\varphi_* : H_*(X; \mathbb{R}) \rightarrow H_*(Y; \mathbb{R}).$$

*Exercise 18.* For continuous maps  $\tilde{\varphi} : X \rightarrow Y$  and  $\varphi : Y \rightarrow Z$ , prove that  $\varphi_* \circ \tilde{\varphi}_* = (\varphi \circ \tilde{\varphi})_*$ .  $\square$

**Corollary 1.**  $H\varphi_* \circ H\tilde{\varphi}_* = H(\varphi \circ \tilde{\varphi})_*$ .  $\square$

Since  $(\text{id})_*$  is the identity map, we immediately obtain that if  $\varphi : X \rightarrow Y$  is a homeomorphism then  $H\varphi_* : H_*(X; \mathbb{R}) \rightarrow H_*(Y; \mathbb{R})$  is a linear isomorphism.

Hence singular homology can distinguish non-homeomorphic topological spaces (it does not have to!) Actually singular homology is in general only sensitive to the homotopy equivalence class. Let's explain this.

**Recall:**  $\bullet f, g : X \xrightarrow{cts} Y$  are called homotopic if there exists a continuous

$$F : X \times [0, 1] \rightarrow Y \text{ s.t.}$$

$$F|_{\{0\}} = f \text{ \& } F|_{\{1\}} = g.$$

$\bullet$  A continuous  $f : X \rightarrow Y$  is called a homotopy equivalence if  $\exists g : Y \rightarrow X$  such that  $f \circ g$  and  $g \circ f$  are homotopic to identity maps (of  $Y$  and  $X$ ).

*Exercise 19.* Let  $U \subset \mathbb{R}^n$  be star shaped with respect to  $x_0$ . Prove the map  $\text{pt.} \rightarrow U$  with image  $x_0$  is a homotopy equivalence.  $\square$

**Theorem 2.** *If  $f$  &  $g : X \rightarrow Y$  are homotopic, then  $Hf_* = Hg_*$ .*

I don't want to spend time proving this homotopy invariance theorem, but it is quite important and the proof is not too difficult. If you want we can discuss during office hours. You are responsible from the statement, not the proof. The key idea in the proof is essentially the one that we used on proving  $H_*(\text{star shaped}) \cong H_*(\text{pt.})$ .

The corollary below can be proved using the same logic that showed that homeomorphisms induce isomorphisms on homology but this time using Theorem 2.

**Corollary 2.** *If  $f : X \rightarrow Y$  is a homotopy equivalence, then  $Hf : H_*(X; \mathbb{R}) \rightarrow H_*(Y; \mathbb{R})$  is an isomorphism.*  $\square$

*Exercise 20.* Prove this corollary assuming Theorem 2.  $\square$

## 7. MARCH 17, 2022: MAYER-VIETORIS PROPERTY, SPECIAL PROPERTIES OF SINGULAR HOMOLOGY FOR MANIFOLDS, SINGULAR COHOMOLOGY

*Definition 20.* An exact sequence is a sequence of vector spaces  $V_n$ ,  $n \in \mathbb{Z}$  and maps  $V_n \xrightarrow{f_n} V_{n-1}$  such that  $\forall n \in \mathbb{Z} \quad \ker(f_n) = \text{im}(f_{n+1})$

$$\begin{array}{ccccccc} V_{n-1} & \xleftarrow{f_n} & V_n & \xleftarrow{f_{n+1}} & V_{n+1} & \xleftarrow{f_{n+2}} & V_{n+2} \\ & \searrow & \uparrow & \nwarrow & \uparrow & \searrow & \\ & \text{ker} & = & \text{Im} & & & \end{array}$$

$\square$

*Remark 7.* This is the same data as chain complex with 0-Homology.

