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07: Variance, Bernoulli, Binomial

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[Lecture Discussion on Ed]

Variance

Average temperatures

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$

Washington, DC

$E[\text{high}] = 67°\text{F}$

$E[\text{low}] = 51°\text{F}$

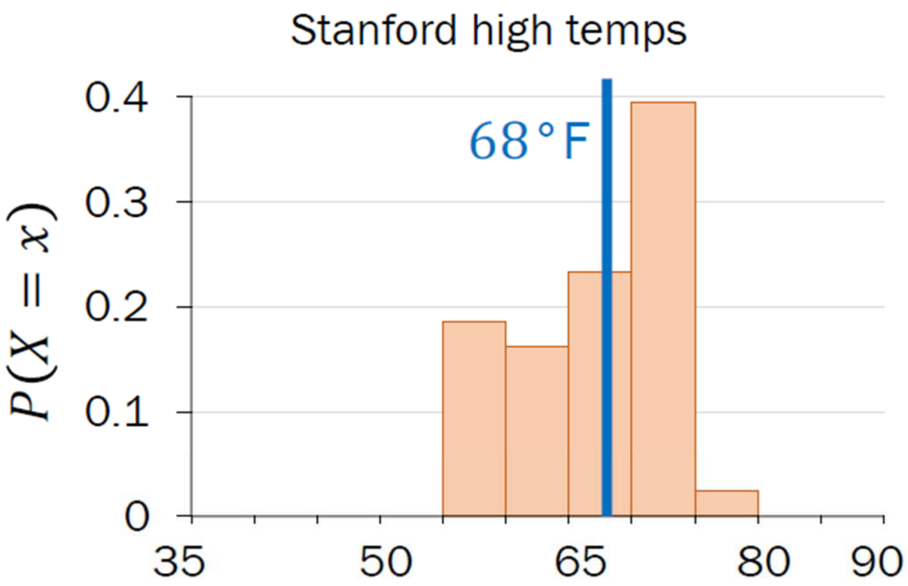
Is $E[X]$ enough?

Average temperatures

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$



Washington, DC

$E[\text{high}] = 67°\text{F}$

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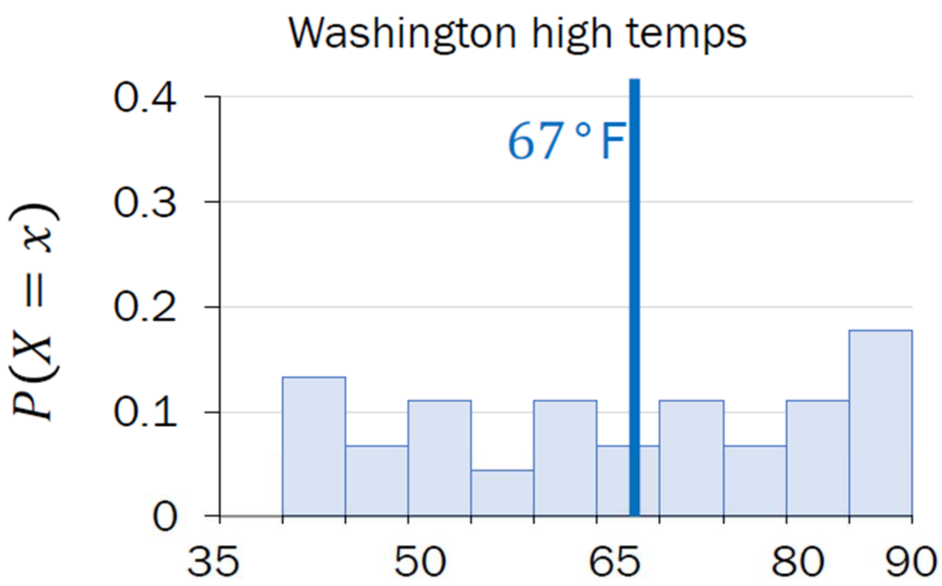
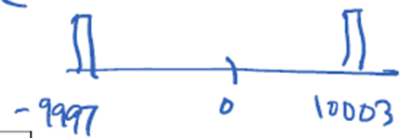


Normalized histograms are approximations of PMFs.

Variance = "spread"

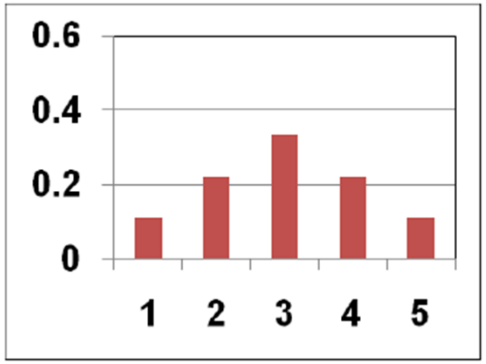
Consider the following three distributions (PMFs):

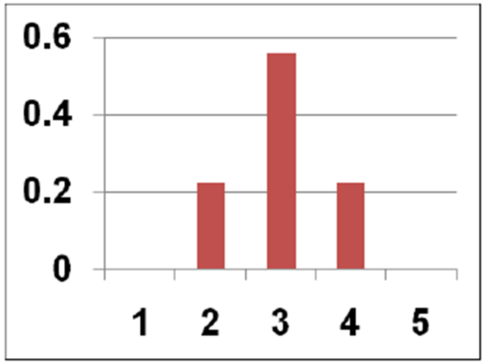
extreme example:



$E[X] = 3$

$\mathrm{Var}(X) = \text{huge}$





- Expectation: $E[X] = 3$ for all distributions
- But the shape and spread across distributions are very different!
- Variance, $\mathrm{Var}(X)$: a formal quantification of spread

Variance

The variance of a random variable X with mean $E[X] = \mu$ is

$$\text{Var}(X) = E[(X - \mu)^2]$$

- Also written as: $E[(X - E[X])^2]$
- Note: $\text{Var}(X) \geq 0$
- Other names: **2nd central moment**, or square of the standard deviation

	$\text{Var}(X)$	Units of X^2
def standard deviation	$\text{SD}(X) = \sqrt{\text{Var}(X)}$	Units of X

