Table of Contents



# 11: Joint (Multivariate) Distributions

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Lecture Discussion on Ed

# Normal Approximation

# Normal Random Variables

$$X \sim \mathcal{N}(\underbrace{\mu}_{}^{\text{mean}}, \sigma^2)$$

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

colloquially, this means it commits to the least amount of structure for the given mean and variance

- Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance.

- Also useful for approximating the Binomial random variable!

focus on this for a few slides!

# Website testing

- 100 people are presented with a new website design.
- $X = \#$ people whose time on site increases
- PM assumes design has no effect, so assume P(stickier) = 0.5 independently.
- CEO will endorse the new design if $X \geq 65$.

What is $P(\text{CEO endorses change})$? *Give a numerical approximation*.

## Approach 1: Binomial

### Define

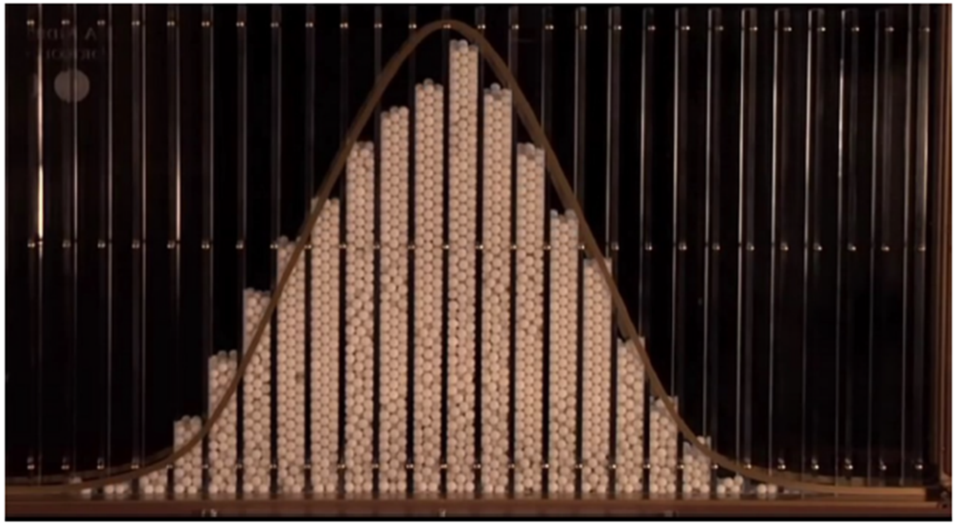
$$X \sim \text{Bin}(n = 100, p = 0.5)$$

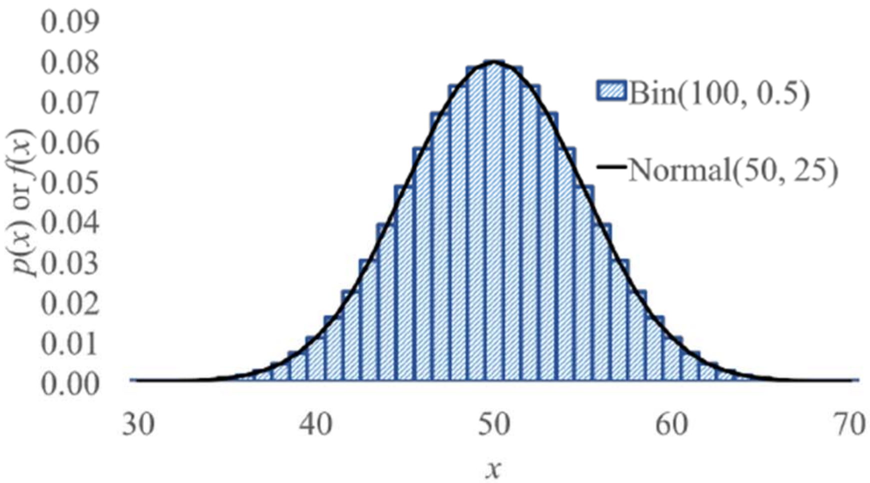
Want: $P(X \geq 65)$


### Solve

$$P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} 0.5^i (1 - 0.5)^{100-i}$$

# Don’t worry, Normal approximates Binomial






Galton Board

(We’ll explain **why** in 2 weeks)

# Website testing

```
jerry$ python
>>> from scipy.stats import binom, norm
>>> binom.pmf(range(65, 101), n, p).sum()
0.0017588208614850058
>>> 1 - norm(50, 5).cdf(65)
0.0013498980316301035
```

- 100 people are given a new website design.
- $X$ = # people whose time on site increases
- PM assumes design has no effect, so P(stickier) = 0.5 independently.
- CEO will endorse the new design if $X \geq 65$.