$\square$

*Exercise 21.* Let  $V_n$  be an exact sequence. Assume that  $\sum \dim V_n < \infty$ . Show that  $\sum_{n \text{ even}} \dim V_n = \sum_{n \text{ odd}} \dim V_n$ .  $\square$

**Theorem 3** (Mayer-Vietoris Theorem). *Let  $X$  be a topological space and  $U, V \subseteq X$  open subsets.*

*There are canonical maps*

$$H_{n+1}(U \cup V) \xrightarrow{\text{c}_{n+1}} H_n(U \cap V)$$

*called connecting maps that makes the following graded vector space*



$$\begin{array}{ccccccc}
 \cdots & \cdots & \longrightarrow & 0 \oplus 0 & \longrightarrow & H_2(S^1) & \\
 & & & \searrow c_2 & & \nearrow & \\
 \longrightarrow & 0 \oplus 0 & \longrightarrow & 0 \oplus 0 & \longrightarrow & H_1(S^1) & \\
 & & & \searrow c_1 & & \nearrow & \\
 \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{-i_0} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{-j_0} & H_0(S^1) = \mathbb{R} & \\
 & & & \searrow c_0 & & \nearrow & \\
 \longrightarrow & 0 & \cdots & & & & 
 \end{array}$$

Simply by observing the diagram and counting dimensions we get  $H_i(S^1) = 0$  for  $i > 1$  and  $H_1(S^1) \cong \mathbb{R}$ . The only non-canonical isomorphism we have here is  $H_1(S^1) \cong \mathbb{R}$ . To understand this isomorphism better we have to inspect this sequence further.

Since  $\ker c_1 = 0$ , we may identify  $H_1(S^1) = \text{im } c_1 = \ker i_0$ . Let's introduce some notation to communicate better, denote by  $W_1 \sqcup W_2 = U \cap V$  where  $W_i$  are the two disjoint intervals. Let  $p_i : \Delta^0 \rightarrow W_i$  be any specific map. It's clear that each element of  $H_0(U \cap V)$  is represented uniquely by a cycle of the form  $ap_1 + bp_2$ , where  $a, b \in \mathbb{R}$ . Also, we have  $i_*^{U \cap V \subset U} p_1 = i_*^{U \cap V \subset U} p_2 \in H_0(U)$  and  $i_*^{U \cap V \subset V} p_1 = i_*^{U \cap V \subset V} p_2 \in H_0(V)$ , so  $\ker i_0 = \{ap_1 - ap_2 \in H_0(U \cap V)\}$ . There are two natural bases for this space:  $p_1 - p_2$  and  $p_2 - p_1$ . These two choices give us two choices of isomorphisms  $H_1(S^1) \cong \mathbb{R}$  and correspond to the two choices of orientation we have on  $S^1$ .

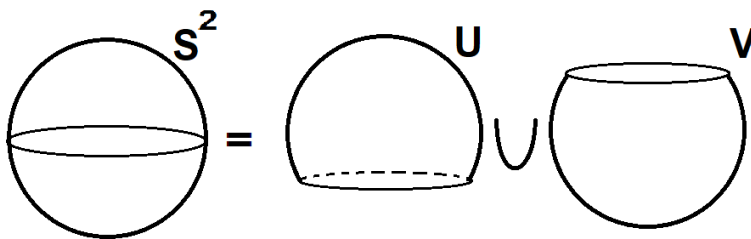
9

*Exercise 22.* Compute  $H_*(S^n, \mathbb{R})$ .

9

*Example 6.* Sketch for  $H_*(S^2)$

Consider the following open sets  $U, V \subset S^2$ .



Since  $U, V \cong \mathbb{R}^2$ , we have  $H_*(U) \cong H_*(V) \cong H_*(\mathbb{R}^2) \cong H_*(pt)$ . Notice that the circular belt  $U \cap V$  can be retracted onto the equator of the sphere, which is homeomorphic to  $S^1$ . Since this is a homotopy equivalence, by Theorem 2 we have  $H_*(U \cap V) \cong H_*(S^1)$ . Plugging all we know into the Mayer-Vietoris sequence



$$\begin{array}{rcl}
0 \rightarrow 0 \oplus 0 \rightarrow H_3(S^2) & \xrightarrow{\quad} & \cong 0 \\
\swarrow & & \\
0 \rightarrow 0 \oplus 0 \rightarrow H_2(S^2) & \xrightarrow{\quad} & \cong \mathbb{R} \\
\swarrow & & \\
\mathbb{R} \rightarrow 0 \oplus 0 \rightarrow H_1(S^2) & \xrightarrow{\quad} & \cong 0 \\
\swarrow & & \\
\mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow H_0(S^2) & \xrightarrow{\quad} & \cong \mathbb{R}
\end{array}$$

we get that  $H_i(S^2) = 0$  for  $i < 2$ ,  $H_1(S^2) = 0$  and  $H_2(S^2) \cong H_0(S^2) \cong \mathbb{R}$ .  $\square$

**7.2. Singular Homology Of Manifolds.** From now on, we'll assume that our manifolds are Hausdorff. When we write manifold we will mean a topological manifold below.

