Pseudo Landau Levels of Dirac Electrons in Strained Graphene

Charlie Glass
Center for Theoretical Physics, 77 Massachusetts Ave., Cambridge, MA 02139-4307
(Dated: May 3, 2025)

The quantized energy levels of electrons in graphene under the influence of a magnetic field can be recreated through a specific type of strain. Here, we calculate the Landau levels of relativistic Dirac electrons in graphene in a uniform perpendicular magnetic field. We then show how a specific strain field can create a vector potential which produces a pseudo magnetic field and calculate the pseudo Landau levels of this vector potential.

I. INTRODUCTION

Understanding the physical properties of solids is an exciting application of quantum mechanics. This understanding of the behavior of electrons in solids can lead to new nanotechnologies which utilize quantum phenomena. Graphene exhibits many interesting electronic properties which could be applied in a variety of technological applications [1]. We explore the fascinating behavior that arises in graphene when it is put under elastic strain.

Graphene is a 2-dimensional sheet of carbon atoms that form a hexagonal lattice through the σ bonds of its electrons. The carbon's $2p_z$ orbitals which are not involved with the σ bonds are oriented perpendicular to the graphene lattice. We are particularly interested in the behavior of these $2p_z$ electrons when experiencing a vector potential, $\mathbf{A}$, because their energy levels become quantized.

We get our magnetic field, $\mathbf{B}$, from taking the curl of our vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

An interpretation of this equation is that you can start with a vector potential first, take the curl, and then get a magnetic field. In fact, this means that if you have a system and you want to create the effects of a magnetic field, you don't need a physical magnetic field. All you need is for a vector potential term to appear within your Hamiltonian.

$$\hat{H} = \frac{1}{2m}(\mathbf{p} + \frac{q}{c}\mathbf{A})^2 + \hat{V}(\mathbf{r}) \tag{2}$$

The curl of such a vector potential is referred to as a pseudo magnetic field.

We examine how such a pseudo magnetic field can be created in graphene by elastic strain and how this pseudo magnetic field can create pseudo Landau levels for the $2p_z$ electrons. We first study the case of graphene in a real magnetic field and work backwards to create a strain profile that creates the vector potential necessary for a pseudo magnetic field that matches our real field. Finally, we solve the Schrödinger Equation in order to calculate the pseudo Landau levels of the $2p_z$ electrons.

II. GRAPHENE IN A MAGNETIC FIELD

A. Tight Binding Hamiltonian

Since graphene is a periodic array of atoms, we may define a unit cell in graphene. The unit cell is a unit of area created by the parallelogram formed from two vectors known as lattice vectors. The square of the amplitude of the wavefunction is invariant under translations of integer multiples of the lattice vectors. This manifests as a wavefunction which satisfies the following equation.

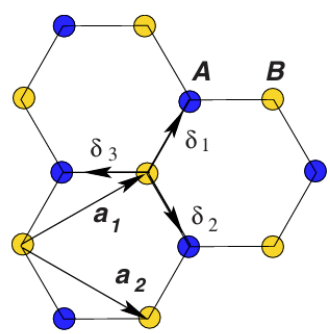

FIG. 1: Graphical representation of lattice vectors and nearest neighbor vectors imposed onto the 2 dimensional structure of graphene [2].

$$|\psi(\mathbf{r} + \mathbf{R})|^2 = |\psi(\mathbf{r})|^2 \tag{3}$$
$$\mathbf{R} = n\mathbf{a}_1 + m\mathbf{a}_2$$

Here, $\mathbf{a}_1$ and $\mathbf{a}_2$ are the two lattice vectors which define the 2D crystal lattice of graphene and n and m are arbitrary integers.

The lattice vectors and vectors giving the location of the 3 closest neighboring atoms are well-known experi-

mentally obtained values [2].

$$\mathbf{a_1} = \frac{a}{2}(3, \sqrt{3})$$
$$\mathbf{a_1} = \frac{a}{2}(3, -\sqrt{3})$$
$$\delta_\mathbf{1} = \frac{a}{2}(1, \sqrt{3})$$
$$\delta_\mathbf{2} = \frac{a}{2}(1, -\sqrt{3})$$
$$\delta_\mathbf{3} = a(-1, 0)$$

And from this formulation, we can see that the variable, a, is the length of the σ bonds between the carbon atoms.