What is $P$(CEO endorses change)? *Give a numerical approximation*.

## Approach 1: Binomial

Define

$$X \sim \text{Bin}(n = 100, p = 0.5)$$

Want: $P(X \geq 65)$

Solve

$$P(X \geq 65) \approx 0.0018$$

## Approach 2: approximate with Normal

Define

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = np = 50$$
$$\sigma^2 = np(1-p) = 25$$
$$\sigma = \sqrt{25} = 5$$

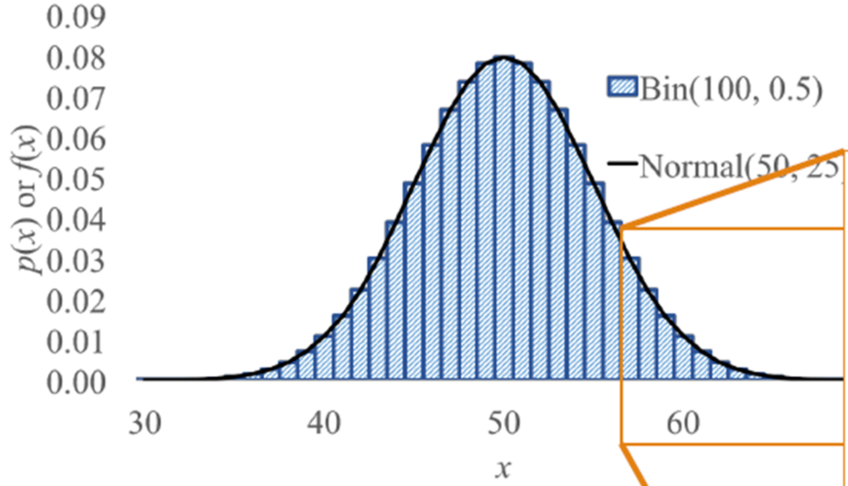
Solve

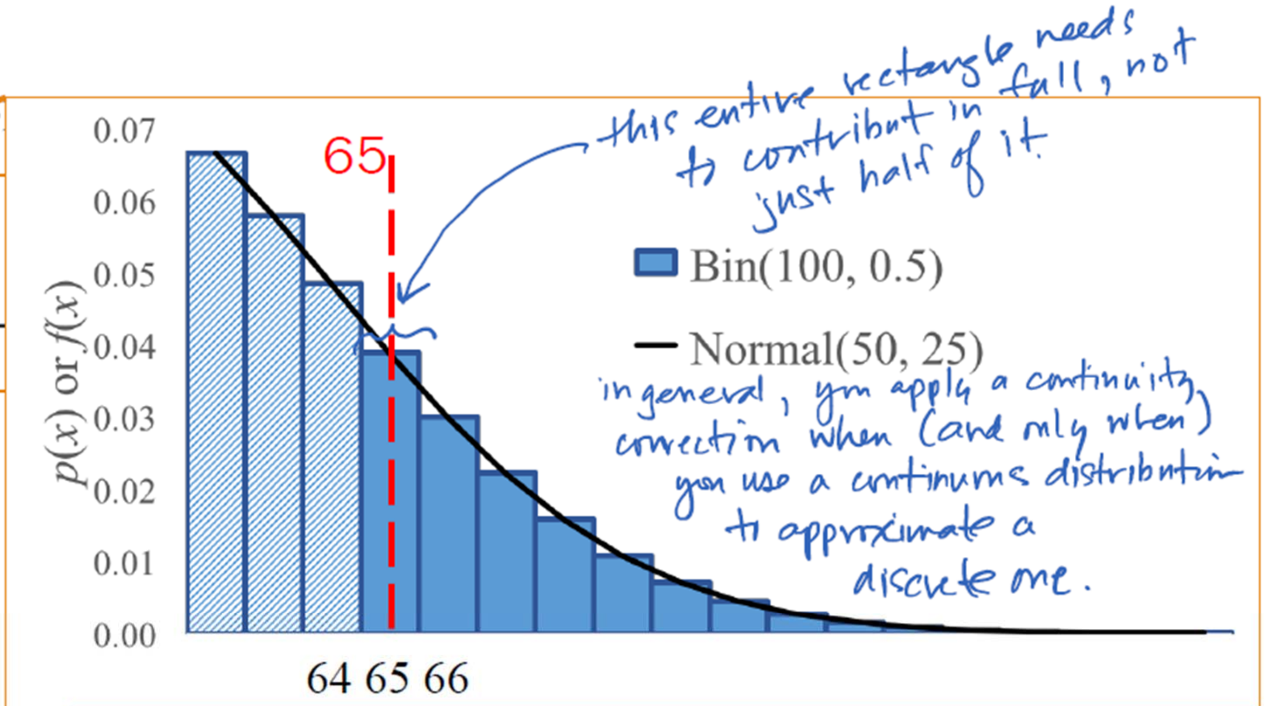
$$P(X \geq 65) \approx P(Y \geq 65) = 1 - F_Y(65)$$
$$= 1 - \Phi\left(\frac{65-50}{5}\right) = 1 - \Phi(3) \approx 0.0013\ ?$$