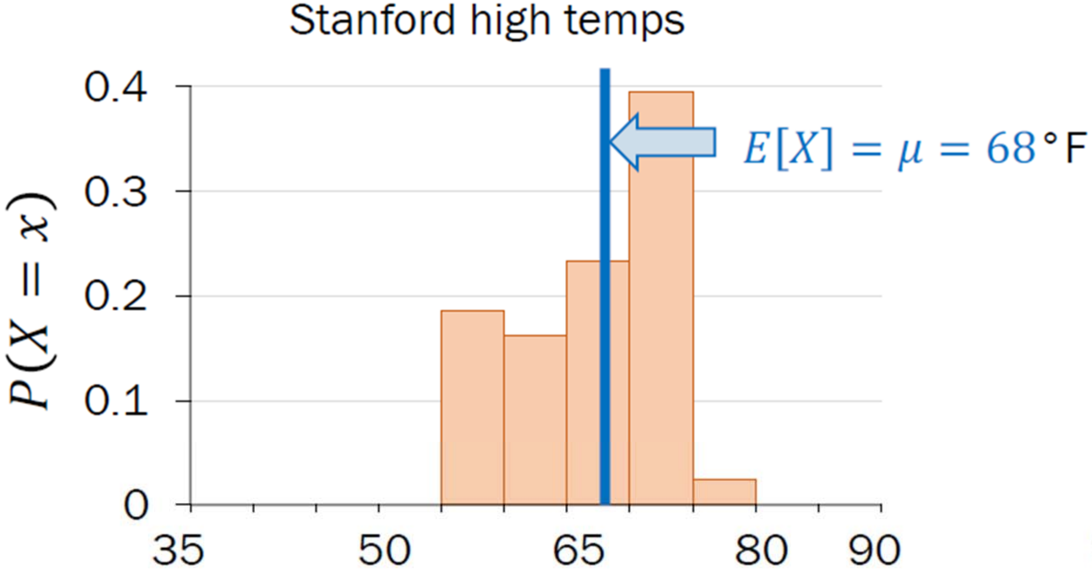
Variance of Stanford weather

$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$



X	$(X - \mu)^2$
57°F	121 (°F)^2
71°F	9 (°F)^2
75°F	49 (°F)^2
69°F	1 (°F)^2
...	...

Variance $E[(X - \mu)^2] = 39\ (°\text{F})^2$

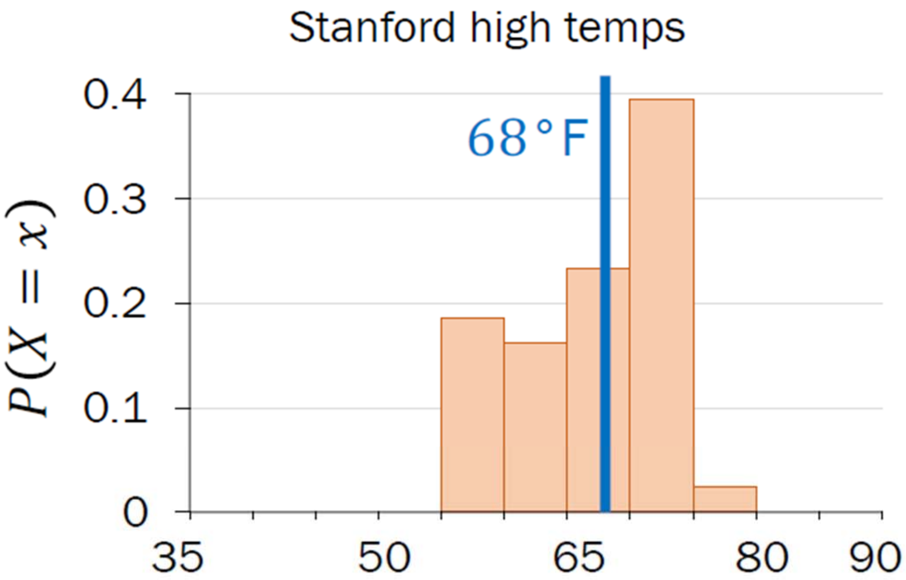
Standard deviation $= 6.2°\text{F}$

Comparing variance

$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA

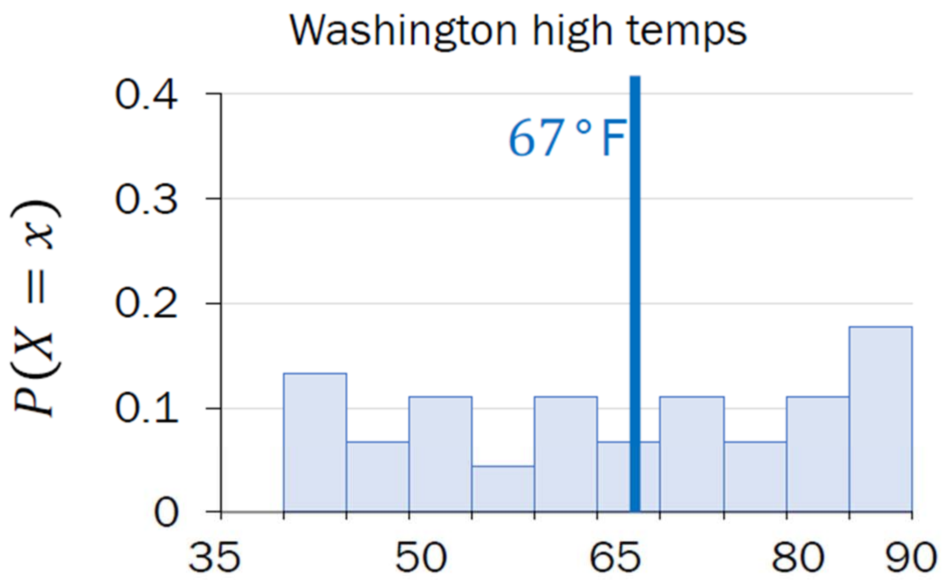
$E[\text{high}] = 68\,°\text{F}$



$\text{Var}(X) = 39\ (°\text{F})^2$

Washington, DC

$E[\text{high}] = 67\,°\text{F}$



$\text{Var}(X) = 248\ (°\text{F})^2$

Properties of Variance

Properties of variance

Definition $\quad \text{Var}(X) = E[(X - E[X])^2]$ $\quad$ Units of X^2

def standard deviation $\quad \text{SD}(X) = \sqrt{\text{Var}(X)}$ $\quad$ Units of X

