Table of Contents

06: Random Variables

Jerry Cain

January 22, 2024

Lecture Discussion on Ed

Conditional Independence

Conditional Paradigm

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

when in doubt, simply assume that E is really just the full sample space S and see if the formula devolves into one you learned during Week 1 and 2.

Axiom 1	$0 \leq P(A \mid E) \leq 1$
Corollary 1 (complement)	$P(A \mid E) = 1 - P(A^C \mid E)$
Transitivity	$P(AB \mid E) = P(BA \mid E)$
Chain Rule	$P(AB \mid E) = P(B \mid E)P(A \mid BE)$
Bayes' Theorem	$P(A \mid BE) = \dfrac{P(B \mid AE)P(A \mid E)}{P(B \mid E)}$

BAE 's theorem?

Conditional Independence

Independent events E and F	$\Leftrightarrow$	$P(EF) = P(E)P(F)$ $P(E\|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$

Conditional Independence

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

each of these might be true, but only the third one conditions everything on E, so it's the only one that's equivalent to the above definition

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$

E is the new sample space, so left and right side must both be conditioned on E.

Netflix and Condition

Review

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is $P(E)$?

$$P(E) \approx \frac{\text{\# people who have watched movie}}{\text{\# people on Netflix}} = \frac{10{,}234{,}231}{50{,}923{,}123} \approx 0.20$$

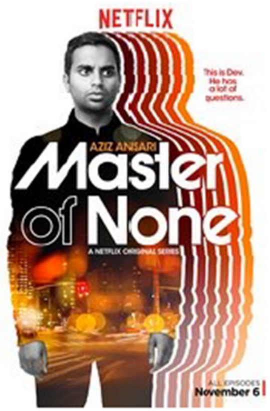
What is the probability that a user watches
Life is Beautiful, given they watched Amelie?

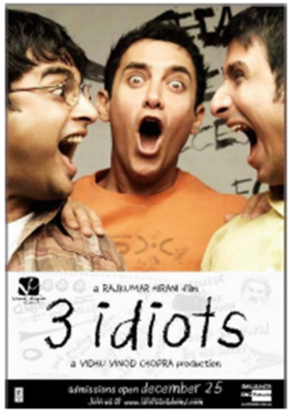
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}} \approx 0.42$$

Netflix and Condition

Review

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.





$P(E) = 0.19$	$P(E) = 0.32$	$P(E) = 0.20$	$P(E) = 0.09$	$P(E) = 0.20$
$P(E\|F) = 0.14$	$P(E\|F) = 0.35$	$P(E\|F) = 0.20$	$P(E\|F) = 0.72$	$P(E\|F) = 0.42$

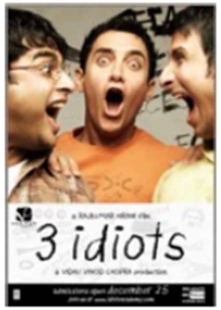
Independent!

Netflix and Condition (on many movies)

Watched:

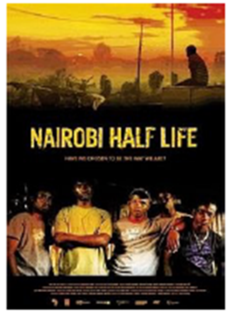
E_1

E_2



E_3

Will they watch?



E_4

What if $E_1 E_2 E_3 E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4 | E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)} = \frac{\text{\# people who have watched all 4}}{\text{\# people who have watched those 3}}$$

We need to keep track of an exponential number of movie-watching statistics

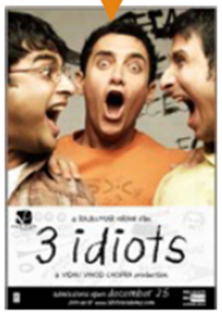
Netflix and Condition (on many movies)

K: likes international emotional comedies

Watched:

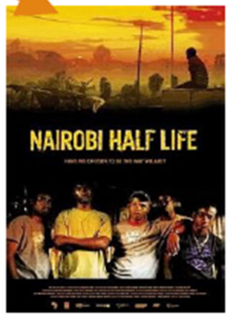
E_1

E_2



E_3

Will they watch?



