Regulation, Capital, & Margin: Quant Angle - III

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Part III: IMM Models

Basel 2 and 2.5 Flashback - (1)

- Regulatory capital under Basel 2.5 is loosely broken into three pieces:
  - A general market risk piece (VaR + stressed VaR) and specific risk
  - IRC and CRM
  - Credit risk capital (the IRB and IMM)
- We recall the Basel 2 Internal Ratings-Based (IRB) formula for credit risk regulatory capital:

$$RC = EAD \cdot RW, \qquad EAD = 1.06 \times \max(\alpha \cdot EEPE(1) - CVA, 0),$$

$$RW = l \cdot \left\{ \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho(p)}\Phi^{-1}(0.001)}{\sqrt{1-\rho(p)}} \right) - p \right\} \cdot k(M, p).$$

Basel 2 and 2.5 Flashback - (2)

- Where:
  - EAD: exposure-at-default (a.k.a. loan-equivalent notional);
  - p: 1-year probability of default (PD);
  - l: loss-given-default percentage (LGD);
  - M: effective maturity;
  - $\rho(p) = 0.24 - 0.12(1 - e^{-50p})$;
  - k: function to incorporate transition risk.
- IMM (Internal Models Methodology) is the regulatory framework for computing EAD and M.
- A model for joint movements of all financial variables "in the world" are needed to compute expected exposure (EE) profiles, EPE, EEPE, and thereby EAD.

Choice of Measure - (1)

- IMM models have to be validated by model validation and by regulators (FRB, OCC, PRA,..)
- The model choice and parameterization is, rightfully, a key area of focus in IMM examinations
- In theory, all simulations should be set in the actual (aka historical, aka real-life, aka statistical) probability measure.
- Yet regulators allow for leeway:

> *"In theory, the expectations should be taken with respect to the actual probability distribution of future exposure and not the risk-neutral one. Supervisors recognize that practical considerations may make it more feasible to use the risk-neutral one. As a result, supervisors will not mandate which kind of forecasting distribution to employ."*

Choice of Measure - (2)

- This leeway can be very substantial (probably more than regulators intended), since there is no such thing as a single risk-neutral measure – there is one for each choice of numeraire asset.
- In rates space, if the numeraire is picked to be, say, a long-dated bond, the drift can be pushed towards $-\infty$!
- All in all, drift terms of stochastic processes used for IMM should be set "pragmatically". For example, it is not uncommon to set $\mu(t, T) = 0$: "roll up forward curve".
- That said, European regulators have recently started being prescriptive about which drifts to use. Some of the specifications are on shaky ground, but should be kept in mind.

Choice of Measure - (3)

- Notice that for CVA applications, we are trying to price credit exposure so here there is no ambiguity
- For instance, let $\mathbb{Q}$ be the measure corresponding to money-market account numeraire $\beta(t)$; and $\mathbb{Q}^*$ be the measure corresponding to a discount bond $P(t, T)$ numeraire (T is bond maturity).
- Then (assuming zero collateral)

$$PV(t) = \mathrm{E}^{\mathbb{Q}} \left(V(t)^+ \frac{1}{\beta(t)} \right) = \mathrm{E}^{\mathbb{Q}^*} \left(V(t)^+ \frac{P(0, T^*)}{P(t, T^*)} \right).$$

- On the other hand, when computing expected exposure (rather than *present value* of exposure)

$$\mathrm{E}^{\mathbb{Q}} \left(V(t)^+ \right) \neq \mathrm{E}^{\mathbb{Q}^*} \left(V(t)^+ \right).$$

Choice of Measure - (4)

- The drift ambiguity, and the use of expected exposure without discounting, are flaws in IMM.
- In addition, there are several securities (e.g. compounding trades) where $\mathrm{E}^{\mathbb{Q}}(V(t)^{+})$ can be extremely large (overflow), whereas $PV(t)$ is perfectly stable.
- Ideally, one should have defined $EE(t) = PV(t)/P(0,t)$.

