Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

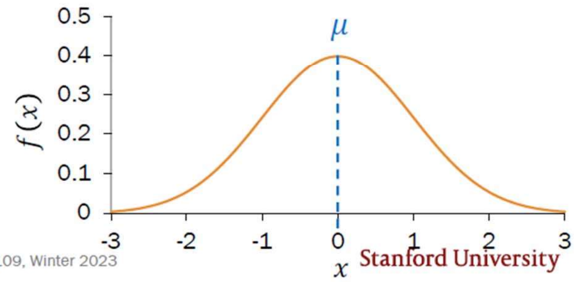
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

symmetric around μ

variance σ^2 manages spread

exponential tail

normalizing constant



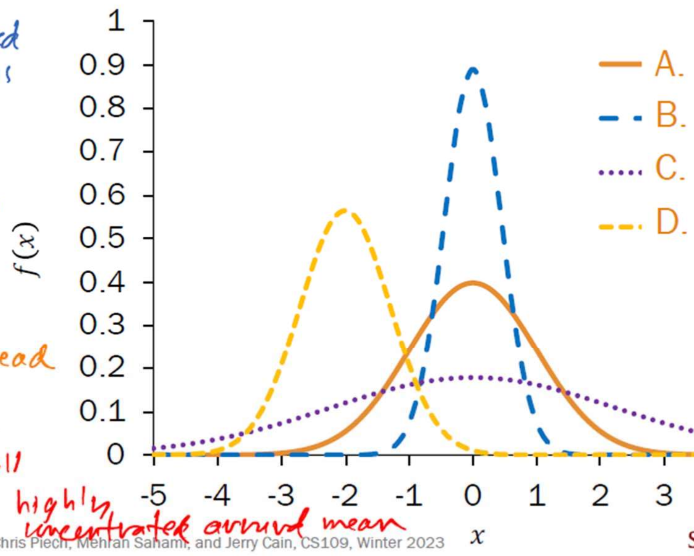
Normal Random Variable

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Match PDF to distribution:

1. $\mathcal{N}(0,1)$ A of three centered at x=0, this has "middle" spread
2. $\mathcal{N}(-2,0.5)$ D only one not centered at x=0
3. $\mathcal{N}(0,5)$ C centered at x=0, $\sigma^2=5$ is largest spread
4. $\mathcal{N}(0,0.2)$ B centered at x=0, small variance means highly concentrated around mean



Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$ ✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_X(6)$$
$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$
$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$
$$\approx 1 - 0.9207$$
$$= 0.0793$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$ $= 1 - P(X \leq 0)$

$$= 1 - F(0)$$
$$= 1 - \phi\left(\frac{0-3}{4}\right)$$
$$= 1 - \phi\left(\frac{-3}{4}\right)$$
$$= 1 - \left(1 - \phi\left(\frac{3}{4}\right)\right)$$
$$= 1 - 1 + \phi\left(\frac{3}{4}\right)$$
$$= \phi\left(\frac{3}{4}\right) = 0.7734$$

how?. where did this come from?

I looked it up in the table!

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$ $= F(5) - F(2)$

$= \Phi\left(\frac{5-3}{4}\right) - \Phi\left(\frac{2-3}{4}\right)$

$= \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{4}\right)$

$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right)$

$= \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{4}\right) - 1$

$= 0.2902$

look these up in the table!

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Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
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- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
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Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

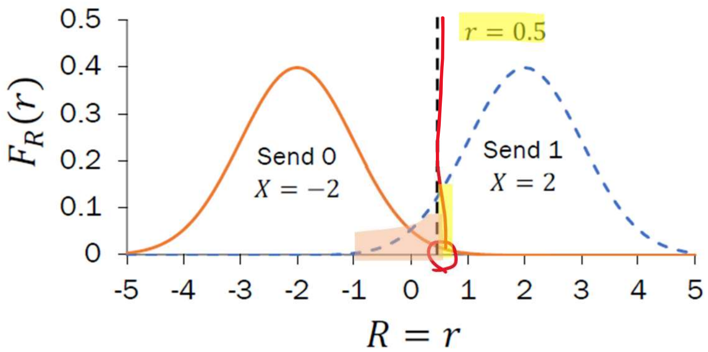
$$\approx 0.1337$$ yay!

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.



1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?

$$P(R < 0.5 \mid X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$ Y is Standard Normal

$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

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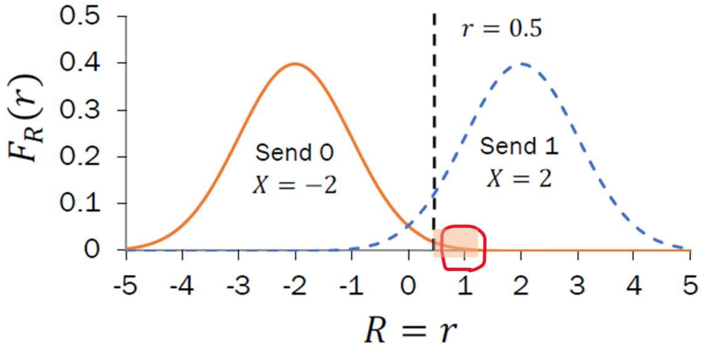
- $R = X + Y$ voltage received.

Decode: $\begin{cases} 1 & \text{if } R \geq 0.5 \\ 0 & \text{otherwise.} \end{cases}$



1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

0.0668

2. What is P(decoding error | original bit is 0)?

$$P(R \geq 0.5 | X = -2) = P(-2 + Y \geq 0.5) = \overbrace{P(Y \geq 2.5)}^{1-\Phi(2.5)} \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.