LECTURE NOTES - MATH 58J (SPRING 2022)

UMUT VAROLGUNES

1. Feb 24, 2022: Introduction, cohomology of $U \subset \mathbb{R}^n$

Consider the n dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, 1 \in [n]\},$$

where $[n] := \{1, \ldots, n\}$. Abusing notation x_i's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$ has a metric given by

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

This induces a topology on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on U and see how these can be used to analyze the topology of U.

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of U (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different

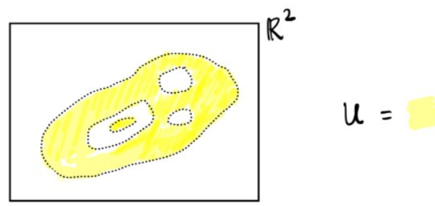

Figure 1. An example of an open subset of Euclidean space

ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.

- Up to many technical details, you can intuitively think about a degree k cohomology class β on U as a way of associating a real number $\beta(Z)$ to each compact, boundariless[1], oriented submanifold[2] Z of dimension k such that the following condition $(\star)$ holds.

If Z is the oriented boundary of a $(k+1)-$dimensional submanifold with boundary, then $\beta(Z) = 0$.

  Let's refer to such Z as "$k-$cycles" - in quotation marks because we will use this word with a different meaning later
- The main operation that one does with a differential $k-$form is to integrate them along $k-$dimensional oriented submanifolds and we use this to associate real numbers to "$k-$cycles".
- Property $(\star)$ will only hold if the differential form is closed.

1.1. Cohomology of $U \subset \mathbb{R}^n$.

Let $\pi_0(U)$ be the set of all connected components of U.

Definition 1. $H^0(U, \mathbb{R})$ is defined as the vector space of all maps from $\pi_0(U)$ to $\mathbb{R}$. $\square$

Let $b \in U$ and $\pi_1(U, b)$ be the fundamental group of U with base point b. Recall that

$$\pi_1(U, b) := \frac{\{(S^1, *) \to (U, b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where $S^1 = \frac{[0,1]}{0\sim 1}$ and $* = [0] \in S^1$.
Here are some properties

- $\pi_1(U, b)$ is a group.
- Choosing a continuous path $\gamma : [0, 1] \to U$ from b to b' gives rise to a group isomorphism $f_\gamma : \pi_1(U, b) \to \pi_1(U, b')$.

Definition 2. Assuming that U is connected we define $H^1(U, \mathbb{R})_b$ as the vector space of group homomorphisms

$$\pi_1(U, b) \to \mathbb{R}.$$

$\square$

Exercise 1. Prove that for any $b, b' \in U$, as long as U is connected, there is a canonical isomorphism $H^1(U, \mathbb{R})_b \to H^1(U, \mathbb{R})_{b'}$. $\square$

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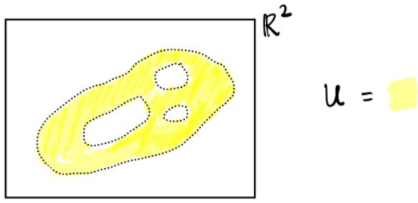
[1] I write this only to stress the point

[2] submanifold here means a subset that locally looks like a k dimensional Euclidean space

As a consequence of this exercise we can write $H^1(U, \mathbb{R})$ without any ambiguity.

Example 1. Let U be defined as below.



Then, $\dim(H^1(U, \mathbb{R})) = 3$. □

2. Feb 24, 2022: Differential Forms on $U \subset \mathbb{R}^n$

- A differential 0-form on U is a smooth[3] function $U \to \mathbb{R}$.
- A differential 0-form f is called closed if $\frac{\partial f}{\partial x_i} \equiv 0, \forall i \in [n]$

Proposition 1. *There is a canonical linear isomorphism*

$$H^0_{dR} := \{\textit{closed differential 0-form on } U\} \simeq H^0(U; \mathbb{R})$$

Remark 1. In general $H^k_{dR} := \dfrac{\{\text{closed differential k-form on U}\}}{\{\text{exact differential k-form on U}\}}$ □

- A differential 1-form on U is an expression $f_1 dx_1 + ... + f_n dx_n$ where $f_i : U \to \mathbb{R}$ are smooth functions
- A differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ is called exact, if for some smooth $V : U \to \mathbb{R}$,
$$f_i = \frac{\partial V}{\partial x_i}, \forall i \in [n].$$
In this case we write $\alpha = dV$.
- A differential 1-form is closed if for all $i \neq j \in [n]$,
$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0.$$

Lemma 1. *If $\alpha = \sum_{i=1}^{n} f_i dx_i$ is exact, then it is closed.*

Proof. Since it is exact, $\exists V : U \to \mathbb{R}$, such that $f_i = \frac{\partial V}{\partial x_i}$, so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

[3]note that this is a condition much stronger than differentiable, it means that all iterated partial derivatives exist. please read the wikipedia page if you are not familiar.

