Zilber's Principle in Practice

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August 2019

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1 Introduction

1.1 Towards appreciating contemporary model theory

To the beginner, model theory seems like an attempt to *study mathematics*, as opposed to actually *doing mathematics*. Concrete theories and structures can be eschewed, and thus one can ask more abstract questions that seem to describe the features of mathematics itself. As an example, consider the Löwenheim-Skolem theorems, which loosely state that first-order theories have little control over the size of their infinite structures. Philosophically, this is a result of great interest - especially when one considers the mindbending Skolem's paradox of set theory:

If ZFC is consistent, then it has a model. In particular, by the Löwenheim-Skolem theorems, it has a countable model. But in ZFC there exists an uncountable set.

These types of problems can lead to some fruitful discussions about the nature of mathematics, but these problems and their solutions can hardly be considered actual mathematics. But something like the Löwenheim-Skolem theorem does inspire some truly mathematical questions. Indeed, an important question that mathematicians working in a particular theory ask themselves is: how can one classify the structures of this theory? Löwenheim-Skolem tells us that theories with infinite structures have structures of all infinite sizes. Classification theory tells us how many non-isomorphic structures of a given cardinal exist in a theory, based on some hypotheses of the theory. The paradigm shift that led to real work of this sort being done in the field is due to Morley, who demonstrated that countable theories that are categorical in some uncountable cardinal i.e. have only one model of that size up to isomorphism, are in fact categorical in every uncountable cardinal [11]. Morley's methods were innovative as they transcended the set-theoretic analysis that had dominated model theory in the past, instead focusing on more geometric concepts such as the space of types of a theory and the rank of a type. Further innovations in the field were lead by Shelah, who introduced the notions of forking and stability (combinatorial concepts with geometric interpretations) to (almost) classify the number of non-isomorphic models of a given cardinal, (more details in [1]). While results such as these avoid the speculation of Skolem's paradox above, it is not a stretch to say that this content is purely model-theoretic.

That withstanding, these new geometric techniques have proved to be powerful tools outside of the realm of pure model theory, as the model-theoretic proof of the funciton field Mordell-Lang conjecture by Hrushovski, and many other applications of these to number theory, algebra, and geometry demonstrate. Knowing this history, it becomes clear that model theory is indeed a way of doing mathematics. This paper is an attempt to explain one of the ways one 'does mathematics' using the abstract algebraic geometry of modern pure model theory. In particular, we will study Zilber's conjecture for its abstract model-theoretic content, as well as a principle that can be applied when studying common mathematical structures.

1.2 How this is going to go

Modern model theory can roughly be divided into two streams: pure model theory and applied model theory. Pure model theory studies theories with certain tameness conditions, using (often) abstracted algebro-geometric techniques (this is especially the case in stable theories, which is what we will encounter here). Applied model theory instead takes the results of pure model theory, and examines structures whose theories have these same tameness conditions to draw conclusions with concrete mathematical consequences from this model theoretic analysis. Like the discipline, this paper will be divided into two parts. Structurally, we hope that this emphasizes the manner in which the two streams interact with one another. Most importantly, we hope it demonstrates how it is that model theory is a way of doing mathematics.

The first part will feature the pure model theoretic content of the paper. In it, we will take the formal statement of Zilber's conjecture, strip back all of its components, and build up our theory in a focused and expository fashion, using concrete examples of the theory in simple cases such as infinite vector spaces and algebraically closed fields. The goal of this section is to present just enough content for a graduate of an introductory model theory course to understand the statement and implications of Zilber's conjecture. We will end the section with a brief discussion of the conjecture as a principle for applied model theory, as well as a discussion of its importance.

The second part will feature the applied content of the paper. In it, we study two theories: differentially closed fields of characteristic zero (DCF_0) and compact complex analytic spaces (CCA). For each, we briefly recount their basic theory and model -theoretic properties. This will be followed by a discussion of the Zilber dichotomy, as it is realized in these theories.

1.3 Prerequisites and conventions

We assume throughout that the reader has taken a graduate level course in model theory, so that they are familiar with compactness, quantifier elimination, elimination of imaginaries, saturation and homogeneity, Morley rank, and some stability theory.

Throughout, we let T be a complete first-order theory with infinite models in a language L. We let $\mathbb{M}$ denote the *monster model* of the theory T, whose construction using Gödel-Bernay's set theory can be found in [18]. We note the following properties of $\mathbb{M}$:

1. If $M \models T$ then M is an elementarily embeddable in $\mathbb{M}$, denoted $M \preceq \mathbb{M}$.
2. If $M \models T$ and $p(x)$ is a type defined over $A \subseteq M$, then $p(x)$ is realized in $\mathbb{M}$, i.e. $\mathbb{M}$ is κ-saturated for all cardinals κ.
3. If $f : A \to \mathbb{M}$ is a partial elementary map for some set of parameters $A \subseteq \mathbb{M}$, then f can be extended to an automorphism of $\mathbb{M}$, i.e. $\mathbb{M}$ is strongly κ-homogeneous for all cardinals κ.

Unless stated otherwise, we will use:

- $M, N, \ldots$ to refer to models of T,
- $A, B, \ldots$ to refer to sets of parameters contained in $\mathbb{M}$,
- $a, b, \ldots$ to refer to finite tuples of parameters (from $A, B, \ldots$ respectively),
- $\phi, \psi, \ldots$ to refer to $L_A, L_B, \ldots$-formulas (whenever $A, B, \ldots$ are specified),
- $p, q, \ldots$ to refer to complete types over $A, B, \ldots$ (whenever $A, B, \ldots$ are specified),
- $\phi(\mathbb{M})$, and $p(\mathbb{M})$ to refer to the class of realizations in $\mathbb{M}$ of a formula or type,
- $\mathrm{Aut}(\mathbb{M}/A)$ to refer to the group of automorphisms of $\mathbb{M}$ which also pointwise fix the set A,
- $\mathrm{acl}(A)$ to refer to the set of all tuples a such that $a \in \phi(\mathbb{M})$ and $|\phi(\mathbb{M})| < \omega$ for some L_A-formula ϕ,
- $\mathrm{dcl}(A)$ to refer to the set of all tuples a such that $a \in \phi(\mathbb{M})$ and $|\phi(\mathbb{M})| = 1$ for some L_A-formula ϕ.

We remark here that $a \in \mathrm{acl}(A)$ if and only if the orbit of a under $\mathrm{Aut}(\mathbb{M}/A)$ is finite, and further $a \in \mathrm{dcl}(A)$ if and only if a is fixed under $\mathrm{Aut}(\mathbb{M}/A)$. By the strong homogeneity of $\mathbb{M}$ this also implies that $p(x) = \mathrm{tp}(a/A)$ has only finitely many realizations, or respectively a is the unique realization of $p(x)$ in $\mathbb{M}$ when $a \in \mathrm{dcl}(A)$.

We also recall that if $X = \phi(\mathbb{M})$ is a definable set, then X is a A-definable (i.e. there is some L_A-formula ψ with $\psi(\mathbb{M}) = X$) if and only if $\alpha(X) = X$ for all $\alpha \in \mathrm{Aut}(\mathbb{M}/A)$.

We will also use the convention of avoiding the union symbol whenever taking unions of parameter sets and tuples. To elaborate, if A and B are parameters sets, $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ finite tuples, then $AB := A \cup B$, $Aa := A \cup \{a_1, \ldots, a_n\}$, and $ab := \{a_1, \ldots, a_n, b_1 \ldots, b_m\}$.

2 Zilber's Conjecture

In 1984, after studying uncountably categorical structures, Zilber conjectured that the geometries induced by these theories fall into a strict trichotomy. The modern statement of the conjecture (written as a dichotomy) is as follows

Zilber's Conjecture. Let X be a strongly minimal definable set in $\mathbb{M}$. If (X, acl) is not locally modular, then an infinite field F is interpretable in $\mathbb{M}$ on X.

The purpose of this section is to strip the conjecture back to its barest, then build ourselves back up to the conjecture's statement. To do these, we will discuss the notions of forking and U-rank, strongly minimal sets and modularity, and interpretability and orthogonality. Following this exposition, we will discuss the conjecture as a principle of model theory.

2.1 Forking and U-rank

While this section is not strictly necessary for one's understanding of Zilber's conjecture, it does allow for a deeper understanding of certain concepts that will follow in other sections, such as acl-dimension and orthogonality. It also provides a wider context for which we can understand Zilber's conjecture to exist within.

Throughout we suppose that T is a stable theory, i.e. $|S(A)| \leq \kappa$ whenever $|A| \leq \kappa$ for some fixed cardinal κ. Recall the definition of dividing:

Definition 2.1. Let $\phi(x, b)$ be an L_A-formula. We say that $\phi(x, b)$ *divides over* A if there is a sequence $(b_i)_{i<\omega}$ with $\mathrm{tp}(b/A) = \mathrm{tp}(b_i/A)$, and $\phi(x, b_i)$ is k-inconsistent for some $k < \omega$. A type $p(x)$ is said to divide over A if $p(x)$ implies an A-dividing formula $\phi(x, b)$.

The definition is not easy to grasp at first look, however there is a way to reinterpret it as a geometric statement about $\mathbb{M}$. Let $X := \phi(\mathbb{M}, b)$. As $\mathrm{tp}(b/A) = \mathrm{tp}(b_i/A)$ for all $i < \omega$, it follows that there are automorphisms $\alpha_i \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\alpha_i(b) = b_i$. Then taking $X_i = \phi(x, b_i)$, we also see that $\alpha_i(X) = X_i$. Then in some sense, the Ab-definable set X divides over A if there are symmetries of $\mathbb{M}$ which fix A, but translate X in such a way that every large finite collection of these translates has an empty intersection.

Forking is a similar, but looser notion. A type $p(x)$ forks over A if and only if the set $X = p(\mathbb{M})$ can be covered by finitely definable many sets X_i, (i.e. $X \subseteq \bigcup_{i=1}^{n} X_i$), all of which divide over A. It is clear that if a type divides over a parameter set, it must also fork over it. The converse is not true in general, but in a stable theory these two notions coincide, see [18]. We adopt the following definition for convenience of the discussion of forking:

Definition 2.2. We write

$$A \underset{C}{\perp} B$$

to denote that for all finite tuples a from A, $\mathrm{tp}(a/BC)$ does not fork over C. In such a situation, we say that A *is independent of* B *over* C. We write $A \not\perp_C B$ when this is not the case.

The language used in the above definition is rather evocative - it suggests that forking captures some abstract notion of dependence. Referring back to the geometric picture of forking, we can see that this is indeed the case: the type $\mathrm{tp}(a/BC)$ collects all the information that BC can say about the tuple $a \in A$. If $\mathrm{tp}(a/BC)$ forks over C, then there is some tuple $b \in B$ and some L_C-formula $\phi(x, y)$ such that $\models \phi(a, b)$ and $\phi(x, b)$ forks over C. In terms of the geometry, this says that b can define a subclass fine enough that a can be separated from other members of $\mathrm{tp}(a/C)$ by taking an infinite set of translates of the class $\phi(\mathbb{M}, b)$ under the symmetries which fix C. When this is not the case, it says that there is no such tuple that can separate the members of $\mathrm{tp}(a/C)$, so BC tells us no more information about A than what C could. To see this notion of forking as dependence in action, let us examine forking in vectors spaces over infinite fields, where model-theoretic independence coincides exactly with linear algebraic independence.

Proposition 2.3. Let $(V, 0, +, (\lambda_k)_{k\in K})$ be a vector space over an infinite field K. The following are equivalent in V:

1. $A \perp_C B$,
2. $\dim(A_0/BC) = \dim(A_0/C)$ for all finite subsets $A_0 \subseteq A$,
3. $\mathrm{span}(AC) \cap \mathrm{span}(BC) = \mathrm{span}(C)$.

