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13: Statistics on Multiple Random Variables

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Lecture Discussion on Ed

Expectation of Common RVs

Linearity of Expectation is useful

Expectation is a linear mathematical operation. If $X = \sum_{i=1}^{n} X_i$:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

- Even if you don't know the **distribution** of X (e.g., because the joint distribution of $(X_1, \ldots, X_n)$ is unknown), you can still compute **expectation** of X.

- Problem-solving key: Define X_i such that $X = \sum_{i=1}^{n} X_i$

Most common use cases:
- $E[X_i]$ easy to calculate
- Sum of dependent RVs

Coupon Collecting

Coupon collecting and server requests

The coupon collector's problem in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

1. How many coupons do you expect after buying n boxes of cereal?

Servers

requests

k servers

request to server i

What is the expected number of servers utilized after n requests?

* 52% of Amazon profits
** more profitable than Amazon's North America commerce operations
source

Computer cluster utilization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a computer cluster with k servers. We send n requests.

- Requests independently go to server i with probability p_i
- Let X = # servers that receive ≥ 1 request.

What is $E[X]$?

1. Define additional random variables.

Let: A_i = event that server i receives ≥ 1 request

X_i = indicator for A_i

$X_i = \begin{cases} 1 & \text{if } A_i \geq 1 \\ 0 & \text{if } A_i = 0 \end{cases}$ $\quad E[X_i] = P(A_i \geq 1)$

$P(A_i) = 1 - P(\text{no requests to } i)$

$= 1 - (1 - p_i)^n$

Note: A_i are dependent!

2. Solve.

$$E[X_i] = P(A_i) = 1 - (1 - p_i)^n$$

$$E[X] = E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} (1 - (1 - p_i)^n)$$

$$= \sum_{i=1}^{k} 1 - \sum_{i=1}^{k} (1 - p_i)^n = k - \sum_{i=1}^{k} (1 - p_i)^n$$

does this result make sense for $n=0$? how about $n=1$?

Coupon collecting problems: Hash tables

The coupon collector's problem in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

Servers	Hash Tables
requests	strings
k servers	k buckets
request to server i	hashed to bucket i

1. How many coupons do you expect after buying n boxes of cereal? → What is the expected number of utilized servers after n requests?

2. How many boxes do you expect to buy until you have one of each coupon? → What is the expected number of strings to hash until each bucket has ≥ 1 string?

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

assume perfect hashing, so that $p_i = \frac{1}{k}$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

Y_0 = # trials needed until first bucket gets a string
Y_1 = # trials beyond Y_0 until second bucket gets a string
Y_2 = # trials beyond Y_1 until third bucket sees a string

1. Define additional random variables.

Let: Y_i = # of trials needed to get success after i-th success

- Success: hash string to previously empty bucket
- If i non-empty buckets: $P(\text{success}) = \frac{k-i}{k}$ ← # empty buckets

$$P(Y_i = n) = \left(\frac{i}{k}\right)^{n-1}\left(\frac{k-i}{k}\right)$$

Equivalently, $Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right)$ $\quad E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$

2. Solve.

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket $\geq$ 1 string.

What is $E[Y]$?

1. Define additional random variables.

Let: Y_i = # of trials to needed get success after i-th success

$$Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right), \qquad E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$$

2. Solve. $Y = Y_0 + Y_1 + \cdots + Y_{k-1}$

$$E[Y] = E[Y_0] + E[Y_k] + \cdots + E[Y_{k-1}]$$

$$= \frac{k}{k} + \frac{k}{k-1} + \frac{k}{k-2} + \cdots + \frac{k}{1} = k\left[\frac{1}{k} + \frac{1}{k-1} + \cdots + 1\right] = O(k \log k)$$

$$\sum_{m=1}^{k} \frac{1}{m} \approx \int_1^k \frac{1}{m}\, dm = \ln k$$

Covariance

공분산(共分散, 영어: covariance)은 2개의 확률변수의 선형 관계를 나타내는 값이다.[1] 만약 2개의 변수중 하나의 값이 상승하는 경향을 보일 때 다른 값도 상승하는 선형 상관성이 있다면 양수의 공분산을 가진다.[2] 반대로 2개의 변수중 하나의 값이 상승하는 경향을 보일 때 다른 값이 하강하는 선형 상관성을 보인다면 공분산의 값은 음수가 된다. 이렇게 공분산은 상관관계의 상승 혹은 하강하는 경향을 이해할 수 있으나 2개 변수의 측정 단위의 크기에 따라 값이 달라지므로 상관분석을 통해 정도를 파악하기에는 부적절하다. 상관분석에서는 상관관계의 정도를 나타내는 단위로 모상관계수로는 그리스 문자 ρ를, 표본상관계수로는 알파벳 s를 사용한다.

[공분산 - 위키백과, 우리 모두의 백과사전 (wikipedia.org)](https://ko.wikipedia.org/wiki/공분산)
https://ko.wikipedia.org/wiki/공분산

Statistics of sums of RVs

For any random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

$$\text{Var}(X + Y) = \quad ?$$

But first, a new statistic!

