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# 17: Continuous Joint Distributions II

Jacob Mejia, Kanu Grover
February 16, 2024

Lecture Discussion on Ed

# Convolution: Sum of independent Uniform RVs

# Today's lecture

Take what we've seen in discrete joint distributions...

...and translate them to continuous joint distributions!

For the most part, this is easy. For example:

| | | |
|---|---|---|
| Marginal distributions | $p_X(a) = \sum_y p_{X,Y}(a, y)$ | $f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)dy$ |
| Independent RVs | $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ | $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ |

But some concepts, while mathematically accessible, are difficult to implement in practice.

We'll focus on some of these today.

Goal of CS109 continuous joint distributions unit:
**build mathematical maturity**

# Dance, Dance, Convolution

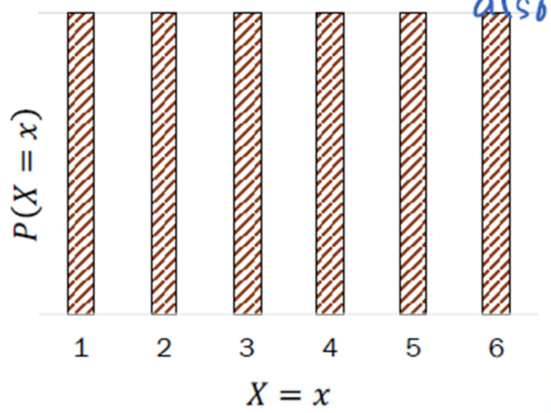
Review

Recall that for independent discrete random variables $X$ and $Y$:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of $p_X$ and $p_Y$

also written as $p_{X+Y}(n) = \sum_k p_X(k)\, p_Y(n-k)$ as well




Independent $X, Y$

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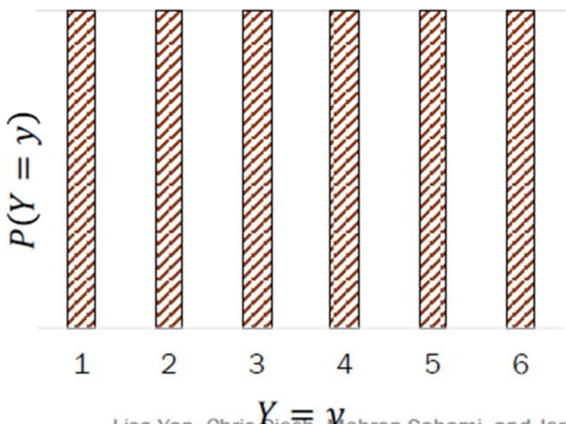


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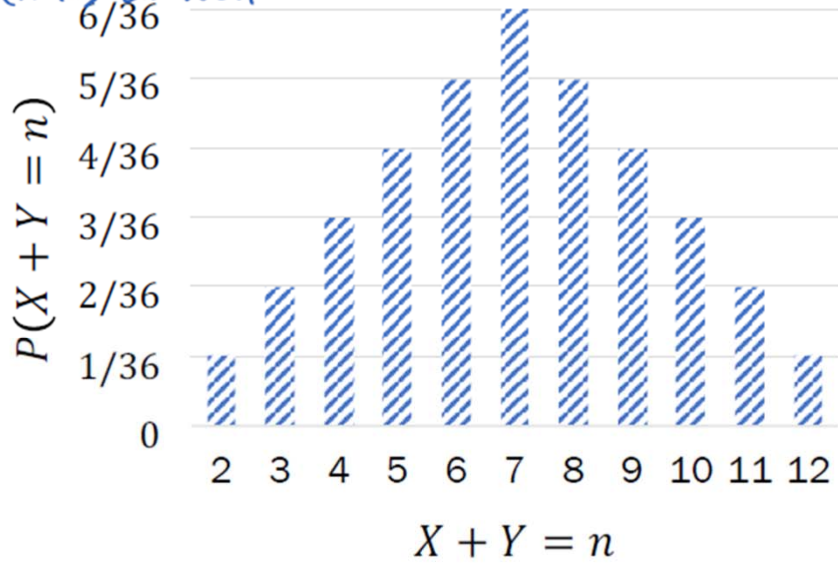


# Dance, Dance, Convolution

Recall that for independent discrete random variables $X$ and $Y$:

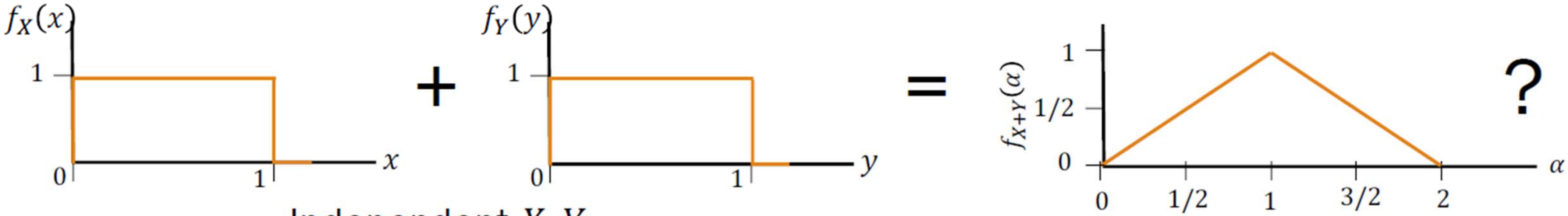
$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of $p_X$ and $p_Y$

For independent continuous random variables $X$ and $Y$:

$$f_{X+Y}(\alpha)=\int_{-\infty}^{\infty} f_X(x) f_Y(\alpha-x)\,dx$$

the convolution of $f_X$ and $f_Y$

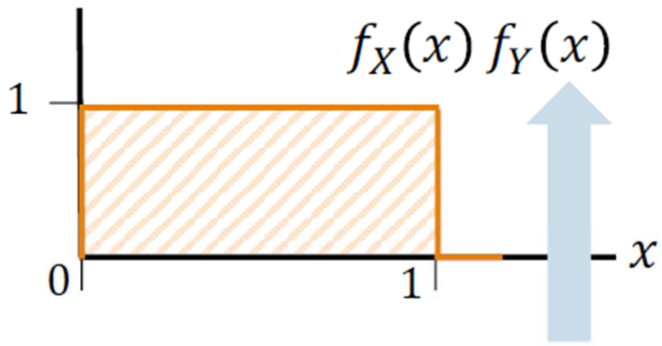



Independent $X, Y$

# Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.
What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$




Not so fast...

