Discussion 3: Recursion, Tree Recursion **disc03.pdf (disc03.pdf)**

This is an online worksheet that you can work on during discussions. Your work is not graded and you do not need to submit anything.

Walkthrough Videos

Feel free to try these problems on the worksheet in discussion or on your own, and then come back to reference these walkthrough videos as you study.

> To see these videos, you should be logged into your berkeley.edu email.

Note: Since the contents of each discussion changes semester by semester, the Discussion and Question numbers in the videos may not always align with what is in the worksheet.

YouTube link (https://youtu.be/playlist?list=PLx38hZJ5RLZfSvyai8czk8oRYajROjQyg)

Recursion

A *recursive* function is a function that is defined in terms of itself.

Consider this recursive `factorial` function:

```
def factorial(n):
    """Return the factorial of N, a positive integer."""
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Inside of the body of `factorial`, we are able to call `factorial` itself, since the function body is not evaluated until the function is called.

When `n` is 1, we can directly return the factorial of 1, which is 1. This is known as the *base case* of this recursive function, which is the case where we can return from the function call directly, without having to first recurse (i.e. call `factorial`) and then returning. The base case is what prevents `factorial` from recursing infinitely.

Since we know that our base case `factorial(1)` will return, we can compute `factorial(2)` in terms of `factorial(1)`, then `factorial(3)` in terms of `factorial(2)`, and so on.

There are three main steps in a recursive definition:

1. **Base case.** You can think of the base case as the case of the simplest function input, or as the stopping condition for the recursion.
   In our example, `factorial(1)` is our base case for the `factorial` function.
2. **Recursive call on a smaller problem.** You can think of this step as calling the function on a smaller problem that our current problem depends on. We assume that a recursive call on this smaller problem will give us the expected result; we call this idea the "recursive leap of faith".
   In our example, `factorial(n)` depends on the smaller problem of `factorial(n-1)`.
3. **Solve the larger problem.** In step 2, we found the result of a smaller problem. We want to now use that result to figure out what the result of our current problem should be, which is what we want to return from our current function call.

In our example, we can compute `factorial(n)` by multiplying the result of our smaller problem `factorial(n-1)` (which represents `(n-1)!`) by `n` (the reasoning being that `n! = n * (n-1)!`).

Q1: Warm Up: Recursive Multiplication

These exercises are meant to help refresh your memory of the topics covered in lecture.

Write a function that takes two numbers `m` and `n` and returns their product. Assume `m` and `n` are positive integers. Use **recursion**, not `mul` or `*`.

> Hint: `5 * 3 = 5 + (5 * 2) = 5 + 5 + (5 * 1)`.

For the base case, what is the simplest possible input for `multiply`?

If one of the inputs is one, you simply return the other input.

For the recursive case, what does calling `multiply(m - 1, n)` do? What does calling `multiply(m, n - 1)` do? Do we prefer one over the other?

The first call will calculate a value that is `n` less than the total, while the second will calculate a value that is `m` less. Either recursive call will work, but only `multiply(m, n - 1)` is used in this solution.

Your Answer

