Studies of ERM Models with Correlated Disorder

by Tom Folgmann

Bachelor Thesis Presentation, 2024

What is ERM?

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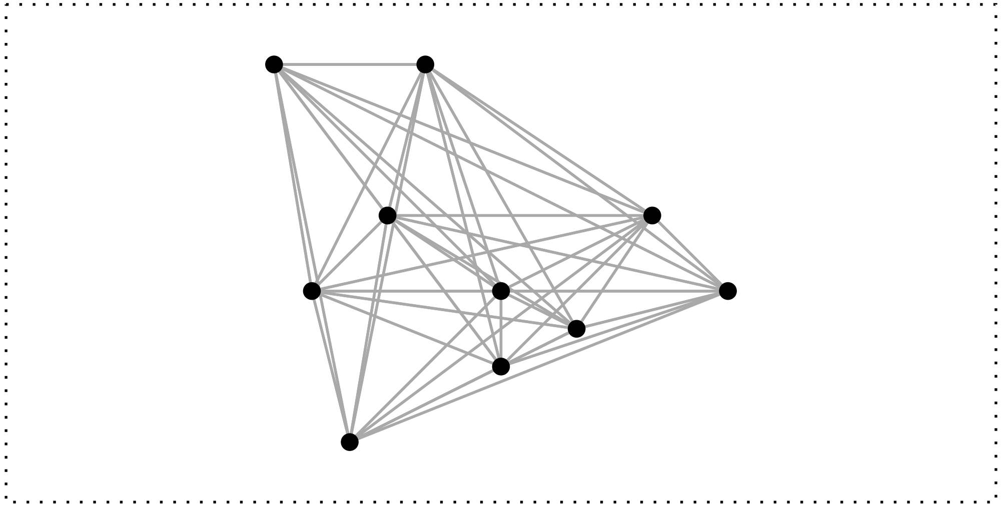

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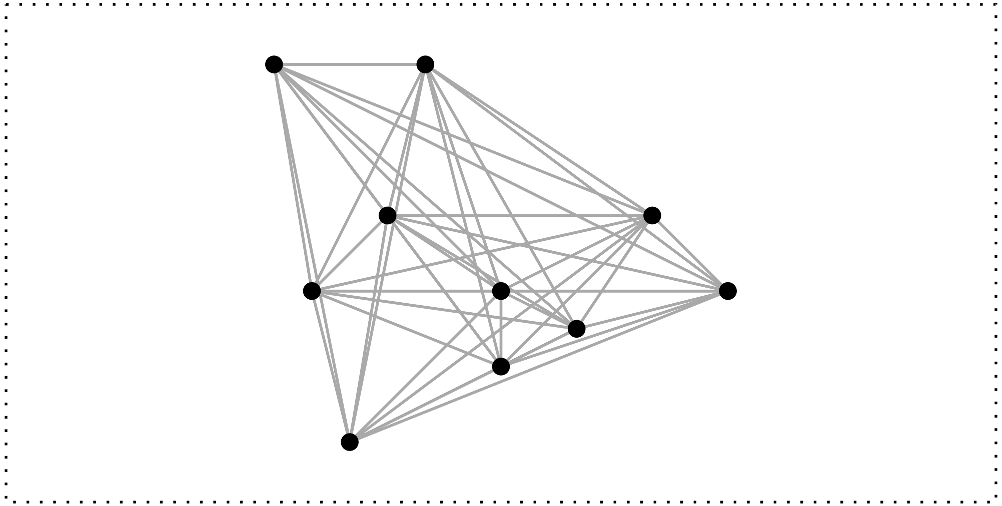

A system with $N \in \mathbb{N}$ (related) particles can be described by a mathematical *Graph*.

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.. special case is the *Adjacency Matrix* A, where $w_{i,j} \in \{0, -1, 1\}$.

Definition of the ERM Laplacian Matrix

In the ERM model the Laplacian matrix is defined as

$$\tilde{U}(f,r) := \begin{pmatrix} \Sigma(f,1) & -f_{12} & \dots & -f_{1N} \\ -f_{21} & \Sigma(f,2) & \dots & -f_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{N1} & -f_{N2} & \dots & \Sigma(f,N) \end{pmatrix},$$

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- *Self-interaction* given by $\Sigma(f,i) \overset{\mathrm{m}}{=} \sum_{j\in[N]\setminus\{i\}} f_{ij}$

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Let $\Lambda : [p] \to \sigma_P\Big(\tilde{U}(f, r)\Big)$ map bijectively into the *point spectrum* of the ERM Laplacian.

