Introduction to Machine Learning

Lecture 17&18: Elementary Reinforcement Learning

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- Learning Scenarios
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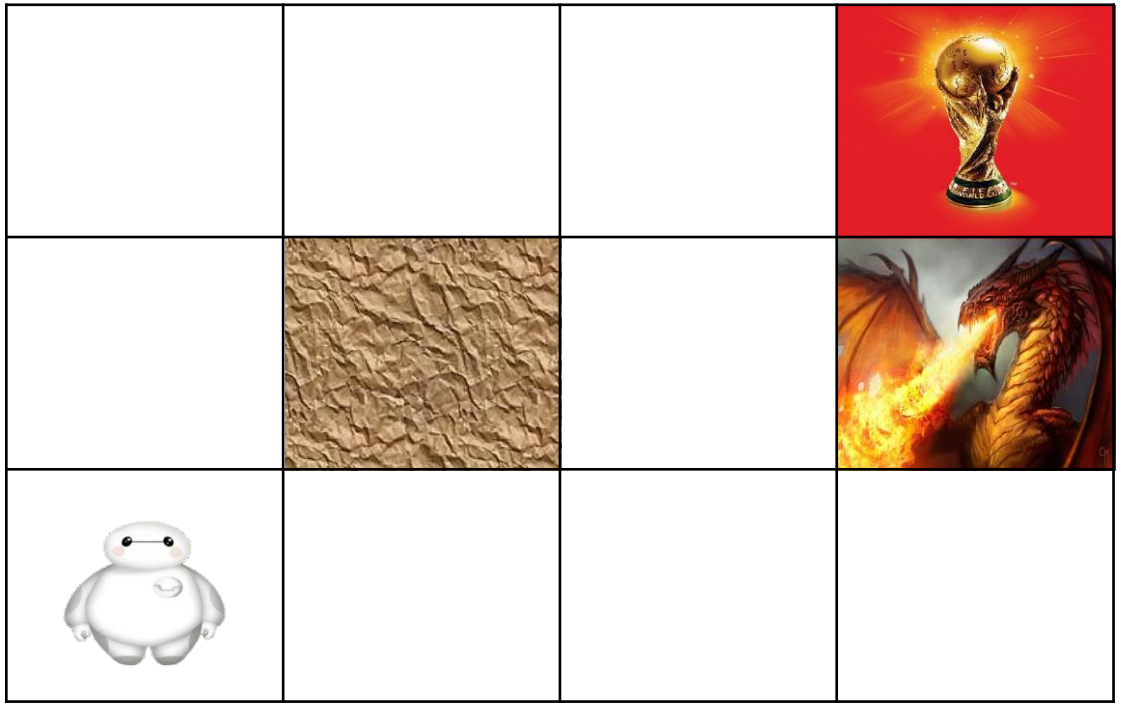
- Planning Algorithms
- Learning Algorithms

Learning Scenarios

Grid World



Snooker


Ladbrokes
Ladbrokes
Ladbrokes
Ladbrokes
Ladbrokes
优酷
HAWKINS
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25
DING
FRI
2016斯诺克巡回赛总决赛第四天五佳球
TV
丁俊晖 5

Three Kingdoms

Agent

Environment

Three Kingdoms

Environment

Agent

S_t

R_t

Three Kingdoms

$\pi(A_t|S_t) \rightarrow A_t$

Environment

Agent

S_t

R_t

Three Kingdoms

$\pi(A_t|S_t) \rightarrow A_t$

Environment

Agent

$P(S_{t+1}, R_{t+1}|S_t, A_t)$

↓

S_{t+1}, R_{t+1}

R_{t+1}

S_{t+1}

Three Kingdoms

$\pi(A_t|S_t) \rightarrow A_t$

Environment

Agent

S_t

R_t

R_{t+1}

S_{t+1}

$P(S_{t+1}, R_{t+1} | S_t, A_t)$

↓

S_{t+1}, R_{t+1}

Agent & Environment

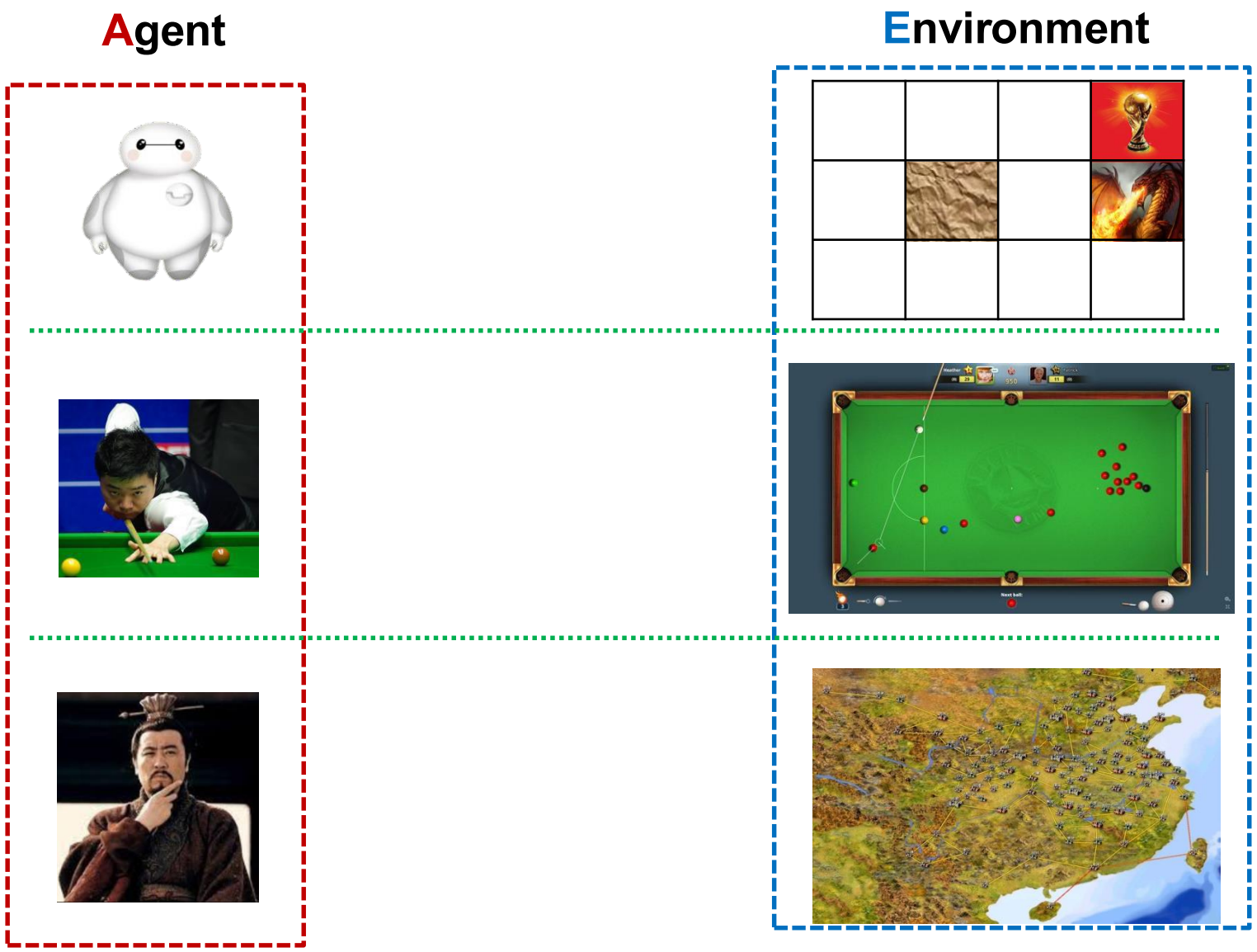
Agent & Environment

Agent

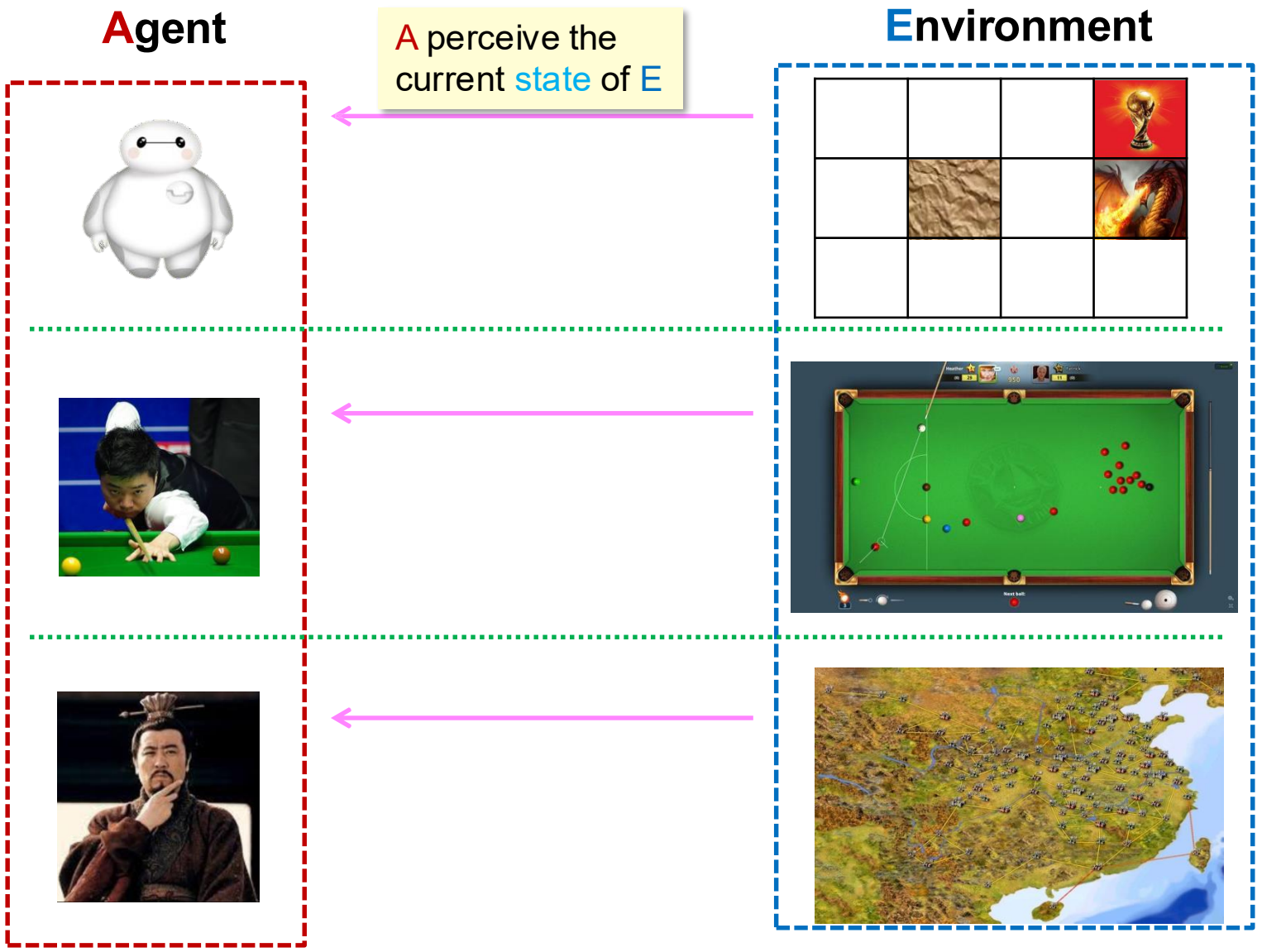
Agent & Environment

Agent

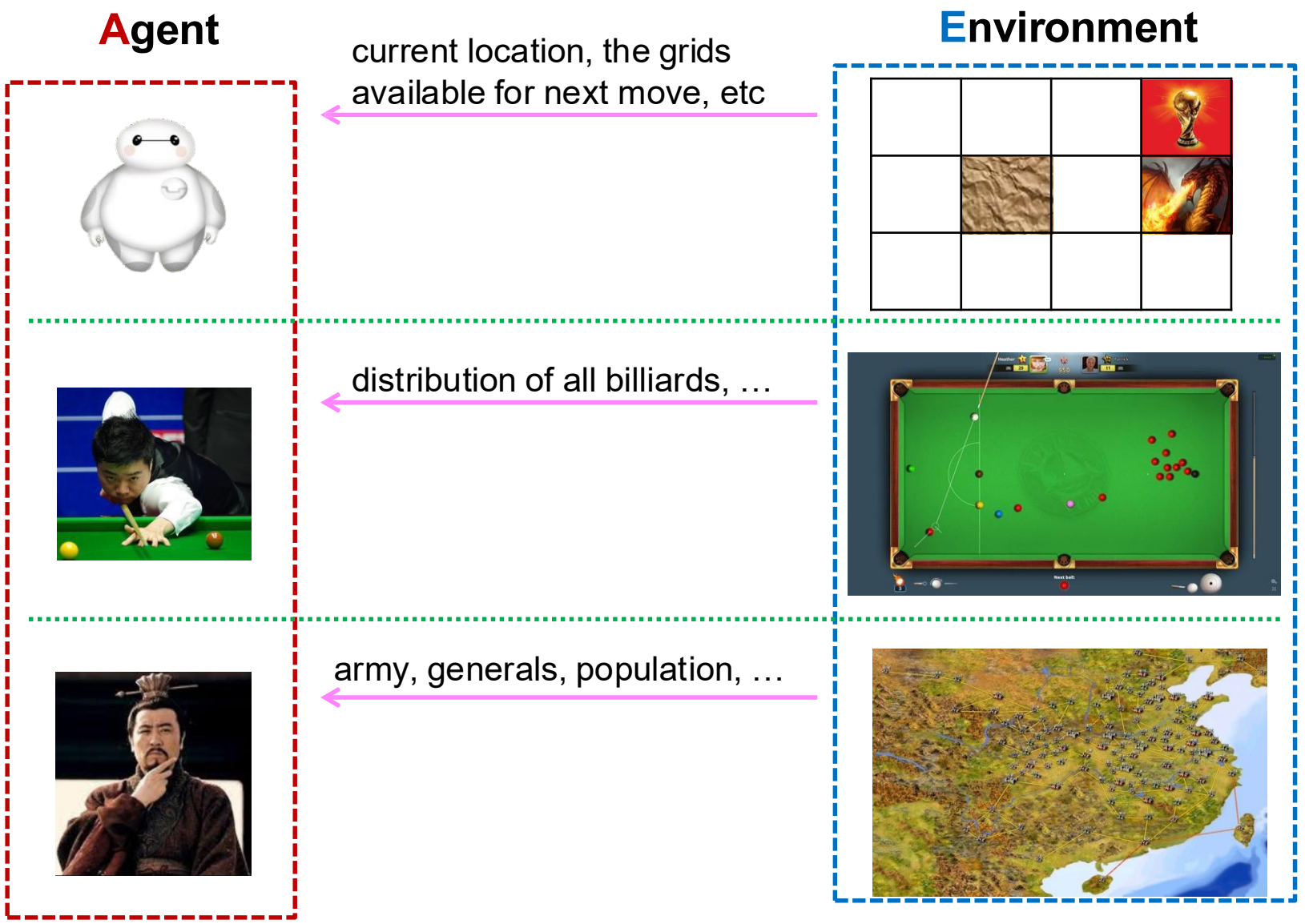
Agent & Environment



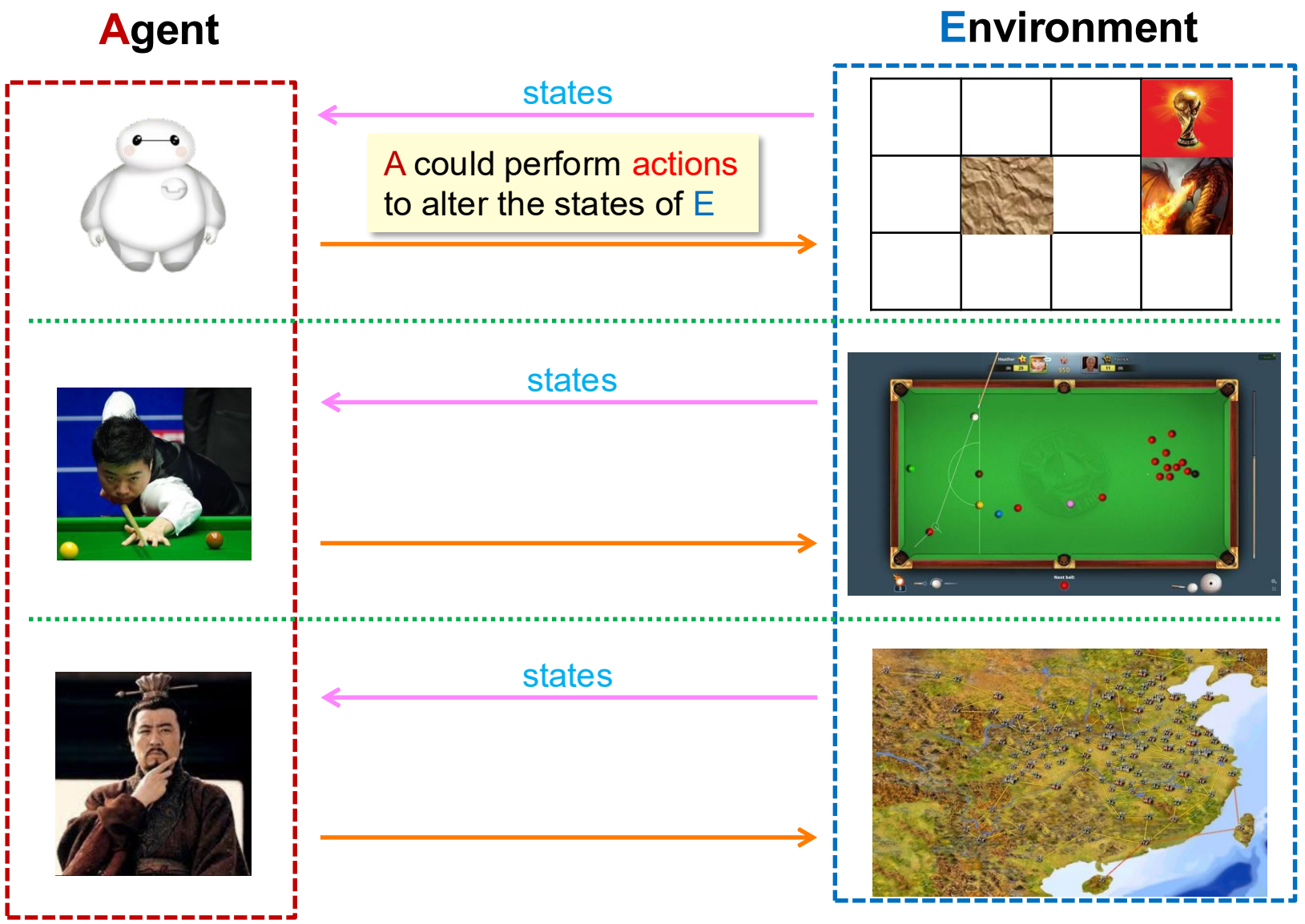
Interactions between Agent & Environment



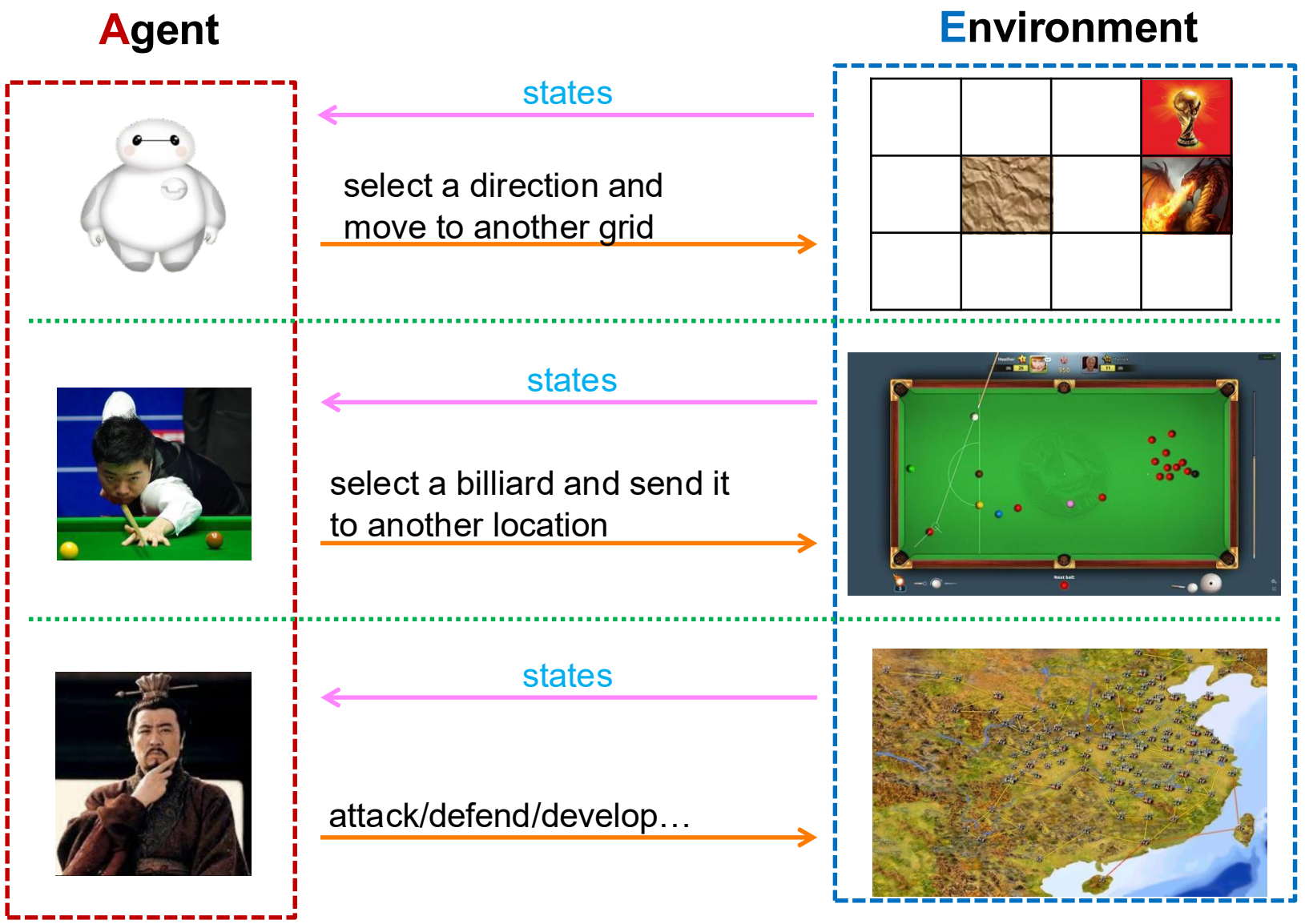
Interactions between Agent & Environment



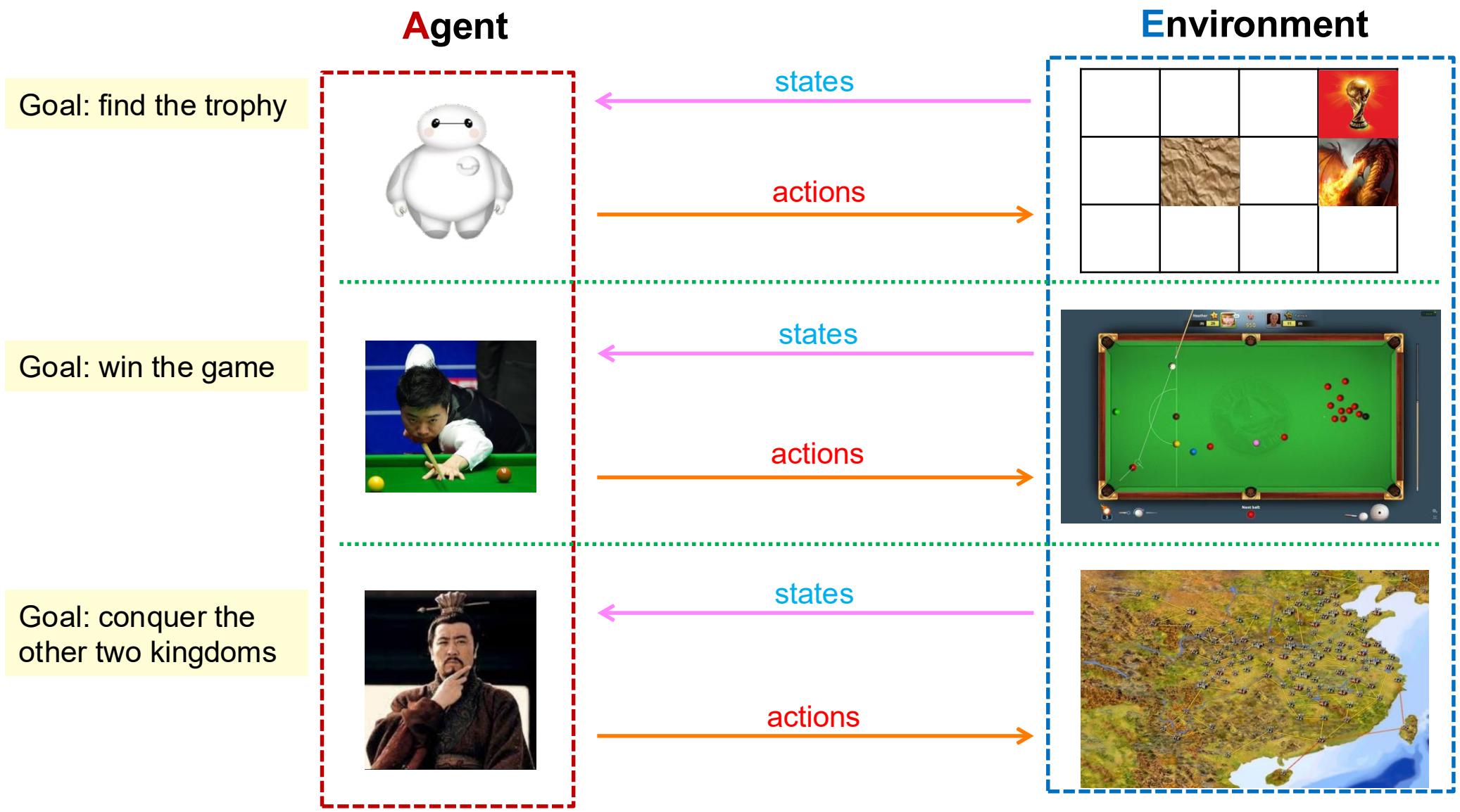
Interactions between Agent & Environment



Interactions between Agent & Environment



Goal State of the Agent



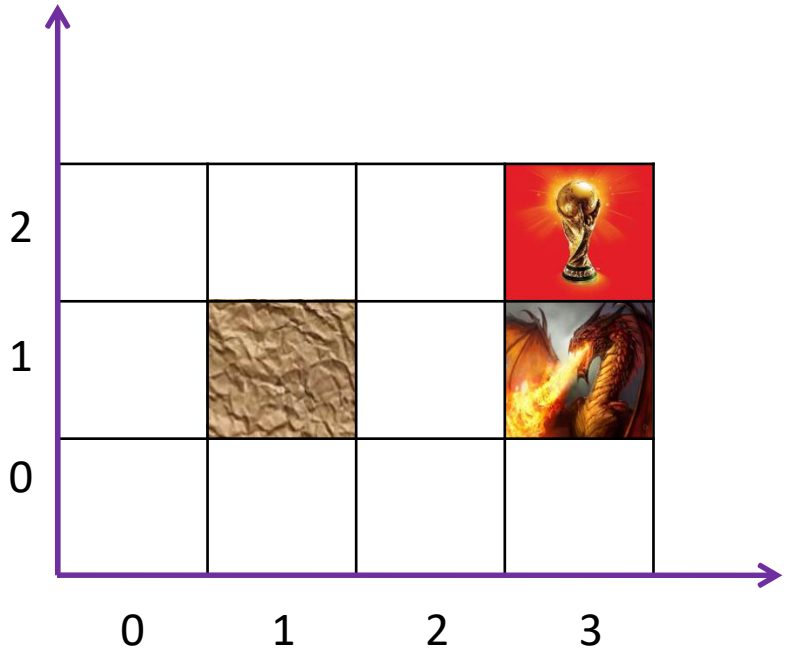
Markov Decision Process

Agent & Environment

- The system consists of an agent (may be more) and an environment, interacting with each other.

