1 Introduction

Why do we care about statistic and probability in physics?

- Experimental uncertainty: how confident are you about your measurements? Is your experimental data consistent with your model?
- Physics is statistical: statistical mechanics (thermodynamics) and quantum mechanics (statistical ensembles).

2 Events, Sample Spaces, RVs

- EVENT: a possible result of a measurement or experiment. Probability describes the statistics of an event happening.
- SAMPLE SPACE: the set of all events. For example, for a coin toss, the event space E is given by $E = \{H, T\}$.
- RANDOM VARIABLE: a variable that represents the numerical result of a random process. A RV takes on different values by chance. Each value the RV can take represents an event in the sample space. For example: if X is the result of a dice roll, then after rolling a 5, the RV has value $X = 5$
- Function of random variables: suppose X, Y are random variables. We can define another random variable as a function of X, Y,

$$Z = f(X, Y) \tag{2.1}$$

NOTE: the probability distribution of functions of random variables are not trivial. For example, let X, Y be independent RVs for a coin toss (0 tails, 1 heads) and $Z = X + Y$.

$$P(Z = 0) = P(X = 0 \cap Y = 0) = 1/4 \tag{2.2}$$
$$P(Z = 1) = P(X = 0 \cap Y = 1) + P(X = 1 \cap Y = 0) = 1/2 \tag{2.3}$$
$$P(Z = 2) = P(X = 1 \cap Y = 1) = 1/4 \tag{2.4}$$

In general,

$$P(Z = n) = \sum_{x,y:f(X=x,Y=y)=n} P(X = x \cap Y = y) \tag{2.5}$$

3 Probability

- For equally likely outcomes, the probability of event A is,

$$P(A) = \frac{\#A}{\#\text{total}} \tag{3.1}$$

- Complements: $P(A^C) = 1 - P(A)$ (draw diagram)

- Union ($\cup$) and intersection ($\cap$)
- Inclusion-exclusion rule: (for 2 sets)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{3.2}$$

This can be generalized: the cardinality of the union of n sets:

1. Include the cardinalities of the sets.
2. Exclude the cardinalities of the pairwise intersections.
3. Include the cardinalities of the triple-wise intersections.
4. Exclude the cardinalities of the quadruple-wise intersections.
5. ... continue, until the n-tuple-wise intersection

- Conditional probability: $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$
- Bayes rule (from above):

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} \tag{3.3}$$

- INDEPENDENCE: if events A, B are independent,

$$P(A \cap B) = P(A)P(B) \tag{3.4}$$

4 Expectation and uncertainty

- The EXPECTATION value of a RV is the average value of the RV. For RV X,

$$E(X) = \sum_{x'} x' P(X = x') \tag{4.1}$$

- Properties:

1. $E(X + Y) = E(X) + E(Y)$ where X, Y do not have to be independent (there's a cool proof just by rearranging terms in the sum pg 6)
2. $E(a \cdot C) = a \cdot P(X)$ where a is a constant.
3. In general, $E[f(X, Y)] \neq f(E(Y), E(Y))$. For example, $E(X \cdot Y) \neq E(X)E(Y)$.

- UNCERTAINTY: Variance is expectation value of the squared deviation from the mean μ.

$$Var(X) = E[(X - \mu)^2] = E(X^2) - [E(X)]^2 \tag{4.2}$$

Properties of variance:

1. $Var(aX + b) = a^2 \cdot Var(X)$

In physics, we call the standard deviation the uncertainty,

$$SD(x) = \sqrt{Var(X)} \tag{4.3}$$

5 Distributions

- BERNOULLI trial: tossing a weighted coin with probability p of getting heads,

$$\text{Bernoulli}(p) = \begin{cases} 1, & p \\ 0, & 1-p=q \end{cases} \tag{5.1}$$

- BINOMIAL distribution: the probability of k successes in n Bernoulli(p) trials.

$$\text{Binomial}(p) = \binom{n}{k} p^k q^{n-k} \tag{5.2}$$

  n *choose* k: the number of ways to choose k successful trials among n total trials.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{5.3}$$

  Mean: np, Variance: npq

- The binomial theorem:

$$(p+q)^n = \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k} \tag{5.4}$$

- The binomial distribution can be generalized:

	2 outcomes per trial	**multiple outcomes per trial**
Independent trials (w/replacement)	binomial	multinomial
Dependent trails (wo/replacement)	hypergeometric	multivariate hypergeometric

- NORMAL distribution. Q: how tall is the Campanille? If I were to give you a meter stick, ladder, and a well-written liability release form, you could measure for yourself! If you were to measure the height 20 times, we would expect a slightly different value each time. If we were to plot the measured values, we would expect a distribution that looks like a normal distribution.

  ~~Data is normally distributed if the rate that the probability density falls is proportional to the distance from the mean.~~ (Almost)

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -k(x-\mu) \quad \Longrightarrow \quad f(x) = -\frac{k}{2}(x-\mu)^2 \tag{5.5}$$

  The probability density here is a parabola, and will go negative. Instead,

  Data is normally distributed if the *rate* that the probability density falls is *proportional* to the distance from the mean *and* the probability density at that location.

