Proof book

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Lemma 1.1: Equivalent states s_1 and s_2 of machines M_1 and M_2 (machines with same input and output alphabets but different states) go into equivalent states under each input.

Proof:

Suppose for some $s_1'=\delta(s_1,x)$ and $s_2'=\delta(s_2,x)$, we have $\bar{\lambda_1}(s_1',\bar{w})\neq\bar{\lambda_2}(s_2',\bar{w})$ for some \bar{w} . Then, we have found an input sequence $a\bar{w}$ such that $\bar{\lambda_1}(s_1,a\bar{w})\neq\bar{\lambda_2}(s_2,a\bar{w})$ which is a contradiction. \Box

Theorem 1.1: Given a machine M_1 , there is a reduced machine M equivalent to M_1 . Furthermore, if M_2 is any machine equivalent to M_1 , then there exists a state homomorphism which maps M_2 onto M.

Proof:

a. For any unreduced machine M_0 where there exists different but equivalent states s_1 and s_2 , we can group these two states into one new state s such that all inward transitions to s_1 or s_2 now go into s, and all outward transitions from s_1 or s_2 now go out of s. Additionally, s is equivalent to both s_1 and s_2 . This reduces the number of equivalent states (in terms of s_1/s_2) by one and does not introduce new equivalent states. We can repeat this process until every state has no other equivalent states but itself. Hence, by definition of reduced, we have a reduced machine, and it is shown that it must exist for every machine.

b. Since M_2 is equivalent to M, for every state s_2 in S_2 of M_2 , there exists an (and only one as M is reduced) equivalent state s in S of M. Hence, we can define $h:S_2\to S$ if and only if $s_2\equiv s$. This mapping is also onto since equivalence goes both ways. By Lemma 1.1 and $s_2\equiv h(s_2)$ for all s_2 , we have $\delta_2(s_2,a)\equiv \delta(h(s_2),a)$ for all s_2 and s_2 . By the definition of s_2 in s_2 and s_2 in s_2 and s_3 in s_2 in s_3 in s_2 and s_3 in s_3 in s_3 in s_3 in s_4 in s_3 in s_3 in s_4 in s_3 in s_4 in s_4 in s_3 in s_4 i

Corollary 1.1: If two reduced machines are equivalent, then they are isomorphic.

Justification:

Because $rM_1\equiv rM_2$, there is a two-way state homomorphism, which is an isomorphism.

Theorem 1.4: If M' is a realization of M, then $ar{\lambda}(s,ar{x})=\xi[ar{\lambda}'(s',\iota(ar{x}))].$

Proof.

Lemma 1.4.a: $ar{\delta}'[lpha(s),\iota(ar{x})]\subseteqlpha[ar{\delta}(s,ar{x})]$

Proof

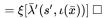
We induce on \bar{x} . The base case holds by definition.

Inductively, we have $\bar{\delta}'[\alpha(s),\iota(\bar{x}')]\subseteq \alpha[\bar{\delta}(s,\bar{x}')]$. Assume $v\in\bar{\delta}'[\alpha(s),\iota(\bar{x}')]$, then $v\in\alpha[\bar{\delta}(s,\bar{x}')]$. Hence, $\delta'[v,\iota(x)]\in\delta'[\alpha[\bar{\delta}(s,\bar{x}')],\iota(x)]\subseteq\alpha[\bar{\delta}(s,\bar{x}')]$. By the property of assginment, $\delta'[\alpha[\bar{\delta}(s,\bar{x}')],\iota(x)]\subseteq\alpha[\bar{\delta}(s,\bar{x}'),x]]=\alpha[\bar{\delta}(s,\bar{x})]$. That is, $v\in\alpha[\bar{\delta}(s,\bar{x})]$, and that $\bar{\delta}'[\alpha(s),\iota(\bar{x}')]\subseteq\alpha[\bar{\delta}(s,\bar{x})]$. Since in general we have $\bar{\delta}'[\alpha(s),\iota(\bar{x})]\subseteq\bar{\delta}'[\alpha(s),\iota(\bar{x}')]$, thus $\bar{\delta}'[\alpha(s),\iota(\bar{x})]\subseteq\alpha[\bar{\delta}(s,\bar{x})]$. \square

Evaluate the left hand side

 $\bar{\lambda}(s,\bar{x})$ [By definition of $\bar{\lambda}$]

- $=\lambda(\bar{\delta}(s,\bar{x}'),x)$ [By property 2 of realization]
- $=\xi[\lambda'[\alpha(\bar{\delta}(s,\bar{x}')),x]]$ [By Lemma 1.4.a]
- $=\xi[\lambda'[\bar{\delta}'[\alpha(s),\iota(\bar{x}')],x]]$ [By definition of $\bar{\lambda}$]



Theorem 1.5: If M and M' are equivalent machines, then M is realized by M' and M' is realized by M.

Proof:

Because $M \equiv M'$, we have isomorphism h such that $h[\delta(s,a)] = \delta'[h(s),a]$ for all s and a. Based on h, we can have $\alpha: S \to \{S'\}$ where all s is mapped to a singleton set containing h(s). Moreover, ι and ξ are identity functions. Therefore, $\alpha[\delta(s,a)] = \{h[\delta(s,a)]\} = \{\delta'[h(s),a]\} \supseteq \{\delta'[h(s),a]\} = \delta'(\alpha(s),a)$. That is, $\alpha[\delta(s,a)] \supseteq \delta'[\alpha(s),\iota(a)]$, so the first property is satisfied. The proof of the second property is trivial. \square