States

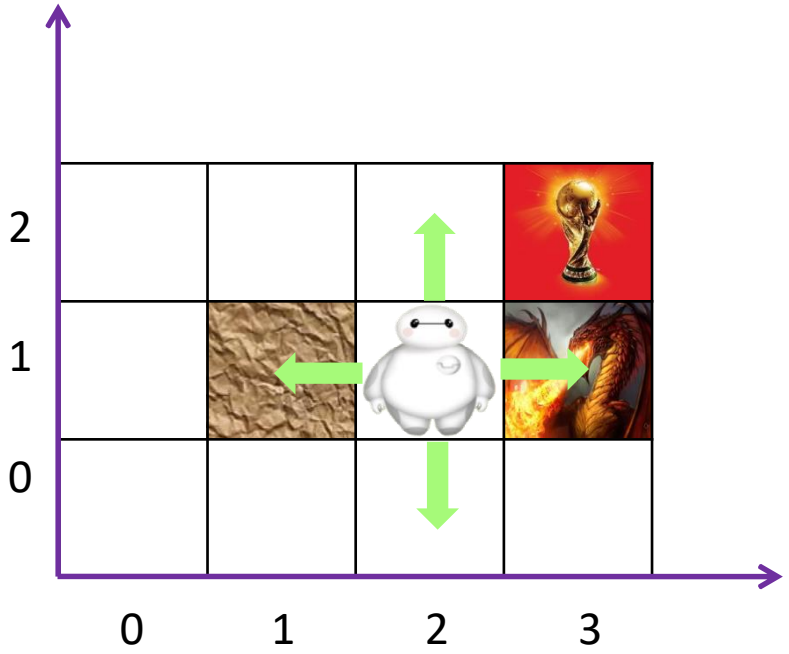
- From the perspective of the agent, the environment is described by a set of states.



States: $\mathcal{S} = \{(i, j) : i = 0, 1, 2, 3, j = 0, 1, 2\}$

Actions

- At each state, the agent can pick and perform certain action to alter the state.



$$\delta((2,1),\text{up}) = (2,2)$$

$$\delta((2,1),\text{down}) = (2,0)$$

$$\delta((2,1),\text{left}) = (2,1)$$

$$\delta((2,1),\text{right}) = (3,1)$$

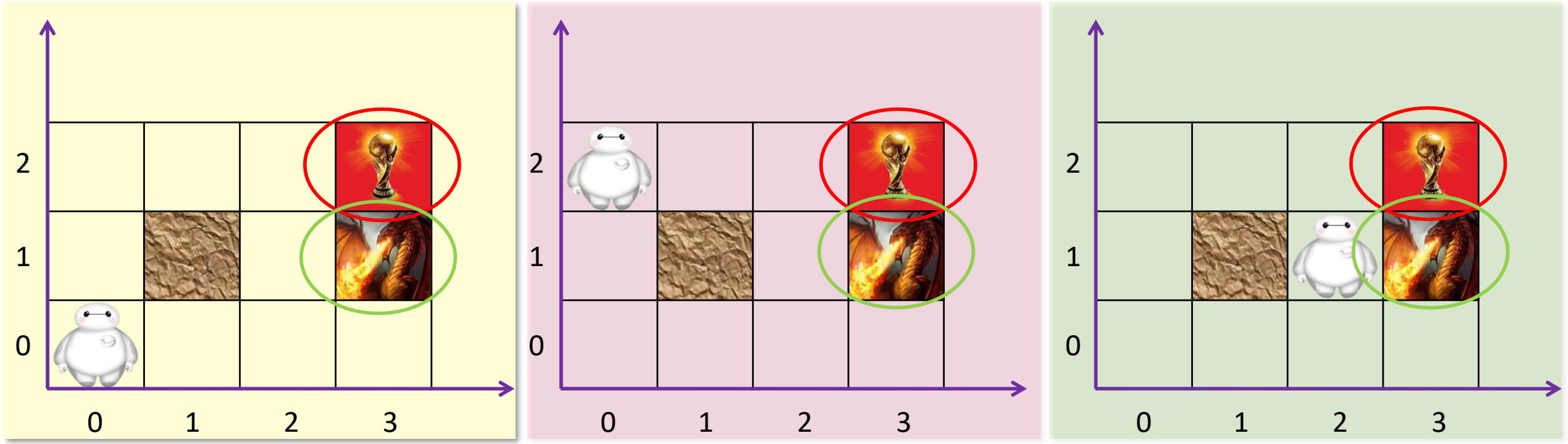
$\delta : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$: state transition function

deterministic environment

Action space: $\mathcal{A} = \{\text{up}, \text{down}, \text{left}, \text{right}\}$

Goal State

- No matter starting from which state, the agent would like to achieve certain goal state.



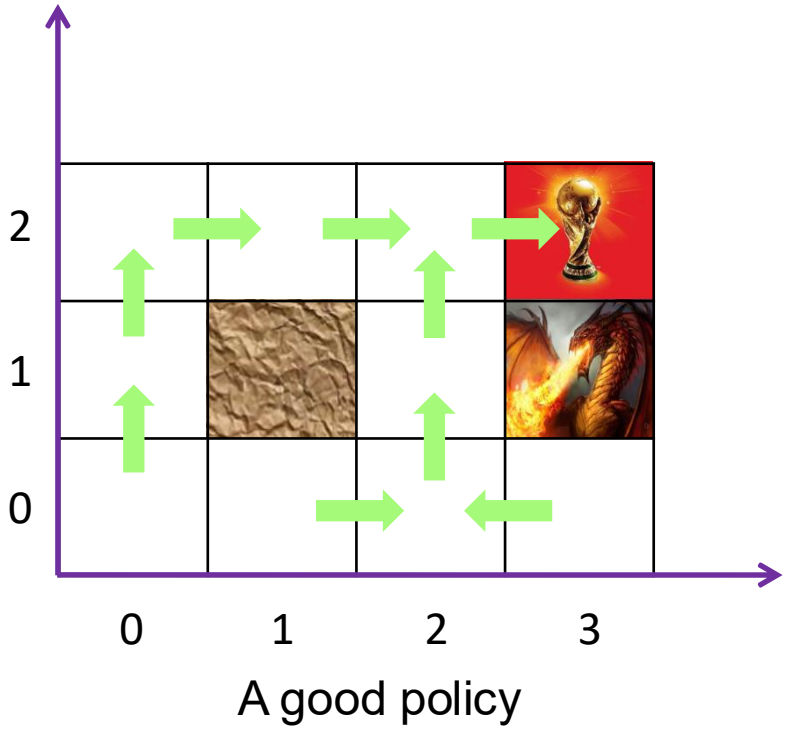
The game will terminate if the agent arrives at $(3, 2)$ (win) or $(3, 1)$ (lose).

The states $(3, 2)$ and $(3, 1)$ are also called absorbing states

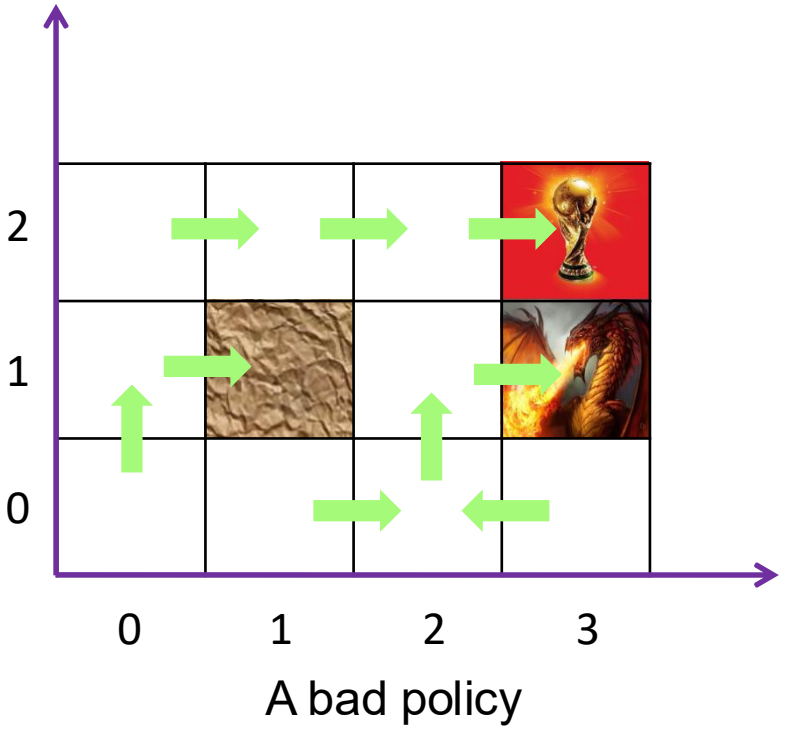
In some cases, there is NO goal state.

Policy

- To achieve the goal state, the agent needs to pick and perform a sequence of actions according to the observed states.



A good policy

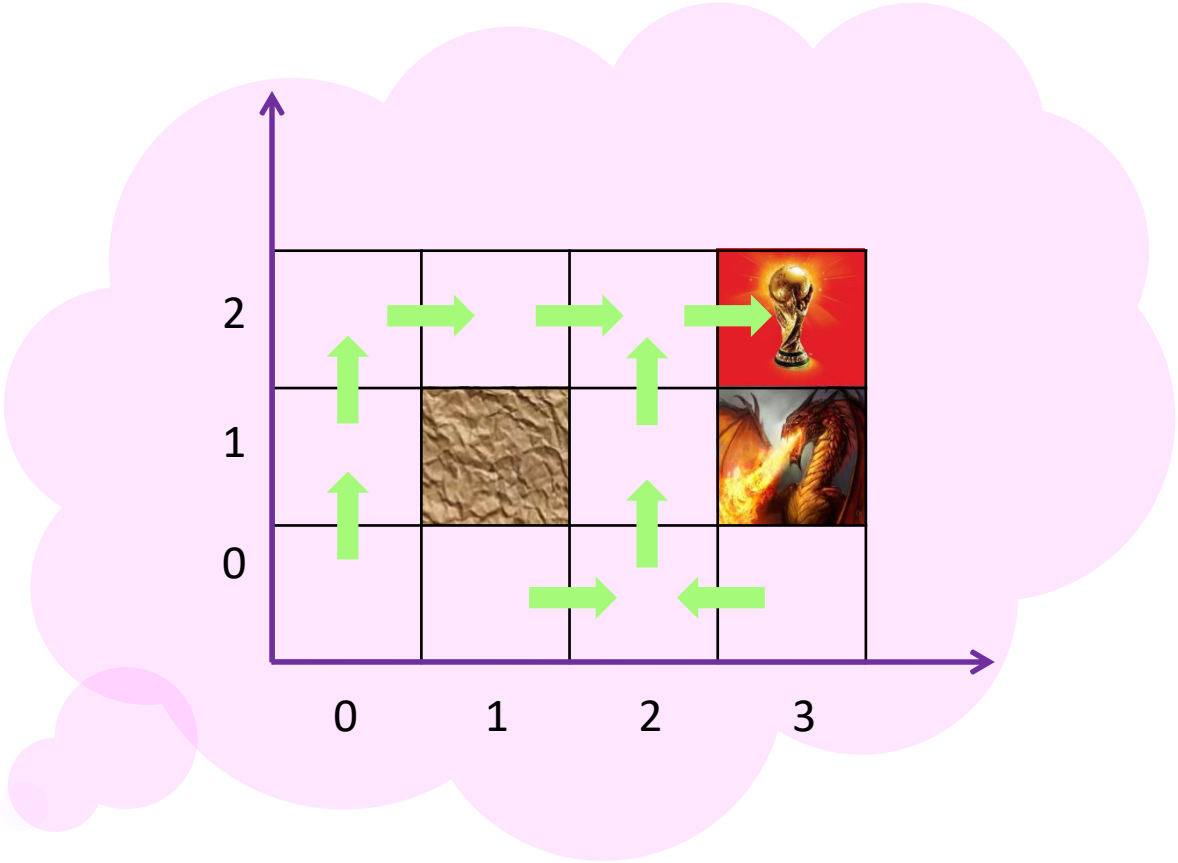


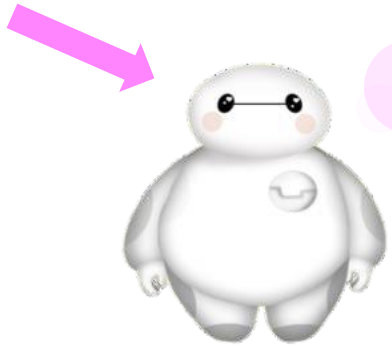
A bad policy

Policy: $\pi : \mathcal{S} \rightarrow \mathcal{A}$

The Learning Task

- Find a policy that can direct the agent to its goal state no matter which state the agent would have been at the very first beginning.





The Learning Task

How can we find a desired policy to direct the agent's move?

Reward

- We assume that the goal of the agent can be encoded by a reward function

$$r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$$

The reward function is not always available. For some applications, you need to define it properly.

- Starting from an arbitrary state, the desired policy would pick for the agent the actions that maximize the reward accumulated over time.

Looking for a policy that would pick for the agent the actions to achieve its goals

starting from an arbitrary state

Looking for a policy to pick for the agent the actions that will maximize the accumulated reward over time

Reward

- We assume that the goals of the agent can be encoded by a reward function

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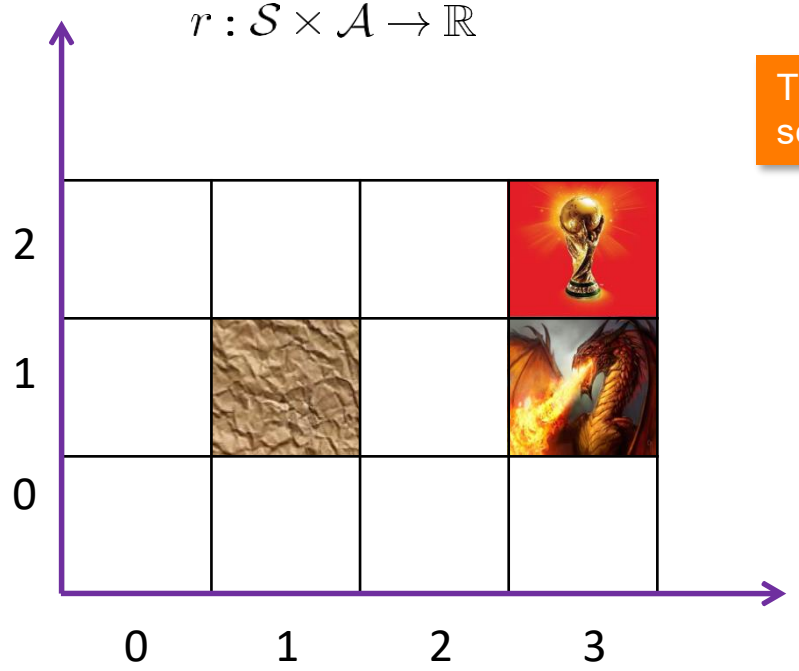


The reward function is not always available. For some applications, you need to define it properly.

$$r(s,a) = \begin{cases} 100, & \text{if } \delta(s,a) = (3,2) \\ -100, & \text{if } \delta(s,a) = (3,1) \\ 0, & \text{otherwise} \end{cases}$$

Markov Decision Process (MDP)

- Indeed, we have already introduced the so-called MDP, which is defined (rigorously) by
  - a set of states $\mathcal{S}$, possibly infinite
  - a set of actions $\mathcal{A}$, possibly infinite
  - an initial state $s_0 \in \mathcal{S}$
  - a transition probability $\Pr[s'|s,a]$: distribution over destination states $s' = \delta(s,a)$
  - a reward probability $\Pr[r|s,a]$: distribution over rewards $r' = r(s,a)$

MRT Chapter 14

- This model is Markovian because the transition and reward probabilities only depend on the current state and the action picked and performed at the current state, instead of the previous sequence of states and actions performed.

$$\Pr[S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}, \ldots, S_0 = s_0, A_0 = a_0] = \Pr[S_{t+1} = s' | S_t = s_t, A_t = a_t]$$

$$\Pr[R_{t+1} = r | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}, \ldots, S_0 = s_0, A_0 = a_0] = \Pr[R_{t+1} = r | S_t = s_t, A_t = a_t]$$

- In this lecture, we assume that
  - the states and the actions are finite
  - the environment is **deterministic**, i.e., the destination state and the reward are completely determined by the current state and the action performed at the current state

The Optimal Policy

Under a MDP, we shall look for the (optimal) policy that leads to the greatest (expected) accumulated reward no matter which state the agent begins with.

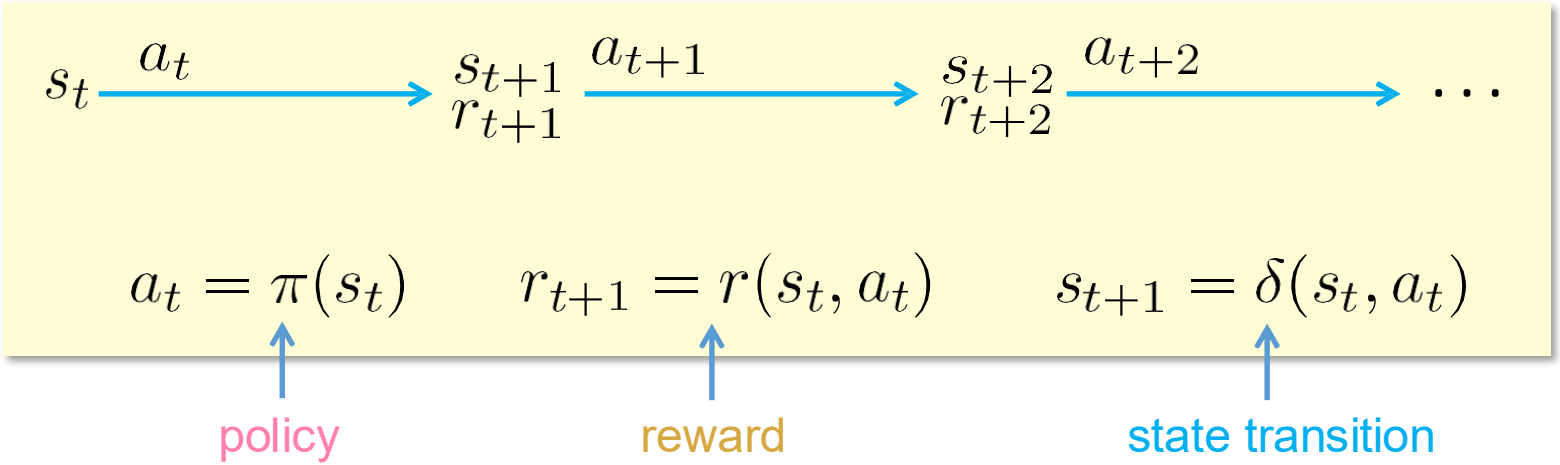
Accumulated Reward

- Suppose that a policy π is given.