E_4

Assume: $E_1 E_2 E_3 E_4$ are *conditionally independent given* K

simplifying assumption that works well in practice, even if it's not 100% accurate.

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)}$$

$$P(E_4|E_1E_2E_3K) = \underbrace{P(E_4|K)}_{\text{An easier probability to store and compute!}}$$

Conditional independence is a Big Deal

Conditional independence is a practical, real-world way of decomposing hard probability questions.

> "Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory."

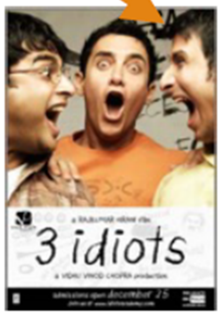
–Judea Pearl wins 2011 Turing Award [video of speech],
"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"

Netflix and Condition

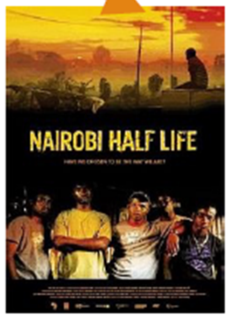
K: likes international emotional comedies

E_1

E_2



E_3



E_4

Challenge: How do we determine K? Stay tuned in 6 weeks' time!

$E_1 E_2 E_3 E_4$ are dependent

$E_1 E_2 E_3 E_4$ are conditionally independent given K

Dependent events can be conditionally independent.
(**And vice versa**: Independent events can be conditionally dependent.)

Random Variables

Random variables are like typed variables

type name value

```
int a = 5;
```

A is the number of Pokemon we bring to our *future* battle.

$$A \in \{1, 2, \ldots, 6\}$$

```
double b = 4.2;
```

B is the amount of money we get *after* we win a battle.

$$B \in \mathbb{R}^+$$

```
bit c = 1;
```

C is 1 *if* we successfully beat the Elite Four. 0 otherwise.

$$C \in \{0,1\}$$

CS variables

Random variables

Random variables are like typed variables (with uncertainty)

Random Variable

A random variable is a real-valued function defined on a sample space.

Experiment → Outcome → $X = k$

Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

1. What is the value of X for the outcomes:
   - (T,T,T)? $X=0$
   - (H,H,T)? $X=2$
2. What is the event (set of outcomes) where $X = 2$?

   $\{(H,H,T),(H,T,H),(T,H,H)\}$
3. What is $P(X = 2)$? $\frac{3}{8}$ ← cardinality of this set ← really $2^3 = 8$

Random variables are NOT events!

It is confusing that random variables and events use the same notation.

- **Random variables $\neq$ events.**
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

$$X = 2$$

event

$$P(X = 2)$$

probability
(number b/t 0 and 1)

Random variables are NOT events!

It is confusing that random variables and events use the same notation.

- Random variables $\neq$ events.
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

$X = x$	Set of outcomes	$P(X = k)$
$X = \mathbf{0}$	{(T, T, T)}	1/8
$X = \mathbf{1}$	{(H, T, T), (T, H, T), (T, T, H)}	3/8
$X = \mathbf{2}$	{(H, H, T), (H, T, H), (T, H, H)}	3/8 ← consistent with earlier slide
$X = \mathbf{3}$	{(H, H, H)}	1/8
$X \geq 4$	{ }	0

PMF/CDF

So far

3 coins are flipped. Let X = # of heads. X is a random variable.

could be any value in the support of X.

Experiment → Outcome (flip 2 heads) → $X = \underline{2}$ → $P(X = \underline{2})$

$X = x$	$P(X = k)$	Set of outcomes
$X = \mathbf{0}$	1/8	{(T, T, T)}
$X = \mathbf{1}$	3/8	{(H, T, T), (T, H, T), (T, T, H)}
$X = \mathbf{2}$	3/8	{(H, H, T), (H, T, H), (T, H, H)}
$X = \mathbf{3}$	1/8	{(H, H, H)}
$X \geq 4$	0	{ }

Can we get a "shorthand" for this last step?
Seems like it might be useful!

Probability Mass Function

PMF : Probabilty Mass Function (확률 질량 함수)
https://ko.wikipedia.org/wiki/확률_질량_함수

3 coins are flipped. Let $X = \#$ of heads. X is a random variable.

parameter/input k

A function on k
with range [0,1]

$$P(X = k)$$

return value/output
number between
0 and 1

What would be a *useful* function to define?