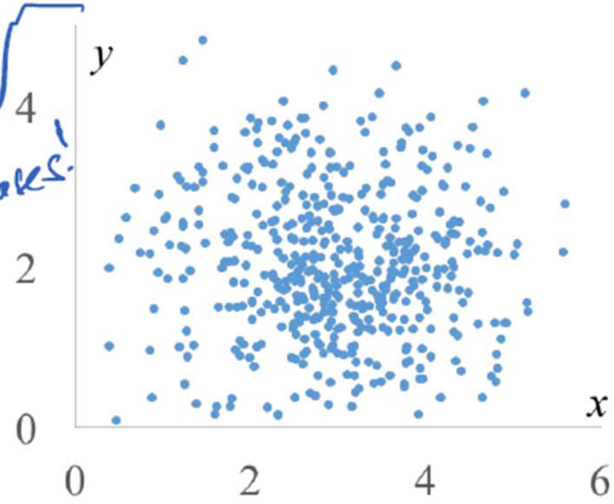
Spot the difference

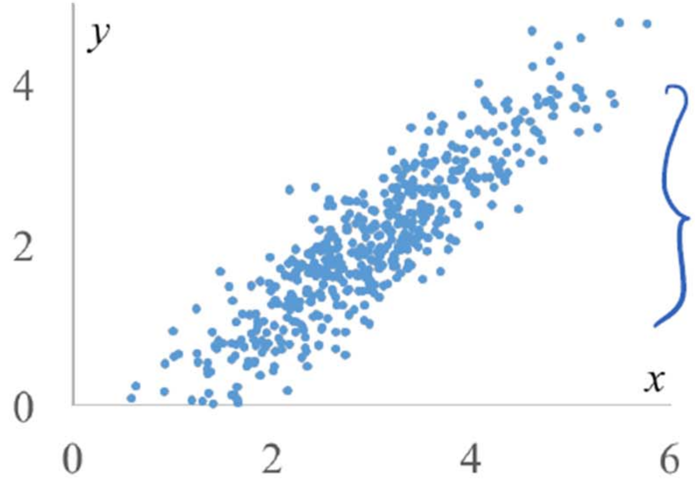
Compare/contrast the following two distributions:

Assume all points are equally likely.

$$P(X = x, Y = y) = \frac{1}{N}$$

can't predict how y will change as x increases!





can predict, at least roughly, how y will change as x increases!

these four statistics don't capture how x and y are coupled!

Both distributions have the same $E[X]$, $E[Y]$, $\mathrm{Var}(X)$, and $\mathrm{Var}(Y)$

Difference: how the two variables vary with **each other**.

Covariance

The covariance of two variables X and Y is:

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Proof of second part (rewriting $E[X]$, $E[Y]$ as μ_X, μ_Y to emphasize the fact they're each constants):

$$\begin{aligned}
\text{Cov}(X,Y) &= E[(X - E[X])(Y - E[Y])] = E[(X - \mu_X)(Y - \mu_Y)] \\
&= E[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y] \\
&= E[XY] - E[\mu_Y X] - E[\mu_X Y] + E[\mu_X \mu_Y] \\
&= E[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\
&= E[XY] - \mu_X \mu_Y = E[XY] - E[X]E[Y]
\end{aligned}$$

(linearity of expectation)

(μ_X, μ_Y are constants)

Covariance

The covariance of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Covariance measures how one random variable varies with a second.

- Outside temperature and utility bills have a negative covariance.
  공공요금
- Handedness and musical ability have near zero covariance.
- Product demand and price have a positive covariance.

Feel the covariance

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

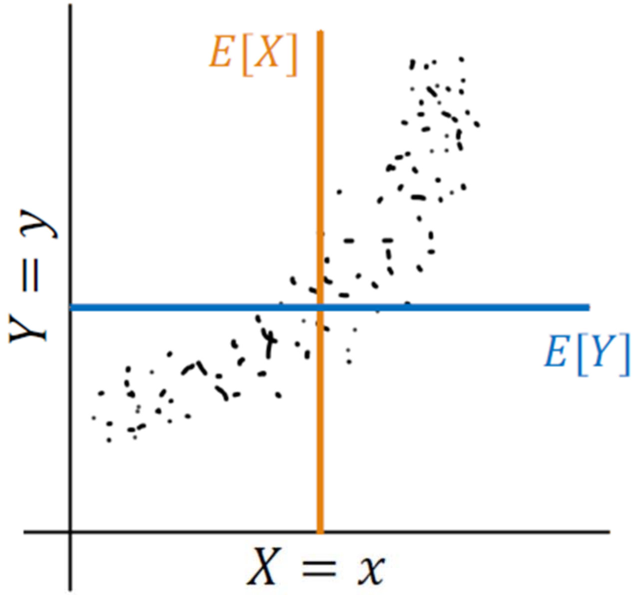
Is the covariance positive, negative, or zero?

