




# maze-dataset: Maze Generation with Algorithmic Variety and Representational Flexibility

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## Summary

Solving mazes is a classic problem in computer science and artificial intelligence, and humans have been constructing mazes for thousands of years. Although finding the shortest path through a maze is a solved problem, this very fact makes it an excellent testbed for studying how machine learning algorithms solve problems and represent spatial information. We introduce maze-dataset, a user-friendly Python library for generating, processing, and visualizing datasets of mazes. This library supports a variety of maze generation algorithms providing mazes with or without loops, mazes that are connected or not, and many other variations. These generation algorithms can be configured with various parameters, and the resulting mazes can be filtered to satisfy desired properties. Also provided are tools for converting mazes to and from various formats suitable for a variety of neural network architectures, such as rasterized images, tokenized text sequences, and various visualizations. As well as providing a simple interface for generating, storing, and loading these datasets, maze-dataset is extensively tested, type hinted, benchmarked, and documented.

```
cfg = MazeDatasetConfig(
    name = "test",
    grid_n = 5,
    n_mazes = 1,
    maze_ctor = gen_dfs,
    ... # many, many options
)
```

m:

```
ds = MazeDataset.from_config(cfg)
```

```
<ADJLIST_START> (2,4) <--> (3,4) ; (1,3) <--> (1,4) ;
(1,4) <--> (0,4) ; (2,0) <--> (3,0) ; (4,4) <--> (4,3)
(3,1) <--> (3,2) ; (3,0) <--> (4,0) ; (4,1) <--> (4,2)
(0,2) <--> (0,3) ; (2,2) <--> (1,2) ; (0,2) <--> (0,1)
(3,3) <--> (3,2) ; (1,0) <--> (0,0) ; (0,2) <--> (1,2)
(4,0) <--> (4,1) ; (2,1) <--> (2,0) ; (3,2) <--> (2,2)
(3,4) <--> (4,4) ; <ADJLIST_END>
```

```
<ORIGIN_START> (1,3) <ORIGIN_END> <TARGET_START> (2,3)
```

```
<PATH_START> (1,3) (0,3) (0,2) (1,2) (2,2) (2,1) (2,0)
(4,2) (4,3) (4,4) (3,4) (2,4) (2,3) <PATH_END>
```

**Figure 1:** Usage of maze-dataset. We create a MazeDataset from a MazeDatasetConfig. This contains SolvedMaze objects which can be converted to and from a variety of formats. Code in the image contains clickable links to [documentation](#). A variety of generated examples can be viewed [here](#).

## Statement of Need

While maze generation itself is straightforward, the architectural challenge comes from building a system supporting many algorithms with configurable parameters, property filtering, and representation transformation. This library aims to greatly streamline the process of generating and working with datasets of mazes that can be described as subgraphs of an  $n \times n$  lattice with boolean connections and, optionally, start and end points that are nodes in the graph. Furthermore, we place emphasis on a wide variety of possible text output formats aimed at evaluating the spatial reasoning capabilities of Large Language Models (LLMs) and other text-based transformer models.

For interpretability and behavioral research, algorithmic tasks offer benefits by allowing systematic data generation and task decomposition, as well as simplifying the process of circuit discovery (Räuker et al., 2023). Although mazes are well suited for these investigations, we found that existing maze generation packages (Cobbe et al., 2019; Ehsan, 2022; Harries et al., n.d.; Németh, 2019; Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021) lack support for transforming between multiple representations and provide limited control over the maze generation process.

## Related Works

A multitude of public and open-source software packages exist for generating mazes (Ehsan, 2022; Németh, 2019; Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021). However, nearly all of these packages produce mazes represented as rasterized images or other visual formats rather than the underlying graph structure, and this makes it difficult to work with these datasets.

