Table of Contents

09: Continuous RVs

Jerry Cain

January 29, 2024

Lecture Discussion on Ed

Continuous RVs

People heights

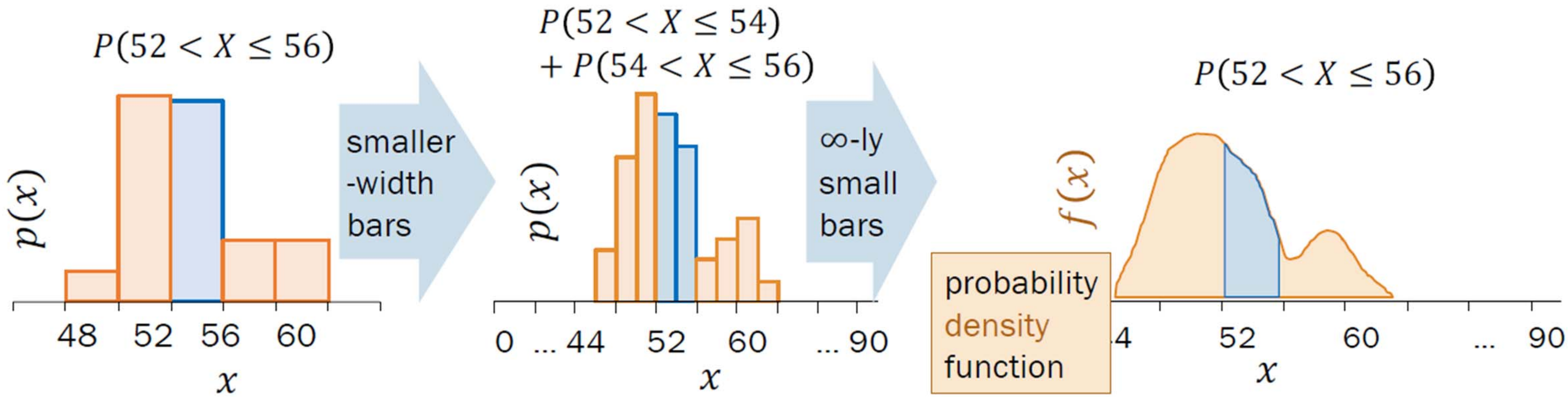
You are volunteering at the local elementary school.

- To buy a t-shirt for your friend Chendi, you need to know her height.

1. What is the probability that your friend is 54.0923857234 inches tall?

Essentially 0

2. What is the probability Chendi is between 52–56 inches tall?



Continuous RV definition

A random variable X is continuous if there is a **probability density function** $f(x) \geq 0$ such that for $-\infty < x < \infty$:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

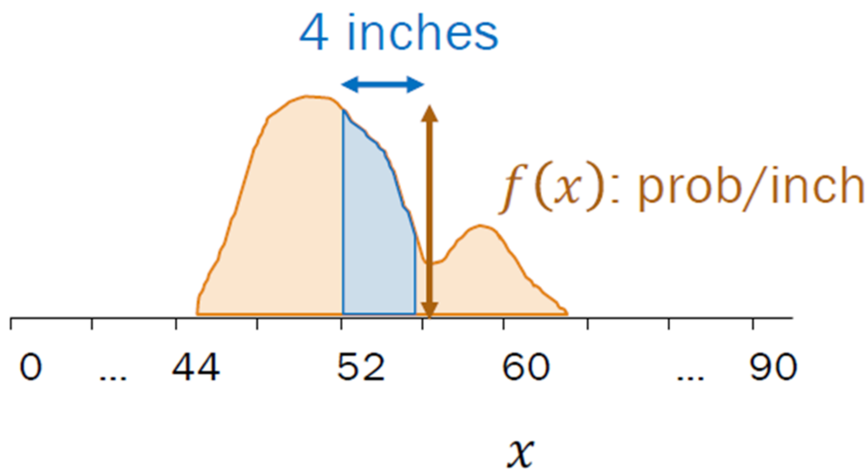
Integrating a PDF must always yield a valid probability, no matter the values of a and b. The PDF must also satisfy:

$$\int_{-\infty}^{\infty} f(x)\, dx = P(-\infty < X < \infty) = 1$$

Note: $f(x)$ is sometimes written as $f_X(x)$ to be clear the random variable is X.

Main takeaway #1

Integrate $f(x)$ to get probabilities.



PDF Units: probability per units of X

$$P(52 \le X \le 56) = \int_{52}^{56} f(x)\, dx$$

PMF vs PDF

Discrete random variable X

Probability mass function (PMF):

$$p(x)$$

To get probability:

$$P(X = x) = p(x)$$

Continuous random variable X

Probability density function (PDF):

$$f(x)$$

To get probability:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

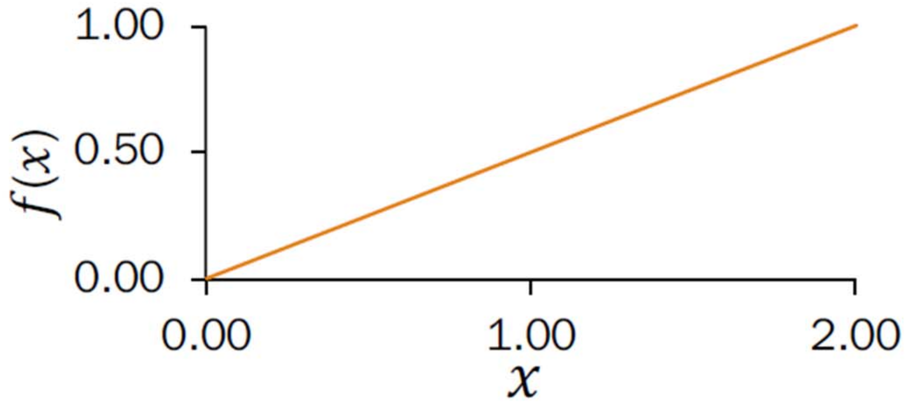
Both are measures of how likely X is to take on a value or some range of values.