Proof. We first characterize the consistent formulas $\phi(x, b)$ that divide over C. By quantifier elimination and basic manipulation of linear equations, any L_C-formula $\phi(x, b)$ will be equivalent to a conjunction of literals of the form

$$\left(\bigwedge_{j=1}^{m_1} \sum_{i=1}^{n} \alpha_{j_i} x_i = b_j\right) \wedge \left(\bigwedge_{\ell=1}^{m_2} \sum_{i=1}^{n} \beta_{\ell_i} x_i \neq b'_\ell\right),$$

where the $b_j, b'_\ell \in \mathrm{span}(Cb)$. We will show that such formulas divide over C if and only if some $b_j \notin \mathrm{span}(C)$. Observe that the set P of realizations of $\left(\bigwedge_{j=1}^{m_1} \sum_{i=1}^{n} \alpha_{j_i} x_i = b_j\right)$ defines an intersection of affine subspaces of V^n, so it is itself an affine subspace of V^n. If all $b_j \in \mathrm{span}(C)$, then P is fixed by automorphisms of V^n that fix C pointwise. It is a fact that an affine space over an infinite field cannot be covered by finitely many proper affine subspaces. Towards a contradiction, suppose that $\phi(x, b)$ divides over C, we show that this would imply P can be covered by finitely many proper affine subspaces.

Indeed, the set H of realizations of $\bigwedge_{\ell=1}^{m_2} \sum_{i=1}^n \beta_{\ell_i} x_i \neq b'_\ell$ is equivalent the compliment of a union of affine subspaces of V^n, i.e. $H = V^n \setminus (R_1 \cup \cdots \cup R_{m_2})$ where the R_i are affine subspaces of V^n. Furthermore, if $H^{(1)}, \ldots, H^{(k)}$ are C-conjugates of H, then

$$H^{(1)} \cap \cdots \cap H^{(k)} = V^n \setminus (R_1^{(1)} \cup \cdots R_{m_2}^{(1)} \cup \cdots \cup R_1^{(k)} \cup \cdots \cup R_{m_2}^{(k)}).$$

Thus, if $P \cap (H^{(1)} \cap \cdots \cap H^{(k)}) = \emptyset$, then $P \subseteq R_1^{(1)} \cup \cdots R_{m_2}^{(1)} \cup \cdots \cup R_1^{(k)} \cup \cdots \cup R_{m_2}^{(k)}$. From the fact above, it follows that for some (i, j), $P \subseteq R_i^{(j)}$, and thus $P \cap H^{(i)} = \emptyset$. But this is a C-conjugate of $P \cap H \neq \emptyset$, so this is impossible.

On the other hand, if for some b_j as above, $b_j \in \operatorname{span}(Cb) \setminus \operatorname{span}(C)$, then clearly the affine subspace P' defined by $\sum_{i=1}^n \alpha_{j_i} x_i = b_j$ divides over C. Indeed, let $(\gamma_i)_{i<\omega}$ be a sequence of distinct non-zero members of K, and $(\gamma_i b_j)_{i<\omega}$ is a sequence of C-conjugates of b_j. It is easy to see that the equations

$$\sum_{i=1}^n \alpha_{j_i} x_i = \gamma_k b_j, \ \ \sum_{i=1}^n \alpha_{j_i} x_i = \gamma_\ell b_j$$

cannot have a simultaneous solution whenever $k \neq \ell$. Therefore, P' divides over C, and as $P \cap H \subseteq P'$, it follows $P \cap H$ divides as well.

Now, suppose that $A \not\downarrow_C B$, so there is some L_C-formula $\phi(x, y)$, $a \in A$ and $b \in B$ such that $\models \phi(a, b)$ and $\phi(x, b) \to \bigvee_{j=1}^m \phi_j(x, b_j)$, where each $\phi_i(x, b_i)$ divides over C. By our work above, we may assume that each $\phi_i(x, b_i)$ is a formula of the form

$$\sum_{i=1} \alpha_{j_i} x_i = b_j,$$

where $b_j \notin \operatorname{span}(C)$. Let $P \cap H$ be the set of realizations of $\phi(x, b)$ as above, and let P'_i be the affine subspaces defined by $\phi_i(x, b_i)$. Then we see that $P \subseteq \bigcup_{i=1}^m P'_i \cup H^c$, and H^c is a finite union of affine subspaces. Thus, P must be contained in one of these affine spaces, and moreover it cannot be contained in H^c for otherwise $P \cap H = \emptyset$. Thus, $P \cap H \subseteq P \subseteq P'_i$ for some i, so $\phi(x, b)$ divides over C. By our characterization above, this implies that $\phi(x, b)$ includes a literal

$$\sum_{i=1}^n \alpha_{j_i} x_i = b_j$$

where $b_j \in \operatorname{span}(BC) \setminus \operatorname{span}(C)$. Then $\operatorname{span}(AC) \cap \operatorname{span}(BC) \neq \operatorname{span}(C)$, as some $a \in A$ satisfies this equation.

If $\operatorname{span}(AC) \cap \operatorname{span}(BC) \neq \operatorname{span}(C)$, then there is some non-trivial equation

$$\sum_{i=1}^n \alpha_i a_i + \beta_i b_i + \gamma_i c_i = 0$$

where $\alpha_i, \beta_i, \gamma_i \in F$, $a_i \in A$, $b_i \in B$, $c_i \in C$, $\alpha_1 \neq 0 \neq \beta_1$, and where we may suppose that the a_i are linearly independent over C. But the above shows that there is a dependence relationship between these a_i over BC, hence $\dim(a_1 \cdots a_n/BC) < \dim(a_1 \cdots a_n/C)$.

Lastly, if $\dim(a_1 \cdots a_n/BC) < \dim(a_1 \cdots a_n/C) = n$ for some $a_i \in A$, so for some $k < n$ we have

$$a_{k+1} = \sum_{i=1}^{k} \alpha_i a_i + \sum_{j=1}^{m} \beta_i b_i + \gamma_i c_i$$

where $\alpha_i, \beta_i, \gamma_i \in F$, $b_i \in B$ and $c_i \in C$, where we may suppose that the b_i are linearly independent over C as well. Note that some $\beta_i \neq 0$. We show that

$$x_{k+1} = \sum_{i=1}^{k} \alpha_i x_i + \sum_{j=1}^{m} \beta_i b_i + \gamma_i c_i$$

divides over C. The type $\mathrm{tp}(b_1 \cdots b_m/C)$ is realized by all $v_1 \cdots v_m$ such that

$$\mathrm{span}(v_1, \ldots, v_m) \cap \mathrm{span}(C) = \{0\}\,,$$

i.e. it is the unique type that defines the linearly independent sets of size m with respect to C. Now consider the infinite sequence of $b_1 \cdots b_m$ conjugates given by $((\delta_i b_1)(\delta_i b_2) \cdots (\delta_i b_n))_{i<\omega}$, where the δ_i are distinct non-zero scalars in K. It is easy to see that if $i_0 \neq j_0$, then

$$\sum_{i=1}^{k} \alpha_i x_i + \sum_{j=1}^{m} \beta_i(\delta_{i_0} b_i) + \gamma_i c_i = \sum_{i=1}^{k} \alpha_i x_i + \sum_{j=1}^{m} \beta_i(\delta_{j_0} b_i) + \gamma_i c_i$$

is inconsistent, as the b_i are linearly independent and some $\beta_i \neq 0$. □

This phenomena is not unique to vector spaces over infinite fields. One can verify, for example, that in the case of algebraically closed fields, model theoretic independence coincides with the usual field-theoretic algebraic independence.

We also prove here a lemma that will be used in the next section:

Lemma 2.4. $A \mathop{\smile\!\!\!\!\!\!\mid}_C B \iff A \mathop{\smile\!\!\!\!\!\!\mid}_C \mathrm{acl}(BC)$

Proof. If $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$, then then there is some finite tuple a from A and a formula in $\mathrm{tp}(a/BC)$ that forks over C. As $B \subseteq \mathrm{acl}(BC)$, this formula will also belong to $\mathrm{tp}(a/\mathrm{acl}(BC))$, so $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C \mathrm{acl}(BC)$.

Conversely, suppose that $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C \mathrm{acl}(BC)$. Let a and b' be a finite tuples in A and $\mathrm{acl}(BC)$ respectively, and $\phi(x, y')$ an L-formula such that $\phi(x, b') \in \mathrm{tp}(a/\mathrm{acl}(BC))$, and it forks over C. As $b' \in \mathrm{acl}(BC)$, it follows that there is some $b \in BC$ and an L-formula $\psi(x, y)$ such that $\phi(y', b)$ defines precisely the BC-conjugates $b_1, \ldots, b_k$ of b'. Note that $b_1, \ldots, b_k$ are

C-conjugates of b'. Then the formula $\theta(x,b) := \exists y'(\phi(x,y') \wedge \psi(y',b))$ is equivalent to the conjunction $\bigvee_{i=1}^{k} \phi(x,b_i)$, and $\models \theta(a,b)$. It is clear that C-conjugation preserves forking, so $X = \theta(\mathbb{M}, b)$ is covered by sets that fork over C, so further X is covered by sets that divide over C. Thus X forks over C, so $a \not\perp\!\!\!\perp_C B$. □

We may use forking to define the following foundation rank on the types of a theory:

Definition 2.5. Let $p(x)$ be a complete type over A. The U-rank of $p(x)$, denoted $U(p)$, is defined recursively on ordinals as follows:

- $U(p) \geq 0$.
- $U(p) \geq \alpha + 1$ if $p(x)$ has a forking extension $q(x)$ such that $U(q) = \alpha$.
- $U(p) \geq \gamma$ if $U(p) \geq \alpha$ for all $\alpha < \gamma$ whenever γ is a limit ordinal.

If $U(p) \geq \alpha$ but $U(p) \not\geq \alpha + 1$, we say that $U(p) = \alpha$. If $U(p) \geq \alpha$ for all $\alpha \in \mathrm{Ord}$, we say that the U-rank of p is undefined, and write $U(p) = \infty$.

One can see that $U(p) = 0$ if and only if $p(x)$ is an algebraic type. Indeed, if $p(x) = \mathrm{tp}(a/A)$ then $a \in \mathrm{acl}(A)$, then there is an L_A-formula $\phi(x)$ that defines the finite set of realizations of $p(x)$. Now suppose that b is a tuple such that $\psi(x,b) \wedge \phi(x)$ is consistent. By the pigeonhole principle, $\psi(x,b)$ cannot divide over A, as every A-conjugate $\psi(\mathbb{M}, b)$ will intersect the finite set $\phi(\mathbb{M})$. So $p(x)$ does not have any forking extensions. Conversely, suppose that $p(x)$ is not algebraic, i.e. $p(x) = \mathrm{tp}(a/A)$ and $a \notin \mathrm{acl}(A)$. Then we see that $\mathrm{tp}(a/Aa)$ forks over A. This follows from the fact that $x = a \in \mathrm{tp}(a/Aa)$ divides over A, since there exists an infinite sequence $(a_i)_{i<\omega}$ of conjugates of a over A.

So $U(p) = 0$ tells us that $p(x)$ defines a finite set. In this spirit, one should view the U-rank as a rough definition of dimension for a type, where higher dimensional types have longer descending chains of types which introduce dependencies at each intermediary step.

For the reader familiar with Morley rank, let us include here the following proposition:

Proposition 2.6. For all types p, $U(p) \leq RM(p)$.

Proof. To see this, we first note that if q is a forking extension of $p \in S_n(A)$, and $RM(p)$ is ordinal valued, then $RM(q) < RM(p)$. As a special case, let us first show that if $\phi(x)$ is a Morley A-formula, $\psi(x,b) \to \phi(x)$, and $\psi(x,b)$ divides over A, then $RM(\psi) < RM(\phi)$. It is easy to see that Morley rank is preserved by automorphisms of $\mathbb{M}$, hence $RM(\psi(x,b_i)) = RM(\psi(x,b))$ for all A-conjugates of b and also $\psi(x,b_i) \to \phi(x)$. Let q_i be $A(b_i)_{i<\omega}$ types such that $\psi(x,b_i) \in q_i$, and $RM(q_i) = RM(\psi(x,b_i))$. Since the formulas $\psi(x,b_i)$ are k-inconsistent for some k, it follows that there are infinitely many such q_i, and these all contain $\phi(x)$. But there can only be a finite number of types over a model containing ϕ which also share its Morley rank. Thus, $RM(q_i) < RM(\phi)$ for all q_i, hence $RM(\psi) < RM(\phi)$.