The probability of the event that a random variable X takes on the value k!

For **discrete random variables**, this is a **probability mass function**.

Probability Mass Function

3 coins are flipped. Let X = # of heads. X is a random variable.

A function on k with range [0,1]

2

parameter/input k: a value of X

$$P(X = 2) \rightarrow 0.375$$

return value/output: probability of the event $X = 2$

```
N = 3
P = 0.5

def prob_event_y_equals(k):
  n_ways = scipy.special.binom(N, k)
  p_way = np.power(P, k) * np.power(1 - P, N-k)
  return n_ways * p_way
```

probability mass function

seems like $p(X=k) = \binom{3}{k} 0.5^3$

for all reasonable values of k.

Discrete RVs and Probability Mass Functions

A random variable X is discrete if it can take on countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \dots\}$

The probability mass function (PMF) of a discrete random variable is

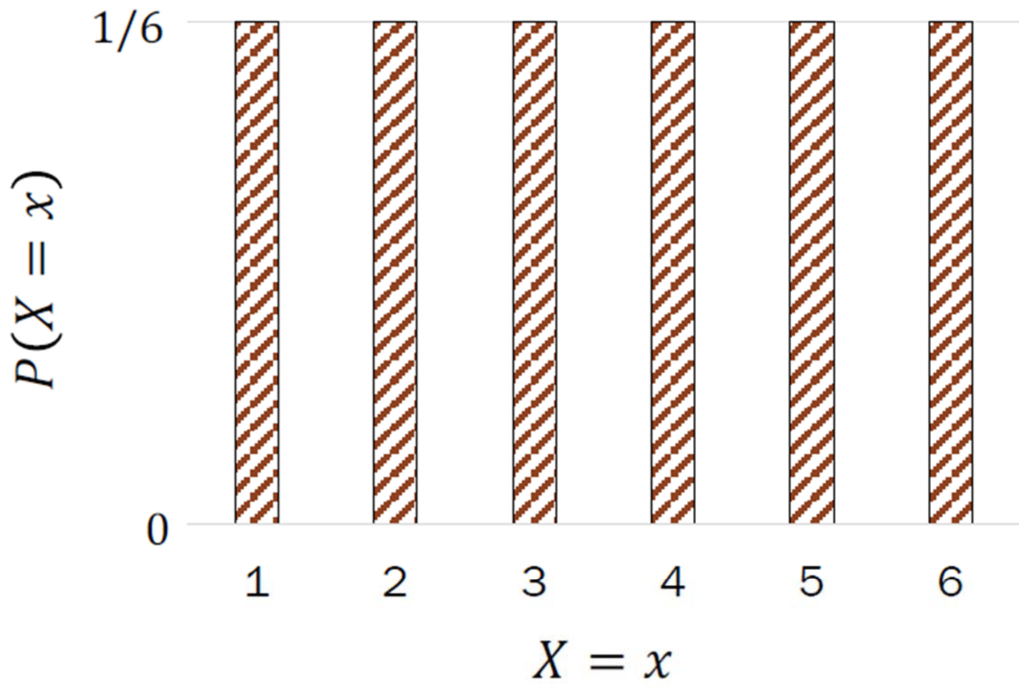
$$P(X = x) = \underbrace{p(x) = p_X(x)}_{\text{shorthand notation}}$$

- Probabilities must sum to 1: $\displaystyle\sum_{i=1}^{\infty} p(x_i) = 1$

This last point is a good way to verify any PMF you create is valid

PMF for a single 6-sided die

Let X be a random variable that represents the result of a single dice roll.