So with the idea of a unit cell in mind, we may create a Hamiltonian for the unit cell which means that the Hamiltonian also has periodic behavior throughout the lattice. Because the potential that the electrons experience in graphene is very complex and involves interactions between many other electrons and nuclei, our approach to realistically calculate the electron's energy levels is to approximate their Hamiltonian as that of atomic orbitals that are slightly perturbed by the potential of the neighboring nuclei closest to them. So we ignore interactions from all other atoms and electrons in the system. This is known as the tight binding approximation.

$$\hat{H} = \hat{H}_{at} + \Delta\hat{U} \tag{4}$$

Here, $\Delta\hat{U}$ represents the interactions of the potential for the electron's three nearest-neighbor atomic potentials, and $\hat{H}_{at}$ gives the energy of each atomic orbital.

$$\hat{H}_{at} \left|\mathbf{R}\right\rangle = \epsilon \left|\mathbf{R}\right\rangle \tag{5}$$

Here, ϵ is the energy of the $2p_z$ electrons orbiting one of the carbon atoms in the unit cell, and $\langle\mathbf{r}|\mathbf{R}\rangle$ is the atomic orbital wavefunction in position space for A lattice sites as shown in Fig. 1. The $\mathbf{R}$ in the ket labels which unit cell we're working with by the lattice vectors. The atomic orbital wavefunction for the B lattice sites is defined as $\langle\mathbf{r}|\mathbf{R} + \delta_3\rangle$, so it's the same position in the unit cell but displaced by the nearest neighbor vector. $\Delta\hat{U}$ describes the off-diagonal terms of the entire tight binding matrix. It can be seen from this formulation that $\Delta\hat{U}$ contains information about the probability that the electron transitions from the A orbital eigenstate to the nearest B orbital eigenstate.

$$\left|\mathbf{R}\right\rangle \rightarrow \left|\mathbf{R} + \delta_i\right\rangle, i = 1, 2, 3 \tag{6}$$

This is the probability of the electron transitioning to any of its three nearest neighbors, two of which are not in the unit cell, but still turn out to be the B atom of the neighboring unit cell. These off-diagonal terms are referred to as the hopping terms because we think of the electron as jumping from its original atom to its neighbor. So with these properties in mind, we can write out the form of our Hamiltonian.

$$\hat{H} = \sum_{\mathbf{R}} (\epsilon \left|\mathbf{R}\right\rangle \left\langle\mathbf{R}\right| + t \sum_i \left|\mathbf{R}\right\rangle \left\langle\mathbf{R} + \delta_i\right| + H.c.) \tag{7}$$

H.c. is the shorthand notation for Hermitian conjugate and is included to ensure that the Hamiltonian itself is always Hermitian. The tight binding model is key to developing an intuition for a pseudo magnetic field, because it is the alteration of this hopping term which gives rise to a vector potential in the Hamiltonian.

To solve for the energy eigenvalues of our tight binding Hamiltonian, we construct basis states that satisfy the condition set by equation (3). The general form of this state is known as a Bloch state. The Bloch state is an extremely important idea in condensed matter physics because forming Bloch states allows us to easily solve for the energy levels of our Hamiltonian by taking advantage of the periodicity system. The idea is that if our Bloch state is periodic across unit cells, then if we solve our Schrodinger equation in one unit cell, the solution works across the whole space of the lattice [5]. Because there are only two atoms in each unit cell and we only care about electrons in the $2p_z$ orbitals of one unit cell, we can work in the basis of only two Bloch states.

$$\left|\mathbf{k}_A\right\rangle = \tfrac{1}{\sqrt{N}} \textstyle\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \left|\mathbf{R}\right\rangle \tag{8}$$
$$\left|\mathbf{k}_B\right\rangle = \tfrac{1}{\sqrt{N}} \textstyle\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \left|\mathbf{R} + \delta_\mathbf{3}\right\rangle$$

We summed over all possible atomic orbital states and normalized each state by dividing by the root of the total number of lattice sites, N. There is a Bloch state for each atom in the unit cell and each state is labeled by a reciprocal lattice vector, $\mathbf{k} = (k_x, k_y)$. The value $\hbar\mathbf{k}$ is analogous to the value of momentum, $\mathbf{p}$, for a particle in free space, so we refer to it as the crystal momentum of the electron. This means that our Hamiltonian is a function of crystal momentum since it is a function of $\mathbf{k}$.