as x increases, so does y : positive covariance

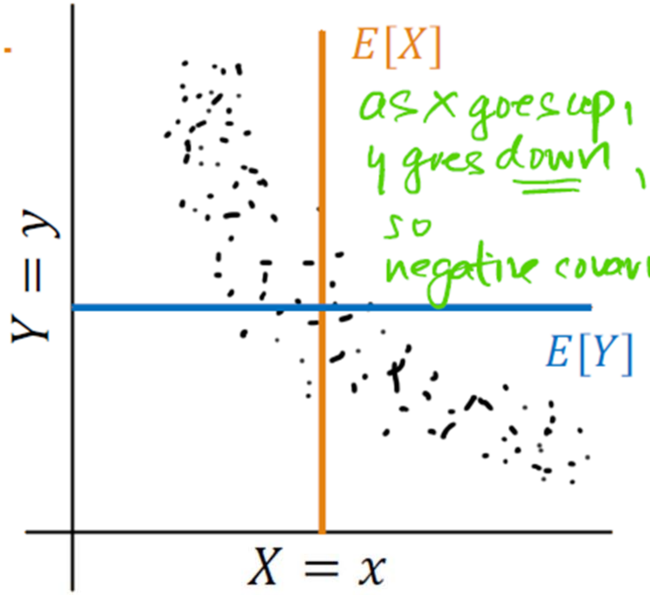
no obvious pattern in how y changes as x increases.

~0 covariance

1.

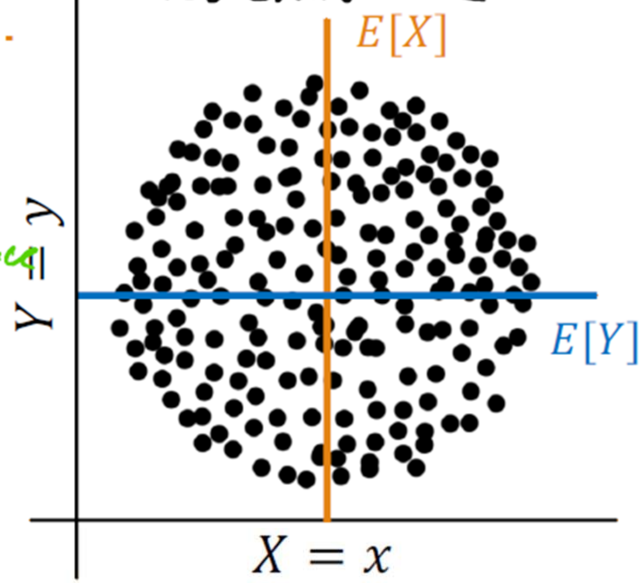


2.



as x goes up, y goes down, so negative covariance

3.



positive

negative

zero

Covarying humans

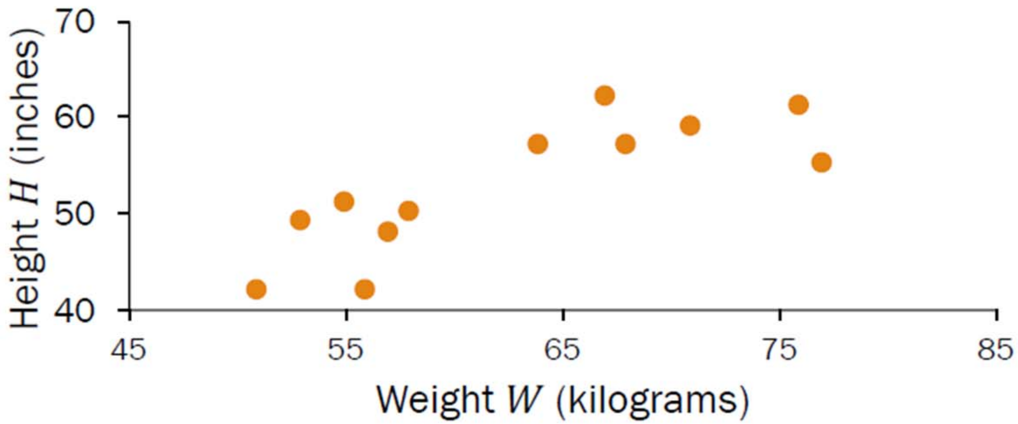
$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Weight (kg)	Height (in)	W · H
64	57	3648
71	59	4189
53	49	2597
67	62	4154
55	51	2805
58	50	2900
77	55	4235
57	48	2736
56	42	2352
51	42	2142
76	61	4636
68	57	3876

$E[W] = 62.75$ $E[H] = 52.75$ $E[WH] = 3355.83$

What is the covariance of weight W and height H?

$$\begin{aligned}\text{Cov}(W,H) &= E[WH] - E[W]E[H] \\ &= 3355.83 - (62.75)(52.75) \\ \text{(positive)} \quad &= 45.77\end{aligned}$$



Covariance > 0: one variable ↑, other variable ↑

Properties of Covariance

The **covariance** of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Properties:

1. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
2. $\text{Var}(X) = E[X^2] - (E[X])^2 = E[XX] - E[X]E[X] = \text{Cov}(X, X)$
3. Covariance of sums = sum of all pairwise covariances (proof left to you)
   $\text{Cov}(X_1 + X_2, Y_1 + Y_2) = \text{Cov}(X_1, Y_1) + \text{Cov}(X_2, Y_1) + \text{Cov}(X_1, Y_2) + \text{Cov}(X_2, Y_2)$
4. Covariance under linear transformation: $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)$
   recall that $\text{Var}(aX+b) = a^2\,\text{Var}(X)$

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$
with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

What is the joint PMF of X and Y?