Computing probability

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Let X be a continuous RV with PDF:

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



What is $P(X \geq 1)$?

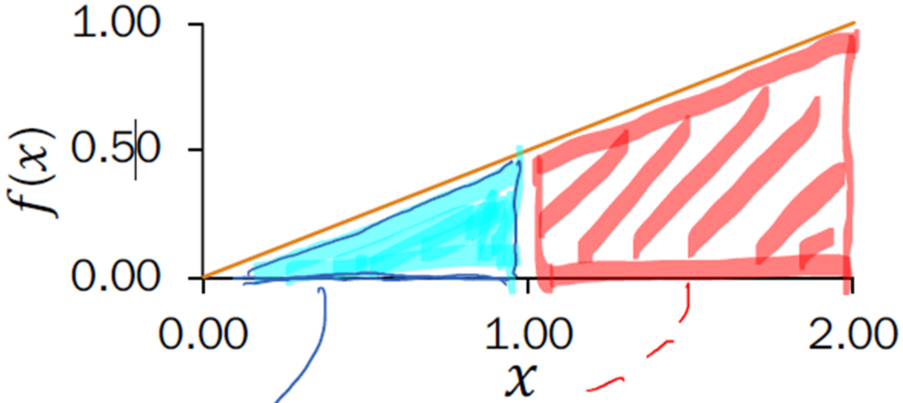
Computing probability

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Let X be a continuous RV with PDF:

confirm: $\int_0^2 \frac{x}{2} dx = 1$, so valid pdf

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



What is $P(X \ge 1)$?

Strategy 1: Integrate

$$P(1 \le X < \infty) = \int_1^\infty f(x)dx = \int_1^2 \frac{1}{2} x dx$$

$$= \frac{1}{2}\left(\frac{1}{2}x^2\right)\Big|_1^2 = \frac{1}{2}\left[2 - \frac{1}{2}\right] = \frac{3}{4}$$

Strategy 2: Know triangles

$$1 - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

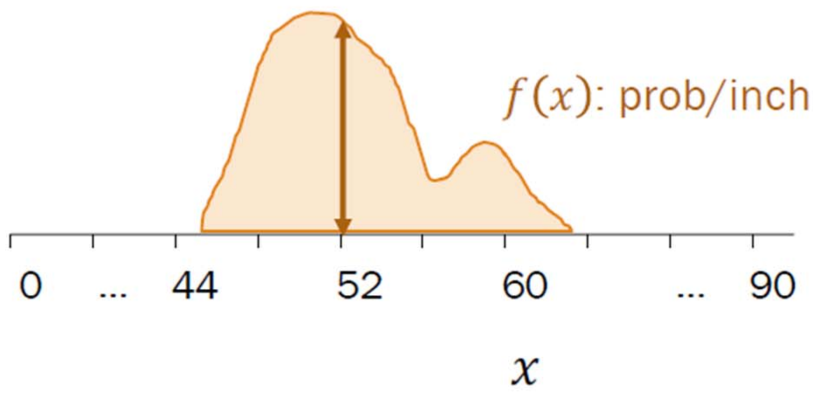
Wait! Is this even legal?

$$P(0 \le X < 1) = \int_0^1 f(x)dx\,??$$

Main takeaway #2

For a continuous random variable X with PDF $f(x)$,

$$P(X = c) = \int_c^c f(x)dx = 0.$$

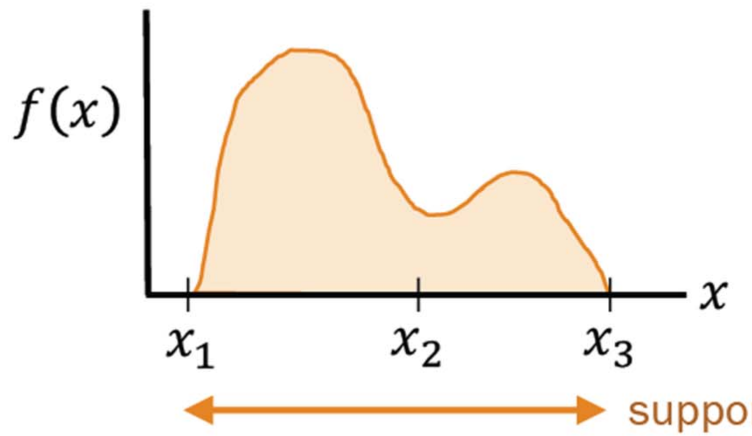


Contrast with PMF in discrete case: $P(X = c) = p(c)$

PDF Properties

For a continuous RV X with PDF f,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$



True/False:

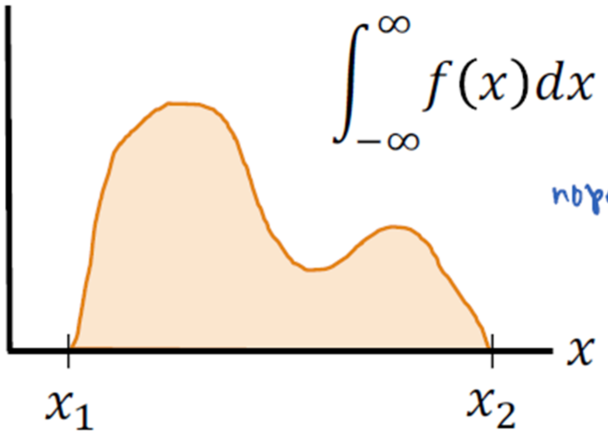
1. $P(X = c) = 0$ — Interval width $dx \to 0$
2. $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$
3. $f(x)$ is a probability — It's a probability density!
4. In the graphed PDF above, $P(x_1 \leq X \leq x_2) > P(x_2 \leq X \leq x_3)$ — Compare area under the curve

Determining valid PDFs

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Which of the following functions are valid PDFs?

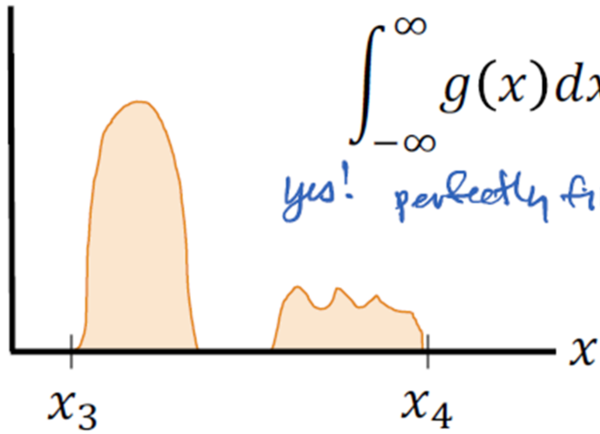
1. $f(x)$



$$\int_{-\infty}^{\infty} f(x)dx = 0.5$$

nope! this integral needs to be 1.0, not 0.5

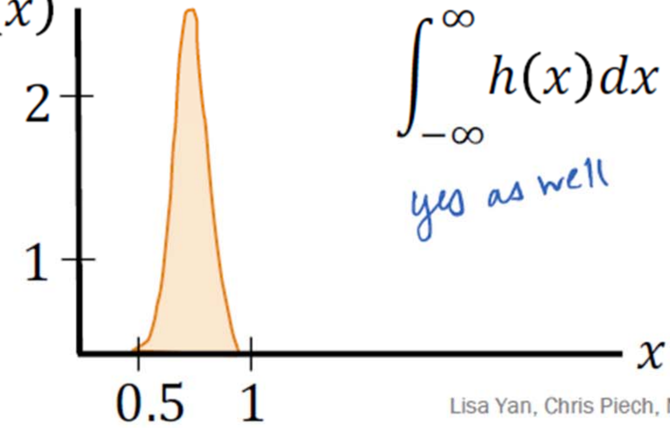
2. $g(x)$



$$\int_{-\infty}^{\infty} g(x)dx = 1$$

yes! perfectly fine

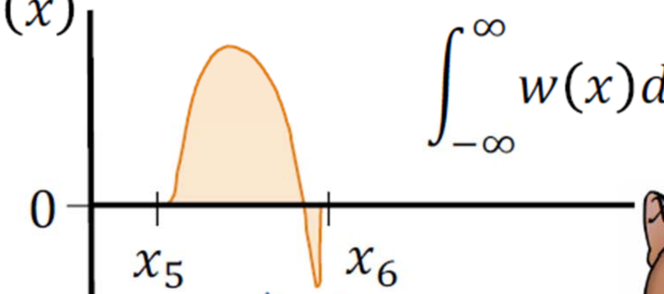
3. $h(x)$



$$\int_{-\infty}^{\infty} h(x)dx = 1$$

yes as well

4. $w(x)$



$$\int_{-\infty}^{\infty} w(x)dx = 1$$

no! f(x) >0 for valid pdf's

Uniform RV

Uniform Random Variable

def A **Uniform** random variable X is defined as follows:

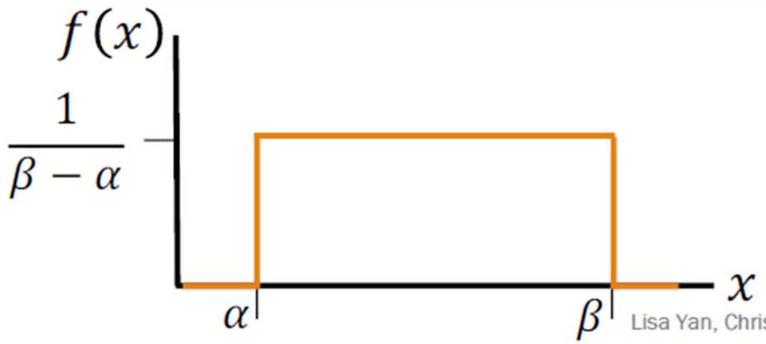
$$X \sim \text{Uni}(\alpha, \beta)$$

Support: $[\alpha, \beta]$
(sometimes defined over (α, β))

PDF $$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Expectation $$E[X] = \frac{\alpha + \beta}{2}$$

