Studies of ERM Models with Correlated Disorder

by Tom Folgmann

Bachelor Thesis Presentation, 2024

Upfront: Goals.

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Foundations

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- What did we **conclude**?

What is ERM?

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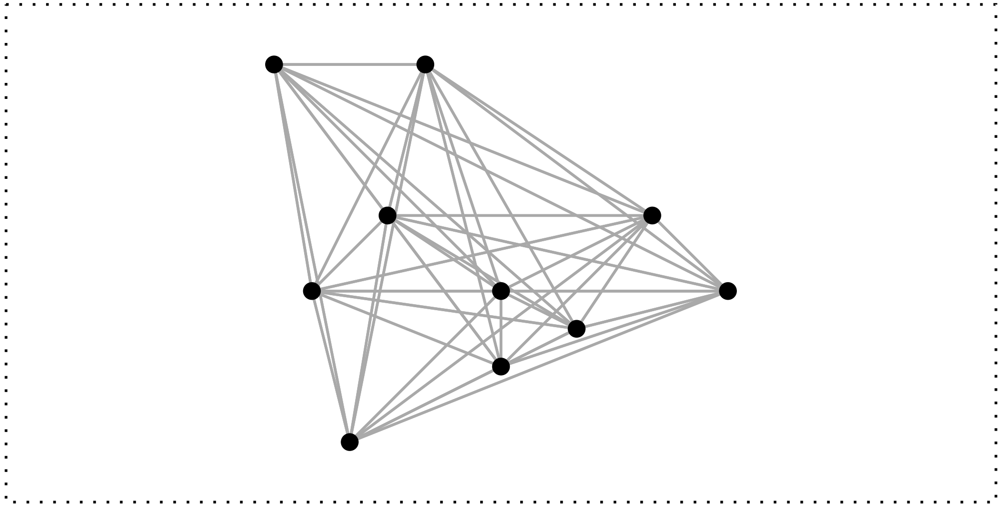

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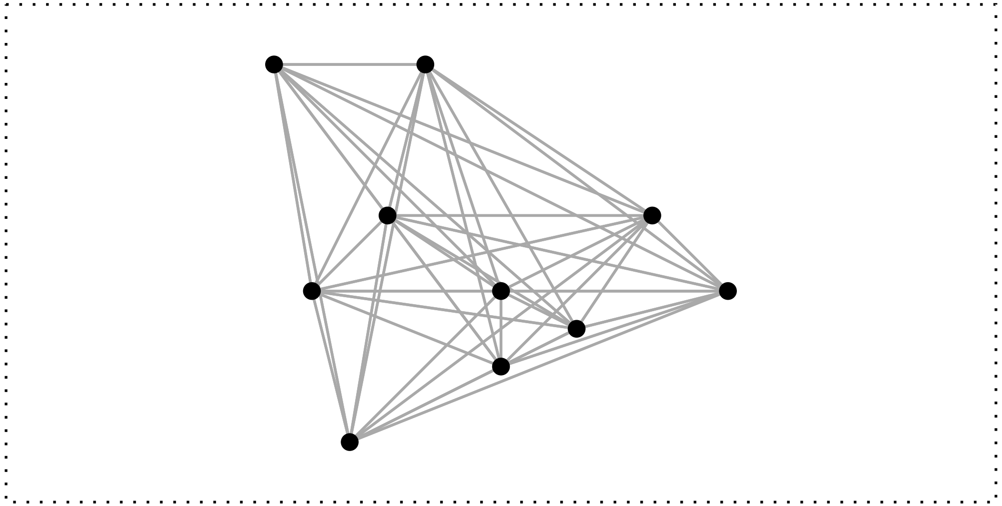

A system with $N \in \mathbb{N}$ (related) particles can be described by a mathematical *Graph*.

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- *Self-interaction* given by $\Sigma(f,i) \stackrel{\mathrm{m}}{=} \sum_{j\in[N]\setminus\{i\}} f_{ij}$

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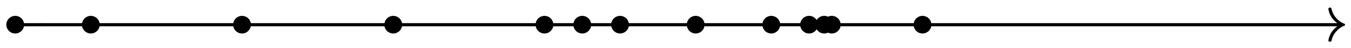

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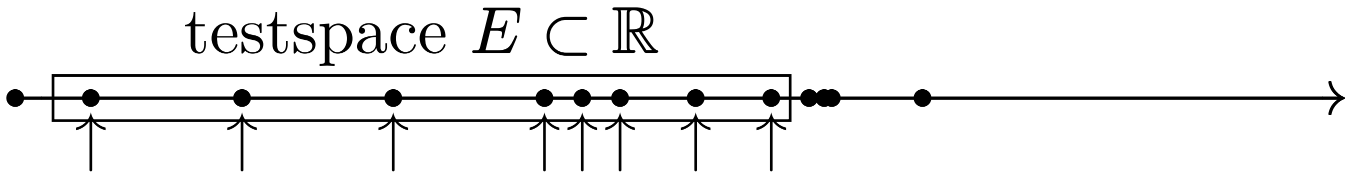

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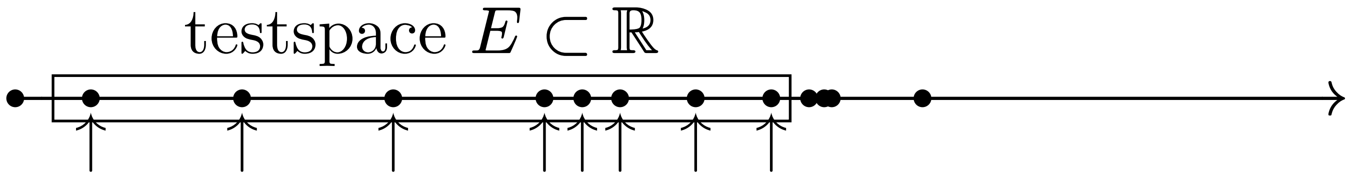