This means that by working in our basis of Bloch states, we only have to work in a 2 dimensional space of states labeled by $\mathbf{k}$. We construct each term of our 2 dimensional Hamiltonian using our Bloch basis states. The diagonal terms are still the energy of the $2p_z$ orbitals.

$$\left\langle\mathbf{k}_A\right| \hat{H} \left|\mathbf{k}_A\right\rangle = \epsilon \tag{9}$$
$$\left\langle\mathbf{k}_B\right| \hat{H} \left|\mathbf{k}_B\right\rangle = \epsilon$$

And we compute our off-diagonal terms of our matrix.

$$\left\langle\mathbf{k}_A\right| \hat{H} \left|\mathbf{k}_B\right\rangle = \sum_{\mathbf{R}\mathbf{R}'} e^{i\mathbf{k}\cdot(\mathbf{R}'-\mathbf{R})} \left\langle\mathbf{R}\right| \hat{H} \left|\mathbf{R}' + \delta_3\right\rangle \tag{10}$$

We use our nearest neighbor approximation to set all terms of the sum that are not inner products of the nearest neighbors equal to zero.

$$\left\langle\mathbf{R}\right| \hat{H} \left|\mathbf{R} + \delta_i\right\rangle = t \tag{11}$$

The variable, t, in this expression is our hopping term. Each nonzero term in equation (10) gains a phase depending on the dot product of $\mathbf{k}$ and δ_i.

$$\left\langle\mathbf{k}_A\right| \hat{H} \left|\mathbf{k}_B\right\rangle = t(1 + e^{i\mathbf{k}\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2}) \tag{12}$$

The same process is applied for the other off-diagonal element, yielding a matrix.

$$\hat{H}(\mathbf{k}) = \begin{bmatrix} \epsilon & t(1+e^{i\mathbf{k}\cdot\mathbf{a}_1}+e^{i\mathbf{k}\cdot\mathbf{a}_2}) \\ t(1+e^{-i\mathbf{k}\cdot\mathbf{a}_1}+e^{-i\mathbf{k}\cdot\mathbf{a}_2}) & \epsilon \end{bmatrix} \quad (13)$$

We then solve the eigenvalues of this matrix to get our equation for the energy of the electrons as a function of crystal momentum.

$$E(\mathbf{k}) = \epsilon \pm t\sqrt{1+4\cos\frac{3k_x a}{2}\cos\frac{\sqrt{3}k_y a}{2}+4\cos^2\frac{\sqrt{3}k_y a}{2}} \quad (14)$$

B. Effect of Magnetic Field

Now we may introduce a magnetic field into our system. We choose the magnetic field to be constant and perpendicular to our graphene plane, which we define as the x-y plane.

$$\mathbf{B} = B\ \hat{\mathbf{z}} \quad (15)$$

There are many vector potentials that work for this B field, but we choose to work with the Landau vector potential or Landau gauge.

$$\mathbf{A} = -By\ \hat{\mathbf{x}} \quad (16)$$

This vector potential is introduced into our tight binding Hamiltonian through the Peierls substitution [2]. This substitution involves adding phases to the hopping terms on the off-diagonal elements of our Hamiltonian. The idea behind this phase is that generally in quantum mechanics, when a charged particle experiencing a vector potential travels on a path, its amplitude is altered by a phase proportional to the line integral over that vector potential [6].

$$t\,|\mathbf{R}\rangle\langle\mathbf{R}+\delta_i| \rightarrow e^{\frac{iq\theta_i}{\hbar c}}t\,|\mathbf{R}\rangle\langle\mathbf{R}+\delta_i| \quad (17)$$

$$\theta_i = \int_{\mathbf{R}}^{\mathbf{R}+\delta_i}\mathbf{A}(\mathbf{r})\cdot d\mathbf{r} \quad (18)$$

We then go through the same process before to find the elements of our tight binding Hamiltonian. This leads to new path dependent phases being added to each term in equation (12).

$$\langle\mathbf{k}_A|\,\hat{H}\,|\mathbf{k}_B\rangle = t(e^{\frac{iq}{\hbar c}B_0 ya}+e^{i\mathbf{k}\cdot\mathbf{a}_1-\frac{iq}{2\hbar c}B_0 ya}+e^{i\mathbf{k}\cdot\mathbf{a}_2-\frac{iq}{2\hbar c}B_0 ya}) \quad (19)$$

So because the vector potential is not uniform in the directions of the nearest neighbor vectors, the hopping terms will be influenced in specific directions. This will influence the motion of an electron through graphene and is the intuition for how it is really the vector potential that is changes the motion of the electron. We use these new matrix elements and create a new equation for the electron's energy which involves the position value of the electrons position, y. The equation we've created is a new time independent Schrodinger equation. We derived the electron's energy as a function of its reciprocal wave vector, $E(\mathbf{k})$, which is the crystal momentum up to the constant, $\hbar$, and now the Peierls substitution transformed the equation for energy into a Hamiltonian operator. The new Hamiltonian is a function the new gauge invariant momentum operator, $\mathbf{p}-\frac{e}{c}\mathbf{A}$.