*Remark 8.* We needed this condition for the existence of a triangulation as well.  $\square$

Later when we go back to smooth manifolds we'll add another condition, being second countable.

*Exercise 23.* Find a non-Hausdorff manifold which can be equipped with a smooth atlas.  $\square$

**Theorem 4.** *Let  $M$  be an  $n$ -dimensional manifold, then  $H_i(M; \mathbb{R}) = 0$  for  $i > n$ .*

Hence the singular homology of  $M$  can only live in degrees  $i = 0, 1, \dots, n$ .

If we assume that  $M$  is also connected, then  $H_0(M; \mathbb{R}) \cong \mathbb{R}$ . This isomorphism is canonical, where we identify any map from a point to  $M$  with  $1 \in \mathbb{R}$ . It turns out that we know quite a bit about the top degree as well

**Theorem 5.** *Let  $M$  be an  $n$ -dimensional manifold. Then*

$$H_n(M; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is compact and orientable} \\ 0 & \text{otherwise.} \end{cases}$$

Here the isomorphism with  $\mathbb{R}$  depends on our choice of orientation.

**Theorem 6** (Poincaré Duality). *Let  $M$  be a compact, oriented<sup>8</sup>  $n$ -dimensional manifold. Then we have the canonical isomorphisms*

$$H_{n-k}(M; \mathbb{R}) \cong (H_k(M; \mathbb{R}))^\vee$$

*for all  $k \in \mathbb{Z}$ .*

<sup>8</sup>That is, we have chosen a specific orientation and so we are equipped with an isomorphism  $H_n(M; \mathbb{R}) \cong \mathbb{R}$ .

*Exercise 24.* Recall that the linear dual  $V^\vee$  of a vector space  $V$  is the vector space of linear maps  $V \rightarrow \mathbb{R}$ . Prove the following:  $V^\vee \cong V$  if and only if  $\dim V < \infty$ . The only if part is optional, similar to Exercise 11  $\square$

*Exercise 25.* Let  $f : V \rightarrow W$  be a linear map. Define  $f^\vee : W^\vee \rightarrow V^\vee$  by  $f^\vee \alpha = \alpha \circ f$ . Show that  $f^\vee$  is a linear map. Describe  $\ker f^\vee$  and  $\operatorname{im} f^\vee$  in terms of  $\ker f$  and  $\operatorname{im} f$ .

Let  $g : W \rightarrow P$  be another linear map, show that  $(g \circ f)^\vee = f^\vee \circ g^\vee$ .  $\square$

### 7.3. Singular Cohomology.

*Definition 21.* Let  $X$  be a topological space. The singular cochain complex  $C^*(X; \mathbb{R})$  is defined by

$$C^n(X; \mathbb{R}) = (C_n(X; \mathbb{R}))^\vee$$

and the coboundary maps  $\delta_n = \partial_{n+1}^\vee$  are given by

$$\begin{aligned} \delta_n : C^n(X; \mathbb{R}) &\rightarrow C^{n+1}(X; \mathbb{R}) \\ \alpha &\mapsto \alpha \circ \partial_{n+1}. \end{aligned}$$

The cohomology of this complex is called the singular cohomology of  $X$ . We denote it by

$$H^*(X; \mathbb{R}) = H^*(C^*(X; \mathbb{R})).$$

$\square$

*Exercise 26.* Prove that the above defined graded vector space is indeed a cochain complex, viz  $\delta_{n+1} \circ \delta_n = 0$ .  $\square$

*Exercise 27.* Prove that  $H^n(X; \mathbb{R}) \cong (H_n(X; \mathbb{R}))^\vee$ . Hint: Start by constructing a map.  $\square$

*Exercise 28.* Deduce a Mayer-Vietoris sequence for singular cohomology from the Mayer-Vietoris sequence for singular homology. Be careful!  $\square$

## 8. MARCH 21, 2022: SMOOTH MANIFOLDS, SMOOTH MAPS, GLUING

Recall that for a topological space  $X$ , we call  $U \subset \mathbb{R}^n$ , an open subset  $V \subset X$  and a homeomorphism  $\varphi : V \rightarrow U$  a (coordinate) chart.

