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10: Normal Distributions

Jerry Cain

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Lecture Discussion on Ed

Normal Random Variables

Normal Random Variable

https://en.wikipedia.org/wiki/Normal_distribution

def A **Normal** random variable X is defined as follows:

expectation, center of gravity

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

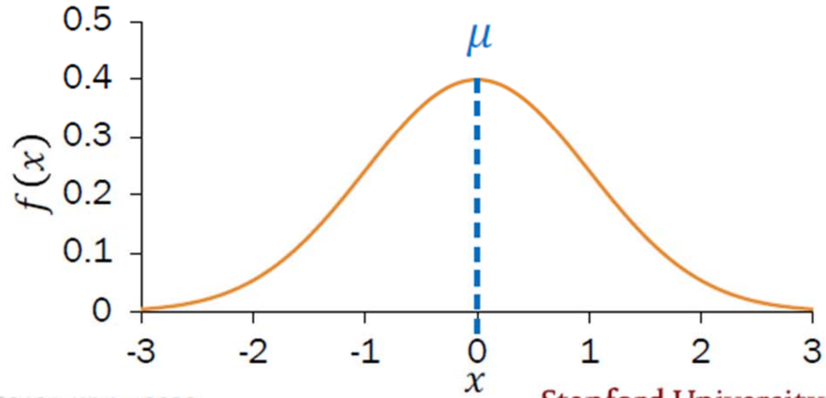
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\text{Var}(X) = \sigma^2$

Other names: **Gaussian** random variable

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

That's what they want you to believe...

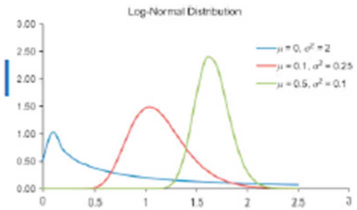
Why the Normal?

- Common for natural phenomena: height, weight, etc.

$\ln X \sim \mathcal{N}(\mu, \sigma^2)$

Actually log-normal

https://ko.wikipedia.org/wiki/로그_정규_분포



- Most noise in the world is Normal

Just an assumption

- Often results from the sum of many random variables

Only if equally weighted

- Sample means are distributed normally

(okay this one is true, we'll see this in 3 weeks)

The sample mean is the average of the values of a variable in a sample, which is the sum of those values divided by the number of values. Using mathematical notation, if a sample of N observations on variable X is taken from the population, the sample mean is:

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i.$$

https://en.wikipedia.org/wiki/Sample_mean_and_covariance

Okay, so why the Normal?

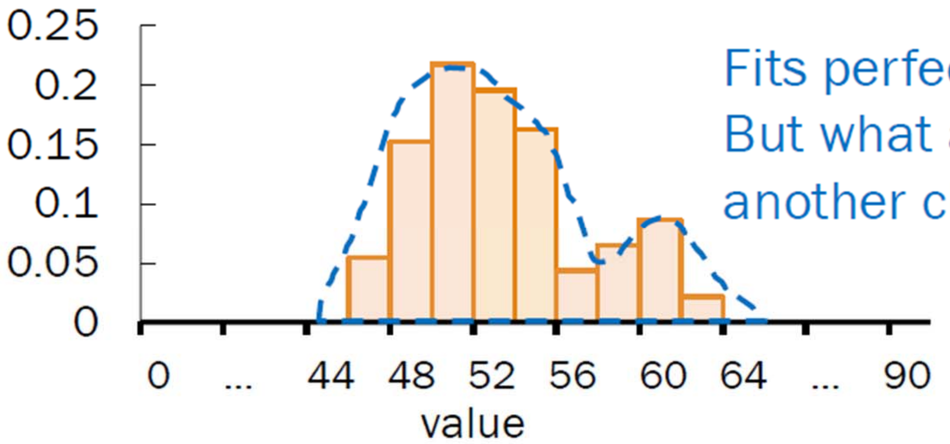
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: model real life situations with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



Fits perfectly!
But what about in another classroom?

Okay, so why the Normal?

maximizes entropy? what?
it means it is as reasonably generic a distribution as we can choose and still do a good job modelling some probability problem.

