최적화 수학

1. Matrix

Def. 1-1

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

* Square matrix: $m = n$

Def. 1-2

If $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ then

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$

$$\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$$

$$c\mathbf{A} = (ca_{ij})$$

*zero matrix: $\mathbf{O}$ if $a_{ij} = 0 \quad \forall i, j$

Theorem. 1-1

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

Def. 1-3

If $\mathbf{A} = (a_{ij}) \in \mathrm{Mat}(m, p; \mathbb{R})$

$\mathbf{B} = (b_{ij}) \in \mathrm{Mat}(p, n; \mathbb{R})$ then

$$\mathbf{AB} = (c_{ij}) \in \mathrm{Mat}(m, n; \mathbb{R})$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Theorem. 1-2

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AB + BC$$

Ex. 1-3

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{AB} =$$

$$\mathbf{BC} =$$

Ex. 1-3

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{AB} =$$

$$\mathbf{BA} =$$

Def. 1-4 Identity Matrix

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = (\delta_{ij})$$

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Ex. 1-4

$$\mathbf{A}^2 - \mathbf{I} = (\mathbf{A} + \mathbf{I})(\mathbf{A} - \mathbf{I})$$

$$\mathbf{A}^3 + \mathbf{I} = (\mathbf{A} + \mathbf{I})(\mathbf{A}^2 - \mathbf{A} + \mathbf{I})$$

Def. 1-5 Transpose

$$\mathbf{A}^{\mathrm{T}} = (a_{ji})$$

Theorem 1-3

$$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$$

$$(\mathbf{A} \pm \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \pm \mathbf{B}^{\mathrm{T}}$$

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

$$(c\mathbf{A})^{\mathrm{T}} = c\mathbf{A}^{\mathrm{T}}$$

Def. 1-6 Symmetric Matrix

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

Def. 1-7 Inverse Matrix $\mathbf{A}^{-1}$

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

*$\mathbf{A}$ is invertible <-> $\exists \mathbf{A}^{-1}$

Ex. 1-6

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -2 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

Show $\mathbf{B} = \mathbf{A}^{-1}$

* $\mathbf{AB} = \mathbf{I}$ 로 충분한가?

Theorem. 1-4

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(c\mathbf{A})^{-1} = c^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{k}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{k}$$

Theorem. 1-5

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Ex. 1-7

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{A}^{-1} =$$

$$\mathbf{B}^{-1} =$$

Def. 1-8 Determinant

For $\mathbf{A} = (a_{ij})$ $\qquad$ $\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det \mathbf{A}_{ij}$

minor

cofactor

* $\det(\mathbf{I}) = 1$

$\det[\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_n] = -\det \mathbf{A}$

$\det[k\mathbf{v}_1 + l\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$

$= k \det[\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n] + l \det[\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$

*Sarrus's method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Ex. 1-9

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 3 & 1 & 5 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 4 & 2 & 3 \end{bmatrix}$$

$$\det \mathbf{A} =$$

$$\det \mathbf{B} =$$

Theorem. 1-6

If $\mathbf{A}=[\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_i,\cdots,\mathbf{v}_j,\cdots,\mathbf{v}_n]$ then

$$\det[\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_i+c\mathbf{v}_j,\cdots,\mathbf{v}_j,\cdots,\mathbf{v}_n]=\det\mathbf{A}$$

$$\det[\mathbf{v}_1,\mathbf{v}_2,\cdots,k\mathbf{v}_i,\cdots,\mathbf{v}_j,\cdots,\mathbf{v}_n]=k\det\mathbf{A}$$

$$\det[\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_j,\cdots,\mathbf{v}_i,\cdots,\mathbf{v}_n]=-\det\mathbf{A}$$

$$\det[\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_i,\cdots,\mathbf{v}_i,\cdots,\mathbf{v}_n]=0$$

$$\det[\mathbf{AB}]=\det\mathbf{A}\det\mathbf{B}$$

$$\det[\mathbf{A}^T]=\det\mathbf{A}$$

Theorem. 1-7

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21} & \cdots & \mathbf{C}_{n1} \\ \mathbf{C}_{12} & \mathbf{C}_{22} & \cdots & \mathbf{C}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{1n} & \mathbf{C}_{2n} & \cdots & \mathbf{C}_{nn} \end{bmatrix}}$$