*Definition 22.* With the same notation,  $V$  is called the domain of the chart and the functions  $x_1, \dots, x_n$  defined by  $x_i : V \xrightarrow{\varphi} U \xrightarrow{pr_i} \mathbb{R}$  are called the coordinates of the chart. Here  $pr_i$  is the projection from  $U \subset \mathbb{R}^n$  to the  $i^{\text{th}}$  Euclidean coordinate.  $\square$

*Definition 23.* Let  $\varphi_1 : V_1 \rightarrow U_1$  and  $\varphi_2 : V_2 \rightarrow U_2$  be two charts of the topological space  $X$ . The map  $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(V_1 \cap V_2) \rightarrow \varphi_2(V_1 \cap V_2)$  is called the transition map from the chart  $\varphi_1$  to the chart  $\varphi_2$ .

The charts  $\varphi_1$  and  $\varphi_2$  are called smoothly compatible if the transition maps  $\varphi_2 \circ \varphi_1^{-1}$  and  $\varphi_1 \circ \varphi_2^{-1}$  are both smooth.  $\square$

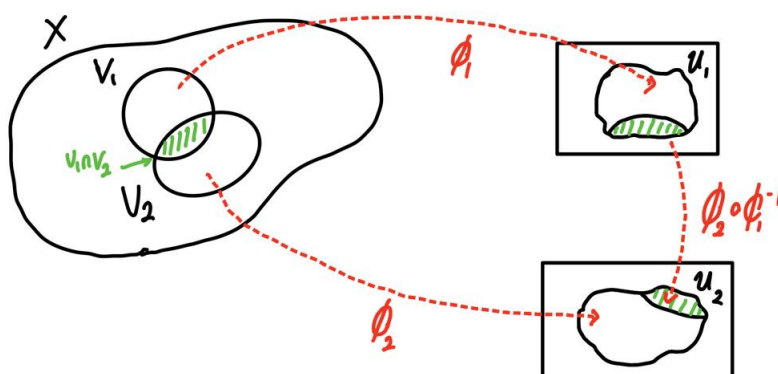


FIGURE 15. Transition map between charts.

Also recall that a smooth atlas on a topological space  $X$  is a collection of charts  $\varphi_i : V_i \rightarrow U_i |_{i \in I}$  (of the same dimension) such that

- $\cup_{i \in I} V_i = X$
- For any  $i, j \in I$ ,  $\varphi_i$  and  $\varphi_j$  are smoothly compatible.

*Warning:* There is no such thing as a smooth chart unless we have a smooth atlas.

*Definition 24.* We call a smooth atlas  $\mathcal{A}$  maximal if any chart that is smoothly compatible with  $\mathcal{A}$  is already in  $\mathcal{A}$ .  $\square$

*Exercise 29.* Prove that every smooth atlas is contained in a unique maximal smooth atlas.  $\square$

*Definition 25.* A smooth manifold is a second countable and Hausdorff topological space equipped with a maximal smooth atlas.  $\square$

*Warning:* A maximal smooth atlas is still extra data. We will call it the smooth structure. Let us also call the charts that are in the maximal smooth atlas the smooth charts.

*Exercise.* Prove that every open subset of  $\mathbb{R}^n$  is second countable.

### 8.1. Examples of smooth manifolds.

- $\mathbb{R}^n$
- $S^n \subset \mathbb{R}^{n+1}$
- $\{x^2 + y^3 = 1\} \subset \mathbb{C}^2$
- $Gr(2, 4) := \{2 \text{ dimensional linear subspaces of } \mathbb{R}^4\}$

What we really mean here is that these have natural topologies which are second countable and Hausdorff and they are equipped with a standard smooth structure.

*Exercise 30.* Describe each of these topologies and smooth structures. As long as you are correct, you don't need to prove anything.  $\square$

- Open subsets of smooth manifolds
- Products of smooth manifolds

*Exercise 31.* Explain what these mean precisely and actually prove what you wrote.  $\square$

**8.2. Smooth maps.** Recall that for an open subset  $U \subset \mathbb{R}^n$ , a map  $U \rightarrow \mathbb{R}$  being smooth means the existence of all iterated partial derivatives; and  $U \rightarrow V \subset \mathbb{R}^n$  being smooth means that each component is smooth.