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

		X = -1	0	1	Marginal PMF of Y, $p_Y(y)$
Y	0	1/3	0	1/3	2/3
	1	0	1/3	0	1/3
Marginal PMF of X, $p_X(x)$		1/3	1/3	1/3	

1. $E[X] = -1\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = 0$ $\quad E[Y] = 0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = 1/3$

2. $E[XY] = (-1 \cdot 0)\left(\frac{1}{3}\right) + (0 \cdot 1)\left(\frac{1}{3}\right) + (1 \cdot 0)\left(\frac{1}{3}\right)$
   $= 0$

3. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$
   $= 0 - 0(1/3) = 0$ ⚠️ does not imply independence!

4. Are X and Y independent? ❌
   $P(Y = 0|X = 1) = 1$
   $\neq \quad P(Y = 0) = 2/3$

Variance of sums of RVs

Statistics of sums of RVs

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

Variance of general sum of RVs

For any random variables X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

Proof:

$$\begin{aligned}\text{Var}(X+Y) &= \text{Cov}(X+Y, X+Y) \\ &= \text{Cov}(X,X) + \text{Cov}(X,Y) + \text{Cov}(Y,X) + \text{Cov}(Y,Y) \\ &= \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)\end{aligned}$$

$\text{Var}(X) = \text{Cov}(X,X)$

covariance of all pairs

Symmetry of covariance + $\text{Cov}(X,X) = \text{Var}(X)$

More generally:

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\underline{\sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{Cov}\left(X_i, X_j\right)}$$

(proof in extra slides)

Statistics of sums of RVs

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

For independent X and Y,

$$E[XY] = E[X]E[Y]$$

(Lemma: proof in extra slides)

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Variance of sum of independent RVs

For independent X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Proof:

1. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$ def. of covariance
   $= E[X]E[Y] - E[X]E[Y]$ X and Y are independent
   $= 0$

2. $\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \overbrace{\text{Cov}(X,Y)}^{\text{this is zero when } X \text{ and } Y \text{ are independent}} + \text{Var}(Y)$
   $= \text{Var}(X) + \text{Var}(Y)$

NOT bidirectional:
$\text{Cov}(X,Y) = 0$ does NOT imply independence of X and Y!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{k \ge 0}^{n} k^2 \binom{n}{k} p^k q^{n-k} && \text{Definition of Binomial Distribution: } p + q = 1 \\
&= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} && \text{Factors of Binomial Coefficient: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} && \text{Change of limit: term is zero when } k - 1 = 0 \\
&= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j} && \text{putting } j = k - 1,\ m = n - 1 \\
&= np\left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right) && \text{splitting sum up into two} \\
&= np\left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right) && \text{Factors of Binomial Coefficient: } j\binom{m}{j} = m\binom{m-1}{j-1} \\
&= np\left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right) && \text{Change of limit: term is zero when } j - 1 = 0 \\
&= np\left((n-1)p(p+q)^{m-1} + (p+q)^m\right) && \text{Binomial Theorem} \\
&= np((n-1)p + 1) && \text{as } p + q = 1 \\
&= n^2 p^2 + np(1-p) && \text{by algebra}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= np(1-p) + n^2 p^2 - (np)^2 && \text{Expectation of Binomial Distribution: } \mathrm{E}(X) = np \\
&= np(1-p)
\end{aligned}
$$

as required.

proofwiki.org

Let's instead prove this using independence and variance!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

Let $\quad X = \sum_{i=1}^{n} X_i$

Let X_i = ith trial is heads

$X_i \sim \text{Ber}(p)$

$\text{Var}(X_i) = p(1-p)$

X_i are independent (by definition)

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i)$$

yay!

X_i are independent, therefore variance of sum = sum of variance

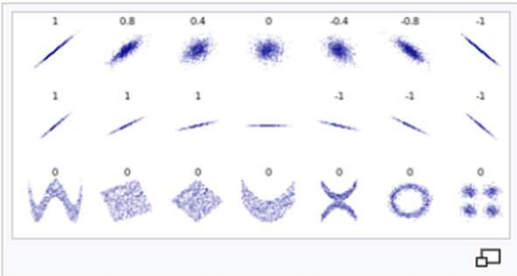
$$= \sum_{i=1}^{n} p(1-p)$$

Variance of Bernoulli

$$= np(1-p)$$

Correlation

상관 분석(相關 分析, Correlation analysis)은 확률론과 통계학에서 두 변수 간에 어떤 선형적 관계를 갖고 있는 지를 분석하는 방법이다. 두 변수는 서로 독립적인 관계이거나 상관된 관계일 수 있으며 이때 두 변수간의 관계의 강도를 상관관계(Correlation, Correlation coefficient)라 한다. 상관분석에서는 상관관계의 정도를 나타내는 단위로 모상관계수로 ρ를 사용하며 표본 상관 계수로 r 을 사용한다.