- Most prior works provide mazes in visual or raster formats, and we provide a variety of similar output formats:
  - `RasterizedMazeDataset`, utilizing `as_pixels()`, which can exactly mimic the outputs provided in `easy-to-hard-data` (Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021) and can be configured to be similar to the outputs of Németh (2019)
  - `as_ascii()` provides a format similar to (Oppenheim, 2018; Singla, 2023)
  - `MazePlot` provides a feature-rich plotting utility with support for multiple paths, heatmaps over positions, and more. This is similar to the outputs of (Alance AB, 2019; Ehsan, 2022; Guo et al., 2011; Nag, 2020)
- The text format provided by `SolvedMaze(...).as_tokens()` is similar to that of (Liu & Wu, 2023) but with many more options, detailed [section: Tokenized Output Formats](#).
- Preserving metadata about the generation algorithm with the dataset itself is essential for studying the effects of distributional shifts. Our package efficiently stores the dataset along with its metadata in a single human-readable file (M. Ivanitskiy, n.d.). As far as we are aware, no existing packages do this reliably.
- Storing mazes as images or adjacency matrices is not only difficult to work with, but also inefficient. We use a highly efficient method detailed in [section: Implementation](#).
- Our package is easily installable with source code freely available. It is extensively tested, type hinted, benchmarked, and documented. Many other maze generation packages lack this level of rigor and scope, and some (Ayaz et al., 2008) appear to simply no longer be accessible.

## Features

We direct readers to our [examples](#), [docs](#), and [notebooks](#) for more information.

## Generation and Basic Usage

Our package can be installed from [PyPi](#) via `pip install maze-dataset`, or directly from the [git repository](#) (Michael I. Ivanitskiy et al., 2023a).

To create a dataset, we first create a [MazeDatasetConfig](#) configuration object, which specifies the seed, number, and size of mazes, as well as the generation algorithm and its corresponding parameters. This object is passed to a [MazeDataset](#) class to create a dataset. Crucially, this [MazeDataset](#) mimics the interface of a PyTorch (Paszke et al., 2019) [Dataset](#), and can thus be easily incorporated into existing data pre-processing and training pipelines, e.g., through the use of a `DataLoader` class.

```
from maze_dataset import (
    MazeDataset, MazeDatasetConfig, LatticeMazeGenerators
)
# create a config
cfg: MazeDatasetConfig = MazeDatasetConfig(
    name="example", # names need not be unique
    grid_n=3, # size of the maze
    n_mazes=32, # number of mazes in the dataset
    maze_ctor=LatticeMazeGenerators.gen_dfs, # many algorithms available
    # (optional) algorithm-specific parameters
    maze_ctor_kwargs={"do_forks": True, ...},
    # (optional) many options for restricting start/end points
    endpoint_kwargs={"deadend_start": True, ...},
)
# create a dataset
dataset: MazeDataset = MazeDataset.from_config(
    cfg, # pass the config
    ..., # other options for disk loading, parallelization, etc.
)
```

When initializing a dataset, options which do not affect the mazes themselves can be specified through the `from_config()` factory method as necessary. These options allow for saving/loading existing datasets instead of re-generating, parallelization options for generation, and more. Available maze generation algorithms are static methods of the [LatticeMazeGenerators](#) namespace class and include generation algorithms based on randomized depth-first search, Wilson's algorithm (Wilson, 1996), percolation (Duminil-Copin, 2017; Fisher & Essam, 2004), Kruskal's algorithm (Kruskal, 1956), and others.

Furthermore, a dataset of mazes can be filtered to satisfy certain properties. Custom filters can be specified, and some filters are included in [MazeDatasetFilters](#). For example, we can require a minimum path length of three steps from the origin to the target:

```
dataset_filtered: MazeDataset = dataset.filter_by.path_length(min_length=3)
```

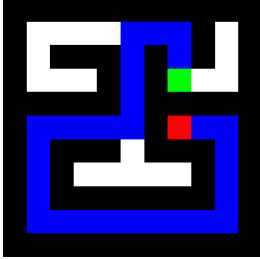
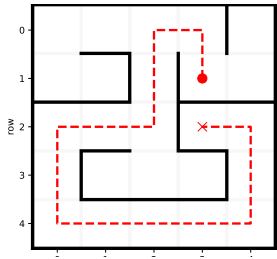
All implemented maze generation algorithms are stochastic by nature. For reproducibility, the seed parameter of [MazeDatasetConfig](#) may be set. In practice, using provided deduplication filters, we find that exact duplicate mazes are generated very infrequently, even when generating very large datasets.