- Support of X : $\{1, 2, 3, 4, 5, 6\}$
- Therefore X is a discrete random variable.
- PMF of X:

$$p(x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

Cumulative Distribution Functions

CDF : 누적 분포 함수
https://ko.wikipedia.org/wiki/누적_분포_함수

For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{ where } -\infty < a < \infty$$

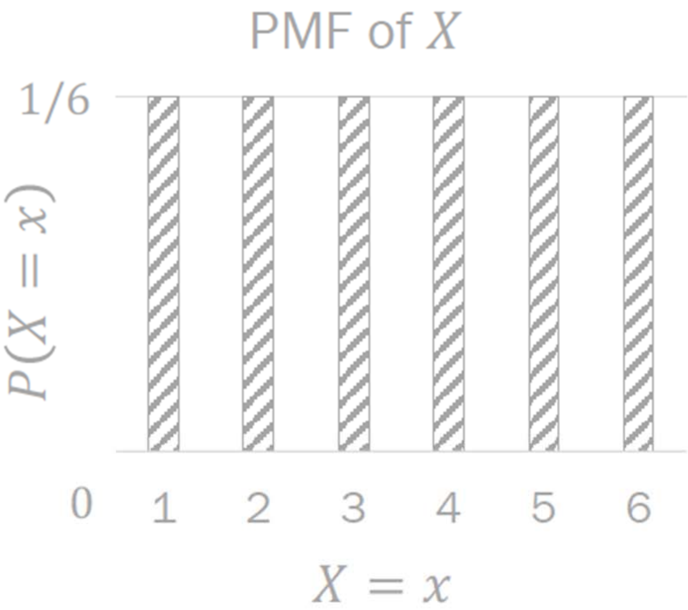
For a discrete RV X, the CDF is:

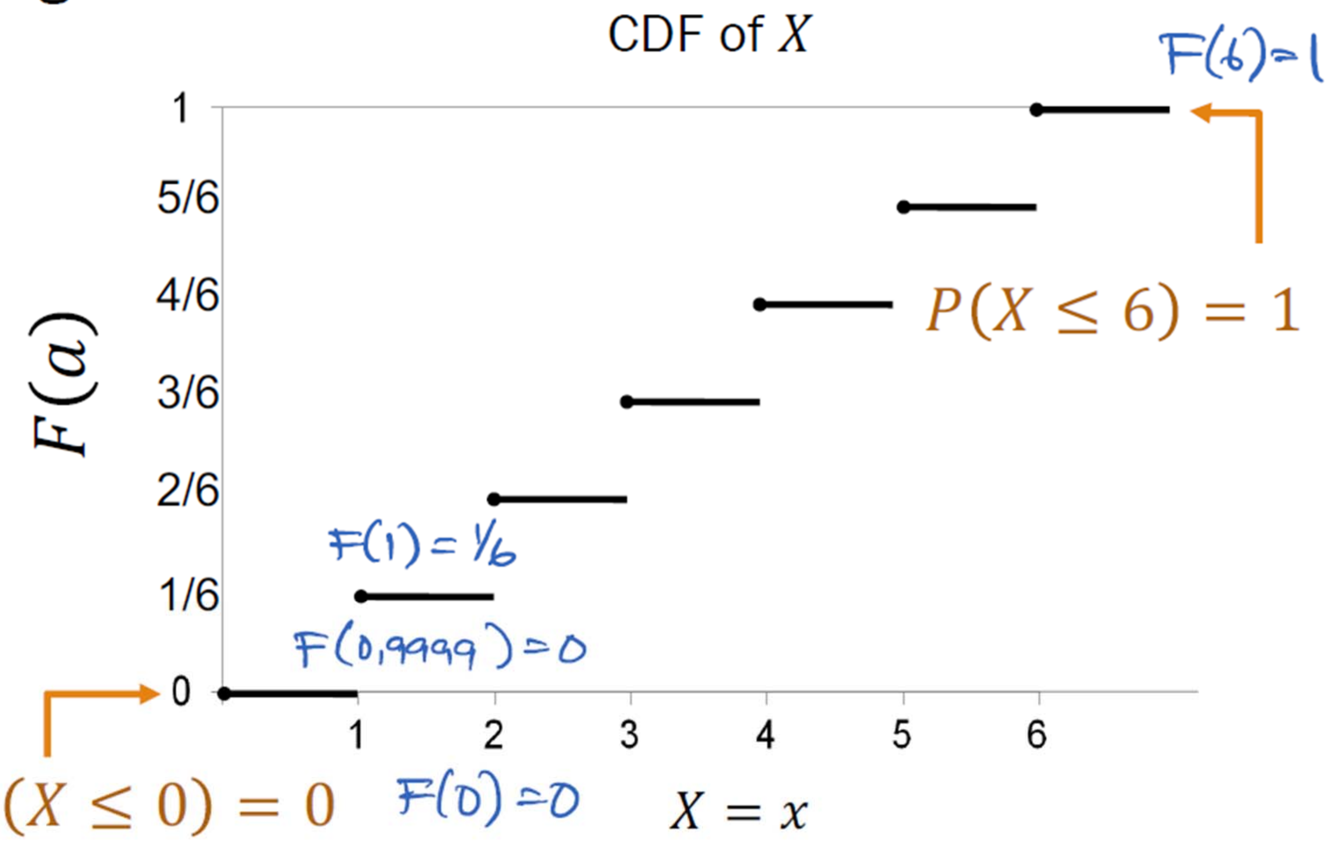
$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