[상관 분석 - 위키백과, 우리 모두의 백과사전 (wikipedia.org)](https://ko.wikipedia.org/wiki/상관_분석)
https://ko.wikipedia.org/wiki/상관_분석

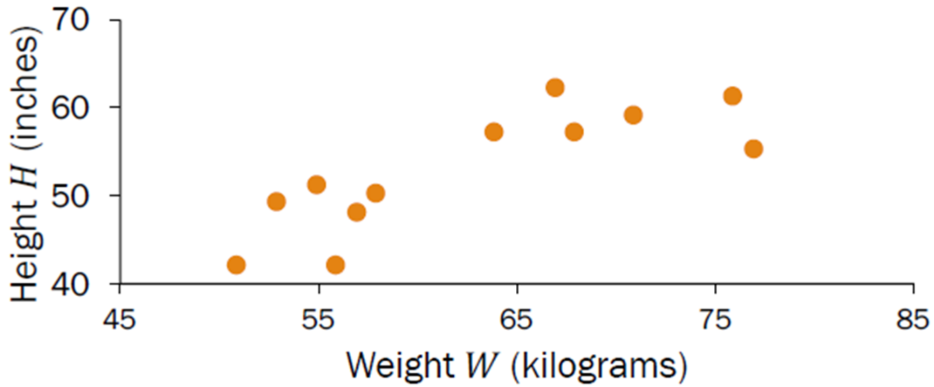
상관관계의 정도를 파악하는 상관 계수(相關係數, Correlation coefficient)는 두 변수간의 연관된 정도를 나타낼 뿐 인과관계를 설명하는 것은 아니다. 두 변수간에 원인과 결과의 인과관계가 있는지에 대한 것은 회귀분석을 통해 인과관계의 방향, 정도와 수학적 모델을 확인해 볼 수 있다.

Covarying humans

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

What is the covariance of weight W and height H?

$$\text{Cov}(W,H) = E[WH] - E[W]E[H]$$
$$= 3355.83 - (62.75)(52.75)$$
$$= 45.77 \quad \text{(positive)}$$

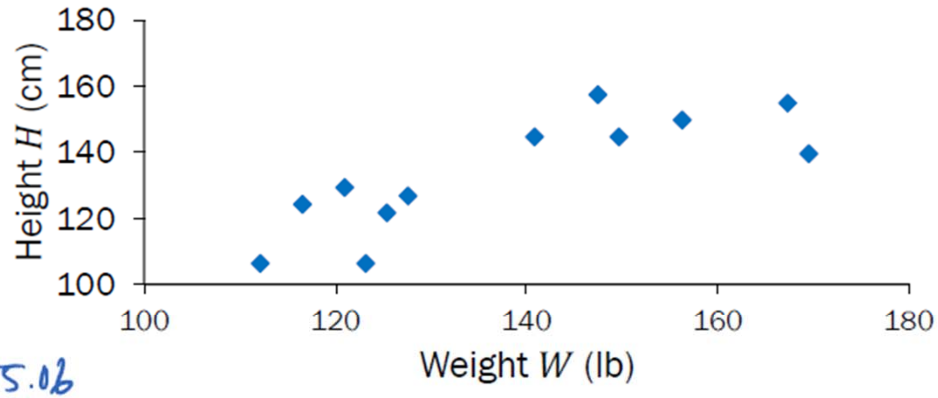


What about weight (lb) and height (cm)?

$$\text{Cov}(2.20W, 2.54H)$$
$$= E[2.20W \cdot 2.54H] - E[2.20W]E[2.54H]$$
$$= 18752.38 - (138.05)(133.99)$$
$$= 255.06 \quad \text{(positive)}$$

$2.20 \cdot 2.54 \cdot 45.77 \approx 255.06$

⚠ Covariance depends on units!



Sign of covariance (+/–) more meaningful than magnitude

Correlation

The **correlation** of two variables X and Y is:

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \, \sigma_Y}$$

$\sigma_X^2 = \text{Var}(X)$,
$\sigma_Y^2 = \text{Var}(Y)$

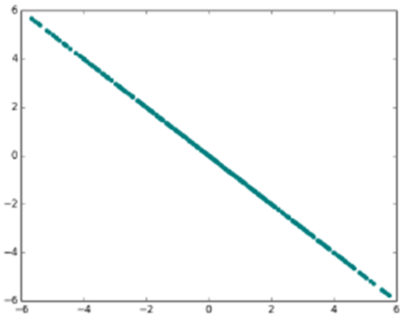
- Note: $-1 \leq \rho(X,Y) \leq 1$
- Correlation measures the **linear relationship** between X and Y:

$$\begin{aligned} \rho(X,Y) &= 1 && \Rightarrow Y = aX + b, \text{where } a = \sigma_Y/\sigma_X \\ \rho(X,Y) &= -1 && \Rightarrow Y = aX + b, \text{where } a = -\sigma_Y/\sigma_X \\ \rho(X,Y) &= 0 && \Rightarrow \text{uncorrelated (absence of linear relationship)} \end{aligned}$$

Correlation reps

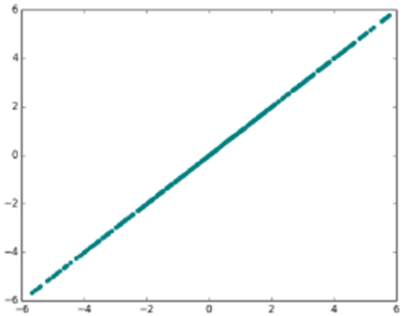
A. $\rho(X,Y) = 1$
B. $\rho(X,Y) = -1$
C. $\rho(X,Y) = 0$
D. Other

What is the correlation coefficient $\rho(X,Y)$?