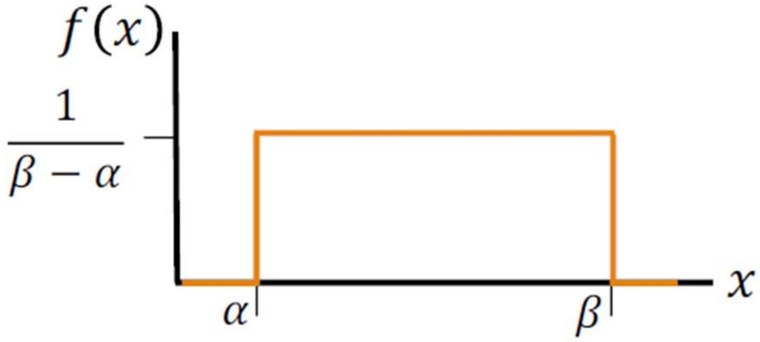
Variance $$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$



Quick check

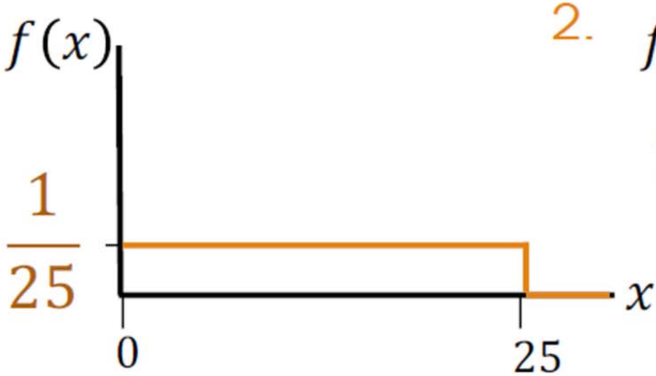
If $X \sim \text{Uni}(\alpha, \beta)$, the PDF of X is:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

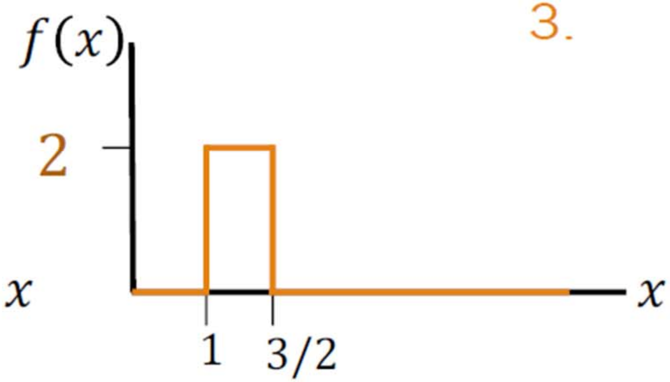


What is $\frac{1}{\beta-\alpha}$ if the following graphs are PDFs of Uniform RVs X?

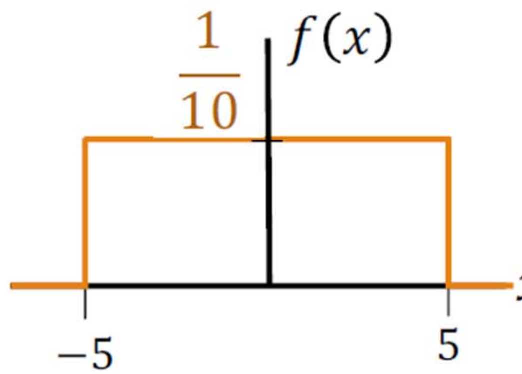
1.



2.



3.



Expectation and Variance

<u>Discrete</u> RV X

$$E[X] = \sum_x x\, p(x)$$

$$E[g(X)] = \sum_x g(x)\, p(x)$$

<u>Continuous</u> RV X

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

<u>Both</u> continuous and discrete RVs

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Linearity of Expectation

Properties of variance

TL;DR: $\sum_{x=a}^{b} \Rightarrow \int_a^b$

Uniform RV expectation

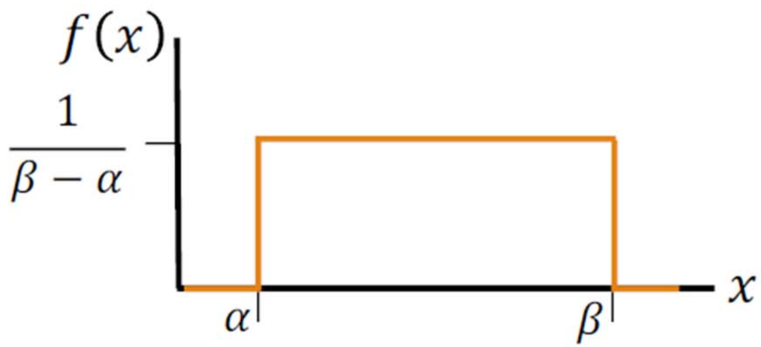
$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

$$= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} dx$$

$$= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} x^2 \Big|_{\alpha}^{\beta}$$

$$= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} (\beta^2 - \alpha^2)$$

$$= \frac{1}{2} \cdot \frac{(\beta + \alpha)(\beta - \alpha)}{\beta - \alpha} = \frac{\alpha + \beta}{2}$$



Interpretation:
Average the start & end

Uniform Random Variable

def An **Uniform** random variable X is defined as follows:

$$X \sim \text{Uni}(\alpha, \beta)$$

Support: $[\alpha, \beta]$ (sometimes defined over (α, β))

