1. Relation and Partitions

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Relation

Definition

A relation between set S and T is a subset of $S \times T$. Whether (s, t) has such "relationship" depends on if the pair is in this subset. Hence we have the definition

$$R = \{(s, t) \mid s\ R\ t\}$$

Properties

A relation can have some properties. For example,

- if $\forall s \in S : s\ R\ s$, then R is *reflexive* (if all elements are related to themselves)
- if $s\ R\ t \Rightarrow t\ R\ s$, then R is *symmetric* (if the relation "goes both ways")
- if $s\ R\ t, t\ R\ u \Rightarrow s\ R\ u$, then R is *transitive* (if the relation propagates)

Equivalence

Equivalence relation

If R is reflexive, symmetric, and transitive, then R is called an equivalence relation.

Equivalence class

An equivalence class of s ($s \in S$) about R (R is on S) is the set in which all elements are equal to s.

$$B_R(s) = \{t \mid s\ R\ t\}$$

where $t \in S$.

Partition

Definition

A partition π of S is the set containing all possible equivalence classes of S about some relation. That is

$$\pi = \{B_\alpha\}$$

(where α is the index), such that

$$\alpha \neq \beta \Rightarrow B_\alpha \cap B_\beta = \emptyset$$
$$\cup \{B_\alpha\} = S$$

In other words, π is an **unambiguous**, **complete** division of S.

Block notations

If s and t are in the same block of π, we denote this as

$$s \equiv t\ (\pi)$$

> Note here $t\ (\pi)$ is not a functional application. The parenthesis is read as "concerns π".

Obviously,

$$s \equiv t\ (\pi) \Leftrightarrow B_\pi(s) = B_\pi(t)$$

Also, if R defines π, then

$$s\ R\ t \Leftrightarrow s \equiv t\ (\pi)$$

That is, if s is R-equivalent to t, they are in the same partition block. Conversely, if s, t are in the same partition block, they must equal under some relation R.

Partition comparison

Definition

We say that $\pi_1 \leq \pi_2$ if and only if for all B_{π_1}, there exists (and can only exists) one B_{π_2} such that $B_{\pi_1} \subseteq B_{\pi_2}$.

Lemma 1.1: $\pi_1 \leq \pi_2$ **if and only if** $B_{\pi_1}(r) \subseteq B_{\pi_2}(r)$ **for all** r

Theorem 1.1: $\pi_1 \leq \pi_2$ **if and only if** $s \equiv t(\pi_1) \Rightarrow s \equiv t(\pi_2)$

Corollary 1.1: $\leq : \pi_1 \rightarrow \pi_2$ **is a surjective function**