We will now have to solve a new time-independent Schrodinger equation.

$$\hat{H}(\mathbf{p}-\frac{e}{c}\mathbf{A})\psi(\mathbf{r}) = E(\mathbf{p}-\frac{e}{c}\mathbf{A})\psi(\mathbf{r}) \quad (20)$$

Here, $\hat{H}(\mathbf{k})$ is the function of the momentum operator and the vector potential. We are interested in solving this equation for electrons near a specific value of momentum, $\mathbf{k} = (\frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a})$. The reason for choosing this value is explained in a later section. We refer to this specific value of momentum as $\mathbf{K}$ and work with a new shifted momentum, $\mathbf{q} = \hbar(\mathbf{k}-\mathbf{K})$, so that our shifted value will be very small near $\mathbf{K}$ and will allow us to perform a Taylor expansion about $\mathbf{K}$.

So with our new substitutions, we compute the new eigenvalues of our tight-binding Hamiltonian as a function of shifted crystal momentum, $\mathbf{q}$.

$$E(\mathbf{k}-\mathbf{K}) = \epsilon \pm t\sqrt{1+4\cos\frac{3q_x a}{2\hbar}\cos\frac{\sqrt{3}q_y a}{2\hbar}+4\cos^2\frac{\sqrt{3}q_y a}{2\hbar}} \quad (21)$$

Since it doesn't matter what we set as the zero point energy to in this case, we set $\epsilon = 0$. We then square this equation and Taylor expand the cosine terms about the $\mathbf{K}$ point to the second order of their argument. After this, we substitute in our gauge invariant momentum operator for our values of $\mathbf{q}$. We then have a form of the square of the Hamiltonian that will allow us to solve for the square of the eigenvalues.

$$\hat{H}^2 = t^2 + 4t^2[(1-\frac{1}{2}(\frac{3a}{2\hbar}(\hat{q}_x+\frac{eB}{c}\hat{y}))^2)\times (1-\frac{1}{2}(\frac{\sqrt{3}a}{2\hbar}\hat{q}_y)^2)+(1-\frac{1}{2}(\frac{\sqrt{3}a}{2\hbar}\hat{q}_y)^2)^2] \quad (22)$$

This equation above looks unfamiliar, but it can actually take the form of a simple harmonic oscillator Hamiltonian. We approach solving this problem in the same way the Landau levels are calculated for a free electron in section 24.5 of Zwiebach's textbook [4]. We first make use of our convenient choice of vector potential by observing that $[\hat{H}^2, q_x] = 0$. So because of this commutation relation, we may factor out an eigenfunction of momentum in the x direction in our complete spatial wavefunction.

$$\psi(\mathbf{r}) = \phi(y)e^{\frac{iq_x x}{\hbar}} \quad (23)$$

Using this equation has the benefit that the operator, $\hat{q}_x$, may be treated as a constant.

$$\hat{q}_x \rightarrow q_x \quad (24)$$

So when we expand equation (22) and neglect momentum terms that are on the order of the fourth power, we get

the form of a simple harmonic oscillator Hamiltonian.

$$\hat{H}^2 = 9t^2 - \frac{18}{4}[\frac{tBae}{\hbar c}(\hat{y} + \frac{q_x c}{eB})]^2 - \frac{18}{4}(\frac{ta}{\hbar}\hat{q}_y)^2 \quad (25)$$

This is a harmonic oscillator with a shift in position and a ground state energy which is shifted by $+9t^2$. To solve for our eigenenergies, we define a new value called the Fermi velocity and construct ladder operators that work our Hamiltonian.

$$v_f = \frac{3}{2}\frac{ta}{\hbar} \quad (26)$$

$$\hat{a} = -\sqrt{\frac{c}{e\hbar B}}(\hat{q_y} - i(\hbar k_x + \frac{eB}{c}\hat{y})) \quad (27)$$

$$\hat{a}^\dagger = -\sqrt{\frac{c}{e\hbar B}}(\hat{q_y} + i(\hbar k_x + \frac{eB}{c}\hat{y}))$$

