Differential Equations numerical assignment

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Analytical solution

Given function

$$f(x, y) = y' = xy - xy^3$$

with IVP: $y(x_0) = y_0$

Solution:

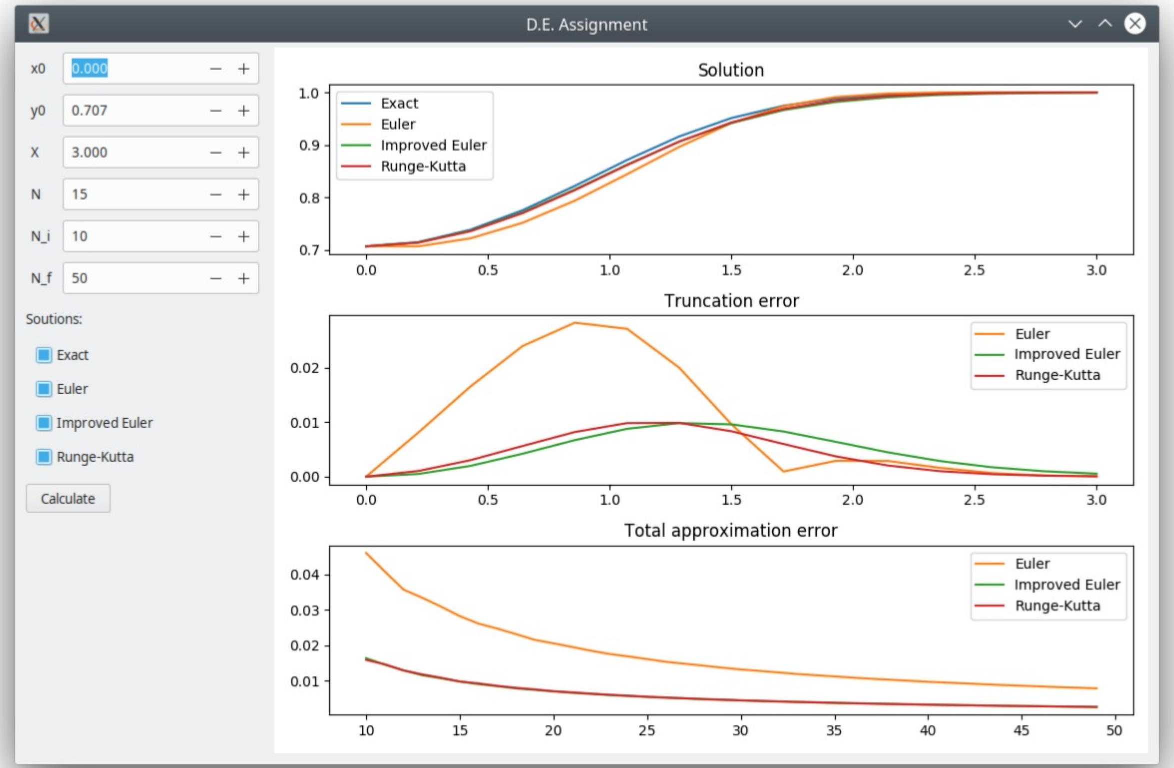
1. Let's transform equation to the first order Bernoulli ODE: $y' - xy = -xy^3$ and we have $p(x) = -x, q(x) = -x, n = 3$.
2. Let's substitute $v = y^{1-n} = y^{-2}$ and obtain $-\frac{v'}{2} - xv = -x$.
3. Solving complementary we obtain $v(x) = k * e^{-x^2}$ and by method of variation of parameter we get the final solution for $v(x) = c_1 e^{-x^2} + 1$
4. Than, substituting back $v = y^{-2}$ we get: $y^{-2} = c_1 \cdot e^{-x^2} + 1$, which is $y = \sqrt{\frac{1}{c_1 \cdot e^{-x^2} + 1}}, y = -\sqrt{\frac{1}{c_1 \cdot e^{-x^2} + 1}}$
5. With given IVP: $y(0) = \sqrt{\frac{1}{2}}$, we can derive formula for $c_1 = \frac{(1 - y_0^2) \cdot e^{x_0^2}}{y_0^2}$ and compute c_1 for given particular case $c_1 = 1$