Now, suppose that $\psi(x, b) \in q$ is such that $\psi(x, b)$ forks over A. Note that we may assume $RM(\psi) = RM(q)$, and moreover may take some $\phi \in p$ such that $RM(\phi) = RM(p)$ and $\psi(x, b) \to \phi(x)$. Then $\psi(x, b)$ is covered by some $\{\psi_i(x, b_i) : 1 \leq i \leq n\}$, all of which divide over A. Moreover, we may assume that all the $\psi_i(x, b_i) \to \phi(x)$. As $RM(\psi) = \max RM(\psi_i) < RM(\phi)$, it follows that $RM(q) < RM(p)$.

Using this fact, we show inductively on α that $U(p) \geq \alpha$ implies $RM(p) \geq \alpha$. We may suppose that $RM(p)$ is ordinal valued, otherwise this implication is vacuous. That this holds when $\alpha = 0$ follows from our observation about algebraic types above, and limit ordinals follow immediately from the definition and the inductive hypothesis. For successors, we see that if $p(x)$ has a forking extension $q(x)$ with $U(q) \geq \alpha$, then $RM(q) \geq \alpha$ by induction. But if $q(x)$ is a forking extension of $p(x)$, and $p(x)$ has finite Morley rank, we know that $RM(q) < RM(p)$. Hence $RM(p) \geq \alpha + 1$. □

This inequality shows that U-rank can be used effectively to study a broader class of theories than Morley rank. In fact, U-rank ordinal valued in a theory precisely when it is superstable, a weakening of ω-stability, which characterizes when Morley rank is ordinal valued.

2.2 Strongly minimal sets and modularity

Definition 2.7. An non-algebraic L_A-formula $\phi(x)$ is said to be *strongly minimal* if for all L_B-formulas $\psi(x)$, $(\phi \wedge \psi)(\mathbb{M})$ is either finite or cofinite in $\phi(\mathbb{M})$. A non-algebraic type $p(x)$ is said to be strongly minimal if it contains a strongly minimal formula. A theory is said to be strongly minimal if $x = x$ is strongly minimal.

We remark here that the theory of infinite sets, $\mathbb{Q}$-vector spaces, and algebraically closed fields are all strongly minimal theories. Indeed, it is not hard to see that their atomic formulas in a single variable either define a finite set or the whole universe, hence their boolean combinations are finite or cofinite. By quantifier elimination in each of these theories, these are all the definable sets of one variable.

It is easy to see that a formula $\phi(x)$ is strongly minimal if and only if $RM(\phi) = 1$ and $dM(\phi) = 1$. Indeed, if $\phi(x)$ is strongly minimal then $X = \phi(\mathbb{M})$ is infinite, and moreover if $X_1, X_2 \subseteq X$ are infinite definable sets, $X_1 \cap X_2 \neq \emptyset$ as their complements are cofinite. This proves that $RM(\phi) = dM(\phi) = 1$. Conversely, if $RM(\phi) = DM(\phi) = 1$ then $X = \phi(\mathbb{M})$ is infinite, and if $Y \subseteq X$ is an infinite definable set then $RM(Y) = 1$ and $X \setminus Y$ is definable and disjoint from Y. Thus $RM(X \setminus Y) < 1$, so Y is cofinite. Thus, strongly minimal sets are in some sense the simplest non-trivial sets in an ω-stable theory.

A strongly minimal type is stationary (i.e. it has a unique non-forking extension to every parameter set) and has U-rank 1. Indeed, if p is strongly minimal then it contains a strongly minimal formula $\phi(x)$, and $RM(\phi) = 1$, $dM(\phi) = 1$. As p is non-algebraic $0 < U(p) \leq RM(p) = 1$, so $U(p) = 1$. Moreover, if $q_1(x)$ and $q_2(x)$ are non-forking extension of $p(x)$, then $\psi(x) \in q_1$ implies that $\psi \wedge \phi(x) \in q_1$, and this is non-algebraic, hence $(\neg\psi) \wedge \phi(x)$ is algebraic, and thus $(\neg\psi) \wedge \phi(x) \notin q_2$, so $\psi(x) \in q_2$, and $q_1 = q_2$, so p is stationary. The converse is not true - there are stationary U-rank 1 types that are not strongly minimal - so we say that a stationary type of U-rank 1 is *minimal*.

Definition 2.8. A *pregeometry* is a tuple (X, cl) where X is a set and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the axioms:

1. (Closure) $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
2. (Monotonicity) if $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
3. (Finiteness) $a \in \mathrm{cl}(A)$ if and only if there is a finite set $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$,
4. (Exchange) $b \in \mathrm{cl}(Aa) \setminus \mathrm{cl}(A)$ if and only if $a \in \mathrm{cl}(Ab)$.

One can view a pregeometry as an abstraction of the linear span operation on subsets of a vector space. The reader will do well to keep this in mind whenever working with a pregeometry, as many related concepts have familiar analogs in the usual setting of linear algebra.

Given a strongly minimal L_A-formula $\phi(x)$, the set $X = \phi(\mathbb{M})$ and the closure operator $\mathrm{cl}(B) := \mathrm{acl}(AB) \cap X$ define a pregeometry [18]. This is also true when X is replaced by the set of realizations of a minimal type. The importance of this naturally occurring pregeometry in strongly minimal sets is that it allows us to study the definable sets of the theory using a dimension that behaves very much like that of a vector space or algebraically closed field.

Definition 2.9. Let (X, cl),be a pregeometry, $Y \subseteq X$, and $A \subseteq X$. Y is cl-*independent over* A if for all $y \in Y$, $y \notin \mathrm{cl}(A \cup Y \setminus \{y\})$. A sequence $(b_i : i \in I)$ is said to be cl-independent over A if for all $i \in I$, $b_i \notin \mathrm{cl}(A \cup \{b_j : j \neq i\})$. We say that $B \subseteq Y$ is a *basis of* Y *over* A if B is cl-independent over A and $Y \subseteq \mathrm{cl}(A \cup B)$. If B is a basis of Y over A then the cl-*dimension of* Y *over* A, denoted cl-dim(Y/A), is given by the cardinality of B. When $A = \emptyset$, we simply write cl-dim(Y).

By Zorn's lemma, bases always exist. By the exchange property all bases have the same cardinality, hence cl-dim is well-defined. It is easy to see that in the case of vector spaces with linear span as the closure operation, cl-dim$(Y) = \dim(\mathrm{span}(Y))$. Indeed, the only cl-dependence relations in this theory are given by linear equations, which are exactly the dependence relations used to define the usual dimension.

In a strongly minimal theory T, for any set of parameters A there is a unique n-type p -called the generic type of T over A - where $(a_1, \dots, a_n) \models p$ if and only if $(a_1, \dots, a_n)$ is acl-independent over A. Indeed, suppose that $a, b \in \mathbb{M}$ such that $a, b \notin \operatorname{acl}(A)$. For any L_A-formula $\phi(x)$, it follows that $\models \phi(a) \iff |\neg\phi(\mathbb{M})| < \omega \iff \models \phi(b)$, thus $\operatorname{tp}(a/A) = \operatorname{tp}(b/A)$. Now suppose that this is true for n, and consider $\{a_1, \dots, a_{n+1}\}, \{b_1, \dots, b_{n+1}\}$ acl-independent over A. Then $\operatorname{tp}(a_1 \cdots a_n/A) = \operatorname{tp}(b_1 \cdots b_n/A)$, hence there is an automorphism α fixing A and $a_i \mapsto b_i$ for $1 \leq i \leq n$. Now, $p = \operatorname{tp}(a_{n+1}/Aa_1 \cdots a_n)$ is the unique non-algebraic type over $Aa_1 \cdots a_n$. Letting $q = \alpha(p)$, we see that $b_{n+1} \models q$, as q will be the unique non-algebraic type over $Ab_1 \cdots b_n$, thus b_{n+1} is $Ab_1 \cdots b_n$-conjugate to $\alpha(a_{n+1})$, so taking β fixing $Ab_1 \cdots b_n$ with $\beta(\alpha(a_{n+1})) = b_{n+1}$, we see that $\gamma = \beta\alpha$ is an automorphism fixing A that sends $a_1 \cdots a_{n+1}$ to $b_1 \cdots b_{n+1}$, so $\operatorname{tp}(a_1 \cdots a_{n+1}/A) = \operatorname{tp}(b_1 \cdots b_{n+1}/A)$.

Using this, we can summarize neatly how Morley rank is determined in strongly minimal theories:

Proposition 2.10. Let T be strongly minimal, and let $p = \operatorname{tp}(a_1 \cdots a_n/A)$. Then

$$RM(p) = \operatorname{acl-dim}(a_1 \cdots a_n/A).$$

Proof. It is sufficient to show this for the case when $\{a_1, \dots, a_n\}$ is acl-indpendent over A. Indeed, if $a_{k+1}, \dots, a_n$ are algebraic over $Aa_1 \cdots a_k$ then $RM(a_1 \cdots a_k/A) = RM(a_1 \cdots a_n/A)$ [18]. When $n = 1$ this follows as a_1 is contained in only cofinite definable sets over A, hence $RM(p) \geq 1$, but also $RM(\mathbb{M}) = 1$, hence $RM(p) = 1$. Now suppose inductively that this holds for n, and consider $\{a_1, \dots, a_{n+1}\}$ acl-independent over A. By the inductive hypothesis $RM(a_1 \cdots a_{n+1}/Aa_{n+1}) = n$, and $a_1 \cdots a_{n+1} \not\perp_A a_{n+1}$, so $RM(a_1 \cdots a_{n+1}) \geq n + 1$. Now if $RM(a_1 \cdots a_{n+1}/A) > n + 1$ it follows that there is $A \subset B$ and two disjoint B-definable subsets X, Y of $\mathbb{M}^n$ with $RM(X), RM(Y) \geq n+1$. By our induction hypothesis, if an $n+1$-tuple $x \in X$ is not acl-independent over B, then $RM(x/B) < n + 1$. Thus there is some x from X that is acl-independent over B, the same is true for some y from Y. But x and y have the same type over B, so X and Y cannot be disjoint, contrary to our assumption. $\square$

Proposition 2.11. Let T be a strongly minimal theory, and let A, B, C be parameter sets.. The following are equivalent

1. $A \perp_C B$,
2. $\operatorname{acl-dim}(A_0/BC) = \operatorname{acl-dim}(A_0/C)$ for any finite subset $A_0 \subseteq A$.

Proof. Suppose (2) holds, and let a be a finite tuple from A. Then $RM(\operatorname{tp}(a/BC)) = \operatorname{acl-dim}(a/BC) = \operatorname{acl-dim}(a/C) = RM(\operatorname{tp}(a/C))$, and it follows from the proof of proposition 2.6 that $a \perp_C B$, hence $A \perp_C B$.

For the converse, suppose that $\mathrm{acl}\text{-}\dim(a/BC) < \mathrm{acl}\text{-}\dim(a/C)$ for some finite tuple a from A, which we may assume to be acl-independent over C. Now it follows that there is some proper $a' \subseteq a$ that is a basis of a over BC, and there is some minimal tuple $b \in B$ that is independent over Ca' and $a \in \mathrm{acl}(Ca'b)$. By exchange it follows that $\mathrm{acl}\text{-}\dim(b/Ca) < \mathrm{acl}\text{-}\dim(b/C)$.

Write $b = b_0 b'$ where $b' \in B$ is a singleton, and $b' \in \mathrm{acl}(Cab_0)$. We see that $D := \mathrm{acl}(b_0 C) \subseteq \mathrm{acl}(BC)$, and $\mathrm{acl}(aD) \cap \mathrm{acl}(BD) \neq \mathrm{acl}(D)$, as this intersection contains b'.