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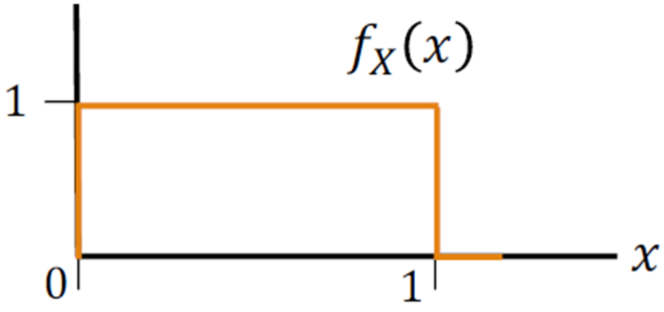


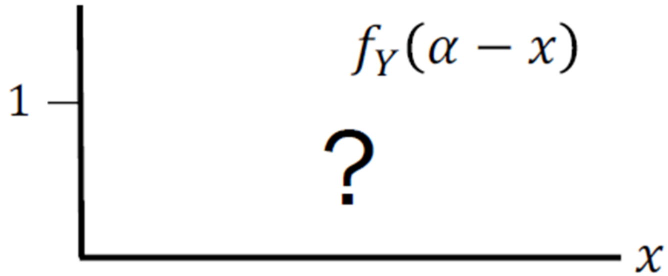


$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } 0 \le \alpha - x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

① subtract $\alpha$ from everything
$-\alpha \le -x \le 1 - \alpha$
② then divide by $-1$

$$= \begin{cases} 1 & \text{if } \alpha - 1 \le x \le \alpha \\ 0 & \text{otherwise} \end{cases}$$

$\alpha$ is a constant in the integral w.r.t. $x$.

# Sum of independent Uniforms

$X$ and $Y$ independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.
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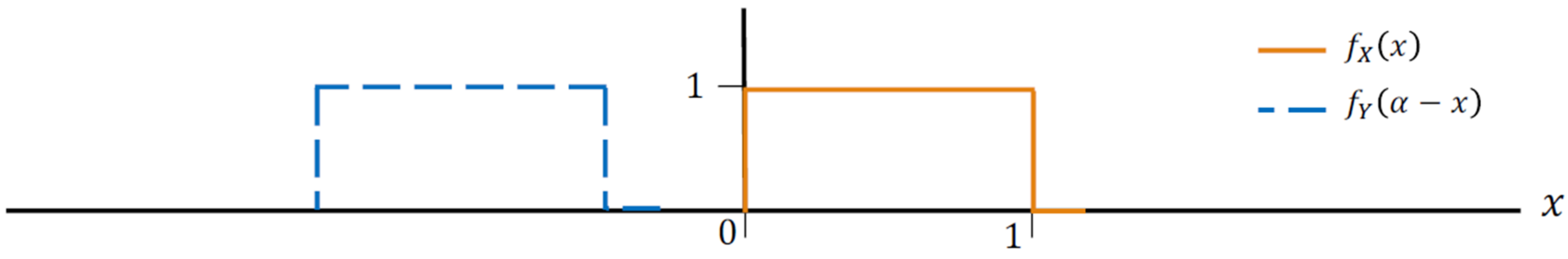

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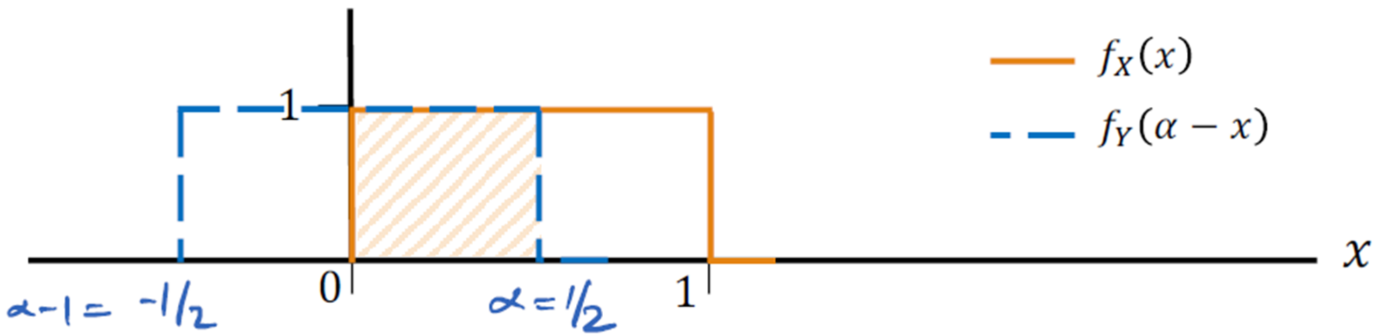


Integral = area under the curve
This curve = product of 2 functions of $x$

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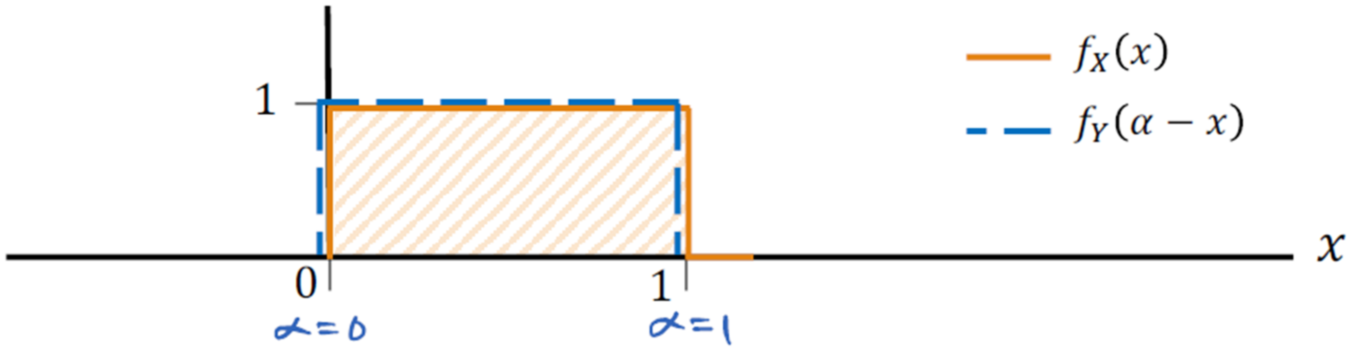