... results in an (unnormalized) density function

$$E \mapsto \sum_{i \in [k]} \delta_{\Lambda_i}(E) \qquad \in \{0, ..., k\}$$

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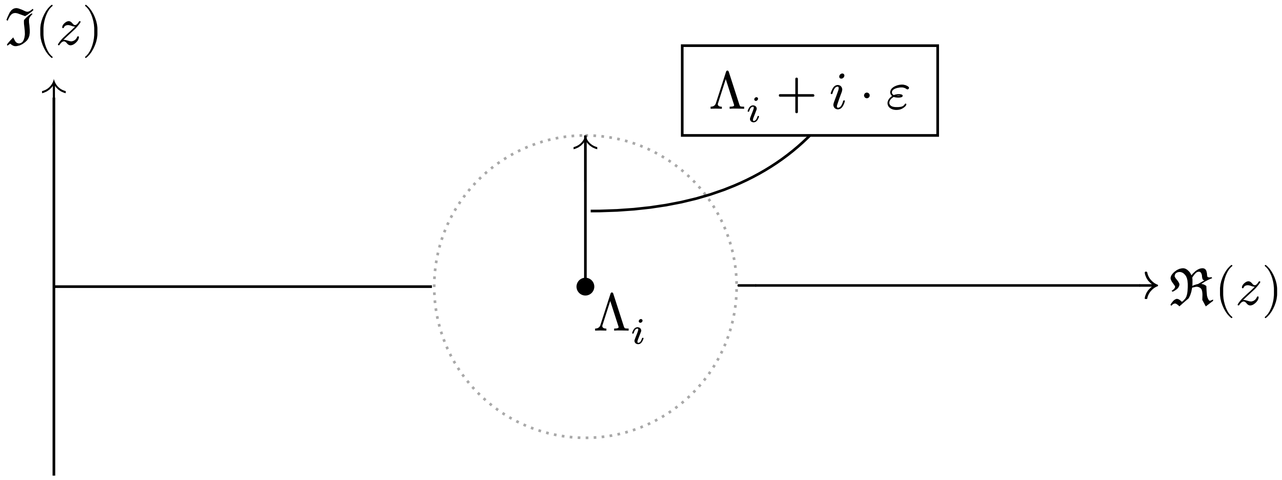

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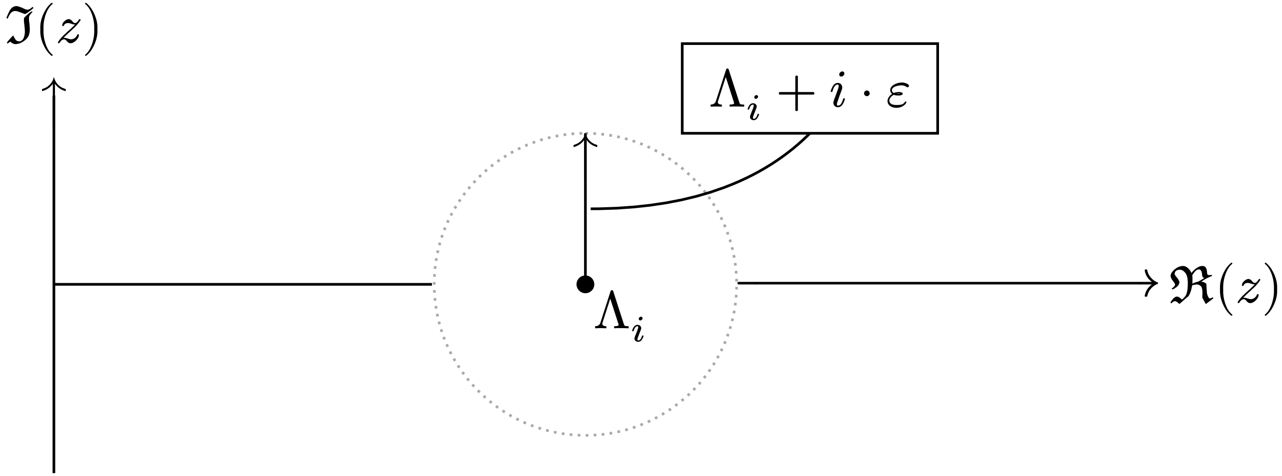

$\hookrightarrow$ Usecase is the resolvent with a singularity at Λ_i.

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This is just $f_{i,j}$!

.. for $i \neq j$.

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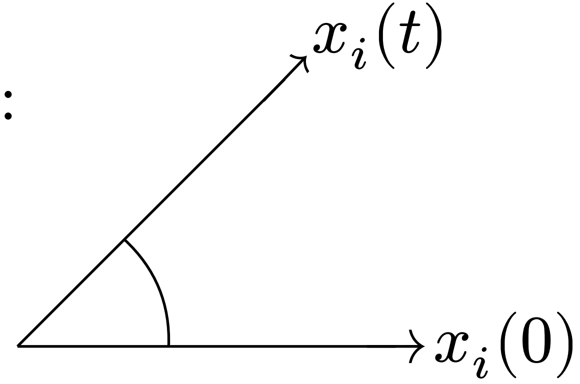

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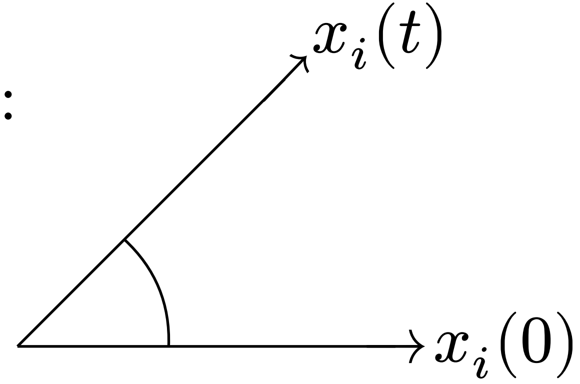