There is **no points of discontinuity** on the given interval $x \in [0, 3]$

Application

Application interface seems to be intuitive. User can change x_0, y_0, X, N to play with parameters of approximation and N_i, N_f to see the dependency of methods' accuracy on N. Changes are reactively propagate and user interface updated accordingly to them

Application GUI



Source Code

Exact solution

is computed with *_exact_at* function

```
def _exact_at(self, x: float):
        return math.e ** (x ** 2 / 2) / \
               math.sqrt(self.coef + math.e ** (x ** 2))
```

which is applied to a vector of points with length n over interval (x_0, X)

```
def solve(self, n: int) -> np.array:
        # Apply vectorized '_exact_at' to each x point
        vfunc = np.vectorize(self._exact_at)
        return vfunc(self.x_list(n))
```

Euler method

is computed like so

```
def solve(self, n: int) -> np.array:
        t = self.x_list(n)
        y = np.full(n, self.y0)
        step = self.step(n)

        for i in range(1, n):
            y[i] = y[i - 1] + step * self.derivative(t[i - 1], y[i - 1])
        return y
```

Improved Euler method

is computed like so

```
def solve(self, n: int) -> np.array:
        t = self.x_list(n)
        y = np.full(n, self.y0)
        step = self.step(n)

        for i in range(1, n):
            y_ = y[i - 1] + step * self.derivative(t[i - 1], y[i - 1])
            y[i] = y[i - 1] + 0.5 * step * (self.derivative(t[i - 1], y[i -
                                            + self.derivative(t[i], y_))
        return y
```

Runge-Kutta method

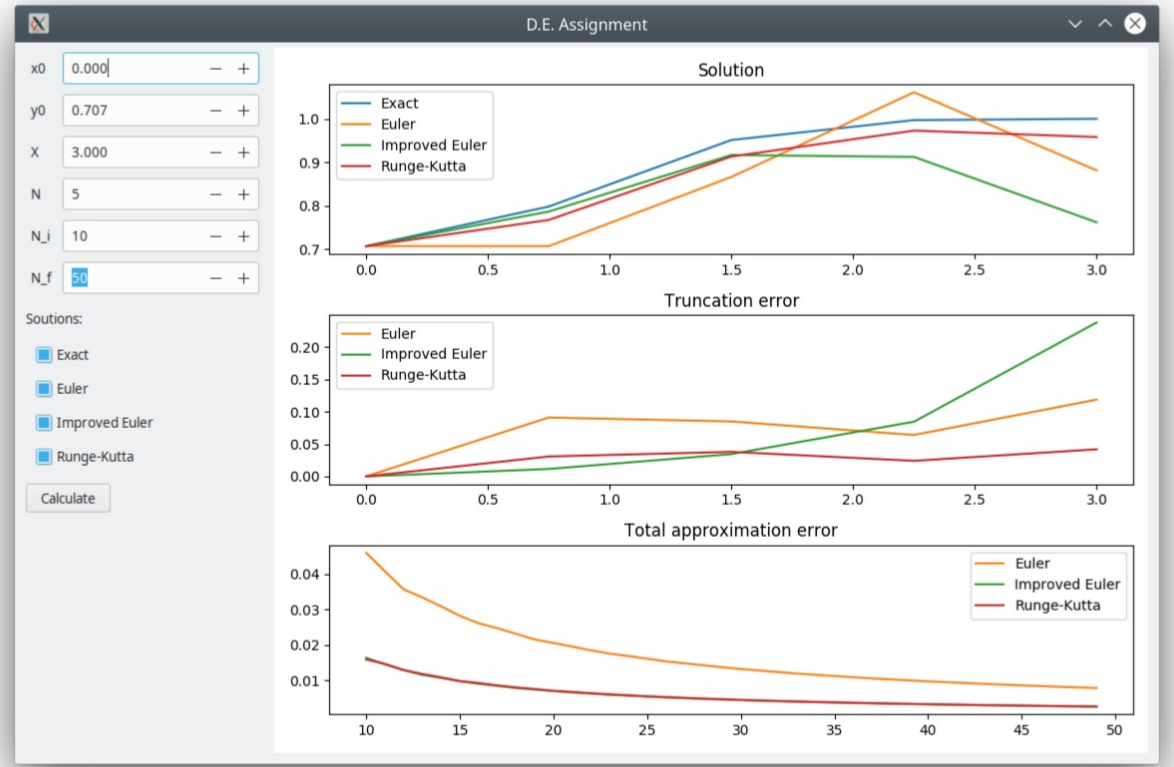
is computed like so

```
def solve(self, n: int) -> np.array:
        t = self.x_list(n)
        y = np.full(n, self.y0)
        step = self.step(n)

        for i in range(1, n):
            k1 = step * self.derivative(t[i - 1], y[i - 1])
            k2 = step * self.derivative(t[i - 1] + step / 2, y[i - 1] + k1
            k3 = step * self.derivative(t[i - 1] + step / 2, y[i - 1] + k2
            k4 = step * self.derivative(t[i - 1] + step, y[i - 1] + k3)

            y[i] = y[i - 1] + 1 / 6 * (k1 + 2 * k2 + 2 * k3 + k4)

        return y
```