⚠️ ⚠️ (this approach is missing something important)

# Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.




$P(X \geq 65)$ Binomial

$\approx P(Y \geq 64.5)$ Normal

$\approx 0.0018$ ✅ the better

Approach 2

You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.

# Continuity correction

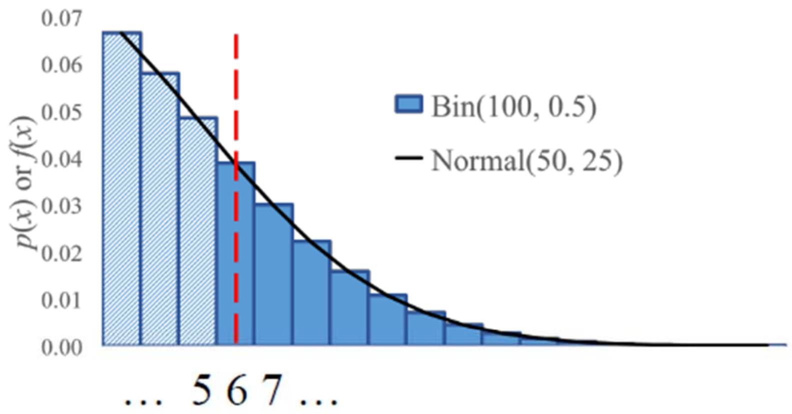
If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

those underlined in green should include $X=6$!

those underlined in red want nothing to do with $X=6$, so be sure to exclude its contribution when approximating.

| Discrete (e.g., Binomial) probability question | Continuous (Normal) probability question |
|---|---|
| $P(X = 6)$ | $P(5.5 \le Y \le 6.5)$ |
| $P(X \ge 6)$ | $P(Y \ge 5.5)$ |
| means $\ge 7$ $P(X > 6)$ | $P(Y \ge 6.5)$ |
| $\le 5$ $P(X < 6)$ | $P(Y \le 5.5)$ |
| $P(X \le 6)$ | $P(Y \le 6.5)$ |

helpful to frame boundaries in terms of $\le$ and $\ge$, avoid $<$ and $>$ for consistency.