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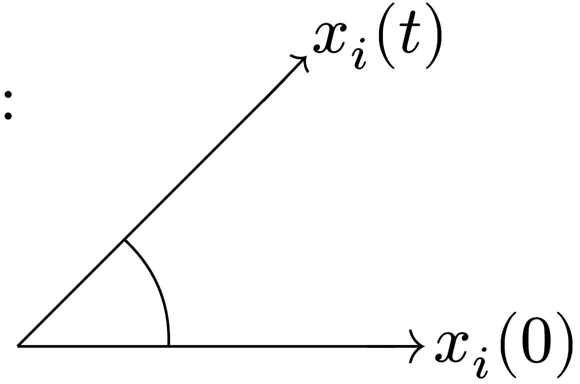

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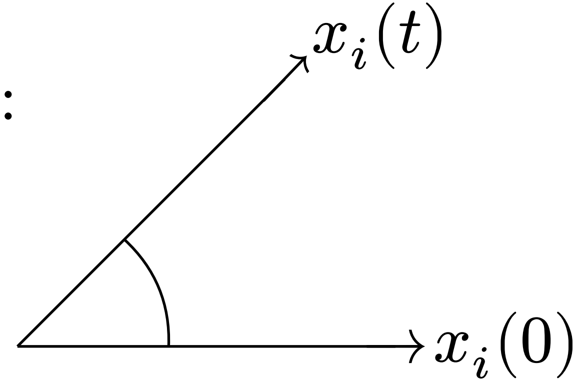

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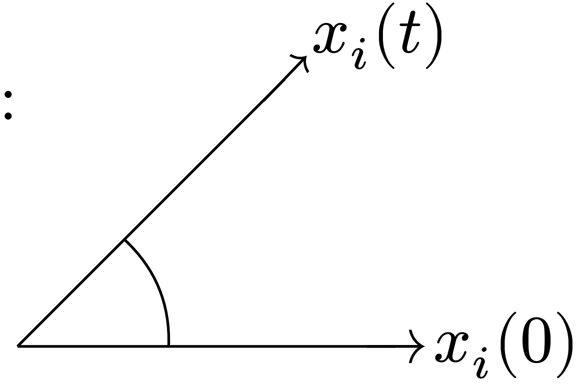

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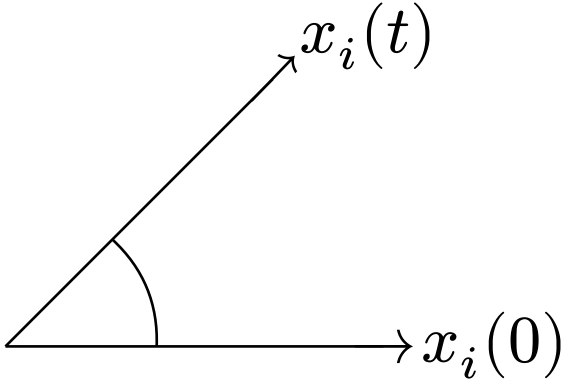

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$$(\mathcal{L}F_{j,i})(s) = \pm \frac{1}{\tilde{U}(f, x^*(0))_{i,j} - \delta_{ij} \cdot \lambda_i^2}.$$

[6]With $x^*(t) = \left(i \mapsto x_{i(t)}\right)$ and $F_{j,i}(t) := \langle x_j(t), x_i(0)\rangle$.

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Q: What are we integrating over?

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Ev. step	**Meaning**
R	Random variable, abstract
$R(\omega)$	Vector of time dep. pos.
$R(\omega)_i$	i-th particle position, time dep. path
$R(\omega)_i(t)$	Position of i-th particle at time t (fixed for us.)

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Value in $\mathbb{R}$ ..
(simplification!)

This is already a good starting point to understand our *Correlated Disorder* modification!

.. missing key elements:

[11] Expansion to a functional can be argued, see thesis p. 19.

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- The *action* (functional) S_{z,R_ω} at a *test point* $z \in \mathbb{C}$ and a particle position vector R_ω.

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- The *moment generating function* $Z_{z,R_\omega}[J]$. It requires the *force field* J.

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.. coming from ..

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$\varphi \to \Phi$

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$\rightarrow$ Looking at different Taylor expansion terms of $\underbrace{\exp\left(\int_{(\mathbb{R}^d)^2} ..\, d\boldsymbol{p}\right)}_{\mathrm{Ex}_{\mathcal{L}_f}}$ yields different powers of integral operators.

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Hereby $G_0(p, z) = \Big(z - \rho_* \cdot \Big(\hat{f}(0) - \hat{f}(p) \Big) \Big)^{-1}$ is the *bare propagator* and $\Sigma(G)$ the *self-energy*.

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.. possible connections of these edges are given by *vertices*:

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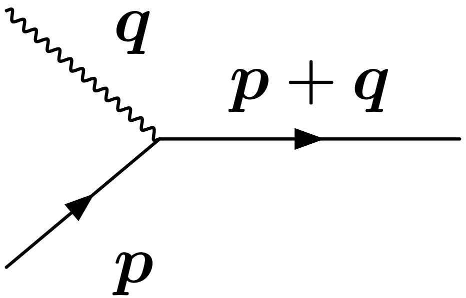

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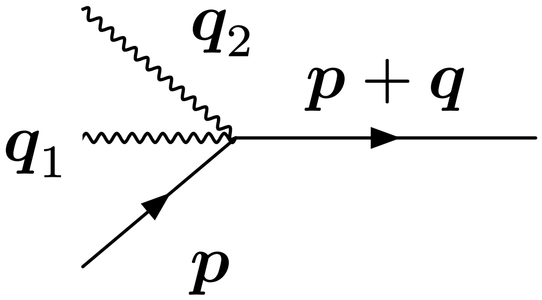