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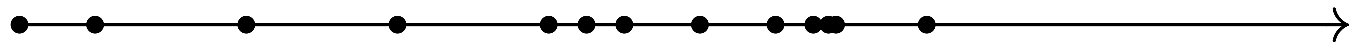

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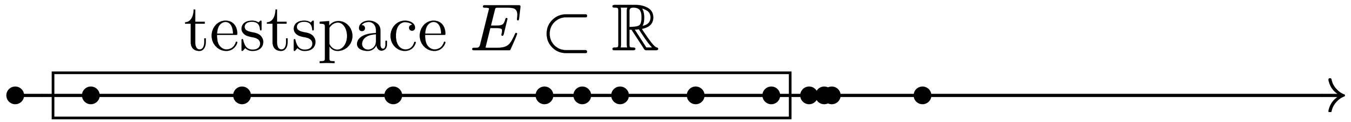

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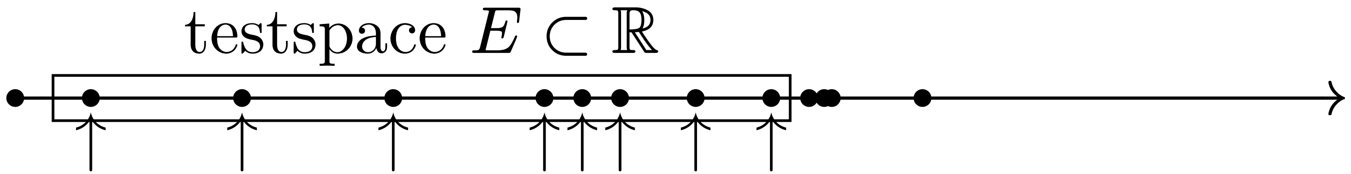

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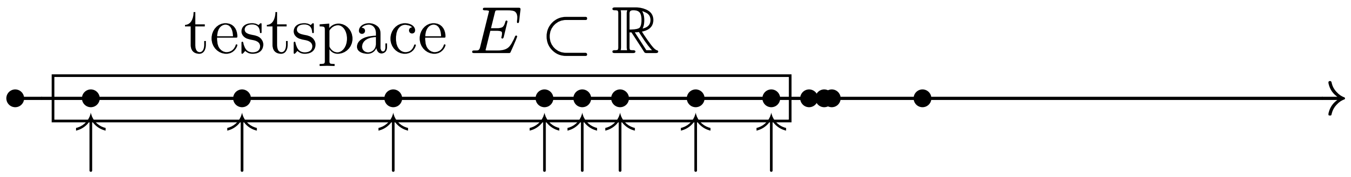

... results in an (unnormalized) density function

$$E \mapsto \sum_{i \in [p]} \delta_{\Lambda_i}(E) \qquad \in \{0, p\}$$

The Resolvent Eigenvalue Approximation

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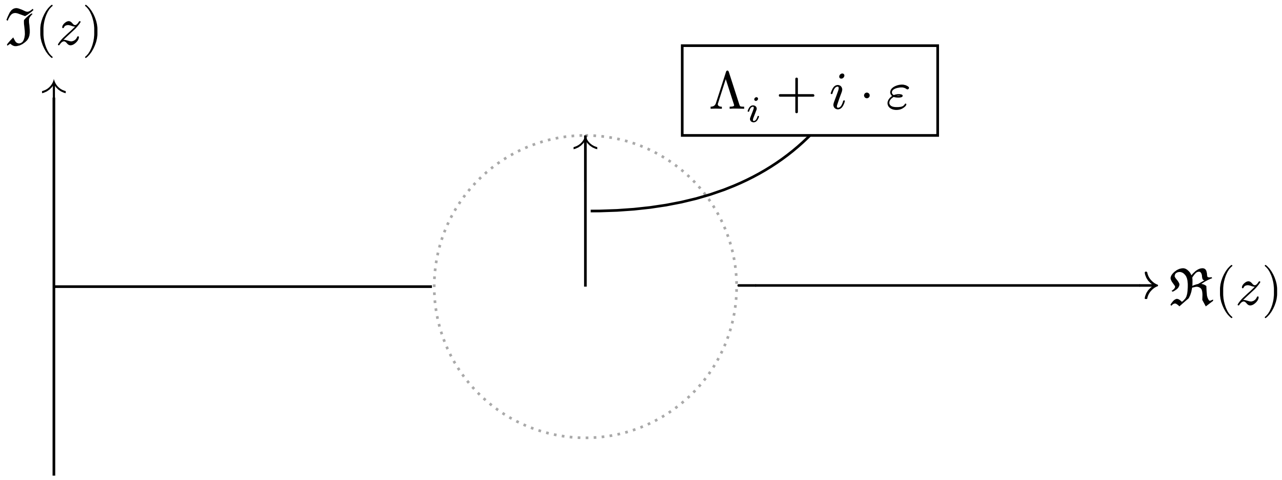

