Introduction to Inequalities

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This is an introduction to inequalities for math competitions, written for the Wayland High School Math Team.

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§1 Squares are Always Greater Than 0

Here is the most basic inequality in algebra, stated without proof:

Proposition 1.1 (Trivial Inequality)
For all real $x, x^2 \geq 0$ with equality if and only if $x = 0$.

This inequality is known as the trivial inequality. Although this inequality may seem basic, it is actually applicable to many problems.

Example 1.2
Prove that
$$a^2 + b^2 + c^2 \geq ab + bc + ac$$
Proof: multiply by 2 then subtract from RHS. We have $a^2 - ab + b^2 + b^2 - bc + c^2 + a^2 - ac + c^2 \geq 0$ Then factor:
$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$
which is true by the trivial inequality. □

Problem 1.3. Prove that for all real x, y, z,
$$10x^2 + 5y^2 + 16xy + 5x^2y^2z^2 \geq 0.$$

Problem 1.4. Find all real numbers a, b, c, d such that
$$\begin{cases} a+b+c+d = 20, \\ ab+ac+ad+bc+bd+cd = 150. \end{cases}$$

Problem 1.5. Prove that $a^3 + b^3 + c^3 \geq 3abc$.

Problem 1.6. Let x, y be real numbers such that $xy \geq 1$. Show that
$$\frac{1}{x^2+1} + \frac{1}{y^2+1} \geq \frac{2}{xy+1}.$$

Problem 1.7. Let x, y, z be real numbers that satisfy $xy + yz + zx = -10$. Show that
$$x^2 + 5y^2 + 8z^2 \geq 40.$$

§2 The Arithmetic Mean-Geometric Mean Inequality

The Arithmetic Mean-Geometric Mean Inequality is powerful method of proving inequalities, and is usually one of the first discussed inequalities in Olympiad math.

We will start with some definitions. Let $S = a_1, a_2 \dots a_n$ be a set of n positive real numbers.

Definition 2.1. Let the arithmetic mean (AM) of S be the sum of the elements of S divided by n:

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

Definition 2.2. Let the geometric mean (GM) of S be the *nth* root of their product:

$$\sqrt[n]{a_1 a_2 \dots a_n}$$

Theorem 2.3 (AM-GM)

For nonnegative reals $a_1, a_2, \dots, a_n$ we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}.$$

Here, equality holds if and only if $a_1 = a_2 = \cdots = a_n$. We will only be proving the case when $n = 2$. The motivated reader is encouraged to research Jensen's inequality or induction for the general case.

Proof. For the case $n = 2$, we will use the fact that squares are greater than 0.

$$\begin{aligned}
(a-b)^2 &\geq 0 \\
a^2 - 2ab + b^2 &\geq 0 \\
a^2 + 2ab + b^2 &\geq 4ab \\
(a+b)^2 &\geq 4ab \\
a + b &\geq 2\sqrt{ab} \\
\frac{a+b}{2} &\geq \sqrt{ab}
\end{aligned}$$

□

There is a very clever proof to the two variable case using geometry that is an example problem in the Geometric Inequalities section. Now, let us see how we can use this inequality:

Problem 2.4. For a positive real number a, show that $a + \frac{1}{a} \geq 2$.

Problem 2.5. Ms. Marton wants to make a rectangular pig pen for her math team to have practice in. She has 80 feet of fencing. What is the maximum area pen she can make?

Problem 2.6. For positive real numbers a, b, and c, show that $a^3b^2c + b^3c^2a + c^3a^2b \geq 3a^2b^2c^2$.

Problem 2.7. Let a, b, c be positive real numbers. Prove that $(a + b)(b + c)(c + a) \geq 8abc$.

Problem 2.8. Let a, b, c be positive real numbers. Prove that $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$

Problem 2.9. Let a, b, c be positive real numbers such that $abc = 8$. Prove that $\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 6$.

Problem 2.10. All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

Problem 2.11. Prove that among all triangles of a given perimeter, the one with the largest area is an equilateral one.

Problem 2.12. Let a, b, c be positive real numbers. Prove that $\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$

Problem 2.13. There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of a the value of b is unique. What is this value of b?

§3 Geometric Inequalities

Geometric Inequalities are a beautiful part of math. While it is easy to get bogged down with lots of abstract concepts and theory, geometric inequalities are beautiful and often require clever insights. We will see some examples of these.

Theorem 3.1

In an arbitrary triangle $\triangle ABC$, $AB + BC \geq AC$, and likewise for the other sides.

This theorem is much like the trivial inequality: they both seem to have little substance, but are often used.

Problem 3.2. The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.

Problem 3.3. Prove that for any triangle with side lengths a, b, c and perimeter 2, the following inequality holds: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{abc} \geq 1$

Problem 3.4. Prove that in a convex quadrilateral $ABCD$, $AB + BC + CD + DA < 2(AC + BD)$

Problem 3.5. Prove that for all $a, b, c > 0$, we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

Problem 3.6. Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$
$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$
$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

§4 Kevin's Collection

Problem 4.1. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \leq abc.$$

Problem 4.2. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2} \geq 3.$$

Problem 4.3. Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2) \geq a^2+b^2+c^2.$$

Problem 4.4. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

§5 Hints

1.3. Where did the z come from?

1.4. Try squaring the first condition. (Source: JBMO)

1.5. This has a really weird factorization...

2.8. Break apart the sum.

2.9. Consider substituting a, b, c as $\frac{2x}{y}, \frac{2y}{z}, \frac{2z}{x}$. (Source: 2016 JBMO)

2.10. See if you can assign some variable names to different sides and come up with a factorization that will help you. (Source: 2016 AMC 10)

2.11. Consider using Heron's formula.

2.12. Prove that $\frac{1}{2a} + \frac{1}{2b} \geq \frac{2}{a+b}$

2.13. Try Vieta's relations. (Source: 2016 AMC 12)

3.2. (Source: 2006 AIME II)

3.4. Use the triangle inequality.

3.6. 2006 AIME II Problem 15

4.2. This is a tricky problem. Hint coming soon. (Source: 2011 USAJMO)

4.3. 2018 USAMO