We know that a Hamiltonian that is proportional to the product of these operators, $\hat{a}^\dagger\hat{a}$, produces an energy spectrum equivalent to the shifted harmonic oscillator. This is because all of the physics of the quantum harmonic oscillator lies in the commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$, which our operators satisfy. Now we write our squared Hamiltonian in the form of our number operator multiplied by a constant with units of energy squared.

$$\hat{H}^2 = \hbar\omega(\hat{a}^\dagger\hat{a} + 1) + 9t^2 \quad (28)$$

$$\omega = \frac{2eBv_f^2}{c} \quad (29)$$

This Hamiltonian leads to a discretized energy spectrum that is proportional to the square root of the electron's quantum number, n.

$$E_n \sim \pm\sqrt{\hbar\omega n} \quad (30)$$

C. Dirac Electrons

So why did we choose to work with a shifted momentum, **q**? This is because the electrons with momentum near the value of **K** are actually described by the relativistic Dirac equation. These electrons are a special case of relativistic particles where their mass is effectively zero and their speed is not the speed of light but actually v_f: the quantity we factored out in our Hamiltonian. Our Dirac Hamiltonian takes the form of a matrix with components of momentum in the off-diagonal elements and zero in the diagonal elements since we set $\epsilon = 0$.

$$\hat{H} = v_F \begin{bmatrix} 0 & q_x - iq_y \\ q_x + iq_y & 0 \end{bmatrix} \quad (31)$$

This equation results in a linear energy dispersion relation from the **K** point known as a Dirac cone. And while we are not using a relativistic equation, we can still discover this behavior with our original, approximate Hamiltonian. The Dirac equation is an extremely interesting topic that is also explained in section 24.7 of Zwiebach's textbook [4].

III. PSEUDO MAGNETIC FIELD FROM STRAIN

We apply elastic strain on graphene to create the effect of a vector potential in the Hamiltonian. Strain on graphene can be thought of as simply the displacement of the Carbon atoms from their original position in the lattice. This is known as the continuum model which is reliant on the tight-binding assumption that we've made for our graphene lattice.

Elastic strain can be created by placing the graphene over a substrate with a specific curvature it can mold to. When the graphene is placed on the substrate, some atoms are displaced more than others as the graphene relaxes to the position of minimum energy.

In the continuum model, we define a vector valued function which gives the displacement of atoms from their original position. This displacement is the consequence of the elastic strain on the graphene.

$$\mathbf{u}(\mathbf{r}) = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix} \quad (32)$$

The hopping parameter in our Hamiltonian depends on the bond length between the nearest neighbor atoms. And these bond lengths will be altered by the displacement of the atoms from their original position. In our case, we only care about the bond length between the A and B, so we will find the displacement in terms of the difference in displacement between the A and B sublattices.

$$\delta u_i(\mathbf{R}) = \frac{\delta_i}{a} \cdot [\mathbf{u}_A(\mathbf{R}) - \mathbf{u}_B(\mathbf{R} + \delta_i)] \quad (33)$$

We do not work with the strain fields specifically for lattice A and lattice B. Instead of working with strain fields specific to each sublattice, we use the relation of the strain for each sublattice to the spatial derivative of our whole strain field [7].

$$\mathbf{u}_A(\mathbf{R}) - \mathbf{u}_B(\mathbf{R} + \delta_i) = \kappa(\delta_i \cdot \nabla)\mathbf{u}(\mathbf{R}) \quad (34)$$

In the equation above, κ, is a microscopic parameter of graphene which is found to be $\sim \frac{1}{3}$ [7]. The change in the hopping term is dependent on the overall change in the distance between the nearest neighbor atoms [2].

$$\delta t_i(\mathbf{R}) = t_0 + \frac{\partial t_0}{\partial a}\delta u_i(\mathbf{R}) \quad (35)$$

We plug in equation (34) into equation (33) and use this to create an expression for the change in the hopping parameter to the ith nearest neighbor in terms of the strain field and the nearest neighbor vectors.

$$\delta t_i(\mathbf{R}) = t_0 + g[\delta_i \cdot (\delta_i \cdot \nabla)\mathbf{u}(\mathbf{R})] \quad (36)$$

$$g = \frac{\kappa}{a}\frac{\partial t_0}{\partial a} \quad (37)$$

We combine all the constant parameters that are intrinsic to graphene into a single parameter, g, for convenience.