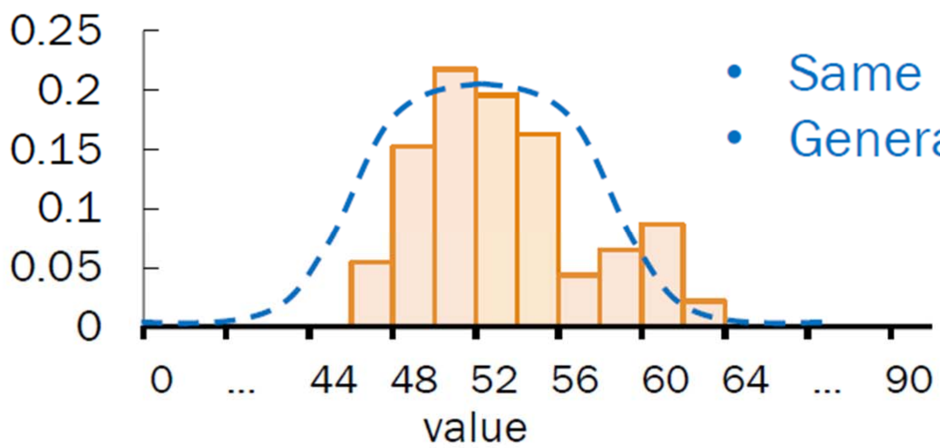
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: model real life situations with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.



- Same mean/var
- Generalizes well

Occam's Razor:

"Non sunt multiplicanda entia sine necessitate."

Entities should not be multiplied without necessity.

A Gaussian maximizes **entropy** for a given mean and variance.

Entropy

- 엔트로피(entropy) : 확률분포가 가지는 정보의 확신도 혹은 정보량을 수치로 표현한 것
  - ✓ 확률분포에서 특정한 값이 나올 확률이 높아지고 나머지 값의 확률은 낮아진다면 엔트로피가 작아진다.
  - ✓ 빈대로 여러가지 값이 나올 확률이 대부분 비슷한 경우에는 엔트로피가 높아진다.
- 엔트로피는 확률분포의 모양이 어떤지를 나타내는 특성값 중 하나로 볼 수도 있다. 확률 또는 확률밀도가 특정값에 몰려있으면 엔트로피가 작다고 하고 반대로 여러가지 값에 골고루 퍼져 있다면 엔트로피가 크다고 한다.

- 확률변수 Y가 카테고리 분포와 같은 이산확률변수이면, 엔트로피 H[Y}는

$$H[Y] = -\sum_{k=1}^{K} p(y_k)\log_2 p(y_k)$$

- K는 X가 가질 수 있는 클래스의 수이고 p(y)는 확률질량함수다. 확률의 로그값이 항상 음수이므로 음수 기호를 붙여서 양수로 만듬

- 확률변수 Y가 정규분포와 같은 연속확률변수이면, 엔트로피 H[Y]는

$$H[Y] = -\int_{-\infty}^{\infty} p(y)\log_2 p(y)\, dy$$

- p(y)는 확률밀도함수다.

Why the Normal?

- Common for natural phenomena: height, weight, etc.

  Actually log-norm

- Most noise in the world is Normal

  …mption

- Often results from … random v…

  Only if equally weighted

- Sampl… …ans are distributed normally

  (okay this one is true, we'll see this in 3 weeks)

because it's well understood

Stay critical of how to model real-world phenomena.

Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

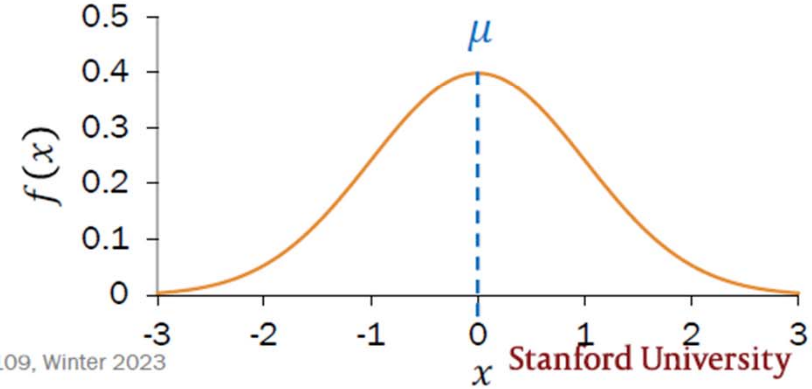
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