Property 1 $\quad \text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\quad \text{Var}(aX + b) = a^2 \text{Var}(X)$

- Property 1 is often easier to manipulate than the original definition
- Unlike expectation, variance is not linear

Properties of variance

Definition $\quad \mathrm{Var}(X) = E[(X - E[X])^2] \quad$ Units of X^2

def standard deviation $\quad \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} \quad$ Units of X

Property 1 $\quad \mathrm{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\quad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Computing variance, a proof

$\text{Var}(X) = E[(X - E[X])^2]$ Variance
$= E[X^2] - (E[X])^2$ of X

$$\text{Var}(X) = E[(X - E[X])^2] = E[(X - \mu)^2]$$

Let $E[X] = \mu$

$$= \sum_x (x - \mu)^2 p(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$

Everyone, please welcome the second moment!

$$= E[X^2] - 2\mu E[X] + \mu^2 \cdot 1$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

Variance of a 6-sided die

$\text{Var}(X) = E[(X - E[X])^2]$ Variance
$= E[X^2] - (E[X])^2$ of X

Let Y = outcome of a single die roll. Recall $E[Y] = 7/2$.

Calculate the variance of Y.

1. Approach #1: Definition

$$\text{Var}(Y) = \frac{1}{6}\left(1 - \frac{7}{2}\right)^2 + \frac{1}{6}\left(2 - \frac{7}{2}\right)^2 + \frac{1}{6}\left(3 - \frac{7}{2}\right)^2 + \frac{1}{6}\left(4 - \frac{7}{2}\right)^2 + \frac{1}{6}\left(5 - \frac{7}{2}\right)^2 + \frac{1}{6}\left(6 - \frac{7}{2}\right)^2$$

$$= 35/12$$

2. Approach #2: A property

2nd moment

$$E[Y^2] = \frac{1}{6}[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2] = 91/6$$

$$\text{Var}(Y) = 91/6 - (7/2)^2 = 35/12$$

Properties of variance

Definition $\quad \text{Var}(X) = E[(X - E[X])^2]$ Units of X^2

def standard deviation $\quad \text{SD}(X) = \sqrt{\text{Var}(X)}$ Units of X

Property 1 $\quad \text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\quad \text{Var}(aX + b) = a^2\text{Var}(X)$

Property 2: A proof

Property 2 $\quad \text{Var}(aX + b) = a^2\text{Var}(X)$

terms cancel
terms cancel

Proof: $\text{Var}(aX + b)$

$$
\begin{aligned}
&= E[(aX+b)^2] - (E[aX+b])^2 && \text{Property 1} \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 && \\
&= a^2E[X^2] + 2abE[X] + b^2 - (a^2(E[X])^2 + 2abE[X] + b^2) && \\
&= a^2E[X^2] - a^2(E[X])^2 && \\
&= a^2(E[X^2] - (E[X])^2) && \\
&= a^2\text{Var}(X) && \text{Property 1}
\end{aligned}
$$

Factoring/ Linearity of Expectation

Bernoulli RV

Bernoulli Random Variable

Consider an experiment with two outcomes: "success" and "failure".

<u>def</u> A **Bernoulli** random variable X maps "success" to 1 and "failure" to 0.
Other names: **indicator** random variable, Boolean random variable

$$X \sim \text{Ber}(p)$$

Support: $\{0,1\}$

PMF $P(X=1) = p(1) = p$
$P(X=0) = p(0) = 1-p = q$

often used instead of $1-p$ ☺

Expectation $E[X] = p$

Variance $\text{Var}(X) = p(1-p) = pq$

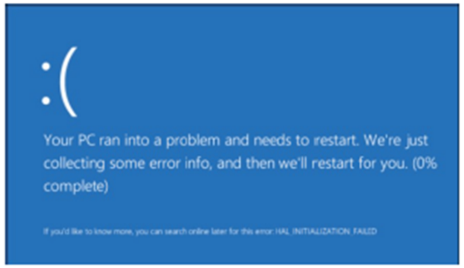
Examples:

- Coin flip
- Random binary digit
- Whether Doris barks

Remember this nice property of expectation.

Defining Bernoulli RVs

$X \sim \text{Ber}(p)$ $\quad p_X(1) = p$
$E[X] = p$ $\quad p_X(0) = 1 - p$



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crash

$$X \sim \text{Ber}(p)$$

$P(X = 1) = p$

$P(X = 0) = 1 - p$

Serve an ad.

- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \text{Ber}(\underline{0.2})$$

$P(X = 1) = \underline{0.2}$

$P(X = 0) = \underline{0,8}$

Roll two dice.

- Success: roll two 6's
- Failure: anything else

Let X : 1 if success

$$X \sim \text{Ber}\left(\underline{\tfrac{1}{36}}\right)$$

$E[X] = \underline{\tfrac{1}{36}}$

Binomial RV

Binomial Random Variable

$E[X] = \sum_{k=0}^{n} k \cdot P(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \dots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = \boxed{np}$
Support: $\{0, 1, \dots, n\}$	Variance	$\text{Var}(X) = np(1-p)$ ← will prove later

If X is a Binomial, then X is really a sum of n Bernoullis

$X = B_1 + B_2 + B_3 + \cdots + B_{n-1} + B_n$

$E[X] = E[B_1] + E[B_2] + \cdots + E[B_n]$

$= p + p + \cdots + p$

$\boxed{= np}$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Reiterating notation

1. The random variable

$$X \sim \text{Bin}(n, p)$$

2. is distributed as a

3. Binomial

4. with parameters

The parameters of a Binomial random variable:

- n: number of independent trials
- p: probability of success on each trial

Reiterating notation

$$X \sim \text{Bin}(n, p)$$

If X is a binomial with parameters n and p, the PMF of X is

$$\underbrace{P(X = k)}_{\substack{\text{Probability that } X \\ \text{takes on the value } k}} = \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{\textbf{Probability Mass Function} \text{ for a Binomial}}$$