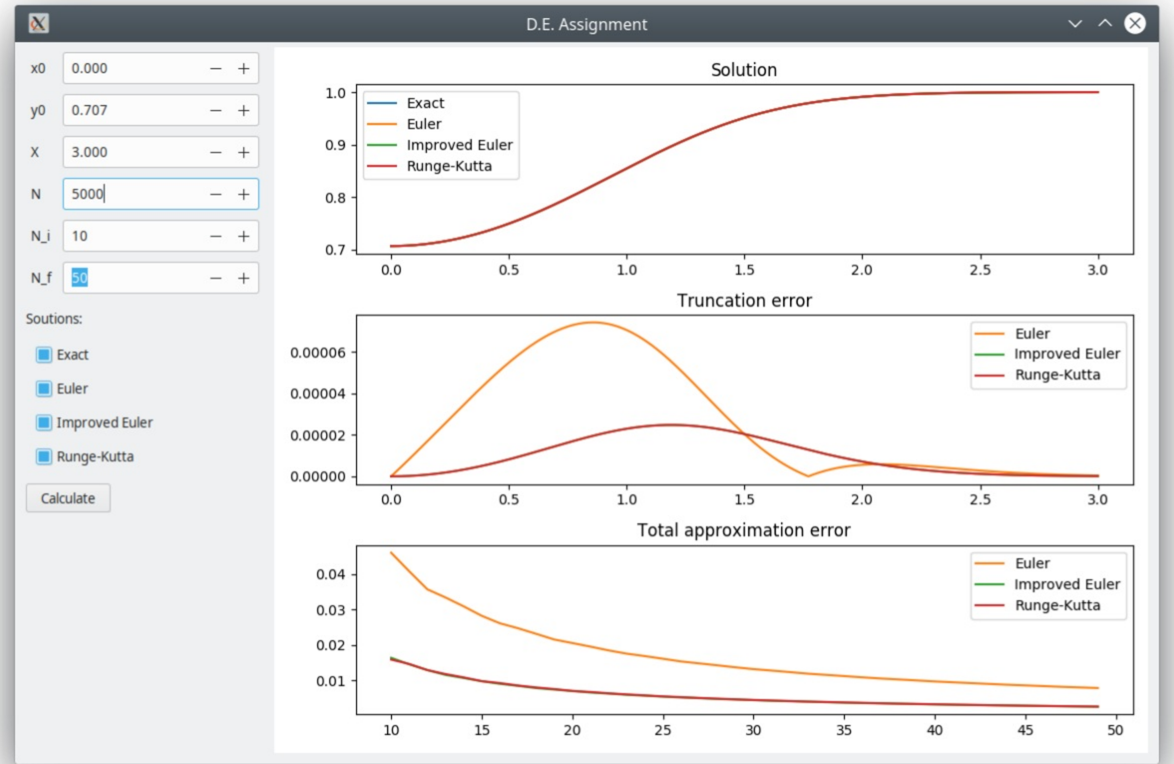
Numerical Investigation

We can see that with increasing N, the error of solutions is decreasing proportionally