symmetric around μ

variance σ^2 manages spread

exponential tail

normalizing constant



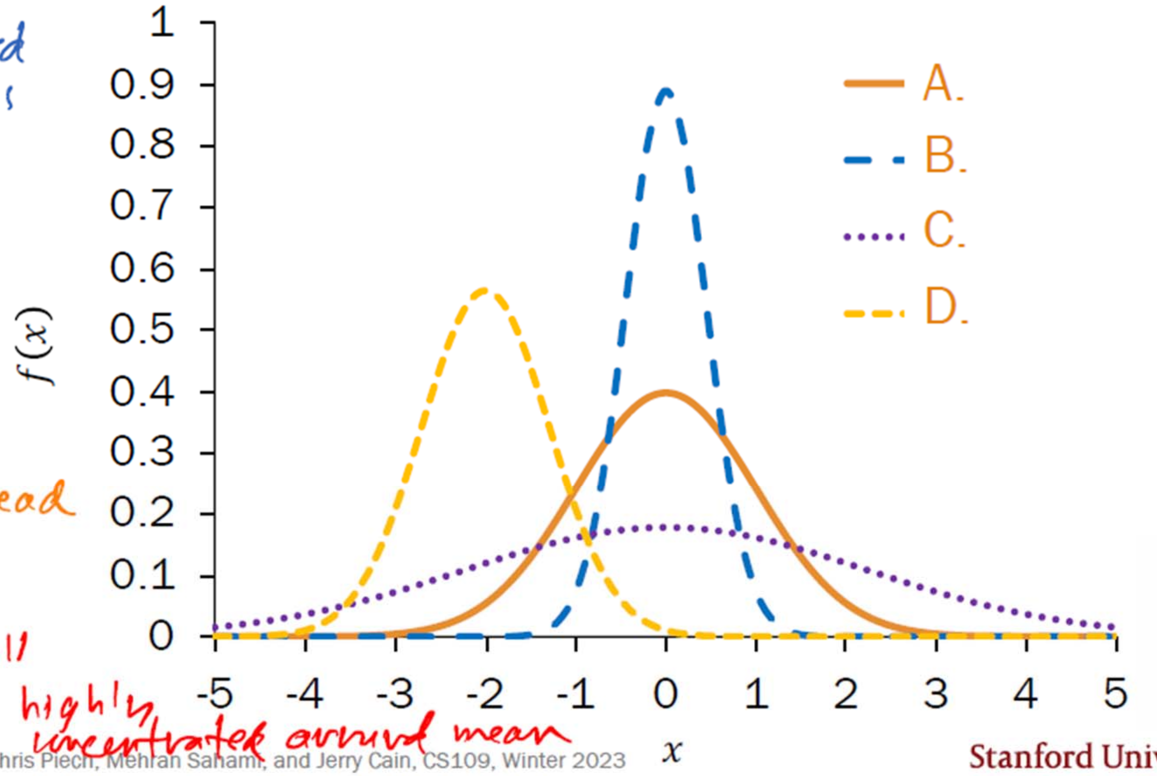
Normal Random Variable

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Match PDF to distribution:

1. $\mathcal{N}(0,1)$ A — of three centered at $x=0$, this has "middle" spread
2. $\mathcal{N}(-2,0.5)$ D — only one not centered at $x=0$
3. $\mathcal{N}(0,5)$ C — centered at $x=0$, $\sigma^2=5$ is largest spread
4. $\mathcal{N}(0,0.2)$ B — centered at $x=0$, small variance means highly concentrated around mean



Getting to class

random variable

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

μ

$$P(X \geq 6) = \int_6^\infty f(x)dx = \int_6^\infty \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-4)^2}{4}} dx$$

$2\sigma^2$

(tell Jerry if you solve this analytically and we'll be famous together)

Love and Anger in the Same Formula

Computing probabilities with Normal RVs

For a Normal RV $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

Cannot be solved analytically

except for a small number of x values!

However, we can solve for probabilities numerically using a function Φ:

argument is number of standard deviations from the mean

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

To get here, we'll first need to know some properties of Normal RVs.

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved **numerically**

Normal RV: Properties

Properties of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \le x) = F(x)$.

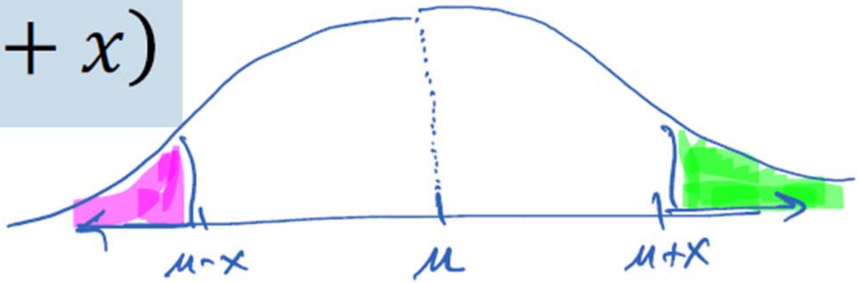
1. Linear transformations of Normal RVs are also Normal RVs.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2).$$

2. The PDF of a Normal RV is symmetric about the mean μ.

$$F(\mu - x) = 1 - F(\mu + x)$$

this says that area in purple
matches area in green



1. Linear transformations of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

Linear transformations of X are also Normal.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Proof:

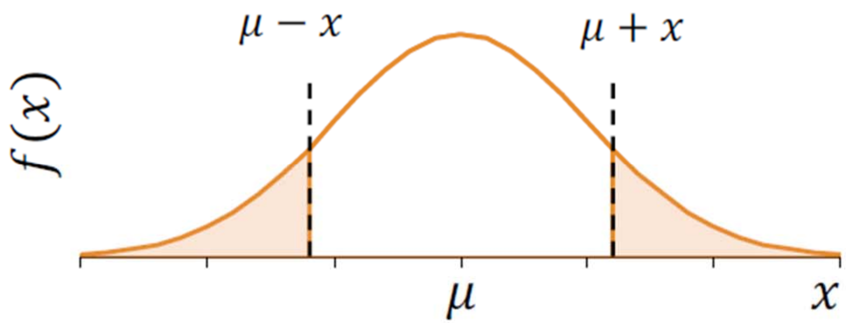
- $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$ — Linearity of Expectation
- $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$ — $\text{Var}(aX + b) = a^2\text{Var}(X)$
- Y is also Normal — Proof in Ross, 10th ed (Section 5.4)

2. Symmetry of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

The PDF of a Normal RV is symmetric about the mean μ.

$$F(\mu - x) = 1 - F(\mu + x)$$

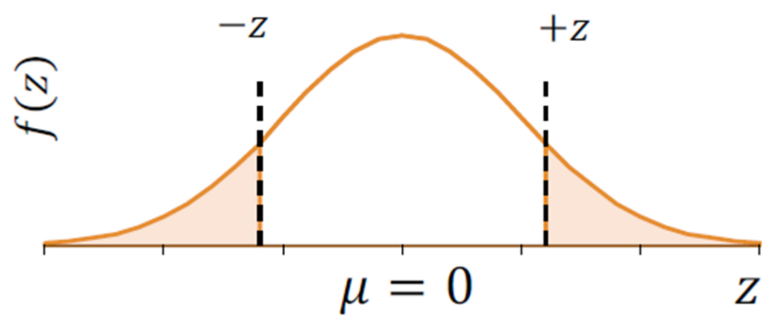


Using symmetry of the Normal RV

$F(\mu - x) = 1 - F(\mu + x)$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $y, z \geq 0$.



How do we compute the following probabilities?

1. $P(Z \leq z) \quad = F(z)$
2. $P(Z < z) \quad = F(z)$
3. $P(Z \geq z) \quad = 1 - F(z)$
4. $P(Z \leq -z) \quad = 1 - F(z)$
5. $P(Z \geq -z) \quad = F(z)$
6. $P(y < Z < z) \quad = F(z) - F(y)$

A. $F(z)$
B. $1 - F(z)$
C. $F(z) - F(y)$

Symmetry is particularly useful when computing probabilities of zero-mean Normal RVs.

Normal RV: Computing probability

Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

To compute the CDF, $P(X \leq x) = F(x)$:

- We cannot analytically solve the integral, as it has no closed form.
- ... but we ***can*** solve numerically using a function Φ:

Precalculated "Phi" function

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of the
Standard Normal, Z

Standard Normal RV, Z

The Standard Normal random variable Z is defined as follows:

$$Z \sim \mathcal{N}(0, 1)$$

Expectation $E[Z] = \mu = 0$

Variance $\text{Var}(Z) = \sigma^2 = 1$

Note: not a new distribution; just a special case of the Normal

Other names: Unit Normal

CDF of Z defined as: $P(Z \leq z) = \Phi(z)$

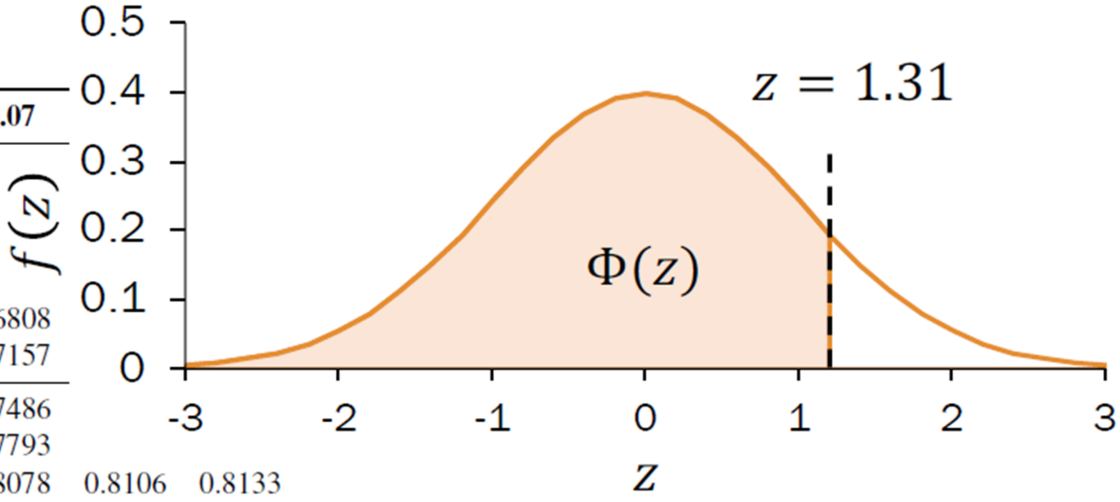
Φ has been numerically computed

Standard Normal Table

An entry in the table is the area under the curve to the left of z, $P(Z \le z) = \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07		
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0		
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0		
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0		
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0		
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808		
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157		
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486		
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793		
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	$\boxed{0.9049}$	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441

$$P(Z \le 1.31) = \Phi(1.31)$$



Standard Normal Table only has probabilities $\Phi(z)$ for $z \ge 0$.

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \le x) = F(x)$, we use Φ, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

define new variable Z to be $\frac{X-\mu}{\sigma}$

$$F(x) = P(X \le x)$$ Definition of CDF

$$= P(X - \mu \le x - \mu) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$ Algebra + $\sigma > 0$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X.
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0,1)$
- In other words, $\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$ with CDF Φ.

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

$$\begin{aligned} F(x) &= P(X \leq x) && \text{Definition of CDF} \\ &= P(X - \mu \leq x - \mu) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) && \text{Algebra} + \sigma > 0 \\ &= P\left(Z \leq \frac{x-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X.

1. Compute $z = (x-\mu)/\sigma$.
2. Look up $\Phi(z)$ in Standard Normal table.

Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_X(6)$$
$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$
$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$
$$\approx 1 - 0.9207$$
$$= 0.0793$$

Is there an easier way? (yes)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

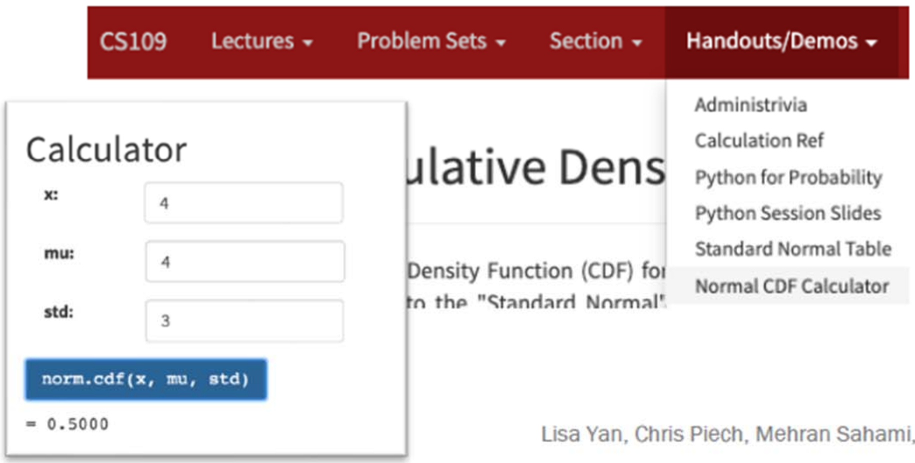
- Use Python

```python
from scipy import stats
X = stats.norm(mu, std)
X.cdf(x)
```

much more accurate!

SciPy reference:
https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html

- Use website tool



Website tool:
https://web.stanford.edu/class/cs109/handouts/normalCDF.html

Exercises

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$ $= 1 - P(X \le 0)$
$= 1 - F(0)$
$= 1 - \phi\left(\frac{0-3}{4}\right)$
$= 1 - \phi\left(\frac{-3}{4}\right)$
$= 1 - \left(1 - \phi\left(\frac{3}{4}\right)\right)$
$= 1 - 1 + \phi\left(\frac{3}{4}\right)$
$= \phi\left(\frac{3}{4}\right) = \underbrace{0.7734}_{\text{how?. where did this come from?}}$

I looked it up in the table!

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.
Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ?

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$ $= F(5) - F(2)$

$$= \Phi\left(\frac{5-3}{4}\right) - \Phi\left(\frac{2-3}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right)$$

$$= \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{4}\right) - 1$$

$$= 0.2902$$

look these up in the table!