□

Exercise 2. For $n=2$ and $n=3$ explain what it means for the differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ to be closed in terms of the vector field $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ using terms from your calculus classes. Recall Green's and Stokes theorems. □

Theorem 1. *Assuming that U is connected, there exists a linear isomorphism,*

$$H^1_{dR} := \frac{\{\textit{closed differential 1-forms on } U\}}{\{\textit{exact differential 1-forms on } U\}} \simeq H^1(U;\mathbb{R}) \tag{1}$$

Proof sketch. First we want to define a linear map

$$\int : \{\text{closed 1-forms}\} \to \{\pi_1(U,b) \to \mathbb{R} \quad \text{group homomorphisms}\}$$

Recall: $X \subset \mathbb{R}^n$ arbitrary subset. A map $g : X \to \mathbb{R}^m$ is called smooth if it extends to a smooth map $N(X) \to \mathbb{R}^m$ where $N(X)$ is an open neighborhood of X.

Fact:

- Any class in $\pi_1(U,b)$ can be represented by a smooth map $(S^1,*) \to (U,b)$.
- Any two smooth maps $S^1 \to U$ that are homotopic continuously are homotopic smoothly.

Recall: Given $\alpha = \sum_{i=1}^{n} f_i dx_i$ and a smooth path $\gamma : [0,1] \to U$, we can define the line integral $\int_\gamma \alpha := \int_0^1 F \cdot \gamma' dt$, where $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$.

- The map $\int$ is independent of the parametrization of γ, meaning, if $\phi : [0,1] \to [0,1]$ is a smooth bijective map with $\phi' \neq 0$, then $\int_\gamma \alpha = \int_{\gamma\circ\phi} \alpha$. So line integral only depends on the image of γ.

Back to the map $\int$: we send a given $\alpha \in \{\text{closed 1-forms}\}$ to the map $\int_\cdot \alpha : \pi_1(U,b) \to \mathbb{R}$ defined by

$$a \mapsto \int_{\bar{\gamma}} \alpha,$$

where $\bar{\gamma}$ is an arbitrary smooth representative of a. In the exercise below you will show that this is well defined. Assuming that for now, it is easy to see that $\int_\cdot \alpha$ is a group homomorphism[4], so an element of $H^1(U;\mathbb{R})$, and the resulting $\int$ is a linear map.

[4]I forgot to say this in class, so please check it for yourself!

Exercise 3. Show that the map $\int$ is well defined by proving if $\alpha = \sum_{i=1}^{n} f_i dx_i$ is closed and $\gamma, \gamma' : S^1 \to U$ are smooth maps that are smoothly homotopic, then $\int_\gamma \alpha = \int'_\gamma \alpha$. (**Hint:** Start by analyzing $n = 2, 3$ and where the smooth homotopy $S^1 \times [0,1] \to U$ is injective, then reduce the statement to Green's theorem. Additionally, you may want to check proof of Stokes theorem.)

□

FIGURE 2. An example of an injective smooth homotopy's image for $n = 2$.

Goals:

(1) Prove that $\int$ sends an exact differential 1-form to zero. We obtain a linear map $\widetilde{\int} : H^1_{dR}(U, \mathbb{R}) \to H^1(U, \mathbb{R})$.
(2) Prove that $\widetilde{\int}$ is injective. ("Construct a potential")
(3) Prove that $\widetilde{\int}$ is surjective.

We start with 1). Let us integrate $\alpha = df$ along a smooth loop γ.

$$\int_\gamma \alpha = \int_0^1 \nabla f \cdot \gamma'(t) dt \overset{\text{FTC}}{=} f(b) - f(b) = 0.$$

Denote the resulting map by

$$\widetilde{\int} : \frac{\text{Closed differential 1-forms}}{\text{Exact differential 1-forms}} \to H^1(U, \mathbb{R}).$$

For 2), we need to show that if $\int_\gamma \alpha = 0 \quad$ for all $\quad \gamma : (S^1, \star) \to (U, b)$, then $\alpha = df$ for some $f : U \to \mathbb{R}$.

Exercise 4. Do this! This is the same task as constructing a potential (recall work integrals, conservative fields etc.) for the corresponding vector field. □

Let us make a simplifying assumption on U to not deal with orthogonal difficulties in 3). Assume there exists $\gamma_1, \gamma_2, .., \gamma_n \in \pi_1(U, b)$ that freely generates the abelianization of $\pi_1(U, b)$. This implies that giving a group homomorphism $\pi_1(U, b) \to \mathbb{R}$ is equivalent to assigning real numbers to each of $\gamma_1, \gamma_2, .., \gamma_n$ (arbitrarily).

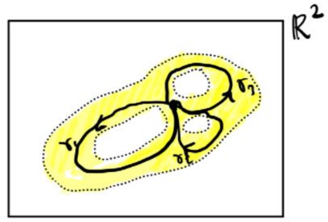


FIGURE 3

Now we need to create a differential 1-form that integrates to any $a_1, a_2, .., a_n \in \mathbb{R}$ along $\gamma_1, \gamma_2, .., \gamma_n$. This is still quite difficult. That will follow from DeRham Theorem, which will be the highlight of our course.

Remark 2. Once we give the general definition of singular cohomology, I will assign a homework exercise which shows that it agrees with what we defined today in degrees 0 and 1. The analogous statement will be automatic for DeRham cohomology. □

Exercise 5. Finish the proof of surjectivity in the case

$$U = \mathbb{R}^2 - \text{finitely many points.}$$

(**Hint:**Start with $\mathbb{R}^2 - (0, 0)$ and use the closed differential 1-form $\alpha = -\frac{ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$.) □