For use cases where mazes of different sizes, generation algorithms, or other parameter variations are required, we provide the [MazeDatasetCollection](#) class, which allows for creating a single iterable dataset from multiple independent configurations.

## Visual Output Formats

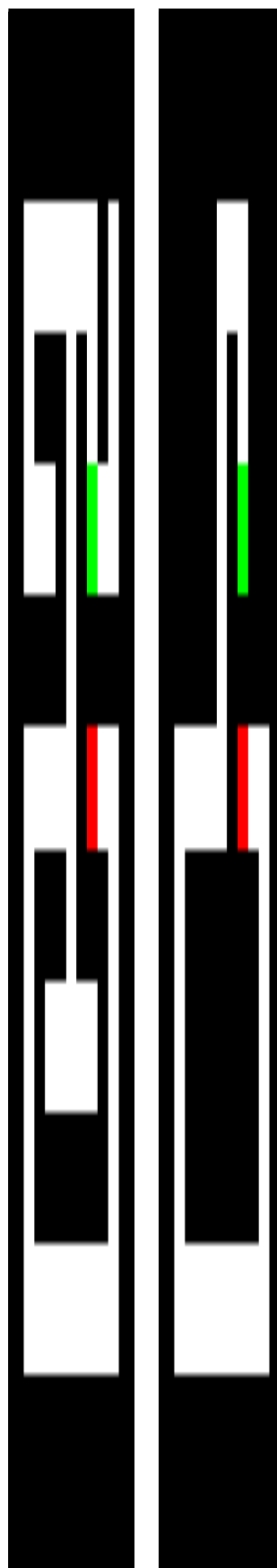
Internally, mazes are [SolvedMaze](#) objects, which have path information and a tensor optimized for storing sub-graphs of a lattice. These objects can be converted to and from several formats

to maximize their utility in different contexts.

<code>as_ascii()</code>	<code>as_pixels()</code>	<code>MazePlot()</code>
Simple text format for displaying mazes, useful for debugging in a terminal environment.	numpy array of dtype=uint8 and shape (height, width, 3). The last dimension is RGB color.	feature-rich plotting utility with support for multiple paths, heatmaps over positions, and more.
<pre>##### #      X X X # #    # # X # X # #      # X # S # # # # # # X # # # # X X X X X # E X X # # X # # # # # X # # X #      # X # # X # # # # # X # # X X X X X X X X # #####</pre>		

**Figure 2:** Various output formats. Top row (left to right): ASCII diagram, rasterized pixel grid, and advanced display tool.

In previous work, maze tasks have been used with Recurrent Convolutional Neural Network (RCNN) derived architectures (Schwarzschild, Borgnia, Gupta, Huang, et al., 2021). To facilitate the use of our package in this context, we replicate the format of (Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021) and provide the `RasterizedMazeDataset` class which returns rasterized pairs of (input, target) mazes as shown in Figure 3 below.



**Figure 3:** Input is the rasterized maze without the path marked (left), and provide as a target the maze

Ivanitskiy et al. (1970). maze-dataset with all but the lowest path from vertex (right) Configuration: Pairs exist in Adj. of whether source is  $\neq$  VOL? (ISSUE?), PAGE? <https://arxiv.org/abs/1904.08484> and if empty cells should be filled in.