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \left(1 - F(9)\right)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

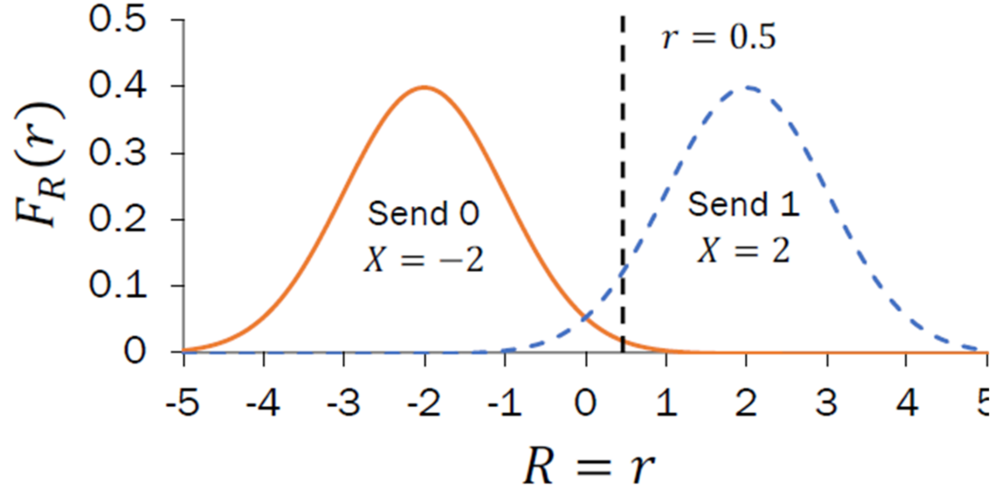
yay!

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: $\begin{cases} 1 & \text{if } R \geq 0.5 \\ 0 & \text{otherwise.} \end{cases}$



1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?
2. What is P(decoding error | original bit is 0)?

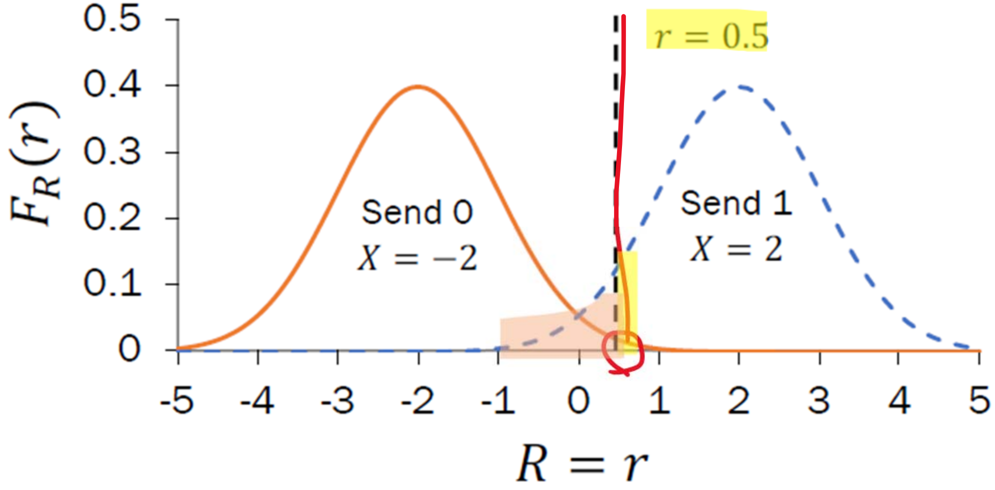
These probabilities are unequal. Why might this be useful?

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.



1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?

$$P(R < 0.5 | X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$

Y is Standard Normal

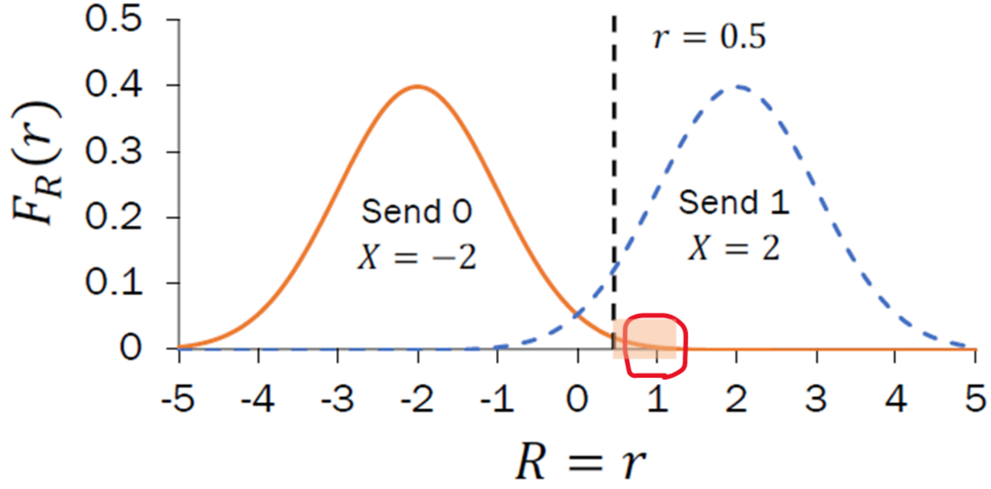
$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.