*Definition 26.* Let  $X$  be a smooth manifold. We call a function  $f : X \rightarrow \mathbb{R}$  smooth if for every smooth chart  $\varphi : V \rightarrow U$ ,  $f \circ \varphi^{-1} : U \rightarrow \mathbb{R}$  is smooth.  $\square$

*Exercise 32.* Prove that it suffices to check the smoothness of  $f$  on an atlas contained in the maximal smooth atlas.  $\square$

*Definition 27.* Let  $X, Y$  be smooth manifolds and  $f : X \rightarrow Y$  be continuous. We say that  $f$  is smooth if for every smooth chart  $\varphi_X : V_X \rightarrow U_X$  in  $X$  and  $\varphi_Y : V_Y \rightarrow U_Y$  such that  $f(V_X) \subset V_Y$ ,  $\varphi_Y \circ f \circ \varphi_X^{-1} : U_X \rightarrow U_Y$  is smooth.  $\square$

**8.3. Gluing.** Let  $\{X_\alpha\}_{\alpha \in I}$  be a collection of topological spaces indexed by a set  $I$ . If we are given open subsets  $X_{\alpha\beta} \subset X_\alpha$  for every  $\alpha \neq \beta \in I$ , and homeomorphisms  $\varphi_{\alpha\beta} : X_{\alpha\beta} \rightarrow X_{\beta\alpha}$  and the following conditions are satisfied, then we call this a gluing data and in particular,  $\varphi_{\alpha\beta}$  gluing maps:

- For every  $\alpha, \beta \in I$ ,  $\varphi_{\alpha\beta} \circ \varphi_{\beta\alpha} = id$ .
- For every  $\alpha, \beta, \gamma$ , pairwise distinct,
 
$$\varphi_{\alpha\beta}(X_{\alpha\beta} \cap X_{\alpha\gamma}) \subset X_{\beta\alpha} \cap X_{\beta\gamma}.$$
- (Cocycle condition)
 
$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma} \text{ on } X_{\alpha\beta} \cap X_{\alpha\gamma}.$$

Under these assumptions, we can define an equivalence relation on  $X = \sqcup_{\alpha \in I} X_\alpha$  by  $a \equiv b$  if  $a \in X_\alpha, b \in X_\beta$  such that  $\alpha \neq \beta$  and  $\varphi_{\alpha\beta}(a) = b$ . We equip  $X$  with its natural topology, that is, the quotient of the disjoint union topology.

**Proposition 4.** *Assume that each  $X_\alpha$  is a smooth manifold, each  $\varphi_{\alpha\beta}$  is smooth (as in Definition 27) and  $X$  is second countable and Hausdorff. Then, there exists a unique smooth structure on  $X$  such that the induced smooth structure on the open subset  $X_\alpha \subset X$  is the given one.*

*Exercise.* Prove this proposition.

*Remark 9.*  $I$  countable  $\Rightarrow X$  is second countable.  $\square$

*Exercise.* Prove that every smooth manifold can be obtained by gluing open subsets of  $\mathbb{R}^n$ .

## 9. MARCH 24, 2022: DIFFEOMORPHISMS, TANGENT BUNDLE OF A MANIFOLD, DIFFERENTIAL OF A SMOOTH MAP

*Definition 28.* Let  $X$  and  $Y$  be smooth manifolds, and  $f : X \rightarrow Y$  be a bijective smooth map. If the inverse map is also smooth, then we call  $f$  a diffeomorphism. We also say that  $X$  and  $Y$  are diffeomorphic.  $\square$

*Exercise.* Let  $X$  be Hausdorff and second countable topological space. Let  $S_1$  and  $S_2$  be two maximal smooth atlases. Prove that the identity map  $X \rightarrow X$  is a diffeomorphism if and only if  $S_1 = S_2$ .

*Exercise 33.* Consider the real line  $\mathbb{R}$  as a topological space. Equip it with (i) its "standard" smooth structure. (ii) smooth structure that admits a chart  $(U, \phi)$  with  $U = \mathbb{R}$  and  $\phi(x) = x^3$ . Prove that (i) and (ii) are not the same smooth structure, but they are diffeomorphic.  $\square$

We have talked about the differentiability of maps between smooth manifolds but we didn't take any actual derivatives yet. Note that the partial derivatives of a function as we learned in calculus courses depend on the coordinates that we are given and we do not have such preferred coordinates in a general smooth manifold. We have to develop the notion of tangent bundle to get a head start. Then we will define the differential of a smooth map.

