13-14-15 Triangles and Metric Relations

Jet Chung and Michael Liu

August 7, 2019

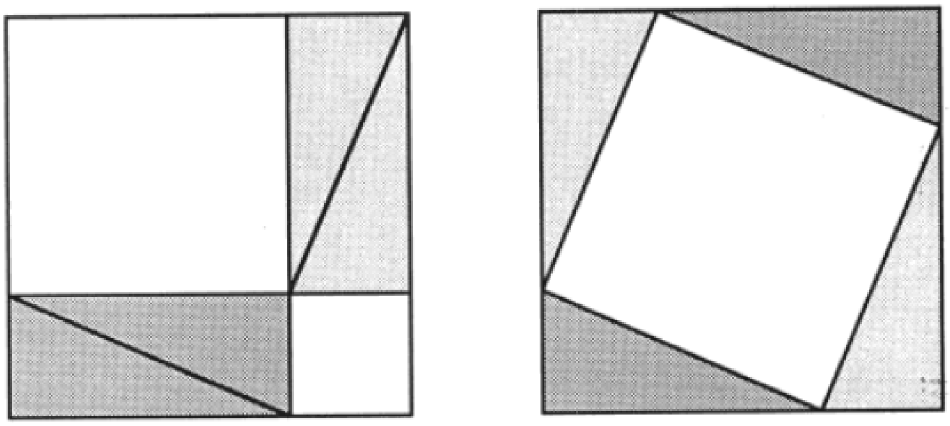
13-14-15 right triangles show up very often in short answer contests such as HMMT, GBML, and MAML. We will develop common relations for areas of triangles and length chasing.

Contents

§1 The Pythagorean Theorem

This is one we all know (I hope?):

Theorem 1.1
Given $\triangle ABC$ with $\angle C = 90°$, $BC = a, CA = b, AB = c$, $a^2 + b^2 = c^2$.



Problem 1.2. Can you find a graphical proof of this fact, using area?

Problem 1.3. In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

Problem 1.4. A triangle with side lengths in the ratio $3 : 4 : 5$ is inscribed in a circle with radius 3. What is the area of the triangle?

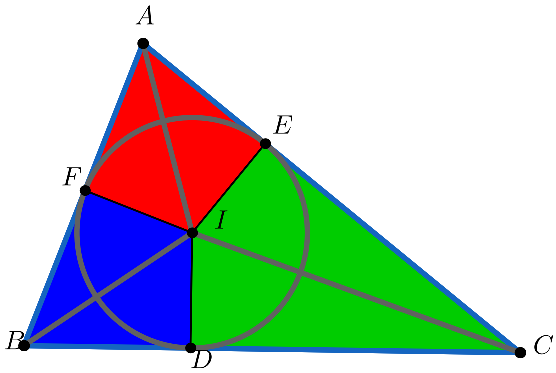
Problem 1.5. Let ABCD be an isosceles trapezoid such that $AD = BC$, $AB = 3$, and $CD = 8$. Let E be a point in the plane such that $BC = EC$ and $AE \perp EC$. Compute AE.

§2 Formulae for area of a triangle

We will develop formulae for finding the area of a triangle given various measurements.

Theorem 2.1 (Triangle Areas)
$K = [ABC] = \frac{bh}{2} = \frac{1}{2}ab\sin C = \frac{abc}{4R} = rs = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ is the semiperimeter of $\triangle ABC$, and r is the inradius of $\triangle ABC$.

Can you fill in the proof that $K = rs$, given this picture?



Problem 2.2. Prove that the area formulae are correct.

Problem 2.3. Can you come up with an alternate proof of the Law of Sines ($\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$) using area?

Problem 2.4. Let $\triangle ABC$ have $AB = 13$, $BC = 14$, $CA = 15$. Find the following:

1. The area
2. The inradius and circumradius
3. The length of altitude AD, length BD, and DC
4. $\cos\angle A, \cos\angle B, \cos\angle C$ as fractions

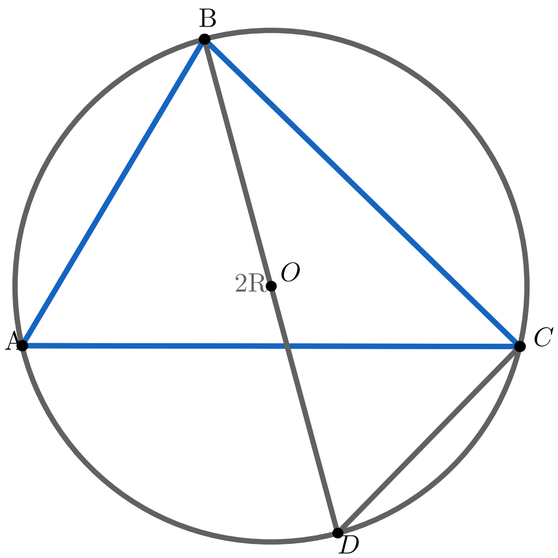
Problem 2.5. Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

Problem 2.6. Jarris the triangle is playing in the (x, y) plane. Let his maximum y coordinate be k. Given that he has side lengths 6, 8, and 10 and that no part of him is below the x-axis, find the minimum possible value of k.

§3 Law of Sines and Law of Cosines

These two relationships give us a way to find either missing angles or missing side lengths of a triangle, given at least three measures (if we only have two measures given, the triangle is not uniquely determined).

Theorem 3.1 (Law of Sines)
In $\triangle ABC$ with side lengths a, b, and c, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ where R is the circumradius of $\triangle ABC$.