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Three fair (with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$

Compute the following event probabilities:

$$P(X = 0)$$

$$P(X = 1)$$

$$P(X = 2)$$

$$P(X = 3)$$

$$P(X = 7)$$

P(event)

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Three fair (with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$

Compute the following event probabilities:

$$P(X=0) \quad = p(0) \quad = \binom{3}{0} p^0 (1-p)^3 = \frac{1}{8}$$

$$P(X=1) \quad = p(1) \quad = \binom{3}{1} p^1 (1-p)^2 = \frac{3}{8}$$

$$P(X=2) \quad = p(2) \quad = \binom{3}{2} p^2 (1-p)^1 = \frac{3}{8}$$

$$P(X=3) \quad = p(3) \quad = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{8}$$

$$P(X=7) \quad = p(7) \quad = 0$$

P(event) PMF

Extra math note:
By Binomial Theorem, we can prove $\sum_{k=0}^{n} P(X=k) = 1$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

<u>def</u> A **Binomial** random variable X is the number of successes in n trials.

$$X \sim \text{Bin}(n, p)$$

Range: $\{0, 1, \dots, n\}$

PMF	$k = 0, 1, \dots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Expectation	$E[X] = np$
Variance	$\text{Var}(X) = np(1-p)$

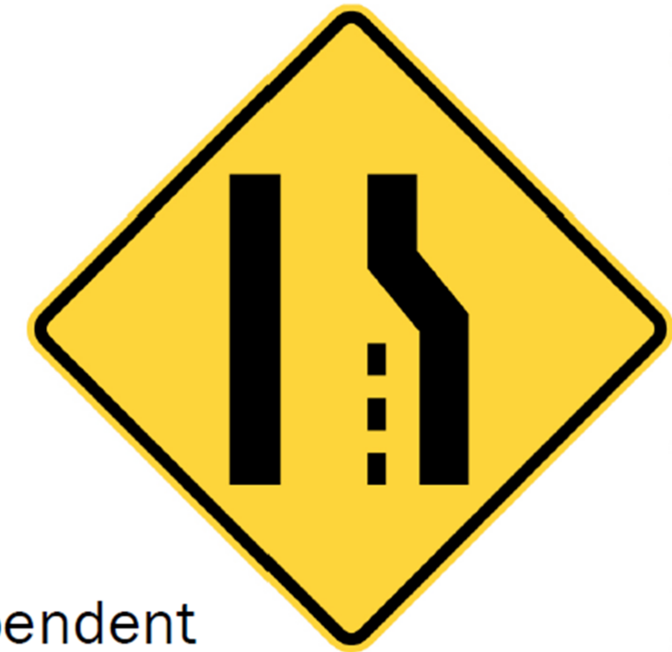
Examples:

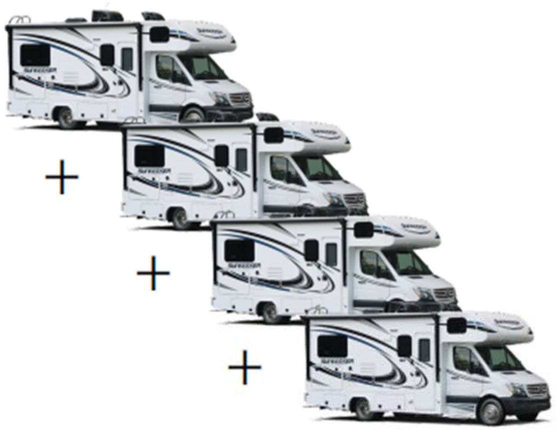
- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Binomial RV is sum of Bernoulli RVs

Bernoulli

- $X \sim \text{Ber}(p)$





Binomial

- $Y \sim \text{Bin}(n, p)$
- The sum of n independent Bernoulli RVs

$$Y = \sum_{i=1}^{n} X_i, \qquad X_i \sim \text{Ber}(p)$$

$$\boxed{\text{Ber}(p) = \text{Bin}(1, p)}$$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Range: $\{0, 1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Proof:

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Range: $\{0, 1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

We'll prove this later in the course

Examples:
- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

No, give me the variance proof right now

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$$\begin{aligned}
\mathrm{E}\left(X^2\right) &= \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k} && \text{Definition of Binomial Distribution: } p + q = 1 \\
&= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} && \text{Factors of Binomial Coefficient: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} && \text{Change of limit: term is zero when } k-1=0 \\
&= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j} && \text{putting } j = k-1,\ m = n-1 \\
&= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{splitting sum up into two} \\
&= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Factors of Binomial Coefficient: } j\binom{m}{j} = m\binom{m-1}{j-1} \\
&= np \left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Change of limit: term is zero when } j-1=0 \\
&= np\left((n-1)p(p+q)^{m-1} + (p+q)^m\right) && \text{Binomial Theorem} \\
&= np((n-1)p+1) && \text{as } p+q=1 \\
&= n^2p^2 + np(1-p) && \text{by algebra}
\end{aligned}$$

proofwiki.org

No, give me the variance proof right now

Then:

$$\begin{aligned}\operatorname{var}(X) &= \operatorname{E}\left(X^2\right) - (\operatorname{E}(X))^2 \\ &= np(1-p) + n^2p^2 - (np)^2 && \text{Expectation of Binomial Distribution: } \operatorname{E}(X) = np \\ &= np(1-p)\end{aligned}$$

as required.

proofwiki.org

Exercises

Statistics: Expectation and variance

If you can identify common RVs, just look up statistics instead of rederiving from scratch.

1. a. Let $X =$ the outcome of a fair 24-sided die roll. What is $E[X]$?
   b. Let $Y =$ the sum of seven rolls of a fair 24-sided die. What is $E[Y]$?

support = $\{1, 2, 3, 4, 5, \ldots, 23, 24\}$

$E[X] = 12.5$, by symmetry

$E[Y] = E[X_1 + X_2 + X_3 + \cdots + X_7]$

$= E[X_1] + E[X_2] + E[X_3] + \cdots$

$= 7E[X_1] = 87.5$

2. Let $Z =$ # of ***tails*** on 10 flips of a biased coin, with p = 0.71. What is $E[Z]$?

$p = 0.71$

$E[Z] = 10p = 7.1$

3. Compare the variances of $B_0 \sim \text{Ber}(0.0)$, $B_1 \sim \text{Ber}(0.1)$, $B_2 \sim \text{Ber}(0.5)$, and $B_3 \sim \text{Ber}(0.9)$.