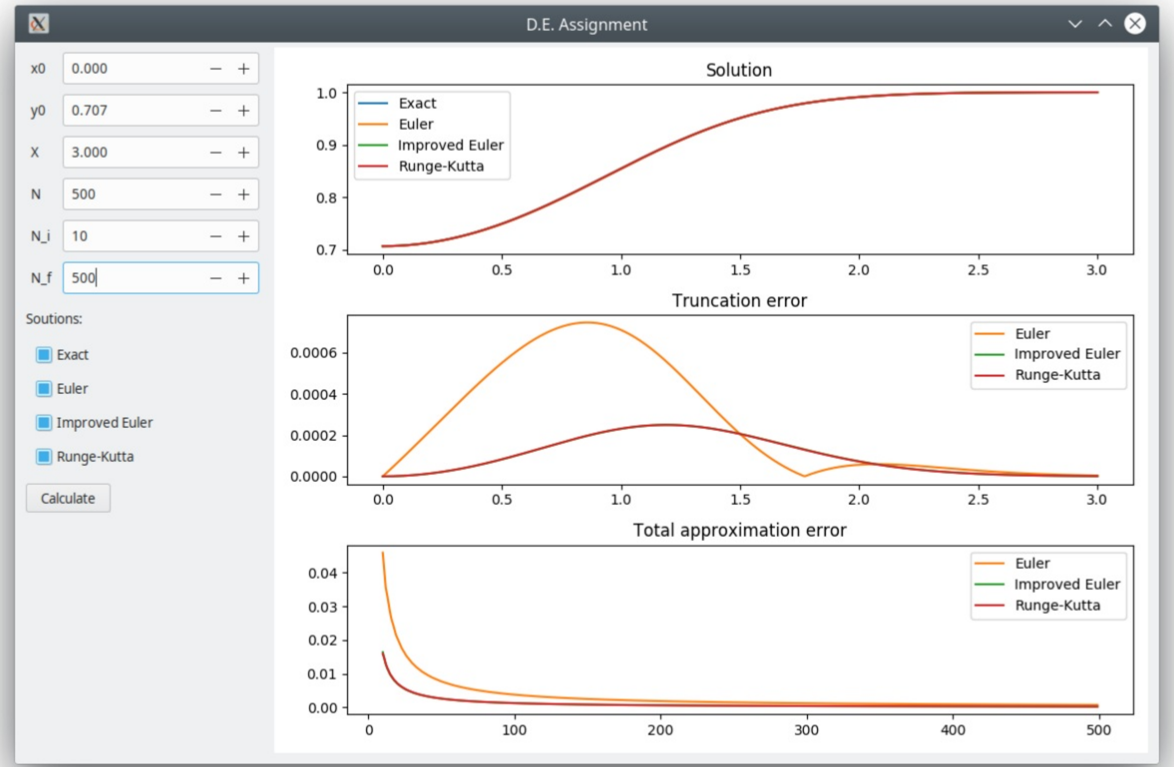
$$x_0 = 0, y = \sqrt{\frac{1}{2}}, X = 3, N = 5, N_i = 10, N_f = 50$$