1. 

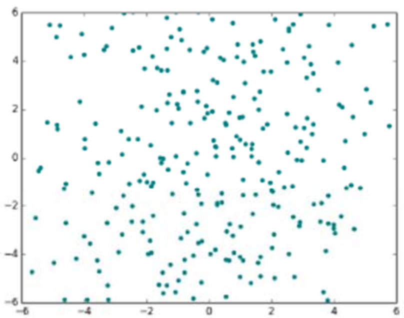
B. $\rho(X,Y) = -1$

$$Y = -aX + b$$
$$a > 0$$

2. 

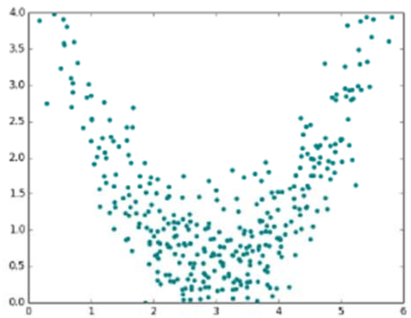
A. $\rho(X,Y) = 1$

$$Y = aX + b$$
$$a > 0$$

3. 

C. $\rho(X,Y) = 0$

"uncorrelated"

4. 

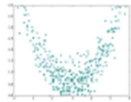
C. $\rho(X,Y) = 0$

$$Y = X^2$$

X and Y can be nonlinearly related even if $\rho(X,Y) = 0$.

Throwback to CS103: Conditional statements

Statement $P \to Q$:	Independence → No correlation	✅
Contrapositive $\neg Q \to \neg P$:	Correlation → Dependence	✅ (logically equivalent)
Inverse $\neg P \to \neg Q$:	Dependence → Correlation	❌ (not always) $Y = X^2$ $\rho(X, Y) = 0$
Converse $Q \to P$:	No correlation → Independence	❌ (not always)



"Correlation does not imply causation"

Spurious Correlation

[Spurious Correlations (tylervigen.com)](https://www.tylervigen.com/spurious-correlations)
https://www.tylervigen.com/spurious-correlations

Spurious Correlations

$\rho(X, Y)$ is used a lot to statistically quantify the relationship b/t X and Y.