Choice of Models: Rates - (1)

- There are many models available for interest rate modeling: BGM, HJM, short-rate, affine, Markov functional,...
- For CVA and IMM purposes, it is convenient to have a model where
$$P(t, T) = F(t, T, x(t))$$
where $x(t)$ is a low-dimensional Markovian state variable vector, and $F(\cdot)$ is a deterministic *reconstitution* formula.
- Many popular models (HJM, BGM,..) do not, in general, satisfy this form, but require one to keep track of every point of the yield curve.
- The class of *Quasi-Gaussian models* fits the bill.
- Background reading exposes you to a simple Gaussian two-factor version of this model.

of Rates Factors - (1)

- In reality, one often wants more than two factors, and also wants to use more sophisticated dynamics.
- We often need more factors to ensure that risk is not being hidden by overly simplistic moves of yield curve. 4 factors is common.
- A quasi-Gaussian model with many factors uses

$$df(t, T) = \sigma_f(t, T)^\top \int_0^T \sigma_f(t, u)\, du + \sigma_f(t, T)^\top\, dW(t)$$

where $W(t)$ is an N-dimensional BM, and

$$\sigma_f(t, T) = a(t)b(T)$$

where $a(t)$ is a $N \times N$ (possibly stochastic) matrix and $b(T)$ is a deterministic $N \times 1$ vector.

of Rates Factors - (2)

- We get, following the same steps as in your reading material,

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-G(t,T)^\top x(t) - \frac{1}{2} G(t,T)^\top y(t) G(t,T)\right)$$
$$f(t,T) = f(0,T) + \mathbf{1}^\top H(T) H(t)^{-1} \left(x(t) + y(t) G(t,T)\right)$$

where

$$dx(t) = \left(y(t)\mathbf{1} - \kappa(t)x(t)\right)\, dt + \sigma_r(t)^\top dW(t), \quad x(0) = 0,$$
$$dy(t) = \left(\sigma_r(t)^\top \sigma_r(t) - \kappa(t)y(t) - y(t)\kappa(t)\right)\, dt, \quad y(0) = 0.$$

- Here $x(t)$ is a N-dimensional vector, $y(t)$ is a $N \times N$ matrix, $\mathbf{1}$ is a vector of N 1's, and, with $H(t) = \mathrm{diag}(b(t))$,

$$\kappa(t) = -\frac{dH(t)}{dt} H(t)^{-1}, \;\; G(t,T) = \int_t^T H(u) H(t)^{-1} \mathbf{1} du, \;\; \sigma_r(t) = a(t) H(t).$$

of Rates Factors - (3)

- As $y(t)$ is symmetric, we have a total of $N + N(N+1)/2$ different variables in the system for x, y. The y's are all locally deterministic.
- If we add M stochastic volatility drivers (see below), we have $M + N + N(N+1)/2$ variables.

Non-Gaussian Rates Model - (1)

- In our exercise, we used Gaussian rates, which in the previous machinery corresponds to using deterministic $a(t)$.
- In reality, Gaussian rates are not ideal – even though it is not inconceivable that rates can be (a little) negative.
- We therefore would pick $a(t)$ to be non-deterministic. To get a Markov system, we could set $a(t) = g(t, x(t), y(t))$. But how do we do this in a meaningful way?
- A useful idea (which we exploited to some extent in the background reading note) involves picking some concrete forward rates $f(t, t+\delta_i)$ for N different δ_i.
- It is easy to see that

$$df(t, t+\delta_i) = O(dt) + \mathbf{1}^\top H(t+\delta_i) H(t)^{-1} \sigma_r(t)^\top dW(t), \quad i = 1, \ldots, N.$$

Non-Gaussian Rates Model - (2)

- We may want, say, a fancy displaced stochastic volatility model where

$$df(t, t+\delta_i) = O(dt) + \sqrt{z(t)}\left(\alpha_i(t) + \beta_i(t) f(t, t+\delta_i)\right) dU_i(t)$$

  where $U_i(t)$ is a set of correlated Brownian motions and $z(t)$ is a stochastic volatility process.
- This can be accomplished by setting $\sigma_r(t) = \sqrt{z(t)} c(t, \{f(t, t+\delta_i)\}_i)$ for a function c to be determined by some matrix algebra (see Andersen/Piterbarg for the details).
- Calibration of these models are challenging, but doable – again, see Andersen/Piterbarg.