- Starting from the t^{th} step, the cumulative reward by following π is given by

$$G_t := R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots$$

discounted factor, $\gamma \in [0, 1)$



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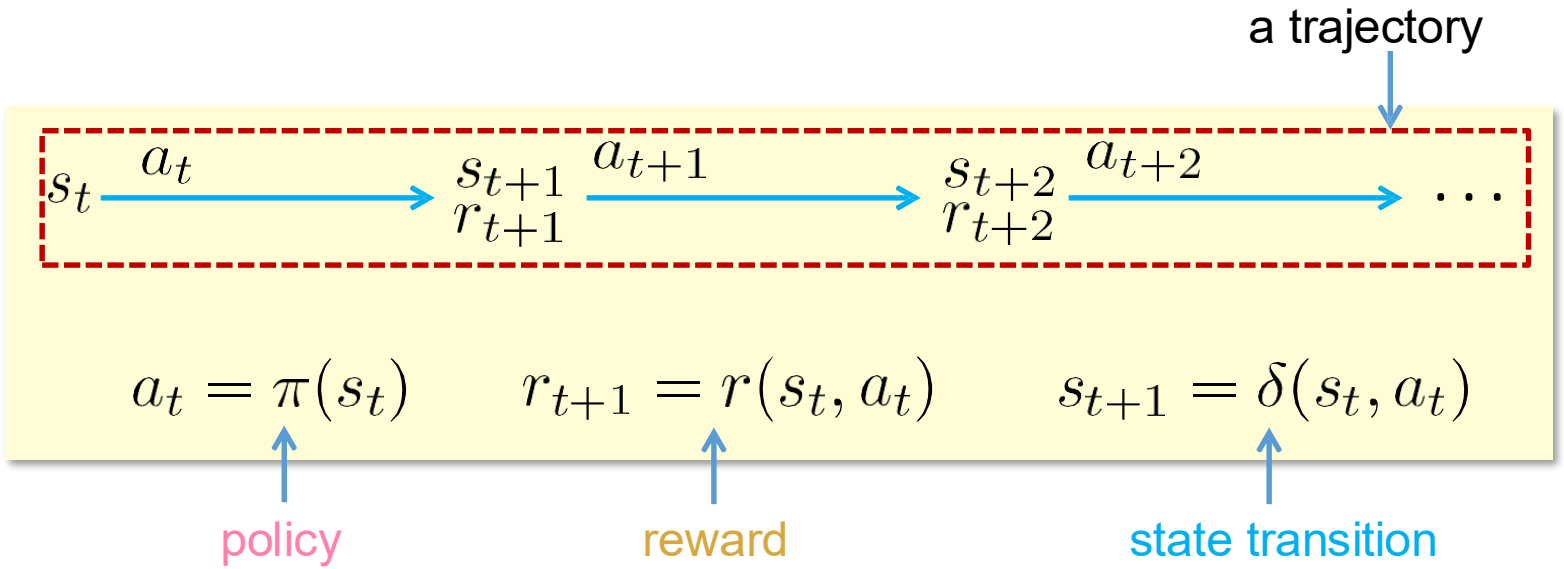
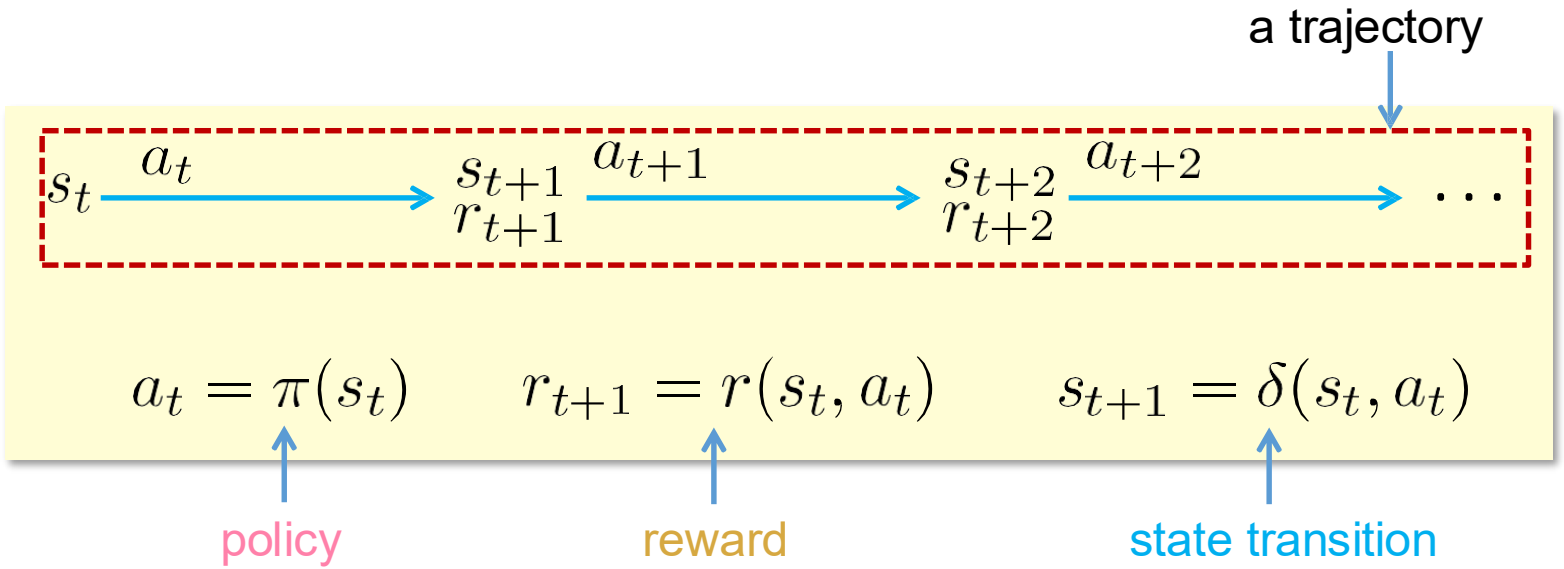


Value Function

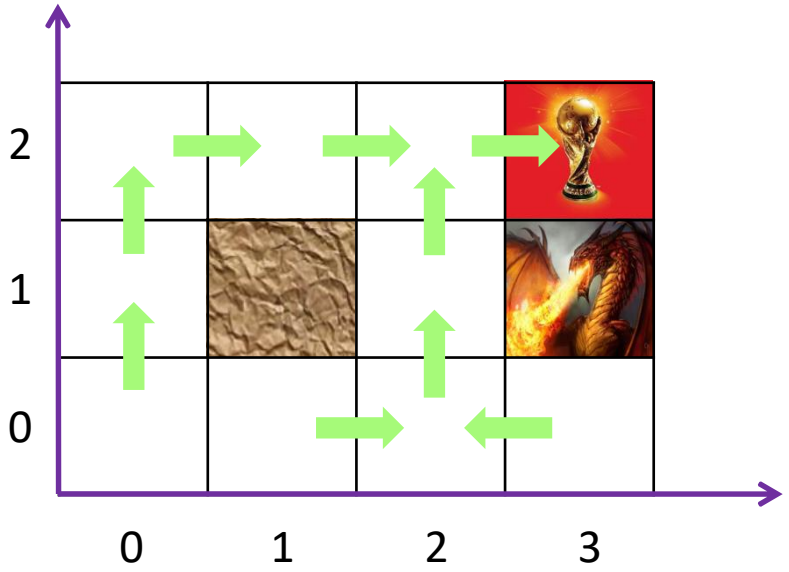
- Suppose that a policy π is given.

- The value function $V^\pi : \mathcal{S} \to \mathbb{R}$ is given by

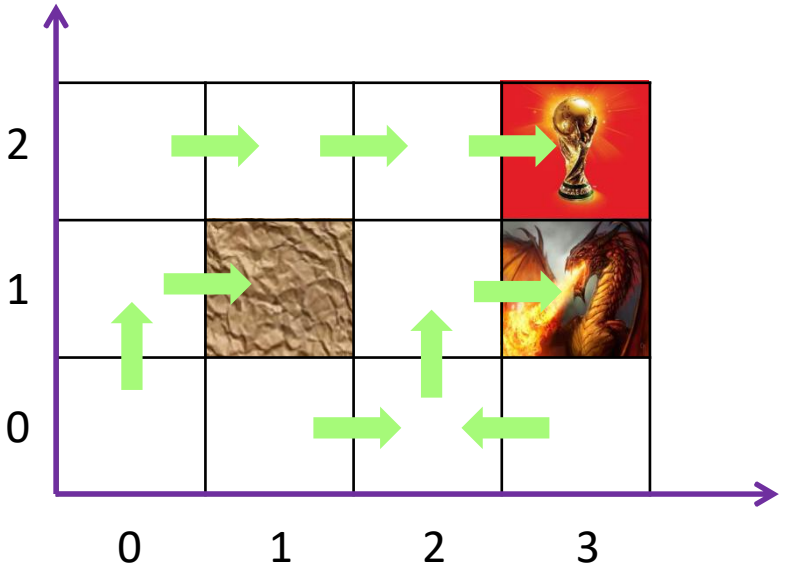
$$V^\pi(s) = \mathbb{E}[G_t | S_t = s]$$



Value Function



A good policy π_1



A bad policy π_2

$\gamma = 0.9$

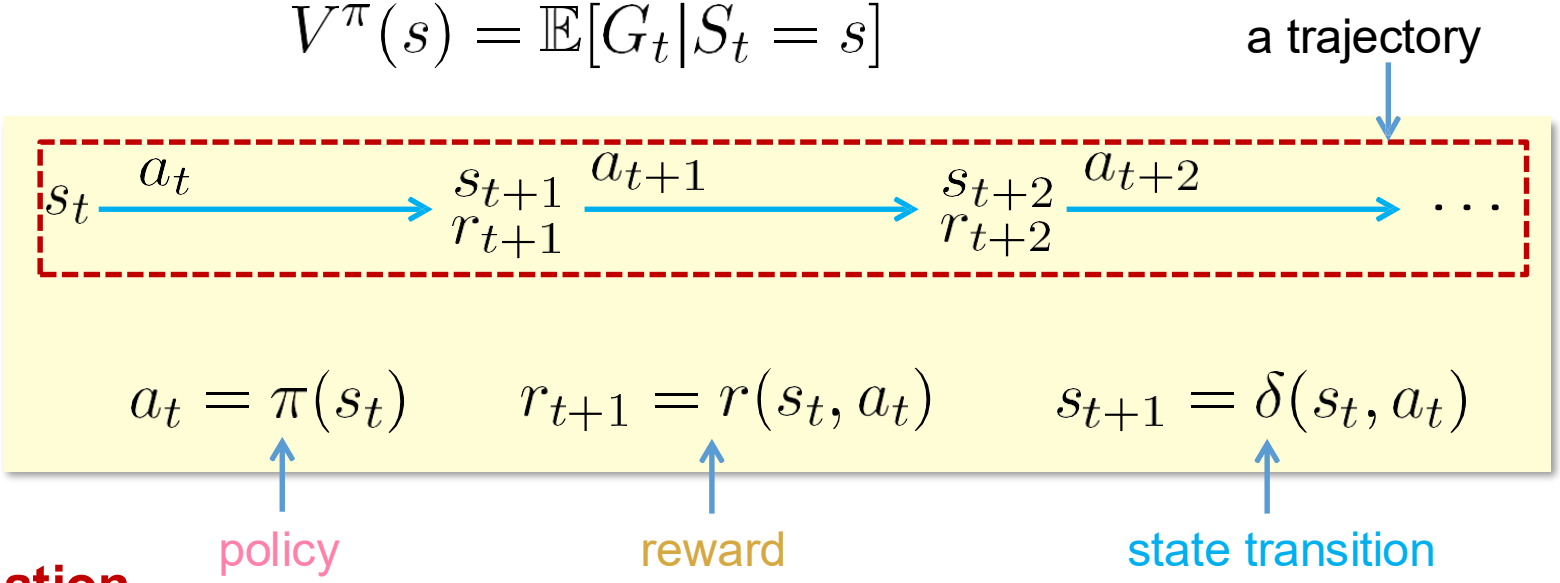
$$V^{\pi_1}((0,0)) = 0.9^4 \times 100 = 65.61$$
$$V^{\pi_1}((1,0)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((2,0)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_1}((3,0)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((0,1)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((2,1)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_1}((0,2)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_1}((1,2)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_1}((2,2)) = 100.0$$

$$V^{\pi_2}((0,0)) = 0$$
$$V^{\pi_2}((1,0)) = 0.9^2 \times (-100) = -81.0$$
$$V^{\pi_2}((2,0)) = 0.9 \times (-100) = -90.0$$
$$V^{\pi_2}((3,0)) = 0.9^2 \times (-100) = -81$$
$$V^{\pi_2}((0,1)) = 0$$
$$V^{\pi_2}((2,1)) = -100.0$$
$$V^{\pi_2}((0,2)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_2}((1,2)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_2}((2,2)) = 100.0$$

Value Function – Bellman Equation

- The value function $V^\pi : \mathcal{S} \to \mathbb{R}$ is given by

$$V^\pi(s) = \mathbb{E}[G_t | S_t = s]$$



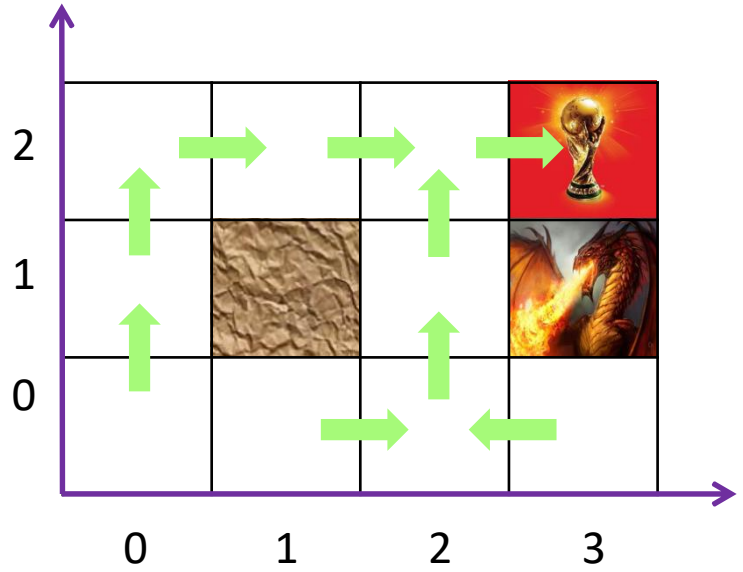
- **Bellman Equation**

$$\begin{aligned} V^\pi(s) &= \mathbb{E}[R_{t+1} + \gamma\,(R_{t+2} + \gamma R_{t+2} + \ldots)] \\ &= \mathbb{E}[R_{t+1}] + \gamma \mathbb{E}[G_{t+1}] \\ &= \mathbb{E}[R_{t+1}] + \gamma V^\pi(\delta(s_t, a_t)) \end{aligned}$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



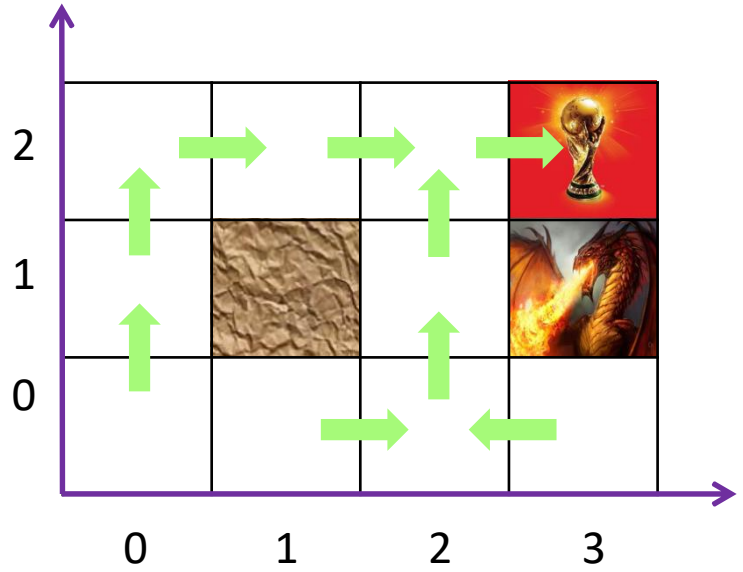
A good policy π_1

$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



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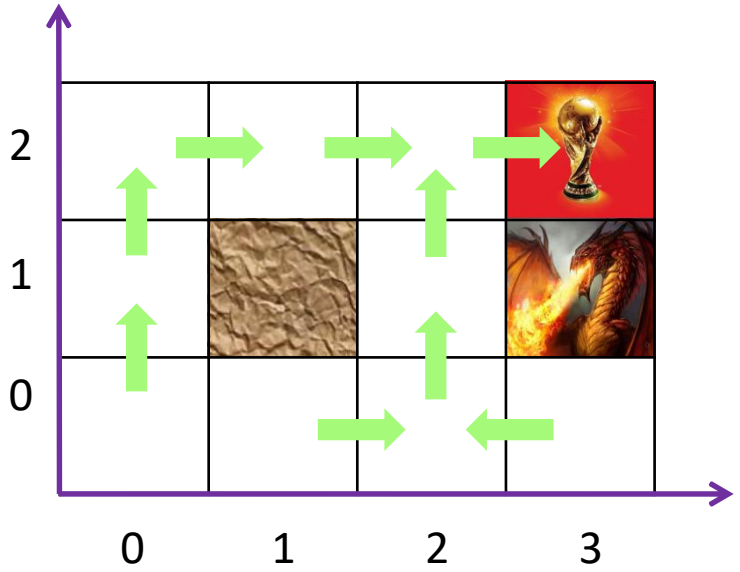
Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



A good policy π_1

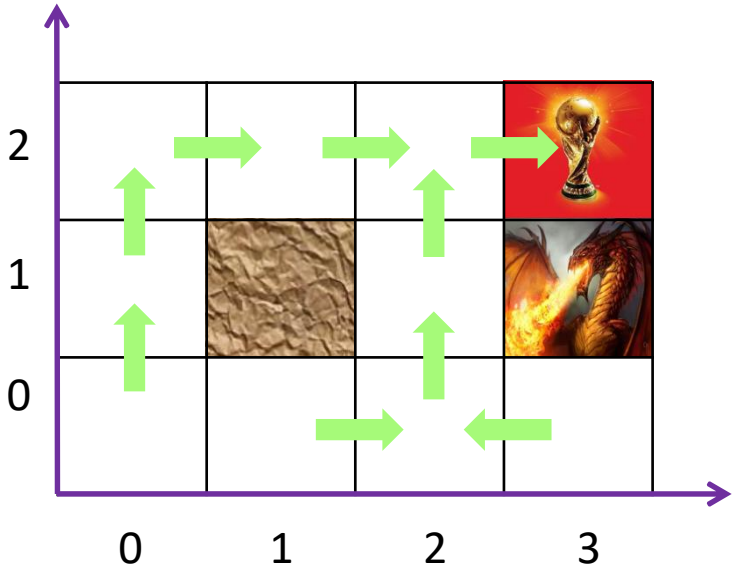
$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

$$V = R + \gamma T V$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



A good policy π_1

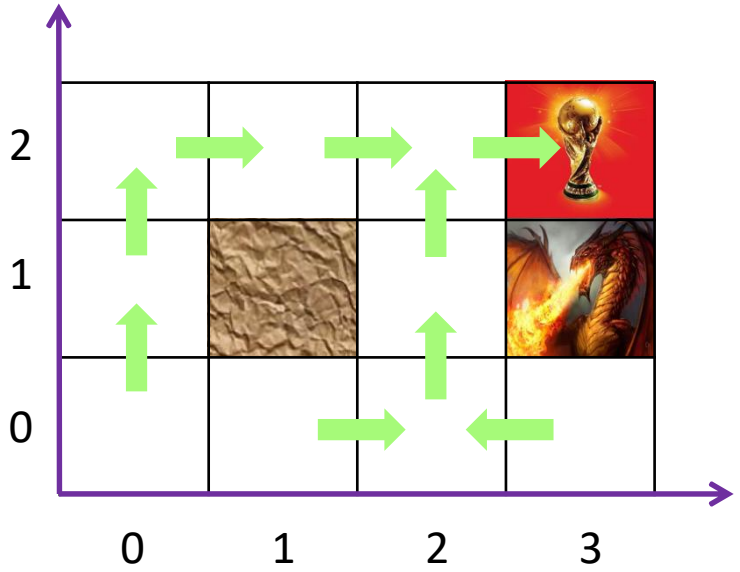
$$V = R + \gamma T V$$

$$V = (I - \gamma T)^{-1} R$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



A good policy π_1

$$V = R + \gamma T V$$

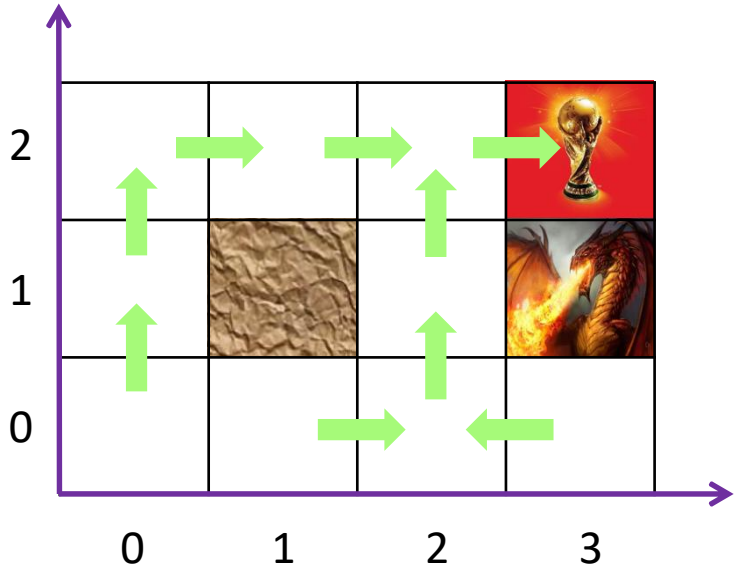
$$V = (I - \gamma T)^{-1} R$$

invertible ?

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



A good policy π_1

Theorem: *For a finite MDP, Bellman's equation admits a unique solution that is given by*

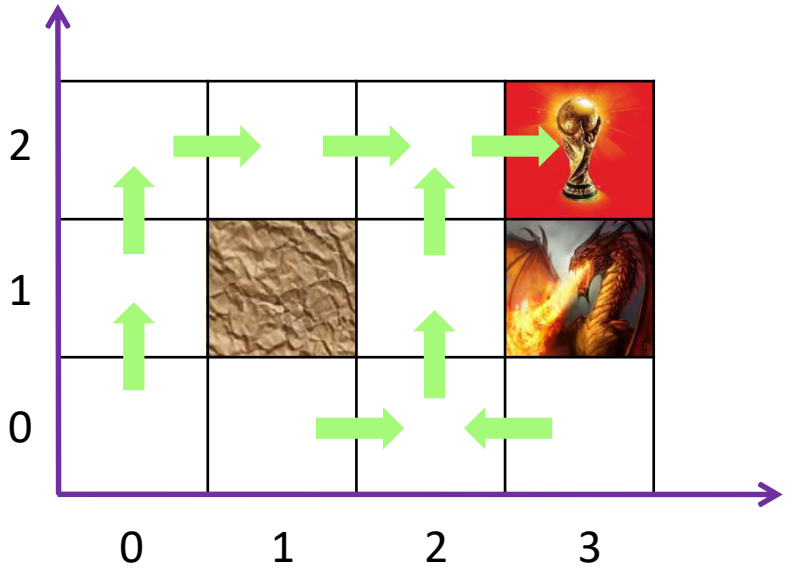
$$V = (I - \gamma T)^{-1} R$$

- The vector R and matrix T depend on the policy

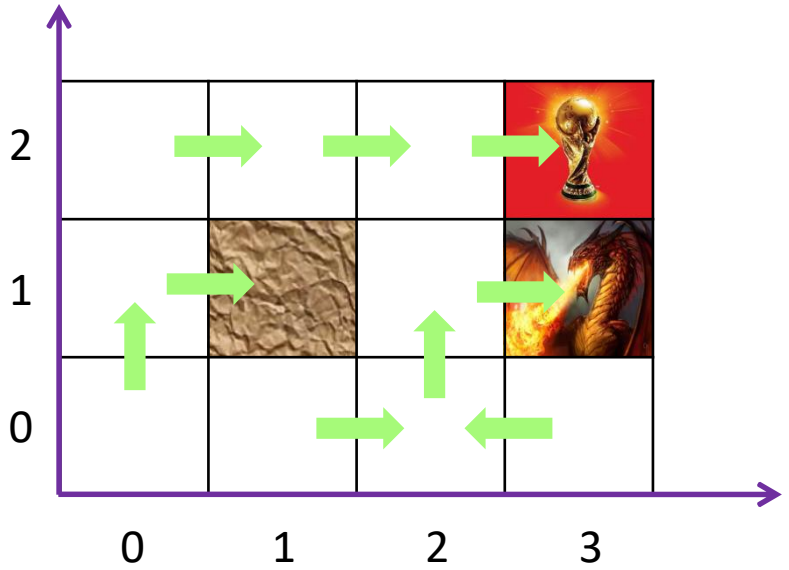
The Learning Task Revisited

- The learning task for RL scenarios is to learn an **optimal policy** in the sense that

$$\pi^* := \text{argmax}_\pi V^\pi(s), \forall s.$$



A good policy π_1



A bad policy π_2

- For π_1 and π_2, we have

$$V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s.$$

- Indeed, π_1 is the optimal policy.