```
def multiply(m, n):
    """ Takes two positive integers and returns their product using recursion.
    >>> multiply(5, 3)
    15
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def multiply(m, n):
    """ Takes two positive integers and returns their product using recursion.
    >>> multiply(5, 3)
    15
    """
    if n == 1:
        return m
    else:
        return m + multiply(m, n - 1)
```

Q2: Recursion Environment Diagram

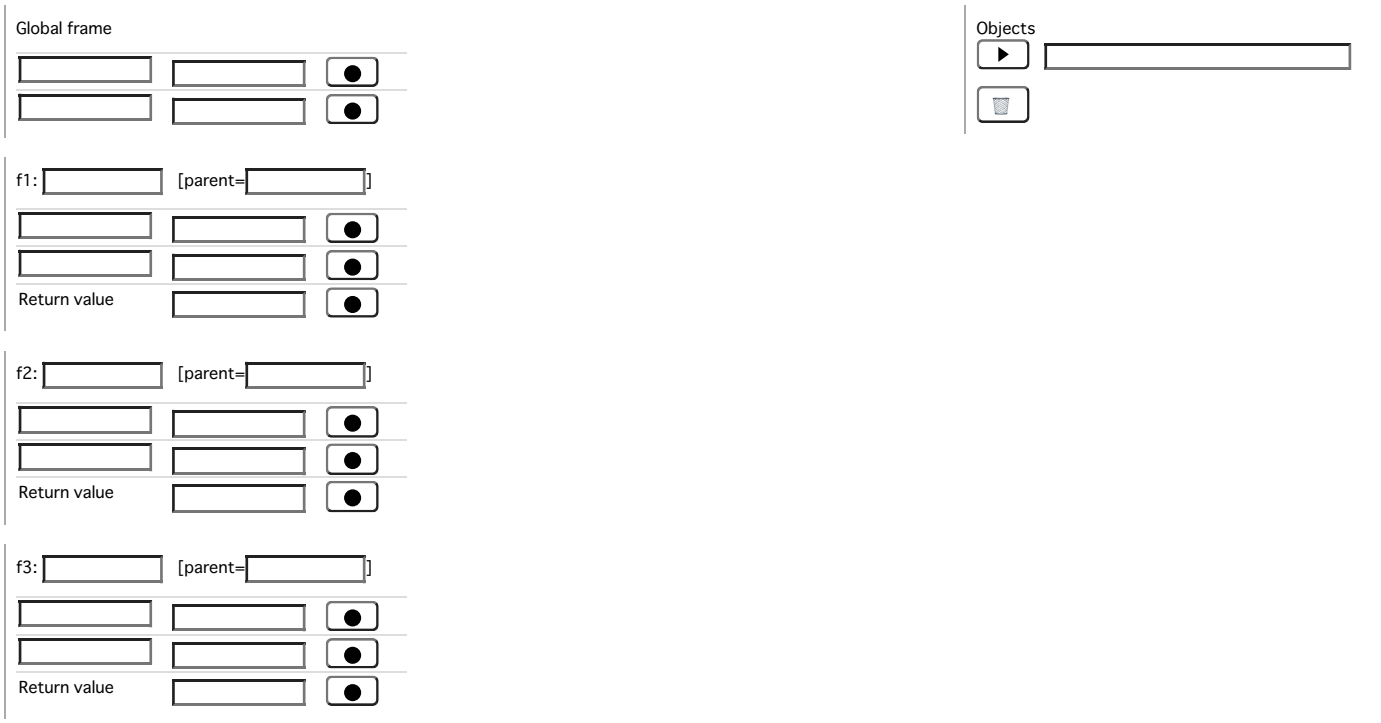
Draw an environment diagram for the following code:

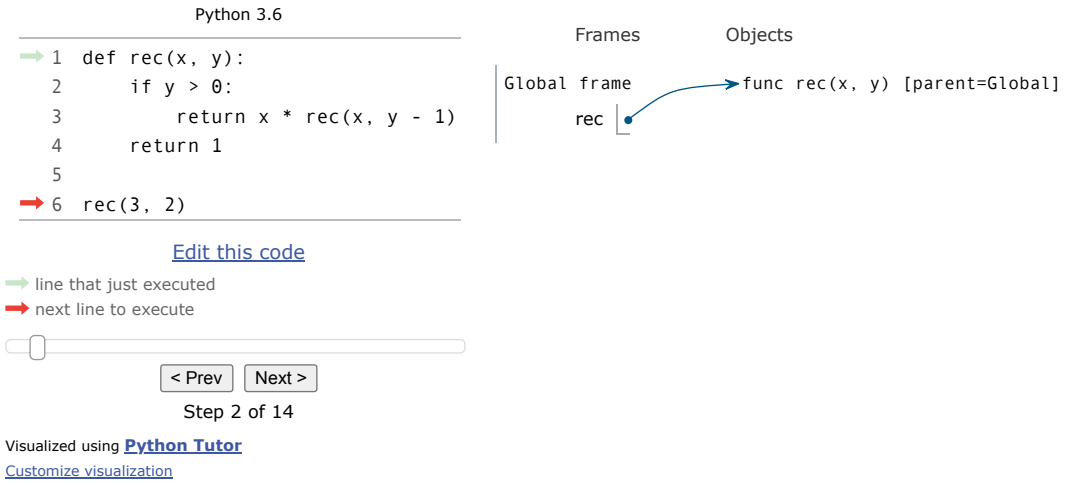
> Note: If you can't move elements around, make sure you're logged in!