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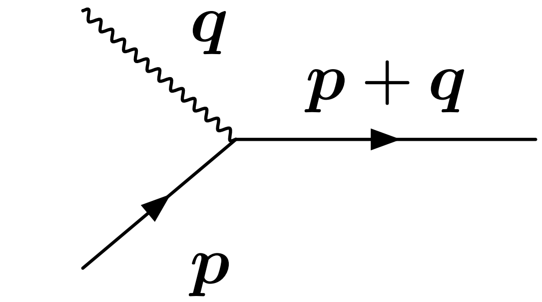

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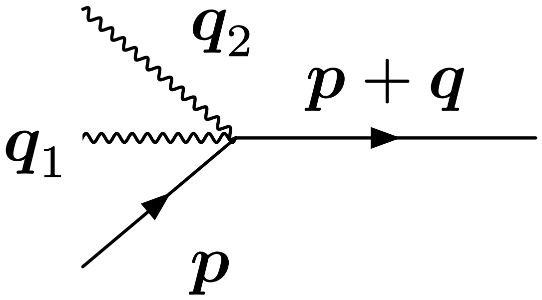

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.. which completes the set of Feynman rules.

How can we use diagrammatics?

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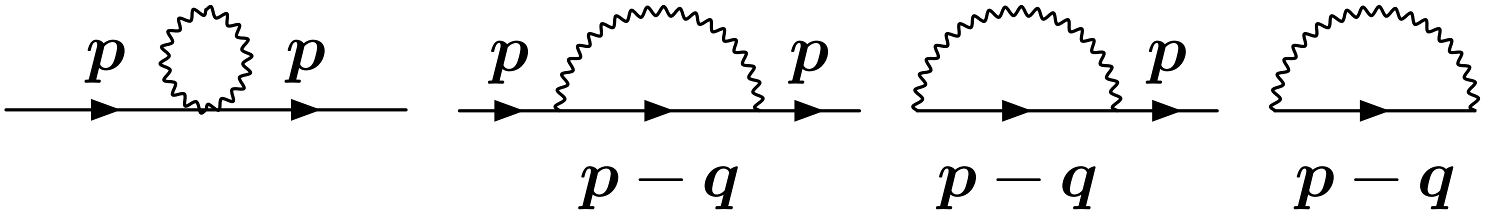

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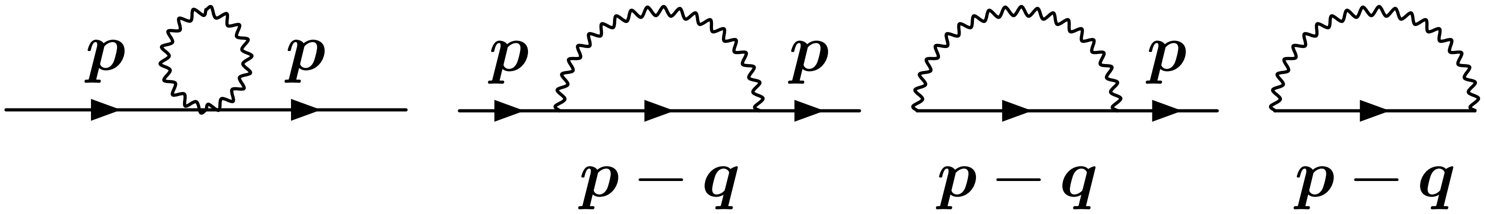

.. represented diagrams are *irreducible*: $Z_{z,R_\omega}[J] \propto \exp\Big(\sum_{C\in\mathcal{C}} C\Big)$.

Integral representations[16]

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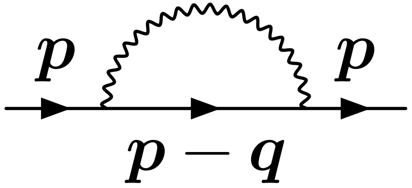

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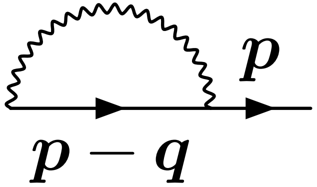
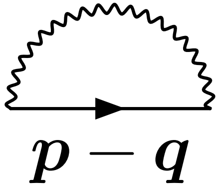


$$= -\frac{2 \cdot G_0(\boldsymbol{p}, z)}{\rho_*} \cdot \int_{\mathbb{R}^d} G_0(\boldsymbol{p}-\boldsymbol{q}, z) \cdot \mu_z(\boldsymbol{p}, -\boldsymbol{q}) \, d\boldsymbol{q},$$



$$= \frac{1}{\rho_*} \cdot \int_{\mathbb{R}^d} G_0(\boldsymbol{p}-\boldsymbol{q}, z) \, d\boldsymbol{q}.$$

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- Utilization of vertex' and propagator symmetries.

What is Correlated Disorder?

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How can we include *structure* in our probability density?

