Localising feedback mechanisms in eroding landscapes

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1 Introduction

Rain falling on a hillside soaks into the soil and what cannot be absorbed trickles down towards the ocean. The path the surface water takes is convoluted, coalescing into a network of increasingly large streams and rivers. The form of this network is an emergent feature of feedback processes that exist between the flowing water and the carving of material from the channel bed.

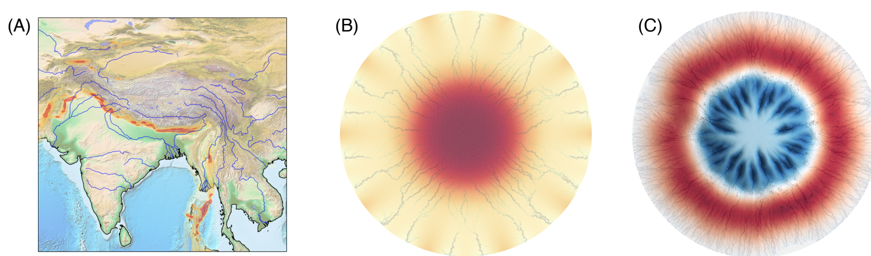

Fig. 1 (A) The India-Eurasia collision zone highlights interactions between tectonic uplift and river evolution. Rivers in the region do not follow the shortest path but ones governed by the history of erosion and tectonic compression. (B) A model landscape with high symmetry that highlights the mesh-dependent flow paths. (C) After a few million years of scattered showers, the topography erodes (blue areas) and deposits material (red areas). Once established, channels become very stable.

Over geological time, the action of flowing water (and ice) on topography is one of the key elements in shaping terrestrial landscapes. At the same time, the material that is eroded by rivers is transported and deposited downstream. Consequently,

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sedimentary basins, the topographic lows which collect the detritus shed from the topographic highs, are the information stores where the geological record of both the high regions and the low regions is found.
Interaction between tectonics and erosion are known to be important in both directions. The transport of material from topographic high to low by fluvial erosion and transport is important in lithospheric dynamics as the redistribution of loads during tectonic events can dramatically change the large-scale deformation. Erosion dissipates the gravitational potential energy of high topography, promotes exhumation and cooling of deep crustal layers, provides additional far field loads on the lithsosphere and supplies an insulating sedimentary blanket to rifted depressions.
We can take as understood the importance of the geological applications of fluvial erosion with sediment transport and focus on the mathematical formulation of the problem and its numerical realisation. We discuss an abstract formulation that can be analysed independently of the underlying discretisation and is amenable to efficient decomposed-domain, parallel implementation.

2 Simple Erosion, Deposition and Transport Models

We follow the approach outlined by Beaumont et al (1992). The contributing processes to vertical landscape evolution consist of local hillslope effects (slumping, creep, small-scale collapse) that suppress steep gradients, incision of channels by flowing surface water, deposition of sediment carried downstream by flowing water, and tectonic uplift of the basement rocks, i.e.

$$\frac{Dh}{Dt} = \dot{h}_{\text{local}} + \dot{h}_{\text{incision}} + \dot{h}_{\text{deposition}} + \dot{h}_{\text{basement}} \tag{1}$$

h is the surface height at each point, the $\dot{h}$ terms are time derivatives in the height, the derivative D/Dt includes transport by horizontal motions of the basement rocks, and $\dot{h}_{\text{basement}}$ accounts for tectonic uplift/subsidence.
The local evolution rate represents small-scale, hill-slope dependent processes which can be represented as a non-linear diffusion equation.

$$\dot{h}(\boldsymbol{x})_{\text{local}} = \nabla \left[\kappa(h, \boldsymbol{x}) \nabla h(\boldsymbol{x}) \right] \tag{2}$$

κ is a non-linear diffusion coefficient which can, for example, be used to enforce a critical hill slope value if it is a strongly increasing function of the local gradient.
The fluvial incision rate is more complicated in that it includes the effect of cumulative rainfall runoff across the landscape. This term depends on the available energy of rivers which in turn is related to both the discharge flux at any given point and the local stream-bed slope. In the so-called stream power form, the incision rate may be written as

$$\dot{h}(\boldsymbol{x})_{\text{incision}} = K(\boldsymbol{x}) q_r(\boldsymbol{x})^m \left| \nabla h(\boldsymbol{x}) \right|^n \tag{3}$$

Where K, m, n are constants, q_r is the runoff flux, and $|\nabla h|$ is the downhill bed slope.

$$q_r(\boldsymbol{x}) = \int_{\text{upstream}} \mathscr{R}(\xi) d\xi \tag{4}$$

This integral computes the accumulated run off for all of the areas which lie upstream of the point $\boldsymbol{x}$. This term is strongly dependent on the geometry of the catchment and the connectivity of the network of tributaries above $\boldsymbol{x}$ and is generally not local, non-uniform in sampling the domain, and will change, potentially discontinuously, as the topography evolves.