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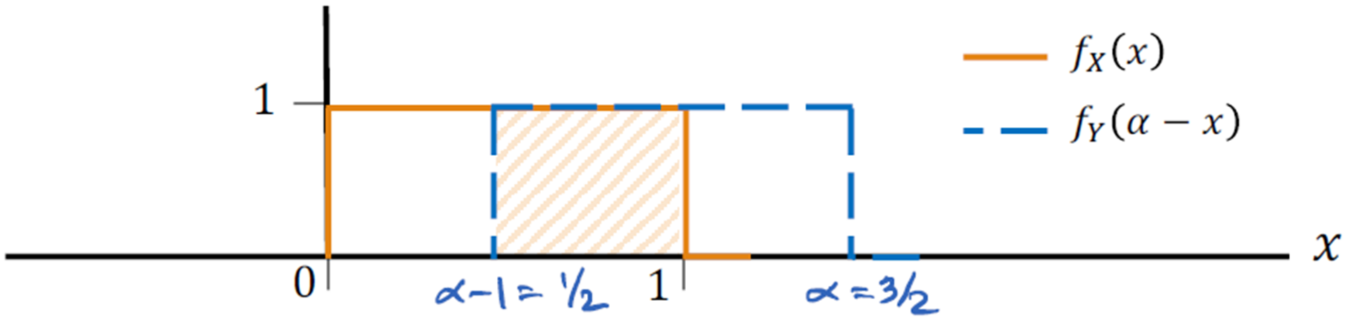

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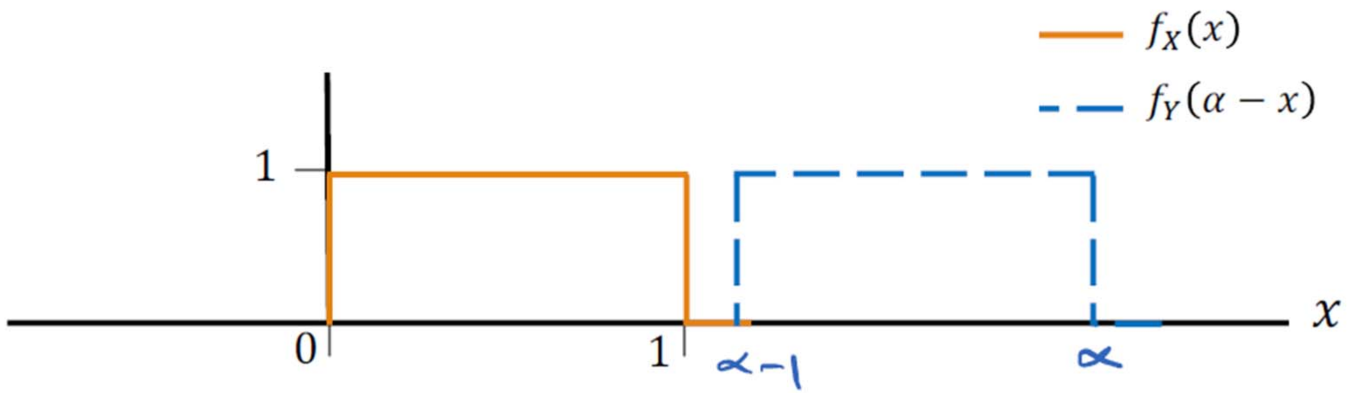

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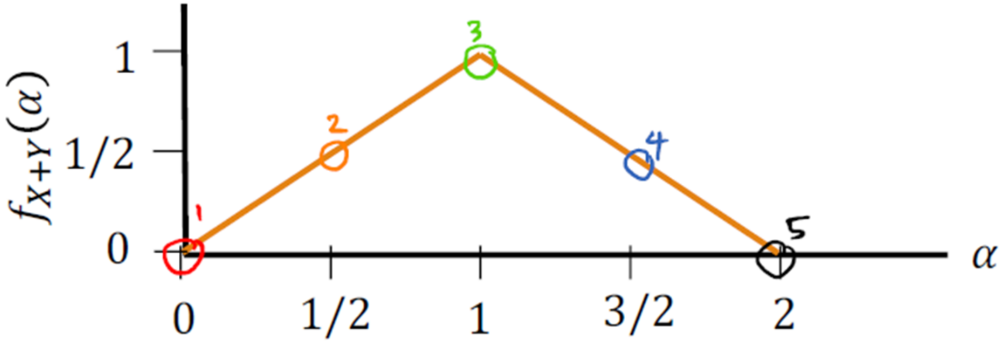
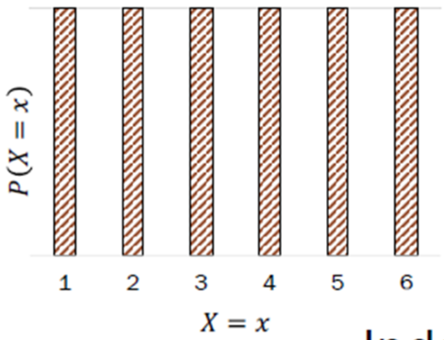



$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \leq \alpha \leq 1 \\ 2 - \alpha & 1 \leq \alpha \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

# Dance, Dance, Convolution Extreme

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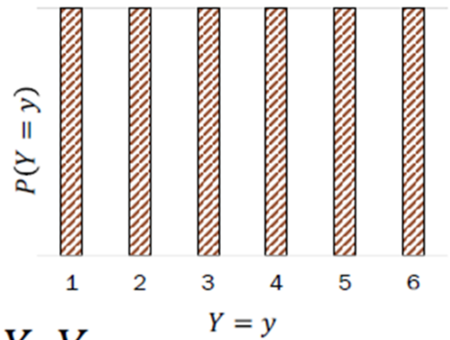


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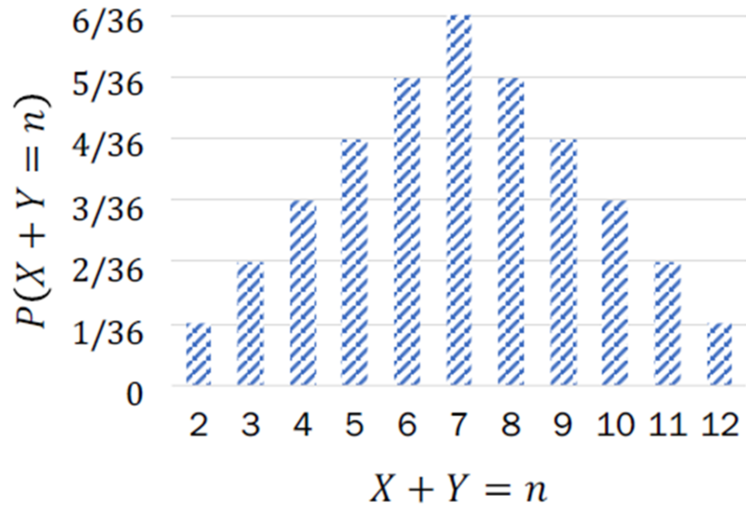
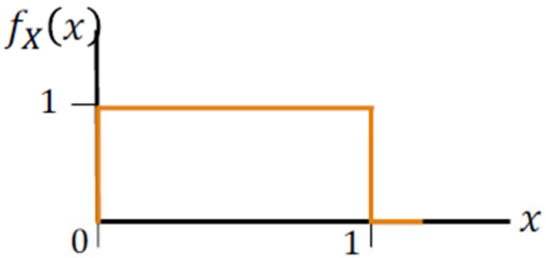