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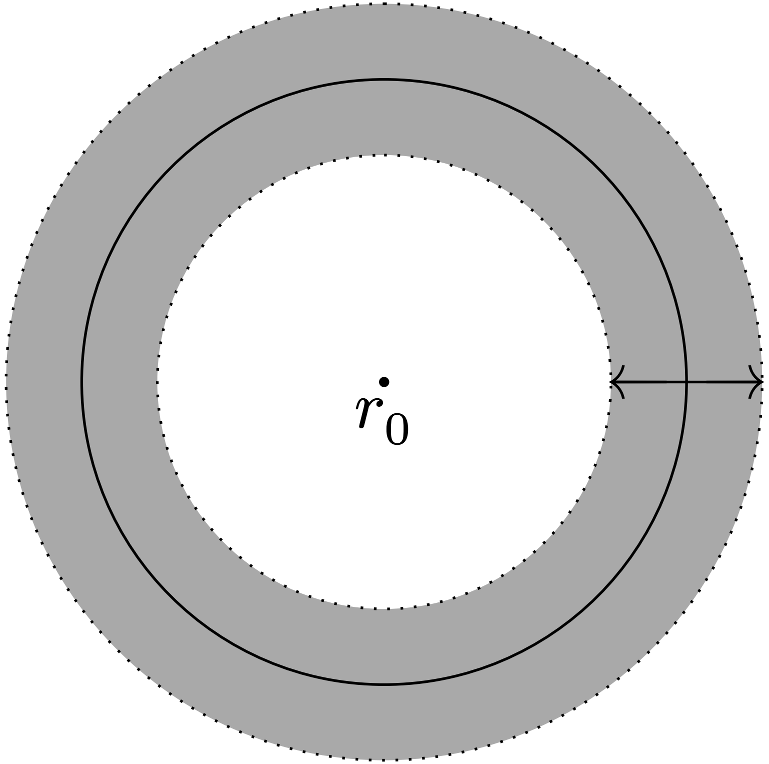

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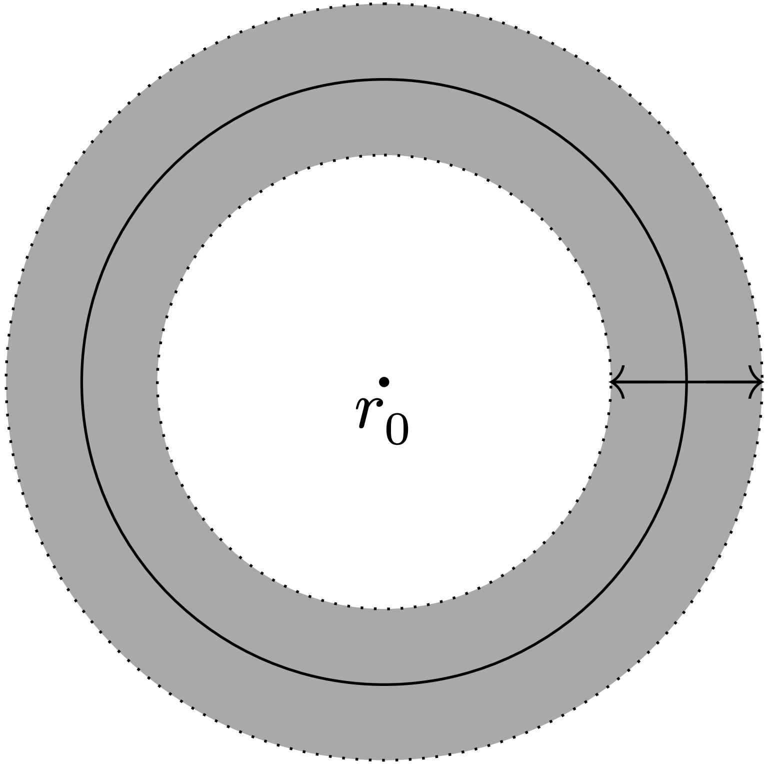
while $\rho_N^{(2)}$ reflects integration of $\exp(-\beta \cdot H(r, \cdot))$ for remaining particles.



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.. namely given by

$$S_*(\boldsymbol{q}) = 1 + \rho_* \cdot \int_{\mathbb{R}^d} (g_0(\boldsymbol{r}) - 1) \cdot e^{\mathring{\imath} \cdot \boldsymbol{q} \cdot \boldsymbol{r}} \, d\boldsymbol{r}.$$

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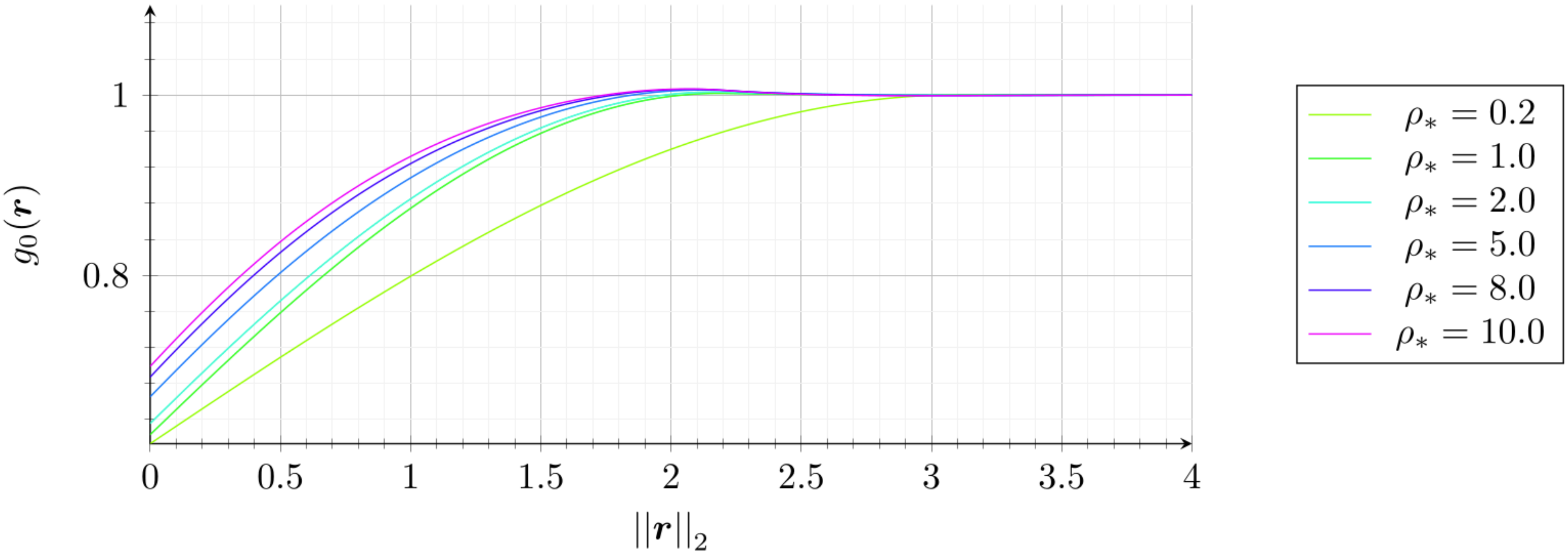