## Tokenized Output Formats

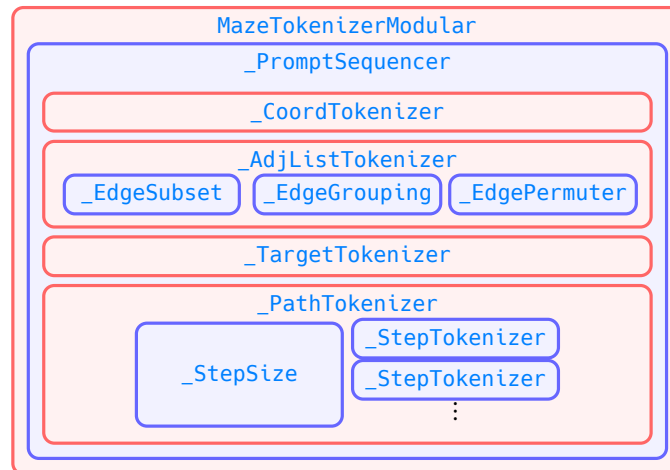
Autoregressive transformer models can be quite sensitive to the exact format of input data, and may even use delimiter tokens to perform reasoning steps (Pfau et al., 2024; Spies et al., 2024). To facilitate systematic investigation of the effects of different representations of data on text model performance, we provide a variety of text output formats. By passing an instance of `MazeTokenizerModular` to `as_tokens(...)`, a maze can be converted to a text sequence. The `MazeTokenizerModular` class contains a rich set of options with class contains 19 discrete parameters, resulting in over 5.8 million unique possible tokenizers.

All output sequences consist of four token regions representing different features of the maze; an example output sequence is shown in Figure 4.

```
<ADJLIST_START> (0,0) <--> (1,0) ; (2,0) <--> (3,0) ; (4,1) <--> (4,0) ; (2,0) <--> (2,1) ;
(1,0) <--> (1,1) ; (3,4) <--> (2,4) ; (4,2) <--> (4,3) ; (0,0) <--> (0,1) ; (0,3) <--> (0,2) ;
(4,4) <--> (3,4) ; (4,3) <--> (4,4) ; (4,1) <--> (4,2) ; (2,1) <--> (2,2) ; (1,4) <--> (0,4) ;
(1,2) <--> (0,2) ; (2,4) <--> (2,3) ; (4,0) <--> (3,0) ; (2,2) <--> (3,2) ; (1,2) <--> (2,2) ;
(1,3) <--> (0,3) ; (3,2) <--> (3,3) ; (0,2) <--> (0,1) ; (3,1) <--> (3,2) ; (1,3) <--> (1,4) ;
<ADJLIST_END> <ORIGIN_START> (1,3) <ORIGIN_END> <TARGET_START> (2,3) <TARGET_END>
<PATH_START> (1,3) (0,3) (0,2) (1,2) (2,2) (2,1) (2,0) (3,0) (4,0) (4,1) (4,2) (4,3) (4,4)
(3,4) (2,4) (2,3) <PATH_END>
```

**Figure 4:** Example text output format with token regions highlighted. **Adjacency list** : text representation of the graph, **Origin** : starting coordinate, **Target** : ending coordinate, **Path** : maze solution sequence

Each `MazeTokenizerModular` is constructed from a set of several `_TokenizerElement` objects, each of which specifies how different token regions or other elements of the stringification are produced.