Independent $X, Y$

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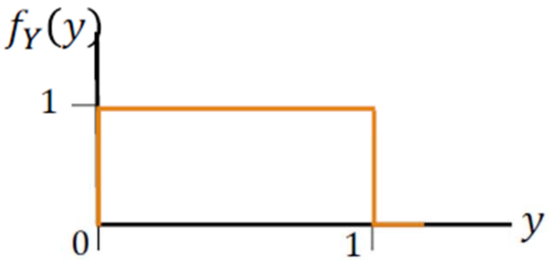


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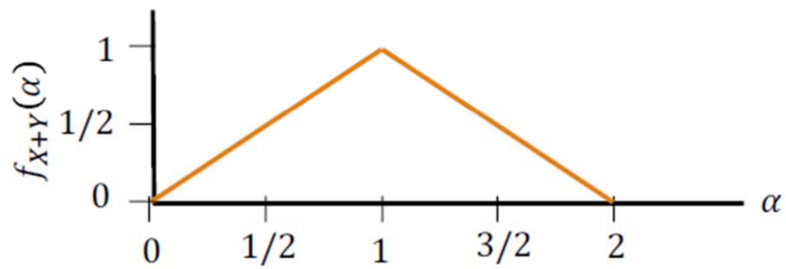



Independent $X, Y$

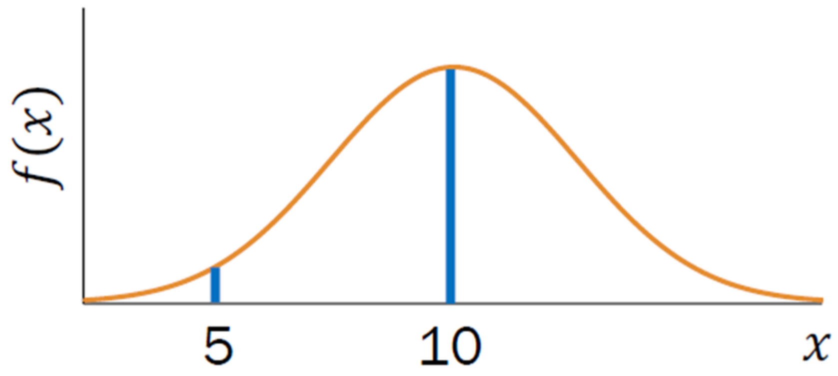
# Ratio of PDFs

# Relative probabilities of continuous random variables

Let $X$ = time to finish problem set 4.

Suppose $X \sim \mathcal{N}(10,2)$.

How much more likely are you to complete in 10 hours than 5 hours?




$$\frac{P(X = 10)}{P(X = 5)} =$$

A. 0/0 = undefined

B. 2

C. $\frac{f(10)}{f(5)}$

D. $\frac{f(2)}{f(1)}$

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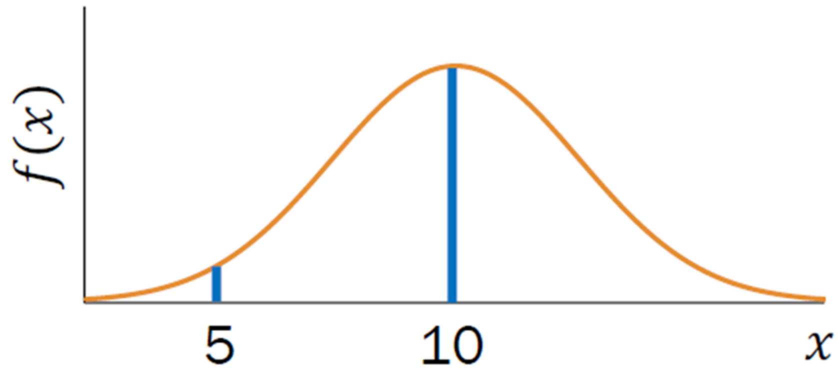


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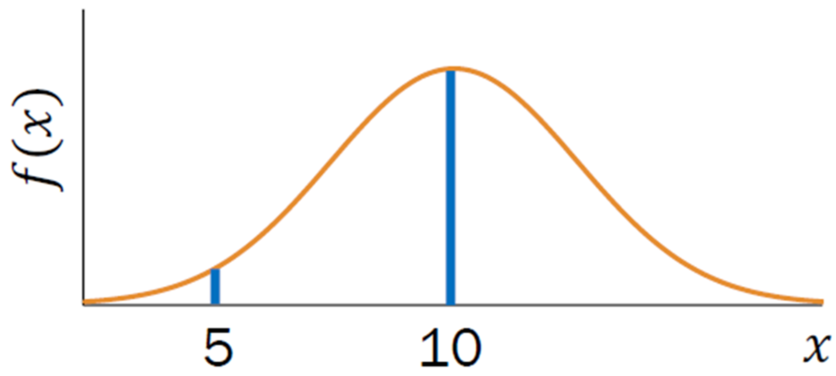



$$\frac{P(X=10)}{P(X=5)} = \boxed{\frac{f(10)}{f(5)}}$$

$$P(X=a) = P\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \approx \varepsilon f(a)$$

Therefore $\frac{P(X=a)}{P(X=b)} = \frac{\varepsilon f(a)}{\varepsilon f(b)} = \frac{f(a)}{f(b)}$

really this: $\lim_{\varepsilon \to 0} \frac{\varepsilon f(a)}{\varepsilon f(b)}$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}} = \frac{e^{-\frac{(10-10)^2}{2\cdot 2}}}{e^{-\frac{(5-10)^2}{2\cdot 2}}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Ratios of PDFs are meaningful!