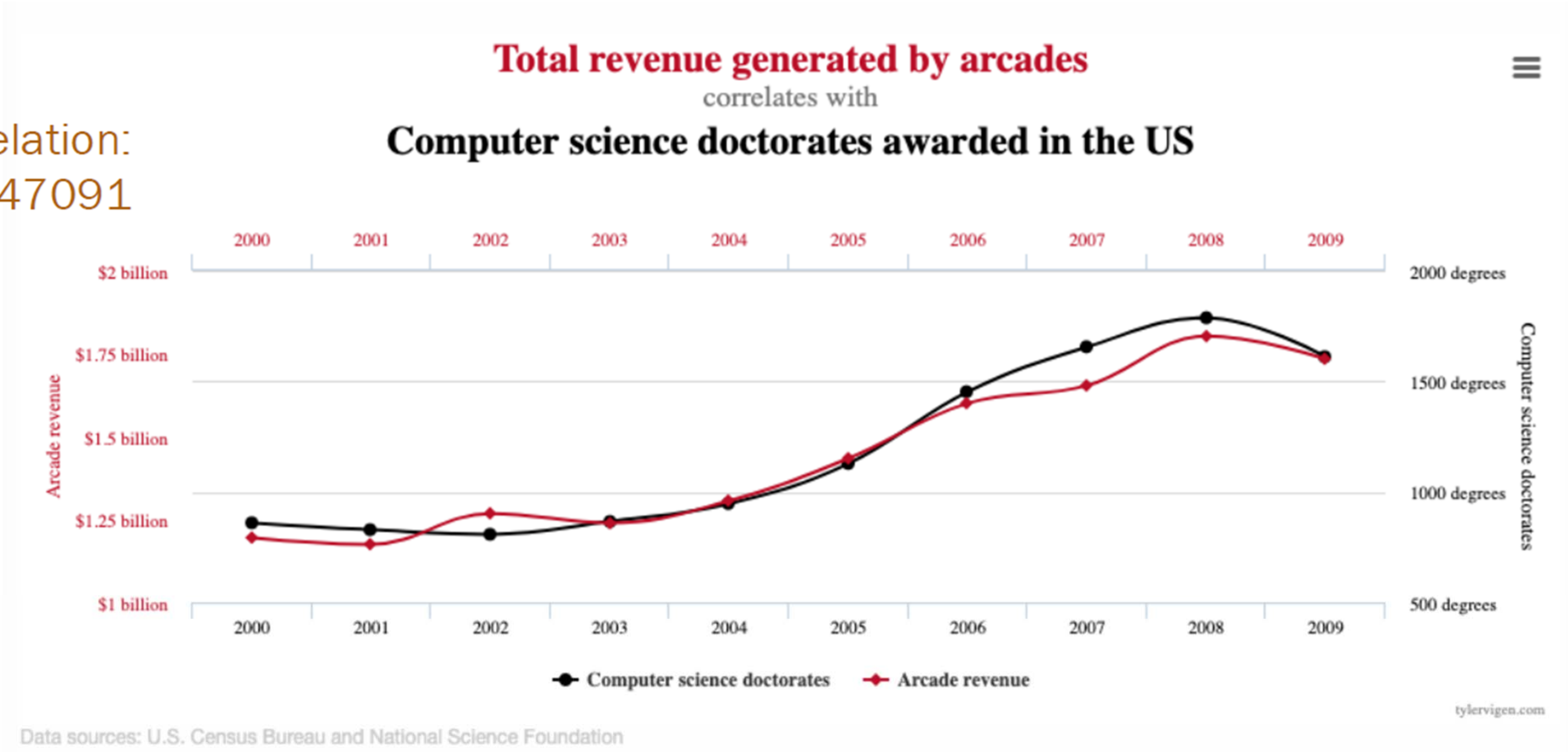
Correlation:
0.947091

Spurious correlations

Arcade revenue vs. CS PhDs

게임 수입

Correlation:
0.947091



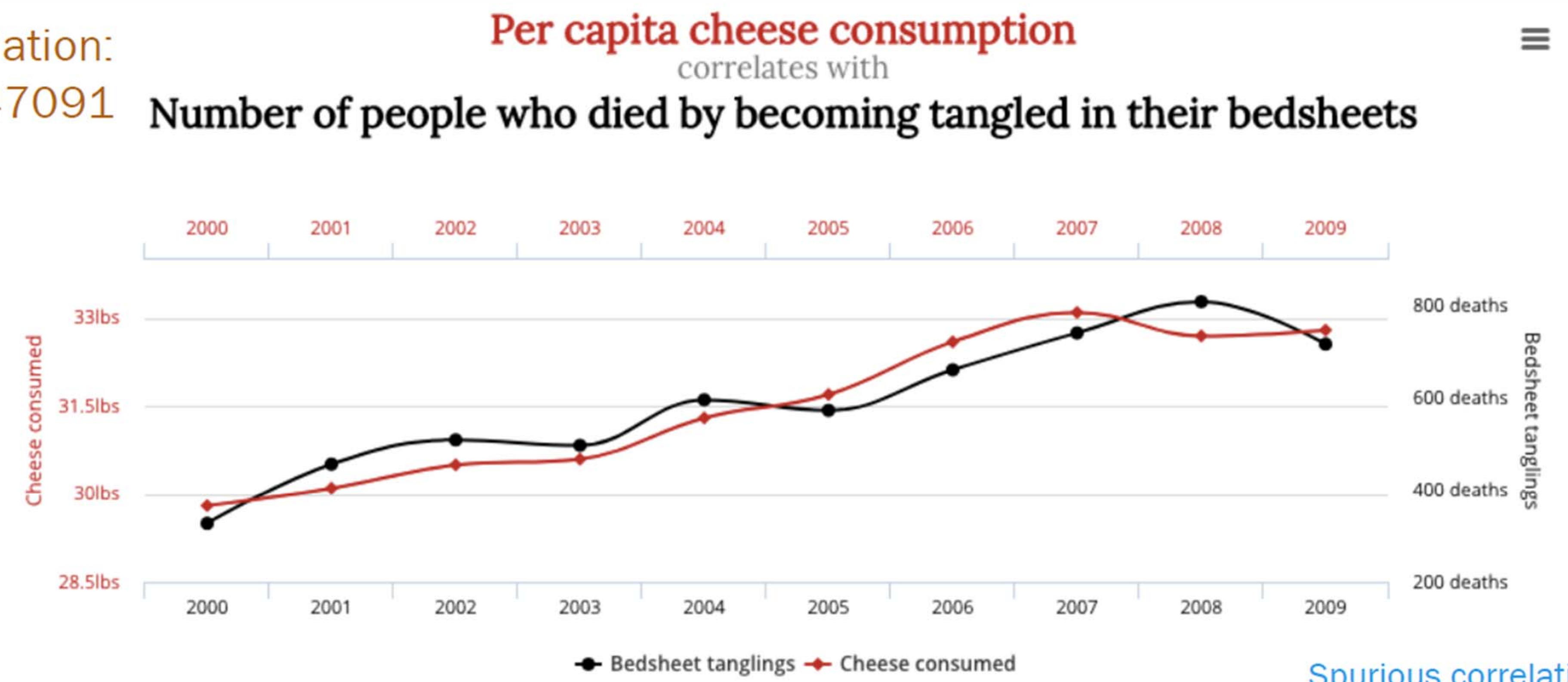
[Spurious correlations]

Spurious Correlations

$\rho(X, Y)$ is used a lot to statistically quantify the relationship b/t X and Y.

