Partially Ordered Sets and Lattices

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Partial ordering

Definition

A binary relation R on S is a partial ordering or S if and only if R is reflexive, *antisymmetric*, and transitive.

> Antisymmetry implies that if aRb but $a \neq b$, then $\neg bRa$. It relates to the nature of inequality in that there are two "sides", and an element cannot be on both sides at the same time except for on a certain point (where they are equal).

Notation

A partial ordering on S is represented by

$$(S, \leq)$$

Example 3.1: The set of all partitions of S is a partially ordered set under $\leq$ previously defined in section 2.

Least upper bound

Definition

Let $(S, \leq)$ be a partially ordered set, and T is a subset of S. s is the least upper bound (**l.u.b.**) of T if and only if for all $t \in T$

$$s \geq t$$

and that for all $s' \in S$,

$$s' \geq t \Rightarrow s' \geq s$$

. The first part is related to the fact that s is an upper bound of T. The second part is related to the fact that s is the smallest among any upper bound s'.

Possible non-existence

Note that l.u.b. doesn't necessarily exists. It doesn't exist either in the case where no higher bound exists for t, or the case where the set of higher bounds doesn't contain a minimal element. The same applies to g.l.b..

Lemma 3.1: When l.u.b/g.l.b exists, it is unique.

Greatest lower bound

The greatest lower bound (**g.l.b.**) is defined similarly in that s is the l.u.b. of T if and only if for all $t \in T$

$$s \leq t$$

and that for all $s' \in S$

$$s' \leq t \Rightarrow s' \leq s$$

Lattice

Definition

A lattice is a partially ordered set $L = (S, \leq)$ where every pair of elements has a l.u.b. *and* a g.l.b..

Theorem 3.1: In a lattice, every non-empty subset has a unique l.u.b. and g.l.b.

Example 3.2: The set of all partitions of S is a lattice such that

$$g.l.b.(\pi_1, \pi_2) = \pi_1 \cdot \pi_2$$
$$l.u.b.(\pi_1, \pi_2) = \pi_1 + \pi_2$$

Algebraic definition of lattice

Definition

Let triplet $L = (S, \cdot, +)$ where $[\cdot]$ and $[+]$ are both idempotent

$$x \cdot x = x$$
$$x + x = x$$

, communicative

$$x \cdot y = y \cdot x$$
$$x + y = y + x$$

, associative

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x + (y + z) = (x + y) + z$$

, and absorptive

$$x \cdot (x + y) = x$$
$$x + (x \cdot y) = x$$

.

Next, we define

$$x \leq y \Leftrightarrow x \cdot y = x$$

. Then, $(S, \leq)$ is a lattice and

$$g.l.b.(x, y) = x \cdot y$$
$$l.u.b.(x, y) = x + y$$

Defining a lattice using (two) operators makes it possible to do algebraic manipulation.

Theorem 3.2: In a lattice $x \leq y \Leftrightarrow x + y = y$.

Theorem 3.3: In a lattice $L = (S, \cdot, +)$,

$$\begin{gathered} x_1 \leq x_2 \\ y_1 \leq y_2 \Rightarrow x_1 \cdot y_1 \leq x_2 \cdot y_2 \\ x_1 + y_1 \leq x_2 + y_2 \end{gathered}$$

I and 0

If lattice L is a finite, then

$$l.u.b.(S) = I$$
$$g.l.b.(S) = 0$$

such that for all $s \in S$, $s \cdot I = s$ and $s \cdot 0 = 0$.

Unique existence of I and 0 is a direct corollary of theorem 3.1.

An analogy

Operator $[\cdot]$ is a "refinement/strengthen" operation that make things "finer/stronger", and operator $[+]$ is a "coarsification/weakenment" operation that make things "coarser/weaker".

$x \leq y \Leftrightarrow x \cdot y = x$ can be interpreted as "a weaker cannot help a stronger become stronger".

Similarly, $x \leq y \Leftrightarrow x + y = y$ can be interpreted as "a stronger cannot make a weaker become weaker".

I in a lattice is the weakest/biggest element, while 0 is the strongest/tiniest element.

This analogy is drawn from the examples of partition lattice and set lattice.

Figure 1. An analogous illustration of lattice structures

Sublattice

$L' = (T, \cdot, +)$ is a sublattice of $L = (S, \cdot, +)$ if and only if $T \subseteq S$ and that $x, y \in T$ implies $x \cdot y \in T$ and $x + y \in T$.

Otherwise, T wouldn't contain g.l.b. or l.u.b. for some x and y, making T not a lattice.

Isomorphic lattices

> Isomorphic means "identical except for names".

Two lattices $L_1 = (S_1, \cdot, +)$ and $L_2 = (S_2, \cdot, +)$ are isomorphic if and only if there exists a *one-to-one correspondence* mapping $h : S_1 \to S_2$ such that

$$h(x \cdot y) = h(x) \cdot h(y)$$
$$h(x + y) = h(x) + h(y)$$

Homomorphic lattices

Two lattices $L_1 = (S_1, \cdot, +)$ and $L_2 = (S_2, \cdot, +)$ are homomorphic if and only if there exists an *onto* mapping $h : S_1 \to S_2$ such that

$$h(x \cdot y) = h(x) \cdot h(y)$$
$$h(x + y) = h(x) + h(y)$$

Distributive lattice

Definition

A lattice L is distributive if and only if

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

Example

The lattice $L = (S, \cap, \cup)$ where S is all the subsets of any set is distributive. Proof of this follows from set theory laws.

Theorem 3.4: In a distributive lattice, the two distributive laws are logically equivalent.

Theorem 3.5: In a distributive lattice, for any x, there can exist at most one y such that $x \cdot y = 0$ and $x + y = I$ (y is called the complement of x).