$\text{Var}(B_0) = 0 \Rightarrow$ no spread, no variation

$\text{Var}(B_1) = 0.1(1-0.1) = 0.09 \leftarrow$ nonzero, but small

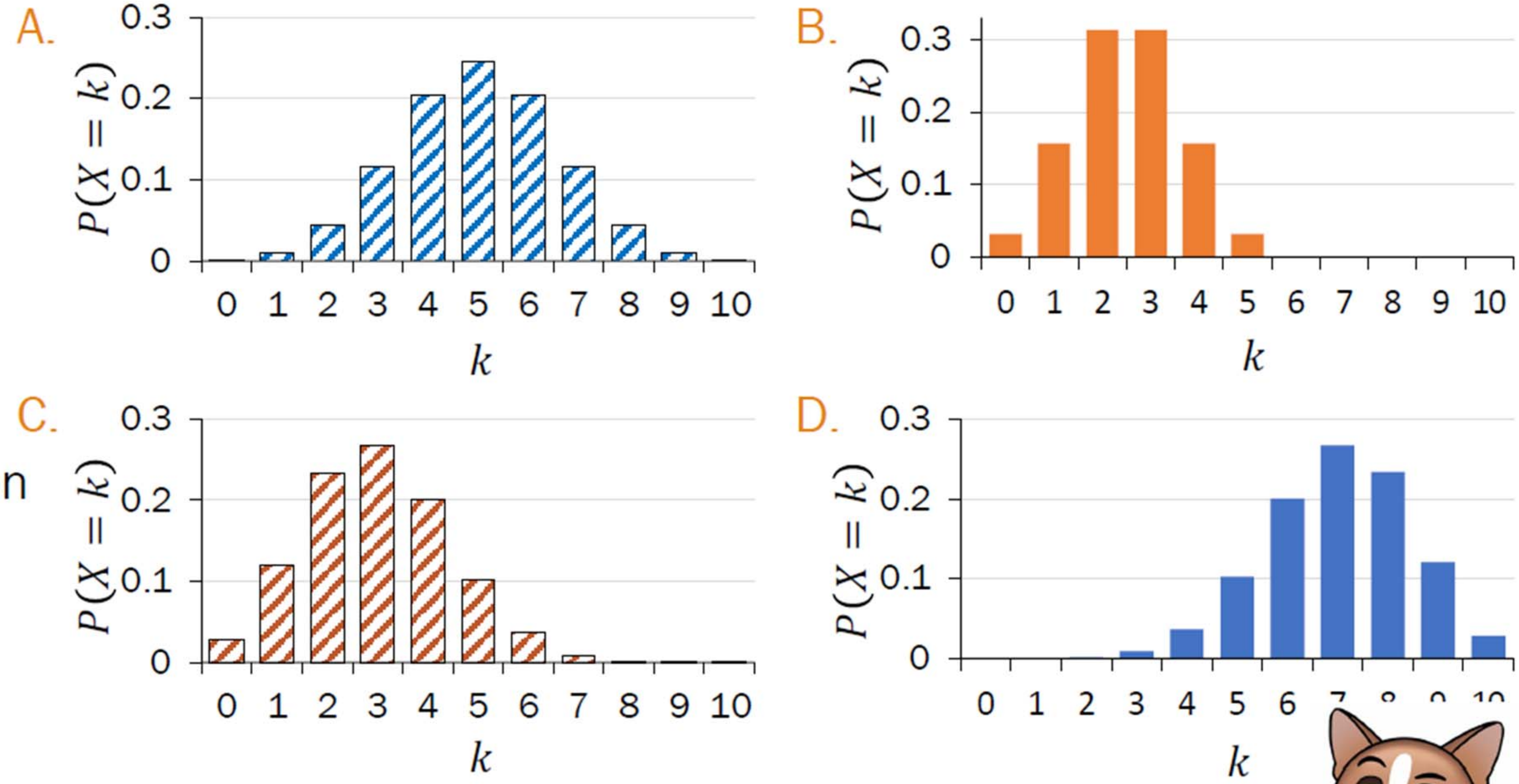
$\text{Var}(B_2) = 0.5^2 = 0.25 \leftarrow$ relatively substantial

$\text{Var}(B_3) = \text{Var}(B_1) = 0.09$

$p = 0.5$ maximizes variance

Visualizing Binomial PMFs

$X \sim \text{Bin}(n, p)$ $\quad E[X] = np$ $\quad p(i) = \binom{n}{k} p^k (1-p)^{n-k}$



Match the distribution of X to the graph:

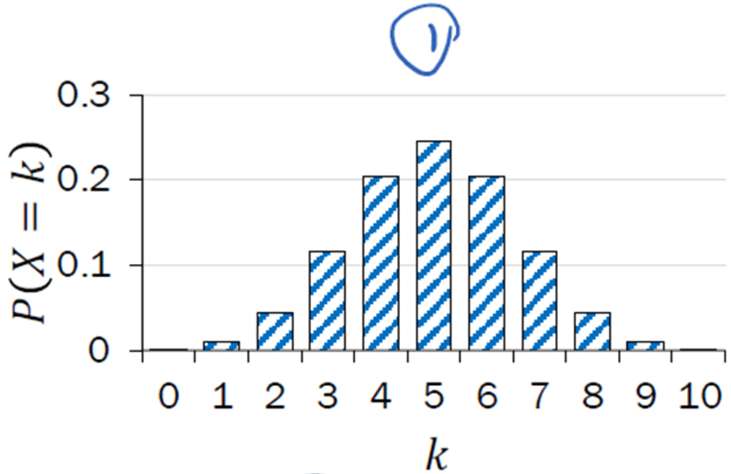
1. Bin(10,0.5)
2. Bin(10,0.3)
3. Bin(10,0.7)
4. Bin(5,0.5)