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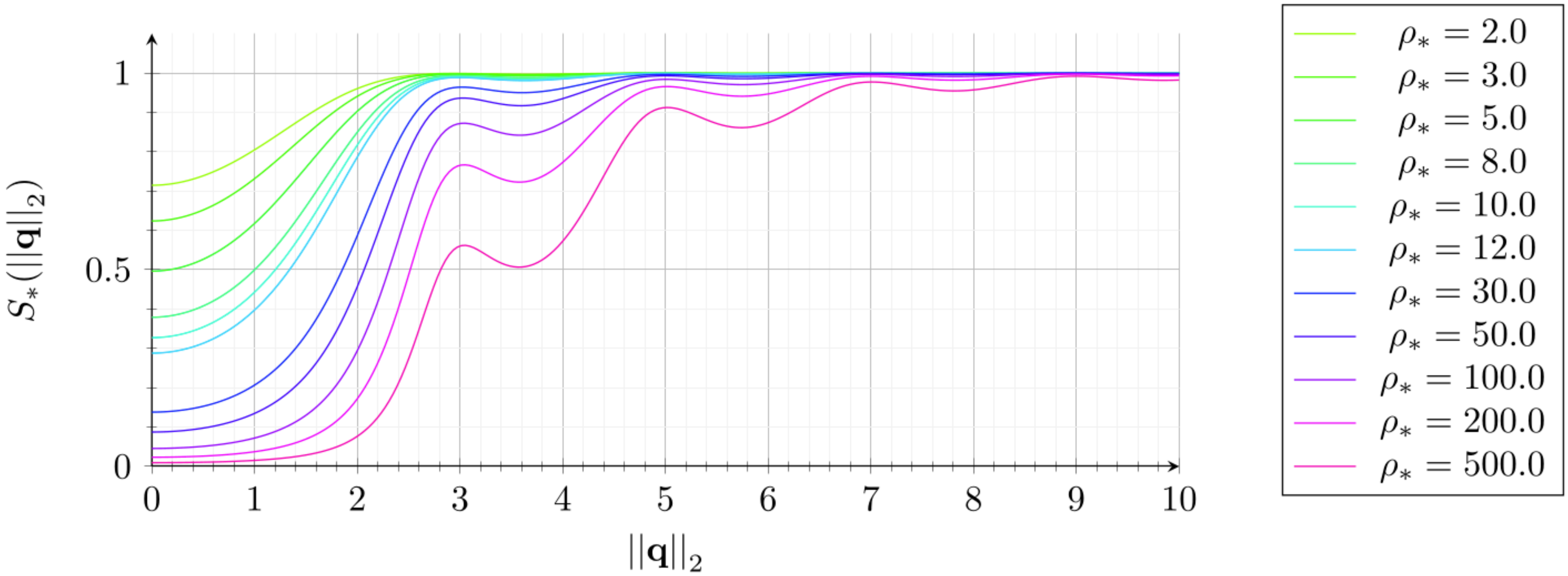

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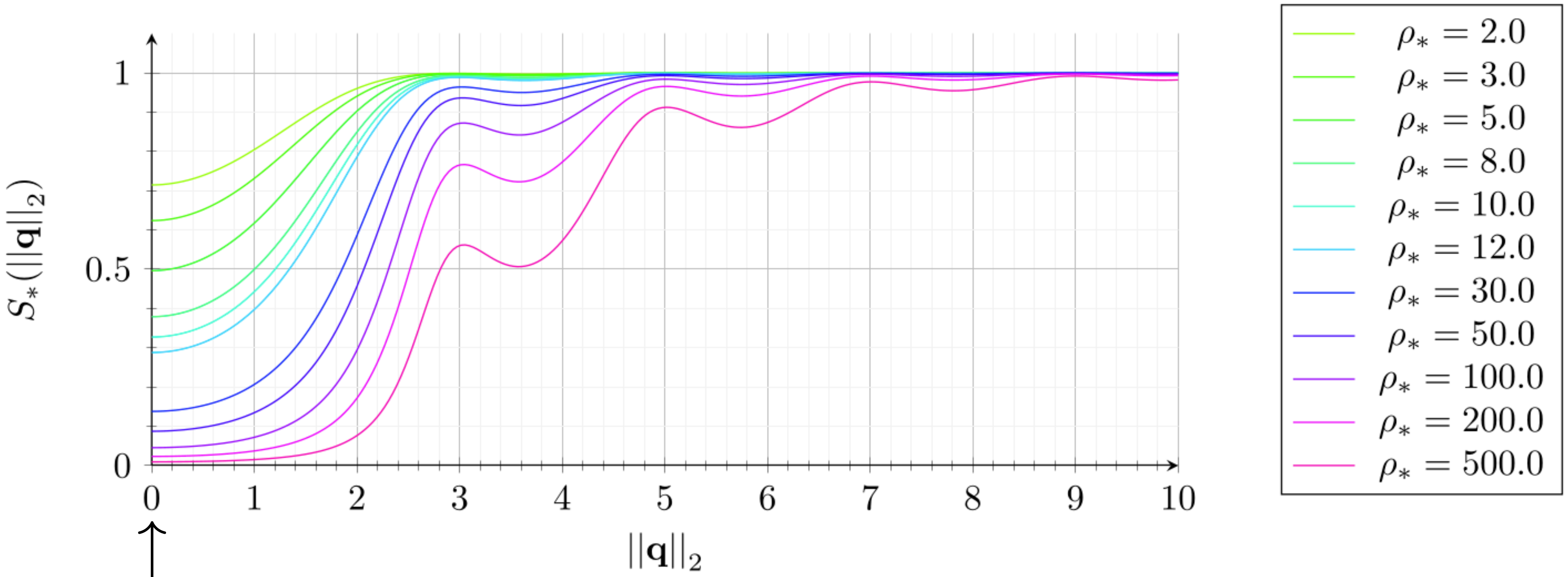

$$S_*(\mathbf{0}) = \rho_* k_B T \cdot \kappa$$

Where did we implement this, what did it change?