First, we need to deal with the case of open subsets of Euclidean spaces. If you remember your multivariable calculus class well, this is at best a reformulation of what you already know.

Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  be open subsets, and  $f : U \rightarrow V$  be a smooth map. Then the Jacobian matrix of  $f$  at a point  $p$  is the

following matrix.

$$Jac_p(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1}(p) & & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}$$

We define the tangent bundle of an open set  $U \subset \mathbb{R}^n$  as

$$TU := U \times \mathbb{R}^n,$$

which is an open subset of  $\mathbb{R}^{2n}$ . It is very important to be able to visualize points of  $TU$  as a point  $p$  in  $U$  and a vector at  $p$  effectively, see Figure 19.

We define the differential

$$df : TU \rightarrow TV$$

of  $f : U \rightarrow V$  (as above) by formula

$$df(p, v) = (f(p), Jac_p(f)v)$$

*Exercise.* Prove that if we have open subsets  $U \subset \mathbb{R}^n$ ,  $V \subset \mathbb{R}^m$  and  $W \subset \mathbb{R}^k$  and smooth maps  $f : U \rightarrow V$ ,  $g : V \rightarrow W$ , we have the following reinterpretation of the chain rule.

$$d(g \circ f) = dg \circ df$$

You can use the multivariable calculus chain rule without proof.

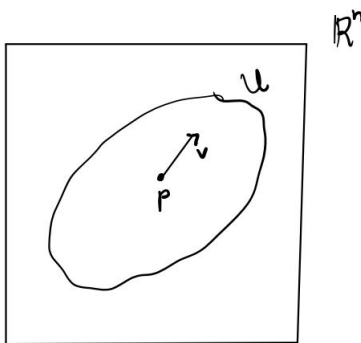


FIGURE 16. A point  $(p, v)$  of  $TU = U \times \mathbb{R}^n$  can be thought of the vector  $v$  at  $p$ .

Now we introduce the tangent bundle of an arbitrary smooth manifold.

*Remark 10.* If  $X \subset \mathbb{R}^n$  is a submanifold (will be defined next lecture but you can imagine  $S^n \subset \mathbb{R}^{n+1}$ ), its tangent bundle is the union of tangent spaces at all points of  $X$ . See Figure 17. This is where the name "tangent bundle" comes from. The description uses the embedding, which we do not want for a definition.  $\square$

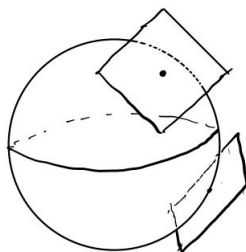


FIGURE 17

*Definition 29.* Let  $M$  be a smooth manifold with maximal atlas  $\{\varphi_\alpha : V_\alpha \rightarrow U_\alpha\}_{\alpha \in I}$ . Define  $U_{\alpha\beta} := \varphi_\alpha(U_\alpha \cap V_\beta)$ .

We define  $TM$  by gluing  $TU_\alpha$  along  $TU_{\alpha\beta} \subset TU_\alpha$  using

$$d(\varphi_\beta \circ \varphi_\alpha^{-1}) : TU_{\alpha\beta} \rightarrow TU_{\beta\alpha}$$

as gluing maps.  $\square$

*Exercise.* Check that this is a valid gluing data.

We also define a smooth map  $\pi : TM \rightarrow M$  by sending  $(p, v) \in TU_\alpha$  to  $\varphi_\alpha^{-1}(p)$ .

*Exercise 34.* Prove that  $\pi$  is well-defined. Prove that  $\forall p \in M$ ,  $\pi^{-1}(p)$  is canonically a  $\dim M$  dimensional vector space.  $\square$

**Proposition 5.** Let  $f : X \rightarrow Y$  be a smooth map between smooth manifolds. Then, there is a canonical smooth map

$$df : TX \rightarrow TY$$

with the following properties:

- (1) For any  $\varphi_X : V_X \rightarrow U_X$  and  $\varphi_Y : V_Y \rightarrow U_Y$  coordinate charts such that  $f(V_X) \subset V_Y$ , the diagram

$$\begin{array}{ccc} TU_X & \longrightarrow & TU_Y \\ \downarrow & & \downarrow \\ TX & \xrightarrow{df} & TY \end{array}$$

commutes. Here

- The top map is  $d(\varphi_Y \circ f \circ \varphi_X^{-1})$  which was already defined.
- The vertical maps are the canonical inclusions coming from the gluing construction.

