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SMAI ASSIGNMENT 2

Q1) A) Eigen value decomposition can only be performed on diagonizable matrices. a diagonizable matrix is diagonizable iff, there exists another square invertible matrix P such that

$D = P \times A \times P^{-1}$ where A is diagonizable and D is Diagonal matrix.

Singular value ~~matrix~~ decomposition can be performed on any $m \times n$ matrix.

Hence Singular value decomposition is more Generalizable.

B) Find singular value decomposition of -

$$A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}, \quad \text{Let } W = A \times A^T$$

$$\Rightarrow W = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \times \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

to find eigen values, solve $|W - \lambda I| = 0$

$$\left| \begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow -\lambda^3 + 450\lambda^2 - 32400\lambda = 0$$

$$\Rightarrow \lambda(\lambda-90)(\lambda-360) = 0$$

$$\Rightarrow \lambda = 0, 90, 360$$

$\therefore$ Eigenvalues $= 0, 90, 360$

normalized eigenvectors —

$$\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}-2\\-1\\2\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$$v_1 = \frac{1}{\sqrt{\sigma_1}} A^T u_1 = \frac{1}{6\sqrt{10}} \times \begin{bmatrix}4 & 11 & 14\\8 & 7 & -2\end{bmatrix} \times \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{\sigma_2}} A^T u_2 = \frac{1}{3\sqrt{10}} \times \begin{bmatrix}4 & 11 & 14\\8 & 7 & -2\end{bmatrix} \times \begin{bmatrix}-2/3\\-1/3\\-2/3\end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix}1\\-3\end{bmatrix}$$

$$U = [u_1 \; u_2 \; u_3] = \frac{1}{3}\begin{bmatrix}1 & -2 & 2\\2 & -1 & -2\\2 & 2 & 1\end{bmatrix}$$

$$\Sigma = \begin{bmatrix}\sqrt{\sigma_1} & 0\\0 & \sqrt{\sigma_2}\\0 & 0\end{bmatrix} = 3\sqrt{10}\begin{bmatrix}2 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

$$V = [v_1 \; v_2] = \frac{1}{\sqrt{10}}\begin{bmatrix}3 & 1\\1 & -3\end{bmatrix}$$

Hence Singular value decomposition —

$$U\Sigma V^T = \frac{1}{3}\begin{bmatrix}1 & -2 & 2\\2 & -1 & -2\\2 & 2 & 1\end{bmatrix} \times \begin{bmatrix}6\sqrt{10} & 0\\0 & 3\sqrt{10}\\0 & 0\end{bmatrix} \times \frac{1}{\sqrt{10}}\begin{bmatrix}3 & 1\\1 & -3\end{bmatrix}$$

Q2) LDA and PCA.

A) (a) Since PCA selects the basis based on the rank of eigen values of covariance matrix ∴ PCA wont be useful if the eigen values of covariance matrix are the same, that is when all elements of D are same.
(a) is wrong.

(b) as values of elements of D are not equal, eigen values of covariance matrix will be unique and PCA will be usefull as explained above.

(c) D can be full rank if even if all points of X lie in a straight line. example :-
Consider line $y = 2x + 1$ and

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 7 \\ -1 & -1 \end{bmatrix}$$

(c) is wrong

(d) V is not full rank if X lies on line, ∴ in SVD $X = UDV^T$
V is unitary matrix hence always Full rank, ∴ (d) is wrong.

(e) D can be full rank if all points of X lie on circle.
$x^2 + y^2 = 1$

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

(e) is wrong.

Answer is (b)

B) True / False.

PCA will project data points on a line which preserves information useful for data classification.

$\boxed{\text{FALSE}}$; the classic LDA example, PCA tries to preserve as much information as possible but does not care about the nature of info it discards.

Q3) A) Prior Probability – probability of an event occuring before any new evidence is introduced.

Posterior Probability – probability of an event occuring after new evidence is introduced

B) Let F (flu) be event of person being infected by flu.
Let S (symptoms) be event of person having headache and sore throat.
We have $P(S) = 0.2$ $P(F) = 0.05$

$P(S|F) = 0.9$.

By Bayes –

$$P(F|S) = \frac{P(S|F) \times P(F)}{P(S)}$$

$$= \frac{0.9 \times 0.05}{0.2} = 0.225$$

Probability of having flu given symptoms is 22.5%