Take $g \in \mathrm{acl}(aD) \cap \mathrm{acl}(BD) \setminus \mathrm{acl}(D)$, then $g \in \phi(a, \mathbb{M})$ for some L_D-formula $\phi(x, y)$ with $|\phi(a, \mathbb{M})| < k$. Now take an arbitrary collection of distinct D-conjugates of g, $(g_i)_{i<\omega}$, which may also assume to be D-indiscernible. Suppose (WLOG) that $a^* \models \phi(x, g_1) \wedge \cdots \wedge \phi(x, g_k)$, and a^* is D-conjugate to a, then it would follow that $|\phi(a, \mathbb{M})| \geq k$, contrary to our assumption. By compactness there is some $\psi(x) \in \mathrm{tp}(a/D)$ such that $\models \psi(x) \rightarrow \neg \bigwedge_{i=1}^{k} \phi(x, g_i)$. By D-indiscernability, it follows that $(\psi(x) \wedge \phi(x, g_i))_{i<\omega}$ is k-inconsistent, so $a \not\perp\!\!\!\perp_D \mathrm{acl}(BD)$.

Since $a \not\perp\!\!\!\perp_D \mathrm{acl}(BD)$, it is clear that $a \not\perp\!\!\!\perp_C \mathrm{acl}(BC)$, as $C \subseteq D \subseteq \mathrm{acl}(BC)$. By lemma 2.4, $A \not\perp\!\!\!\perp_C B$. $\square$

We see that this has the immediate consequence of every type in a strongly minimal theory being of finite Morley rank. Moreover, we see that

Corollary 2.12. If T is strongly minimal, then for all types $U(p) = RM(p)$.

Proof. Let $p = \mathrm{tp}(a_1 \cdots a_m/A)$, and suppose that $(a_1, \ldots, a_n)$ is a basis of $(a_1, \ldots, a_m)$ over A. If $n = 0$ then $RM(p) = 0$, so $U(p) = 0$ as p is algebraic. Now suppose $n \geq 1$, and $RM(q) = U(q)$ for all types whenever $RM(q) < n$. Define $q = \mathrm{tp}(a_1 \cdots a_m/Aa_1)$. Then $RM(q) = n - 1$, hence $U(q) = n - 1$. By propositions 2.10 and 2.11, it follows that q is a forking extension of p, thus $U(p) \geq n$. But $RM(p) \geq U(p)$ for all types, thus $RM(p) = U(p)$. $\square$

We can use the notion of dimension to broadly define some classes of pregeometries:

Definition 2.13. Let (X, cl) be a pregeometry. Then (X, cl) is said to be:

1. *trivial* if for all $A \subseteq X$, $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$
2. *modular* if for all $A, B \subseteq X$, $\mathrm{cl}\text{-}\dim(A \cup B) = \mathrm{cl}\text{-}\dim(A) + \mathrm{cl}\text{-}\dim(B) - \mathrm{cl}\text{-}\dim(A \cap B)$
3. *locally modular* if for all $A, B \subseteq X$ with $A \cap B \not\subseteq \mathrm{cl}(\emptyset)$, $\mathrm{cl}\text{-}\dim(A \cup B) = \mathrm{cl}\text{-}\dim(A) + \mathrm{cl}\text{-}\dim(B) - \mathrm{cl}\text{-}\dim(A \cap B)$.

It is not difficult to see that $1 \implies 2 \implies 3$. Moreover, 3 holds if and only if there is a singleton $a \in X$ such that X is modular with respect to its dimension over a, that is when acl is replaced by $\mathrm{cl}_a(A) := \mathrm{cl}(Aa)$ [2]. Let us see examples and non-examples coming from strongly minimal theories.

In the theory of infinite sets, the universe equipped with $\mathrm{cl} := \mathrm{acl}$ is a trivial pregeometry. Indeed, if $a \in \mathrm{acl}(A)$, then in fact $a \in A$, and so $a \in \mathrm{acl}(\{a\})$.

In the theory of an infinite vector space V over a field K, the universe with acl is a modular pregeometry. Let $\mathcal{B}$ be a basis of $\mathrm{span}(A \cap B)$ and extend these to bases of $\mathrm{span}(A)$ and $\mathrm{span}(B)$, given by $\mathcal{B}_A$ and $\mathcal{B}_B$ respectively. Then $\mathcal{B}_A \cup \mathcal{B}_B$ is a basis of $\mathrm{span}(A \cup B)$. It is easy to see that modularity follows from this. If the dimension of V is at least two then this is non-trivial.

An affine space A is just a reduct of a vector space V in which one excludes 0 from the language, so the algebraically closed sets (affine subspaces) of this theory are just translates of vector subspaces. It is easy to see that an affine space's dimension is preserved under translation, and moreover if X is an affine subspace and $x \in X$ then $\text{acl-dim}_A(X) = \dim_K(X - x)$, where $\dim_K$ is the usual K-vector space dimension. Now consider an infinite affine space A over a field K. It is easy to see that $\mathrm{acl}(\emptyset) = \emptyset$ in this theory, so suppose without loss of generality that X and Y are two affine sets with non-empty intersection. Then for any $a \in X \cap Y$, we see that $X \cap Y - a$ is a vector subspace of A. Thus, translating the whole space by a, the proof of modularity in vector spaces shows that A is locally modular. But, if A contains a plane P and one takes two distinct but parallel lines X and Y in P, it is easy to see that $\dim(X \cup Y) = 3$, $\dim(X) = \dim(Y) = 2$, but $\dim(X \cap Y) = 0$, so that A is not modular.

Lastly, consider an algebraically closed field K with cl given by the usual field-theoretic algebraic closure, i.e. acl in the language of rings. Suppose that K has infinite transcendence degree over its prime subfield F, and take $t \in K$ transcendental over F. Then choose a, b, x that are algebraically independent over $F(t)^{\mathrm{alg}}$. Now take $y = ax + b$, so that $\dim(F(x, y, a, b, t)^{\mathrm{alg}}/F) = 4$. We see that $\dim(F(x, y, t)^{\mathrm{alg}}/F) = \dim(F(a, b, t)^{\mathrm{alg}}/F) = 3$, but $F(x, y, t)^{\mathrm{alg}} \cap F(a, b, t)^{\mathrm{alg}} = F(t)^{\mathrm{alg}}$. Indeed, if not then there is some d algebraic over x, y, t and a, b, t, but not t alone. In particular, $\dim(F(d, x, y, t)^{\mathrm{alg}}/F(d, t)^{\mathrm{alg}}) = 1$, so that there is an irreducible polynomial $P(X, Y)$ over $F(d, t)^{\mathrm{alg}}$ such that $P(x, y) = 0$. Then $P(X, Y)$ remains irreducible over $F(a, b, d, t)^{\mathrm{alg}}$ by model-completeness of ACF. But $Y - aX - b$ divides P, so $P(X, Y) = \alpha(Y - aX - b)$ for some $\alpha \in F(d, t)^{\mathrm{alg}}$. But then $\alpha a, \alpha b \in F(d, t)^{\mathrm{alg}}$, hence $a, b \in F(d, t)^{\mathrm{alg}}$, contrary to the transcendence degree of $F(a, b, t)^{\mathrm{alg}}$ being 3 over F. Thus algebraically closed fields are not locally modular.

2.3 Interpretability

Definition 2.14. Let D be an A-definable set in $\mathbb{M}$. An L'-structure N is *interpretable in* $\mathbb{M}$ *on* D *over* A if there exists an A-definable set $X \subseteq D^n$ for some n, and a surjection $f : X \twoheadrightarrow N$, such that for all $\emptyset$-definable sets $R' \subseteq N^m$ there is an A-definable set $R \subseteq X^m$ in $\mathbb{M}$ such that $(x_1, \ldots, x_m) \in R \iff (f(x_1), \ldots, f(x_m)) \in R'$. When f is a bijection, we say that the interpretation is *definable*.

To elaborate, consider the common construction of *quotients* throughout mathematics. Given some equivalence class that is defined by the theory, one can often form a new structure by taking some set in the theory modulo this equivalence relation. Interpretation roughly captures this idea.

As an example, the fraction ring Q of a commutative ring R is interpretable in R on R over $\emptyset$. Indeed, the regular elements D of a ring are definable by $\forall y(xy \neq 0)$, and taking $X = R \times D \subseteq R^2$, we see that the map $(a, b) \mapsto a/b$ is a surjection $X \twoheadrightarrow Q$. Equality on Q is interpretted in X by $=_Q ((a, b), (c, d)) : ac = bd$, 0 is interpretted by $0_Q((a, b)) : a = 0$, 1 is interpretted by $1_Q((a, b)) : a = b$, addition is interprette by $+_Q((a, b), (c, d), (e, f)) : adf + cbf = ebd$, and multiplication is interpretted by $\cdot_Q((a, b), (c, d), (e, f)) : acf = ebd$. It follows inductively that the ring structure on Q is interpretable in R on X over $\emptyset$.

If a theory has a elimination of imaginaries, then all interpretations are in fact definable. Indeed, the set of equivalence classes X/f induced by the fibres of $f : X \twoheadrightarrow N$ will be definable, and the bijection $X/f \to N$ yields a definable interpretation of N in $\mathbb{M}$ on X/f.

Definition 2.15. Let D be an A-definable set of $\mathbb{M}$. We define the *induced structure of* D *over* A, denoted $(D_A)^{\text{ind}}$, to be the structure with the universe D, and predicates P_X for every A-definable set $X \subseteq D^n$ in $\mathbb{M}$, where P_X is interpreted as X in D.

Note that an L'-structure N is interpretable in $\mathbb{M}$ on D over A if and only if N is interpretable in $(D_A)^{\text{ind}}$ on D over $\emptyset$.

Definition 2.16. An A-definable set D is said to be *stably embedded over* A if for every definable set $X \subseteq D^n$, X may be defined in $(D_A)^{\text{ind}}$ over a set of parameters.

The following is a theorem of Poizat [16], demonstrating the further tameness of stable theories.

Theorem 2.17. Let T be a stable theory, and suppose that D is an A-definable set. Then D is stably embedded over A.

A consequence of this is that if T is stable, D is A-definable, and $A \subseteq A'$, then $\mathbb{M}$ interpets N on D over A' only if $\mathbb{M}$ interpets N on D over A, and $(D_A)^{\text{ind}}$ interprets N over $\emptyset$.

2.4 Orthogonality

Definition 2.18. Let T be a stable theory. Two types p and q over a common parameter set C are said so be *almost orthogonal* if for all realizations a and b of p and q respectively, $a \perp_C b$. Two stationary types p and q are said to be *orthogonal* if all their non-forking extensions to a common parameter set C are almost orthogonal. Two strongly minimal sets P and Q are said to be orthogonal if their generic types p and q are orthogonal.

The term orthogonality evokes concepts from linear algebra. In an inner product space V, one can think of two vectors as being orthogonal when they are geometrically perpendicular. In some sense, this means that any information contained in the vectors is only common in the most trivial way, i.e. their projections onto one another is the zero vector. Likewise with types, two types are orthogonal whenever their generic members (relative to a common set of parameters) have no significant dependence on one another. We see easily that all non-algebraic stationary types $p(x)$ over A are non-orthogonal to themselves, as for all $\models p(a)$, $a \perp_A A$ in a stable theory, and clearly $a \not\perp_A a$. Similarly, a non-forking extension of p is non-orthogonal to p.

It is important to see that two strongly minimal sets P and Q are non-orthogonal if and only if they admit a definable generic finite-to-finite correspondence. By this we mean, there is some definable set $\Gamma \subseteq P \times Q$ and cofinite sets $P_0 \subseteq P$, $Q_0 \subseteq Q$ such that the coordinate projections $\Gamma \cap (P_0 \times Q_0) \to P_0$ and $\Gamma \cap (P_0 \times Q_0) \to Q_0$ are onto and finite-to-one.

Indeed, let C be a common set over which P and Q are defined, and suppose that $a \in P$ and $b \in Q$ are generic over C, but $a \not\perp_C b$. By our characterization of independence in strongly minimal sets (Proposition 2.11), it follows that there is a C formula $\phi(x, y)$ such that $|\phi(a, \mathbb{M})| < N_a$ and $|\phi(\mathbb{M}, b)| < N_b$ for some $N_a, N_b < \omega$, and $\models \phi(a, b)$. Let

$$P_0 := \{a' \in P : |\phi(a', \mathbb{M})| < N_a\},$$

$$Q_0 := \{b' \in Q : |\phi(\mathbb{M}, b')| < N_b\},$$

then P_0 and Q_0 are cofinite as they are infinite and definable (infinitude is due to P_0 and Q_0 containing all the C-conjugates of a and b respectively). Then Γ gives a generic finite-to-finite correspondance of P_0 and Q_0.

