Algebraic Topology knowledge

Betti number

Betti number: the d_{th} Betti number counts the number of d-dimensional holes, It can be used to distinguish between spaces.

β_0 Connected components
β_1 Tunnels
β_2 Voids

Space	β_0	β_1	β_2
Point	1	0	0
Cube	1	0	1
Sphere	1	0	1
Torus	1	2	1

Simplicial complex

Abstract simplicial complex: We call a non-empty family of sets K with a collection of non-empty subsets S an abstract simplicial complex if:
1$\{v\} \in S$ for all $v \in$ K.
2 If $\sigma \in S$ and $\tau \subseteq \sigma$, then $\tau \in$ K.

Simplicial complexes can be decomposed into their skeletons, which only contain simplices of a certain dimension.

Simplices

The elements of a simplicial complex K are called simplices. A k-simplex consists of $k + 1$.

0-simplex: a point

1-simplex: a line

2-simplex: a triangle

3-simplex: a tetrahedron

Group

A group is a set G with a binary operation - that combines two elements to yield another one, such that $(G, \cdot)$ has the following properties:

1 The operation is closed, i.e. $a \cdot b \in G$ for $a, b \in G$.
2 The operation is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in G$.
3 There is an identity element $e \in G$ such that $e \cdot a = a \cdot e$ for $a \in G$.
4 Each $a \in G$ has an inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

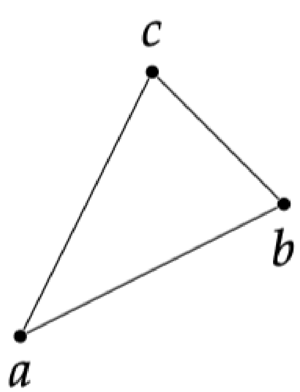
The operation is not required to be commutative. In general, $a \cdot b = b \cdot a$ is not required to hold.

Given a simplicial complex K, the p^{th} **chain group** C_p of K consists of all combinations of p-simplices in the complex. Coefficients are in $\mathbb{Z}_2$, hence all elements of C_p are of the form $\sum_j \sigma_j$, for $\sigma_j \in \mathrm{K}$. The group operation is addition with $\mathbb{Z}_2$ coefficients.

$\mathbb{Z}_2$ is convenient for implementation reasons because addition can be implemented as symmetric difference. Other choices are possible!

We need chain groups to algebraically express the concept of a boundary.

example for chain group

	**Let** $\mathrm{K} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ **Some valid simplicial 1 -chains of** K **are:** - $\{a, b\}$ - $\{a, c\}$ - $\{b, c\}$ - $\{a, b\} + \{a, c\}$ - $\{a, b\} + \{b, c\}$ - $\{a, c\} + \{b, c\}$ - $\{b, c\} +_0 \{a, c\} + \{a, b\}$

Boundary

Boundary homomorphism: Given a simplicial complex K, the p^{th}

boundary homomorphism is a function that assigns each simplex $\sigma = \{v_0, \ldots, v_p\} \in \mathrm{K}$ to its boundary:

$$\partial_p \sigma = \sum_i \{v_0, \ldots, \hat{v}_i, \ldots, v_k\}$$

In the equation above, $\hat{v}_i$ indicates that the set does not contain the i^{th} vertex. e.g. $\partial_1 (\{v_0, v_1\}) = \{v_1\} + \{v_0\}$

The function $\partial_p : C_p \rightarrow C_{p-1}$ is thus a homomorphism between the chain groups.

∂_p is a function that takes each p-simplex in K and assigns it to a combination of its $(p-1)$-dimensional faces, which are the **boundaries** of the p simplex.

also taking the example of triangle

Let $\mathrm{K} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. The boundary of the triangle is non-trivial:

$$\partial_2\{a, b, c\} = \{b, c\} + \{a, c\} + \{a, b\}$$

The boundary of its edges is trivial, though:

$$\partial_1(\{b, c\} + \{a, c\} + \{a, b\}) = \{c\} + \{b\} + \{c\} + \{a\} + \{b\} + \{a\} = 0$$

Chain complex
Chain complex: For all p, we have $\partial_{p-1} \circ \partial_p = 0$: Boundaries do not have a boundary themselves. This leads to the chain complex:

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Kernel
The kernel of a homomorphism $f : A \rightarrow B$ is the set of all elements that are mapped to the zero element, i.e. $\ker f := \{a \in A \mid f(a) = 0\} \subseteq A$.
Kernel means the loss of information or the identity of transformation

Image
The image of f is the set of all its outputs, i.e. $\operatorname{im} f := \{f(a) \mid a \in A\} \subseteq B$.

Cycle group $Z_p = \ker \partial_p$
A cycle is a closed shape, means that when you apply the boundary operation ∂_p, you get zero.
Boundary group $B_p = \operatorname{im} \partial_{p+1}$
The boundaries of $(p+1)$-simplices are exactly the p dimensional "faces" that define the perimeter or surface of the $(p+1)$ simplices.
We have $B_p \subseteq Z_p$ in the group-theoretical sense. In other words, every boundary is also a cycle.

The boundary is something comes from high dimension, The boundary of a p-simplex is formed by its (p-1)-dimensional faces
The cycle is something that is created in same dimension. A cycle is a combination of simplices in the same dimension that together form a closed loop or closed shell

Normal subgroup
Let G be a group and N be a subgroup. N is a normal subgroup if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$.