Visualizing Binomial PMFs

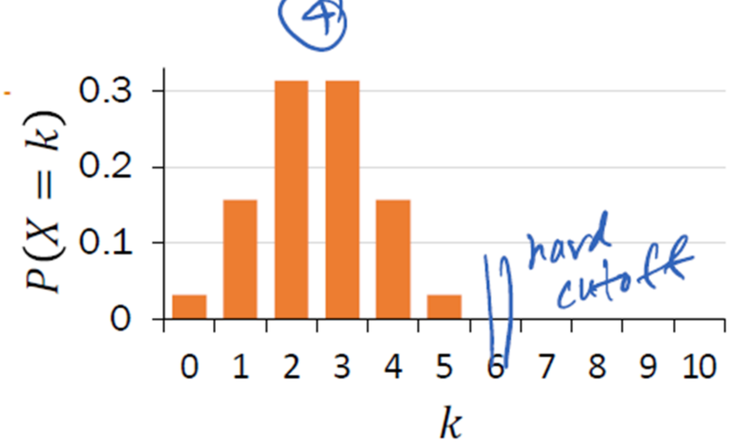
$E[X] = np$

$X \sim \text{Bin}(n, p) \quad p(i) = \binom{n}{k} p^k (1-p)^{n-k}$

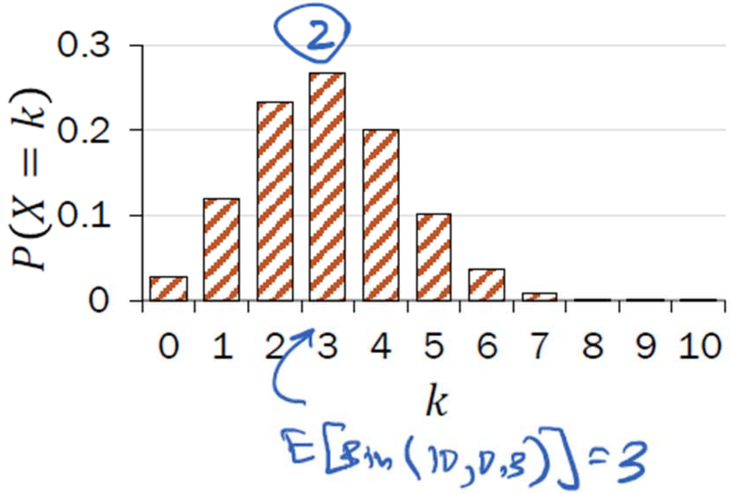
A.



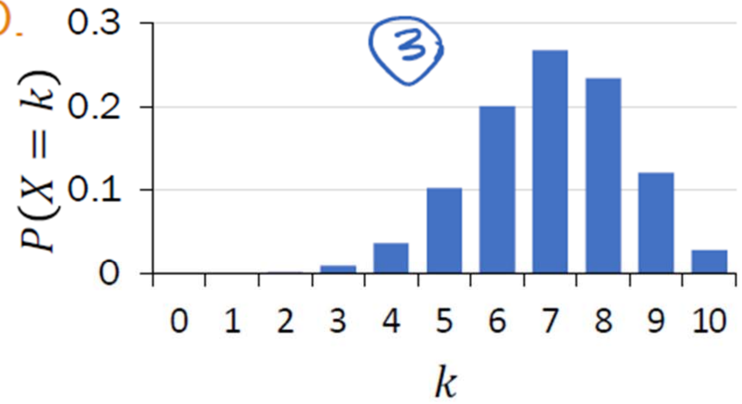
B.



C.



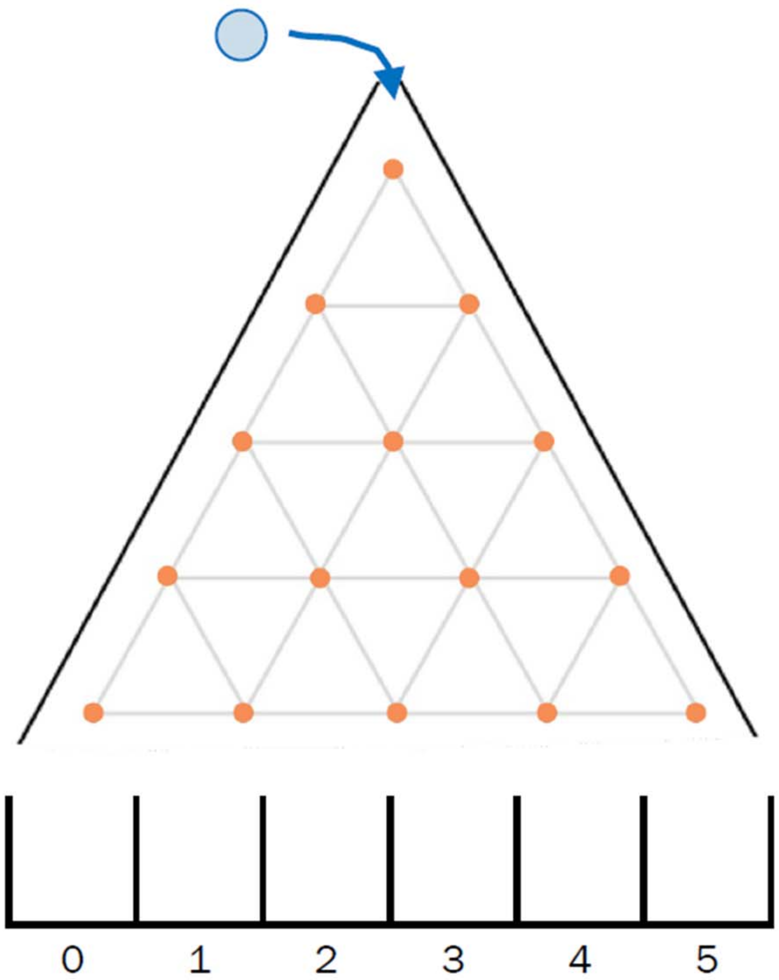
D.