Conversely, if $\Gamma \subseteq P \times Q$ is in generic finite-to-finite correspondance defined over C, then there is some $a \in P_0 \setminus \mathrm{acl}(C)$. Letting $\phi(x, y)$ define $\Gamma \cap (P_0 \times Q_0)$, we see that if $\models \phi(a, b)$ then $b \in Q_0 \setminus \mathrm{acl}(C)$, otherwise taking the fibres of b and its C-conjugates under $\phi(x, y)$ would give the C-algebraicity of a. So a and b are generic in P and Q over C respectively, but clearly $\phi(x, y)$ gives an algebraic dependence between them, so $a \not\perp_C b$.

By this finite-to-finite characterization of non-orthogonality in strongly minimal sets, it can be seen that non-orthogonality is an equivalence relation on strongly minimal sets.

Indeed, the only non-trivial property to verify is transitivity, so suppose that P, Q, R are strongly minimal sets, and that $\Gamma \subseteq P \times Q$, $\Delta \subseteq Q \times R$ are witnesses of their non-orthogonality. We see then that $\Theta \subseteq P \times R$ defined by

$$(x, y) \in \Theta \leftrightarrow \exists z \in Q[(x, z) \in \Gamma \wedge (z, y) \in \Delta]$$

is a finite-to-finite generic correspondence between P and R.

Proposition 2.19. In a strongly minimal theory T, the non-algebraic stationary types are non-orthogonal to each other.

Proof. Let $p \in S_n(A)$ and $q \in S_m(B)$ be non-algebraic types, and let $C = A \cup B$. Let $\models p(a), q(b)$ be such that $a \mathop{\smile\!\!\!\!\!\!|}_A C$ and $b \mathop{\smile\!\!\!\!\!\!|}_B C$. It follows that $a, b \notin \operatorname{acl}(C)$, hence (after re-indexing) we see that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ are such that $a_1, b_1 \notin \operatorname{acl}(C)$. By the uniqueness of non-algebraic types in $S_1(C)$, it follows that there is some $\alpha \in \operatorname{Aut}(\mathbb{M}/C)$ such that $\alpha(a_1) = b_1$. Since p is defined over $A \subseteq C$, p is fixed under C-automorphisms, hence $\models p(\alpha(a))$ and $\alpha(a) \mathop{\smile\!\!\!\!\!\!|}_A C$. But $\alpha(a) \not\mathop{\smile\!\!\!\!\!\!|}_C b$, as $\alpha(a)$ and b share their first coordinate. $\square$

As a final comment on orthogonality of strongly minimal sets, it is true that two non-orthogonal strongly minimal sets have the same geometry. By this we mean that if P and Q are non-orthogonal strongly minimal sets, then the pregeometry of P is locally modular if and only if the pregeometry of Q is locally modular [3]. The same is true for non-orthogonal strongly minimal sets with trivial geometries.

2.5 The conjecture as a principle

Let us restate Zilber's conjecture:

Zilber's Conjecture. Let X be a strongly minimal definable set in $\mathbb{M}$. If (X, acl) is not locally modular, then an infinite field F is interpretable in $\mathbb{M}$ on X.

At this point our exposition has developed more than enough theory to digest the statement of this conjecture for its pure model theoretic content, in fact there is nothing left here to explicate in this vein. Instead, in this section our approach is to explain this conjecture from the perspective of mathematical history and philosophy.

To begin, let us develop an understanding of whence this conjecture comes. It was realized at some point that the strongly minimal sets of a complete theory can be studied not only to provide some local data of the theory, but also a great deal of global data. Recall that a theory T is said to be κ-categorical if T has only one model of size κ up to isomorphism. It was observed by Łos that every known countable theory that was κ_0-categorical for some uncountable κ_0 was also κ-categorical for all uncountable cardinals κ, i.e. uncountably categorical [19]. This was eventually proved by Morley [11]. Prime examples of this are the theories of algebraically closed fields of fixed characteristic, and vector spaces over $\mathbb{Q}$. As the reader might observe, both these theories are strongly minimal. Moreover, the usual proof of this in both cases relies on demonstrating that models of the same dimension are isomorphic, and furthermore that if M is a model of either of these theories, and $|M| > \omega$, then acl-dim$(M) = |M|$. This proof technique generalizes to arbitrary countable strongly minimal theories, so it follows that any countable strongly minimal theory is uncountably categorical.

The Baldwin-Lachlan theorem states that a countable theory T is κ-categorical for an uncountable cardinal κ if and only if T is totally trascendental and contains no Vaughtian pairs. As the latter conditions do not refer to κ this immediately implies Morley's theorem. Proofs of the Baldwin-Lachlan theorem exploit the dimension theory of the strongly minimal definable sets in the prime model of the theory, and can thus be seen as further generalizations of the argument for the uncountable categoricity of strongly minimal theories. Furthermore, it can also be shown that in uncountably categorical theories, the countable models are determined up to isomorphism by the acl dimension of their strongly minimal sets. Details of these arguments can be found in [18].

Another instance of the centrality of strongly minimal sets is that in any stable theory, of which uncountably categorical theories are a special case, every non-algebraic stationary type p of finite U-rank i.e. $1 \leq U(p) < \omega$, is non-orthogonal to some stationary type q_p with $U(q_p) = 1$, see §2.5 of [13]. Recall that such types are said to be *minimal*, and share the geometric properties of strongly minimal sets.

The take away from all this is that in stability theory, understanding the strongly minimal formulas or minimal types of the theory provides a lot of information. We thus derive the following gestaltic principle that we credit in spirit to Zilber:

The structure of strongly minimal sets of a theory have significant consequences for the structure of the models of a theory. Therefore, the study of the theory should attempt to classify the strongly minimal sets up to some practical measure of irredundancy.

Non-orthogonality of strongly minimal sets, or generic finite-to-finite correspondence as discussed in §2.4, turns out to be the right notion of redundancy to work with.

As a first step towards such a classification, it seems natural to ask what types of pregeometries are witnessed by the strongly minimal definable sets (or minimal types) in the given theory under scrutiny.

A trivial strongly minimal set should be thought of as a set without any rich algebraic structure, making their study akin to combinatorics. Non-trivial locally modular strongly minimal sets are module-like: Hrushovski showed in the 1980's that these interpret infinite

Abelian groups, and their pregeometry is that of a projective/affine geometry over a division ring. So their study is akin to linear algebra. Zilber's conjecture states that a non-locally modular strongly minimal set comes from an infinite definable field, so its study is akin to algebraic geometry. The truth of Zilber's conjecture would then give a trichotomy for strongly minimal theories, that would then give a principle along the lines of:

The study of first-order structures can be reduced to the study of combinatorics, linear algebra, or algebraic geometry.

This of course is a very bold claim to make - but given that the only strongly minimal sets to be found in first-order structures of normal mathematical discourse were of one of these types, the conjecture was not absurd either.

Some evidence for this comes from the following theorem of Zilber [20].

Theorem 2.20. Let T be an uncountably categorical theory. Then one and only one of the following holds:

1. The strongly minimal sets of $\mathbb{M}$ are trivial.
2. The strongly minimal sets of $\mathbb{M}$ are non-trivially locally modular.
3. A pseudoplane is definable in $\mathbb{M}$.

For the reader unfamiliar with the notion of a pseudoplane, it is simply a structure $\mathcal{P} = (P \sqcup L, I)$, where $I \subseteq P \times L$, and the fibres of I over P and L are all infinite, but distinct fibres have finite intersections. One should think of P as the points of a plane, L as the lines of the plane, and I as the incidence relation between points and lines. A pseudoplane can be viewed as a generalization of plane geometry over an infinite field.

Now suppose that T is a stable theory, and D is a strongly minimal set of T definable over A. By our work in §2.3, it follows that the theory of $(D_A)^{\mathrm{ind}}$ is strongly minimal, and thus is uncountably categorical. Hence, the results of Zilber's trichotomy apply to this structure, therefore D will be trivial, non-trivially locally modular, or define a pseudo-plane. In Zilber's terminology, D is of *disintegrated type*, *module-like*, or *field-like*, respectively.

That said, Hrushovski, years later in [5], provides a counterexample to Zilber's conjecture: he constructs a strongly minimal set that can define a pseudoplane, however its structure is not rich enough to interpret any higher dimensional analogs of a pseudoplane, the way that an infinite field would. It follows that this set is neither locally modular, nor does it interpret a field. Moreover, Hrushovski demonstrates continuum many distinct such examples.

So Zilber's conjecture is in general false. But as it turns out, for many important mathematical structures the conjecture is satisfied, hence the attendant trichotomy holds as well. In the rest of this essay we will focus on two examples, both expansions of algebraic geometry where the trichotomy does hold.

3 The Principle in Practice: DCF_0

Our first example of Zilber's principle used in practice are the differentially closed fields of characteristic zero (DCF_0). This theory is an expansion of ACF_0, and in many ways is quite similar. That said, this theory is far richer.

The first part will discuss some basic differential algebra. In the next we will summarize the elementary model theoretical properties of DCF_0. The last will discuss how the Zilber dichotomy is actualized in DCF_0. The material in this section comes from David Marker's surveys [7] and [8], unless stated otherwise.

3.1 Differential algebra

A differential ring R is simply a commutative ring (with identity) equipped with a derivation δ, i.e. an additive endomorphism of R which also satisfies the Leibniz rule: for all $x, y \in R$

$$\delta(xy) = x\delta(y) + \delta(x)y.$$

It is easy to see that

$$\delta(1) = 0,$$

$$\forall x[\delta(x^n) = nx^{n-1}\delta(x)],$$

$$\forall x \forall y \left[\exists z(yz = 1) \to \delta\left(\frac{x}{y}\right) = \frac{\delta(x)y - x\delta(y)}{y^2}\right],$$

Some examples of differential rings include:

- Any ring R with $\delta := 0_R$, called the trivial differential ring of R,
- The polynomial ring $R[X]$, where R is a ring and δ is the usual algebraic derivative in $R[X]$,
- $C^\infty(\mathbb{R})$, the smooth functions from $\mathbb{R} \to \mathbb{R}$, where $\delta := \frac{d}{dx}$.

A differential ideal I of a differential ring R is a a ring ideal that is also closed under the derivation, i.e. for all $x \in I$ we have $\delta(x) \in I$. We say that I is a *prime differential ideal* if it is a differential ideal that is also prime as a ring ideal. Liekwise, I is said to be a *radical differential ideal* if it is a differential ideal that is also radical as a ring ideal. We take the *differential radical* of I, denoted $\sqrt[\delta]{I}$, to be the intersection of all radical differential ideals that contain I; it is easy to see that this intersection exists, and is itself a radical differential ideal. The following lemma shows that if $\mathbb{Q} \subseteq R$, then $\sqrt[\delta]{I} = \sqrt{I}$ for all differential ideals.

Lemma 3.1. Let R be a differential ring such that $\mathbb{Q} \subseteq R$. Then the differential radical of a differential ideal I agrees with its ring theoretic ideal, i.e. $\sqrt[\delta]{I} = \sqrt{I}$.

Proof. Clearly $\sqrt{I} \subseteq \sqrt[\delta]{I}$, so we show that $\sqrt{I}$ is a differential ideal. Let $a^n \in I$, we show inductively on $1 \leq k \leq n$ that $a^{n-k}\delta(a)^{2k-1} \in I$. Clearly $na^{n-1}\delta(a) \in I$, so dividing by n we have $a^{n-1}\delta(a) \in I$. Now suppose that $a^{n-k}\delta(a)^{2k-1} \in I$. Differentiating, we see that

$$(n-k)a^{n-k-1}\delta(a)^{2k} + (2k-1)a^{n-k}\delta(a)^{2k-2}\delta^2(a) \in I,$$

and multiplying by $\delta(a)$ gives

$$(n-k)a^{n-k-1}\delta(a)^{2k+1} + (2k-1)a^{n-k}\delta(a)^{2k-1}\delta^2(a) \in I.$$

By our inductive hypothesis $(2k-1)a^{n-k}\delta^{2k-1}\delta^2(a) \in I$, so after the elimination of this term followed by division by $(n-k)$, we see that $a^{n-(k+1)}\delta(a)^{2(k+1)-1} \in I$.