The Q Function

- Learning the optimal policy is challenging
- An alternative approach to find the optimal policy indirectly is by computing the state-action value function (Q function)

$$Q(s,a) = r(s,a) + \gamma V^*(\delta(s,a))$$

$Q(s,a)$ is the accumulated reward by performing the action a first and then following the optimal policy

- The definition of the optimal policy implies that

$$\pi^*(s) = \text{argmax}_a\, Q(s,a) = \text{argmax}_a\, r(s,a) + \gamma V^*(\delta(s,a))$$

- Notice that

$$V^*(s) = \max_a\, Q(s,a) = \max_a\, r(s,a) + \gamma V^*(\delta(s,a))$$

- All together, we have

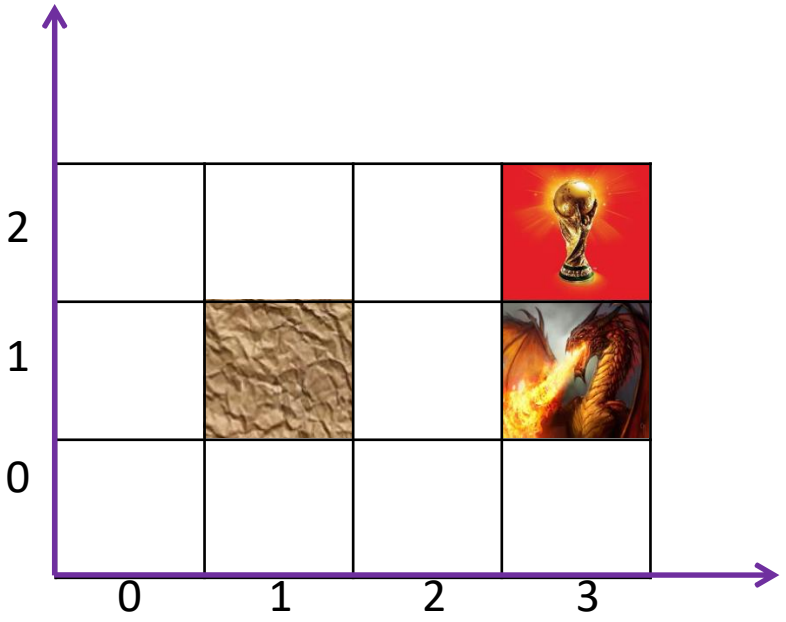
$$Q(s,a) = r(s,a) + \gamma \max_{a'} Q(\delta(s,a), a')$$

Bellman Equations for the optimal policy

Planning Algorithms

Planning

- Planning: we assume that the agent has perfect knowledge of the environment; thus, to find the optimal policy, there is no need for the agent to actually perform actions and interact with the environment (no need to learn)



Known

$\delta : \mathcal{S} \times \mathcal{A} \rightarrow \mathcal{S}$: state transition

$r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$: reward

Value Iteration

- Value iteration aims to find the optimal value function by solving the Bellman equations for the optimal policy
- The key is that the solution to the Bellman equations are indeed a fixed-point, i.e., the unknowns we want to solve for are on both sides of the Bellman equations

$$V^*(s) = \max_a Q(s,a) = \max_a r(s,a) + \gamma V^*(\delta(s,a))$$

Initialize $V(s)$ to arbitrary values

while termination conditions does not hold

For $s \in \mathcal{S}$

For $a \in \mathcal{A}$

$$Q(s,a) \leftarrow r(s,a) + \gamma V(\delta(s,a))$$

$$V(s) \leftarrow \max_a Q(s,a)$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

Example

$$V \leftarrow 0$$
$$Q((0,0), \text{up}) \leftarrow 0 + 0.9 \times V((0,1)) = 0$$
$$Q((0,0), \text{down}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{left}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{right}) \leftarrow 0 + 0.9 \times V((1,0)) = 0$$
$$V((0,0)) \leftarrow \max\{Q((0,0), \text{up}), Q((0,0), \text{down}), Q((0,0), \text{left}), Q((0,0), \text{right})\} = 0$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

Example

$$V \leftarrow 0$$
$$Q((0,0), \text{up}) \leftarrow 0 + 0.9 \times V((0,1)) = 0$$
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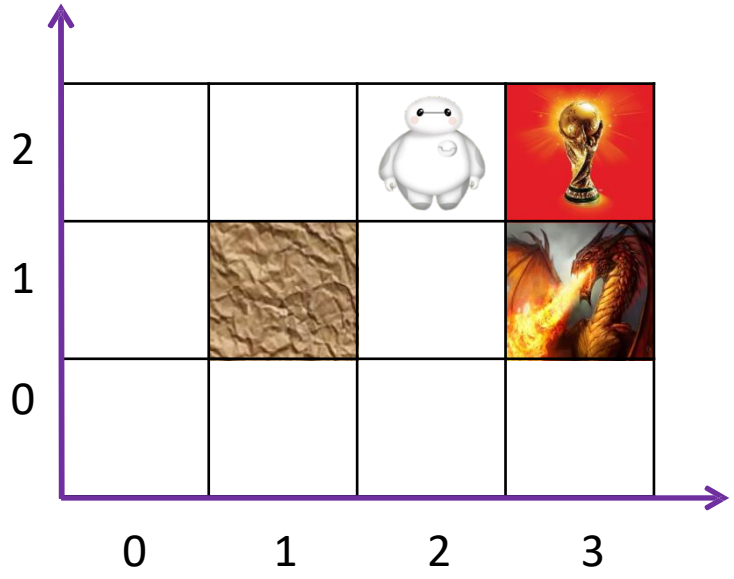
$$Q((0,0), \text{right}) \leftarrow 0 + 0.9 \times V((1,0)) = 0$$
$$V((0,0)) \leftarrow \max\{Q((0,0), \text{up}), Q((0,0), \text{down}), Q((0,0), \text{left}), Q((0,0), \text{right})\} = 0$$

Nothing happens

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy



Example

$$V \leftarrow 0$$
$$Q((2,2),\text{up}) \leftarrow 0 + 0.9 \times V((2,2)) = 0$$
$$Q((2,2),\text{down}) \leftarrow 0 + 0.9 \times V((2,1)) = 0$$
$$Q((2,2),\text{left}) \leftarrow 0 + 0.9 \times V((1,2)) = 0$$
$$Q((2,2),\text{right}) \leftarrow 100 + 0.9 \times V((3,2)) = 100$$
$$V((2,2)) \leftarrow \max\{Q((2,2),\text{up}), Q((2,2),\text{down}), Q((2,2),\text{left}), Q((2,2),\text{right})\} = 100$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

> ***Theorem:*** *For any initial value V, the sequence generated by the value iteration algorithm converges to V^*.*

  - The key to the proof is the contraction mapping theorem

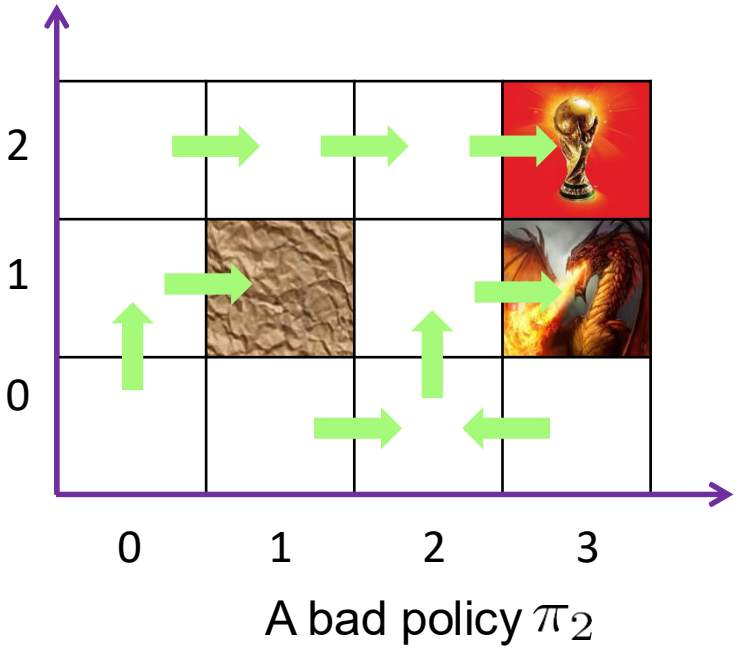
Policy Iteration

- Policy iteration improves the policy directly

$$
\begin{aligned}
&\textbf{Initialize } \pi, \pi' \text{ to two different policies} \\
&\textbf{while}(\pi \neq \pi') \\
&\quad V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} \\
&\quad \pi' \leftarrow \pi \\
&\quad\quad \textbf{For } s \in \mathcal{S} \\
&\quad\quad \pi(s) \leftarrow \text{argmax}_a \, r(s,a) + \gamma V(\delta(s,a))
\end{aligned}
$$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

Initialize $\pi \leftarrow \pi_2, \pi' \neq \pi_2$

while$(\pi \neq \pi')$

1^{st} $V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi}$

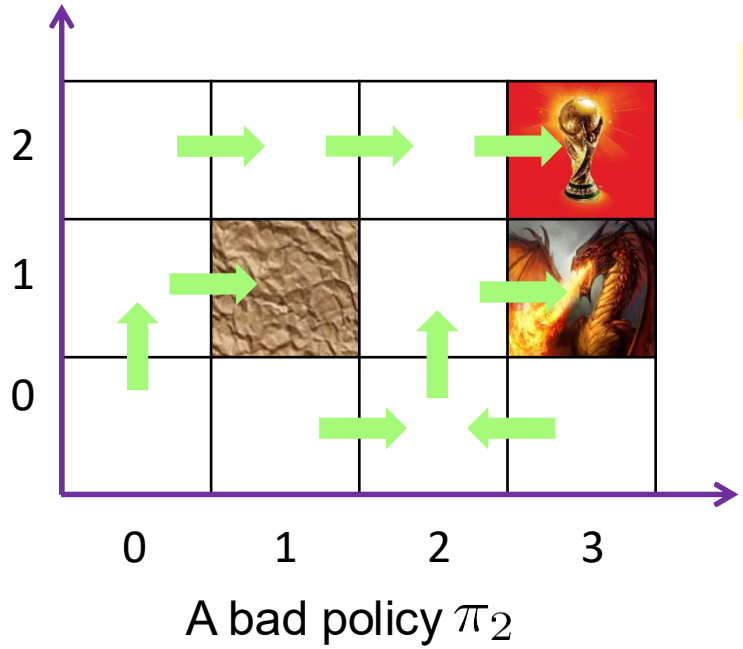
$\pi' \leftarrow \pi$

For $s \in \mathcal{S}$

$\pi(s) \leftarrow \text{argmax}_a \, r(s,a) + \gamma V(\delta(s,a))$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

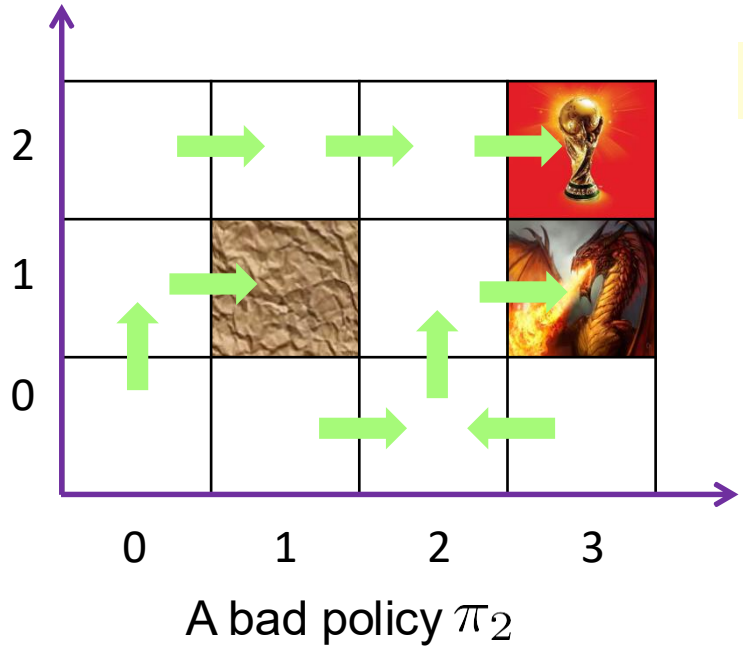
1^{st} iteration

$$V^{\pi} = \begin{pmatrix} V^{\pi}(0,0) \\ V^{\pi}(1,0) \\ V^{\pi}(2,0) \\ V^{\pi}(3,0) \\ V^{\pi}(0,1) \\ V^{\pi}(2,1) \\ V^{\pi}(3,1) \\ V^{\pi}(0,2) \\ V^{\pi}(1,2) \\ V^{\pi}(2,2) \\ V^{\pi}(3,2) \end{pmatrix}$$

11 states in total

Policy Iteration

- Policy iteration improves the policy directly



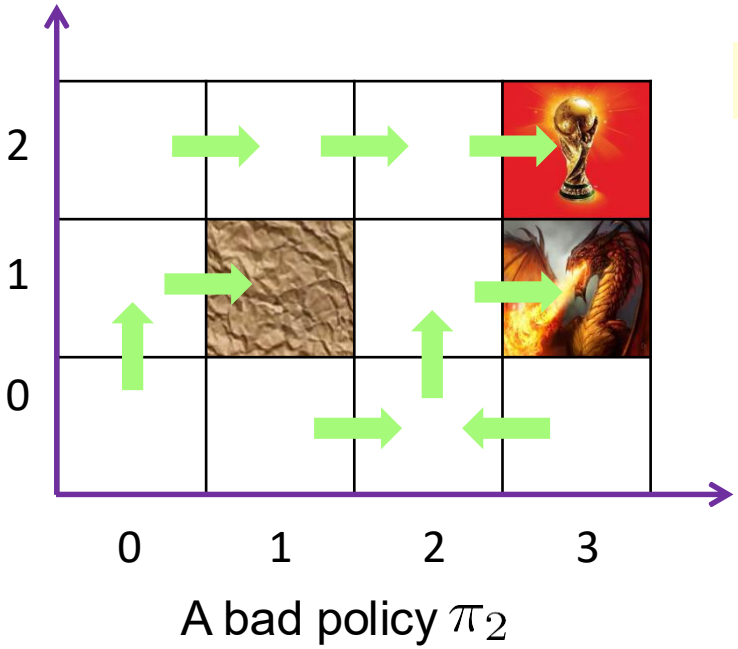
A bad policy π_2

1^{st} iteration

$$R^{\pi} = \begin{pmatrix} r((0,0),\text{up}) \\ r((1,0),\text{right}) \\ r((2,0),\text{up}) \\ r((3,0),\text{left}) \\ r((0,1),\text{right}) \\ r((2,1),\text{right}) \\ r((3,1),\text{END}) \\ r((0,2),\text{right}) \\ r((1,2),\text{right}) \\ r((2,2),\text{right}) \\ r((3,2),\text{END}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly



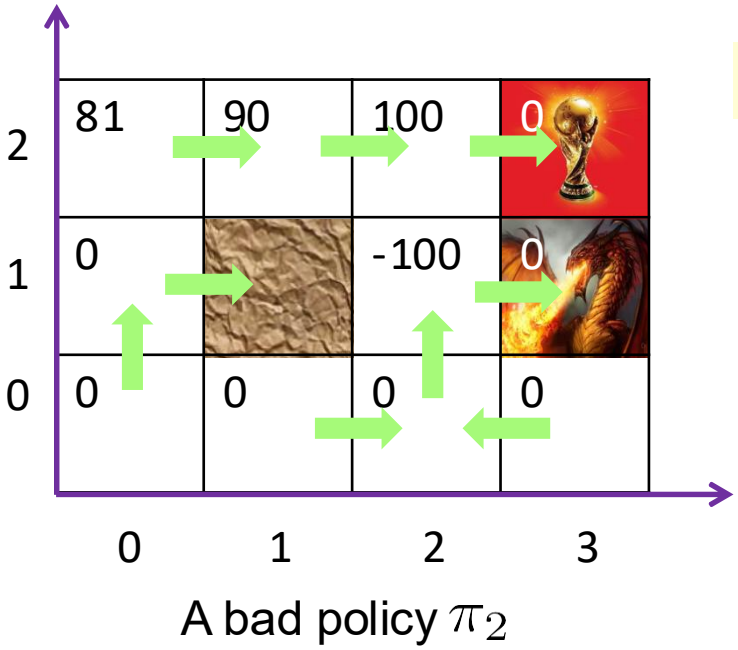
A bad policy π_2

1^{st} iteration

$$T^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly



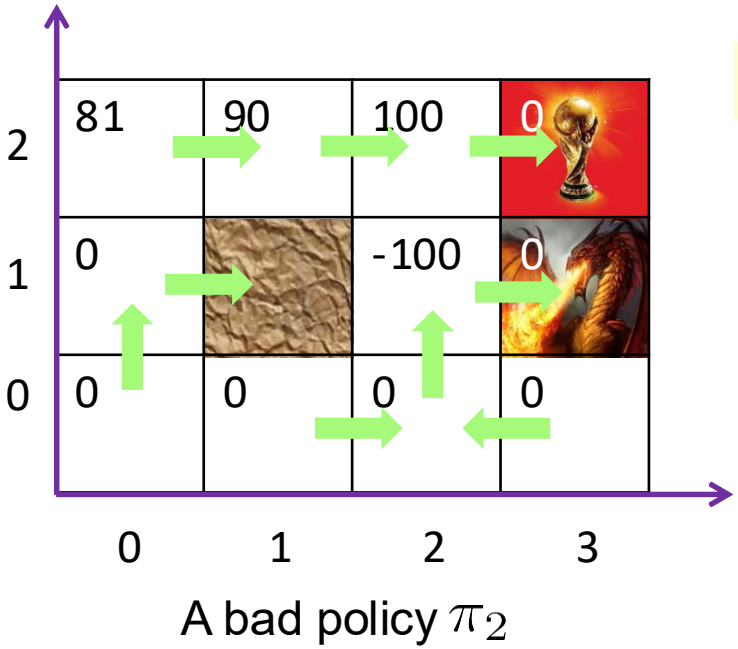
A bad policy π_2

1^{st} iteration

$$V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 81 \\ 90 \\ 100 \\ 0 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

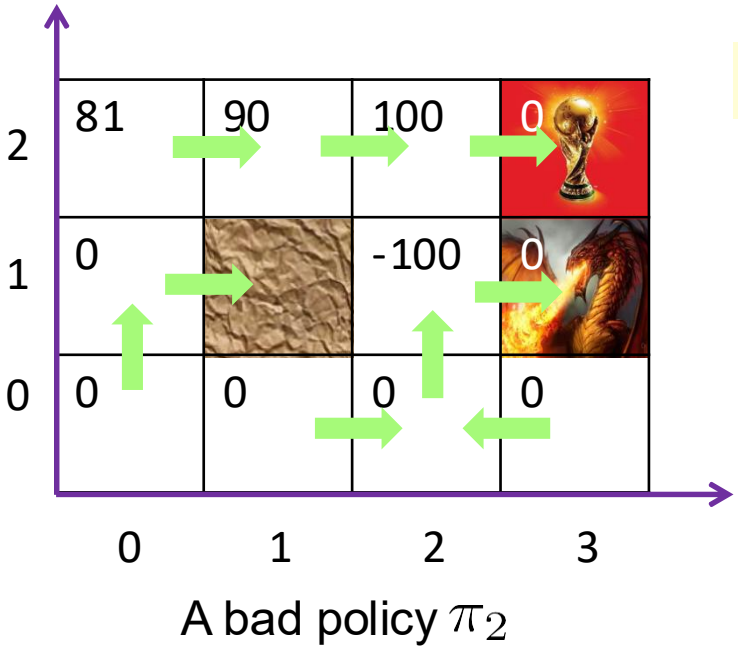
1^{st} iteration: update the policy

$$\pi((0,0)) = \text{argmax}_a\{r((0,0),\text{up}) + \gamma V((0,1)),$$
$$r((0,0),\text{down}) + \gamma V((0,0)),$$
$$r((0,0),\text{left}) + \gamma V((0,0)),$$
$$r((0,0),\text{right}) + \gamma V((1,0))\}$$
$$= \text{argmax}_a\{0,0,0,0\}$$

We can randomly select one action from $\mathcal{A} = \{\text{up}, \text{down}, \text{left}, \text{right}\}$.
However, it is better select one action from up and right (why?).

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

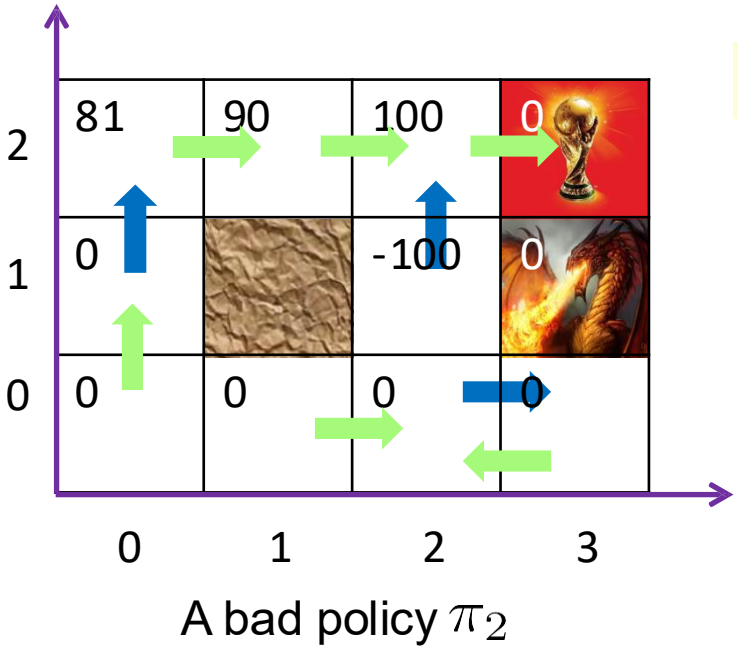
1^{st} iteration: update the policy

$$\pi((0,0)) = \text{argmax}_a\{r((0,0),\text{up}) + \gamma V((0,1)),$$
$$r((0,0),\text{down}) + \gamma V((0,0)),$$
$$r((0,0),\text{left}) + \gamma V((0,0)),$$
$$r((0,0),\text{right}) + \gamma V((1,0))\}$$
$$= \text{argmax}_a\{0,0,0,0\}$$

We can indeed assign negative rewards for actions that will not alter the states when these states are not the goal states. Or, we can simply ignore these actions.

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

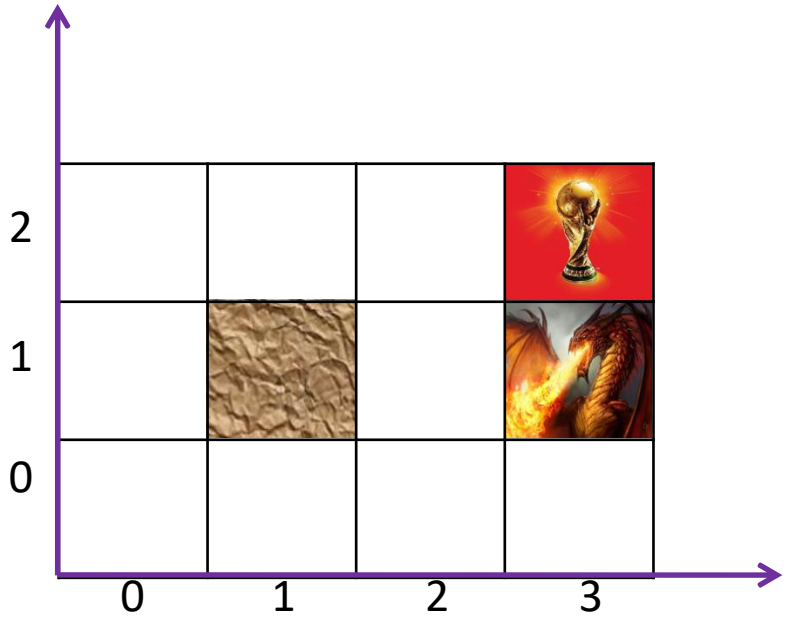
1^{st} iteration: update the policy

$$\pi((0,0)) = \text{up}$$
$$\pi((1,0)) = \text{right}$$
$$\pi((2,0)) = \text{right}$$
$$\pi((3,0)) = \text{left}$$
$$\pi((0,1)) = \text{up}$$
$$\pi((2,1)) = \text{up}$$
$$\pi((3,1)) = \text{END}$$
$$\pi((0,2)) = \text{right}$$
$$\pi((1,2)) = \text{right}$$
$$\pi((2,2)) = \text{right}$$
$$\pi((3,2)) = \text{END}$$

Learning Algorithms

Learning

- Learning: as the environment model, i.e., the transition and reward, is unknown, the agent may need to learn them based on the training information.



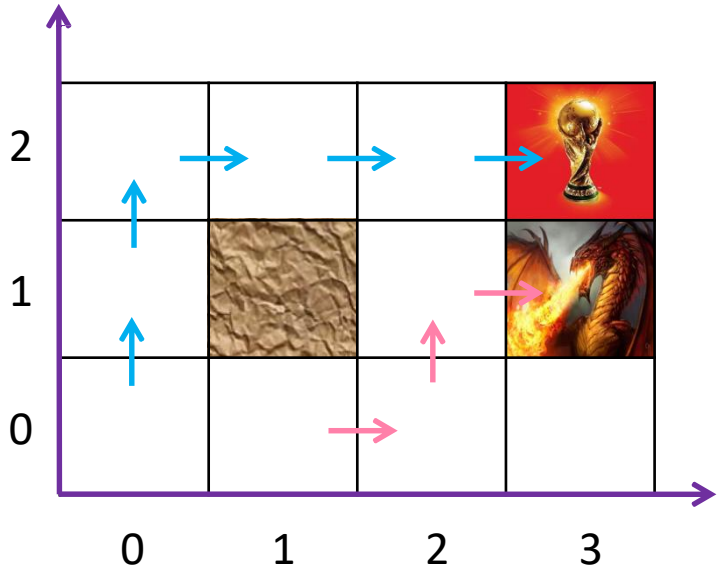
Unknown

$\delta : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$: state transition

$r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$: reward

Learning

- Learning: as the environment model, i.e., the transition and reward, is unknown, the agent may need to learn them based on the training information.
  - Model-free approach: the agent learns the optimal policy directly, e.g., Q-learning
  - Model-based approach: the agent first learns the environment model and then the optimal policy



Examples of training data

(0,0) —up, 0→ (0,1) —up, 0→ (0,2) —right, 0→ (1,2) —right, 0→ (2,3) —right, 100→ (3,2)

(1,0) —right, 0→ (2,0) —up, 0→ (2,1) —right, -100→ (3,1)

The Q-learning Algorithm

- Initialize the matrix $\hat{Q}$ to zero
- Observe the current state s
- Do forever:
  - Pick and perform an action a
  - Receive immediate reward r
  - Observe the new state s'
  - Update

$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a')$$

  - $s \leftarrow s'$

A sufficient condition for $\hat{Q}(s,a)$ to converge is to visit each state-action pair infinitely often

The Q-learning Algorithm

- Initialize the matrix $\hat{Q}$ to zero
- Observe the current state s
- Do forever:
  - Pick and perform an action a
  - Receive immediate reward r
  - Observe the new state s'
  - Update

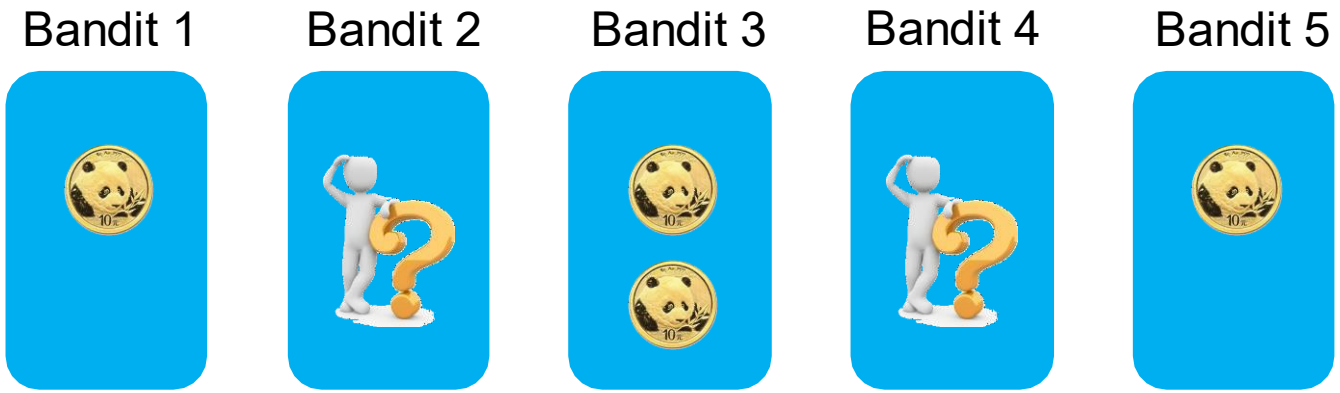
$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a')$$

  - $s \leftarrow s'$

How to pick the action?