Now we compute the changes to the hopping parameter in terms of the stress tensors and take the strain fields dependence on position in the lattice as a given. The strain tensors are constructed with the linear combinations of the partial derivatives of the displacement function with respect to x and y [2].

$$u_{ij} = \tfrac{1}{2}(\tfrac{\partial u_i}{\partial j} + \tfrac{\partial u_j}{\partial i}) \tag{38}$$
$$(i,j) = (x,y)$$

The change in hopping parameter in each direction is a linear combination of these strain tensors.

$$\delta t_1 = g(\tfrac{a}{2})^2[u_{xx} + 2\sqrt{3}u_{xy} + 3u_{yy}] \tag{39}$$
$$\delta t_2 = g(\tfrac{a}{2})^2[u_{xx} - 2\sqrt{3}u_{xy} + 3u_{yy}]$$
$$\delta t_3 = ga^2 u_{xx}$$

Because we are interested in the behavior of electrons which obey the Dirac equation, we use the Dirac Hamiltonian to see how the change in the hopping parameters in this case effectively create a vector potential. The Dirac Hamiltonian contains the terms of the electrons kinetic energy multiplied by the Fermi velocity which we derived previously. And when we applied our uniform perpendicular magnetic field to our graphene, we introduced a vector potential term into the x component of our momentum

$$\hat{H} = v_F \begin{bmatrix} 0 & (q_x + \frac{eBy}{c}) - iq_y \\ (q_x + \frac{eBy}{c}) + iq_y & 0 \end{bmatrix} \tag{40}$$

We know that the Fermi velocity is dependent on the hopping term. Before the strain, each hopping parameter would have the value, t_0. But now that the strain has altered the hopping parameters in a spatially dependent manner, we stop using v_f and distribute the hopping term to the momentum terms. We then form a similar Dirac Hamiltonian for our strained graphene motivated by results from Ando [7].

$$\hat{H} = \begin{bmatrix} 0 & \frac{3at}{2\hbar}(q_x - iq_y) + \delta t_x - i\delta t_y \\ \frac{3at}{2\hbar}(q_x + iq_y) + \delta t_x + i\delta t_y & 0 \end{bmatrix} \tag{41}$$

Here we define the directional dependent change in the hopping parameters in the x and y direction. The change in these hopping terms is given as a linear combination of the changes in the hopping terms from equation (37). The idea is that the same linear combination of the nearest neighbor vectors would create vectors that point in the x and y direction.

$$\delta t_x = \tfrac{1}{2}(\delta t_2 + \delta t_3 - 2\delta t_1) \tag{42}$$
$$\delta t_y = \tfrac{\sqrt{3}}{2}(\delta t_1 - \delta t_2) \tag{43}$$

The new term introduced into the Hamiltonian by the strain is interpreted as a vector potential. This is the case where we have the same problem so we'll get the same solutions. So when we take the curl of this vector potential, we interpret the result as the pseudo magnetic field. Now we need to formulate a strain field that will produce the vector potential that corresponds to a uniform magnetic field in the z direction. From equations (41), (42), and (41), we may formulate the result for the vector potential in terms of the strain tensors [2].

$$A_x^{(s)} = \tfrac{3a^2}{4t_0} g(u_{xx} - u_{yy}) \tag{44}$$
$$A_y^{(s)} = \tfrac{3a^2}{2t_0} g u_{xy} \tag{45}$$

We see from the equation above that the vector potential is proportional to the spatial derivatives of the displacement field, and we also know that the pseudo magnetic field is given by the curl of our vector potential. So if we want to create a vector potential that produces a uniform perpendicular pseudo magnetic field, we need a strain field with a second derivative with respect to space that is greater than zero. And we need to make sure that the combinations of the strain tensors in equations (44) and (45) do not cancel out to zero. There are many strain fields which satisfy these constraints and one of these strain fields is calculated here.

$$\mathbf{u}(\mathbf{r}) = \begin{pmatrix} -xy \\ \frac{1}{2}x^2 \end{pmatrix} \tag{46}$$

And when we plug this strain field into our equations for the vector potential and take the curl,