1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

0.0668

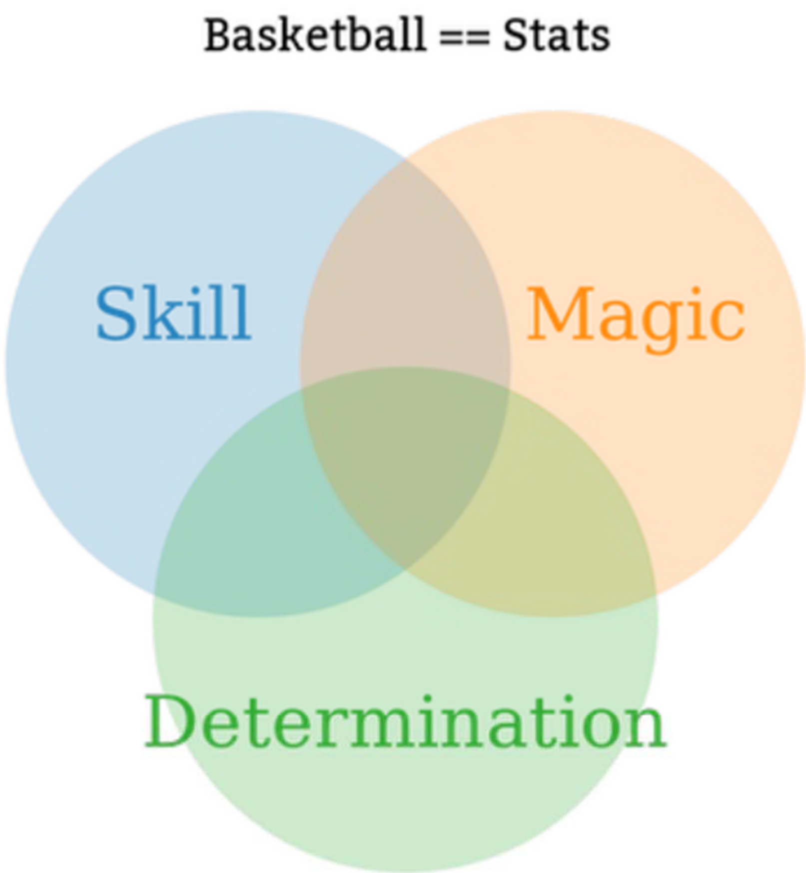
2. What is P(decoding error | original bit is 0)?

$$P(R \geq 0.5 | X = -2) = P(-2 + Y \geq 0.5) = \underbrace{P(Y \geq 2.5)}_{1 - \Phi(2.5)} \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.

Sampling with the Normal RV

ELO ratings



What is the probability that the Warriors win?
More generally: How can you model zero-sum games?

ELO ratings

Each team has an ELO score S, calculated based on its past performance.

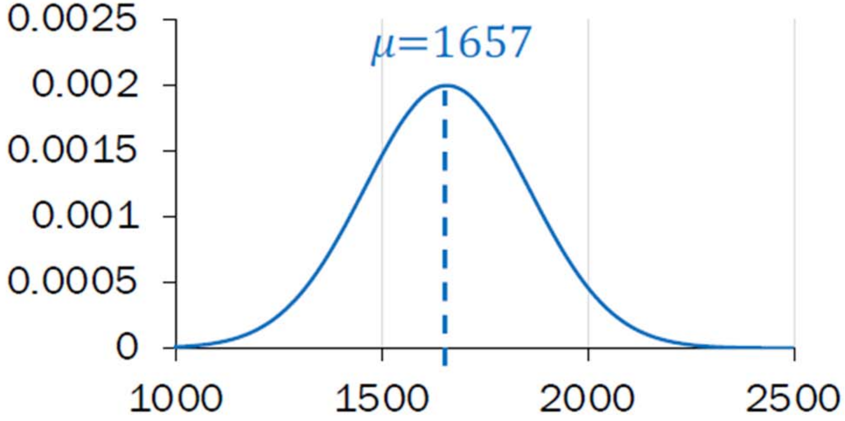
- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