```
def rec(x, y):
    if y > 0:
        return x * rec(x, y - 1)
    return 1

rec(3, 2)
```

Your Answer

```
def rec(x, y):
    if y > 0:
        return x * rec(x, y - 1)
    return 1

rec(3, 2)
```





Solution

This function returns the result of computing X to the power of Y.

Note: This problem is meant to help you understand what really goes on when we make the "recursive leap of faith". However, when approaching or debugging recursive functions, you should avoid visualizing them in this way for large or complicated inputs, since the large number of frames can be quite unwieldy and confusing. Instead, think in terms of the three steps: base case, recursive call, and solving the full problem.

Q3: Find the Bug

Find the bug with this recursive function.

```
def skip_mul(n):
    """Return the product of n * (n - 2) * (n - 4) * ...

    >>> skip_mul(5) # 5 * 3 * 1
    15
    >>> skip_mul(8) # 8 * 6 * 4 * 2
    384
    """
    if n == 2:
        return 2
    else:
        return n * skip_mul(n - 2)
```

Consider what happens when we choose an odd number for `n`. `skip_mul(3)` will return `3 * skip_mul(1)`. `skip_mul(1)` will return `1 * skip_mul(-1)`. You may see the problem now. Since we are decreasing `n` by two at a time, we've completed missed our base case of `n == 2`, and we will end up recursing indefinitely. We need to add another base case to make sure this doesn't happen.

```
def skip_mul(n):
    if n == 1:
        return 1
    elif n == 2:
        return 2
    else:
        return n * skip_mul(n - 2)
```

Q4: Is Prime

Write a function `is_prime` that takes a single argument `n` and returns `True` if `n` is a prime number and `False` otherwise. Assume `n > 1`. We implemented this in Discussion 1 (/disc/disc01/) iteratively, now time to do it recursively!

> *Hint*: You will need a helper function! Remember helper functions are nested functions that are useful if you need to keep track of more variables than the given parameters, or if you need to change the value of the input.

Your Answer

```
def is_prime(n):
    """Returns True if n is a prime number and False otherwise.

    >>> is_prime(2)
    True
    >>> is_prime(16)
    False
    >>> is_prime(521)
    True
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def is_prime(n):
    """Returns True if n is a prime number and False otherwise.

    >>> is_prime(2)
    True
    >>> is_prime(16)
    False
    >>> is_prime(521)
    True
    """
    def helper(i):
        if i > (n ** 0.5): # Could replace with i == n
            return True
        elif n % i == 0:
            return False
        return helper(i + 1)
    return helper(2)
```

Q5: Recursive Hailstone

Recall the `hailstone` function from Homework 1 (/hw/hw01/). First, pick a positive integer `n` as the start. If `n` is even, divide it by 2. If `n` is odd, multiply it by 3 and add 1. Repeat this process until `n` is 1. Write a recursive version of `hailstone` that prints out the values of the sequence and returns the number of steps.

> Hint: When taking the recursive leap of faith, consider both the return value and side effect of this function.

Your Answer

```
def hailstone(n):
    """Print out the hailstone sequence starting at n, and return the number of elements in the se
    >>> a = hailstone(10)
    10
    5
    16
    8
    4
    2
    1
    >>> a
    7
    >>> b = hailstone(1)
    1
    >>> b
    1
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def hailstone(n):
    """Print out the hailstone sequence starting at n, and return the number of elements in the sequence.
    >>> a = hailstone(10)
    10
    5
    16
    8
    4
    2
    1
    >>> a
    7
    >>> b = hailstone(1)
    1
    >>> b
    1
    """
    print(n)
    if n == 1:
        return 1
    elif n % 2 == 0:
        return 1 + hailstone(n // 2)
    else:
        return 1 + hailstone(3 * n + 1)
```

Q6: Merge Numbers

Write a procedure `merge(n1, n2)` which takes numbers with digits in decreasing order and returns a single number with all of the digits of the two, in decreasing order. Any number merged with 0 will be that number (treat 0 as having no digits). Use recursion.

> Hint: If you can figure out which number has the smallest digit out of both, then we know that the resulting number will have that smallest digit, followed by the merge of the two numbers with the smallest digit removed.

Your Answer