If we represent a landscape by heights recorded at discrete points, pathways taken by water flowing over the landscape can be constructed by connecting each point to whichever of its nearby neighbours defines the steepest descent direction. In concave regions of the landscape, these paths generally form a tree-like structure since points usually have a unique downhill neighbour, but several points may share the same downhill neighbour. A small segment taken from such a tree is illustrated in Figure 2 with the direction of the downhill flow indicated by the sense of the arrows.

The erosion terms in the surface evolution equation which depend on the flow of water and sediment across the landscape naturally are localising and tend to focus further erosion in the existing channels. This effect stabilises the topology of the steepest-descent network over time. Where deposition occurs, the reverse is true since channels tend to fill and local gradients reduce over time.

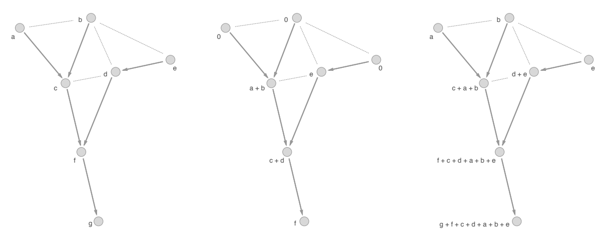

Fig. 2 The progression of information in the downhill direction for a segment of a mesh representing a landscape. In (b) the information is shifted downhill (with summation). In (c), the information is accumulated at each downhill shift and produces a discrete sum of all 'upstream' information.

3 Incremental matrix representation of downhill-flow pathways

Graph-traversal algorithms are common in determining the upstream-summation (4) : a landscape separates naturally into catchments that do not interact, have non-interacting pathways, and can be treated in parallel (e.g. Braun and Willis, 2013). Problems arise in 1) load balancing catchments of very different sizes, 2) handling very high resolutions where multiple processes are needed for a single catchment, and 3) dealing with the non-linear form of (4). Our solution to *all* of these issues is to reformulate the graph representation in matrix form where each has a well-known solution. The directed graph illustrated in Figure 2a can be represented by an adjacency matrix, $\boldsymbol{D}$ that represents the local connectivity of the nodes. This matrix transforms a vector of nodal values, $\boldsymbol{f}_0$ to a new vector, f in which all the information has shifted to its neighbour in the direction of the arrows (which here represents the downhill direction).

$$\boldsymbol{f}_1 = \boldsymbol{D}\boldsymbol{f}_0 \tag{5}$$

This operation is illustrated in Figure 2b with the equivalent matrix form:

$$\begin{bmatrix} 0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \\ e \\ 0 \\ c+d \\ f \end{bmatrix} \tag{6}$$

A second application of $\boldsymbol{D}$ moves all information another increment downhill. $\boldsymbol{D}^k$ moves information by k increments in the downhill direction. Since the graph represents a flow across the landscape and has no closed cycles, all information eventually propagates to an outflow point or a local minimum in the landscape. This means that the vector $\boldsymbol{f}_N = 0$ where N is the length of the longest downhill path in the network. The non-local, upstream integrals can be computed using the adjacency matrix as follows.

$$\boldsymbol{q} = \left[\boldsymbol{I} + \boldsymbol{D} + \boldsymbol{D}^2 + \cdots \boldsymbol{D}^N + \cancel{\boldsymbol{D}^{N+1}} + \cdots + \cancel{\boldsymbol{D}^{\infty}}\right]\mathscr{R} \quad \text{or} \quad \boldsymbol{q} = \sum_{i=0}^{N} \boldsymbol{D}^i \boldsymbol{f} \tag{7}$$

To avoid calculating D^i, we can compute α through a recursion. We define the partial summation of the upstream contributions after i iterations as $\boldsymbol{q}_i$ and find $\boldsymbol{q}_(i+1)$:

$$\boldsymbol{q}_{i+1} = (\boldsymbol{I} + \boldsymbol{D})\boldsymbol{q}_i \quad \text{where} \quad \boldsymbol{q}_i = \sum_{i=0}^{i} \boldsymbol{D}^i \boldsymbol{f} \tag{8}$$

Since $\boldsymbol{I} + \boldsymbol{D}$ is a pre-determined, sparse matrix, this operation can take full advantage of purpose-built, efficient linear algebra routines including those provided by parallel libraries such as PETSc (Balay et al, 1997).
We also note that the extreme sparsity of D can work against computational efficiency due to a high ratio of memory access to calculation and, in parallel, a high ratio of communication to floating point operations. A simple trick to improve this ratio is to use a higher degree of recursion:

$$\boldsymbol{q}_{i+2} = (\boldsymbol{I} + \boldsymbol{D} + \boldsymbol{D}^2)\boldsymbol{q}_i \tag{9}$$

or, more generally,

$$\boldsymbol{q}_{i+s} = \boldsymbol{A}_s \boldsymbol{q}_i \quad \text{where} \quad \boldsymbol{A}_s = \sum_{j=0}^{s} \boldsymbol{D}^j \tag{10}$$

$\boldsymbol{D}^0$ is the identity matrix, and $\boldsymbol{D}^{i+1}$ is always more sparse than $\boldsymbol{D}^i$ as a result of information flowing out of the mesh.