Match the distribution of X to the graph:

1. Bin(10,0.5)
2. Bin(10,0.3)
3. Bin(10,0.7)
4. Bin(5,0.5)

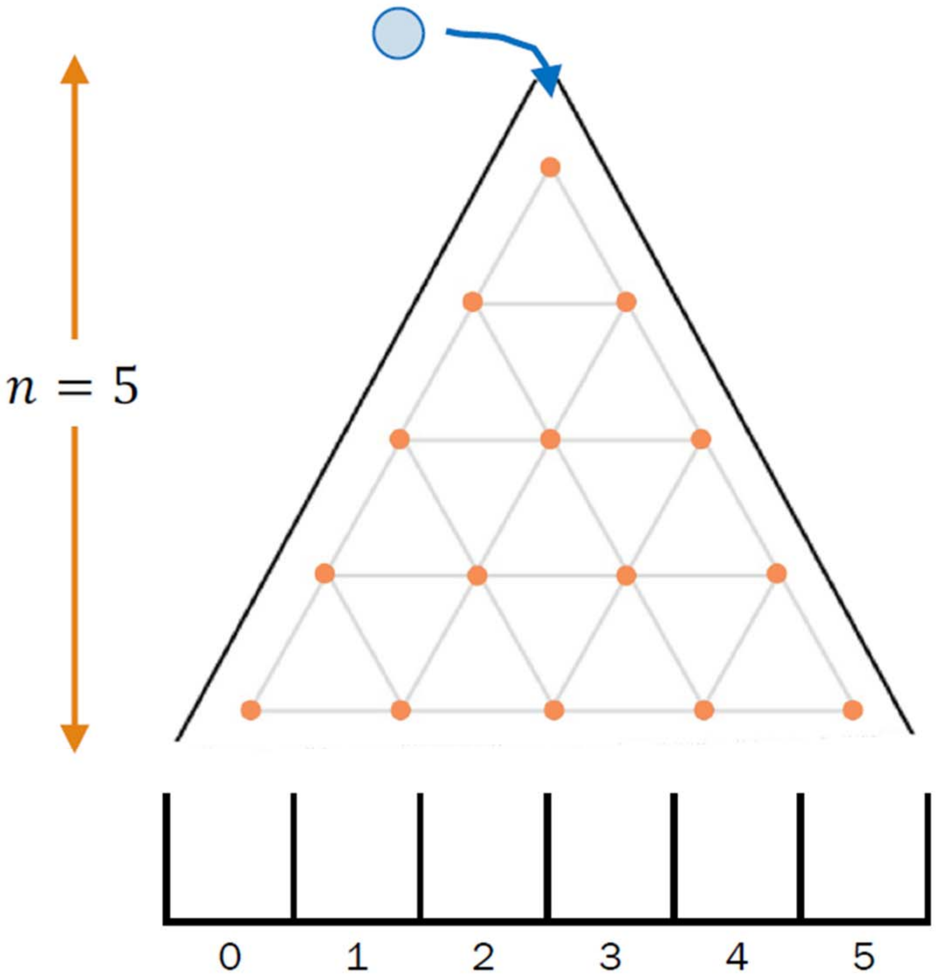
Galton Board



http://web.stanford.edu/class/cs109/demos/galton.html

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

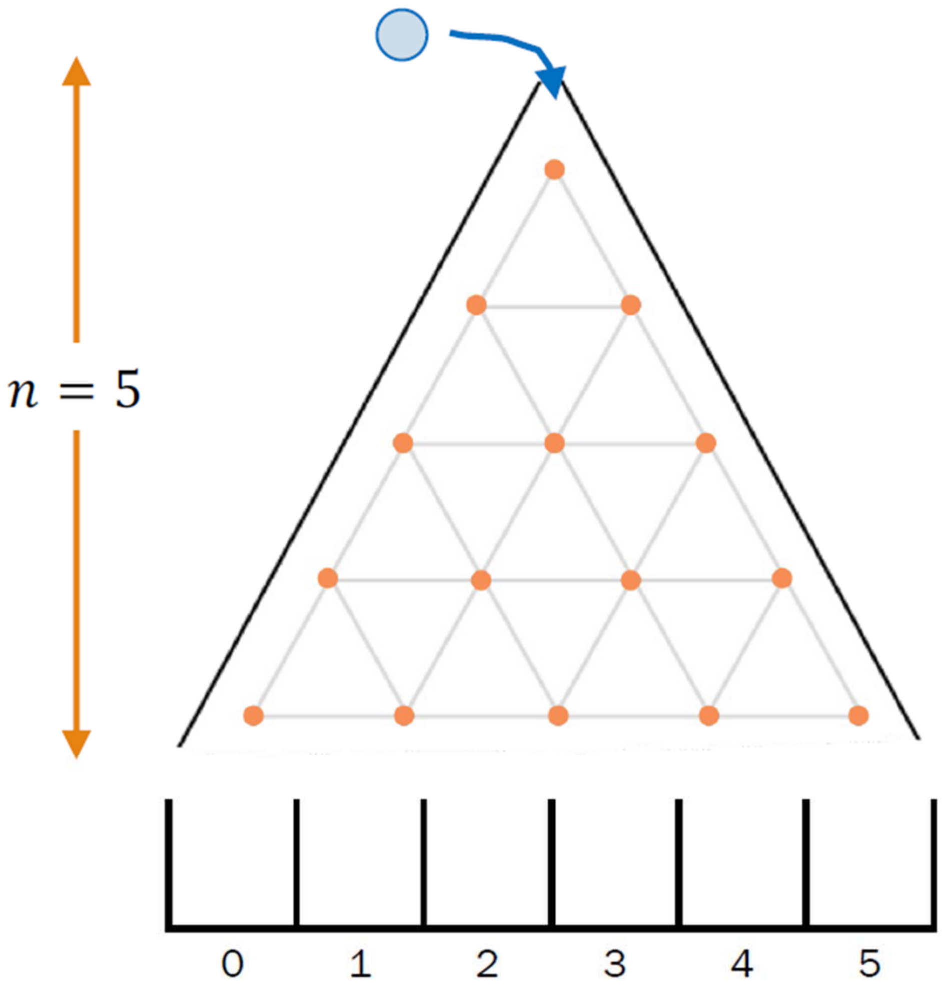


When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
What is the distribution of B?