CDFs as graphs

CDF (cumulative distribution function) $F(a) = P(X \leq a)$

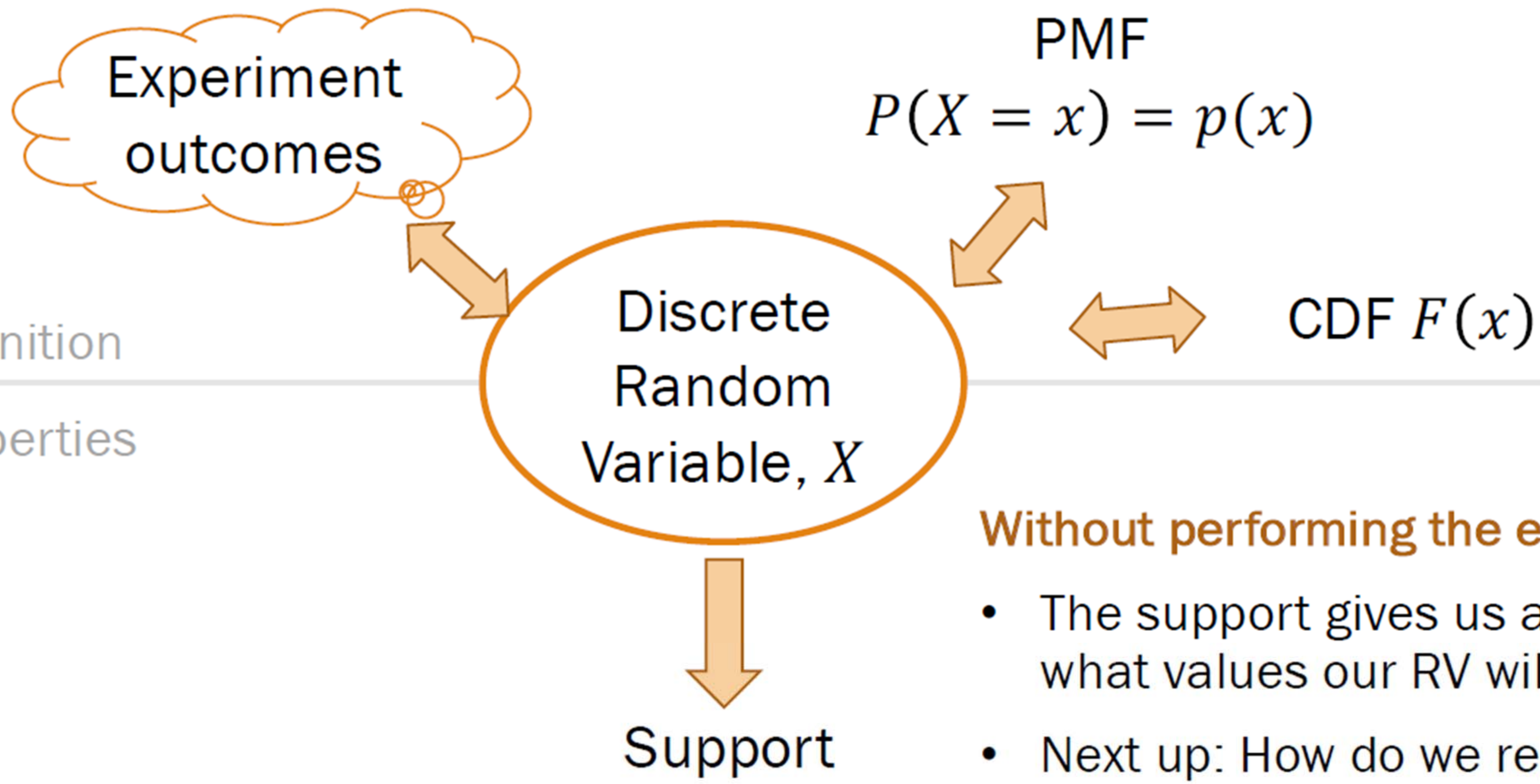
Let X be a random variable that represents the result of a single dice roll.