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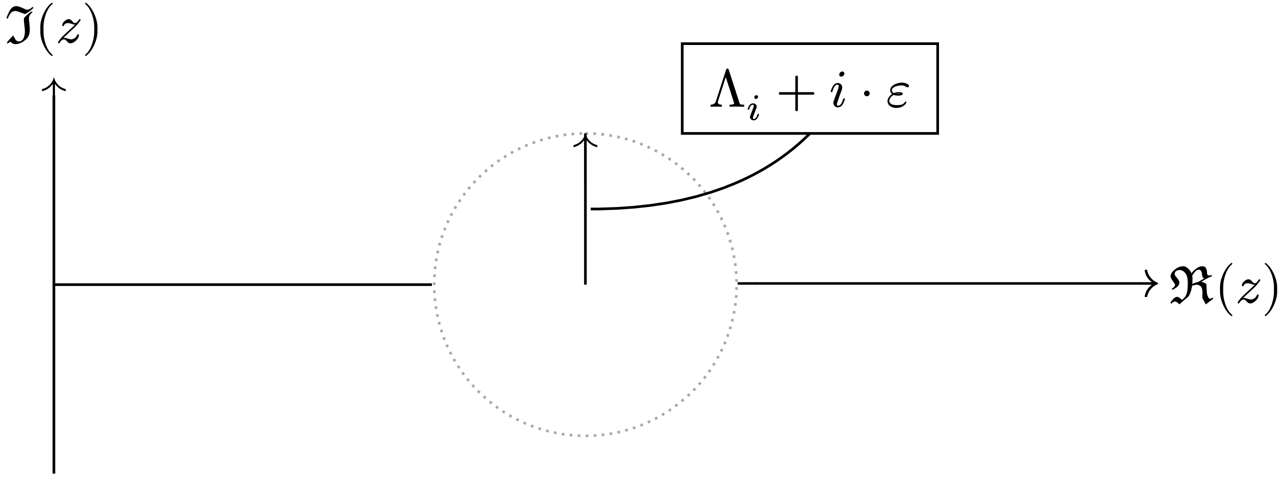

$\hookrightarrow$ Usecase is the resolvent with a singularity at Λ_i.

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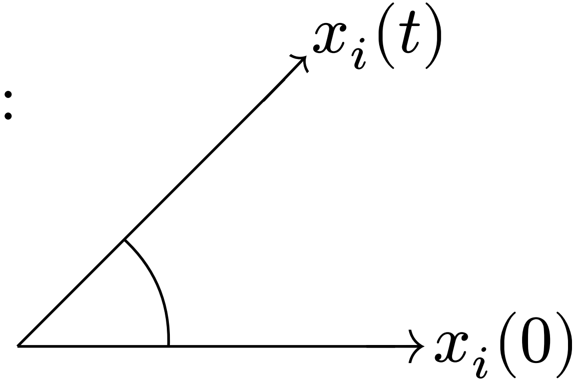
.. looking at the behaviour with regard to the initial conditions:

$$\left(\frac{d}{dt}\right)^2 \langle x_i(t), x_i(0)\rangle = -\tilde{U}(f, i \mapsto x_i(t))_{i,j} \cdot \langle x_j(t), x_i(0)\rangle.$$

In a visual approach $\langle x_i(t), x_i(0) \rangle$ represents:



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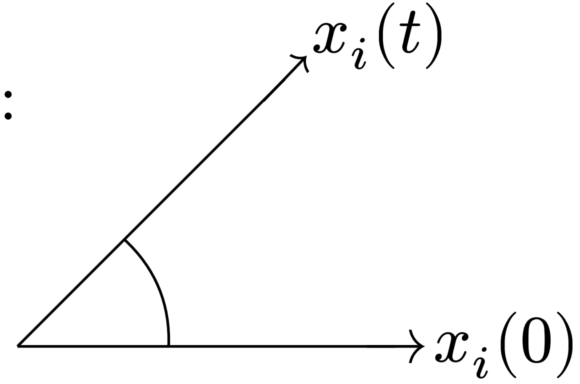

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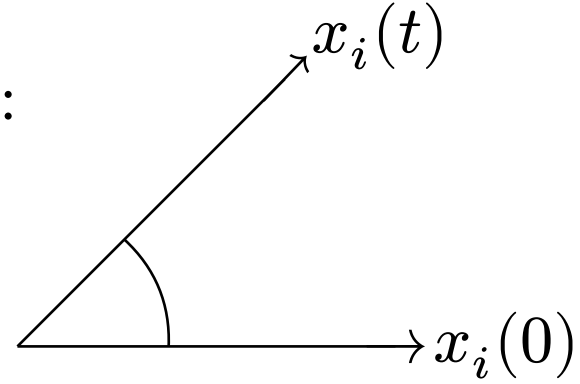

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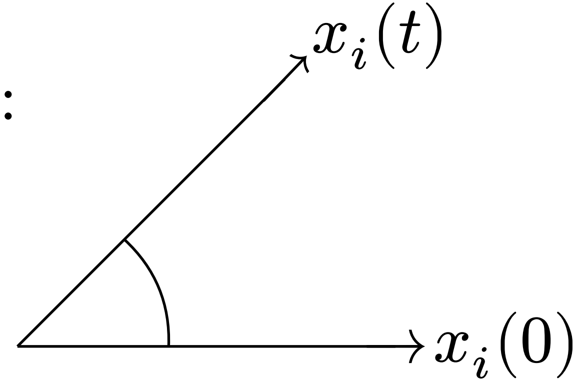

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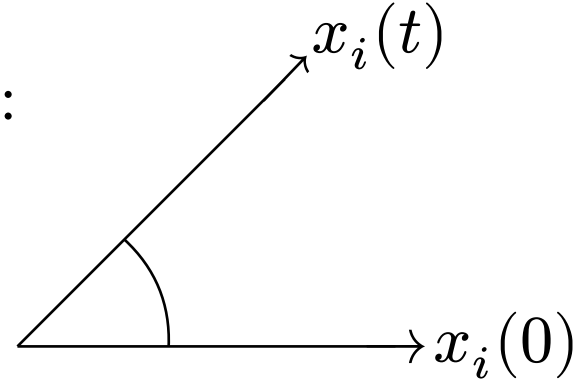

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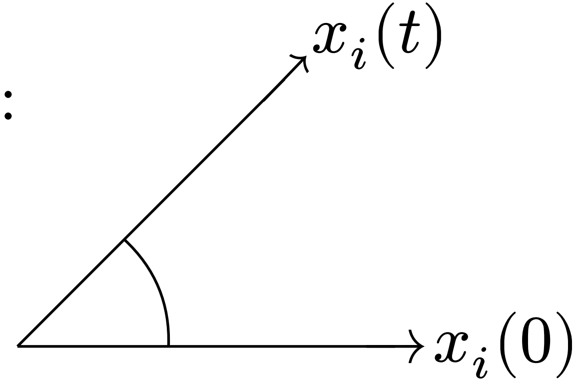

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$$(\mathcal{L}F_{j,i})(s) = \pm \frac{1}{\tilde{U}(f, x^*(t))_{i,j} - \delta_{ij} \cdot \lambda_i^2}.$$

[6] With $x^*(t) = \left(i \mapsto x_{i(t)}\right)$ and $F_{j,i}(t) := \langle x_j(t), x_i(0)\rangle$.

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Ev. step	**Meaning**
R	Random variable, abstract
$R(\omega)$	Vector of time dep. pos.
$R(\omega)_i$	i-th particle position, time dep. path
$R(\omega)_i(t)$	Position of i-th particle at time t

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This is already a good starting point to understand our *Correlated Disorder* modification!

.. missing key elements:

[7]Expansion to a functional can be argued, see thesis p. 19.

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- The *moment generating function* $Z_{z,R_\omega}[J]$. It requires the *force field* J.

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Definition 2.22. External Field Shift.