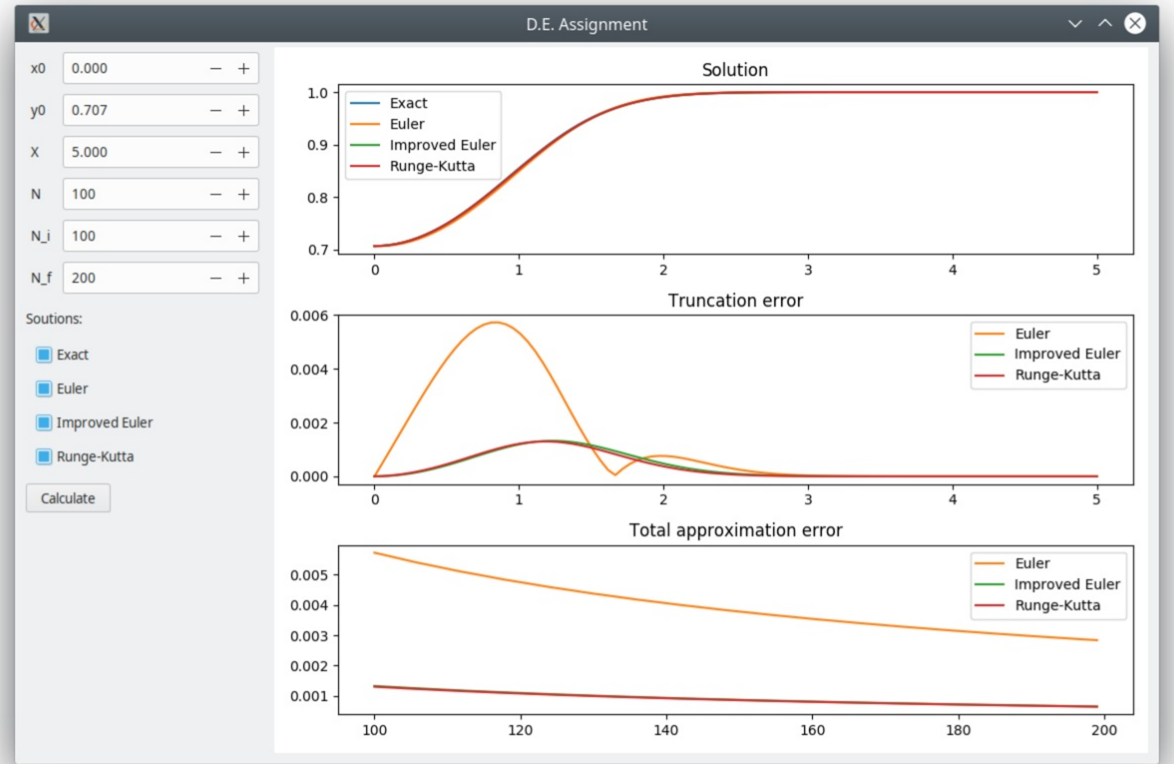
$$x_0 = 0, y = \sqrt{\frac{1}{2}}, X = 3, N = 5000, N_i = 10, N_f = 50$$