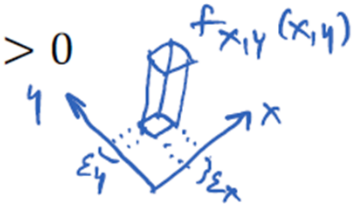
# Continuous conditional distributions

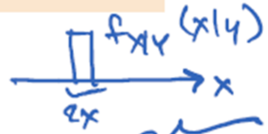
# Continuous conditional distributions

For continuous RVs $X$ and $Y$, the conditional PDF of $X$ given $Y$ is

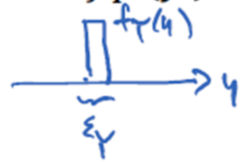
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

where $f_Y(y) > 0$




Intuition: $P(X = x|Y = y) = \dfrac{P(X = x, Y = y)}{P(Y = y)}$ $\Longleftrightarrow$ $f_{X|Y}(x|y)\varepsilon_X = \dfrac{f_{X,Y}(x,y)\varepsilon_X\varepsilon_Y}{f_Y(y)\varepsilon_Y}$




Note that conditional PDF $f_{X|Y}$ is a "true" density:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

analogue to discrete equivalent:

$$\sum_x p_{X|Y}(x|y) = 1$$

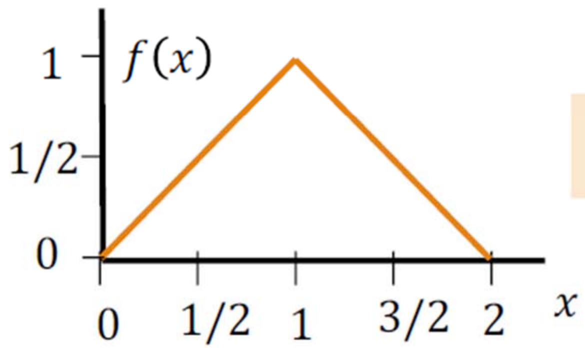
# Why sums of random variables?

Sometimes modeling and understanding a complex RV, $X$, is difficult.

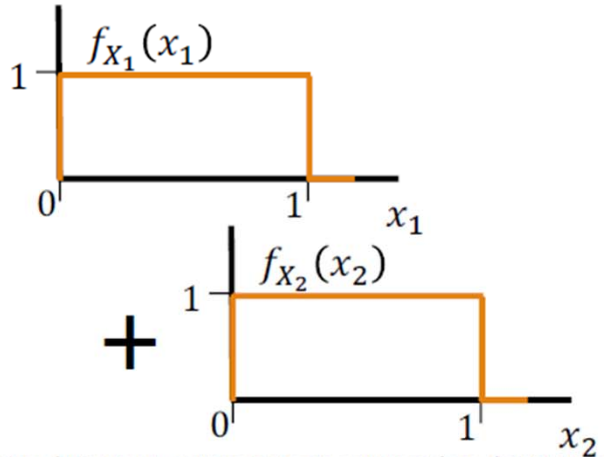
But if we can decompose $X$ into the sum of independent simpler RVs,

- We can then compute distributions on $X$.
- We can then to understand how $X$ changes as its parts change.

What can we model with a triangular PDF?




Sum of uniforms!




We're covering the reverse direction for now; the forward direction will come on Friday

# Everything* in probability is a sum or a product (or both)

*except conditional probability (a ratio)

**Sum** of values that can be considered separately (possibly weighted by prob. of happening)

$$E[X] = \sum_x x\underbrace{p(x)}_{\text{weight}}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \underbrace{f_{X|Y}(x|y)}_{\text{weight}} dx$$

$$P(E) = \sum_{i=1}^{n} P(E|F_i)\underbrace{P(F_i)}_{\text{weight}}$$

Law of Total Probability

$$P(E) = \sum_{i=1}^{n} P(E_i)$$

Axiom 3, $E = E_1 \cup \cdots \cup E_n$

$E_i \cap E_j = \phi \quad i \neq j$

**Product** of values that can each be considered in sequence

$$P(E \cap F \cap G) = P(E)P(F|E)P(G|EF)$$

Chain Rule

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Independent cont. RVs

$$P(X+Y=n) = \sum_k P(X=k)P(Y=n-k)$$

Sum of indep. discrete RVs (convolution)

# Conditional probability and Bayes' Theorem

Definition

$$P(F|E) = \frac{P(E \cap F)}{\underbrace{P(E)}_{\text{Scaling to the correct sample space}}}$$

Independence

$E, F$ independent

$$P(F|E) = \underbrace{P(F)}_{\text{Sample space doesn't need to be scaled}}$$

Bayes' Theorem

Prior: some prob. of event $F$

Likelihood

$$P(F|E) = \frac{P(F)P(E|F)}{\underbrace{P(E)}_{\text{Scaling to the correct sample space}}}$$

Posterior: prob. of $F$ knowing that $E$ happened

# Multiple Bayes' Theorems

with
events

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$

with
discrete RVs

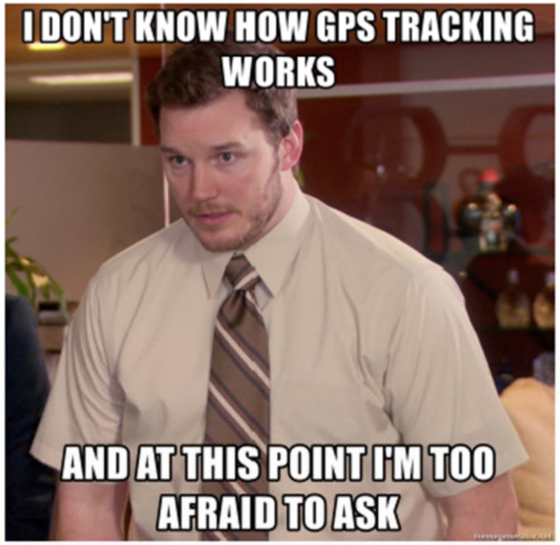
$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)}$$

with
continuous RVs

$$f_{Y|X}(y|\overbrace{x}^{\text{You are given this value...}}) = \frac{f_Y(y)f_{X|Y}(x|y)}{\underbrace{f_X(x)}_{\text{...so this is just a scalar}}}$$