**Figure 5:** Nested internal structure of `_TokenizerElement` objects inside a typical `MazeTokenizerModular`.

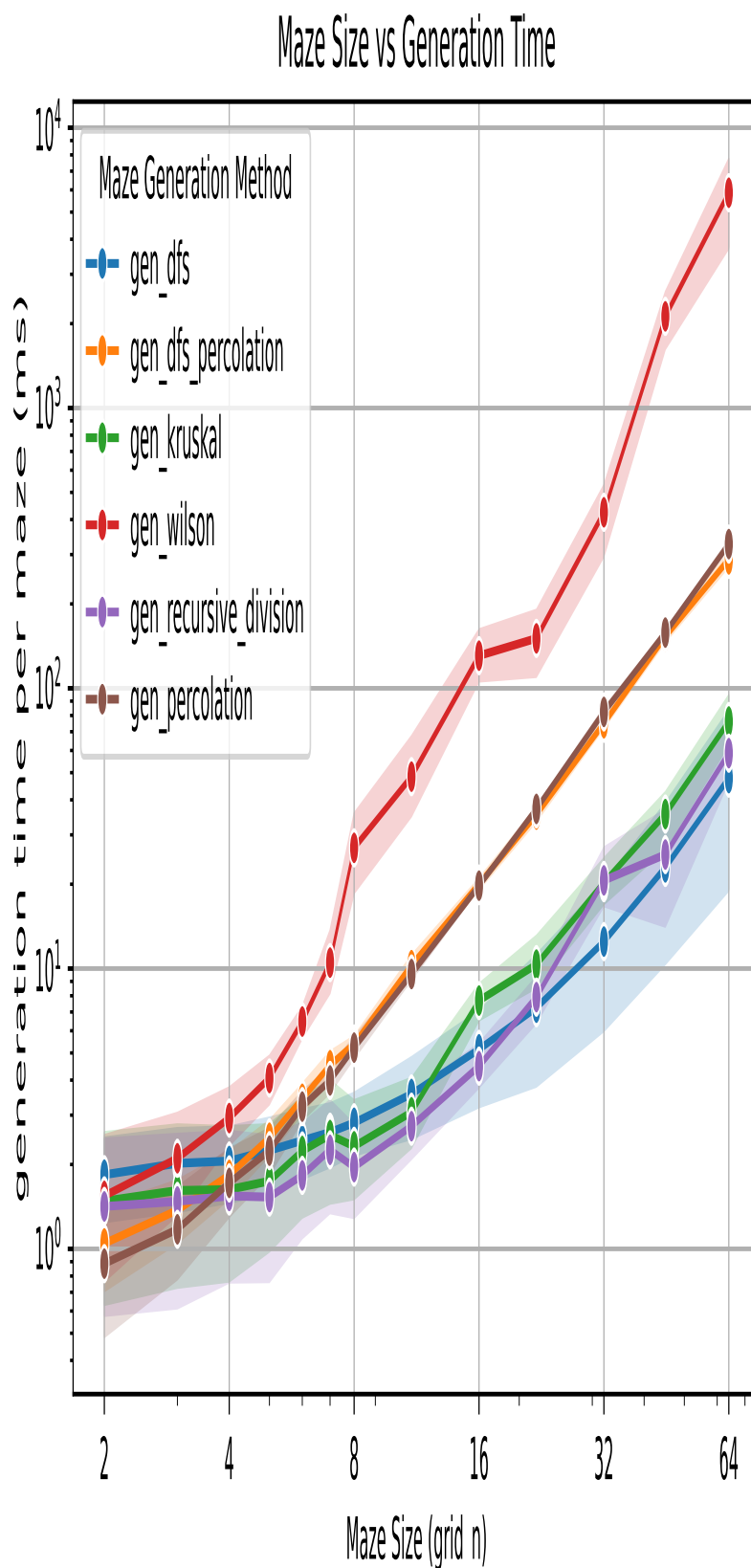
## Benchmarks of Generation Speed

We provide approximate benchmarks for relative generation time across various algorithms, parameter choices, maze sizes, and dataset sizes in Table 1 and Figure 6. Experiments were performed on a standard GitHub runner without parallelism.

maze_ctor	keyword args	all sizes	small $g \leq 10$	medium $g \in (10, 32]$	large $g > 32$
dfs		28.0	2.8	20.3	131.8
dfs	accessible_cells=20	2.3	2.2	2.4	2.2
dfs	do_forks=False	2.7	2.2	3.1	3.5
dfs	max_tree_depth=0.5	2.5	2.0	2.7	4.0
dfs_percolation	p=0.1	43.9	2.8	33.9	208.0
dfs_percolation	p=0.4	48.7	3.0	36.5	233.5
kruskal		12.8	1.9	10.3	55.8
percolation	p=1.0	50.2	2.6	37.2	242.5
recursive_div		10.2	1.7	8.9	42.1
wilson		676.5	7.8	188.6	3992.6
mean		559.9	13.0	223.5	3146.9
median		11.1	6.5	32.9	302.7

**Table 1:** Generation times for various algorithms and maze sizes. More information can be found on the [benchmarks](#) page.





