EMTM 653, Spring 2012: Final Project

(due 5pm on Monday 6/4 – submit via Assignments on Blackboard)

1. An *intensity-based* (or reduced form) model is used to price defaultable bonds. The stochastic intensity process λ is modeled as the exponential of a mean-reverting process X, as described by: $\lambda_t = 0.05\, e^{X_t}$, where

$$dX_t = -aX_t\, dt + \sigma\, dW_t^Q,$$

and W^Q is a standard Brownian motion (in a risk-neutral world Q). Let $X_{n,j}$ denote the values at the node (n, j) on the *trinomial* tree:

$$X_0 = -0.01; \qquad X_{n,j} = X_0 + j\Delta X, \quad j = j_{\min}(n), \cdots, j_{\max}(n).$$

where

$$\Delta X = \sigma\sqrt{3\Delta t}, \qquad \Delta t = 1/250.$$

If the intensity is $\lambda_{n,j} = 0.05\,\exp(X_{n,j})$ at the start of the nth period, then over the time period $(n\Delta t, (n+1)\Delta t)$, the bond defaults with risk-neutral probability $\lambda_{n,j}\Delta t$. If there is no default, the bond pays \$1 at expiration date T, and assume there is no recovery on default. Take the interest rate to be constant $r = 0.04$.

   (a) Write (and turn in) a script to price the defaultable bond at time zero.

   (b) For expirations $T = 1, 2, 3, \cdots, 15$, and parameter values $a = 4, \sigma = 2$, plot the price of the defaultable bond. On the same graph, plot the prices of the equivalent default-free bonds.

   (c) Plot the yield spread curve.

   (d) How does the shape of the yield spread curve vary with the volatility σ? and with the rate of mean-reversion a?

2. Consider a *stochastic volatility* model (under the risk-neutral pricing measure Q) in which volatility σ_t is the exponential of a mean-reverting Ornstein-Uhlenbeck process:

$$\begin{aligned} dS_t &= rS_t\, dt + \sigma_t S_t\, (\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}), \\ \sigma_t &= e^{Y_t} \\ dY_t &= \alpha(m - Y_t)dt + \beta\, dW_t^{(2)}, \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are *independent* Brownian motions. The price of a European call option with strike K and expiration T is given by

$$P_0 = e^{-rT}\mathbb{E}^Q\{(S_T - K)^+\}.$$

   (a) Write a script that uses the Euler discretization of the SDEs

$$\begin{aligned} S_{t_{n+1}} &= S_{t_n}\left(1 + r\Delta t + e^{Y_{t_n}}\sqrt{\Delta t}\,\left(\rho\,\varepsilon_{n+1} + \sqrt{1-\rho^2}\,\eta_{n+1}\right)\right), \\ Y_{t_{n+1}} &= Y_{t_n} + \alpha(m - Y_{t_n})\Delta t + \beta\sqrt{\Delta t}\,\varepsilon_{n+1} \end{aligned}$$

to simulate paths of the process (under the risk-neutral measure) over the time period $0 \le t \le T$, which is discretized into N equal intervals: $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n = n\Delta t$ and $\Delta t = T/N$. Here, the ε_{n+1} and η_{n+1} are independent $\mathcal{N}(0,1)$ random variables.

(b) Using the parameters $r = 0.04, S_0 = 100, Y_0 = \log(0.15), \alpha = 10, \beta = 4, \rho = -0.2, T = 0.5$ and $m = \log 0.15$, use Monte-Carlo simulation to compute the prices of European call options six months from expiry with evenly-spaced strike prices between 90 and 110. Use at least 25,000 paths and 10 strike prices. Compute the implied (Black-Scholes) volatilities and plot them against K. Choose Δt appropriately so that the curve looks relatively smooth.

(c) Repeat for $\rho = 0$ and $\rho = 0.2$ and comment how the implied volatility curve changes with the correlation. What is the economic interpretation?

3. Market options prices on 24 April, 2009 for options on the S&P 500 index (SPX) and Goldman Sachs (GS) are collected from the CBOE website.

As with all data, we need to be careful about errors and missing information in the files. The main way we want to look at the data is by converting option prices into *Black-Scholes implied volatilities*. For European options, like those on the S&P 500 index, this is fairly straightforward (using Matlab's blsimpv command). However all options on named stocks (as opposed to indices) are of American type for which there are no formulas for the option price. For puts and calls when the stock pays a dividend (as GS does), we have to use the binomial tree pricing method.

Download the 4 files from Blackboard: the ones with the .csv extension contain the raw data (put and call options) collected from CBOE (*www.cboe.com*). The .m files contain the call option data after some mild processing to make it readable easily by Matlab. The first file is given as well for the instances where there may be problems in the data. There may be useful additional information that is in the .csv file but not in the other one. The former also contains the ticker symbols for the options which shows which exchange they were traded on.

(a) Convert the S&P 500 European call option prices into implied volatilities. You need to decide how to estimate the appropriate dividend rate and interest rate to use in the Black-Scholes formula. A ballpark estimate of the dividend rate from a broker website should suffice. For the interest rate you might want to find the rate from the appropriate T-bill corresponding to the maturity of the option.

You may also find that options of low or high moneyness K/S are not very liquid and you may decide to restrict your study to options within a range of moneyness.

(b) Having converted to implied vols:

i. How do ATM implied vols differ from the historical volatility of the S&P 500 index at that time? (ATM is the option with strike closest to the current level of the index or stock).

ii. How do implied vols vary with strike? with time to expiration? Is there a skew/smile? What does the term-structure look like?

(c) **American Implied Vol** Write a script to convert American options prices to implied vols? Before trying your procedure on the GS data, generate synthetic American prices using a multi-period binomial tree. Then using a bisection method, see if you recover the volatility you computed this with from some other starting guess.

From the GS options data plot the implied vols as a function of moneyness for the various maturities: is there a skew/smile? Comment on the term-structure.