(Interpret: If B is a common random variable, report it, otherwise report PMF)

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



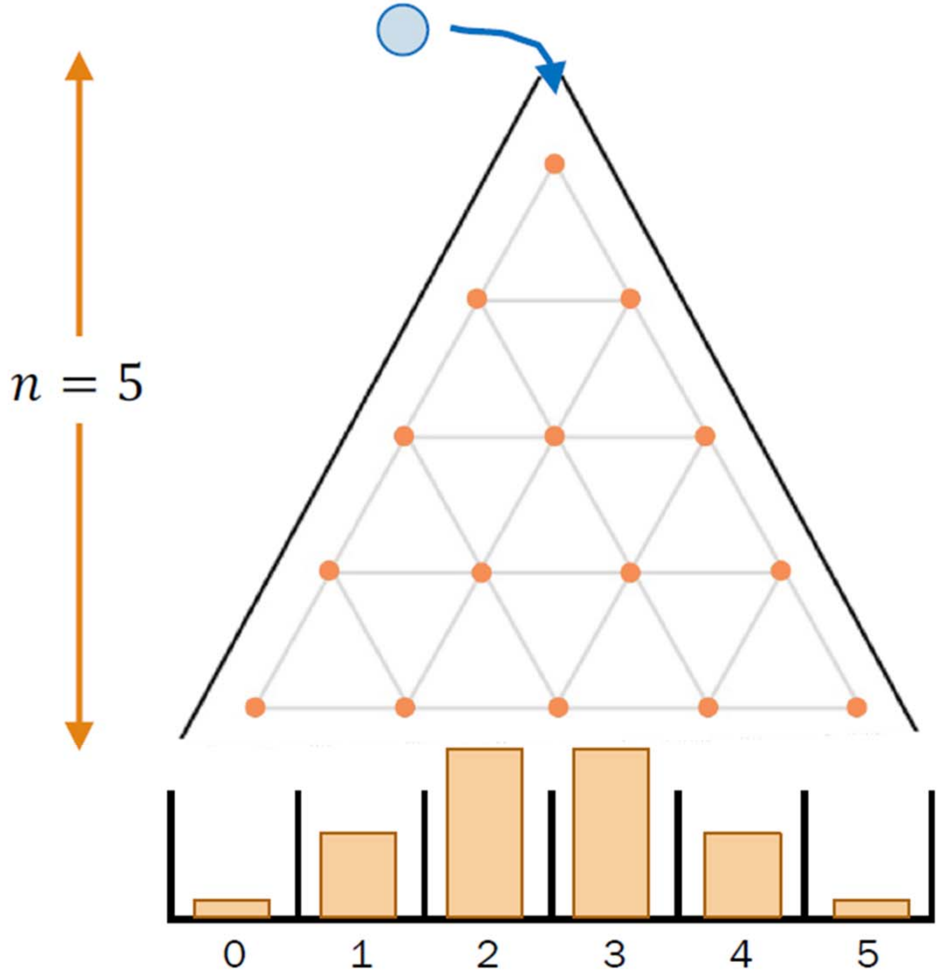
When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
What is the **distribution** of B?

- Each pin is an independent trial
- One decision made for level $i = 1, 2, .., 5$
- Consider a Bernoulli RV with success R_i if ball went right on level i
- Bucket index $B =$ # times ball went right

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
B is distributed as a Binomial RV,

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Calculate the probability of a ball landing in bucket k.

$$P(B = 0) = \binom{5}{0} 0.5^5 \approx 0.03$$

$$P(B = 1) = \binom{5}{1} 0.5^5 \approx 0.16$$

$$P(B = 2) = \binom{5}{2} 0.5^5 \approx 0.31$$

PMF of Binomial RV!

Genetic inheritance

1. Each parent has 2 genes per trait (e.g., eye color).

- Child inherits 1 gene from each parent with equal likelihood.
- Brown eyes are "dominant", blue eyes are "recessive":
  - Child has brown eyes if either or both genes for brown eyes are inherited.
  - Child has blue eyes otherwise (i.e., child inherits two genes for blue eyes)
- Assume parents each have 1 gene for blue eyes and 1 gene for brown eyes.

Two parents have 4 children. What is P(exactly 3 children have brown eyes)?

Big Q: Fixed parameter or random variable?

Parameters What is **common** among all outcomes of our experiment?

$n=4$, $p_{Brown} = 0.75$

Random variable What **differentiates** our event from the rest of the sample space?

$X = \{0,1,2,3,4\}$

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Two parents have 4 children. What is P(exactly 3 children have brown eyes)?

1. Define events/ RVs & state goal

X: # brown-eyed children,

$X \sim \text{Bin}(4, p)$, with $p = 0.75$

p: $P(\text{brown}-\text{eyed child})$

Want: $P(X = 3)$

2. Identify known probabilities

$p = 0.75$

	R	L
R	.25	.25
L	.25	.25

3. Solve

brown = R
blue = L

$$P(X=3) = \binom{4}{3}(0.75)^3(0.25)$$
$$= 0.4219$$

RRRL RLRR
RRLR LRRR

NBA Finals

Let's speculate: the Boston Celtics will play the Denver Nuggets in a 7-game series during the 2023 NBA finals.

- The Celtics have a probability of 58% of winning each game, independently.
- A team wins the series if they win at least 4 games (we play all 7 games).

What is P(Celtics winning)?

1. Define events/ RVs & state goal

X: # games Celtics win
$X \sim \text{Bin}(7, 0.58)$

Want:

Big Q: Fixed parameter or random variable?

Parameters	# of total games prob Celtics winning a game
Random variable	# of games Boston Celtics win
Event based on RV	

NBA Finals

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- A team wins the series if they win at least 4 games (we play all 7 games).

What is P(Celtics winning)?

1. Define events/ RVs & state goal

X: # games Celtics win
$X \sim \text{Bin}(7, 0.58)$

Want: $P(X \geq 4)$

2. Solve

$$P(X \geq 4) = \sum_{k=4}^{7} P(X = k) = \sum_{k=4}^{7} \binom{7}{k} 0.58^k (0.42)^{7-k}$$

$\approx .6706$

Cool Algebra/Probability Fact: this is identical to the probability of winning if we define winning = first to win 4 games