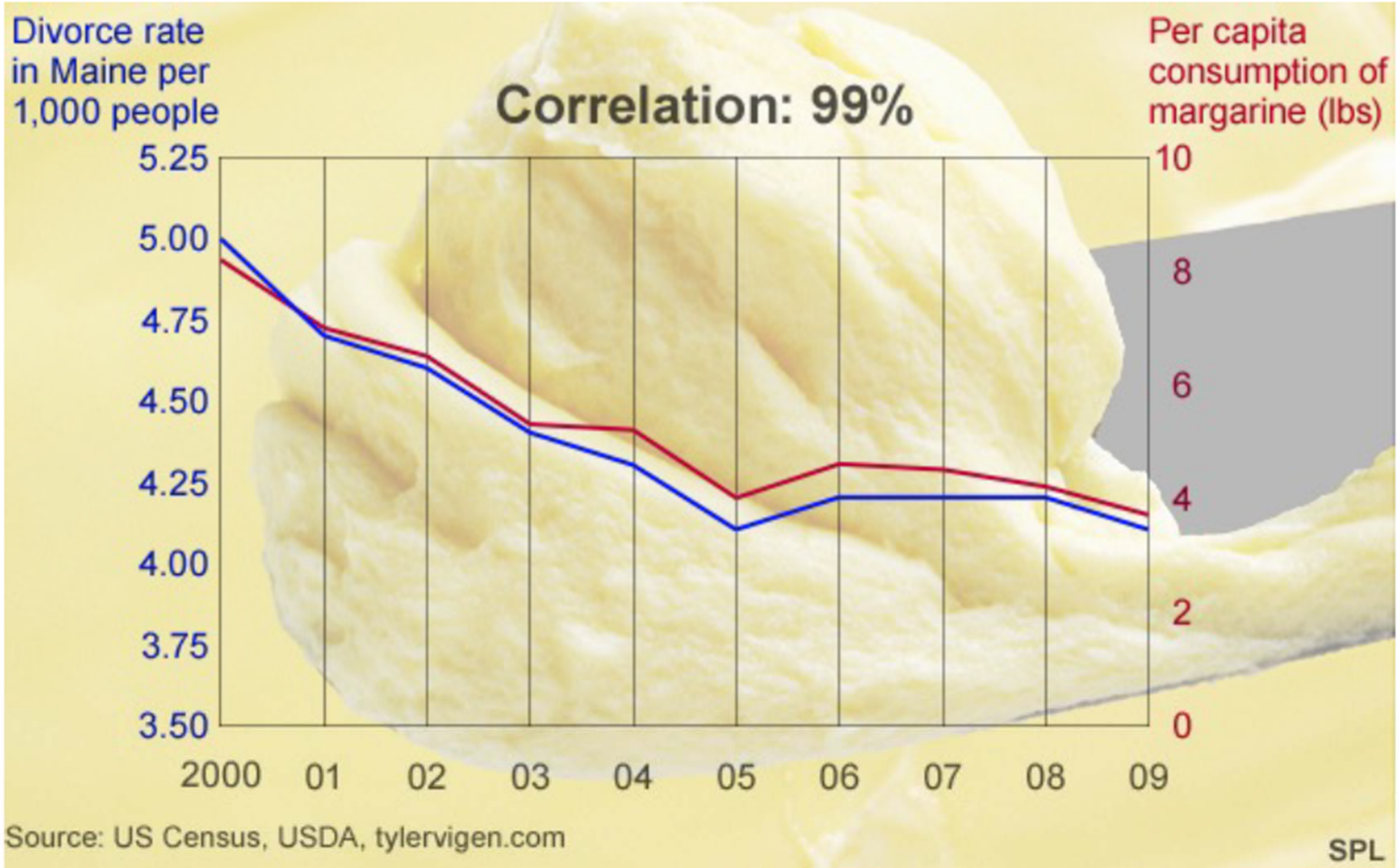
1인당

Correlation: 0.947091



[Spurious correlations]

Divorce vs. Margarine



[Spurious correlations: Margarine linked to divorce? - BBC News](http://www.bbc.com/news/magazine-27537142) http://www.bbc.com/news/magazine-27537142

Extras

Expectation of product of independent RVs

If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$
$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof: $E[g(X)h(Y)] = \sum_y \sum_x g(x)h(y)p_{X,Y}(x,y)$ (for continuous proof, replace summations with integrals)

$= \sum_y \sum_x g(x)h(y)p_X(x)p_Y(y)$ — X and Y are independent

$= \sum_y \left(h(y)p_Y(y) \sum_x g(x)p_X(x) \right)$ — Terms dependent on y are constant in integral of x

$= \left(\sum_x g(x)p_X(x) \right) \left(\sum_y h(y)p_Y(y) \right)$ — Summations separate

$= E[g(X)]E[h(Y)]$

Variance of Sums of Variables

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Proof:

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) \overset{\substack{\text{Var}(X)\\ = \text{Cov}(X,X)}}{=} \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) \overset{\substack{\text{covariance of}\\ \text{all pairs}}}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \text{Cov}\left(X_i, X_j\right)$$

Symmetry of covariance
$\text{Cov}(X,X) = \text{Var}(X)$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Adjust summation bounds