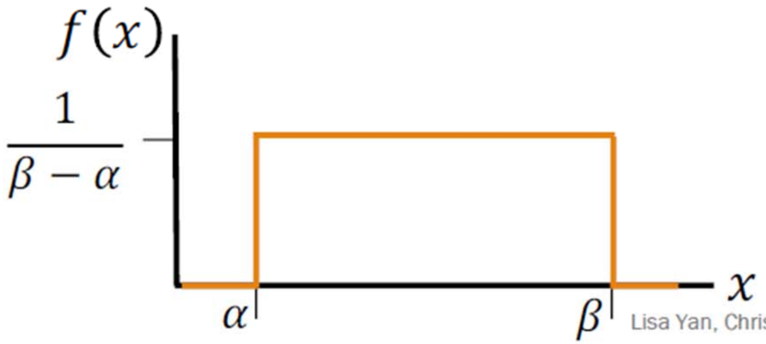
PDF $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$

Expectation $E[X] = \frac{\alpha + \beta}{2}$

Just now

Variance $\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$

On your own time



Exponential RV

Grid of random variables

	Number of successes	Time until success	
One trial	$\text{Ber}(p)$	$\text{Geo}(p)$	One success
	$\uparrow n = 1$	$\uparrow r = 1$	
Several trials	$\text{Bin}(n, p)$	$\text{NegBin}(r, p)$	Several successes
Interval of time	$\text{Poi}(\lambda)$	$\text{Exp}(\lambda)$	Amount of time before first success

Exponential Random Variable

Consider an experiment that lasts a duration of time until success occurs.

def An **Exponential** random variable X is the amount of time until success.

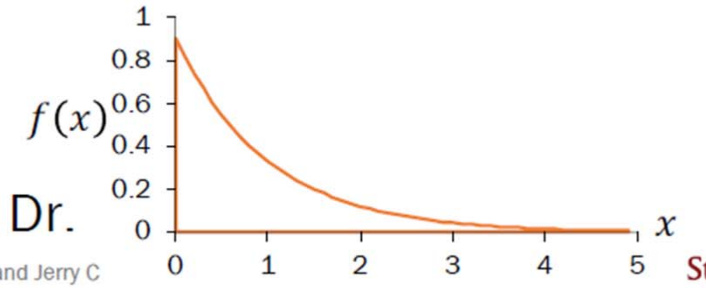
$X \sim \text{Exp}(\lambda)$

Support: $[0, \infty)$

PDF	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
Expectation	$E[X] = \frac{1}{\lambda}$ (in extra slides)
Variance	$\text{Var}(X) = \frac{1}{\lambda^2}$ (on your own)

Examples:

- Time until next earthquake
- Time for request to reach web server
- Time until water main break on Campus Dr.



Interpreting Exp(λ)

def An Exponential random variable X is the amount of time until success.

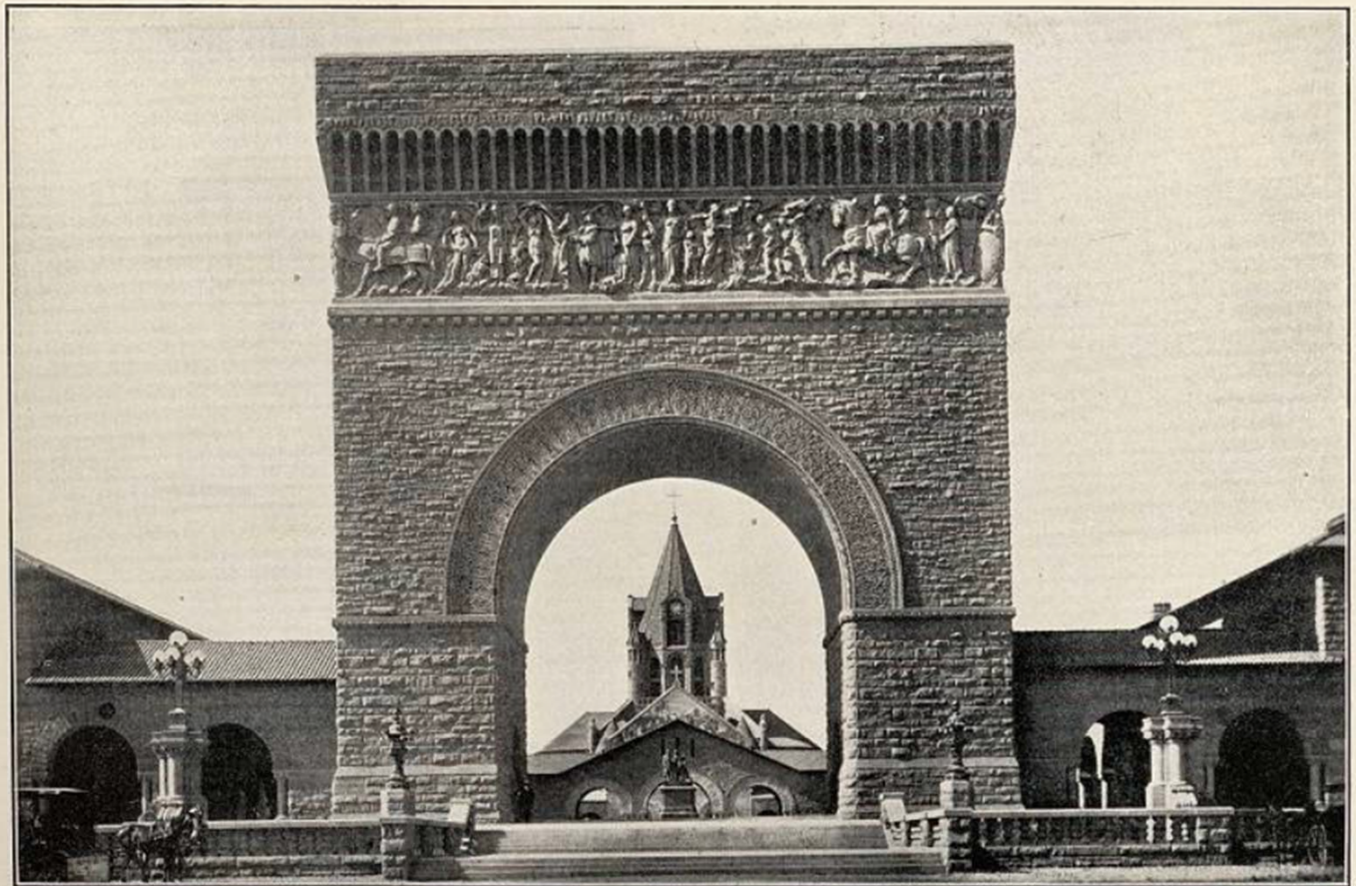
$X \sim \text{Exp}(\lambda)$	Expectation	$E[X] = \frac{1}{\lambda}$