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -k(x-\mu)f(x) \tag{5.6}$$

$$\int \frac{1}{f(x)} \frac{\mathrm{d}f(x)}{\mathrm{d}x} \mathrm{d}x = \int -k(x-\mu)\mathrm{d}x \tag{5.7}$$

$$\ln(f) = -\frac{k}{2}(x-\mu)^2 + \ln(A) \tag{5.8}$$

$$f(x) = Ae^{-\frac{1}{2}(x-\mu)^2} \tag{5.9}$$

By normalization, we will find,

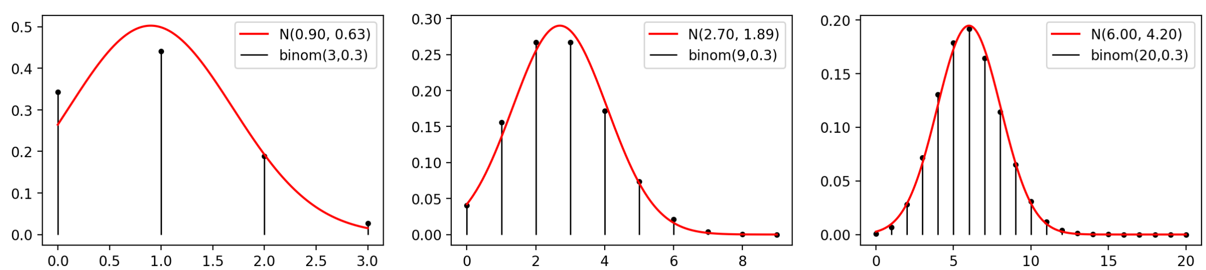
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2} \tag{5.10}$$

Notice that $f(x)$ is the probability density function. This means the probability is given by,

$$P(x \in [x_0, x_1]) = \int_{x_0}^{x_1} f(x)\mathrm{d}x \tag{5.11}$$

We can say the RV X is normally distributed with the notation $X \sim N(\mu, \sigma^2)$

- Normal approximation to the binomial: for n sufficiently large, the normal distribution approximates the binomial with mean np and variance npq.



- POISSON distribution: A Poisson process is a simple stochastic model for arrivals. What do we mean by arrivals?

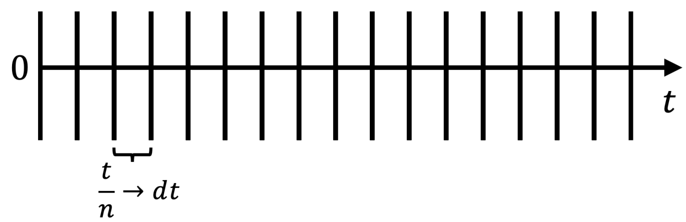
  Suppose you build a muon detector. For the size of the detector, you expect (i.e. on average) to observe λ arrivals per unit time. If you wait t seconds: what is the probability distribution of the number of muons you will have observed? This is modeled by the Poisson distribution.

  Assumptions:

  1. The average rate of an event (i.e. an arrival) is λ
  2. The events are independent
  3. Two events can not occur at the same time

  What is the probability that we observe x events in time t?

  We break up the time t into n intervals. For n large enough (i.e. when we take $n \to \infty$), the time interval is so short that, at most, one event can occur. The probability of this event occurring is $\lambda(t/n)$. Therefore, we can model each small time intervals as a Bernoulli trial with $p = \lambda t/n$



Then, the probability of x arrivals is given by the binomial distribution,

$$P(x \text{ arrivals}) = \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda t}{n}\right)^x \left(1 - \frac{\lambda t}{n}\right)^{n-x} \tag{5.12}$$

$$= \frac{n!}{x!(n-x)!} \frac{(\lambda t)^x}{n^x} \left(1 - \frac{\lambda t}{n}\right)^n \cancelto{1}{\left(1 - \frac{\lambda t}{n}\right)^{-x}} \tag{5.13}$$

$$= \frac{n!}{(n-x)!\; n^x} \frac{(\lambda t)^x}{x!} e^{-\lambda t} \tag{5.14}$$

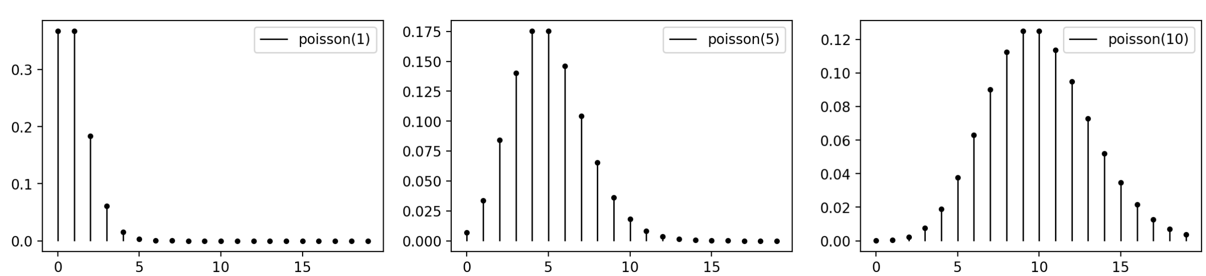
$$= \left[\frac{n(n-1)(n-2)\dots(n-x+1)}{n^x}\right] \frac{(\lambda t)^x}{x!} e^{-\lambda t} \tag{5.15}$$

$$= \cancelto{1}{\left[\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \dots \cdot \frac{n-x+1}{n}\right]} \frac{(\lambda t)^x}{x!} e^{-\lambda t} \tag{5.16}$$

$$\tag{5.17}$$

$$P(x \text{ arrivals}) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} \tag{5.18}$$

The mean and variance of Poisson(λt) is λt



- The Gamma distribution is closely related to the Poisson process. You can read more in a statistics textbook. The GAMMA FUNCTION is derived from the Gamma distribution, and will be relevant when we discus the χ^2 distribution.

- Gamma function:
$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x \tag{5.19}$$
For $z \in \mathbb{N}$, $\Gamma(z) = (z-1)!$. The Gamma function can be considered an analytic continuation of the factorial to a larger domain.

6 Summary

- Today we discussed a basic summary of the formalism of probability, RVs, and some common distributions.
- There is one more important distribution: the χ^2 distribution, which we will discuss next week
- Next time we will also discuss curve fitting and understanding if a model is a good fit.