# Who gets to approximate?

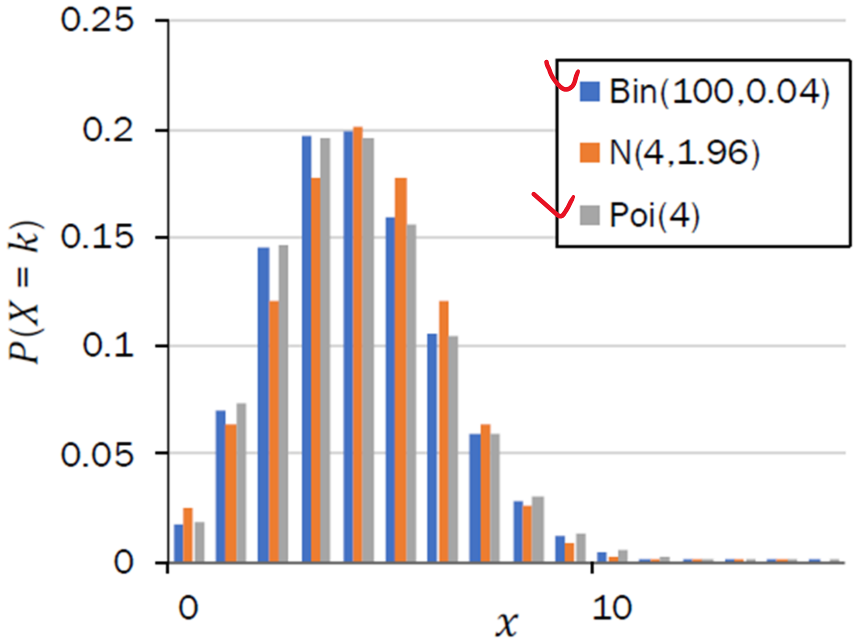
$$X \sim \text{Bin}(n, p)$$
$$E[X] = np$$
$$\text{Var}(X) = np(1-p)$$

$$Y \sim \text{Poi}(\lambda)$$
$$\lambda = np$$

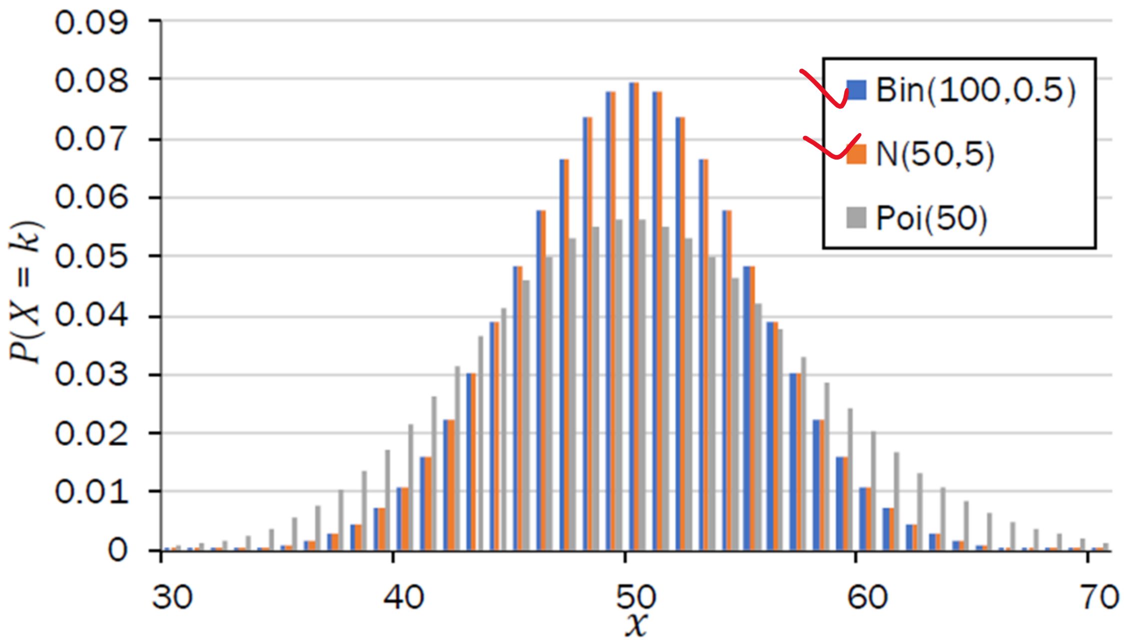
?

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = np$$
$$\sigma^2 = np(1-p)$$

# Who gets to approximate?




Poisson approximation
$n$ large ($> 20$), $p$ small ($< 0.05$)
slight dependence okay




Normal approximation
$n$ large ($> 20$), $p$ mid-ranged ($np(1-p) > 10$)
independence

1. If there is a choice, use Normal to approximate.
2. When using Normal to approximate a discrete RV, use a continuity correction.

# Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates with p=0.68 (independently)
- Let $X$ = # of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

Strategy:

A. Just Binomial — computationally expensive (also not an approximation)
B. Poisson — $p = 0.68$, not small enough
C. Normal — ✅ Variance $np(1-p) = 540 > 10$
D. None/other

Define an approximation

Let $Y \sim \mathcal{N}\left(E[X], \text{Var}(X)\right)$

$E[X] = np = 1686$

$\text{Var}(X) = np(1-p) \approx 540 \rightarrow \sigma = 23.3$

$P(X > 1745) \approx P(Y \geq 1745.5)$ ⚠️ Continuity correction

Solve

$$P(Y \geq 1745.5) = 1 - F(1745.5)$$

$$= 1 - \Phi\left(\frac{1745.5 - 1686}{23.3}\right)$$

$$= 1 - \Phi(2.54) \approx 0.0055$$

# Discrete Joint RVs

# From last slide deck

Review


What is the probability that the Warriors win?
How do you model zero-sum games?

$$P(A_W > A_B)$$

This is a probability of an event involving **two** random variables!

# Joint probability mass functions

Roll two 6-sided dice, yielding values $X$ and $Y$.

$\boxed{X}$

random variable

$P(X = 1)$

probability of an event

$P(X = k)$

probability mass function

# Joint probability mass functions

Roll two 6-sided dice, yielding values $X$ and $Y$.

| $X$ | $P(X = 1)$ | $P(X = k)$ |
|---|---|---|
| random variable | probability of an event | probability mass function |
| $X, Y$ | $P(X = 1 \cap Y = 6)$<br>$P(X = 1, Y = 6)$<br>new notation: the comma<br>probability of the intersection of two events | $P(X = a, Y = b)$<br>joint probability mass function |

# Discrete joint distributions

For two discrete joint random variables $X$ and $Y$, the joint probability mass function is defined as:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

The marginal distributions of the joint PMF are defined as:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a,y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x,b)$$

Use marginal distributions to extract a 1D RV from a joint PMF.

# Two dice

Roll two 6-sided dice, yielding values $X$ and $Y$.

1. What is the joint PMF of $X$ and $Y$?

$$p_{X,Y}(a,b) = 1/36 \qquad (a,b) \in \{(1,1), \dots, (6,6)\}$$

| | | $X$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 |
| | 1 | 1/36 | … | … | … | … | 1/36 |
| | 2 | … | … | … | … | | |
| | 3 | … | … | … | … | … | … |
| $Y$ | 4 | … | … | … | … | … | … |
| | 5 | … | … | … | … | … | … |
| | 6 | 1/36 | … | … | … | … | 1/36 |

$P(X = 4, Y = 3)$

**Probability table**

- All possible outcomes for several discrete RVs
- Not parametric (e.g., parameter $p$ in Ber($p$))

# Two dice

Roll two 6-sided dice, yielding values $X$ and $Y$.

1. What is the joint PMF of $X$ and $Y$?

$$p_{X,Y}(a,b) = 1/36 \qquad (a,b) \in \{(1,1),\dots,(6,6)\}$$

2. What is the marginal PMF of $X$?

$$p_X(a) = P(X=a) = \sum_y p_{X,Y}(a,y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6} \qquad a \in \{1,\dots,6\}$$

# A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has $X$ Macs and $Y$ PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

1. What is $P(X = 1, Y = 0)$, the missing entry in the probability table?

| | | $X$ (# Macs) | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| $Y$ (# PCs) | 0 | .16 | ? | .07 | .04 |
| | 1 | .12 | .14 | .12 | 0 |
| | 2 | .07 | .12 | 0 | 0 |
| | 3 | .04 | 0 | 0 | 0 |

the symmetry above the diagonal here is a coincidence, not a requirement.

