# Proof of Bessel's Correction

**Bessel's correction is the division of the sample variance by $N-1$ rather than $N$. I walk the reader through a quick proof that this correction results in an unbiased estimator of the population variance.**

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Consider $N$ i.i.d. random variables, $x_1, x_2, \dots, x_n$ and a sample mean $\bar{x}$. When computing the sample variance $s^2$, students are told to divide by $N-1$ rather than $N$:

$$s^2 = \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \bar{x})^2.$$

When first learning about this fact, I was shown computer simulations but no mathematical proof of why this must hold. The goal of this post is to provide a quick proof of why this correction makes sense.

The proof outline is straightforward: we need to show that the estimator in Equation $1$ below is biased, and that we can correct this bias by dividing by $N-1$ rather than $N$. For an estimator to be *unbiased*, the expectation of that estimator must equal the population parameter. In our case, if the sample variance is $s^2$ and the population variance is $\sigma^2$, we want

$$\mathbb{E}[s^2] = \sigma^2.$$

Let's begin.

## Proof

Let's prove that the following estimator for the population variance is biased:

$$s^2 = \frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x})^2. \tag{1}$$

First, let's take the expectation of this estimator and manipulate it:

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})^2\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}(x_n^2 - 2x_n\bar{x} + \bar{x}^2)\right] \\
&= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}x_n^2 - 2\bar{x}\frac{1}{N}\sum_{n=1}^{N}x_n + \frac{1}{N}\sum_{n=1}^{N}\bar{x}^2\right] \\
&\stackrel{\star}{=} \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}x_n^2\right] - \mathbb{E}\left[2\bar{x}^2\right] + \mathbb{E}\left[\bar{x}^2\right] \\
&= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}x_n^2\right] - \mathbb{E}\left[\bar{x}^2\right] \\
&\stackrel{\dagger}{=} \mathbb{E}\left[x_n^2\right] - \mathbb{E}\left[\bar{x}^2\right].
\end{aligned}
$$

Note that step $\star$ holds because

$$
\sum_{n=1}^{N} x_n = N\bar{x}.
$$

while step $\dagger$ holds because the data are i.i.d., i.e.

$$
\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}x_n^2\right] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[x_n^2\right] = \mathbb{E}\left[x_n^2\right].
$$

Now note that since $x_n$ is an i.i.d. random variable, any of the $x_n \in \{x_1, x_2, \dots x_N\}$ has the same variance. Furthermore, recall that for any random variable $Y$,

$$
\operatorname{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \quad \Longrightarrow \quad \mathbb{E}[Y^2] = \operatorname{Var}(Y) + \mathbb{E}[Y]^2.
$$

So we can write

$$
\begin{aligned}
\mathbb{E}\left[x_n^2\right] &= \operatorname{Var}(x_n) + \mathbb{E}[x_n]^2 \\
&= \sigma^2 + \mu^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\bar{x}^2\right] &= \operatorname{Var}(\bar{x}) + \mathbb{E}[\bar{x}]^2 \\
&\stackrel{\star}{=} \frac{\sigma^2}{N} + \mu^2.
\end{aligned}
$$

Step $\star$ holds because

$$
\begin{aligned}
\mathrm{Var}(\bar{x}) &= \mathrm{Var}\Big(\frac{1}{N}\sum_{n=1}^{N} x_n\Big) \\
&\stackrel{\text{iid}}{=} \frac{1}{N^2}\sum_{n=1}^{N}\mathrm{Var}(x_n) \\
&= \frac{1}{N^2}\sum_{n=1}^{N}\sigma^2 \\
&= \frac{\sigma^2}{N}.
\end{aligned}
$$

Finally, let's put everything together:

$$
\begin{aligned}
\mathbb{E}[s^2] &= \sigma^2 + \mu^2 - \Big(\frac{\sigma^2}{N} + \mu^2\Big) \\
&= \sigma^2\Big(1 - \frac{1}{N}\Big).
\end{aligned} \tag{3}
$$

What we have shown is that our estimator is off by a constant, $\left(1 - \frac{1}{N}\right) = \left(\frac{N-1}{N}\right)$. If we want an unbiased estimator, we should multiply both sides of Equation $3$ by the inverse of the constant:

$$
\mathbb{E}\Big[\Big(\frac{N}{N-1}\Big)s^2\Big] = \mathbb{E}\Big[\frac{1}{N-1}\sum_{n=1}^{N}(x_n - \bar{x})^2\Big] = \sigma^2.
$$

And this new estimator is exactly what we wanted to prove. Bessel's correction results in an unbiased estimator for the population variance.