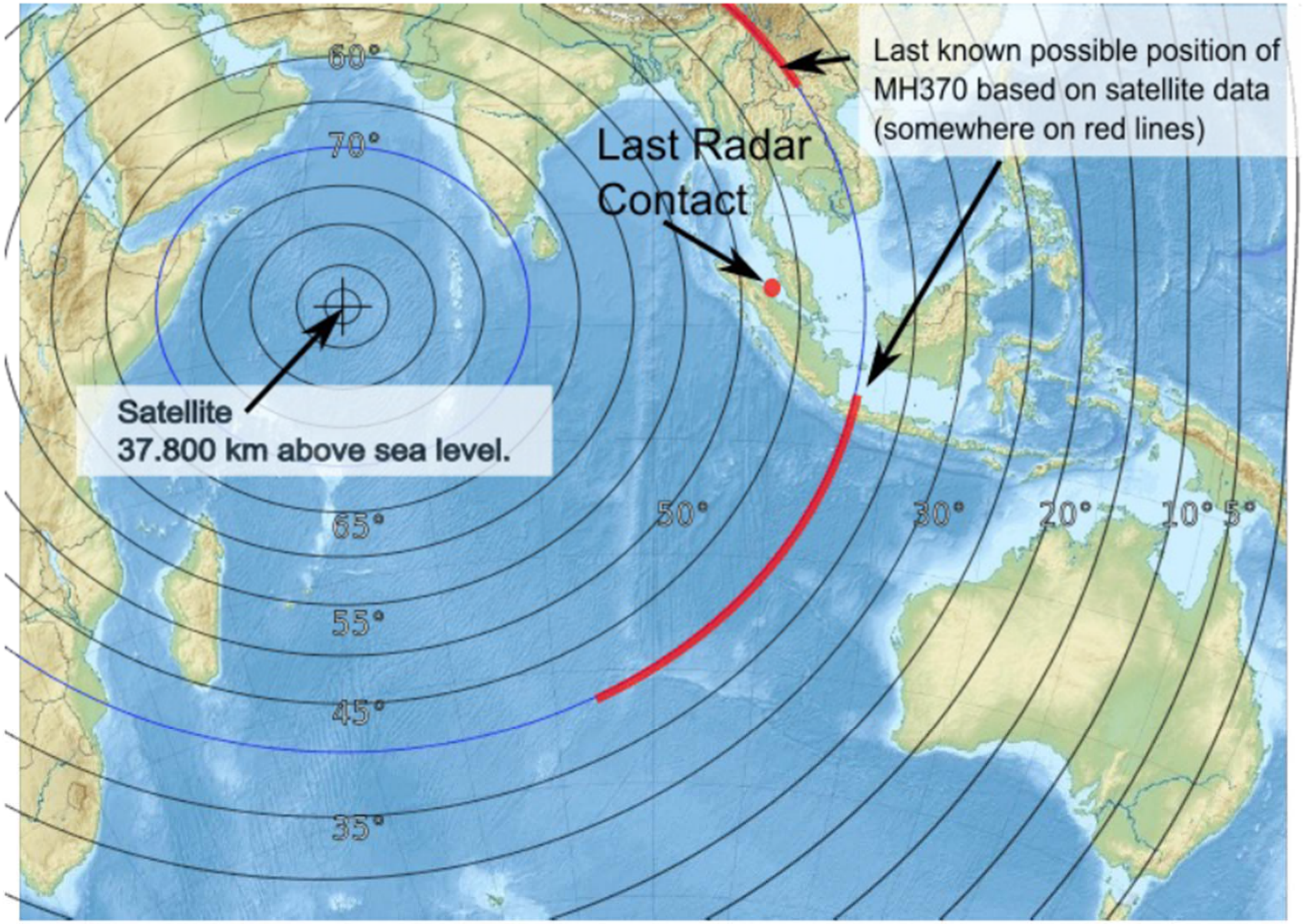
Really all the same idea!

# Intense Exercise


Workout time

# Tracking in 2-D space




You want to know the 2-D location of an object.

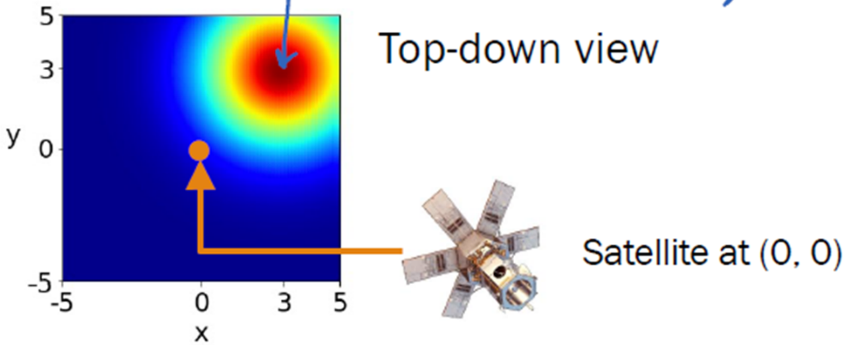
Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

Using the satellite measurement, where is the object?

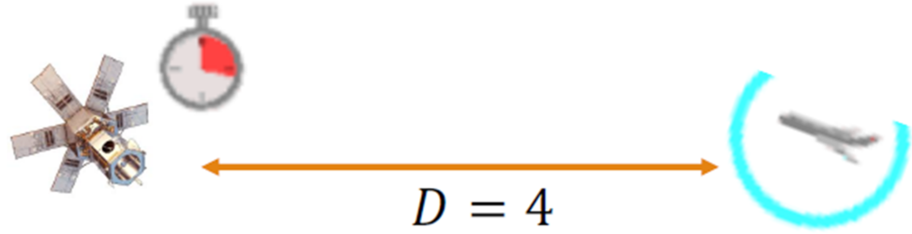
# Tracking in 2-D space

prior belief distribution, centered at (3,3)

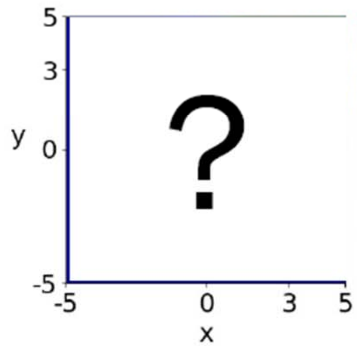
- Before measuring, we have some **prior belief** about the 2-D location of an object, $(X, Y)$.




- We observe some noisy **measurement** $D = 4$, the Euclidean distance of the object to a satellite.




- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?



# Tracking in 2-D space

- You have a prior belief about the 2-D location of an object, $(X, Y)$.
- You observe a noisy distance measurement, $D = 4$.
- What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Recall Bayes terminology:

posterior belief

likelihood (of evidence)

prior belief

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

normalization constant

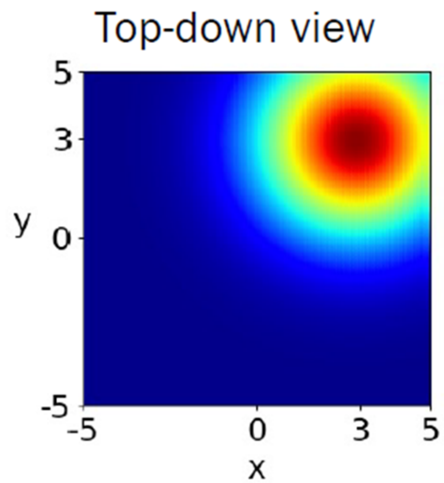
# 1. Define prior

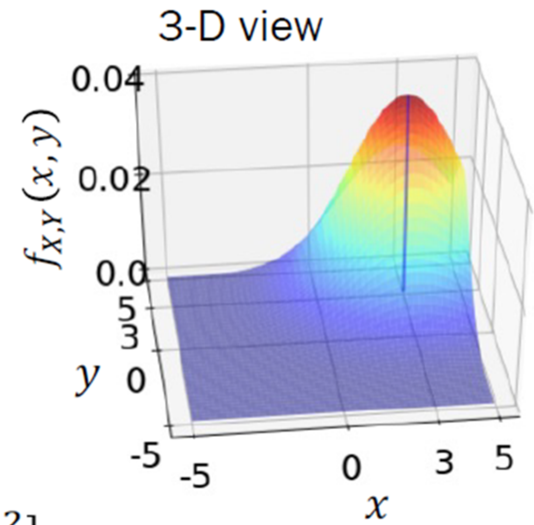
$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\,\boxed{f_{X,Y}(x,y)}}{f_D(d)}$$

You have a prior belief about the 2-D location of an object, $(X, Y)$.