In particular $\delta(a)^{2n-1} \in I$, so that $\delta(a) \in \sqrt{I}$, as desired. □

Given a differential ring R, one may form a differential ring of δ-polynomials in the indeterminate X, $R\{X\} = R[X, X', \ldots, X^{(n)}, \ldots]$. $X^{(k)}$ is by definition the k^{th}-derivative of X. That is, we define $\delta(X^{(k)}) := X^{(k+1)}$, and extend by linearity and the Leibniz rule. Then $R\{X\}$ is itself a differential ring. A δ-polynomial $f(X, X', \ldots, X^{(n)})$ is evaluated at $x \in R$ as $f(x, \delta(x), \ldots, \delta^n(x))$. Inductively we define $R\{X_1, \ldots, X_n, X_{n+1}\}$ as $(R\{X_1, \ldots, X_n\})\{X_{n+1}\}$. We say that the *order* of a differential polynomial $f(X_1, \ldots, X_m)$ is n if f contains a non-zero term with $X_i^{(n)}$ in its expression, and for all $1 \leq j \leq m$, and $n' > n$, $X_j^{(n')}$ does not appear in a non-zero term of f.

Let K be a diffential field in what follows, i.e. a differential ring whose ring structure is a field. To any $V \subseteq K^n$, we may assign a differential ideal $I_\delta(V) \subseteq K\{X_1, \ldots, X_n\}$ that contains all differential polynomials that vanish on V, i.e.

$$I_\delta(V) := \{f(X) \in K\{X_1, \ldots, X_n\} : x \in V \implies f(x) = 0\}.$$

Likewise, if $I \subseteq K\{X_1, \ldots, X_n\}$ be a differential ideal, we may assign to I a set $V_\delta(I) \subseteq K^n$ given by the common zeroes of I, i.e.

$$V_\delta(I) := \{x \in K^n : f \in I \implies f(x) = 0\}.$$

Sets of the form $V_\delta(I)$ for differential ideals $I \subseteq K\{X_1, \ldots, X_n\}$ are called *Kolchin closed sets* of K^n, and the reader will note that like the Zariski closed sets of K^n, these form a toplogy on K^n, called the *Kolchin topology*.

There are differential of Hilbert's basis theorem and nullstellensatz theorems for differential fields. We state them here and mention some consequences.

Theorem 3.2. Let K be a differential field, and let $(I_i)_{i<\omega}$ be an ascending chain of radical differential ideals in $K\{X_1, \ldots, X_n\}$. Then $(I_i)_{i<\omega}$ stabilizes for some $k < \omega$. Moreover, if $I \subseteq K\{X_1 \ldots, X_n\}$ is a radical differential ideal, then $I = \sqrt{f_1, \ldots, f_m}$ for some $f_i \in K\{X_1, \ldots, X_n\}$, and there is a unique finite collection of prime differential ideals $P_1, \ldots, P_m$ such that $I = P_1 \cap \cdots \cap P_m$.

Theorem 3.3. Let K be a *differentially closed field* (defined in §3.2), and let V be a Kolchin closed subset of K^n. Then there is a unique radical differential ideal $I \subseteq K\{X_1, \ldots, X_n\}$ such that $V_\delta(I) = V$, and $I_\delta(V) = I$. Stated another way, the Kolchin closed sets of K^n and the radical diffential ideals of $K\{X_1, \ldots, X_n\}$ are in one-to-one correspondence via V_δ with inverse I_δ.

It is easy to see then that any descending chain of closed sets in the Kolchin topology of a differentially closed field corresponds to an ascending chain of radical differential ideals. Since ascending chains of radical differential ideal in $K\{X_1 \ldots, X_n\}$ stabilize, it follows that descending chains of Kolchin closed sets in K^n stabilize as well, hence the Kolchin topology is Noetherian.

One can also verify that if P is a prime differential ideal of $K\{X_1, \ldots, X_n\}$, then $V_\delta(P)$ is irreducible in the Kolchin topology. Since radical differntial ideals are finite intersections of prime differential ideals, it follows that all Kolchin closed sets are finite unions of irreducible Kolchin closed sets.

3.2 Basic model theory of DCF_0

The language of differential rings is an expansion of the language of fields, given by $L_\delta = \{0, 1, +, \cdot, \delta\}$, where δ is a new unary function symbol. The axioms of DCF_0 are the usual axioms of ACF_0, with the addition of

- $\forall x \forall y[\delta(x + y) = \delta(x) + \delta(y)]$
- $\forall x \forall y[\delta(x \cdot y) = x \cdot \delta(y) + \delta(x) \cdot y]$,
- (for all differential polynomials $f(X)$ and $g(X)$ where the order of f is great than g) $\exists x[f(x) = 0 \wedge g(x) \neq 0]$.

The first two state that the derivation δ is an additive homomorphism of the field, and that it satisfies the usual product rule of derivatives.

Let K be a differentially closed field, the the structure $(K, 0, 1, +, \cdot)$ is of course a model of ACF_0. We refer to $C_K := \ker \delta$ as the *constant field* of K. Note that C_K is a definable set in K, and restricting the field operations to C_K gives it a field structure. Moreover, C_K is algebraically closed. Indeed, suppose $a \in C_K^{\mathrm{alg}}$, and let $p(X) = \sum_{k=0}^{n} b_k X^k$ be its minimal polynomial. Then, as $\delta(b_k) = 0$ for all k we have

$$\sum_{k=0}^{n} b_k a^k = 0 \implies \delta\left(\sum_{k=0}^{n} b_k a^k\right) = \delta(a) \sum_{k=1}^{n} b_k k a^{k-1} = 0$$

but $q(X) = \sum_{k=1}^{n-1} b_{k+1}(k+1) X^k$ does not have a as a root as it is non-trivial and has degree $< n$. Hence $\delta(a) = 0$. Thus there is an algebraically closed field definable in K that is distinct from K. Indeed, note that $C_K \neq K$ as $\delta(X) = 1$ must have a solution x in K. In particular, K is not strongly minimal: C_K is a definable infinite and coinfinite set in K.

We now summarize some basic model theoretic statements regarding $\mathsf{DCF_0}$ with some brief explanations.

Theorem 3.4. The following is a summary of the model theoretic properties of $\mathsf{DCF_0}$:

1. $\mathsf{DCF_0}$ eliminates quantifiers,
2. $\mathsf{DCF_0}$ is complete,
3. Suppose (k, δ) is a differential field, and $P \subseteq k\{X_1, \ldots, X_n\}$ a prime differential ideal. There is a unique n-type $p(x) \in S_n(k)$ with the property that for all $f \in k\{X_1, \ldots, X_n\}$, the formula $f(x) = 0 \in p(x)$ if and only if $f \in P$. Moreover, every complete type over k is of this form.
4. $\mathsf{DCF_0}$ is ω-stable,
5. $\mathsf{DCF_0}$ eliminates imaginaries.

As usual, quantifier elimination implies that the definable sets are constructible from Boolean combinations of Zariski closed sets. That $\mathsf{DCF_0}$ is complete is important as it allows us to study every model of $\mathsf{DCF_0}$ using constructions like the monster model. Since $\mathsf{DCF_0}$ is ω-stable, it is totally transcendental, i.e. all its types have Morley rank. Elimination of imaginaries tells us that every interpretable structure N in $\mathsf{DCF_0}$ is in fact definable in $\mathsf{DCF_0}$.

In the case of $\mathsf{AFC_0}$, there is only one measure of the dimension of a type, which is given by the transcendence degree of its realizations over the field generated by its parameter set. This coincides with what we already know about strongly minimal theories, namely that their U-rank and Morley rank coincide with their acl-dimension (which in $\mathsf{ACF_0}$ is transcendence degree). One also has the notion of dimension in arbitrary Noetherian topologies, where the dimension of an irreducible closed set V is a foundation rank given by $\mathrm{Ndim}(V) \geq 0$ if $X \neq \emptyset$, and $\mathrm{Ndim}(V) \geq \alpha + 1$ if there is a closed and irreducible set $W \subsetneq V$ such that $\mathrm{Ndim}(W) \geq \alpha$. When the topology is the Zariski topology, this dimension is called the

Noetherian dimension of V, and it coincides with the transcendence degree of the generic points of V.

Suppose now that $(\mathbb{K}, \delta) \models \mathsf{DCF}_0$ is the monster model, $k \subseteq \mathbb{K}$ is a δ-subfield, and $V \subseteq k^n$ is an irreducible Kolchin closed subset defined over k. As prime ideals are in bijective correspondence with irreducible Kolchin closed sets, from part 3 of Theorem 3.4, we can associate a complete type $p_V(x) \in S_n(k)$ that contains the statements, $x \in V$ and $x \notin W$ for any proper k-definable Kolchin closed subset $W \subseteq V$. We call p_V the *generic type* of V. We may associate various ranks to p_V:

1. U-rank,
2. Morley rank,
3. $\mathrm{Ndim}(p_V) :=$ the Noetherian dimension of V,
4. $\mathrm{trdeg}_\delta(p_V) := \mathrm{trdeg}_\delta(\mathrm{Frac}(k\{X_1, \ldots, X_n\}/I_\delta(V))/k)$,

where we let $\mathrm{trdeg}_\delta(L/k)$ denote the differential transcendence degree of a δ-field extension $L \supseteq k$. By this we mean the cardinal of a maximal subset of L that does not satisfy any differential polynomial equations in k.

We have the following theorem that relates all these ranks in DCF_0:

Theorem 3.5. Let V be an irreducible Kolchin closed set of a δ-field k. Then

$$U(p_V) \leq RM(p_V) \leq \mathrm{Ndim}(p_V) < \omega \cdot (\mathrm{trdeg}(p_V) + 1),$$

and moreover the strict inequalities are realized.

We already know that in an ω-stable theory $U(p_V) \leq RM(p_V)$. To see $RM(p_V) \leq \dim(p_V)$, it is clear that if $RM(p_V) \geq 0$ then $\mathrm{Ndim}(p_V) \geq 0$. Now if $RM(p_V) \geq \alpha + 1$ then $RM(V) \geq \alpha + 1$ and there are disjoint definable sets $V_i \subseteq V$ all with $RM(V_i) \geq \alpha$. Inductively we assume $\mathrm{Ndim}(V_i) \geq \alpha$ (where we extend the notion of dimension to be the maximal dimension of the components of $\overline{V_i}$, note that this does not invalidate what we have done so far). If there are V_i and V_j disjoint such that $\overline{V_i} = V = \overline{V_j}$, by quantifier elimination, it would follow that V was not irreducible, as it would contain two disjoint open subsets. Hence some $\overline{V_i} \subsetneq V$, and thus $\mathrm{Ndim}(V) \geq \alpha + 1$. For the last inequality, we refer the reader to Corollary 2.6 of [17].

3.3 The trichotomy in DCF_0

Zilber's principle suggests that we attempt to understand the geometry of the strongly minimal sets of our theory, in particular we should see if the geometry of the non-locally modular strongly minimal sets have the geometry of an algebraically closed field. Note that in $\mathsf{DCF_0}$, by quantifier elimination the algebraic closure of a parameter set $\mathrm{acl}(A)$ is given by first taking the differential field $\langle A\rangle$ generated by A and then taking its field-theoretic algebraic closure, $\langle A\rangle^{\mathrm{alg}}$. Let us see some examples of strongly minimal sets in $\mathsf{DCF_0}$, which exhibit the various possible geometries.

To see the existence of trivial strongly minimal sets, we will make use of the following lemma that appears as Corollary 6.3 in [7]:

Lemma 3.6. Let C be a field of constants and consider the equation $X' = X^3 - X^2$. Let K be a differential closure of C, then if non-constant $x_1, \ldots, x_n$ are distinct and satisfy $\delta(x_i) = x_i^3 - x_i^2$, then $x_1, \ldots, x_n$ are algebraically independent over C.