$x_0 = 0, y = \sqrt{\frac{1}{2}}, X = 3, N = 500, N_i = 10, N_f = 500$

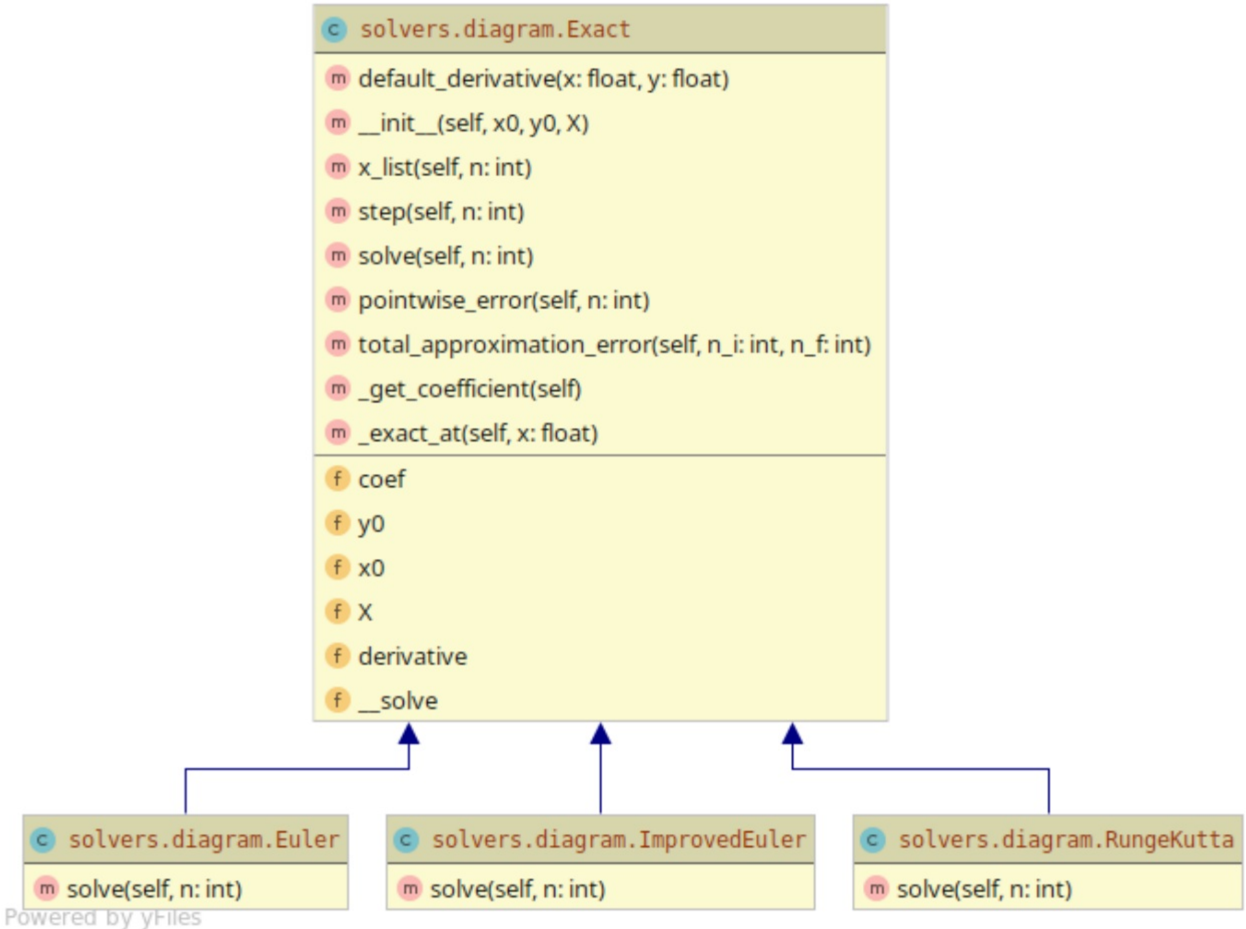


$x_0 = 0, y = \sqrt{\frac{1}{2}}, X = 5, N = 100, N_i = 100, N_f = 200$



As we see, with increasing N the approximation of solution is increasing appropriately

UML Class Diagram


solvers.diagram.Exact
default_derivative(x: float, y: float)
__init__(self, x0, y0, X)
x_list(self, n: int)
step(self, n: int)
solve(self, n: int)
pointwise_error(self, n: int)
total_approximation_error(self, n_i: int, n_f: int)
_get_coefficient(self)
_exact_at(self, x: float)
coef
y0
x0
X
derivative
__solve
solvers.diagram.Euler
solve(self, n: int)
solvers.diagram.ImprovedEuler
solve(self, n: int)
solvers.diagram.RungeKutta
solve(self, n: int)
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