Let $(X, Y)$ = object's 2-D location, assuming satellite is at (0,0)

Suppose the prior distribution is a symmetric bivariate normal distribution:



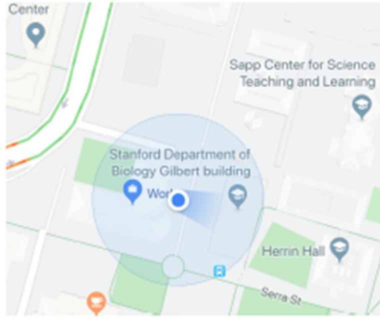

$$f_{X,Y}(x,y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2+(y-3)^2]}{2(2^2)}} = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

normalizing constant

$$\mu = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \Sigma = \begin{bmatrix} 2^2 & 0 \\ 0 & 2^2 \end{bmatrix}$$

# 2. Define likelihood

$$f_{X,Y|D}(x,y|d) = \frac{\boxed{f_{D|X,Y}(d|x,y)}\, f_{X,Y}(x,y)}{f_D(d)}$$

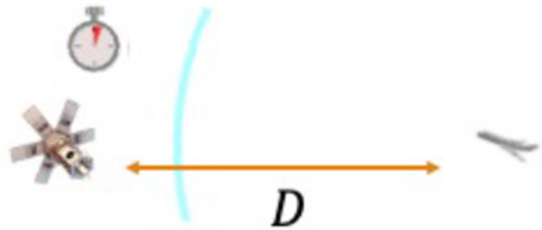



You observe a noisy distance measurement, $D = 4$.

If you knew your actual location $(x, y)$, you could say how likely a measurement $D = 4$ is:

Let $D$ = distance from the satellite (radially).
Suppose you knew your actual position: $(x, y)$.

- $D$ is still noisy! Suppose noise is **standard normal**.
- On average, $D$ is your true Euclidean distance: $\sqrt{x^2 + y^2}$



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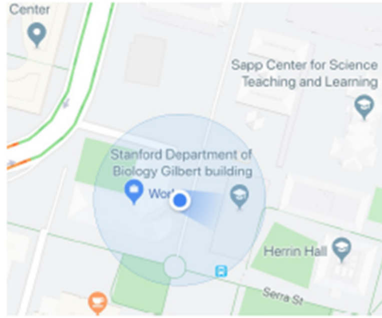


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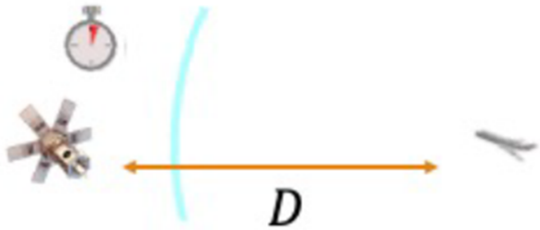


$$D|X,Y \sim N(\mu = \ (A)\ , \sigma^2 = \ (B)\ )$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{(C)\ \sqrt{2\pi}} e^{\{\ \ (D)\ \ \}}$$

# 2. Define likelihood

$$f_{X,Y|D}(x,y|d) = \frac{\boxed{f_{D|X,Y}(d|x,y)}\, f_{X,Y}(x,y)}{f_D(d)}$$

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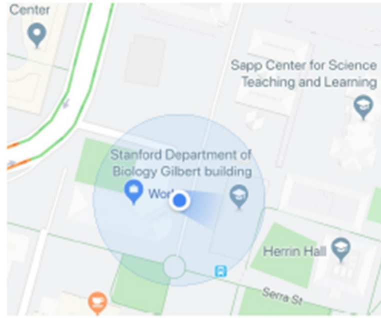


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Suppose you knew your actual position: $(x, y)$.

- $D$ is still noisy! Suppose noise is **standard normal**.
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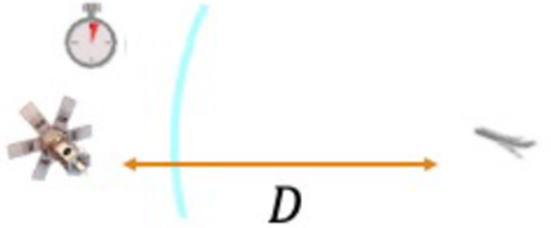
$D = \sqrt{x^2+y^2} + Z$ , $Z \sim N(0,1)$




$$D|X,Y \sim N\left(\mu = \sqrt{x^2 + y^2}, \sigma^2 = 1\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\left(d - \sqrt{x^2+y^2}\right)^2}{2}} \quad = K_2 \cdot e^{\frac{-\left(d - \sqrt{x^2+y^2}\right)^2}{2}}$$

normalizing constant

# 3. Compute posterior

$$\boxed{f_{X,Y|D}(x,y|d)} = \frac{f_{D|X,Y}(d|x,y)\ f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$$

# 3. Compute posterior

$$\boxed{f_{X,Y|D}(x,y|d)} = \frac{f_{D|X,Y}(d|x,y)\ f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

Posterior belief $\quad f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$

Know:

Prior belief $\quad f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$

Observation likelihood $\quad f_{D|X,Y}(d|x,y) = K_2 \cdot e^{\frac{-\left(d-\sqrt{x^2+y^2}\right)^2}{2}}$

Tips
- Use Bayes' Theorem!
- $f_D(4)$ is just a scaling constant. Why?
- How can we approximate the final scaling constant with a computer?