```
def merge(n1, n2):
    """ Merges two numbers by digit in decreasing order
    >>> merge(31, 42)
    4321
    >>> merge(21, 0)
    21
    >>> merge (21, 31)
    3211
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def merge(n1, n2):
    """ Merges two numbers by digit in decreasing order
    >>> merge(31, 42)
    4321
    >>> merge(21, 0)
    21
    >>> merge (21, 31)
    3211
    """
    if n1 == 0:
        return n2
    elif n2 == 0:
        return n1
    elif n1 % 10 < n2 % 10:
        return merge(n1 // 10, n2) * 10 + n1 % 10
    else:
        return merge(n1, n2 // 10) * 10 + n2 % 10
```

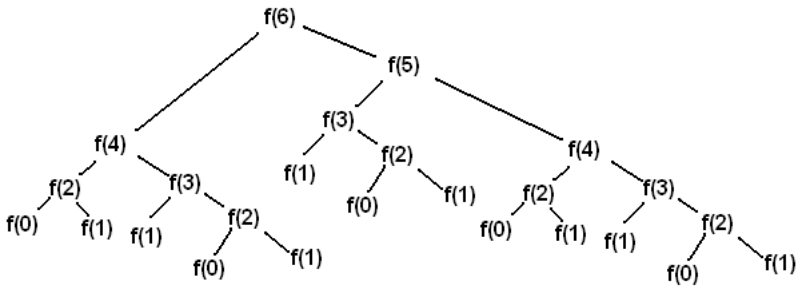
Tree Recursion

A tree recursive function is a recursive function that makes more than one call to itself, resulting in a tree-like series of calls.

For example, let's say we want to recursively calculate the `n` th Virahanka-Fibonacci number (https://en.wikipedia.org/wiki/Fibonacci_number), defined as:

```
def virfib(n):
    if n == 0 or n == 1:
        return n
    return virfib(n - 1) + virfib(n - 2)
```

Calling `virfib(6)` results in the following call structure that looks like an upside-down tree (where `f` is `virfib`):



Each `f(i)` node represents a recursive call to `virfib`. Each recursive call `f(i)` makes another two recursive calls, which are to `f(i-1)` and `f(i-2)`. Whenever we reach a `f(0)` or `f(1)` node, we can directly return `0` or `1` rather than making more recursive calls, since these are our base cases.

In other words, base cases have the information needed to return an answer directly, without depending upon results from other recursive calls. Once we've reached a base case, we can then begin returning back from the recursive calls that led us to the base case in the first place.

Generally, tree recursion can be effective for problems where there are multiple possibilities or choices at a current state. In these types of problems, you make a recursive call for each choice or for a group of choices.

Q7: Count Stair Ways

Imagine that you want to go up a flight of stairs that has `n` steps, where `n` is a positive integer. You can either take 1 or 2 steps each time. How many different ways can you go up this flight of stairs? In this question, you'll write a function `count_stair_ways` that solves this problem. Before you code your approach, consider these questions.

How many different ways are there to go up a flight of stairs with `n = 1` step? How about `n = 2` steps? Try writing out some other examples and see if you notice any patterns.

Solution: When there is only one step, there is only one way to go up the stair. When there are two steps, we can go up in two ways: take a single 2-step, or take two 1-steps.

What's the base case for this question? What is the simplest input?

Solution: Our first base case is when there is one step left. This is, by definition, the smallest input since it is the smallest positive integer. Our second base case is when we have two steps left. We need this base case for a similar reason that `fibonacci` needs 2 base cases: to cover both recursive calls.

Alternate solution: Our first base case is where there are no steps left. This means that we took an action in the previous recursive step that led to our goal of reaching the top. Our second base case is where we have overstepped. This means that the action we took is not valid, as it caused us to step over our goal.

What do `count_stair_ways(n - 1)` and `count_stair_ways(n - 2)` represent?

Solution: `count_stair_ways(n - 1)` represents the number of different ways to go up the last `n-1` stairs (this is the case where we take 1 step as our move). `count_stair_ways(n - 2)` represents the number of different ways to go up the last `n-2` stairs (this is the case where we take 2 steps as our move).

Fill in the code for `count_stair_ways` :

Your Answer