For an Abelian group, every subgroup is normal

Abelian group: group(associativity, identity, inverse, closure) plus commutative.
Lie group: both group and manifold

Quotient group
Let G be a group and N be a normal subgroup of G. Then the quotient group is defined as $G/N := \{gN \mid g \in G\}$, partitioning G into equivalence classes.

$2\mathbb{Z} \subseteq \mathbb{Z}$ is the subgroup of $\mathbb{Z}$ defined by being a multiple of 2 . Hence, $\mathbb{Z}/2\mathbb{Z}$ consists of only 0 and 1.
$0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$ are two cosets in $\mathbb{Z}$.

Why quotient groups?
Quotient groups 'reduce' a group by partitioning it into equivalence classes that are defined by another subgroup.

Homology group

The p^{th} homology group H_p is a quotient group, defined by 'removing' cycles that are boundaries from a higher dimension:

$$H_p = Z_p / B_p = \ker \partial_p / \operatorname{im} \partial_{p+1},$$

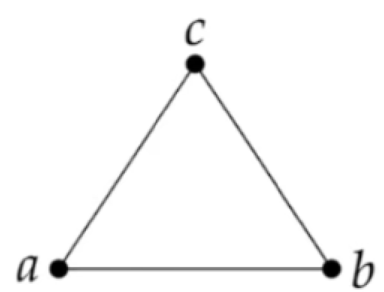
With this definition, we may finally calculate the p^{th} Betti number:

$$\beta_p = \operatorname{rank} H_p$$

Intuition

Calculate all boundaries, remove the boundaries that come from higher-dimensional objects, and count what is left.

Example



$\mathrm{K} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

Notice that K does not contain the 2 -simplex $\{a, b, c\}$, meaning it's the "hollow" triangle with only vertices and edges.

To compute H_0, we need to calculate $Z_0 = \ker \partial_0$ and $B_0 = \operatorname{im} \partial_1$.

Calculating Z_0

We have $Z_0 = \ker \partial_0 = \operatorname{span}(\{a\}, \{b\}, \{c\})$, because each one of these simplices is mapped to zero. Since we cannot express any one of these simplices as a linear combination of the others, we have $Z_0 = (\mathbb{Z}/2\mathbb{Z})^3$,

Calculating B_0

We have $B_0 = \operatorname{im} \partial_1 = \operatorname{span}(\{a\} + \{b\}, \{b\} + \{c\}, \{a\} + \{c\})$. However, since

$\{a\}+\{b\}+\{b\}+\{c\}=\{a\}+\{c\}$, there are only two independent elements, i.e. $\operatorname{im}\partial_1 = \operatorname{span}(\{a\}+\{b\},\{b\}+\{c\})$. Hence, $B_0 = (\mathbb{Z}/2\mathbb{Z})^2$.

By definition, $H_0 = Z_0/B_0 = (\mathbb{Z}/2\mathbb{Z})^3/(\mathbb{Z}/2\mathbb{Z})^2 = \mathbb{Z}/2\mathbb{Z}$.
Hence, $\beta_0 = \operatorname{rank} H_0 = 1$.

Intuition
Our calculation tells us that the simplicial complex has a single connected component!

To compute H_1, we need to calculate $Z_1 = \ker\partial_1$ and $B_1 = \operatorname{im}\partial_2$.

Calculating Z_1
We have $Z_1 = \ker\partial_1 = \operatorname{span}(\{a,b\}+\{b,c\}+\{a,c\})$. This is the only cycle in K; we can verify this by inspection or pure combinatorics. Hence, $Z_1 = \mathbb{Z}/2\mathbb{Z}$.

Calculating B_1
There are no 2-simplices in K, so $B_1 = \operatorname{im}\partial_2 = \{0\}$.

By definition, $H_1 = Z_1/B_1 = (\mathbb{Z}/2\mathbb{Z})/\{0\} = \mathbb{Z}/2\mathbb{Z}$.
Hence, $\beta_1 = \operatorname{rank} H_1 = 1$.

Intuition
Our calculation tells us that the simplicial complex has a single cycle

Smith normal form
Let M be an $n\times m$ matrix with at least one non-zero entry over some field $\mathbb{F}$. There are invertible matrices S and T such that the matrix product SMT has the form

$$SMT = \begin{pmatrix} b_0 & 0 & 0 & & \cdots & 0 \\ 0 & b_1 & 0 & & \cdots & 0 \\ 0 & 0 & \ddots & & & 0 \\ \vdots & & & b_k & & \vdots \\ & & & & 0 & \\ 0 & & & \cdots & & 0 \end{pmatrix},$$

where all the entries b_i satisfy $b_i \geq 1$ and divide each other, i.e. $b_i \mid b_{i+1}$. All b_i are unique up to multiplication by a unit.

Homology calculations in practice
1 Calculate boundary operator matrices.
2 Bring each matrix into Smith normal form (similar to Gaussian elimination).
3 Read off description of p^{th} homology group.

Take-away message
1 Homology groups characterise topological objects.
2 They can be easily expressed as linear operators.
3 The calculation of homology groups boils down to linear algebra.