2. Partition Algebra

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Equivalence multiplication

Definition

Given π_1, π_2 on S, $\pi_1 \cdot \pi_2$ defines a new partition (or equivalence relation) on S such that

$$s \equiv t(\pi_1 \cdot \pi_2) \Leftrightarrow s \equiv t(\pi_1) \wedge s \equiv t(\pi_2)$$

To understand this, consider s and t as objects with multiple properties. π_1 and π_2 are equivalence relations concerning two of these properties. Hence, s and t are respectively put into (generally different) partitions. However, if they happen to be in the same partition in both situations, then a new (stricter) equivalence relation emerges (although in general this formulation is guaranteed) by associating π_1 and π_2, which basically says objects s and t are equal if and only if both of these two properties are equal.

Example

If π_1 denotes "people who own the same number of cars", and π_2 denotes "people who own the same number of houses", then $\pi_1 \cdot \pi_2$ would denote "people who own *both* the same number of cars *and* the same number of houses".

Construction

To construct a set in $\pi_1 \cdot \pi_2$ represented by s, we simply join the set represented by s in π_1 and the set represented by s in π_2. Formally,

$$B_{\pi_1 \cdot \pi_2}(s) = B_{\pi_1}(s) \cap B_{\pi_2}(s)$$

Then, $\pi_1 \cdot \pi_2$ are simply $\{\{B_{\pi_1}(t) \cap B_{\pi_2}(t)\} \mid t \in S\}$

Repeated

The notation for repeated multiplication is

$$\pi_1 \cdot \pi_2 \ldots \pi_n = \prod_{i=1}^{n} \pi_i$$

Lemma 2.1: $s \neq t(\pi_1) \vee s \neq t(\pi_2) \Rightarrow s \neq t(\pi_1 \cdot \pi_2)$

Lemma 2.2: $B_{\pi_1 \cdot \pi_2}(s) \subseteq B_{\pi_1}(s), B_{\pi_2}(s)$

Theorem 2.1: $|\pi_1 \cdot \pi_2| \geq |\pi_1|, |\pi_2|$

Theorem 2.2: $\pi_1 \leq \pi_2 \Rightarrow \pi_1 \cdot \pi_2 = \pi_1$

Theorem 2.3: Partition multiplication is associative.

Equivalence addition

Definition

Notation $\pi_1 + \pi_2$ is slightly more involved. It designates a new equivalence relation such that s and t are equivalent if and only if there exists a path linking s and t where each item in the path is associated by the fact that they satisfy either π_1 or π_2. This formulation is necessary since it conforms to transitivity. Formally,

$$s \equiv t(\pi_1 + \pi_2) \Leftrightarrow s = s_0, s_1, s_2, ..., s_n = t$$

where $s_i \equiv s_{i+1}(\pi_1) \vee s_i \equiv s_{i+1}(\pi_2)$ and $0 \leq i \leq n-1$.

Construction

We construct $B_{\pi_1+\pi_2}(s)$ using the fact that $\pi_1 + \pi_2$ is transitive. First, by definition $B_{\pi_1}(s) \cup B_{\pi_2}(s)$ is in $B_{\pi_1+\pi_2}(s)$ since for all $p \in B_{\pi_1}(s)$ and $q \in B_{\pi_2}(s)$, there is a sequence p, s, q where $p \equiv s(\pi_1)$ and $s \equiv q(\pi_2)$. Intuitively this means s is a "bridge" that associate all items in $B_{\pi_1}(s)$ with all items in $B_{\pi_2}(s)$. The rest is to find other "bridges" in order to "annex" more sets into $B_{\pi_1+\pi_2}(s)$. This is done inductively by

$$B_{\pi_1+\pi_2}(s) \cup \{B \mid B \cap B_{\pi_1+\pi_2}(s) \neq \emptyset, B \in \pi_1 \cup \pi_2\}$$

Do this until no more items can be added.

Repeated

$$\pi_1 + \pi_2 + ... + \pi_n = \sum_{i=1}^{n} \pi_i$$

Theorem 2.4: $\pi_1 \leq \pi_2 \Rightarrow \pi_1 + \pi_2 = \pi_2$

Theorem 2.5: Partition addition is associative

Quotient partition

Definition

Let π and τ be paritititions on S, and $\pi \geq \tau$. The quotient partition is defined as $\overline{\pi}$ such that

$$B_\tau(s) \equiv B_\tau(t)(\overline{\pi}) \Leftrightarrow s \equiv t(\pi)$$

It says that any two blocks in τ (smaller partition) will be placed into the same "meta-block" (a block containing blocks) if and only if their elements are equal under π.

If we consider $\tau \leq \pi$ as a function $f : \tau \to \pi$ (followed from definition), then $\overline{\pi}$ represents the structure of f (i.e. what τ maps to the same π).

Properties 2.1: Given three partitions π_1, π_2 and τ such that $\pi_1 \geq \tau$ and $\pi_2 \geq \tau$, we have

$$\pi_1 \geq \pi_2 \Leftrightarrow \overline{\pi_1} \geq \overline{\pi_2}$$

$$\overline{\pi_1 \cdot \pi_2} = \overline{\pi_1} \cdot \overline{\pi_2}$$

$$\overline{\pi_1 + \pi_2} = \overline{\pi_1} + \overline{\pi_2}$$