Arpad Elo

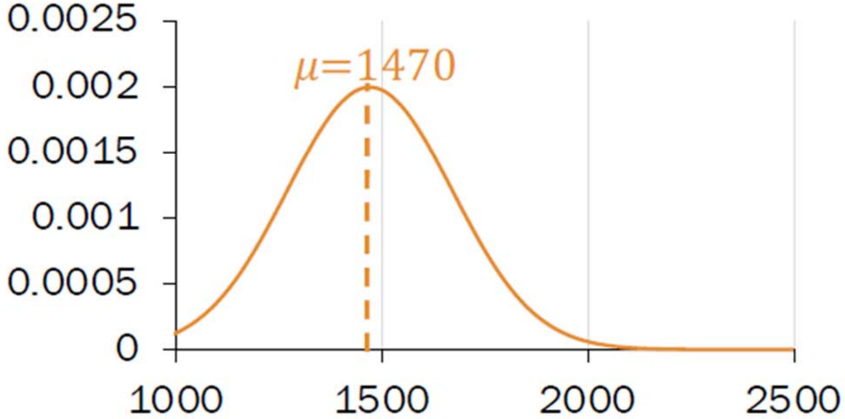
What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



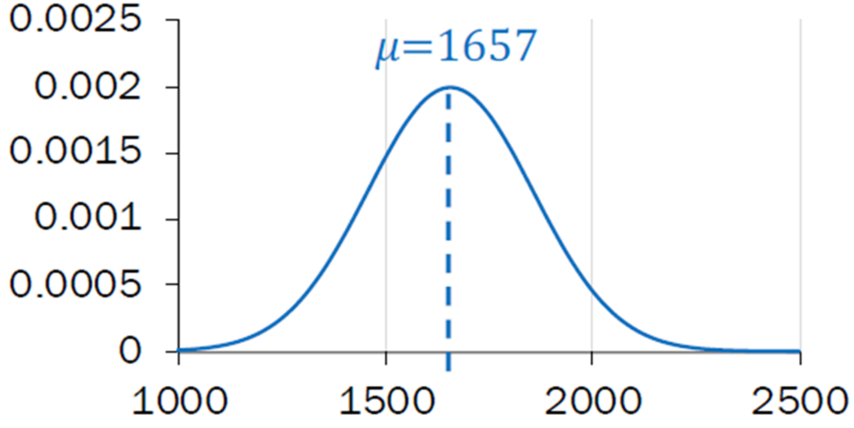
ELO ratings

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

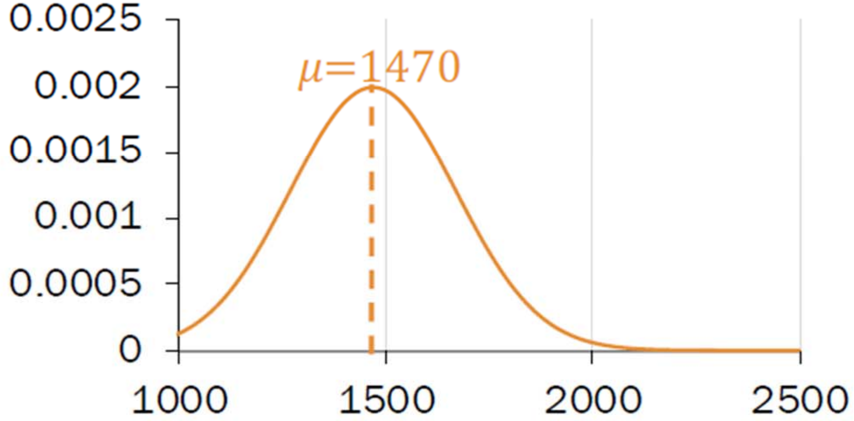
```python
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000

nSuccess = 0
for i in range(NTRIALS):
  w = stats.norm.rvs(WARRIORS_ELO, STDEV)
  o = stats.norm.rvs(OPPONENT_ELO, STDEV)
  if w > o: nSuccess += 1

print("Warriors sampled win fraction",
      float(nSuccess) / NTRIALS)
```

≈ 0.7488, calculated by sampling

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Is there a better way?

$$P(A_W > A_O)$$

actual depiction of someone understanding joint continuous random variables

- This is a probability of an event involving two continuous random variables!
- We'll solve this problem analytically in two weeks' time.

Big goal for next lecture: Events involving two discrete random variables.

Stay tuned!