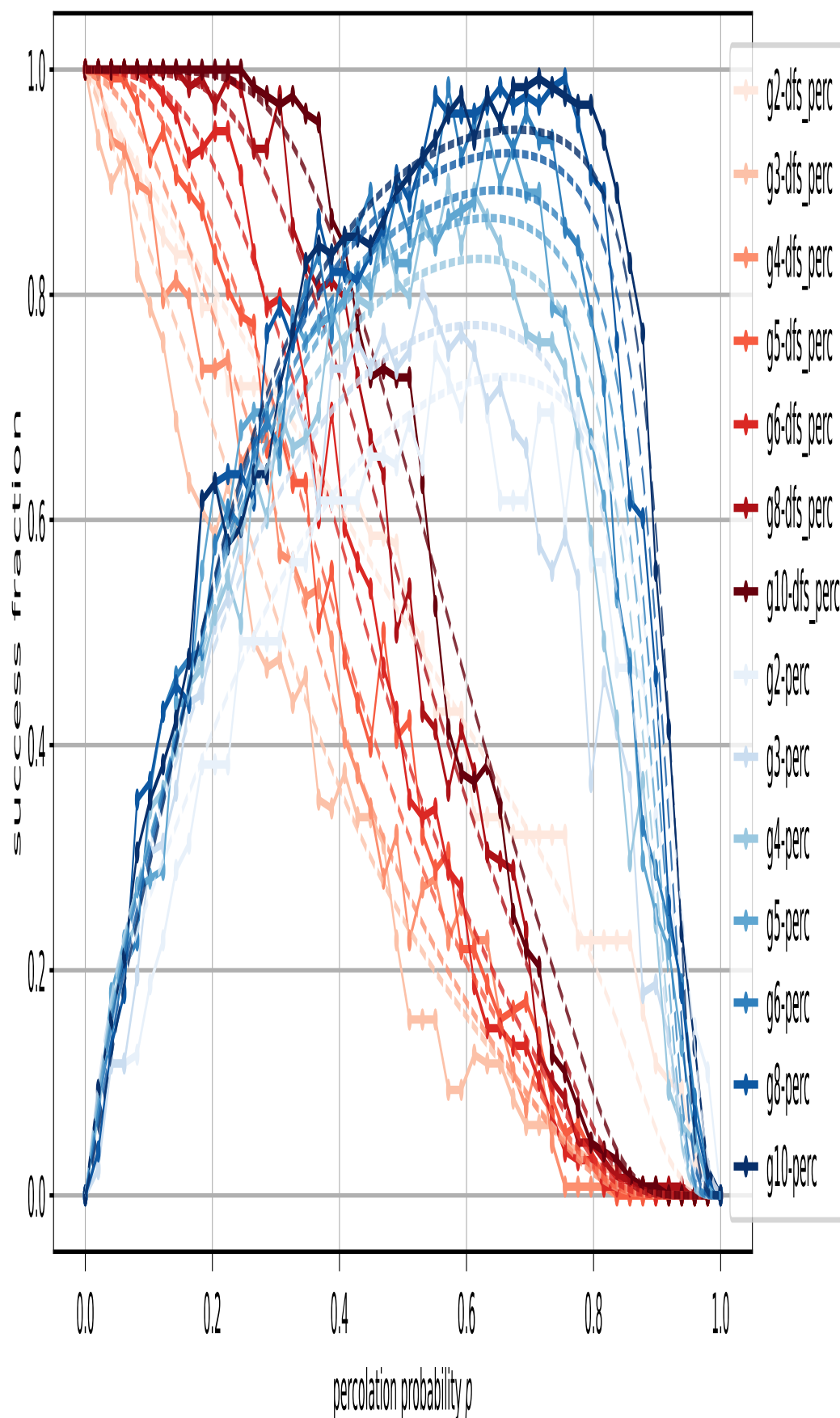
**Figure 6:** Plot of maze generation time. Generation time scales exponentially with maze size for all algorithms. Generation time per maze does not depend on the number of mazes being generated and is minimal overhead to initializing the generation process for a small dataset. Wilson's algorithm is notably less efficient than others and has high variance. Note that values are averaged across all parameter sets for that algorithm. More information can be found on the [benchmarks page](#).