Proof of Theorem 3.1. Let O be the circumcenter of $\triangle ABC$. Extending ray $\overrightarrow{BO}$ to meet the circumcircle of $\triangle ABC$ at D, because BD is a diameter, $\angle DCB = 90°$. Using the definition of sin on $\triangle DCB$, we get $\sin D = \frac{BC}{BD} = \frac{a}{2R}$. Also, because $\angle A$ and $\angle D$ are inscribed in the same arc, $\angle A \cong \angle D$, so $\sin D = \sin A$. Thus, $\sin A = \frac{a}{2R}$, so $\frac{a}{\sin A} = 2R$. We can similarly get $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$, so $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$. □

Typically, most problems can't be solved directly with the Law of Sines, but it can be used as an intermediate step, so be on the lookout for when it appears.

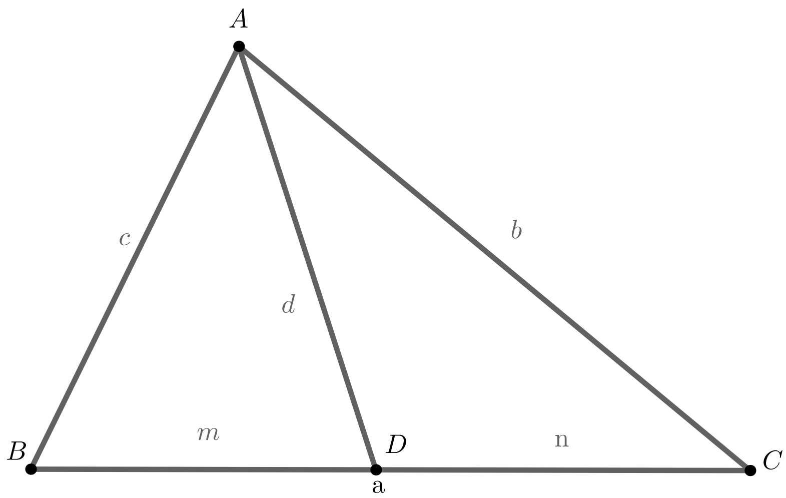
Theorem 3.2 (Law of Cosines)
In $\triangle ABC$ with side lengths a, b, and c, $c^2 = a^2 + b^2 - 2ab\cos C$

Proof of Theorem 3.2. Left as an exercise. □

The Law of Cosines is more common, as it is not always obvious when it might be useful.

Problem 3.3. Prove Theorem 3.2

Problem 3.4. Prove Stewart's theorem: in a triangle $\triangle ABC$, given cevian AD with length d divides side BC into lengths $BD = m$ and $CD = n$, $man + dad = bmb + cnc$ (mnemonic device: a man and his dad put a bomb in the sink).



Problem 3.5. Find the minimum possible value of $\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$, where $-1 \le x \le 1$.

Problem 3.6. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD. Let R_1, R_2, R_3, R_4 be the circumradii of triangles $\triangle APB$, $\triangle BPC$, $\triangle CPD$, $\triangle DPA$ respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4.

Problem 3.7. In triangle ABC, $\angle A = 45°$ and M is the midpoint of BC. AM intersects the circumcircle of ABC for the second time at D, and $AM = 2MD$. Find $\cos \angle AOD$, where O is the circumcenter of ABC.

Problem 3.8. In isosceles triangle ABC with base BC, let M be the midpoint of BC. Let P be the intersection of the circumcircle of $\triangle ACM$ with the circle with center B passing through M, such that $P \neq M$. If $\angle BPC = 135°$, then $\frac{CP}{AP}$ can be written as $a + \sqrt{b}$ for positive integers a and b, where b is not divisible by the square of any prime. Find $a + b$.

Problem 3.9. In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

§4 Problems with 13-14-15 triangles

Triangle ABC has $AB = 13$, $BC = 14$, $CA = 15$...

Problem 4.1. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Problem 4.2. Triangle ABC has $AB = 13, BC = 14$, and $AC = 15$. The points D, E, and F are the midpoints of $\overline{AB}, \overline{BC}$, and $\overline{AC}$ respectively. Let $X \neq E$ be the intersection of the circumcircles of $\triangle BDE$ and $\triangle CEF$. What is $XA + XB + XC$?

Problem 4.3. Let ABC be a triangle with orthocenter H; suppose that $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_A G_B G_C$.

Problem 4.4. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC. Find the distance between the circumcenters of triangles AHB and AHC.

Problem 4.5. In triangle ABC, $AB = 13$, $BC = 14$, $AC = 15$, and point G is the intersection of the medians. Points A', B', and C', are the images of A, B, and C, respectively, after a 180° rotation about G. What is the area of the union of the two regions enclosed by the triangles ABC and $A'B'C'$?

Problem 4.6. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC. Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

Problem 4.7. Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Problem 4.8. The points A, B and C lie on the surface of a sphere with center O and radius 20. It is given that $AB = 13$, $BC = 14$, $CA = 15$, and that the distance from O to $\triangle ABC$ is $\frac{m\sqrt{n}}{k}$, where m, n, and k are positive integers, m and k are relatively prime, and n is not divisible by the square of any prime. Find $m + n + k$.

§5 Kevin's Collection

Problem 5.1. In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.

Problem 5.2. Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 5.3. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

Problem 5.4. Triangle ABC has sidelengths $AB = 14$, $AC = 13$, and $BC = 15$. Point D is chosen in the interior of AB and point E is selected uniformly at random from AD. Point F is then defined to be the intersection point of the perpendicular to AB at E and the union of segments AC and BC. Suppose that D is chosen such that the expected value of the length of EF is maximized. Find AD.