# A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has $X$ Macs and $Y$ PCs.
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1. What is $P(X = 1, Y = 0)$, the missing entry in the probability table?

| $Y$ (# PCs) \ $X$ (# Macs) | 0 | 1 | 2 | 3 | |
|---|---|---|---|---|---|
| 0 | .16 | .12 | .07 | .04 | |
| 1 | .12 | .14 | .12 | 0 | |
| 2 | .07 | .12 | 0 | 0 | |
| 3 | .04 | 0 | 0 | 0 | |
| | | | | | |

A joint PMF must sum to 1:

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

# A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has $X$ Macs and $Y$ PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

2. How do you compute the marginal PMF of $X$?

| $Y$ (# PCs) \ $X$ (# Macs) | 0 | 1 | 2 | 3 | C (sum cols here) |
|---|---|---|---|---|---|
| 0 A | .16 | .12 | .07 | .04 | .39 |
| 1 | .12 | .14 | .12 | 0 | .38 |
| 2 | .07 | .12 | 0 | 0 | .19 |
| 3 | .04 | 0 | 0 | 0 | .04 |
| B (sum rows here) | .39 | .38 | .19 | .04 | |

A. $p_{X,Y}(x, 0) = P(X = x, Y = 0)$

B. Marginal PMF of $X$ $\quad p_X(x) = \sum_y p_{X,Y}(x, y)$

C. Marginal PMF of $Y$ $\quad p_Y(y) = \sum_x p_{X,Y}(x, y)$

To find a marginal distribution over one variable, sum over all other variables in the joint PMF.

# A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has $X$ Macs and $Y$ PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

3. Let $C = X + Y$. What is $P(C = 3)$?

| $Y$ (# PCs) \ $X$ (# Macs) | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | .16 | .12 | .07 | .04 |
| 1 | .12 | .14 | .12 | 0 |
| 2 | .07 | .12 | 0 | 0 |
| 3 | .04 | 0 | 0 | 0 |

$$P(C = 3) = P(X + Y = 3)$$

Law of Total Probability

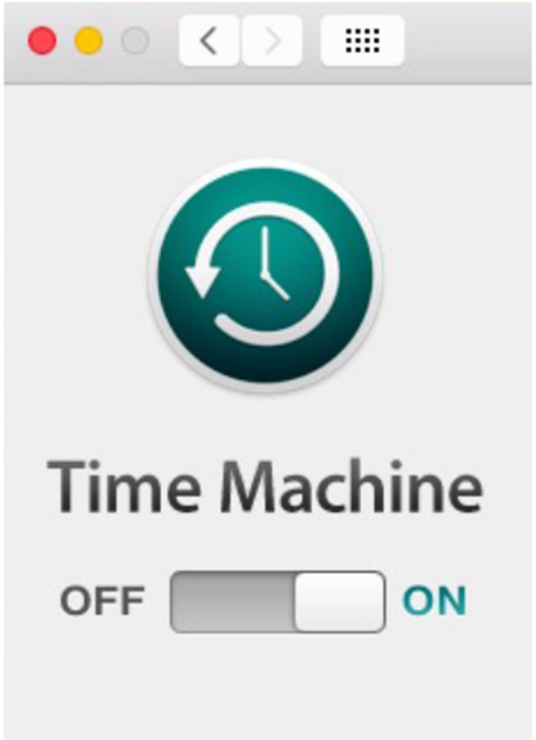
$$= \sum_x \sum_y P(X + Y = 3 | X = x, Y = y) P(X = x, Y = y)$$

$$= P(X = 0, Y = 3) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 3, Y = 0)$$

We'll come back to sums of RVs next lecture!

# Multinomial RV

# Recall the good times





Permutations

$n!$

How many ways are there to order $n$ objects?

# Counting unordered objects

$k$ $n-k$ $n_1$ $n_2$ $n_r$

## Binomial coefficient

How many ways are there to group $n$ objects into two groups of size $k$ and $n-k$, respectively?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Called the binomial coefficient because of something from aLgEbRa

## Multinomial coefficient

How many ways are there to group $n$ objects into $r$ groups of sizes $n_1, n_2, \ldots, n_r$ respectively?

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Multinomials generalize Binomials for counting.

# Probability

## Binomial RV

What is the probability of getting $k$ successes and $n-k$ failures in $n$ trials?

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial # of ways of ordering the successes

Probability of each ordering of $k$ successes is equal + mutually exclusive

## Multinomial RV

What is the probability of getting $c_1$ of outcome 1, $c_2$ of outcome 2, ..., and $c_m$ of outcome $m$ in $n$ trials?