Based on the expectation $E[X]$, what are the units of λ?

e.g., average # of successes per second

For both Poisson and Exponential RVs, λ = # successes/time.

Earthquakes


ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE.

1906 Earthquake
Magnitude 7.8

Earthquakes

$X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$
$f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

Major earthquakes (magnitude 8.0+) occur on average once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?

We know on average:

$$500 \ \frac{\text{years}}{\text{earthquake}}$$

$$0.002 \ \frac{\text{earthquakes}}{\text{year}}$$

if earthquakes are "successes", then $\frac{1}{500} = 0.002$ is our lambda value, since λ is the average number of successes per time unit (in this case, a year).

$$1 \ \frac{\text{earthquakes}}{\text{500 years}} = 1 \ \frac{\text{"success"}}{\text{500 years}}$$

*In California, according to historical data form USGS, 2015

Earthquakes

$X \sim \text{Exp}(\lambda)$ $\quad E[X] = 1/\lambda$ $\quad f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?

Define events/ RVs & state goal

X: when next earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

λ: year^{-1} $= 1/500$

Want: $P(X < 30)$

Solve

$$P(X<30) = \int_0^{30} 0.002\, e^{-0.002x}\, dx$$

$$= 0.002 \frac{-1}{0.002} e^{-0.002x} \Big|_0^{30}$$

$$= -\left(e^{-0.06} - e^{0.00}\right)$$

$$= 1 - e^{-0.06} \approx \boxed{0.058}$$

Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

*In California, according to historical data form USGS, 2015

Earthquakes

$X \sim \text{Exp}(\lambda)$ $\quad E[X] = 1/\lambda$ $\quad f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?
2. What is the standard deviation of years until the next earthquake?

Define events/ RVs & state goal

X: when next earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

λ: year^{-1}

Want: $P(X < 30)$

Solve

$$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(0.002 \text{ year}^{-1})^2} = 250{,}000 \text{ years}^2$$

$$SD(X) = \sqrt{\text{Var}(X)} = \boxed{500 \text{ years}}$$

In general, $SD(X) = E[X] = \frac{1}{\lambda}$

whenever $X \sim \text{Exp}(\lambda)$

*In California, according to historical data form USGS, 2015

Cumulative Distribution Functions

Cumulative Distribution Function (CDF)

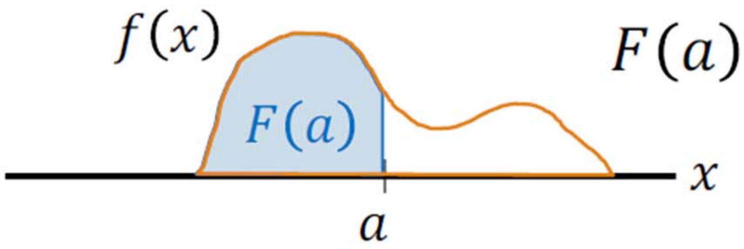
For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{where } -\infty < a < \infty$$

For a discrete RV X, the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

For a continuous RV X, the CDF is:



$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

CDF is a probability, though PDF is not.

If you learn to use CDFs, you can avoid integrating the PDF.

Using the CDF for continuous RVs

For a <u>continuous</u> random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

Matching (choices are used 0/1/2 times)

1. $P(X < a)$
2. $P(X > a)$
3. $P(X \geq a)$
4. $P(a \leq X \leq b)$

A. $F(a)$
B. $1 - F(a)$
C. $F(b) - F(a)$
D. $F(a) - F(b)$

Using the CDF for continuous RVs

For a <u>continuous</u> random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

Matching (choices are used 0/1/2 times)

1. $P(X < a)$
2. $P(X > a)$
3. $P(X \geq a)$
4. $P(a \leq X \leq b)$

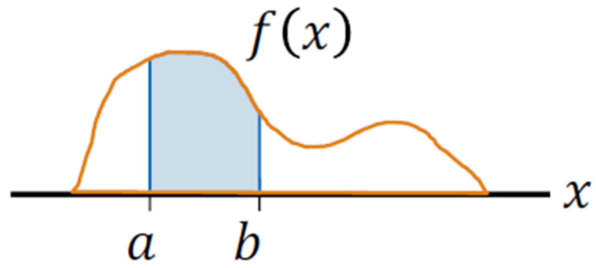
A. $F(a)$
B. $1 - F(a)$
C. $F(b) - F(a)$ (next slide)
D. $F(a) - F(b)$