Now, let D be the set defined by the equation $X' = X^3 - X^2 \wedge X' \neq 0$.

Proposition 3.7. D is a trivial strongly minimal set.

Proof. $X' = X^3 - X^2$ has infinitely many solutions by the axioms of $\mathsf{DCF_0}$, and the only solutions in the constants are 0 and 1, so D is infinite. Any differential polynomial equation restricted to D is equivalent to a polynomial equation, as we may replace instances of X' by X^3-X^2, so it has a finite solution set. By quantifier elimination, it follows that D is strongly minimal. By the defining equations, we see that the differential field generated by arbitrary $x_1, \ldots x_n, y \in D$ is just the field they generate over $\mathbb{Q}$. By Lemma 3.9, these have no field-algebraic relations, so by our characterization of acl, we see that $y \notin \mathrm{acl}(x_1 \cdots x_n) \cap D$. We conclude that $\mathrm{acl}(A) = A$ for all $A \subseteq D$, which is a strict form of triviality for (D, acl). $\square$

This is not the only example of a trivial strongly minimal set in $\mathsf{DCF_0}$. Indeed, in Proposition 2.1 of [4], Hrushovski and Itai give a wide class of trivial strongly minimal sets coming from algebraic curves defined over the constants. They classify these up to orthogonality. That said, there is no known classification of the trivial strongly minimal sets up to orthogonality. However, the situation is much clearer in the case of the non-trivial locally modular strongly minimal sets.

Recall that an abelian variety is a complete variety with a group structure defined by regular functions. Consider when A is an abelian variety of a differentially closed field K. Define $A^{\#}$ to be the Kolchin closure of the torsion elements of A. We have the following dichotomy due to Hrushovski and Sokolovic (Proposition 2.6 of [6]):

Proposition 3.8. Let A be a simple abelian variety over K, a differentially closed field. Then either:

1. A is isomorphic to an abelian variety B defined over C_K, or
2. $A^{\#}$ is strongly minimal and non-trivially locally modular.

In fact, a converse statement is also is true: if D is a strongly minimal non-trivially locally modular set in K, then D is non-orthogonal to $A^{\#}$ for some abelian variety A over K. Moreover, for two abelian varieties A and B defined over K, we have that $A^{\#}$ is non-orthogonal to $B^{\#}$ if and only if A and B are isogenous (Theorem 2.12 of [6]). Thus, the non-trivial locally modular sets in DCF_0 are characterized up to orthogonality by the isogeny classes of simple abelian varieties not definable over the constants.

Finally, the constant field itself is an example of a non-locally modular strongly minimal set in DCF_0. This follows from the following proposition:

Proposition 3.9. Let C_K be the constant field of a differentially closed field K. If $D \subseteq C_K^n$ is definable in K, then D is definable in $(C_K)^{\text{ind}}$ using only its field structure.

Proof. Firstly, since DCF_0 is ω-stable, it follows from Theorem 2.17 that D can be defined using only parameters from C_K. D then is a boolean combination of zero sets of differential polynomials in C_K. But a differential polynomial $f(X)$ may decomposed as $g(X) + h(X)$ where $g(X)$ is an algebraic polynomial, and $h(X)$ contains only higher order terms. But for all $x \in C_K$, it follows that $h(x) = 0$. Thus for all $x \in C_K$, $f(x) = 0$ if and only if $g(x) = 0$. It follows that D can be defined in C_K using only field definable sets with parameters from C_K. $\square$

Note that an immediate consequence of this is that C_K is a strongly minimal set, and since it is an algebraically closed field, it follows that it is non-locally modular in a sufficiently saturated model.

We have the following dichotomy for definable fields in K, which combines the result in Section 5 and Corollary 1.6 of [12]:

Theorem 3.10. Let F be an infinite field definable in K. Then

1. F is definably isomorphic to K if $RM(F)$ is infinite, otherwise
2. F is definably isomorphic to C_K if $RM(F)$ is finite.

We can take this statement to say that there are essentially only two definable fields in DCF_0, and remarkably, they are arguably the only natural candidates.

As we have remarked in §2.5, Zilber's conjecture is not true in general. However, we do have:

Theorem 3.11. (Hrushovski-Sokolovic [6]) The Zilber conjecture holds for DCF_0.

By elimination of imaginaries, it follows that any strongly minimal non-locally modular set D defines an infinite field F. Now, since $F \subseteq D^n$, and D is of finite rank, it follows that F has finite rank as well. By Proposition 2.19 of §2.4, D is non-orthogonal to F. By Theorem 3.10, F is definably isomorphic to C_K. Thus, we obtain the following manifestation of the Zilber conjecture in DCF_0.

Corollary 3.12. Let D be a strongly minimal set in DCF_0. Then D is either locally modular, or D is non-orthogonal to the field of constants.

4 The Principle in Practice: CCA

Our second example of Zilber's principle used in practice are the compact complex analytic spaces (CCA). Unlike other theories we have encountered thus far, CCA does not come from some algebraic axioms, so our study will be perhaps slightly jarring to the reader at first.

The first part will discuss the basics of complex analytic geometries. In the next we will summarize some elementary model theoretic properties of CCA. In the last section we discuss how the Zilber trichotomy is realized in CCA. The material in this section comes from Moosa's thesis and survey, [10] and [9] respectively, unless stated otherwise.

4.1 Complex analytic geometry

Let D be a domain (i.e. a connected and open subset) of $\mathbb{C}^n$, and suppose that $f_1, \ldots, f_m$ are holomorphic functions on D. We write $V = V(f_1, \ldots, f_m)$ to denote their common zeroes, and call such sets *analytic sets in D*. We let $\mathcal{O}_D$ represent the sheaf of holomorphic functions on D. Taking the quotient $\mathcal{O}_V := \mathcal{O}_D/\mathcal{I}_V$, where

$$\mathcal{I}_V := \{f \text{ is a holomorphism on } D : f(V) = 0\},$$

we obtain a structure sheaf on V, and associate to every analytic set the $\mathbb{C}$-ringed space $(V, \mathcal{O}_V)$. The sections on of $\mathcal{O}_V$ induce $\mathbb{C}$-valued functions on V, and such a functions f is called a *holomorphic function on V*. If W is also an analytic set of some domain, then a *holomorphic map $V \to W$* is a morphism of the ringed spaces $(V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$. If it is an isomorphism of ringed spaces, then we call it a *biholomorphism.*

A *complex analytic space* $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-ringed space that is locally modelled on complex analytic sets. Precisely, we say that $(X, \mathcal{O}_X)$ is a complex analytic space if the topology on X is Hausdorff and second-countable, and moreover there is an open cover $\{X_i : i < \omega\}$ of X and analytic sets $\{V_i : i < \omega\}$ such that $(X_i, \mathcal{O}_{X_i})$ is isomorphic to $(V_i, \mathcal{O}_{V_i})$.

If $(X, \mathcal{O}_X)$ is a complex analytic space, then $\mathcal{O}_X$ is called the *sheaf of holomorphic functions on X*. A map between complex analytic spaces $X \to Y$ is said to be a *holomorphic* *map* (*biholomorphic map*) if it is a morphism (isomorphism) of their ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Given a complex analytic space X and a some point $x \in X$, we may define $\dim_x X$, the *dimension of X at x*, to be the least $d \geq 0$ such that there is a finite-to-one holomorphic map $f : U \to D$, where U is a neighbourhood of x and D is a domain of $\mathbb{C}^d$. If X is connected then it is clear that $\dim_x X$ is constant as x varies, hence we may define $\dim X$ to be the value of $\dim_x X$ at any point.

The model theory of analytic spaces is primarily concerned with sets that are defined by holomorphic functions; we introduce these here. Given a complex analytic space, we say that a set $A \subseteq X$ is *analytic* if for all $x \in X$ there is a neighbourhood $x \in U$ and finitely many holomorphisms $f_1, \ldots, f_m \in \mathcal{O}_X$ such that $A \cap U$ is the common set of zeroes of $f_1, \ldots, f_m$. Note that an analytic set is closed in X, and moreover inherits a structure from X making it into a complex analytic space itself. Also, the analytic spaces are closed under Cartesian products, hence we may consider analytic subsets of powers of X, or more generally products $X_1 \times \cdots \times X_n$ of complex analytic spaces.

Analytic sets are analogous to the Zariski closed sets of algebraic geometry, and we will often refer to analytic subsets of a complex analytic space as Zariski closed. In particular, it is true that the analytic sets form the closed sets of a topology, which we refer to as the *Zariski topology.* A complex analytic space is said to be *irreducible* if it is irreducible in the Zariski topology, i.e. it cannot be written as the union of two proper Zariski closed sets. Proper analytic subsets are nowhere dense.

This topology is particularly well-behaved when we consider compact complex analytic spaces. I this case, Zariski closed subset may be written as an irredundant finite union of irreducible Zariski closed subsets. These are called the *irreducible components.* We may then define the dimension of a Zariski closed set of X to be the maximum dimension of its irreducible components (irreducibles are in particular connected). If A is an irreducible Zariski closed set, and $B \subseteq A$ is Zariski closed, then $\dim A = \dim B$ if and only if $A = B$. This implies that the Zariski topology on a compact complex analytic space is Noetherian.

Every complex projective algebraic variety is a compact complex analytic space. Indeed, as polynomials are holomorphic, projective varieties are analytic subsets of the compact complex manifolds $\mathbb{P}_n(\mathbb{C})$, projective n-space. Chow's theorem states that every analytic set in $\mathbb{P}_n(\mathbb{C})$ is given globally by the zero set of some homogeneous polynomials, so all the analytic subsets of a projective variety are algebraic. That is, on projective varieties the complex analytic Zariski topology coincides with the usual algebraic Zariski topology.

The Riemann existence theorem states that if X is an irreducible compact complex analytic space of dimension 1, then X is a projective algebraic curve. So to obtain non-algebraic examples of compact complex analytic spaces, we must consider higher dimensional examples.

A real $2n$-dimensional *lattice* Λ of $\mathbb{C}^n$ is an additive subgroup of $\mathbb{C}^n$ that has the form $\Lambda = \{m_1\alpha_1 + \cdots m_{2n}\alpha_{2n} : m_i \in \mathbb{Z}\}$, where the the set $\{\alpha_1, \ldots, \alpha_{2n}\}$ is an $\mathbb{R}$-basis of $\mathbb{C}^n$. An n-dimensional *complex torus* T is a quotient of $\mathbb{C}^n$ by a $2n$-dimensional lattice Λ, where T inherits an analytic structure from $\mathbb{C}^n$. The complex tori are compact complex manifolds, and moreover come equipped with a definable group operation that is holomorphic. It is a fact that if $n > 1$ and $\alpha_1, \ldots, \alpha_{2n}$ are chosen generally, then the corresponding torus will not be a projective algebraic variety.

4.2 Basic model theory of CCA

Given a structure M in any language L, one may consider the L-theory

$$\mathrm{Th}(M) := \{\sigma \text{ an } L\text{-sentence} : M \models \sigma\},$$

called the theory of the structure M, which is clearly complete. As an example, the additive group structure $(\mathbb{Q}, 0, +)$ yields the theory of torsion-free divisible Abelian groups, or $\mathbb{Q}$-vector spaces. The consideration of an infinite structure's theory allows one to study a particular structure of interest using model theoretic techniques. This will usually require considering (saturated) elementary extensions of M. M is often referred to as the standard model of $\mathrm{Th}(M)$.

This is our philosophy here with compact complex analytic spaces; rather than providing some *a priori* first-order theory that is satisfied by a complex analytic space (a challenge that seems impossible given that the usual axioms of complex analytic spaces are not first-order), we make the compact complex analytic spaces into a (relational) first-order structures and then, *a posteriori*, consider the theory of their structures. As a first-order structure is just a set (the universe) paired with a collection of symbols that define subsets of its Cartesian powers (constants, functions, and relations), then we may consider the language given by taking the Zariski closed sets of the (finite) Cartesian products of compact complex analytic spaces. The way we make a compact complex analytic space X into a first order structure is by taking the basic relations to be the analytic subsets of X^n for all $n > 0$. Note that X^n is also a compact complex analytic space. It turns out that it is useful to consider *all* compact complex analytic spaces at once, by considering one massive many-sorted structure. This allows for us to work model-theoretically with the interactions between the different compact complex analytic spaces.