Clearly, if satisfied, this property determines  $df$  uniquely.

(2) The diagram

$$\begin{array}{ccc} TX & \longrightarrow & TY \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \longrightarrow & Y. \end{array}$$

commutes. For all  $x \in X$ , the induced map  $\pi_X^{-1}(x) \rightarrow \pi_Y^{-1}(f(x))$  is linear.

- (3) If  $g : Y \rightarrow Z$  is another smooth map, then  $dg \circ df = d(g \circ f)$ . Finally  $d(id_X) = id_{TX}$ .

*Exercise 35.* Prove part (1) of this theorem. □

*Exercise.* Prove the remaining parts of this theorem.

**Hint:** To prove part (1), construct the map  $df$  by gluing the maps

$$TU_\alpha \rightarrow TU_\beta$$

with  $f(V_\alpha) \subset V_\beta$ . You need to check that these are compatible with each other. If you can do this one, the other two will be easy.

*Remark 11.* The map  $df$  contains the information of first order derivatives of  $f$ . □

*Definition 30.*  $X^n$  smooth manifold

- $TX \xrightarrow{\pi} X$  is called the tangent bundle of  $X$ .
- For  $x \in X$ , the  $n$  dimensional real vector space  $\pi^{-1}(x)$  is called the tangent space of  $x$  and is defined by  $T_x X$ .

Let  $Y$  be a smooth manifold and  $f : X \rightarrow Y$  smooth map

- The map  $df : TX \rightarrow TY$  is called the differential of  $f : X \rightarrow Y$ .
- We obtain linear maps called  $df_x : T_x X \rightarrow T_{f(x)} Y$  for all  $x \in X$ .

□

## 10. MARCH 29, 2022: SUBMANIFOLDS, REGULAR VALUE THEOREM

*Definition 31.* A subset  $Z$  of a smooth manifold  $X^n$  is called a submanifold of dimension  $k \geq 0$  if for every  $z \in Z$ , there exists a smooth



chart  $\varphi : V \rightarrow U$  of  $X$  such that

$$\varphi(V \cap Z) = U \cap (\mathbb{R}^k \times \{0\}) \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n.$$

□

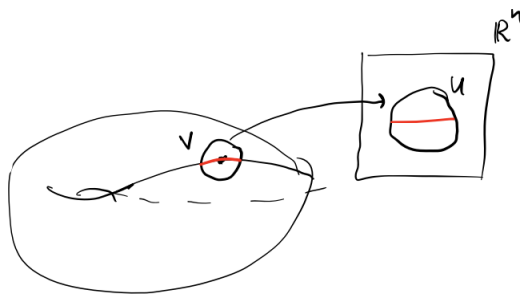


FIGURE 18

*Exercise.* Prove that a submanifold  $Z \subset X$  with its subspace topology can be equipped with a natural smooth structure such that the inclusion map  $Z \hookrightarrow X$  is smooth.

There are two main ways of obtaining submanifolds:

- (1) As solutions of smooth equations, i.e. preimages of points in

$$\begin{array}{ccc} \text{the domain of maps such as } & X & \xrightarrow{\quad} \mathbb{R}^2 \\ & & \searrow \\ & & Y \end{array}$$

- (2) As subsets parametrized by other manifolds, e.g.  $S^2 \rightarrow X$ .

Let's start with (1). Consider a smooth map  $f : X \rightarrow Y$ . We call  $y \in Y$  a *regular value* if for every  $x \in X$  such that  $f(x) = y$ , the linear map  $df_x : T_x X \rightarrow T_y Y$  is surjective.

**Theorem 7** ( Regular Value Theorem ). *Let  $f : X \rightarrow Y$  be a smooth map. If  $y \in Y$  is a regular value, then  $f^{-1}(y) \hookrightarrow X$  is a submanifold. Moreover, there are canonical isomorphisms*

$$T_x f^{-1}(y) \cong \ker(df_x),$$

for every  $x \in f^{-1}(y)$ .