```
def count_stair_ways(n):
    """Returns the number of ways to climb up a flight of
    n stairs, moving either 1 step or 2 steps at a time.
    >>> count_stair_ways(4)
    5
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def count_stair_ways(n):
    """Returns the number of ways to climb up a flight of
    n stairs, moving either 1 step or 2 steps at a time.
    >>> count_stair_ways(4)
    5
    """
    if n == 1:
        return 1
    elif n == 2:
        return 2
    return count_stair_ways(n-1) + count_stair_ways(n-2)
```

Here's an alternate solution corresponding to the alternate base case presented above:

```
def count_stair_ways_alt(n):
    """Returns the number of ways to climb up a flight of
    n stairs, moving either 1 step or 2 steps at a time.
    >>> count_stair_ways_alt(4)
    5
    """
    if n == 0:
        return 1
    elif n < 0:
        return 0
    return count_stair_ways_alt(n-1) + count_stair_ways_alt(n-2)
```

You can use recursion visualizer (https://www.recursionvisualizer.com/?function_definition=def%20count_stair_ways(n)%3A%0A%20%20%20%20if%20n%20%3D%3D%201%3A%0A%20%20%20%20%20%20%20%20ret 1)%20%2B%20count_stair_ways(n-2)&function_call=count_stair_ways(4)) to step through the calls made to `count_stair_ways(4)` for the original approach.

Q8: Count K

Consider a special version of the `count_stair_ways` problem, where instead of taking 1 or 2 steps, we are able to take up to and including `k` steps at a time. Write a function `count_k` that figures out the number of paths for this scenario. Assume `n` and `k` are positive.

> *Hint*: Your solution will follow a very similar logic to what you did for `count_stair_ways`.

Your Answer

```
def count_k(n, k):
    """ Counts the number of paths up a flight of n stairs
    when taking up to and including k steps at a time.
    >>> count_k(3, 3) # 3, 2 + 1, 1 + 2, 1 + 1 + 1
    4
    >>> count_k(4, 4)
    8
    >>> count_k(10, 3)
    274
    >>> count_k(300, 1) # Only one step at a time
    1
    """
    "*** YOUR CODE HERE ***"


```

Solution

```
def count_k(n, k):
    """ Counts the number of paths up a flight of n stairs
    when taking up to and including k steps at a time.
    >>> count_k(3, 3) # 3, 2 + 1, 1 + 2, 1 + 1 + 1
    4
    >>> count_k(4, 4)
    8
    >>> count_k(10, 3)
    274
    >>> count_k(300, 1) # Only one step at a time
    1
    """
    if n == 0:
        return 1
    elif n < 0:
        return 0
    else:
        total = 0
        i = 1
        while i <= k:
            total += count_k(n - i, k)
            i += 1
        return total
```

We need to include the while loop from the `count_k` solution and keep track of a running total for the number of successful ways because we can take *up to* k steps. The while loop will count how many successful ways if we take 1, 2, 3, ... k steps. We also need to keep track of how many successful ways there are for each value of k, so we use the `total` variable to remember how many successful ways there are so far.

You can use recursion visualizer (https://www.recursionvisualizer.com/?function_definition=def%20count_k(n%2C%20k)%3A%0A%20%20%20%20if%20n%20%3D%3D%200%3A%0A%20%20%20%20%20%20%20%20re%20i%2C%20k)%0A%20%20%20%20%20%20%20%20%20%20%20%20i%20%2B%3D%201%0A%20%20%20%20%20%20%20%20return%20total8) to step through the calls made to `count_k(3, 3)`.