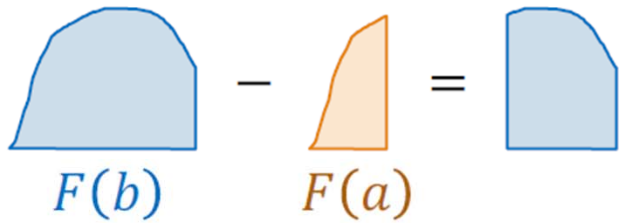
Using the CDF for continuous RVs

For a continuous random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

4. $P(a \le X \le b) = F(b) - F(a)$





Proof:

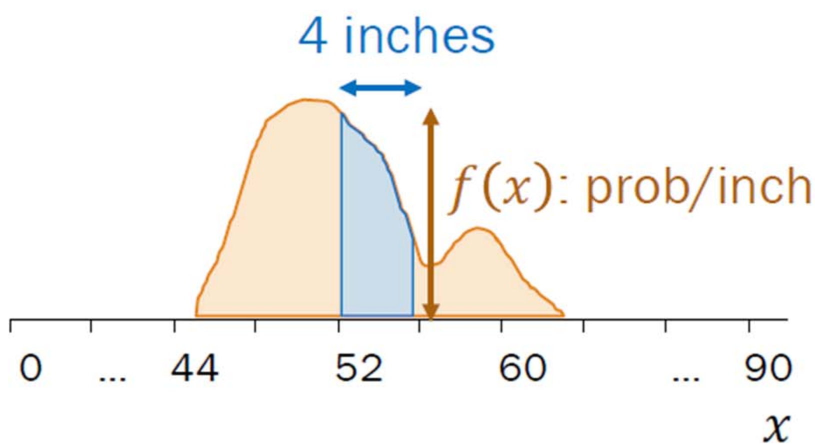
$$F(b) - F(a) = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= \left(\int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx\right) - \int_{-\infty}^{a} f(x)dx$$

$$= \int_{a}^{b} f(x)dx$$

Addendum to main takeaway #1

Integrate $f(x)$ to get probabilities.*



$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

*If you have $F(a)$, you already have probabilities, since $F(a) = \int_{-\infty}^{a} f(x)dx$

CDF of an Exponential RV

$X \sim \text{Exp}(\lambda)$ $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

$$X \sim \text{Exp}(\lambda) \qquad F(x) = 1 - e^{-\lambda x} \text{ if } x \geq 0$$

Proof:

remember that $f(y) = 0$ for $y < 0$

Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

$$F(x) = P(X \leq x) = \int_{y=-\infty}^{x} f(y) dy = \int_{y=0}^{x} \lambda e^{-\lambda y} dy$$

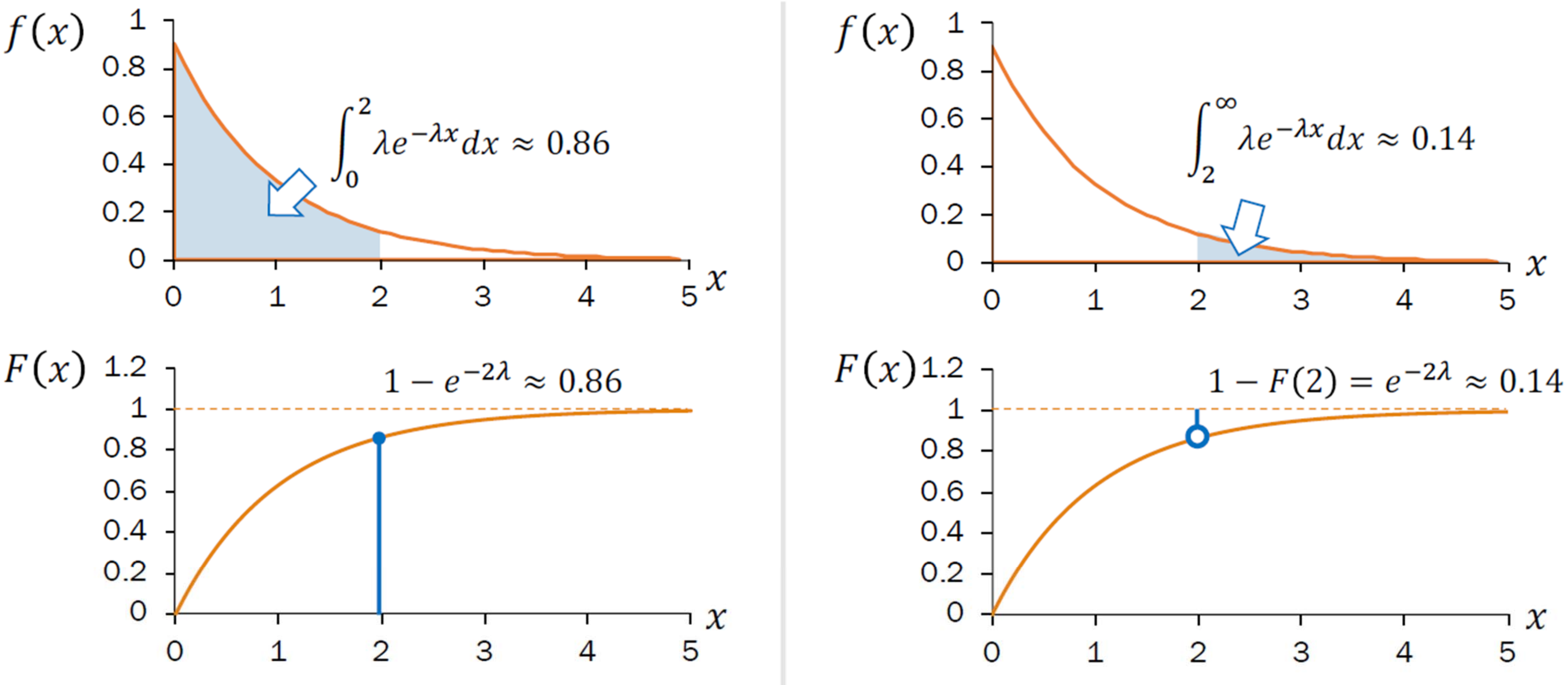
$$= \lambda \frac{1}{-\lambda} e^{-\lambda y} \Big|_0^x$$