Expectation

Discrete random variables



Definition

Properties

Without performing the experiment:

- The support gives us a ballpark of what values our RV will take on
- Next up: How do we report the "average" value?

Expectation

The expectation of a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

- Note: sum over all values of $X = x$ that have non-zero probability.
- Other names: mean, expected value, weighted average, center of mass, first moment

Expectation of a die roll

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$ Expectation of X

What is the expected value of a 6-sided die roll?

1. Define random variables

X = RV for value of roll

$$P(X = x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

Important properties of expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

- Let X = 6-sided dice roll, $Y = 2X - 1$.
- $E[X] = 3.5$
- $E[Y] = 6$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

Sum of two dice rolls:

- Let X = roll of die 1, Y = roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

These properties let you avoid defining difficult PMFs.

Linearity of Expectation proof

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[aX + b] = aE[X] + b$$

Proof:

$$\begin{aligned} E[aX + b] &= \sum_x (ax + b)p(x) = \sum_x axp(x) + bp(x) \\ &= a \sum_x xp(x) + b \sum_x p(x) \\ &= a\, E[X] + b \cdot 1 \end{aligned}$$

Expectation of Sum intuition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[X+Y] = E[X] + E[Y]$$

we'll prove this in a few lectures

Intuition for now:

X	Y	$X+Y$
3	6	9
2	4	6
6	12	18
10	20	30
-1	-2	-3
0	0	0
8	16	24

Average:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \quad + \quad \frac{1}{n}\sum_{i=1}^{n} y_i \quad = \quad \frac{1}{n}\sum_{i=1}^{n} (x_i + y_i)$$

$$\frac{1}{7}(28) \quad + \quad \frac{1}{7}(56) \quad = \quad \frac{1}{7}(84)$$

LOTUS proof

$$E[g(X)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

Let $Y = g(X)$, where g is a real-valued function.

$$E[g(X)] = E[Y] = \sum_j y_j p(y_j)$$

assume that $p(y_j) = \sum p(x_i)$ i such that $g(x_i) = y_j$

$$= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} y_j\, p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} \left[g(x_i)\, p(x_i)\right]$$

no dependence on j

$$= \sum_i g(x_i)\, p(x_i)$$

For you to review so that you can sleep tonight!

Exercises

A Whole New World with Random Variables

Event-driven probability

- Relate only binary events
  - Either something happens (E)
  - or it doesn't happen (E^C)
- Can only report probability
- Lots of combinatorics