Non-Gaussian Rates Model - (3)

- In CVA systems, we normally do not activate stochastic volatility, but we do work with skew functions (local volatility). $\alpha, \beta \neq 0$.
- In IMM systems, regulators have recently started to require stochastic volatility for some asset classes.
- The model above is for interest rates. For credit intensity processes, normally log-normal type specifications are used for the intensity. There is limited information about skews in credit spread options.
- The number of credit factors is kept low (normally one or two factors per credit). Sometimes jumps are added – we'll discuss this later.

Multiple Currencies - (1)

- So far, we have only looked at interest rates in a single currency. Most banks, however, transact in multiple currencies.
- This adds no particular complications for portfolios that are denominated in a single currency – just build a bunch of separate models for interest rates in various currencies.
- The complication arises when a single counterparty has transacted in interest rate securities in multiple currencies.
- The models then need to be tied together by the FX rate dynamics.
- To see how this can work, set $X(t)$ be the domestic/foreign FX rate. Assume that its dynamics in the domestic risk-neutral measure is

$$dX(t)/X(t) = (r_d(t) - r_f(t))\, dt + \sigma_X(t)\, dW_X(t),$$

where r_d and r_f are domestic and foreign short rates, respectively.

Multiple Currencies - (2)

- For notational simplicity, let us just use one-factor models for each rate. Then

$$dx_d(t) = (y_d(t) - \kappa_d(t)x_d(t))\ dt + \sigma_r^d(t)dW_d(t)$$
$$dx_f(t) = (y_f(t) - \kappa_f(t)x_f(t))\ dt + \sigma_r^f(t)\left(dW_f(t) - \rho_{X,f}\sigma_X(t)\,\sigma_r^f(t)\,dt\right)$$

- The domestic forward rates are unchanged, but the foreign forward rate dynamics needs to be “quanto” adjusted due to the shift into domestic risk neutral measure.
- Also notice that we need three correlations: $\mathrm{corr}(W_X, W_d), \mathrm{corr}(W_X, W_f)$, $\mathrm{corr}(W_f, W_d)$.
- The implied volatility of the FX rate gets impacted by stochastic rates; needs to be considered in calibration.

Models Other Asset Classes

- So far, we have only looked at interest rate portfolios.
- In most banks, swap (and swap-like) instruments generate 80-90% of the CVA and capital.
- Nevertheless, many other asset classes (equity, commodities, credit, FX,..) exist and need to be accounted for when computing counterparty portfolio exposure.
- We discussed adding FX above. Equity and commodities can be added roughly the same way; there is need to maintain a **lot** of correlations.
- Adding credit derivatives (e.g., CDS) is a special challenge, since we need to model the "credit" correlation between the reference firms underlying the instruments AND the bank AND its counterparty.

Simulations of Joint Default Risk - (1)

- This can get complex, as CDO markets suggest that default time correlations, due to contagion fears, are much stronger than what one can obtain with diffusive processes for default intensities.
- One way to build in stronger default time correlation is to use processes where spreads can jump up. If two spread processes can jump together, this will mimic contagion phenomena.
- Specifically, a common choice is (for firm j)

$$\lambda_j(t) = \lambda_{j,f}(0,t) + z_j(t) + \sum_{k=1}^{K} a_{jk} y_k(t)$$

where z_j is affine diffusive

$$dz_j(t) = (\theta_j(t) - \kappa_j z_j(t))\, dt + \sigma_j(t)\sqrt{\beta_j(t) + \gamma_j(t) z_j(t)}\, dW_j(t).$$

Simulations of Joint Default Risk - (2)

- And where the y_k are mean reverting compound Poisson jump processes

$$dy_k(t) = (\omega_k(t) - q_k y_k(t))\ dt + v_k dN_k(t),$$

with the v_k being the distribution of the kth jump magnitude (often exponential distributions are used).