$$= -1\left(e^{-\lambda x} - e^{-\lambda 0}\right)$$

$$= 1 - e^{-\lambda x}$$

PDF/CDF $X \sim \text{Exp}(\lambda = 1)$

$X \sim \text{Exp}(\lambda)$ $\quad x \geq 0$: $f(x) = \lambda e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$



$P(X \leq 2)$ $\qquad$ $P(X > 2)$

Memoryless Property

Memorylessness: Hurry Up and Wait

A continuous probability distribution is said to be memoryless if a random variable X on that probability distribution satisfies the following for all $s, t \geq 0$:

$$P(X \geq s + t \mid X \geq s) = P(X \geq t)$$

- Here, s represents the time you've already spent waiting.
- The above states that after you've waited s time units, the probability you'll need to wait an additional t time units is equal to the probability you'd have to wait t time units without having waited those s time units in the first place.
- Example: If train arrival is guided by a memoryless random variable, the fact that you've waited 15 minutes doesn't obligate the train to arrive any faster!

Memorylessness: Hurry Up and Wait

A continuous probability distribution is said to be memoryless if a random variable X on that probability distribution satisfies the following for all $s, t \geq 0$:

$$P(X \geq s + t \mid X \geq s) = P(X \geq t)$$

Using the definition of conditional probability, we can show that our Exponential distribution possesses the memoryless property. Just let $X \sim \text{Exp}(\lambda)$ and trust the math:

$$P(X \geq s + t \mid X \geq s) = \frac{P(X \geq s + t)}{P(X \geq s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t)$$

Exercises

Earthquakes

Major earthquakes (magnitude 8.0+) occur independently on average once every 500 years.*

What is the probability of zero major earthquakes next year?

Strategy 1: Exponential RV

Define events/RVs & state goal

T: when first earthquake happens

$T \sim \text{Exp}(\lambda = 0.002)$

Want: $P(T > 1) = 1 - F(1)$

Solve

$$P(T > 1) = 1 - \left(1 - e^{-\lambda \cdot 1}\right) = e^{-\lambda}$$

$$= e^{-0{,}002} \approx 0{,}998$$

*In California, according to historical data form USGS, 2015

Earthquakes

$Y \sim \text{Poi}(\lambda) \quad p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Major earthquakes (magnitude 8.0+) occur independently on average once every 500 years.*

What is the probability of zero major earthquakes next year?

Strategy 1: Exponential RV

Define events/RVs & state goal

T: when first earthquake happens

$T \sim \text{Exp}(\lambda = 0.002)$

Want: $P(T > 1) = 1 - F(1)$

Solve

$$P(T > 1) = 1 - \left(1 - e^{-\lambda \cdot 1}\right) = e^{-\lambda}$$

Strategy 2: Poisson RV

Define events/RVs & state goal

N: # earthquakes next year

$N \sim \text{Poi}(\lambda = 0.002)$

$\lambda: \frac{\text{earthquakes}}{\text{year}}$

Want: $P(N = 0)$

Solve

$$P(N = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} \approx 0.998$$

*In California, according to historical data form USGS, 2015

Replacing your laptop

$X \sim \text{Exp}(\lambda)$ $\quad E[X] = 1/\lambda$
$F(x) = 1 - e^{-\lambda x}$

Let X = # hours of use until your laptop dies.

- X is distributed as an Exponential RV, where
- On average, laptops die after 5000 hours of use.
- You use your laptop 5 hours a day.

What is P(your laptop lasts 4 years)?

Define

X: # hours until laptop death

$X \sim \text{Exp}(\lambda = 1/5000)$

Want: $P(X > 5 \cdot 365 \cdot 4)$

Solve

$$P(X > 7300) = 1 - F(7300)$$
$$= 1 - \left(1 - e^{-7300/5000}\right) = e^{-1.46} \approx 0.2322$$

Better plan ahead if you're co-terming!

- 5-year plan: $P(X > 9125) = e^{-1.825} \approx 0.1612$
- 6-year plan: $P(X > 10950) = e^{-2.19} \approx 0.1119$

Extra

Expectation of the Exponential

$X \sim \text{Exp}(\lambda)$ $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

$X \sim \text{Exp}(\lambda)$ Expectation $E[X] = \dfrac{1}{\lambda}$

Proof:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty}$$

$$= [0 - 0] + \left[0 - \left(\frac{-1}{\lambda}\right)\right]$$

$$= \frac{1}{\lambda}$$

Integration by parts

$$\int x \lambda e^{-\lambda x} dx = \int u \cdot dv$$

$u = x$ $\quad dv = \lambda e^{-\lambda x} dx$

$du = dx$ $\quad v = -e^{-\lambda x}$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$-x e^{-\lambda x} - \int -e^{-\lambda x} dx$$