Adjoint matrix = $\text{adj}\ \mathbf{A}$

* Cramer's rule

$$\mathbf{A} \cdot \text{adj}\ \mathbf{A} = (\det \mathbf{A})\mathbf{I}$$

Ex. 1-11

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} =$$

Ex. 1-12

$$\begin{aligned} 5x_1 + 3x_2 &= 1 \\ 3x_1 + 2x_2 &= -2 \end{aligned}$$

$$\begin{aligned} 3x_1 - 7x_2 &= 5 \\ 6x_1 - 14x_2 &= 10 \end{aligned}$$

Def. 1-10 Row echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*Gauss Elimination

$$x_1 + 3x_2 + x_3 + 5x_4 = 10$$

$$2x_2 + x_3 + x_4 = 1$$

$$2x_3 = -2$$

$$4x_2 + 2x_3 + 3x_4 = 4$$

Def. 1-11 Linear Transformation

$$T(u+v)=T(u)+T(v)$$

$$T(cu)=cT(u)$$

Ex. 1-15

$$T(x, y) = (x + 2y, 3x - y)$$

$$T(x, y, z) = (x^2 + y - z, x - yz + 1)$$

Theorem 1.8

If $T:\mathbb{R}^m \to \mathbb{R}^n$ is linear transformation,

then $\exists \mathbf{A},\ T(\mathbf{x}) = \mathbf{A}\mathbf{x}$

Ex. 1-16

(a) $T(x, y) = (x - 3y, 2x + y, x - y)$

(b) $U(\mathbf{e}_1) = (1, 2)$

$U(\mathbf{e}_2) = (2, 1)$

$U(\mathbf{e}_3) = (-1, 2)$

$U(\mathbf{e}_4) = (3, 3)$

* Linear transformation

$$\mathbf{A}\mathbf{x}$$

* Homogeneous transformation

$$\mathbf{A}\mathbf{x}+\mathbf{b}$$

Def. 1-13 Scaling

$$T:(x,y)\to(kx,ky)=(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Def. 1-14 Symmetric

$$T:(x,y)\rightarrow(x,-y)=(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T:(x,y)\rightarrow(-x,y)=(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Def. 1-14 Symmetric

$$T:(x,y)\to(-x,-y)=(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T:(x,y)\to(y,x)=(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Def. 1-15 Rotation

$$T:(x,y)\rightarrow(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Def. 1-16 Composite transformation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \mathbf{y}$$

$$U(\mathbf{y}) = \mathbf{B}\mathbf{y}$$

$$U \circ T(\mathbf{x}) = \mathbf{B}\mathbf{A}\mathbf{x}$$

Ex. 1-19

$$\mathbf{A} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Def. 1-17 Inverse transformation

If $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ then $T^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x}$

Ex. 1-20

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, x_1 - x_2, -2x_1 + x_3)$$

Def. 1-18 Vector space

$$\mathbf{u}+\mathbf{v}\in\mathbf{V}$$

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$$

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

$$\mathbf{u}+\mathbf{0}=\mathbf{u},\ \ \mathbf{0}\in\mathbf{V}$$

$$\mathbf{u}+(-\mathbf{u})=0,\ \ -\mathbf{u}\in\mathbf{V}$$

$$c\mathbf{u}\in\mathbf{V}$$

$$c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$$

$$(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$$

$$(cd)\mathbf{u}=c(d\mathbf{u})$$

$$1\cdot\mathbf{u}=\mathbf{u}$$

Ex. 1-21 Polynomial set

$$P(t) = \sum_{i=0}^{n} a_n x^n$$

* $\{f \mid f : [0,1] \to \mathbb{R}\}$

* $\{f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}\}$ Fourier's series

Def. 1-19 Subspace

$$\mathbf{H} \subset \mathbf{V}$$

$$\mathbf{u} + \mathbf{v} \in \mathbf{H}$$

$$\mathbf{0} \in \mathbf{H}$$

$$c\mathbf{u} \in \mathbf{H}$$

Ex. 1-22

$$\mathbf{H} = \left\{ (a, b, 0) \in \mathbb{R}^3 \right\}$$

Def. 1-20 Linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$$

* Spanning set

$$span\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n\}$$

Theorem 1-10

$span\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is subspace of V

Ex. 1-23

$H=\{(3a,2a-b,4b)\}$ is subspace of $\mathbb{R}^3$

Def. 1-21 Linearly independent

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n = 0$$

iff $c_1 = c_2 = c_3 = \cdots = c_n = 0$

Ex. 1-24

(a) $\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (-1,0,2) \quad \mathbf{v}_3 = (0,2,4)$