Exploitation vs Exploration

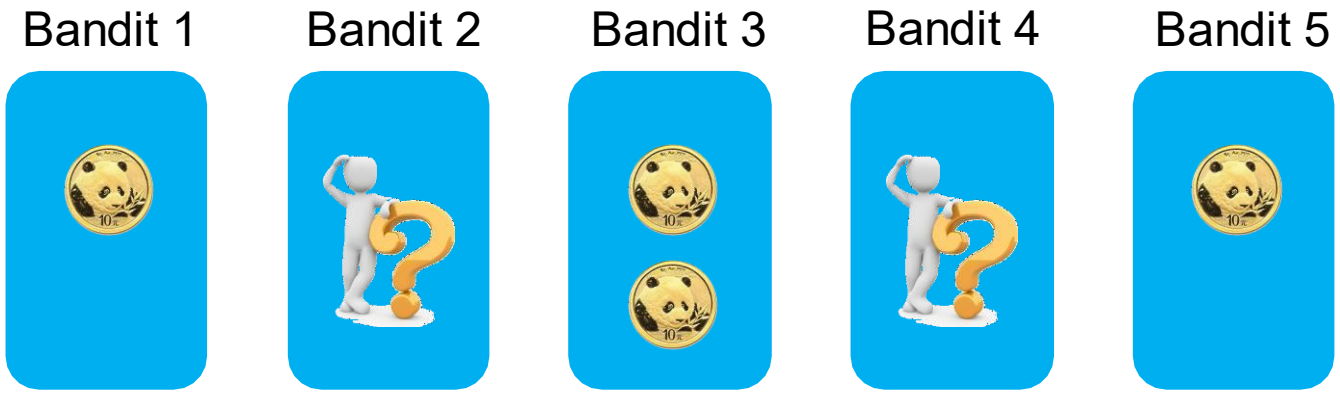
- Multi-armed bandit



- Which machine next?
  - Exploitation: the machine with the largest reward at present
  - Exploration: randomly select a machine

Exploitation vs Exploration

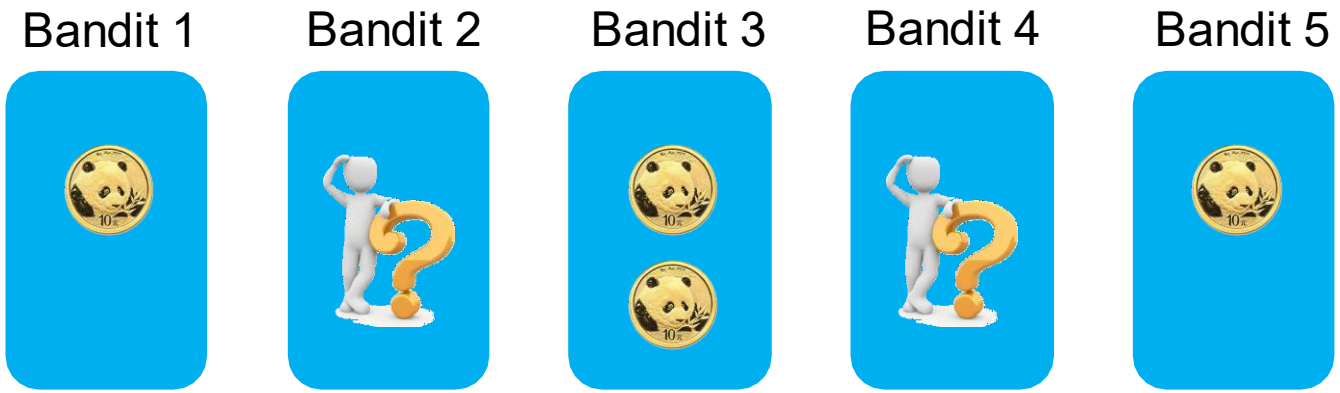
- Multi-armed bandit



- ϵ-greedy
  - with probability $1 - \epsilon$, we do exploitation
  - with probability ϵ, we do exploration, i.e., we uniformly randomly select an action from all possible actions
- Tips for ϵ-greedy
  - At the beginning, the agent does not know the environment very well. Thus, it need to do more exploration and a large value of ϵ is needed.
  - When the environment model is well explored, the agent can do more exploitation. Thus, we favor a small value of ϵ.

Exploitation vs Exploration

- Multi-armed bandit



- A soft sampling strategy
  - Given a state, we can choose action probabilistically

$$\mathrm{P}[a|s] = \frac{e^{\hat{Q}(s,a)/T}}{\sum_{a'} e^{\hat{Q}(s,a')/T}}$$

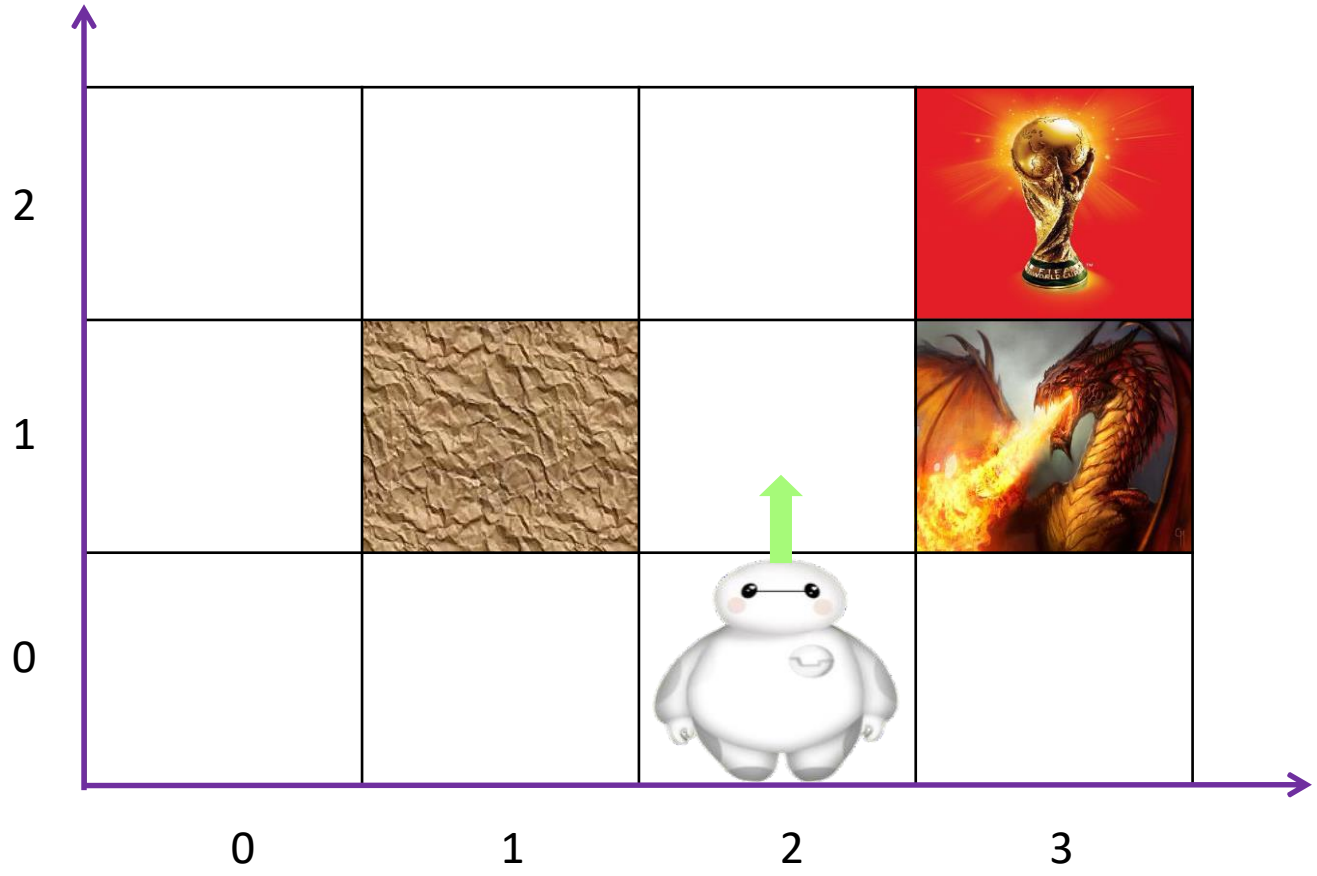
  - Smaller values of T will assign higher probabilities for actions with high $\hat{Q}$ leading to an exploitation strategy.
  - Larger values of T will encourage the agent to explore actions that do not currently have high $\hat{Q}$ values.

Contents

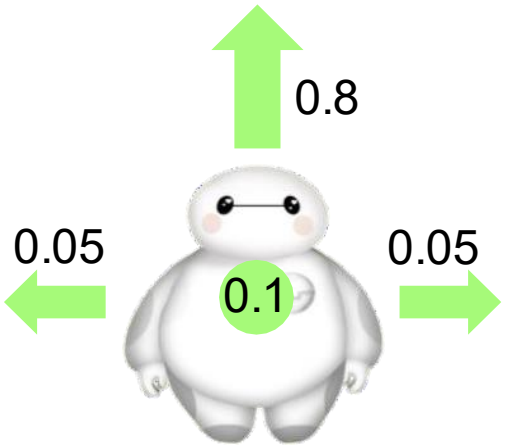
- Stochastic Environment
- Planning Algorithms
- Learning Algorithms

Stochastic Environment

Grid World



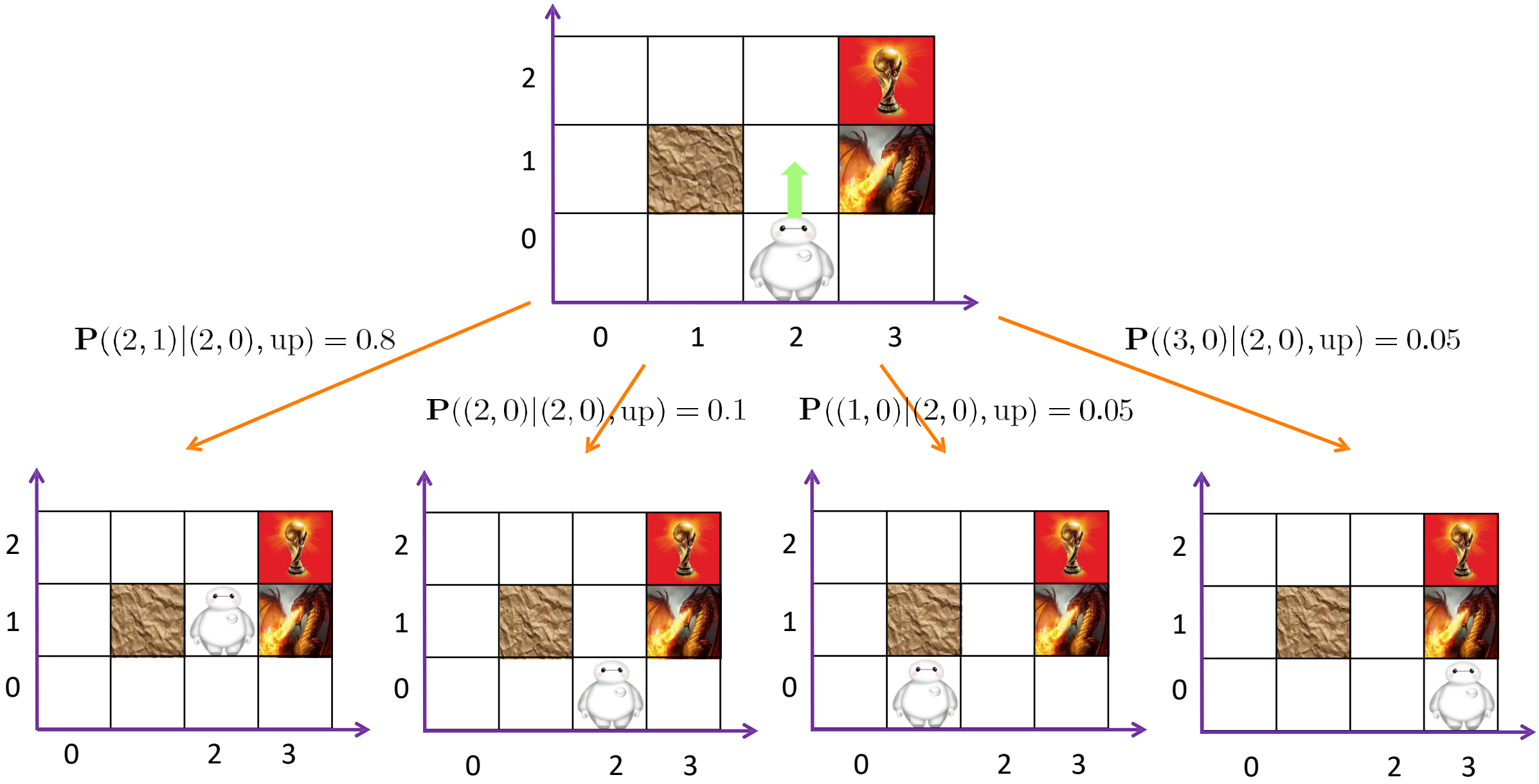
State Transition



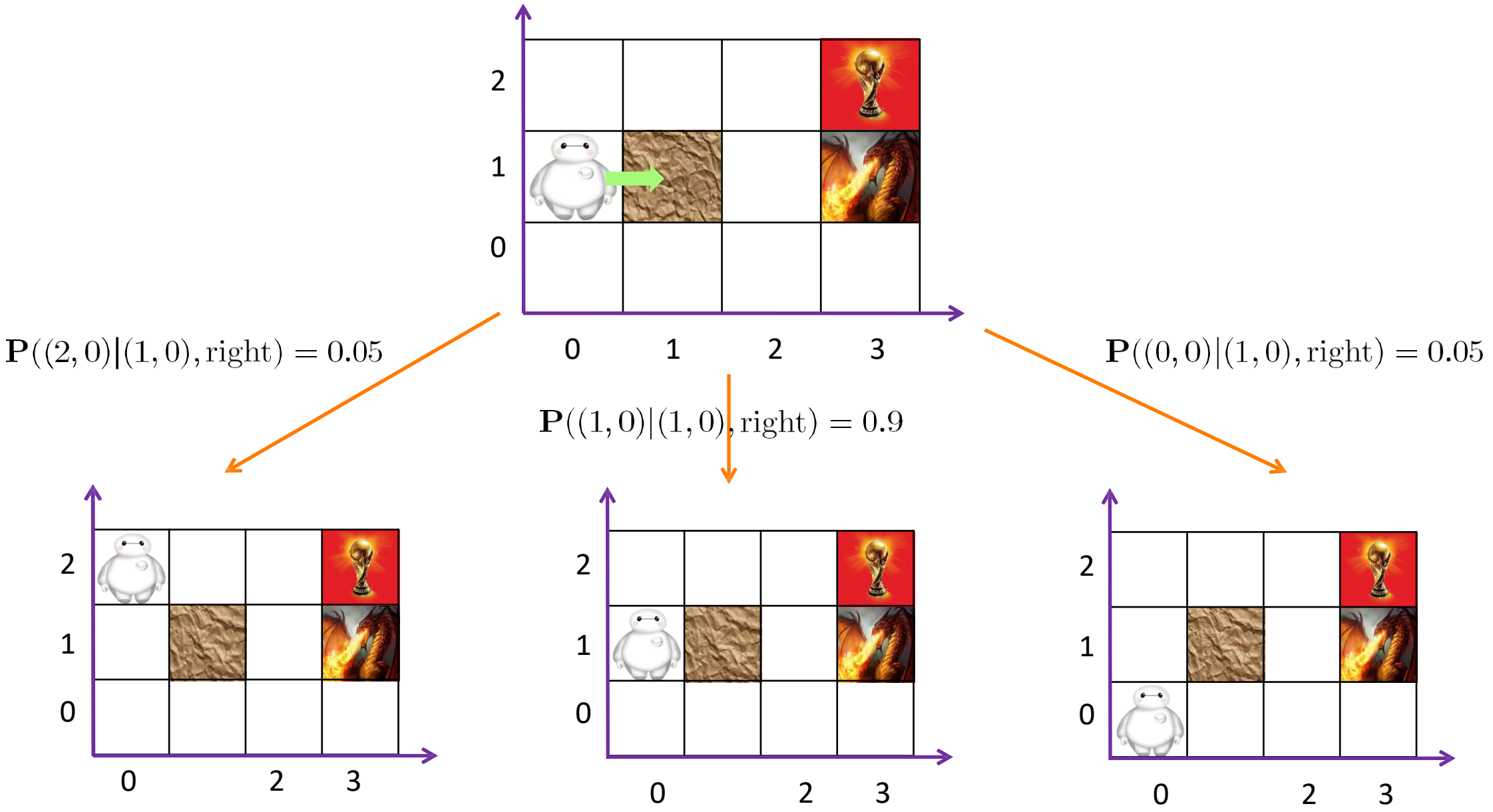
State transition probabilities:

After the agent picks and performs a certain action, there are four possibilities for the next state: the destination state, the current state, the states to the right and left of the current state. If the states are reachable, the corresponding probabilities are 0.8, 0.1, 0.05, and 0.05, respectively; otherwise, the agent stays where it is.

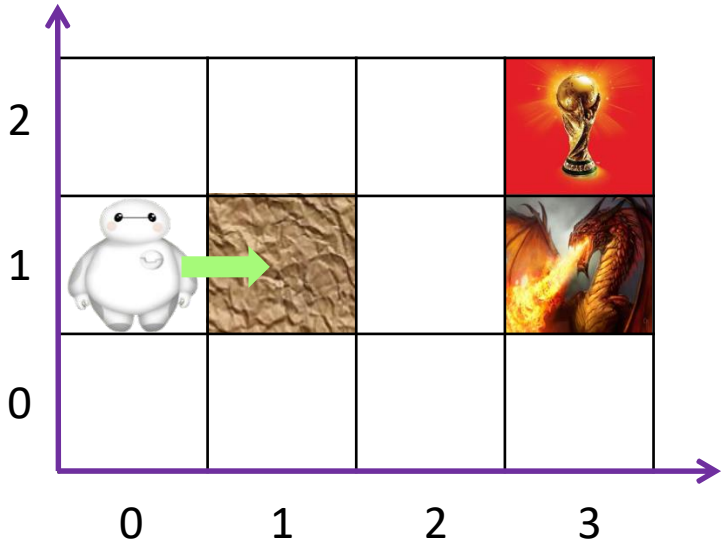
State Transition

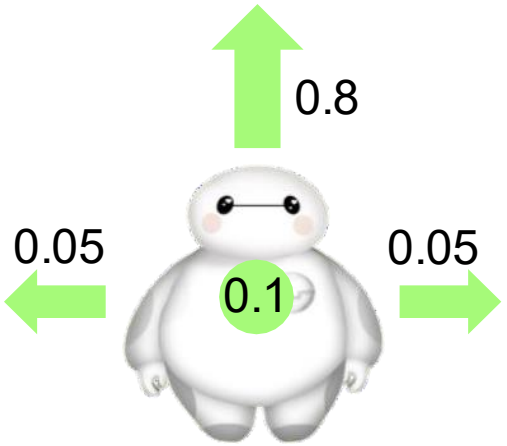


State Transition



Reward

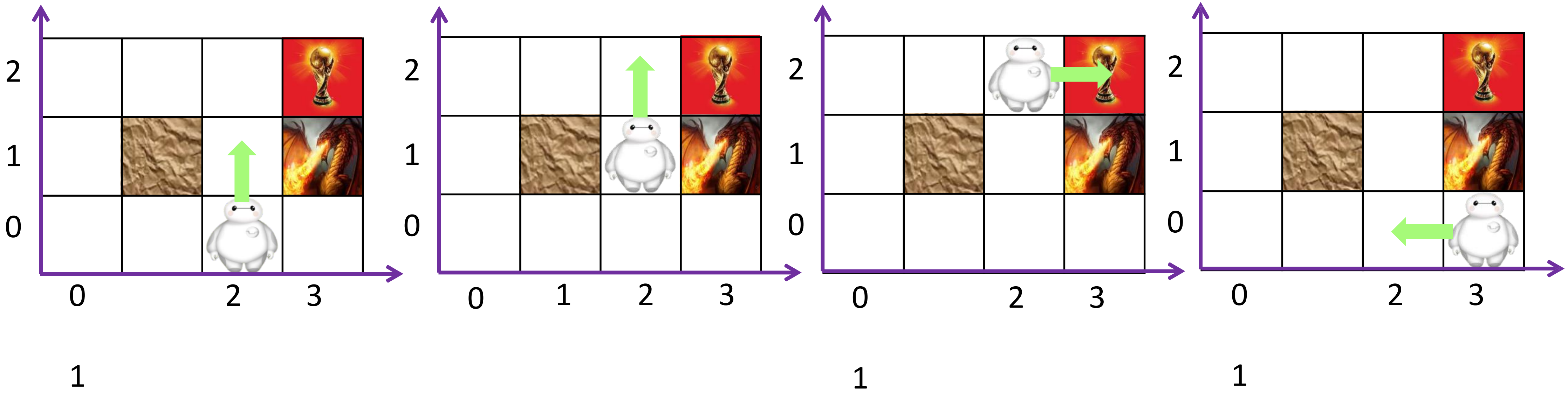




Reward:

After the agent picks and performs a certain action at its current state, it receives rewards of 100, -100, and 0, if it arrives at states (3,2), (2,2), and all the other states, respectively.

Reward



$$\mathbf{E}[r((2,0),\text{up})] = 0.8 \times 0 + 0.1 \times 0 + 0.05 \times 0 + 0.05 \times 0 = 0$$

$$\mathbf{E}[r((2,1),\text{up})] = 0.8 \times 0 + 0.1 \times 0 + 0.05 \times -100 + 0.05 \times 0 = -5$$

$$\mathbf{E}[r((2,2),\text{right})] = 0.8 \times 100 + 0.1 \times 0 + 0.05 \times 0 + 0.05 \times 0 = 80$$

$$\mathbf{E}[r((3,0),\text{left})] = 0.8 \times 0 + 0.1 \times 0 + 0.05 \times -100 + 0.05 \times 0 = -5$$

Markov Decision Process (MDP)

- Indeed, we have already introduced the so-called MDP, which is defined (rigorously) by
  - a set of states $\mathcal{S}$, possibly infinite
  - a set of actions $\mathcal{A}$, possibly infinite
  - an initial state $s_0 \in \mathcal{S}$
  - a transition probability $\mathbf{P}[s'|s,a]$: distribution over destination states $s' = \delta(s,a)$
  - a reward probability $\mathbf{P}[r|s,a]$: distribution over rewards $r' = r(s,a)$

MRT Chapter 14

- This model is Markovian because the transition and reward probabilities only depend on the current state and the action picked and performed at the current state, instead of the previous sequence of states and actions performed.

- In this lecture, we assume that
  - the states and the actions are finite

Value Function

- Suppose that a policy π is given.

- Starting from an arbitrary state s_t, the expected cumulative reward by following π is

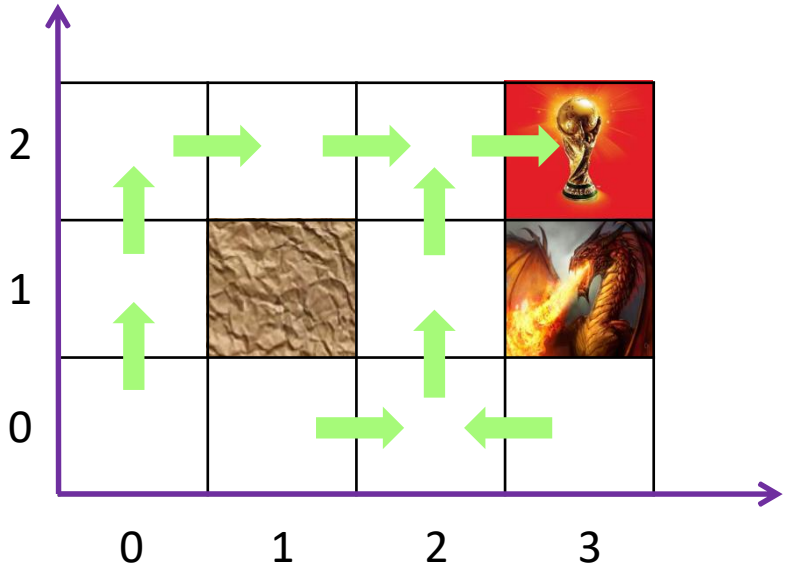
$$V^{\pi}(s_t) := \mathbf{E}[R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + \ldots | S_t = s_t] = \mathbf{E}\left[\sum_{i=0}^{\infty} \gamma^i R_{t+i} | S_t = s_t\right]$$

$$s_t \xrightarrow[r_t]{a_t} s_{t+1} \xrightarrow[r_{t+1}]{a_{t+1}} s_{t+2} \xrightarrow[r_{t+2}]{a_{t+2}} \cdots$$

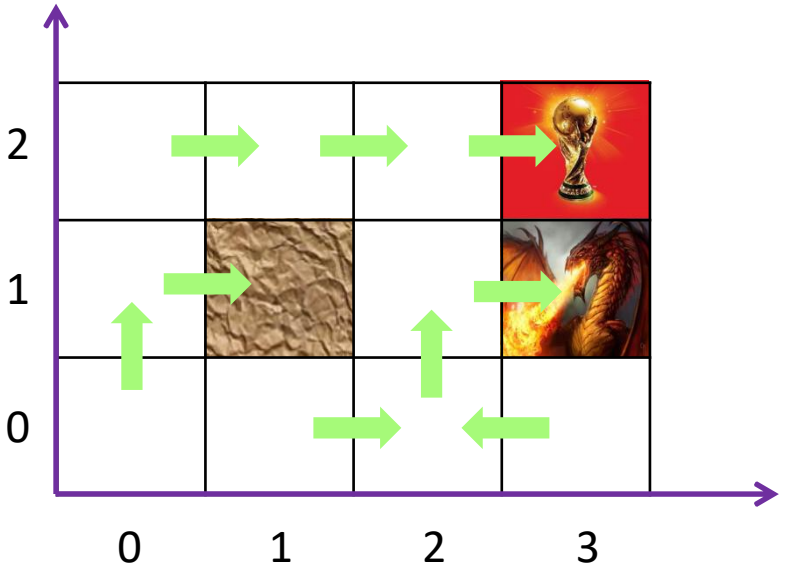
$$a_t = \pi(s_t) \quad r_t = r(s_t, a_t) \quad s_{t+1} = \delta(s_t, a_t)$$

random variable

Value Function



A "good" policy π_1



A bad policy π_2

$\gamma = 0.9$

How to find V^{π_1} and V^{π_2}?