Definition 4.1. Let $\mathbf{A}$ be the many sorted universe of all the compact complex analytic spaces, and let L be the many-sorted language where there is a predicate P_A for each analytic subsets $A \subseteq X_1 \times \cdots \times X_n$ where $X_1, \ldots, X_n$ are among the sorts of $\mathbf{A}$. We define $\mathcal{A}$ to be the L-structure with universe $\mathbf{A}$, where the predicates in L are interpreted in $\mathcal{A}$ as the Zariski closed set they represent. We write CCA in place of $\mathrm{Th}(\mathcal{A})$.

Remark that up until this point, we have only encountered theories that have an algebraic origin. CCA on the other hand is far more geometric. This said, it is not absurd to contend that morally, the Zariski closed sets of complex analytic spaces are very similar to the Zariski closed sets of an algebraically closed field. Their model theoretic behaviours are also similar.

One of the sorts in $\mathcal{A}$ is $\mathbb{P}(\mathbb{C})$. Since each $\mathbb{P}_n(\mathbb{C})$ embeds in a Cartesian power of $\mathbb{P}(\mathbb{C})$, every projective algebraic variety is a definable set in $\mathcal{A}$ on the sort $\mathbb{P}(\mathbb{C})$. Now $\mathbb{C}$ is definable in $\mathbb{P}(\mathbb{C})$ by removing a point, and moreover its field structure is al definable in $\mathcal{A}$ as the graphs of " $+$ " and " $\cdot$ " in $\mathbb{C}^3$ extend to projective varieties in $\mathbb{P}(\mathbb{C})^3$. So the algebraically closed field $(\mathbb{C}, +, \cdot)$ is definable in $\mathcal{A}$ on the sort $\mathbb{P}(\mathbb{C})$, and as a consequence of Chow's theorem as discussed above, the induced structure on $\mathbb{C}$ is the pure field $(\mathbb{C}, +, \cdot)$. In this way, we can see that CCA is an expansion of ACF_0.

The following theorem of Remmert for complex analytic spaces helps to get a handle on what the definable sets of $\mathcal{A}$ are.

Theorem 4.2. Let $f : X \to Y$ be a proper holomorphic map. If $V \subseteq X$ is a Zariski closed, then $f(V)$ is Zariski closed.

Since we only consider when X is compact, it follows that every holomorphic map is proper, in particular the coordinate projections are holomorphic maps. From this fact, as well as a theorem describing the dimension of fibres of surjective holomorphic maps, one can obtain quantifier elimination for CCA. Its proof is far more direct than the proof in §3.2 for DCF_0, which relies on a purely model theoretic criterion. We do not include the proof here, but state:

Proposition 4.3. CCA has quantifier elimination.

By quantifier elimination, every definable set F in $\mathcal{A}$ is of the form

$$F = \bigcup_{i=1}^{k} V_i \setminus W_i,$$

where the V_i are irreducible Zariski closed sets, and the W_i are proper Zariski closed subsets of the V_i. If $k = 1$ we say that F is *irreducible*. Moreover, the Zariski closure of F is given by $\overline{F} = \bigcup_{i=1}^{k} V_i$, and $\dim F = \dim \overline{F} = \max\{\dim V_i : 1 \leq i \leq k\}$. Thus we obtain a notion of dimension for all the definable sets. From this dimension we can show the following:

Proposition 4.4. CCA is ω-stable. Moreover, the dimension of definable sets in $\mathcal{A}$ bounds the Morley rank from above, hence Morley rank in $\mathcal{A}$ is finite.

Proof. Let F be a definable set, and that $RM(F) \geq 0$, then clearly $\dim F \geq 0$. Proceeding forward we may assume that F is irreducible, as Morley rank is a max function with respect to unions. Now suppose inductively that $RM(F') \geq n \implies \dim(F') \geq n$ for arbitrary definable sets F'. Now if $RM(F) \geq n+1$ this implies that there are infinitely many disjoint $F_i \subseteq F$ with $RM(F_i) \geq n$. Then $\dim(F_i) \geq n$, and moreover $\overline{F_i} \subseteq \overline{F}$. If $\overline{F_i} = \overline{F}$, by irreducibility of the latter and quantifier elimination, F_i must contain a Zariski open subset of $\overline{F}$. But any two Zariski pen subsets of $\overline{F}$ intersect non-trivially, so at least some $\overline{F_i} \subseteq \overline{F}$ is proper, whence $\dim(F) > \dim(F_i)$, so $\dim(F) \geq n+1$. $\square$

Remark that our proof glossed over a detail: it is not strictly true that in general model theoretic constructions one can compute Morley rank without passing to an elementary extension. This said, it is the case that $\mathcal{A}$ is ω_1-compact, that is, every countable collection of definable sets with the finite intersection property has a non-trivial intersection. This model theoretic property is enough to compute the Morley rank of definable sets without passing to an elementary extension, validating our proof above.

Proposition 4.5. $\mathcal{A}$ is ω_1-compact.

Note that the definable sets in $\mathcal{A}$ are all $\emptyset$-definable, as every point in $\mathcal{A}$ is an analytic set, and is hence named by a predicate. This means that $\mathcal{A}$ has little saturation: given an infinite sort X and a variable x belonging to the sort X, the partial type

$$\pi_X(x) := \{x \neq a : a \in X\}$$

has no realizations. Thus the ω_1-compactness of $\mathcal{A}$ is not a consequence of $\aleph_1$-saturation.

By quantifier elimination and Noetherianiety, the complete types of $\mathcal{A}$ are given by the generic types of irreducible Zariski closed sets, i.e. every type $p(x)$ in the sort X is the unique completion of the partial type

$$\pi_V(x) := \{x \in V\} \cup \{x \notin Y : Y \subsetneq V \text{ is Zariski closed}\},$$

for some irreducible Zariski closed set $V \subseteq X$. We write that $\dim(p) = \dim(V)$.

Although we have made attempts to justify why we work within the many sorted theory of compact complex analytic spaces, the reader may question why we have not made the choice to restrict our attention to just compact complex manifolds. One justification is the obvious: complex analytic spaces are generalizations of complex manifolds, so we can obtain a more general theory. However, our second justification is based in model-theoretic pragmatism. It is a fact that the compact complex analytic spaces are all holomorphic images of compact complex manifolds (this is resolution of singularities). It follows then that $\mathcal{M}$, the many-sorted structure of all compact complex manifolds, with predicates given by their analytic subsets, interprets every structure in $\mathcal{A}$. Moreover, these types of quotients are always compact complex analytic spaces, so the compact complex analytic subsets are already imaginary sorts of the compact complex manifolds. As the abscence of imaginaries is a hindrance to model theoretic analysis, one usually wishes to pass to the structure that contains the imaginaries. By our remarks above, $\mathcal{M}^{eq} = \mathcal{A}$, so we are already working in the correct structure. Since $(\mathcal{M}^{eq})^{eq} = \mathcal{M}^{eq}$ for all structures $\mathcal{M}$, we have that:

Proposition 4.6. CCA admits elimination of imaginaries.

We end this section by making some more comments on $\mathbb{P}(\mathbb{C})$. As we have already remarked, $\mathbb{P}(\mathbb{C})$ is a sort in $\mathcal{A}$. By quantifier elimination and Chow's theorem, the definable subsets of $\mathbb{P}(\mathbb{C})$ are the (algebraic)-Zariski constructible sets. As $\dim(\mathbb{P}(\mathbb{C})) = 1$, these are all finite or cofinite, so $\mathbb{P}(\mathbb{C})$ is a strongly minimal set in $\mathcal{A}$. In fact, it is bi-interpretable with the pure field $(\mathbb{C}, +, \cdot)$.

4.3 The trichotomy in CCA

Zilber's principle suggests that we attempt to understand the geometry of the strongly minimal sets of our theory, in particular we should see if the geometry of the non-locally modular strongly minimal sets have the geometry of an algebraically closed field. By quantifier elimination and ω_1-compactness, it can be seen that the compact complex curves are all strongly minimal. Indeed, by definition a compact complex curve X is an irreducible one-dimensional Zariski closed set. Thus, the only Zariski closed subsets of X are finite collections of points, so their Boolean combinations are either finite or cofinite. By the Riemann existence theorem, these are all definably isomorphic to projective algebraic curves. But $\mathcal{A}$ also has examples of strongly minimal sets X for which $\dim(X) > 1$, and which are therefore not coming from $\mathbb{P}(\mathbb{C})$.

For a plethora of such examples, one can look at the *generic tori*. These are n-dimensional tori of the form $T = \mathbb{C}^n/\Lambda$ where Λ is a $2n$-dimensional lattice generated by $a_1 + b_1 i, \ldots, a_{2n} + b_{2n} i$, such that the set $\{a_1, \ldots, a_{2n}, b_1, \ldots, b_{2n}\} \subseteq \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. It can be shown that if $n \geq 2$, these tori contain no proper infinite Zariski closed sets, and are thus strongly minimal. Moreover, they are non-trivial locally modular. In fact, every strongly minimal non-trivial locally modular set in $\mathcal{A}$ is non-orthogonal to such a generic complex torus. Note that this is analogous to Proposition 3.8 in §3.3 for DCF_0, where the generic tori are replaced by $A^{\#}$.

There also exist in CCA trivial strongly minimal sets. As in DCF_0, there are many examples are known, but their classification up to non-orthogonality remains open.

So it remains to study the non locally modular strongly minimal sets in CCA. We have already given an example, the complex field definable on $\mathbb{P}(\mathbb{C})$. In fact, it can be shown that the fields definable in $\mathcal{A}$ are all strongly minimal (and non-locally modular). Indeed, this follows from a very strong classification result:

Theorem 4.7. Any infinite field interpretable in $\mathcal{A}$ is definably isomorphic to the field $\mathbb{C}$ in the sort $\mathbb{P}(\mathbb{C})$.

We try to give a rough idea of why this is true, following Pillay in [14]. Firstly, K can be defined in $\mathcal{A}$ be elimination of imaginaries. It follows from ω-stability of $\mathcal{A}$ that K is algebraically closed (Macintyre's theorem). Now, we can definably endow K with a complex manifold structure so that its field operations are holomorphisms, in particualar, K is an algebraically closed locally compact field, hence is in topological isomorphism with $\mathbb{C}$ by the classification of locally compact fields. Moreover, this isomorphism can be taken to be holomorphic, so that in particular K has complex dimension 1. Then by the Riemann existence theorem it follows that K is interdefinable with a curve in $\mathbb{P}(\mathbb{C})$, which we may take to be $\mathbb{C}$.

The above only considers infinite fields definable in $\mathcal{A}$. To extend this result to infinite fields definable in elementary extensions of $\mathcal{A}$, one relies on the *non-standard Riemann existence theorem* established in Moosa's thesis.

In CCA the Zilber conjecture is also known:

Theorem 4.8. (Pillay [15]) The Zilber conjecture holds in $\mathcal{A}$.

By elimination of imaginaries it follows that in every strongly minimal non-locally modular set D of CCA, an infinite field F is definable. By Theorem 4.7, F is definably isomorphic to $\mathbb{C}$ in $\mathbb{P}(\mathbb{C})$. Since D is non-orthogonal to F by Proposition 2.19 in §2.4, we obtain the following manifestation of the Zilber conjecture in CCA:

Corollary 4.9. If D is a strongly minimal set in $\mathcal{A}$ then either it is locally modular or it is non-orthogonal to $\mathbb{C}$.

It follows from this result that if a strongly minimal set X is such that $\dim(X) > 1$, then X must be locally modular, as it is certainly not a curve up to finitely many points.

This is essentially the same result we saw in $\mathsf{DCF_0}$. But actually in CCA we can do better, essentially because there are no non-algebraic finite covers of algebraic curves. We actually have:

Theorem 4.10. If D is a strongly minimal non-locally modular set in $\mathcal{A}$, then D is a projective curve up to finitely many points.

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