(b) $p_1(t) = 1 \quad p_2(t) = t - t^2 \quad p_3(t) = 2t^2 - 2t + 5$

Def. 1-22 Basis

$$span\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = H$$

$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent

Ex. 1-25

$\{1, t, t^2, \cdots, t^n\}$ is a basis set of $p_n(t)$

Def. 1-23 Eigenvalue, Eigenvector

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$\mathbf{x}$ is eigenvector, characteristic vector

λ is eigenvalue, characteristic value

Theorem 1-13

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

λ is eigenvalue of $\mathbf{A}^T$

$$\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$$

$$\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$$

Theorem 1-14

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \qquad \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \qquad \cdots \qquad \mathbf{A}\mathbf{v}_n = \lambda_n \mathbf{v}_n$$

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_n$$

then,

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n$$

is linearly independent

Def. 1-24 Similar transformation

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Theorem. 1-15

If $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

then,

eigenvalue of A =eigenvalue of B

Def. 1-25 Diagonalization

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

Theorem 1-17

$\mathbf{A}$ is diagonalizable, iff
$\mathbf{A}$ has n linearly independent eigenvectors.

Def. 1-26

$\mathbf{u} \cdot \mathbf{v} = 0$ <-> u and v are orthogonal

Theorem 1-18

$$\mathbf{A} = \mathbf{A}^T$$

Eigenvectors of *A* are orthogonal

Theorem 1-19

If $\mathbf{A} = \mathbf{A}^T$

then $\mathbf{A} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T$

Def. 1-27 symmetric positive definite matrix

If $\mathbf{A} = \mathbf{A}^T$

and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$

$\mathbf{A} > 0$

Theorem 1-20

If $\mathbf{A} = \mathbf{A}^T$

and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$

then eigenvalue of A is positive.

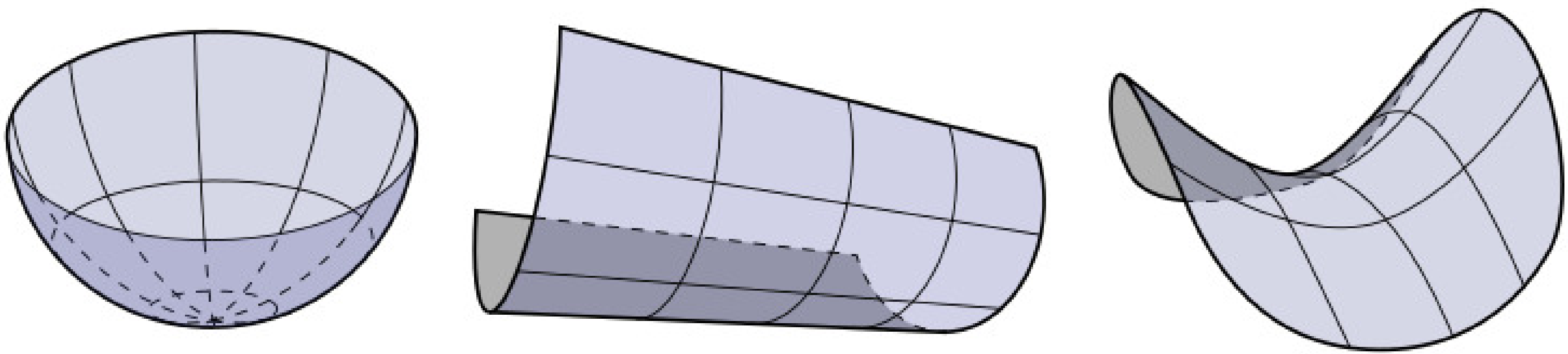
Def. 1-28 Quadratic form

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

s.t. $\mathbf{A} = \mathbf{A}^T$

Theorem 1-21.

If $\mathbf{A}$ is symmetric positively definite matrix, then $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x}$ is convex function.



$x^2 + y^2$ (definite) | x^2 (semidefinite) | $x^2 - y^2$ (indefinite)

Question?