## Success Rate Estimation

In order to replicate the exact dataset distribution of ([Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021](#)), the parameter `MazeDatasetConfig.endpoint_kwargs: EndpointKwargsType` allows for additional constraints, such as enforcing that the start or end point be in a “dead end” with only one accessible neighbor cell. However, combining these constraints with cyclic mazes<sup>1</sup> can lead to an absence of valid start and end points. To deal with this, our package provides a way to estimate the success rate of a given configuration using a symbolic regression model trained with PySR ([Cranmer, 2023](#)). More details on this can be found in [estimate\\_dataset\\_fractions.ipynb](#).

---

<sup>1</sup>Such as those generated with percolation, as was required for the work in ([Knutson et al., 2024](#)).



**Figure 7:** An example of both empirical and predicted success rates as a function of the percolation

probability  $p$  for various maze sizes. Percolation represents a world with out depth first search. open source software 11  
Ivanitskiy et al. (1970). maze-dataset-problem-generation-on-mazes. percolation represents a world with out depth first search. open source software 11  
¿VOL?(¿ISSUE?), ¿PAGE? <https://doi.org/10.21203/rs.3.rs-1120311/v1>  
requiring that both the start and end be in unique dead ends. Empirical measures derived from a sample  
of 128 mazes. More information can be found on the [benchmarks page](#).

## Implementation

Using an adjacency matrix for storing mazes would be memory inefficient by failing to exploit the highly sparse structure, while using an adjacency list could lead to a poor lookup time. This package utilizes a simple, efficient representation of mazes as subgraphs of a finite lattice, which we call a [LatticeMaze](#).

We describe mazes with the following representation: for a 2-dimensional lattice with  $r$  rows and  $c$  columns, we initialize a boolean array

$$A = \{0, 1\}^{2 \times r \times c}$$

which we refer to in the code as a [connection\\_list](#). The value at  $A[0, i, j]$  determines whether a *downward* connection exists from node  $[i, j]$  to  $[i + 1, j]$ . Likewise, the value at  $A[1, i, j]$  determines whether a *rightward* connection to  $[i, j + 1]$  exists. Thus, we avoid duplication of data about the existence of connections and facilitate fast lookup time, at the cost of requiring additional care with indexing.

## Usage in Research

This package was originally built for the needs of the (Michael I. Ivanitskiy et al., 2023b) project, which aims to investigate spatial planning and world models in autoregressive transformer models trained on mazes (Michael Igorevich Ivanitskiy, Spies, et al., 2023; Michael Igorevich Ivanitskiy, Shah, et al., 2023; Spies et al., 2024). It was extended for work on understanding the mechanisms by which recurrent convolutional and implicit networks (Fung et al., 2022) solve mazes given a rasterized view (Knutson et al., 2024), which required matching the pixel-padded and endpoint constrained output format of (Schwarzschild, Borgnia, Gupta, Bansal, et al., 2021). Ongoing work using maze-dataset aims to investigate the effects of varying the tokenization format on the performance of pretrained LLMs on spatial reasoning.

This package has also been utilized in work by other groups:

- By (Nolte et al., 2024) to compare the effectiveness of transformers trained with the MLM- $\mathcal{U}$  (Kitouni et al., 2024) multistep prediction objective against standard autoregressive training for multi-step planning on our maze task.
- By (Wang et al., 2024) and (Chen et al., 2024) to study the effectiveness of imperative learning.
- By (Zhang et al., 2025) to introduce a novel framework for reasoning diffusion models.
- By (Dao & Vu, 2025) to improve spatial reasoning in LLMs with GRPO.

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