# Tracking in 2-D space

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

likelihood of $D = 4$ prior belief

$$f_{X,Y|D}(X = x, Y = y|D = 4) = \frac{f_{D|X,Y}(D = 4|X = x, Y = y)f_{X,Y}(x, y)}{f(D = 4)}$$

Bayes' Theorem

$$= \frac{K_2 \cdot e^{-\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}} \cdot K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}}{f(D = 4)}$$

$$= \frac{K_3 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]}}{f(D = 4)}$$

$$= K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

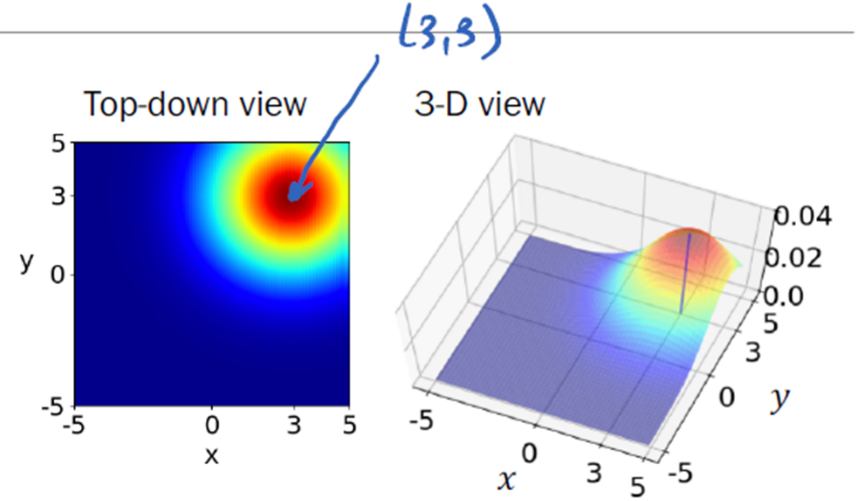
Key: Once we know the part dependent on $x, y$, we can computationally approximate $K_4$ so that $f_{X,Y|D}$ is a valid PDF.

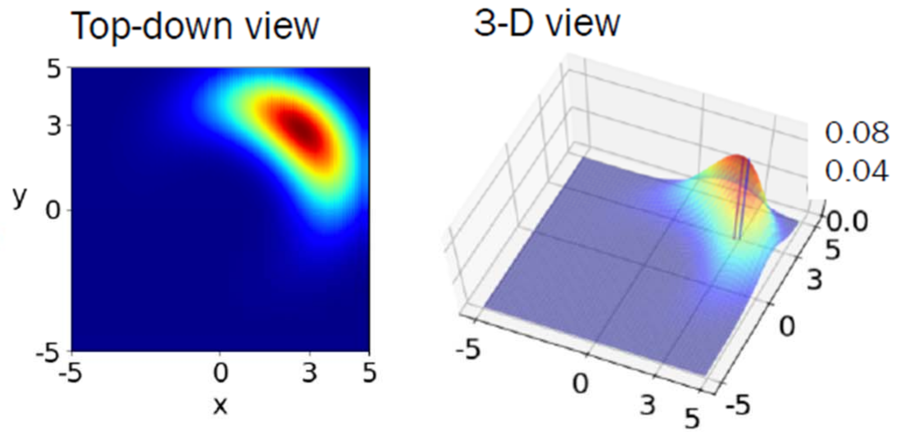
# Tracking in 2-D space

(3,3) is at $3\sqrt{2} \simeq 4.2$ from origin

With this continuous version of Bayes' theorem, we can explore new domains.

- Before measuring, we have some **prior belief** about the 2-D location of an object, $(X, Y)$.
- We observe some noisy **measurement** of the distance of the object to a satellite.
- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?



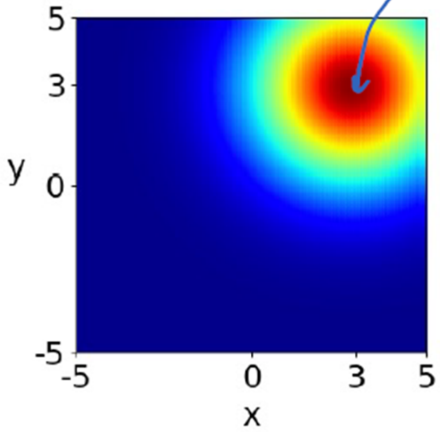



# Tracking in 2-D space: Posterior belief

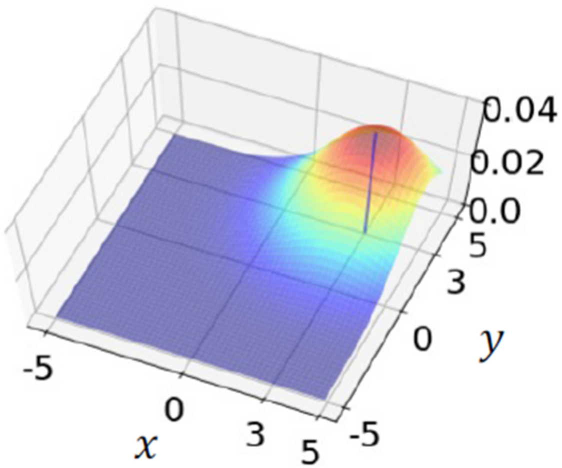
## Prior belief

(3,3), $3\sqrt{2} \approx 4.2$ from origin

Top-down view



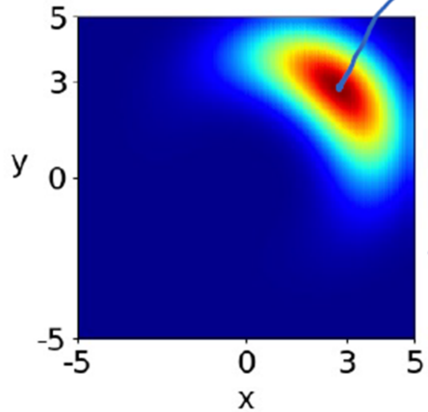

3-D view




$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

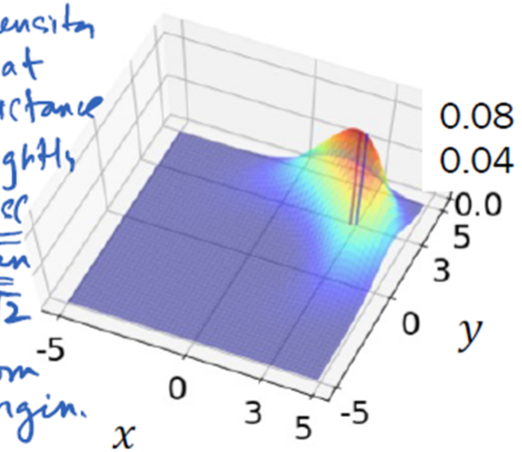
## Posterior belief

Top-down view




max density is at distance slightly less than $3\sqrt{2}$ from origin.

3-D view




observing a 4 indicates prior center at distance of $3\sqrt{2} \simeq 4.2$ was too big and should be adjusted

$$f_{X,Y|D}(x,y|4) = K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

# How'd you compute that $K_4$?

To be a valid conditional PDF, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y|D}(x,y|4)\,dx\,dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]} dx\,dy = 1$$

$$\frac{1}{K_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]} dx\,dy$$

(pull out $K_4$, divide)

Approximate:

$$\frac{1}{K_4} \approx \sum_y \sum_x e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]} \Delta x \Delta y$$

Use a computer!