*Proof Sketch:*

- Can easily reduce to  $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^k$  open subsets.

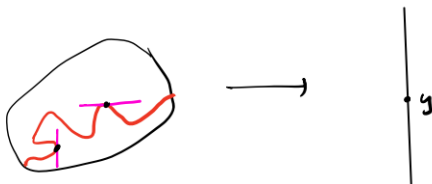


FIGURE 19

- **Implicit Function Theorem:** We can reorder the coordinates of  $\mathbb{R}^n$  such that in an open neighborhood  $x \in U$ ,  $f^{-1}(y) \cap U$  is the graph of a smooth map

$$\varphi = (\varphi_1, \dots, \varphi_k) : W \rightarrow \mathbb{R}^k$$

where  $W = pr_{n-k}(U) \subset \mathbb{R}^{n-k}$  is open and  $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ .

- *Remark* ( $\implies$ ) This already gives a smooth atlas on  $f^{-1}(y)$
- Use the map  $U \rightarrow \mathbb{R}^n$   
 $(x_1, \dots, x_{n-k}, x_{n-k+1} - \varphi_1(x_1, \dots, x_{n-k}), \dots, x_n - \varphi_k(x_1, \dots, x_{n-k}))$
- This gives a chart by the inverse function theorem.
- For the statement with tangent spaces, note the diagram of smooth maps

$$\begin{array}{ccc} f^{-1}(y) & \longrightarrow & * \\ \downarrow & & \downarrow \downarrow \\ x & \longrightarrow & Y \quad y \end{array}$$

and use Exercise 36.

*Exercise.* Make this statement intuitive for yourself.

**Definition 32.** A smooth map  $f : X \rightarrow Y$  is called an immersion if  $df_x : T_x X \rightarrow T_x Y$  is injective for all  $x \in X$ .  $\square$

**Exercise 36.** Prove that inclusions of submanifolds into smooth manifolds are injective immersions.  $\square$

**Example 7.** Consider  $T^2 = S^1 \times S^1$ , which can be represented as the  $[0, 1] \times [0, 1]$  with the ends identified as indicated in Figure 20, and draw on it a line with irrational slope which gives an injective immersion  $\mathbb{R} \rightarrow T^2$ . The image is a dense subset and is not a submanifold.  $\square$

**Proposition 6.** Assume that  $f : Z \rightarrow X$  is an injective immersion. Then  $f(Z)$  is a submanifold if and only if  $Z \rightarrow f(Z)$  is a homeomorphism, where  $f(Z)$  has the subspace topology.

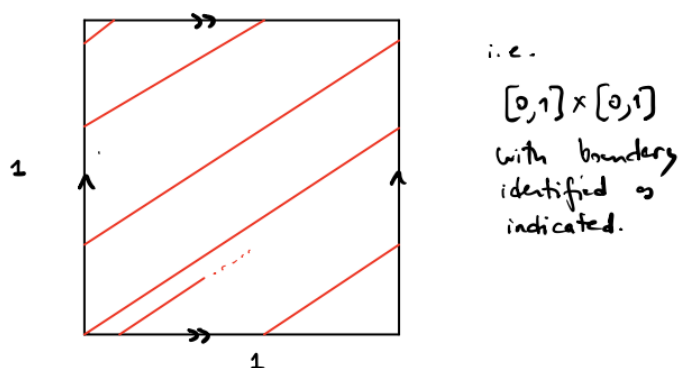


FIGURE 20

*Exercise 37.* Prove this.  $\square$

**Proposition 7.** Assume that  $f : Z \rightarrow X$  is an injective immersion. Then  $f(Z)$  is a submanifold and a closed subset if and only if  $f$  is a proper map.

*Remark 12.* The properness condition in the proposition is automatic if  $Z$  is compact.  $\square$

*Exercise.* Find a submanifold of a smooth manifold which is not a closed subset.

**Theorem 8** (Whitney Embedding Theorem). Any smooth manifold  $X^n$  can be injectively immersed into  $\mathbb{R}^N$  by a smooth proper map for some  $N > 0$ .

*Remark 13.* We can take  $N = 2n$ , but this is sharp, e.g.  $\mathbb{RP}^2$  and Klein bottle. Whitney embedding theorem is not as useful as it might seem because the embeddings are usually inexplicit and complicated.  $\square$