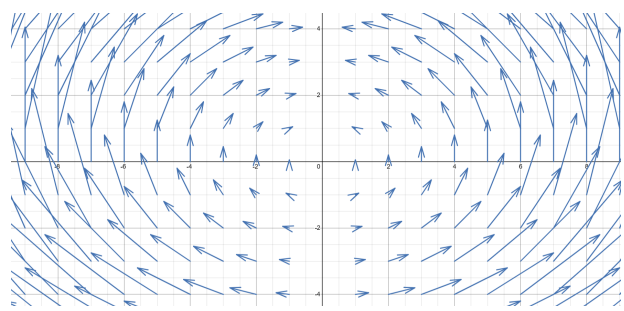

FIG. 2: Vector field representing the in-plane displacement of graphene which creates the uniform pseudo magnetic field.

it produces a spatially uniform pseudo magnetic field.

$$\mathbf{B} = \frac{3a^2}{4t_0} g\ \hat{\mathbf{z}} \tag{47}$$

So we have created the effect of a magnetic field just through the in-plane displacement of our graphene atoms. This was done by changing the bond lengths between the carbon atoms which altered the hopping terms between each atom. And we view the changing of hopping parameters as changing the path of the Dirac electrons and creating motion that looks like a magnetic field is present.

As done previously, we calculate the quantized energy levels of electrons with this kind of magnetic field. In this case, we get the pseudo Landau levels with a different

frequency term and a modified term for velocity replacing the Fermi velocity.

$$E_n^{(s)} \sim \sqrt{\hbar\omega^{(s)} n} \tag{48}$$

$$\omega^{(s)} = \frac{2ev_{(s)}^2}{c}\frac{3a^2}{4t_0}g \tag{49}$$

IV. DISCUSSION

So it has been shown that the elastic strain created can create a pseudo magnetic field which results in quantized energy levels known as pseudo Landau levels. We did this in the special case of electrons with a certain range of momentum, and we discovered that electrons in this momentum range obeyed the Dirac equation. And we calculated the energy dispersion relation in graphene using the tight binding model which allowed us to reveal how the hopping parameter connected the cases of a real magnetic field and elastic strain.

Although the strain field we found did produce a pseudo magnetic field, it is not realistically producible in an experimental setting. However, this strain field was helpful for building an intuition for how the graphene needs to be strained in order to create a pseudo magnetic field. The next steps for these calculations is to work with an experimentally realistic substrate which graphene can be molded over and strained in the necessary manner. Such a strain field could come from a substrate sinusoidally varying in height because the second spatial derivative would be nonzero. Realistically engineering strain could be a powerful method when using graphene in technological applications.

V. ACKNOWLEDGMENTS

I would like to express my appreciation to my peer editor, Rujuta Sane, for greatly improving the clarity of my paper with her suggestions, my writing assistant, Abijith Krishan, for providing great explanations and critiques of the drafts of my paper, and to Jacob Pettine for the very engaging discussions and crucial guidance that led to the creation of this paper. I would also like to thank Prof. Ivanov for a great semester of learning in 8.06.

[1] Lee, C., Wei, X., Kysar, J. W., Hone, J. (2014). Measurement of the elastic properties and intrinsic strength of monolayer graphene. Materials Today, 17(4), 152–160. https://doi.org/10.1016/j.mattod.2014.03.003

[2] Castro Neto, A. H., Guinea, F., Peres, N. M. R., Novoselov, K. S., Geim, A. K. (2009). The electronic properties of graphene. Reviews of Modern Physics, 81(1), 109–162. https://doi.org/10.1103/RevModPhys.81.109

[3] Luttinger, J. M. (1951). The effect of a magnetic field on electrons in a periodic potential. Physical Review, 84(4), 814–817. https://doi.org/10.1103/PhysRev.84.814

[4] Zwiebach, Barton. Mastering Quantum Mechanics: Essentials, Theory, and Applications. Cambridge, MA: The MIT Press, 2022.

[5] Griffiths, D. J. (2005). Introduction to quantum mechanics (3rd ed.). Pearson Prentice Hall.

[6] Feynman, R. P., Leighton, R. B., Sands, M. (1965). The Schrödinger equation in a classical context: A seminar on superconductivity. In The Feynman lectures on physics (Vol. 3, Ch. 21). Addison-Wesley.

[7] Suzuura, H., Ando, T. (2002). Phonons and electron-phonon scattering in carbon nanotubes. Physical Review B, 65(23), 235412. https://doi.org/10.1103/PhysRevB.65.235412