- The bank can here be $j = 1$, the counterparty $j = 2$, and all the reference names for the credit derivatives $j \geq 3$.
- The credit correlation structure in the resulting model is a complicated function of all variables in the model. A decent-sized impact comes from the correlations between the W_j, but the jump structure is more important.

Simulations of Joint Default Risk - (3)

- In particular, if two firms tend to share jumps (especially big jumps), their correlation will be high.
- The affine nature of the model above makes survival probability computation analytically tractable, but rather complicated (Riccati equations).
- Calibration is also complex; normally FTD contracts are used to gauge correlation levels. But setting the a_{jk} (and all the other jump parameters) is not easy – combinatorics can explode for large number of firms.
- Simulation of the model is also non-trivial. In CVA applications we normally don't want to explicitly simulate default, but rather want to work with hazard-rate discount operations. This gets convoluted with many hazard rate processes.

Simulations of Joint Default Risk - (4)

- Let us take a look at this. Consider a portfolio $V(t)$ of CDS-type contracts with underlying reference firms $j = 3, 4, \ldots, J$. The bilateral CVA is, for a T-maturity trade,

$$CVA = \mathrm{E}\left(e^{-\int_0^{\tau_2} r(u)\,du} 1_{\tau_1 > \tau_2} 1_{\tau_2 < T} (1 - R_2) V(\tau_2)^+\right).$$

$$V(\tau_2) = \sum_{j=3}^{J} 1_{\tau_j > \tau_2} U_j(\tau_2)$$

  where U_j is the value of the jth CDS.
- Rather than simulating all indicator functions, in the Cox process setting, we can condition on the paths of z and y processes, to rewrite

$$CVA = (1 - R_2)\mathrm{E}\left(\int_0^T e^{-\int_0^u r(u)\,du} \lambda_2(u) e^{-\int_0^u \lambda_2(u)\,du} e^{-\int_0^u \lambda_1(u)\,du}\, V(u)^+\, du\right).$$

Simulations of Joint Default Risk - (5)

- Now, we only need to simulate defaults for $j \geq 3$.
- We can also write

$$CVA = (1 - R_2)\mathrm{E}\left(\int_0^T e^{-\int_0^u r(u)\,du} \lambda_2(u) e^{-\int_0^u \lambda_2(u)\,du} e^{-\int_0^u \lambda_1(u)\,du} H(u)\,du\right)$$

where

$$H(u) = \mathrm{E}\left(\left(\sum_{j=3}^{J} 1_{\tau_j > u} U_j(u)\right)^+\right).$$

- Fourier transform methods can be used to establish $H(u)$.
- Phew!

Simulations of Joint Default Risk - (6)

- Another situation where jump processes may be necessary are when the counterparty is a sovereign. In that case, one would expect FX rates to jump at the time of counterparty default, which will have a strong impact on exposures.
- FX-credit correlation is sometimes visible in "quanto CDS" markets.

Credit Mitigants - (1)

- In simulating exposure, it is important to take into account collateral agreements.
- A prototypical collateral agreement will typical use *thresholds* on the collateral:

$$C(t) = 1_{V(t)>0}\left(V(t) - H_C(t)\right)^+ - 1_{V(t)<0}\left(-V(t) - H_B(t)\right)^+$$

  where H_B and H_C are the thresholds for posting by banks B and counterparty C, respectively.
- There can also be thresholds ("Minimum Transfer Amount" or MTA) on incremental postings.
- As we shall discuss later, there can also be *Initial Margin* IM (in addition to ordinary Variation Margin, as above).

Credit Mitigants - (2)

- Sometimes collateral terms are dependent on the public credit rating of the bank and/or the counterparty.
- Idea is tighten credit terms if the rating deteriorates – e.g., by lowering thresholds, lowering MTA, increasing IM.
- Sometimes there are also termination requirements: if the rating slips below a certain grade, the portfolio will be liquidated outright.
- In your assignment, the public rating is assumed to be a function of the credit intensity process: high intensity is mapped to a low rating; low intensity is mapped to a high rating.
- A little simplistic, but good enough for our purposes.
- A more sophisticated model might use a Markov chain for the rating.