Value Function

- Tower property

$$\mathbf{E}[X|Y] = \mathbf{E}[\mathbf{E}[X|Y,Z]|Y]$$

- A simpler version

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Z]]$$

- Example: how to find the average height of the men in China?

$$\mathbf{E}[\text{height}] = \mathbf{E}\left[\mathbf{E}[\text{height}|\text{province}]\right] = \sum_{\text{province}} \mathbf{P}(\text{province})\mathbf{E}[\text{height}|\text{province}]$$

Value Function – Bellman Equation

- Starting from an arbitrary state s_t, the expected cumulative reward by following π is

$$V^{\pi}(s_t) := \mathbf{E}[R_t + \gamma R_{t+1} + \gamma^2 R_{t+2} + \ldots | S_t = s_t] = \mathbf{E}\left[\sum_{i=0}^{\infty} \gamma^i R_{t+i} | S_t = s_t\right]$$

$$s_t \xrightarrow[r_t]{a_t} s_{t+1} \xrightarrow[r_{t+1}]{a_{t+1}} s_{t+2} \xrightarrow[r_{t+2}]{a_{t+2}} \cdots$$

$$a_t = \pi(s_t) \quad r_t = r(s_t, a_t) \quad s_{t+1} = \delta(s_t, a_t)$$

random variable

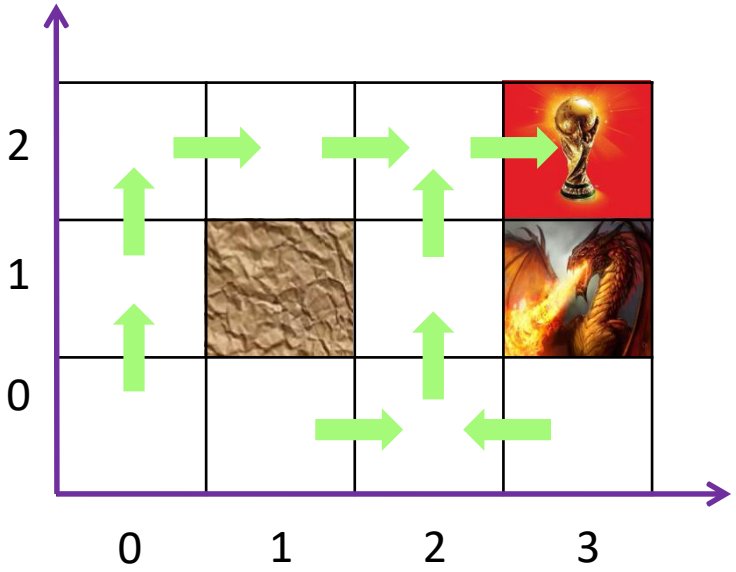
- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s' | s, \pi(s)) V^{\pi}(s')$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s'|s, \pi(s)) V^{\pi}(s')$$



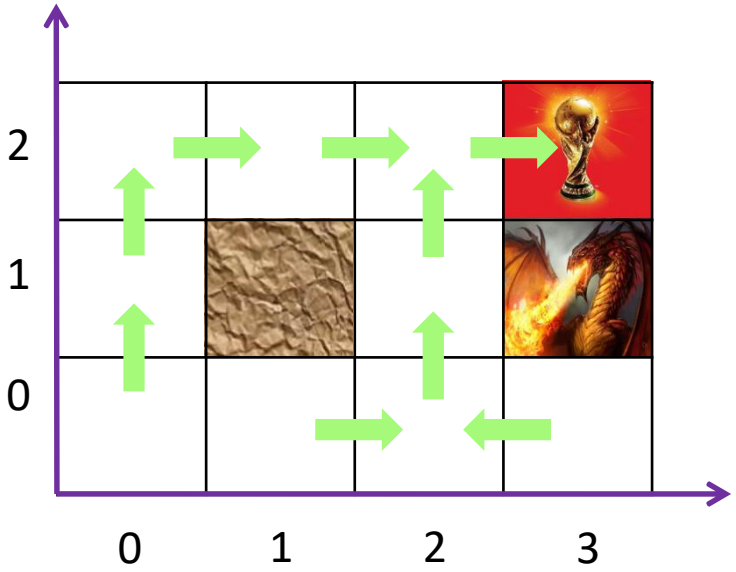
A "good" policy π_1?

$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 80 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0.15 & 0.05 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0.1 & 0.05 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.15 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.05 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.15 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0.15 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s'|s, \pi(s)) V^{\pi}(s')$$



A "good" policy π_1

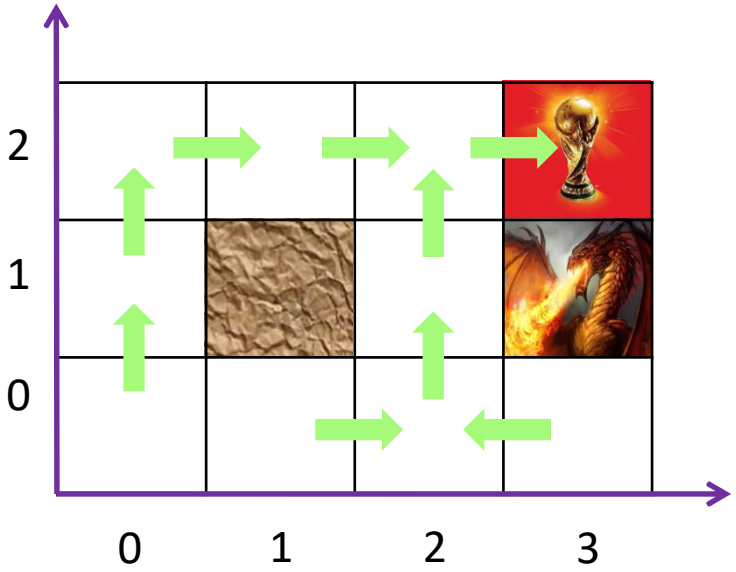
$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 80 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0.15 & 0.05 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0.1 & 0.05 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.15 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.05 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.15 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0.15 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

$$V = R + \gamma T V$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s'|s, \pi(s)) V^{\pi}(s')$$



A "good" policy π_1

$$V = R + \gamma T V$$

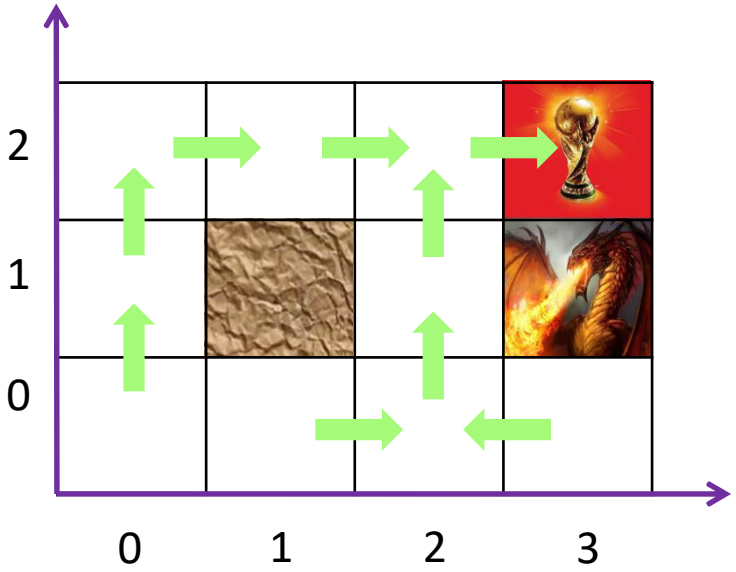
$$\Downarrow$$

$$V = (I - \gamma T)^{-1} R$$

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s'|s, \pi(s)) V^{\pi}(s')$$



A "good" policy π_1

$$V = R + \gamma T V$$

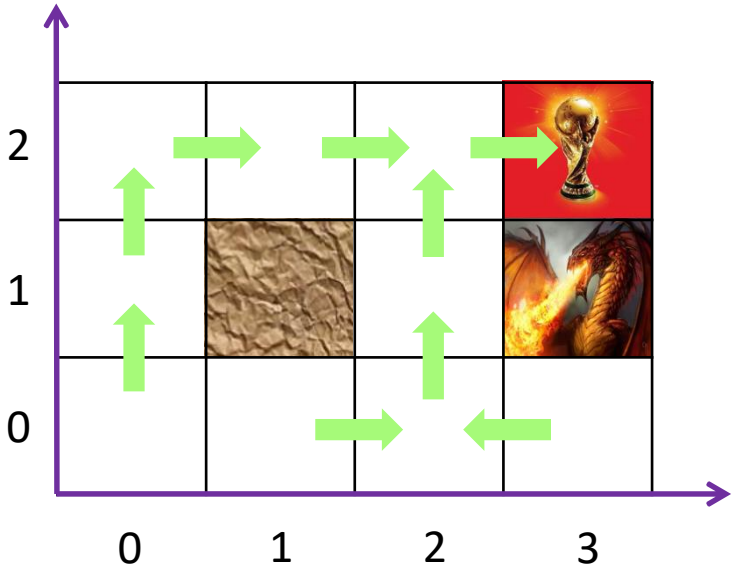
$$V = (I - \gamma T)^{-1} R$$

invertible?

Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = \mathbf{E}[r(s, \pi(s))] + \gamma \sum_{s'} \mathbf{P}(s'|s, \pi(s)) V^{\pi}(s')$$



A "good" policy π_1

Theorem: *For a finite MDP, Bellman's equation admits a unique solution that is given by*

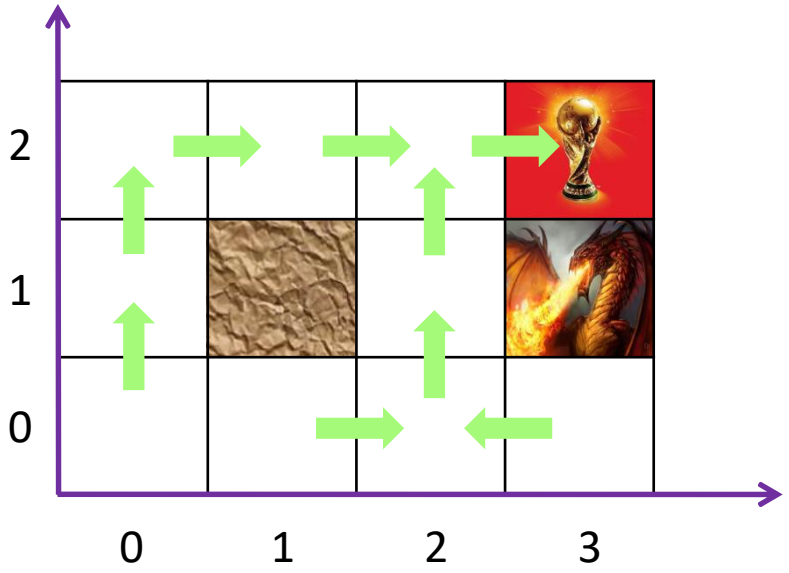
$$V = (I - \gamma T)^{-1} R$$

- The vector R and matrix T depend on the policy

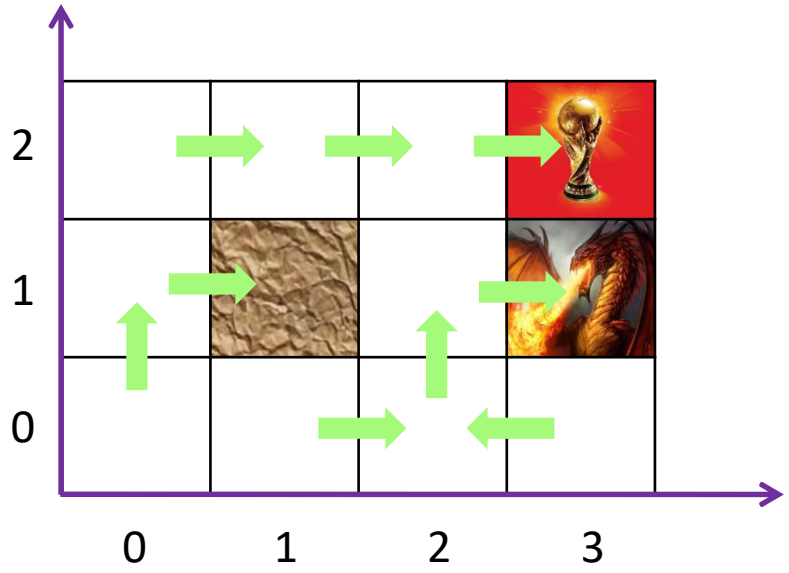
The Learning Task Revisited

- The learning task for RL scenarios is to learn an optimal policy in the sense that

$$\pi^* := \text{argmax}_\pi V^\pi(s), \forall s.$$



A good policy π_1



A bad policy π_2

- For π_1 and π_2, we have

$$V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s.$$

- Indeed, π_1 is the optimal policy.

The Q Function

- Learning the optimal policy is challenging
- An alternative approach to find the optimal policy indirectly is by computing the state-action value function (Q function)

$$Q(s,a) = \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \mathbf{P}(s'|s,a) V^*(s')$$

$Q(s,a)$ is the expected accumulated reward by performing the action a first and then following the optimal policy

- The definition of the optimal policy implies that

$$\pi^*(s) = \text{argmax}_a \, Q(s,a)$$

- Notice that

$$V^*(s) = \max_a \, Q(s,a)$$

- All together, we have

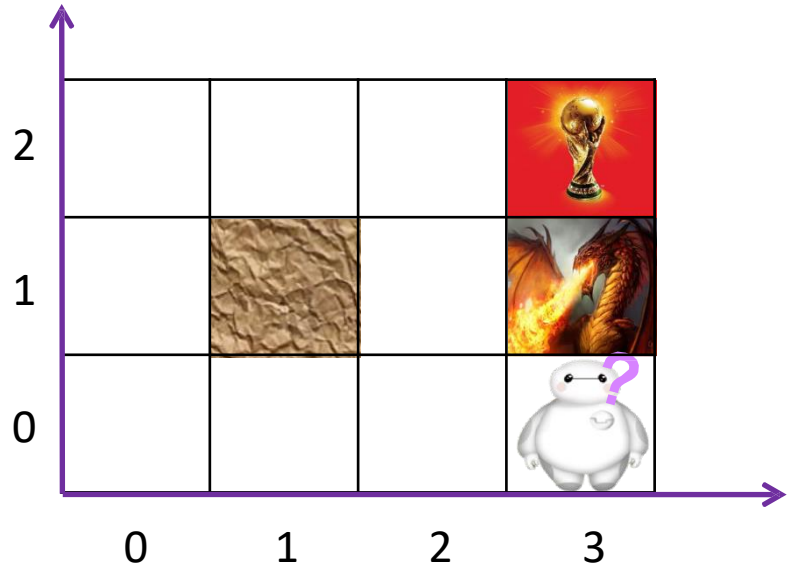
$$Q(s,a) = \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \left[\mathbf{P}(s'|s,a) \max_{a'} Q(s',a') \right]$$

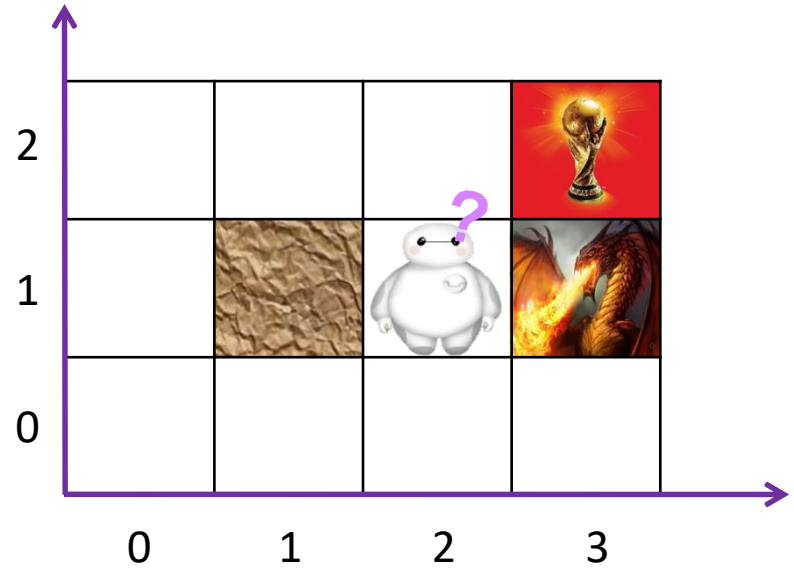
Bellman Equations

Quiz

- The learning task for RL scenarios is to learn an optimal policy in the sense that

$$\pi^* := \text{argmax}_\pi V^\pi(s), \forall s.$$



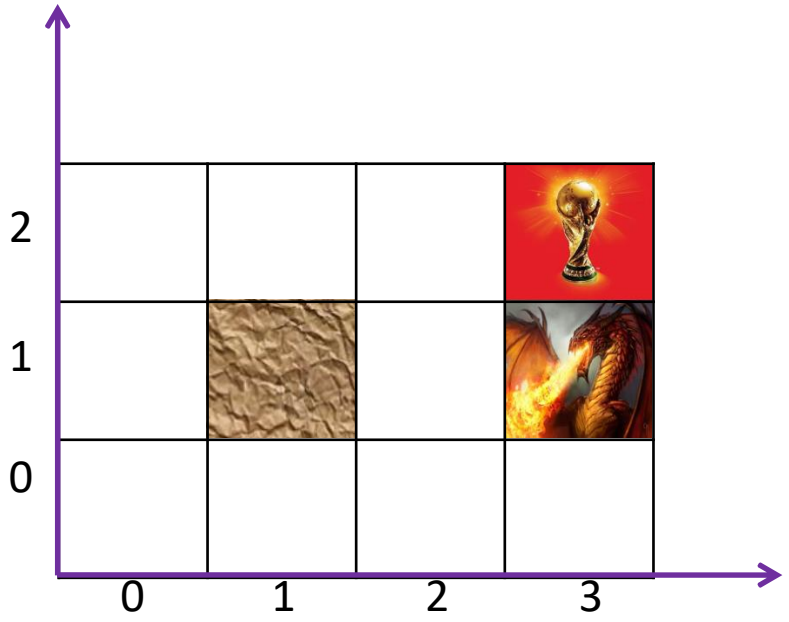


What are the best actions at states (3,0) and (2,1), i.e., $\pi^*((3,0))$ and $\pi^*((2,1))$?

Planning Algorithms

Planning

- Planning: we assume that the agent has perfect knowledge of the environment; thus, to find the optimal policy, there is no need for the agent to actually perform actions and interact with the environment



Known

$\mathbf{P}(s'|s,a)$: state transition

$\mathbf{P}(r|s,a)$: reward

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

Initialize $V(s)$ to arbitrary values

while termination conditions does not hold

For $s \in \mathcal{S}$

For $a \in \mathcal{A}$

$$Q(s,a) \leftarrow \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \mathbf{P}(s'|s,a)V(s')$$

$$V(s) \leftarrow \max_a Q(s,a)$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

Example

$$V \leftarrow 0$$

$$Q((0,0),\text{up}) \leftarrow 0 + 0.9 \times (0.8 \times V((0,1)) + 0.1 \times V((0,0)) + 0.05 \times V((0,0)) + 0.05 \times V((1,0))) = 0$$

$$Q((0,0),\text{down}) \leftarrow 0 + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((1,0))) = 0$$

$$Q((0,0),\text{left}) \leftarrow 0 + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((0,1))) = 0$$

$$Q((0,0),\text{right}) \leftarrow 0 + 0.9 \times (0.8 \times V((1,0)) + 0.15 \times V((0,0)) + 0.05 \times V((0,1))) = 0$$

$$V((0,0)) \leftarrow \max\{Q((0,0),\text{up}), Q((0,0),\text{down}), Q((0,0),\text{left}), Q((0,0),\text{right})\} = 0$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

Example

$$V \leftarrow 0$$

$$Q((0,0),\text{up}) \leftarrow 0 + 0.9 \times (0.8 \times V((0,1)) + 0.1 \times V((0,0)) + 0.05 \times V((0,0)) + 0.05 \times V((1,0))) = 0$$

$$Q((0,0),\text{down}) \leftarrow 0 + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((1,0))) = 0$$

$$Q((0,0),\text{left}) \leftarrow 0 + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((0,1))) = 0$$

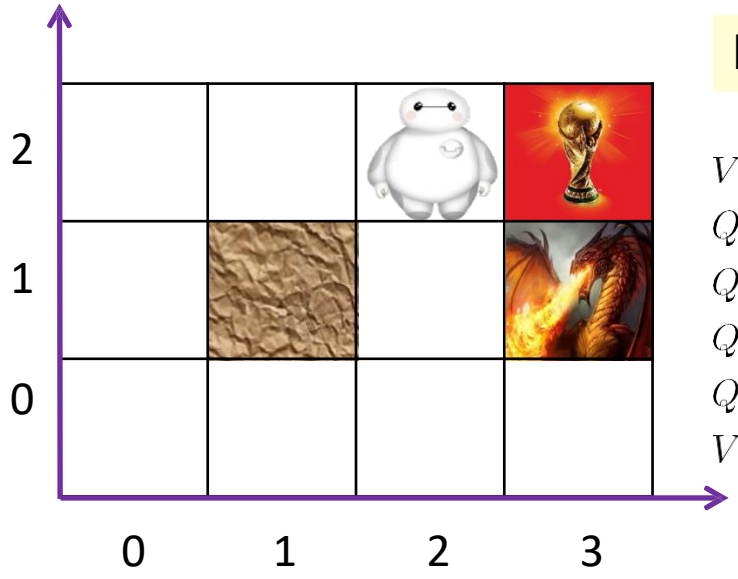
$$Q((0,0),\text{right}) \leftarrow 0 + 0.9 \times (0.8 \times V((1,0)) + 0.15 \times V((0,0)) + 0.05 \times V((0,1))) = 0$$

$$V((0,0)) \leftarrow \max\{Q((0,0),\text{up}), Q((0,0),\text{down}), Q((0,0),\text{left}), Q((0,0),\text{right})\} = 0$$

Nothing happens

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

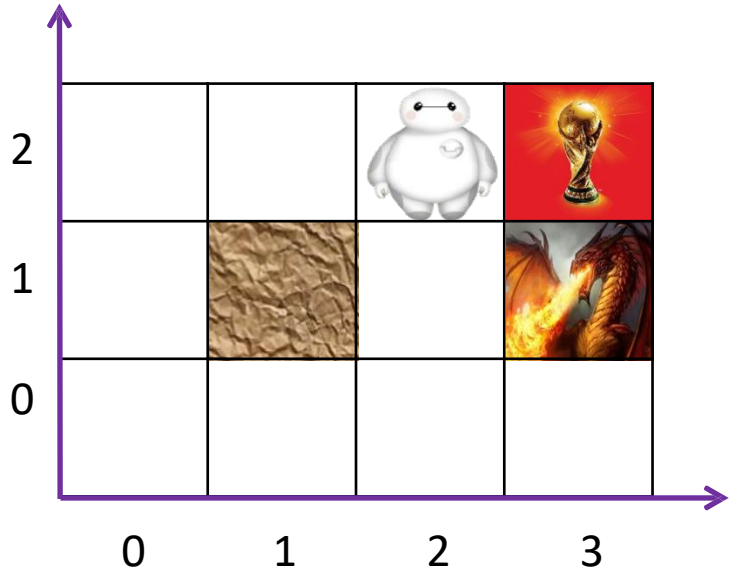


Example

$$V \leftarrow 0$$
$$Q((2,2),\text{up}) \leftarrow 5 + 0.9 \times (0.9 \times V((2,2)) + 0.05 \times V((1,2)) + 0.05 \times V((3,2))) = 5$$
$$Q((2,2),\text{down}) \leftarrow 5 + 0.9 \times (0.8 \times V((2,1)) + 0.1 \times V((2,2)) + 0.05 \times V((1,2)) + 0.05 \times V((3,2))) = 5$$
$$Q((2,2),\text{left}) \leftarrow 0 + 0.9 \times (0.8 \times V((1,2)) + 0.15 \times V((2,2)) + 0.05 \times V((2,1))) = 0$$
$$Q((2,2),\text{right}) \leftarrow 80 + 0.9 \times (0.8 \times V((3,2)) + 0.15 \times V((2,2)) + 0.05 \times V((2,1))) = 80$$
$$V((2,2)) \leftarrow \max\{Q((0,0),\text{up}), Q((0,0),\text{down}), Q((0,0),\text{left}), Q((0,0),\text{right})\} = 80$$

Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy



Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

> ***Theorem:*** *For any initial value V, the sequence generated by the value iteration algorithm converges to V^*.*

  - The key to the proof is the contraction mapping theorem

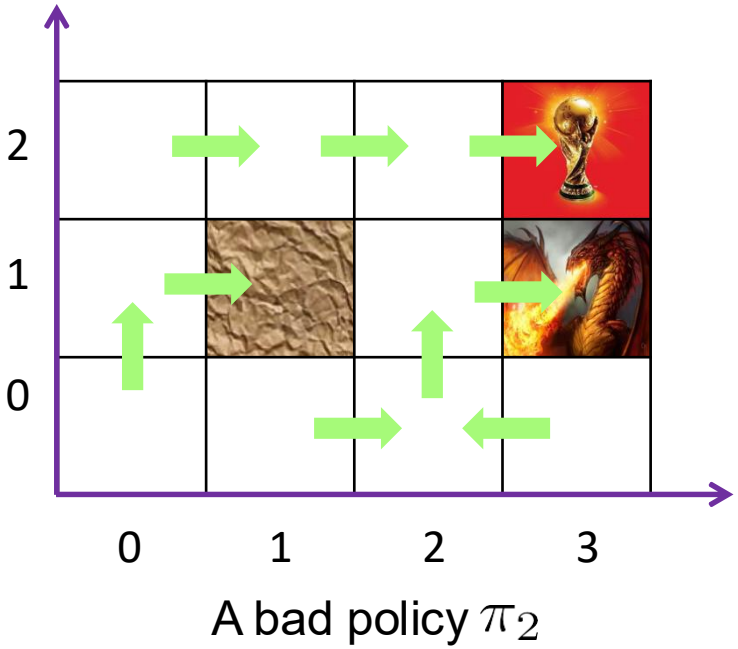
Policy Iteration

- Policy iteration improves the policy directly

$$
\begin{aligned}
&\textbf{Initialize } \pi \leftarrow \pi_2, \pi' \neq \pi_2 \\
&\textbf{while}(\pi \neq \pi') \\
&\quad V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} \\
&\quad \pi' \leftarrow \pi \\
&\quad\quad \textbf{For } s \in \mathcal{S} \\
&\quad\quad \pi(s) \leftarrow \text{argmax}_a \, \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \mathbf{P}(s'|s,a) V(s')
\end{aligned}
$$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

Initialize $\pi \leftarrow \pi_2, \pi' \neq \pi_2$

while$(\pi \neq \pi')$

1^{st} $V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi}$

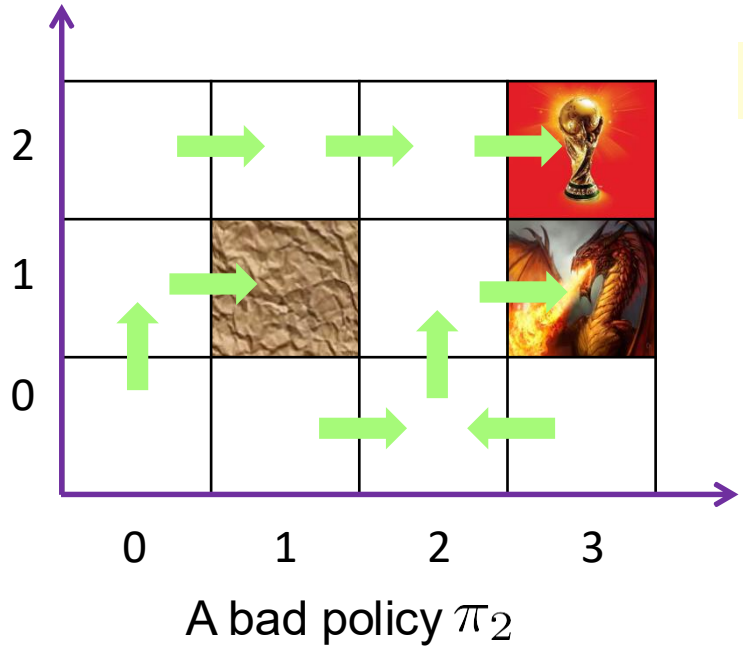
$\pi' \leftarrow \pi$

For $s \in \mathcal{S}$

$\pi(s) \leftarrow \text{argmax}_a \, \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \mathbf{P}(s'|s,a) V(s')$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

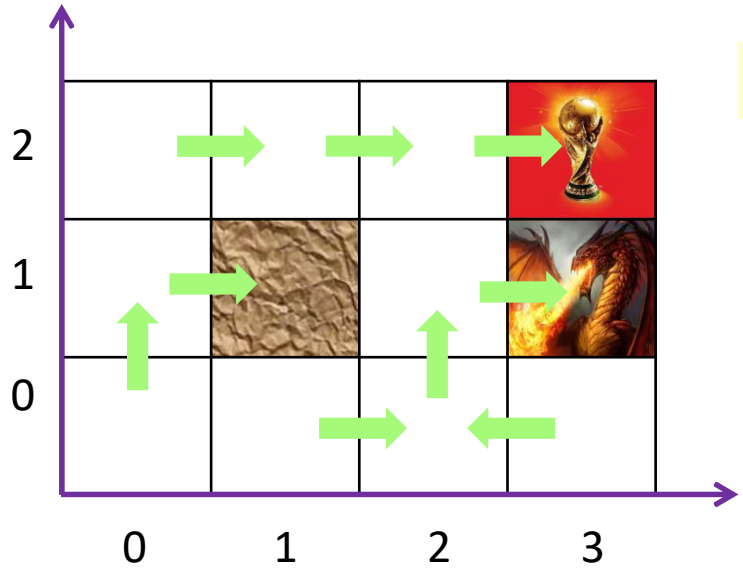
1^{st} iteration

$$V^{\pi} = \begin{pmatrix} V^{\pi}(0,0) \\ V^{\pi}(1,0) \\ V^{\pi}(2,0) \\ V^{\pi}(3,0) \\ V^{\pi}(0,1) \\ V^{\pi}(2,1) \\ V^{\pi}(3,1) \\ V^{\pi}(0,2) \\ V^{\pi}(1,2) \\ V^{\pi}(2,2) \\ V^{\pi}(3,2) \end{pmatrix}$$

11 states in total

Policy Iteration

- Policy iteration improves the policy directly



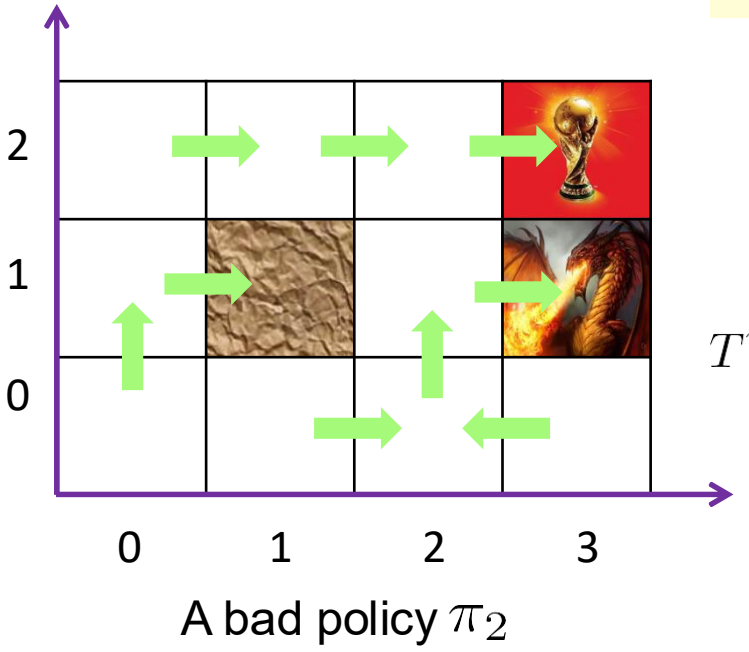
A bad policy π_2

1^{st} iteration

$$R^{\pi} = \begin{pmatrix} r((0,0),\text{up}) \\ r((1,0),\text{right}) \\ r((2,0),\text{up}) \\ r((3,0),\text{left}) \\ r((0,1),\text{right}) \\ r((2,1),\text{right}) \\ r((3,1),\text{END}) \\ r((0,2),\text{right}) \\ r((1,2),\text{right}) \\ r((2,2),\text{right}) \\ r((3,2),\text{END}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ -80 \\ 0 \\ 0 \\ 0 \\ 80 \\ 0 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly



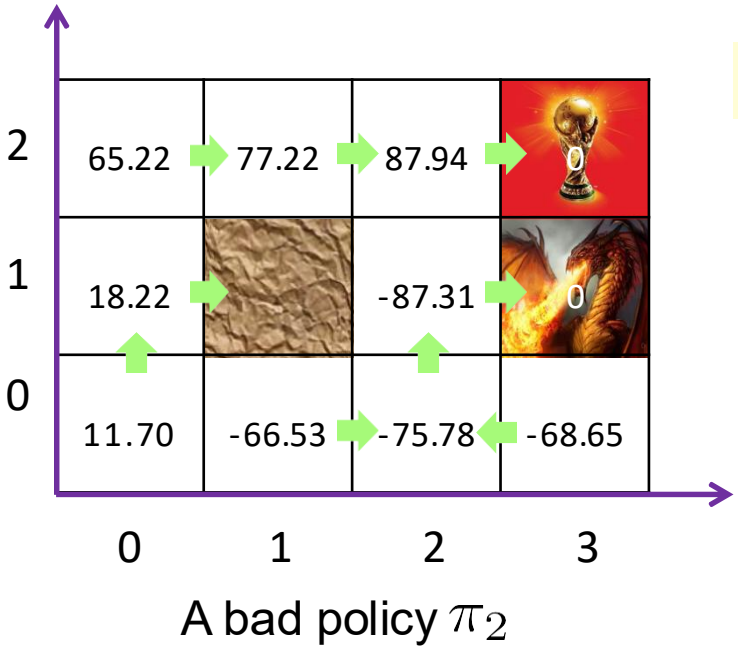
A bad policy π_2

1^{st} iteration

$$T^{\pi} = \begin{pmatrix} 0.15 & 0.05 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0.1 & 0.05 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.15 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0.05 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 & 0.1 & 0.8 & 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0.15 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0.15 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly



A bad policy π_2

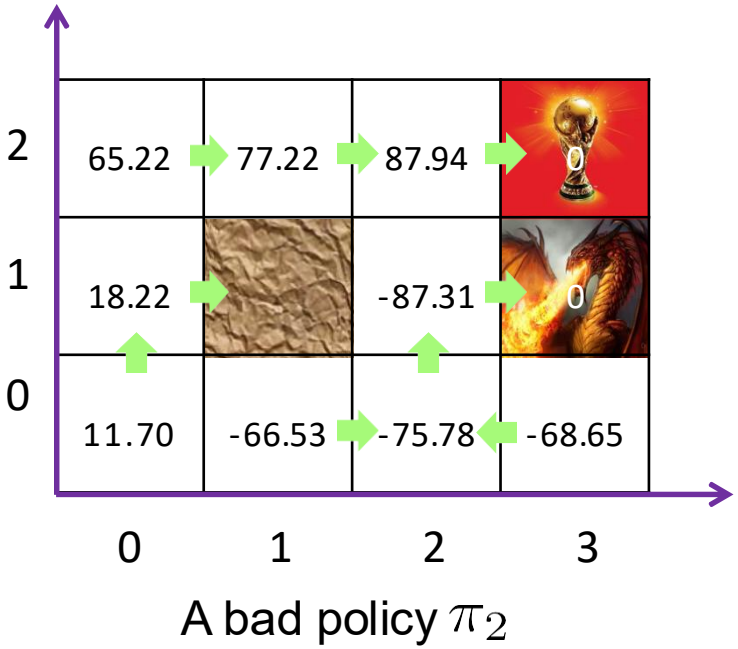
1^{st} iteration

$$V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} = \begin{pmatrix} 11.70 \\ -66.53 \\ -75.78 \\ -68.85 \\ 18.22 \\ -87.31 \\ 0 \\ 65.22 \\ 77.22 \\ 87.94 \\ 0 \end{pmatrix}$$

Policy Iteration

- Policy iteration improves the policy directly

1^{st} iteration: update the policy



A bad policy π_2

$$Q((0,0),\text{up}) = \mathbf{E}[r((0,0),\text{up})] + 0.9 \times (0.8 \times V((0,1)) + 0.15 \times V((0,0)) + 0.05 \times V((1,0)))$$
$$= 0 + 0.9 \times (0.8 \times 18.22 + 0.15 \times 11.70 + 0.05 \times -66.53)$$
$$= 11.70$$

$$Q((0,0),\text{down}) = \mathbf{E}[r((0,0),\text{down})] + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((1,0)))$$
$$= 0 + 0.9 \times (0.95 \times 11.70 + 0.05 \times (-66.53))$$
$$= 7.01$$

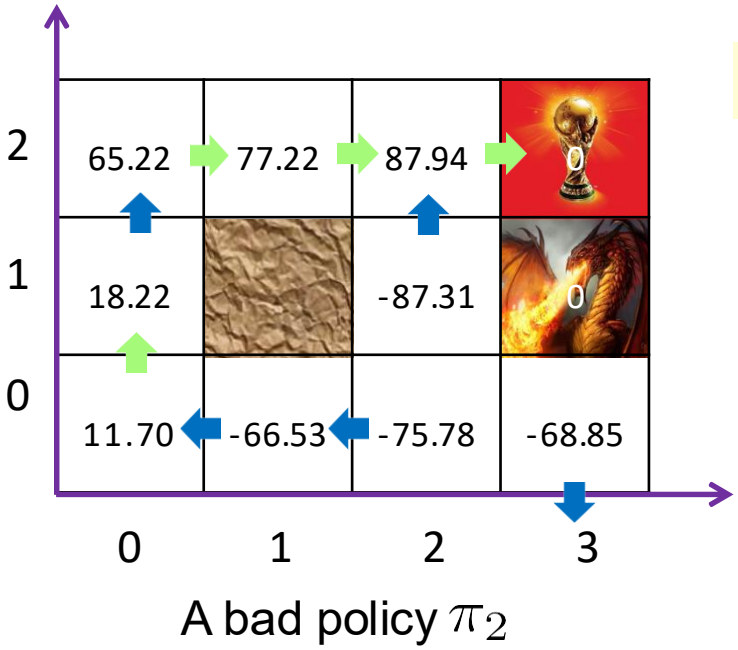
$$Q((0,0),\text{left}) = \mathbf{E}[r((0,0),\text{left})] + 0.9 \times (0.95 \times V((0,0)) + 0.05 \times V((0,1)))$$
$$= 0 + 0.9 \times (0.95 \times 11.70 + 0.05 \times 18.22)$$
$$= 10.82$$

$$Q((0,0),\text{right}) = \mathbf{E}[r((0,0),\text{right})] + 0.9 \times (0.8 \times V((1,0)) + 0.15 \times V((0,0)) + 0.05 \times V((0,1)))$$
$$= 0 + 0.9 \times (0.8 \times (-66.53) + 0.15 \times 11.70 + 0.05 \times 18.22)$$
$$= -45.50$$

$$\pi((0,0)) = \text{argmax}_{\{\text{up, down, left, right}\}}\{Q((0,0),\text{up}), Q((0,0),\text{down}), Q((0,0),\text{left}), Q((0,0),\text{right})\}$$
$$= \text{up}$$

Policy Iteration

- Policy iteration improves the policy directly

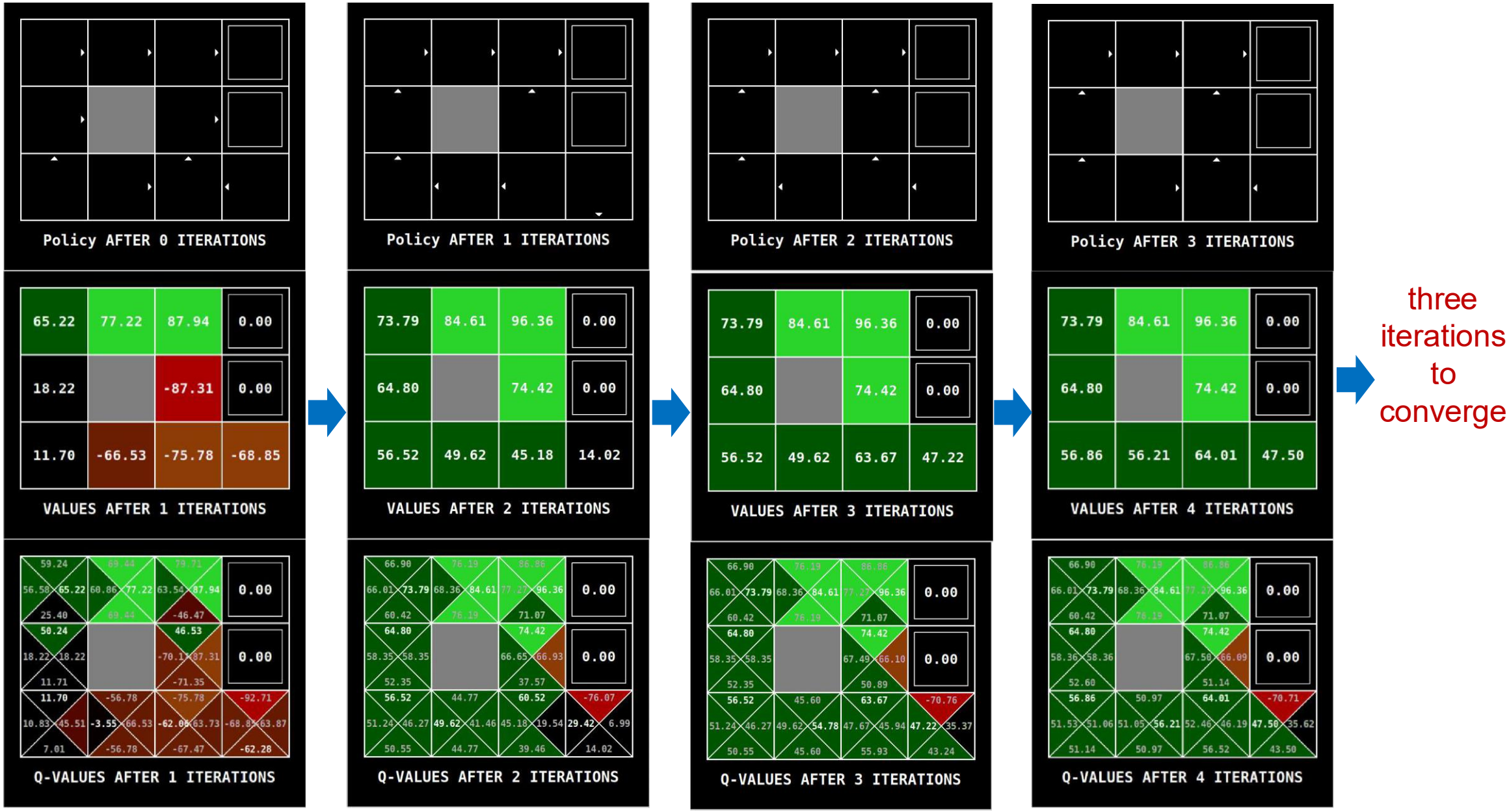


A bad policy π_2

1^{st} iteration: update the policy

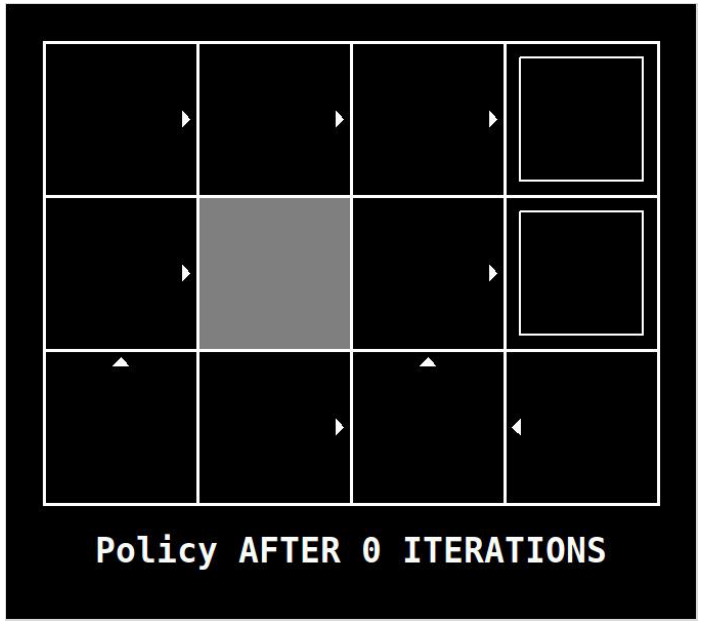
$$\pi((0,0)) = \text{up}$$
$$\pi((1,0)) = \text{left}$$
$$\pi((2,0)) = \text{left}$$
$$\pi((3,0)) = \text{down}$$
$$\pi((0,1)) = \text{up}$$
$$\pi((2,1)) = \text{up}$$
$$\pi((3,1)) = \text{END}$$
$$\pi((0,2)) = \text{right}$$
$$\pi((1,2)) = \text{right}$$
$$\pi((2,2)) = \text{right}$$
$$\pi((3,2)) = \text{END}$$

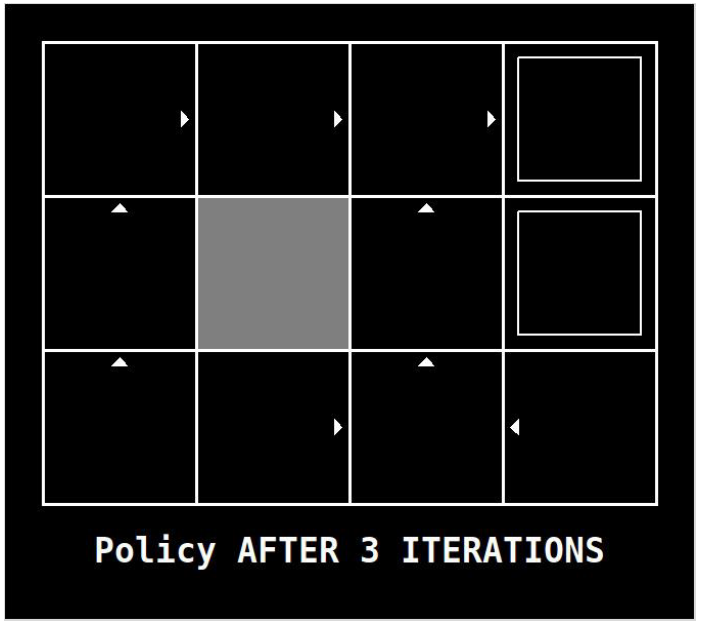
Policy Iteration



Policy Iteration

The optimal policy

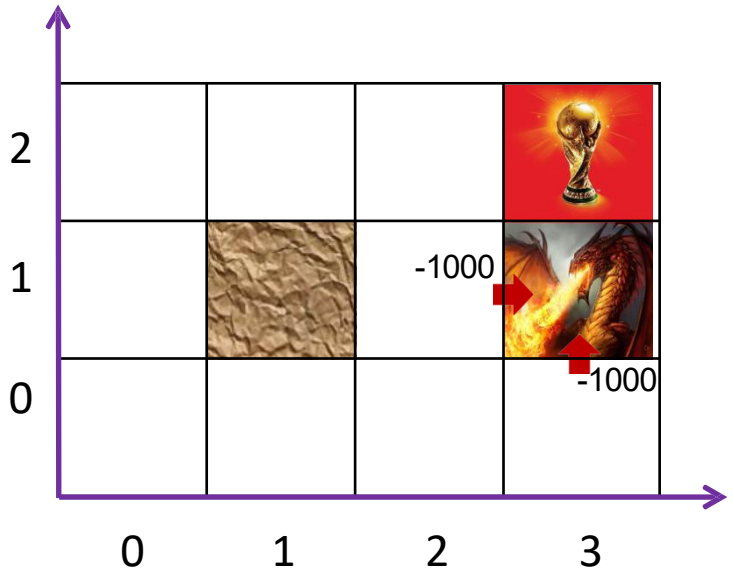




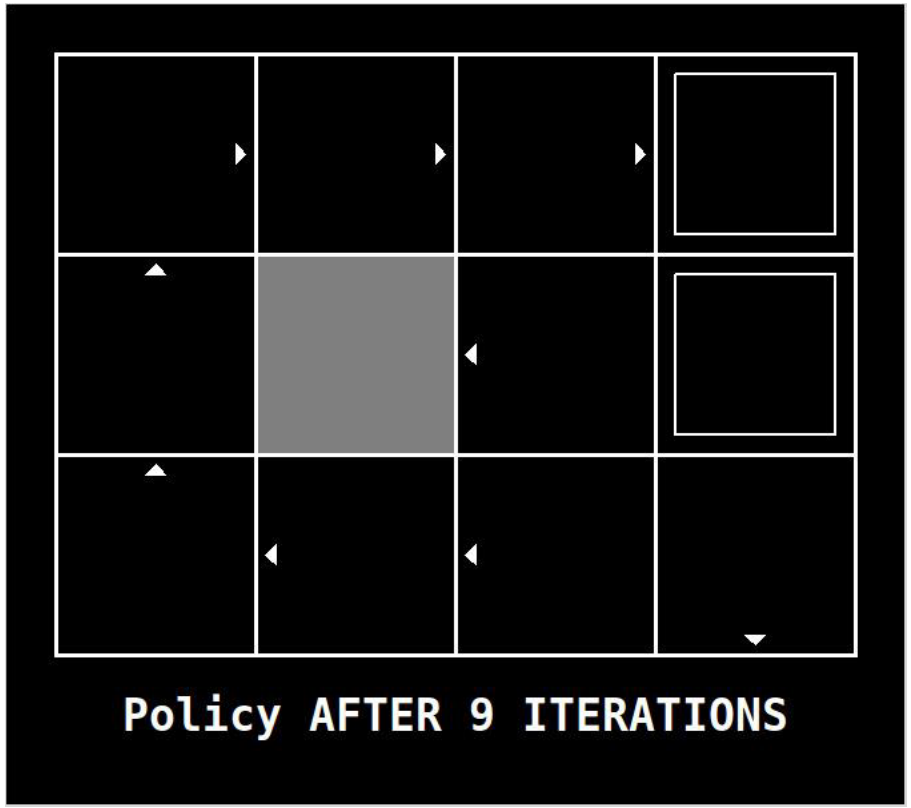
Is this an always winning policy?