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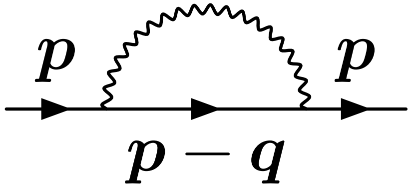

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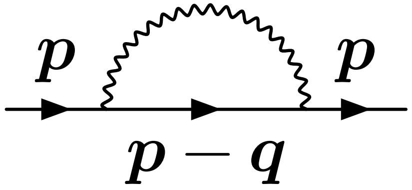

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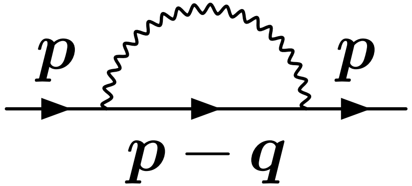

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Can we in any way compare our results?

The Approach of Martin-Mayor

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Here, a *superposition approximation* was used:

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This has an explicit approximation built into the spring function!

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- We did not change the zeroth order term in the propagator.

What did a numerical model show?

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$$V_{d,N} \ni R \mapsto U_a^{(num)}(R) = \sum_{(i,j)\in[N]^2} \begin{cases} \frac{1}{2} \cdot \left(\|R_i - R_j\| - a\right)^2 \text{ if } \|R_i - R_j\| < a, \\ 0 \text{ else.} \end{cases}$$

Results using the Hypernetted Chain

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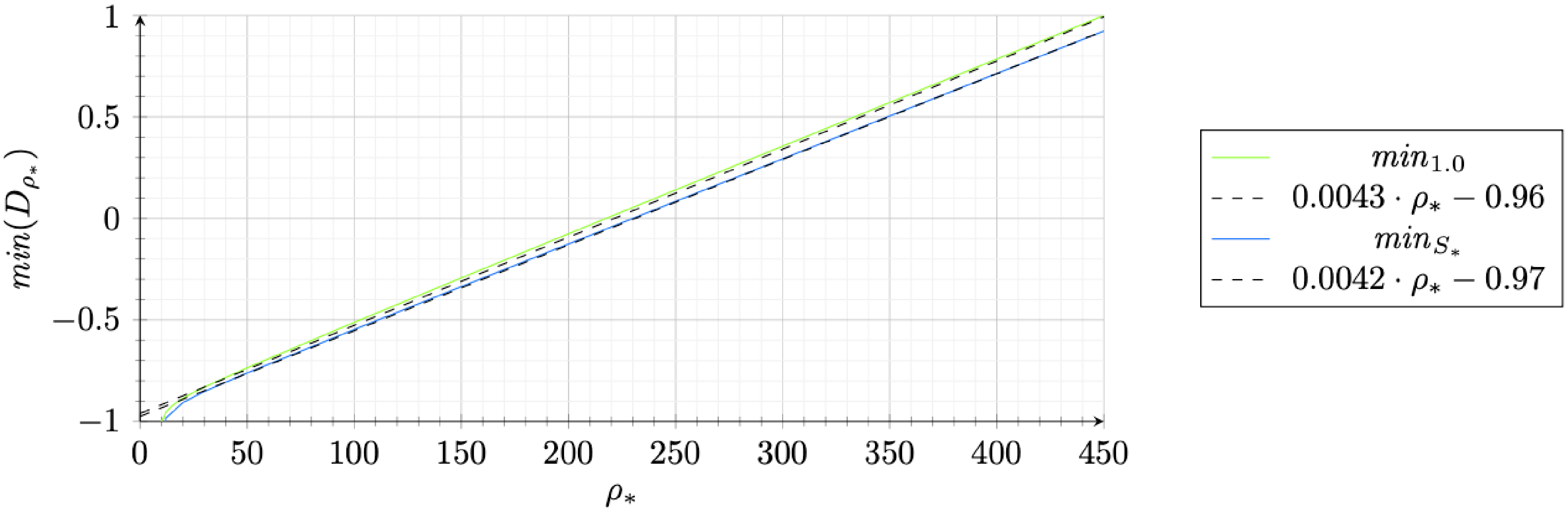

→ Physical applicability at $\rho_* > 225$.