Multinomial RVs also generalize Binomial RVs for probability!

# Multinomial Random Variable

Consider an experiment of $n$ independent trials:

- Each trial results in one of $m$ outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^{m} p_i = 1$
- Let $X_i$ = # trials with outcome $i$

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\sum_{i=1}^{m} p_i = 1$

**Multinomial** # of ways of ordering the outcomes

**Probability** of each ordering is equal + mutually exclusive

# Hello dice rolls, my old friends

A fair, six-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

# Hello dice rolls, my old friends

A fair, six-sided die is rolled 7 times.
What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, \overbrace{X_6 = 3}^{\text{\# of times a six appears}})$$

$$= \binom{7}{1,1,0,2,0,\underbrace{3}_{\text{choose where the sixes appear}}} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \Big(\underbrace{\frac{1}{6}}_{\text{probability of rolling a six}}\Big)^{\overset{\uparrow}{3}\,\text{this many times}} = 420 \left(\frac{1}{6}\right)^7$$

# Probabilistic text analysis

Ignoring the order of words...

What is the probability of any given word that you write in English?

- $P(\text{word} = \text{"the"}) > P(\text{word} = \text{"susurration"})$
- $P(\text{word} = \text{"Stanford"}) > P(\text{word} = \text{"Cal"})$

Probabilities of **counts** of words = Multinomial distribution


A document is a large multinomial.

(according to the Global Language Monitor, there are 988,968 words in the English language used on the internet.)

# Probabilistic text analysis

Probabilities of **counts** of words = multinomial distribution

Example document:

#words: $n = 48$

"When my late husband was alive he deposited some amount of Money with overseas Bank in which the amount will be declared to you once you respond to this message indicating your interest in helping to receive the fund and use it for Heavens work as my wish."

sum of exponents is also 48

$$P\left(\begin{array}{l}\text{bank} = 1\\ \text{fund} = 1\\ \text{money} = 1\\ \text{wish} = 1\\ \cdots\\ \boxed{\text{to} = 3}\end{array}\middle|\ \text{spam}\right) = \frac{48!}{\underbrace{1!\,1!\,1!\,1!\cdots \boxed{3!}}_{\text{these sum to 48}}} p_{\text{bank}}^{1}\, p_{\text{fund}}^{1} \cdots p_{\text{to}}^{\boxed{3}}$$

not concerned with order, just frequency

Note: $P\left(\text{bank}\middle|\begin{array}{c}\text{spam}\\\text{writer}\end{array}\right) \gg P\left(\text{bank}\middle|\begin{array}{c}\text{writer=}\\\text{you}\end{array}\right)$

# Old and New Analysis

Authorship of the Federalist Papers

- 85 essays advocating ratification of the US constitution
- Written under the pseudonym “Publius” (really, Alexander Hamilton, James Madison, John Jay)

Who wrote which essays?

- Analyze probability of words in each essay and compare against word distributions from known writings of three authors
- Curious what the analysis is? You’ll love Problem Set 4!

# Statistics of Two RVs

# Expectation and Covariance

In real life, we often have many RVs interacting at once.

- We've seen some simpler cases (e.g., sum of independent Bernoullis).
- Come Monday, we'll discuss sums of Binomials, sums of Poissons, etc.
- In general, manipulating joint PMFs is difficult.
- Fortunately, you don't need to model joint RVs completely all the time.

Instead, we'll focus next on reporting statistics of multiple RVs:

- Expectation of sums (you've seen some of this, more on Monday)
- Covariance: measure of how two random variable vary with each other (more on Wednesday and Friday)

# Properties of Expectation, extended to two RVs

1. Linearity:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

(we've seen this; we'll prove today!)

3. Unconscious statistician:

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

True for both independent and dependent random variables!

# Proof of expectation of a sum of RVs

$E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \sum_x \sum_y (x+y) p_{X,Y}(x,y)$$

LOTUS,
$g(X,Y) = X + Y$

$$= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

Linearity of summations
(cont. case: linearity of integrals)

$$= \sum_x x p_X(x) + \sum_y y p_Y(y)$$

Marginal PMFs for $X$ and $Y$

$$= E[X] + E[Y]$$