Random Variables

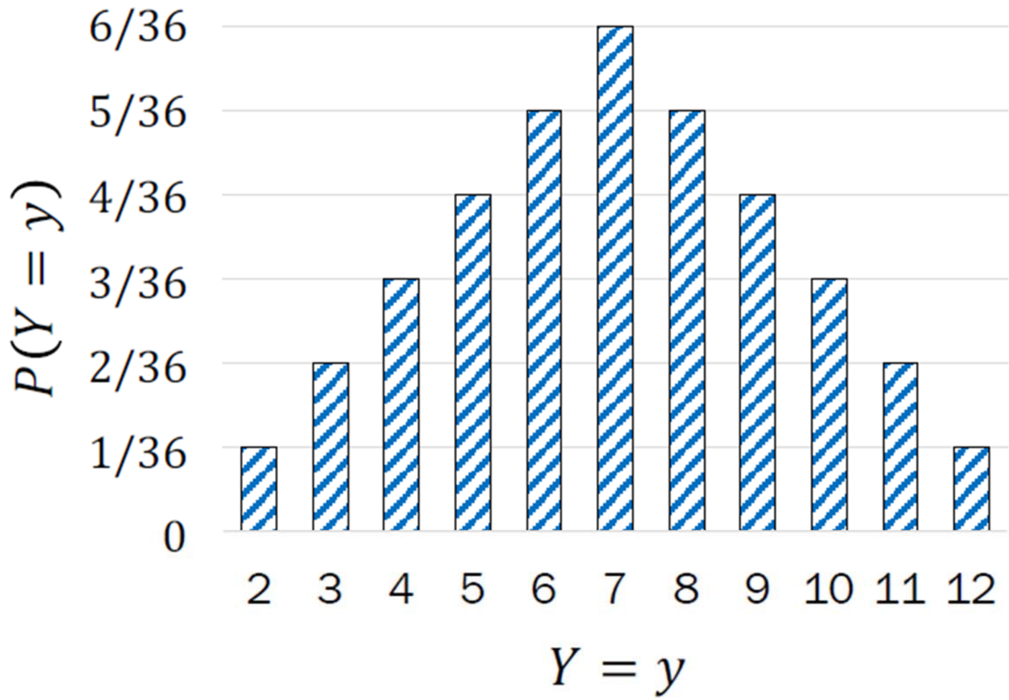
- Link multiple similar events together ($X = 1,\ X = 2, \dots, X = 6$)
- Can compute statistics: report the "average" outcome
- Once we have the PMF (for discrete RVs), we can do regular math

PMF for the sum of two dice

Let Y be a random variable that represents the sum of two independent dice rolls.

Support of Y: $\{2, 3, \dots, 11, 12\}$

$$p(y) = \begin{cases} \frac{y-1}{36} & y \in \mathbb{Z}, 2 \leq y \leq 6 \\ \frac{13-y}{36} & y \in \mathbb{Z}, 7 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$



Sanity check: $$\sum_{y=2}^{12} p(y) = 1$$

Example random variable

Consider 5 flips of a coin which comes up heads with probability p. Each coin flip is an independent trial. **Let $\boldsymbol{Y}$ = # of heads on 5 flips.**

1. What is the support of Y? In other words, what are the values that Y can take on with non-zero probability?

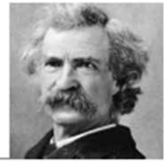
   $\{0, 1, 2, 3, 4, 5\}$

2. Define the event $Y = 2$. What is $P(Y = 2)$? $P(Y = 2) = \binom{5}{2} p^2 (1-p)^3$

3. What is the PMF of Y? In other words, what is $P(Y = k)$, for k in the support of Y?

   $P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$

Lying with statistics



A school has 3 classes with 5, 10, and 150 students.

What is the average class size?

1. Interpretation #1
   - Randomly choose a class with equal probability.
   - X = size of chosen class

$$E[X] = 5\left(\frac{1}{3}\right) + 10\left(\frac{1}{3}\right) + 150\left(\frac{1}{3}\right)$$

$$= \frac{165}{3} = 55$$

What alumni relations usually reports

2. Interpretation #2
   - Randomly choose a student with equal probability.
   - Y = size of chosen class

$$E[Y] = 5\left(\frac{5}{165}\right) + 10\left(\frac{10}{165}\right) + 150\left(\frac{150}{165}\right)$$

$$= \frac{22635}{165} \approx 137$$

Average student perception of class size

Being a statistician unconsciously

$E[g(X)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 \quad = 0$

B. $E[Y] = E[0] \quad = 0$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \quad = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3}|1| = \frac{2}{3}$

E. C and D

Being a statistician unconsciously

$E[g(X)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 \quad = 0$ ✗ $E[X]$

B. $E[Y] = E[0] \quad = 0$ ✗ $E[E[X]]$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \quad = \frac{2}{3}$

1. Find PMF of Y: $p_Y(0) = \frac{1}{3}, p_Y(1) = \frac{2}{3}$
2. Compute $E[Y]$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3}|1| = \frac{2}{3}$

Use LOTUS by using PMF of X:

1. $P(X = x) \cdot |x|$
2. Sum up

E. C and D