For $R : \Omega \to V_{d,N}$ and $\Phi \in \mathbb{F}_{d,N}$ we define

$$J \mapsto -\frac{1}{2} \cdot S^{(0)}_{z,R_\omega}(\Phi) + \int_{\mathbb{R}^d} J(x) \cdot \Phi(-x) + J(-x) \cdot \Phi(x)\,\lambda(dx)$$

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$$\hookrightarrow \quad \frac{\delta}{\delta J(x)} S^{(0)}_{z,R_\omega}[\Phi] = \mathring{i} \cdot \Phi(-x).$$

The Generative Operator

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$$Z_{z,R_\omega}[J] = \int_{\mathbb{F}_{d,N}} e^{\left(S^{(0)}_{z,R_\omega}\Phi + S^{(int)}_{z,R_\omega}\Phi\right)[J]} d\Phi = \underbrace{\left[\mathrm{Ex}_{\mathcal{L}_f}\left[\int_{\mathbb{F}_{d,N}} e^{\left(S^{(0)}_{z,R_\omega}\Phi\right)[\cdot]} d\Phi\right]\right]}_{\text{Generative Part}}[J].$$

$\rightarrow$ Looking at different Taylor expansion terms yields different integrals.

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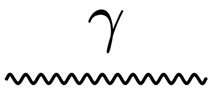

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.. possible connections of these edges are given by *vertices*:

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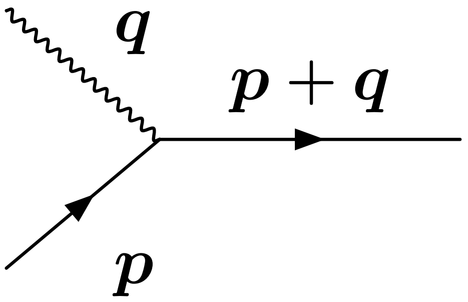
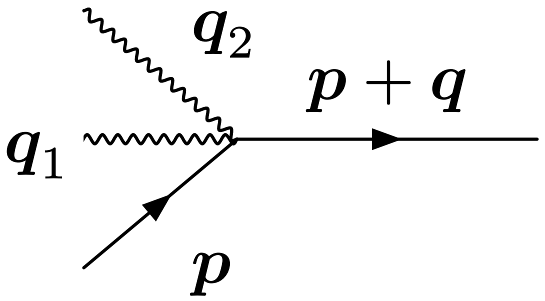


$:= \mu_z(\boldsymbol{p}, \boldsymbol{q}) \quad \Rightarrow$ Three-point Vertex



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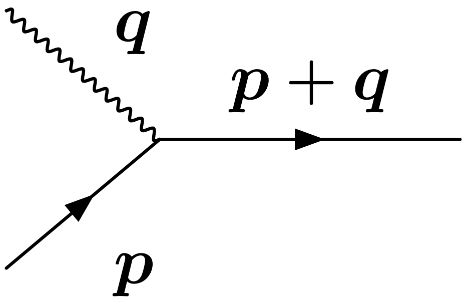

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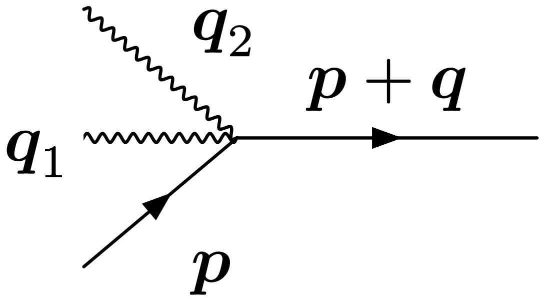

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.. which completes the set of Feynman rules.

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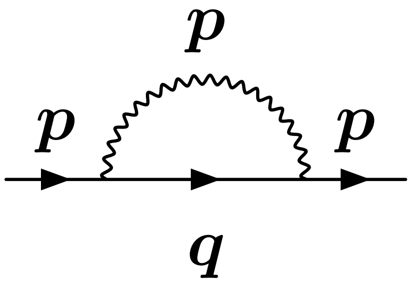
We begin with **one** loop diagrams:



$$= \frac{G_0(\boldsymbol{p}, z)^2}{\rho_*} \cdot \int_{\mathbb{R}^d} G_0(\boldsymbol{q} - \boldsymbol{p}, z) \cdot \mu_z(\boldsymbol{p}, -\boldsymbol{q})^2 \, d\boldsymbol{q}$$

What is Correlated Disorder?

What did we implement?