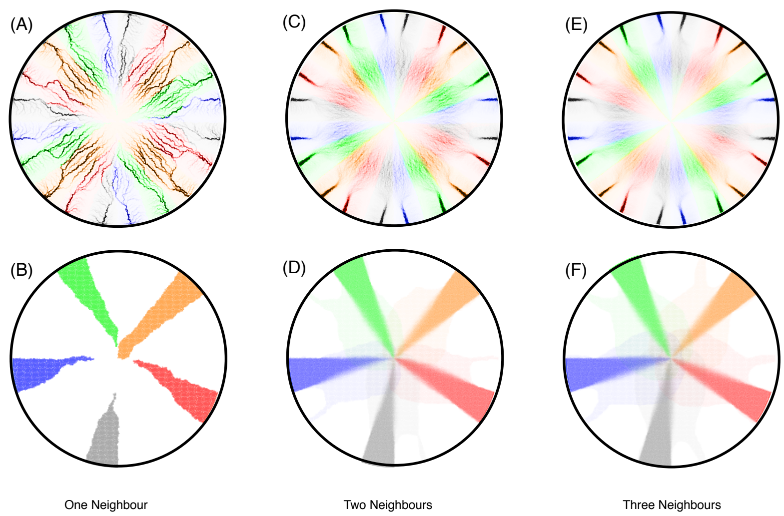

Fig. 3 Illustration of mesh-sensitivity. The landscape is from Figure 1B which has radial symmetry in the central, high regions and a series of outflow channels in the distal, low-lying regions. Irregularities in the flow in A and catchments in B, are entirely due to random locations of the nodes in the triangulation of the surface. In C,D this is mitigated by constructing a second graph of next-lowest neighbours and in E,F a third graph. Appropriately weighted, these alternative pathways reduce the compelling-but-erroneous symmetry breaking seen in A,B

Figure 3 illustrates a fundamental difficulty associated with the discrete form of the localising, stream-power erosion term (4), namely the sensitivity of the emergent network to the underlying mesh structure. Although the stream-network in Figure 3A) has a compelling dendritic form, this is actually an artifact of the triangulation of a rotationally symmetric surface. This effect can be reduced markedly by considering not only the graph representing the single downhill direction, but also one to represent a second or third direction (appropriately weighted by slope). Such approaches have been proposed in the past for meshes of points that are completely regular, it may also be helpful to route flow to more than one destination node in order to prevent locking of erosion pathways along the cardinal directions and for flat regions where deposition promoted delocalisation of channels into multiple branches (Tucker and Hancock, 2010). Note that the catchments are no longer *strictly isolated* if the landscape has regions of low slope where fluid pathways can diverge.

Let us define a matrix $\boldsymbol{D}_2$ to represent the adjacency relationship between a node and its second-steepest connection. The matrix that propagates information a single step in the downhill direction is now defined as

$$\boldsymbol{D}' = \boldsymbol{W}_1\boldsymbol{D}_1 + \boldsymbol{W}_2\boldsymbol{D}_2 \tag{11}$$

Where $\boldsymbol{W}_1, \boldsymbol{W}_2$ are diagonal matrices of weights that satisfy $\boldsymbol{W}_1 + \boldsymbol{W}_2 = \boldsymbol{I}$ (see Tucker and Hancock (2010) for typical approaches to determining $\boldsymbol{W}$). The subscript 1 is used to represent the adjacency matrix for the steepest connection, previously identified as $\boldsymbol{D}$. The results defined above all apply if $\boldsymbol{D}$ is replaced by $\boldsymbol{D}'$. In the matrix formulation, only the overall sparsity is affected by adding additional pathways to $\boldsymbol{D}$ and the blending of catchments does not affect the parallel decomposition.

4 Discussion

We have described a matrix-based formalism for the upstream-integral term in the stream-power equation that is a key component of surface process modelling algorithms. As a result, the entire equation system can be written in matrix form and the parallelism and non-linearity can be treated with standard tools. The non-local nature of this term in the equation means that the necessary matrix representation in dense. However, our formulation can be broken down into a recursive application of very sparse matrices on the field to be integrated. Matrix algebra packages such as PETSc are able to substitute this recursion wherever a matrix-vector product is required and this allows us to access the entire suite of linear and non-linear parallel solvers.
This algorithm has been implemented into the parallel,surface-process code `Quagmire` using the PETSc matrix library to handle matrices and vectors across a decomposed domain. `Quagmire` is an open-source code and can be obtained from the authors.

5 References

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