Policy Iteration

- What if the reward for getting into (3,1) is -1000?



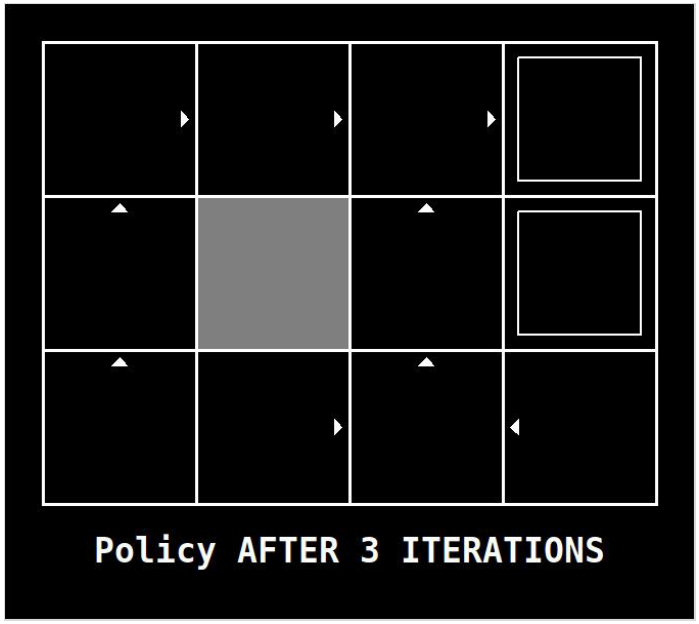
The optimal policy



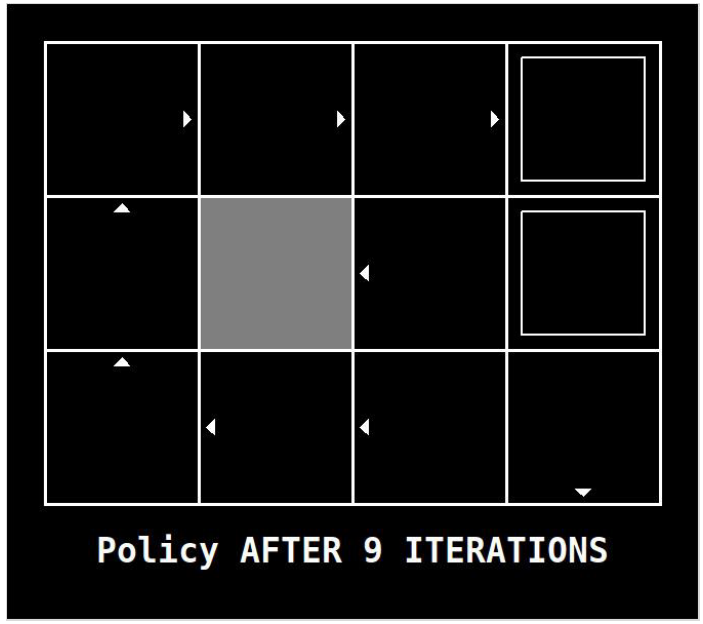
This is an always winning policy (why?).

Policy Iteration

A mostly winning policy



A never lose policy

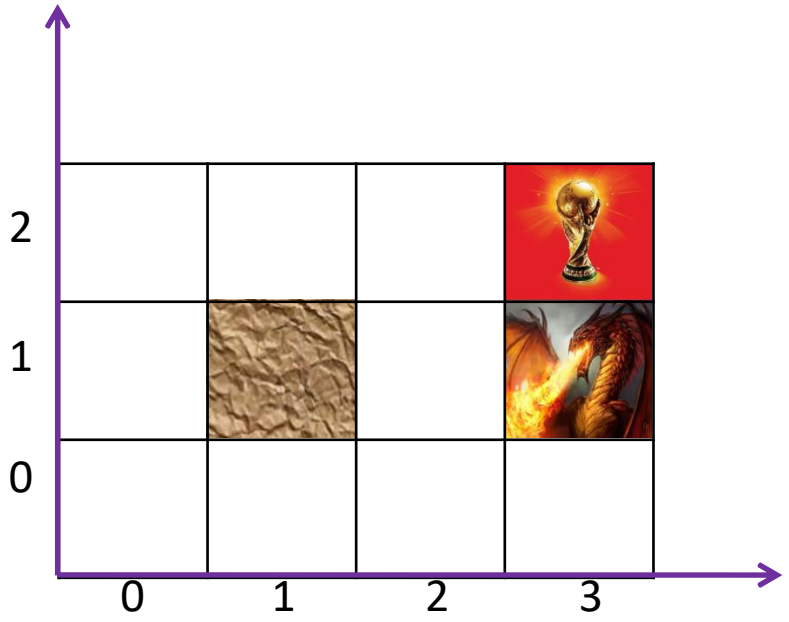


- For the same task, different reward strategies lead to different optimal policy
- According to your preference, you need to carefully design your reward strategy

Learning Algorithms

Learning

- Learning: as the environment model, i.e., the transition and reward, is unknown, the agent may need to learn them based on the training information.



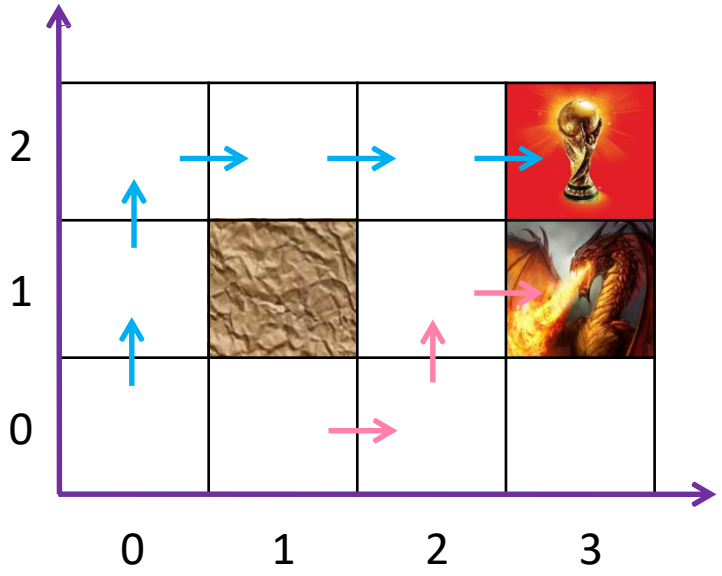
Unknown

$\mathbf{P}(s'|s,a)$: state transition

$\mathbf{P}(r|s,a)$: reward

Learning

- Learning: as the environment model, i.e., the transition and reward probabilities, is unknown, the agent may need to learn them based on the training information.
  - Model-free approach: the agent learns the optimal policy directly, e.g., Q-learning
  - Model-based approach: the agent first learns the environment model and then the optimal policy

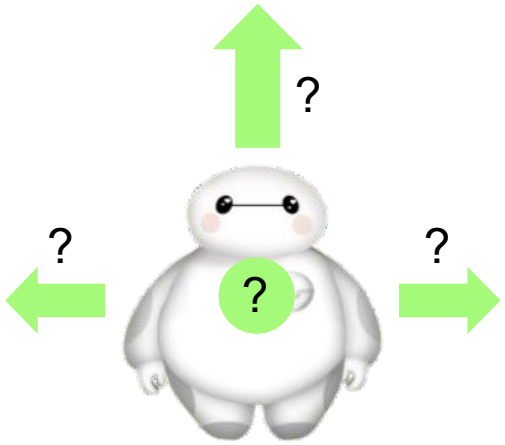


Examples of training data

(0,0) —up, 0→ (0,1) —up, 0→ (0,2) —right, 0→ (1,2) —right, 0→ (2,3) —right, 100→ (3,2)

(1,0) —right, 0→ (2,0) —up, 0→ (2,1) —right, -100→ (3,1)

Nondeterministic Rewards and Actions



Unknown

$\mathbf{P}(s'|s,a)$: state transition

$\mathbf{P}(r|s,a)$: reward

How to find the optimal policy without the state transition and reward probabilities?

The Q-learning Algorithm

Recall the Q-learning algorithm for the deterministic environment

- Initialize the matrix $\hat{Q}$ to zero
- Observe the current state s
- Do forever:
  - Pick and perform an action a
  - Receive immediate reward r
  - Observe the new state s'
  - Update

$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a')$$

What if r and s' become random variables?

- $s \leftarrow s'$

A sufficient condition for $\hat{Q}(s,a)$ to converge is to visit each state-action pair infinitely often

The Q-learning Algorithm

- For the stochastic environment, we replace the random variables with their expectations in the definition of Q values.

$$Q(s,a) = \mathbf{E}[r(s,a)] + \gamma \sum_{s'} \left[\mathbf{P}(s'|s,a) \max_{a'} Q(s',a') \right]$$

expectations

Q: How to find the expectation of a random variable X?

A: Keep sampling and recording its running average

$$\frac{x_1 + x_2 + \ldots + x_{n+1}}{n+1} = \frac{x_1 + x_2 + \ldots + x_n}{n} + \frac{1}{n+1}\left(x_{n+1} - \frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

$$\hat{X} \leftarrow \hat{X} + \frac{1}{n+1}(x_{n+1} - \hat{X})$$

the running average on n+1 samples

the running average on n samples

the estimation of the expectation of X

The Q-learning Algorithm

Initialize $\hat{Q}$ arbitrarily

For all episodes

Initialize s

Alpaydin 2014, Chapter 18

Repeat

Choose a using policy derived from Q, e.g., ϵ-greedy

Take action a, observe r and s'

Update $\hat{Q}(s,a)$:

$$\alpha_n = \frac{1}{1 + n((s,a))}$$

the number of visits of (s, a)

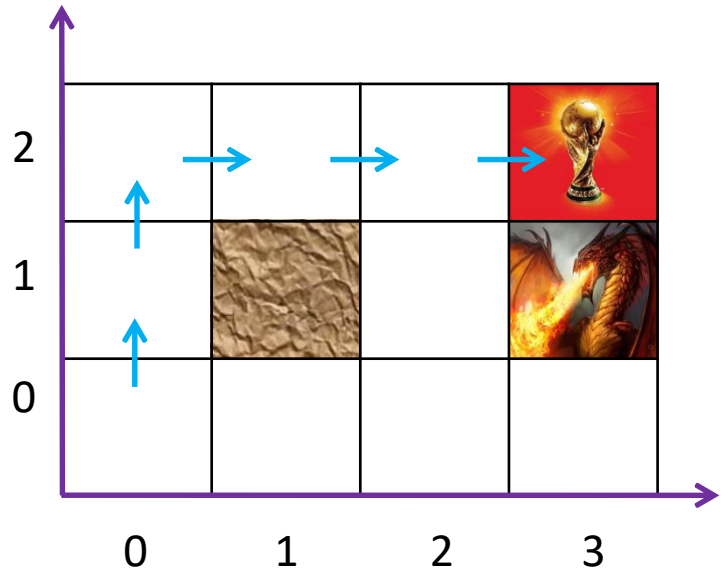
$$\hat{Q}(s,a) \leftarrow \hat{Q}(s,a) + \alpha_n(r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a))$$

$s \leftarrow s'$

Until s is goal state

There are other ways to select α_n to guarantee that $\hat{Q}$ converges to its optimal value. Mitchell 1997, Chapter 13

The Q-learning Algorithm

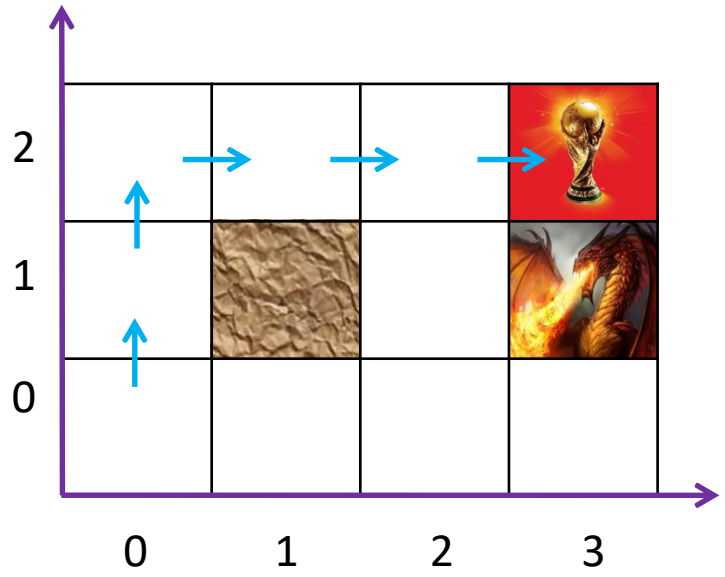


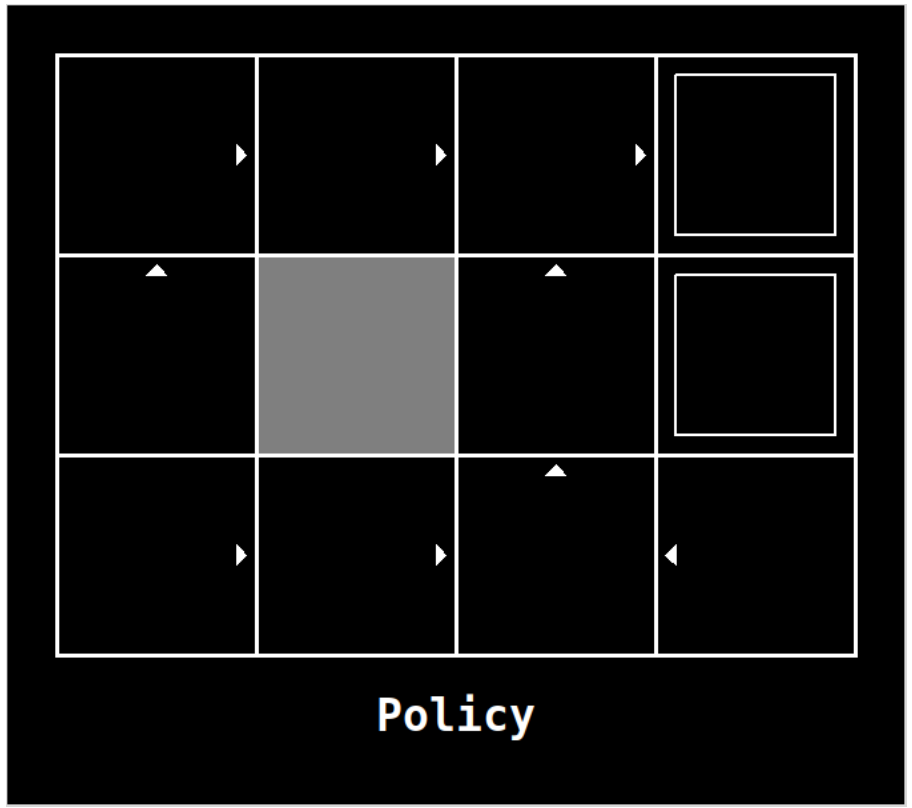
$$(0,0) \xrightarrow[0]{\text{up}} (0,1) \xrightarrow[0]{\text{up}} (0,2) \xrightarrow[0]{\text{right}} (1,2) \xrightarrow[0]{\text{right}} (2,3) \xrightarrow[100]{\text{right}} (3,2)$$

- an example episode
- the initial state in each episode could NOT be fixed (why?)

$\epsilon = 0.3$

The Q-learning Algorithm





$$(0,0) \xrightarrow[0]{\text{up}} (0,1) \xrightarrow[0]{\text{up}} (0,2) \xrightarrow[0]{\text{right}} (1,2) \xrightarrow[0]{\text{right}} (2,3) \xrightarrow[100]{\text{right}} (3,2)$$

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- an example episode
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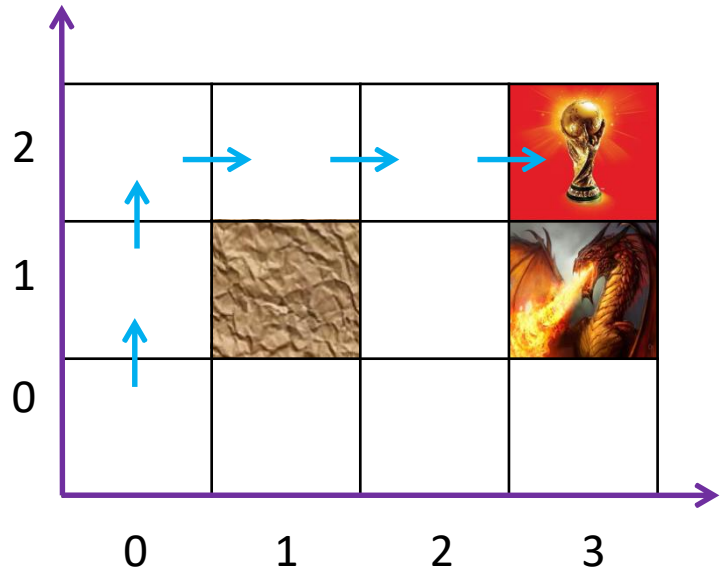
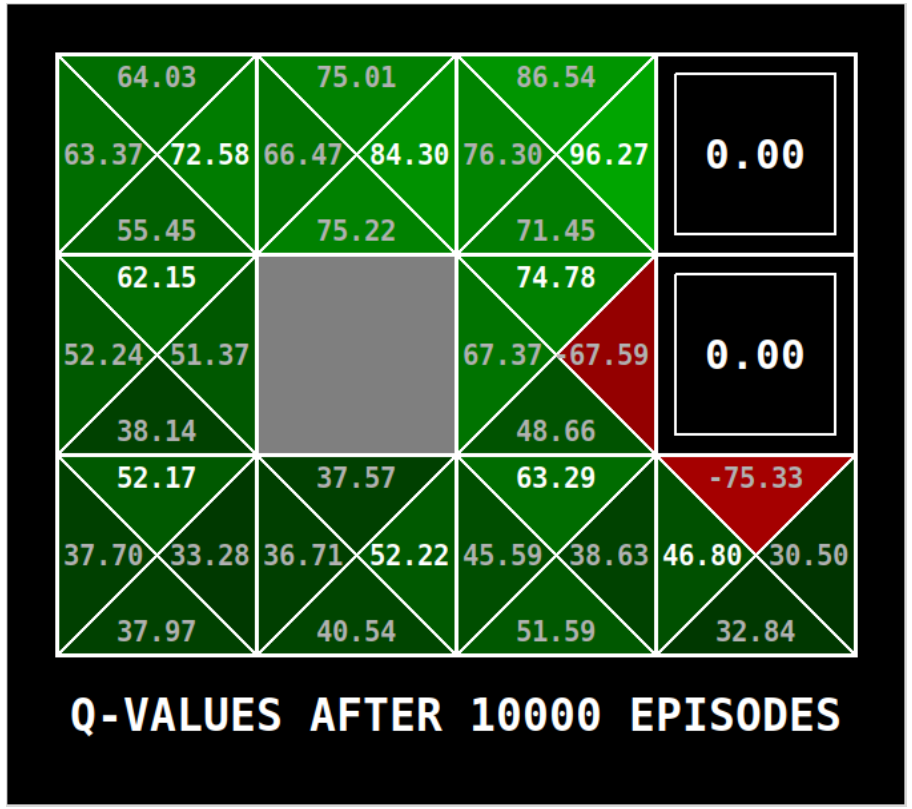
The Q-learning Algorithm





$$(0,0) \xrightarrow[0]{up} (0,1) \xrightarrow[0]{up} (0,2) \xrightarrow[0]{right} (1,2) \xrightarrow[0]{right} (2,3) \xrightarrow[100]{right} (3,2)$$

- an example episode
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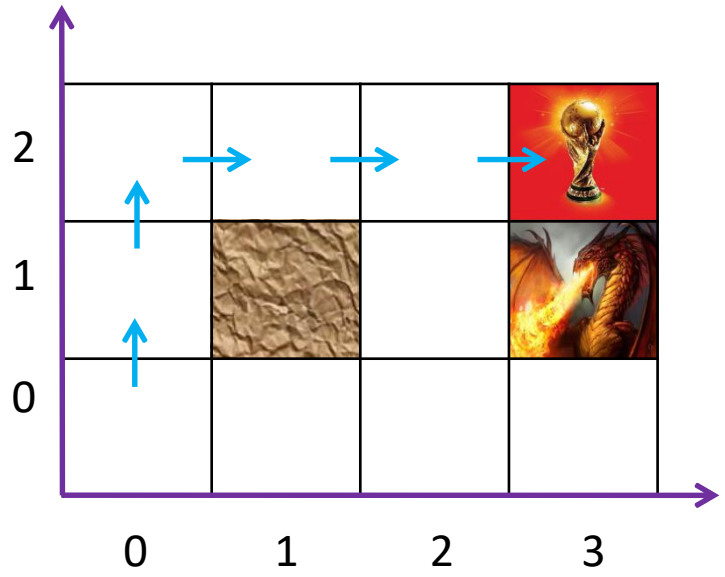
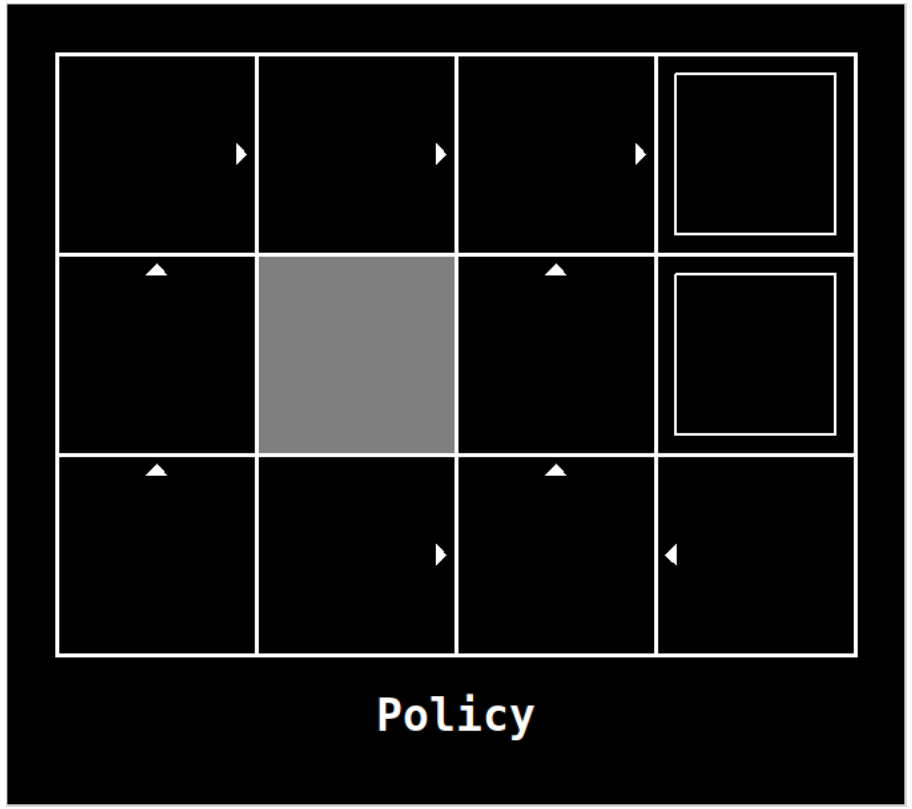
The Q-learning Algorithm





$$(0,0) \xrightarrow[0]{\text{up}} (0,1) \xrightarrow[0]{\text{up}} (0,2) \xrightarrow[0]{\text{right}} (1,2) \xrightarrow[0]{\text{right}} (2,3) \xrightarrow[100]{\text{right}} (3,2)$$

ϵ = 0.3

- an example episode
- the initial state in each episode could NOT be fixed (why?)

Questions