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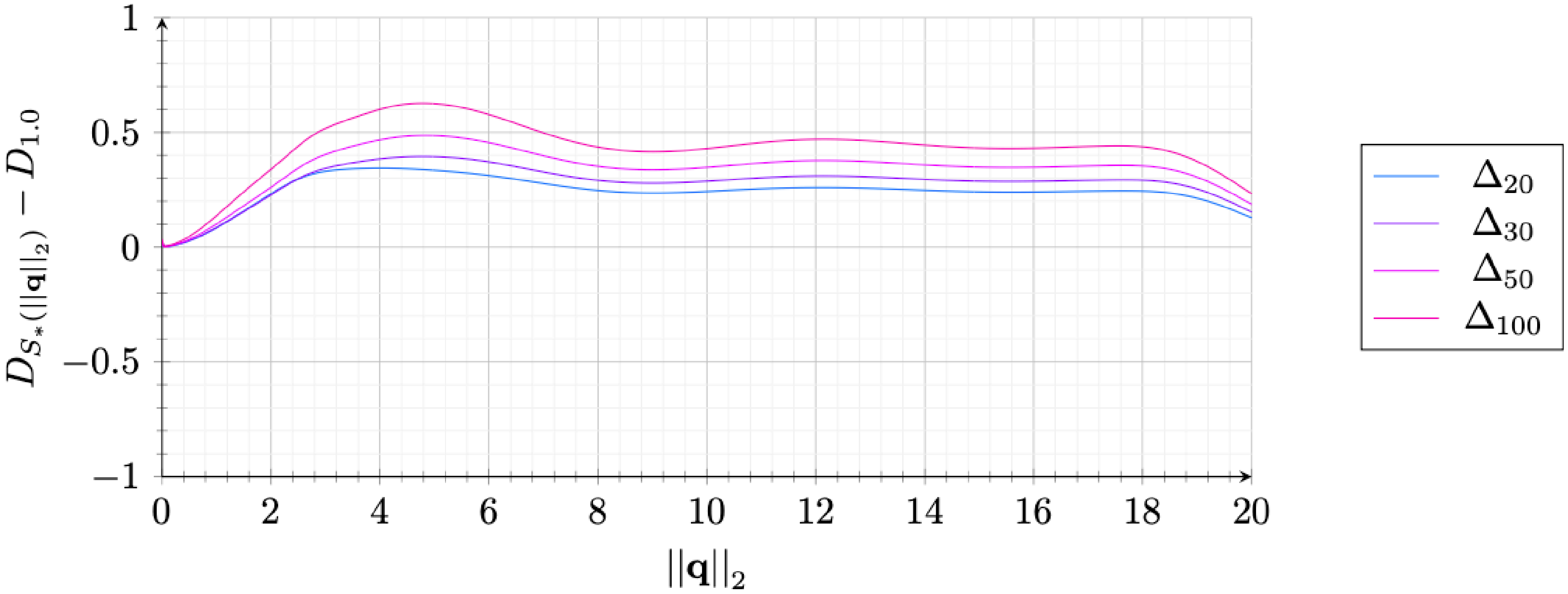

→ $S_* \neq 0$ yields slightly higher amplitudes.

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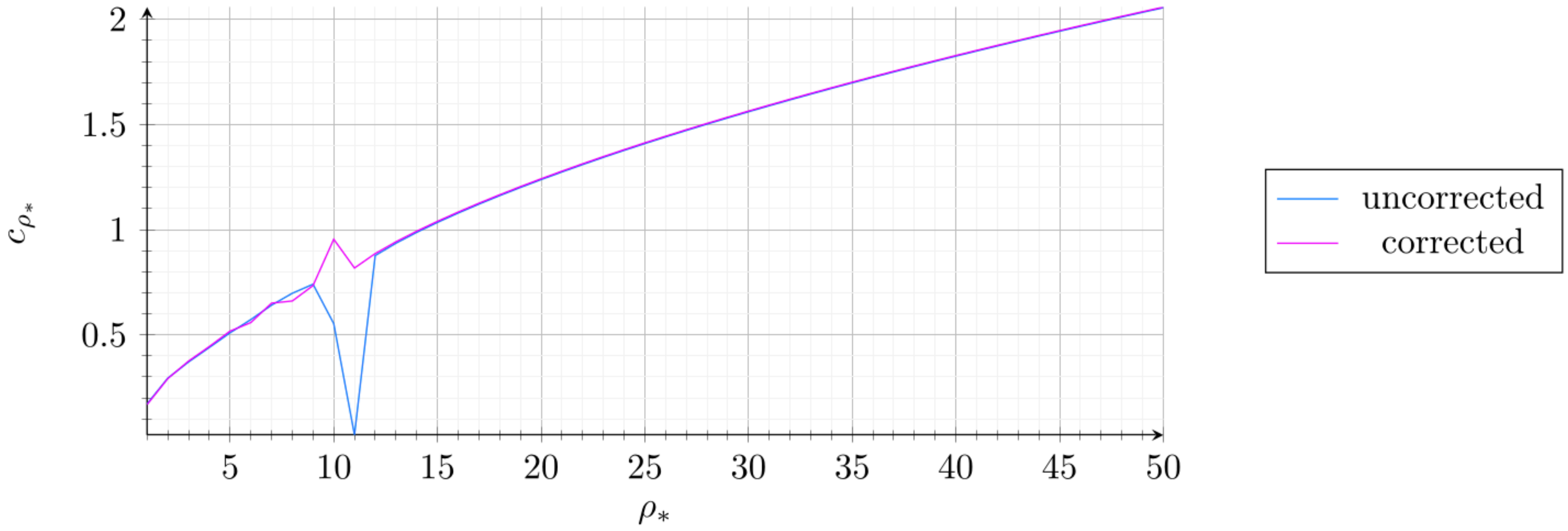

→ No *major* differences in the velocity of sound noticeable.

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There is a possibility of extention onto higher loop orders.

Concluding terms..

Adoption of the radial probability density and static structure factor introduced an integral correction in c_{ρ_*}:

$$G \mapsto \left(G_0(\boldsymbol{p}, 0)^{-1} - \rho_* \cdot \int_{\mathbb{R}^3} S_*(\boldsymbol{p} - \boldsymbol{q}) \cdot G(\boldsymbol{q}, z) \cdot V(\boldsymbol{q}, \boldsymbol{p}) \, d\boldsymbol{q} \right)^{-1}.$$

There is a possibility of extention onto higher loop orders.

$$\exp(-\beta \cdot U(r)) \approx \exp\Big(-\beta \cdot (r - \nu)^{\perp} \cdot A \cdot (r - \nu)\Big), (\rho \text{ mediocre})$$

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`unb3rechenbar/BA24-CorDis.git`

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