Fitting the Weibull model

SURVIVAL ANALYSIS IN PYTHON

Shae Wang
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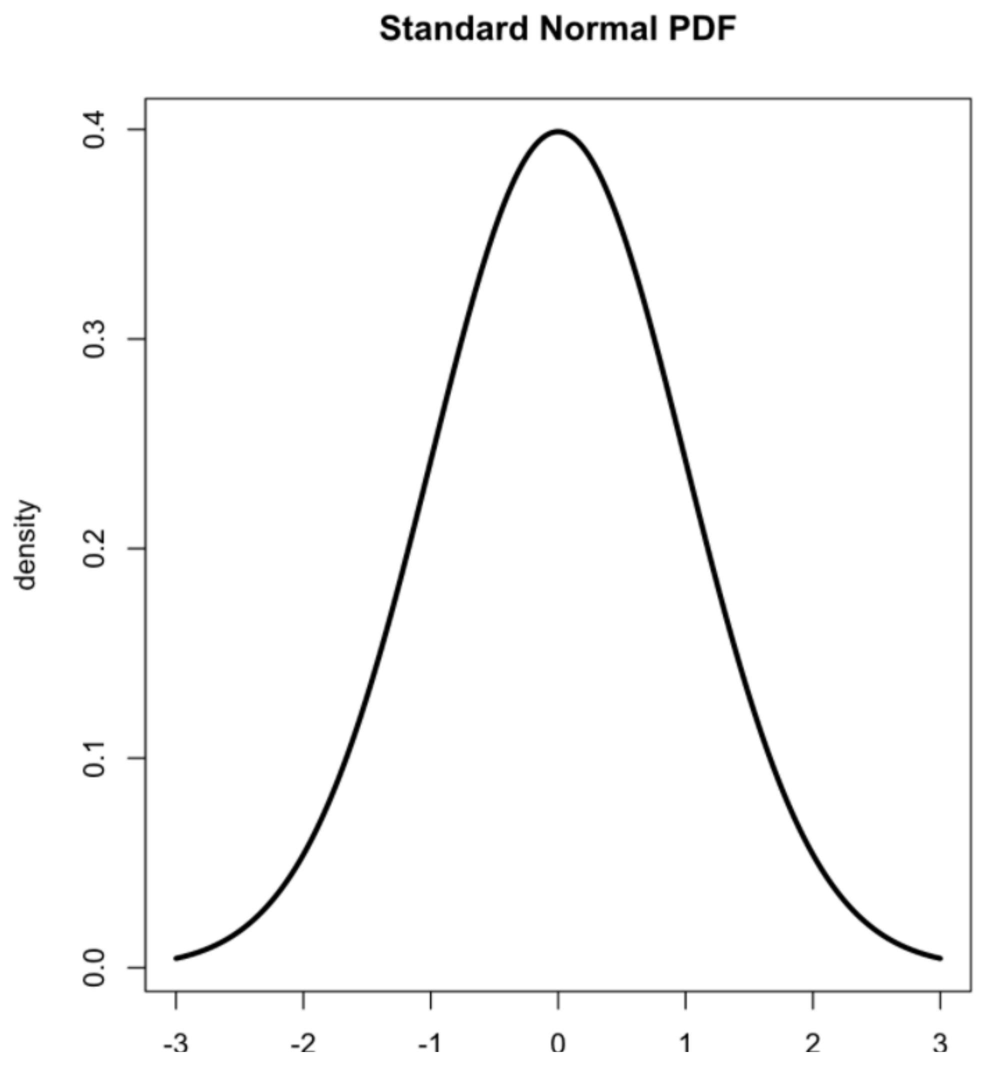
Probability distributions

A probability distribution

A mathematical function that describes the probability of different event outcomes.



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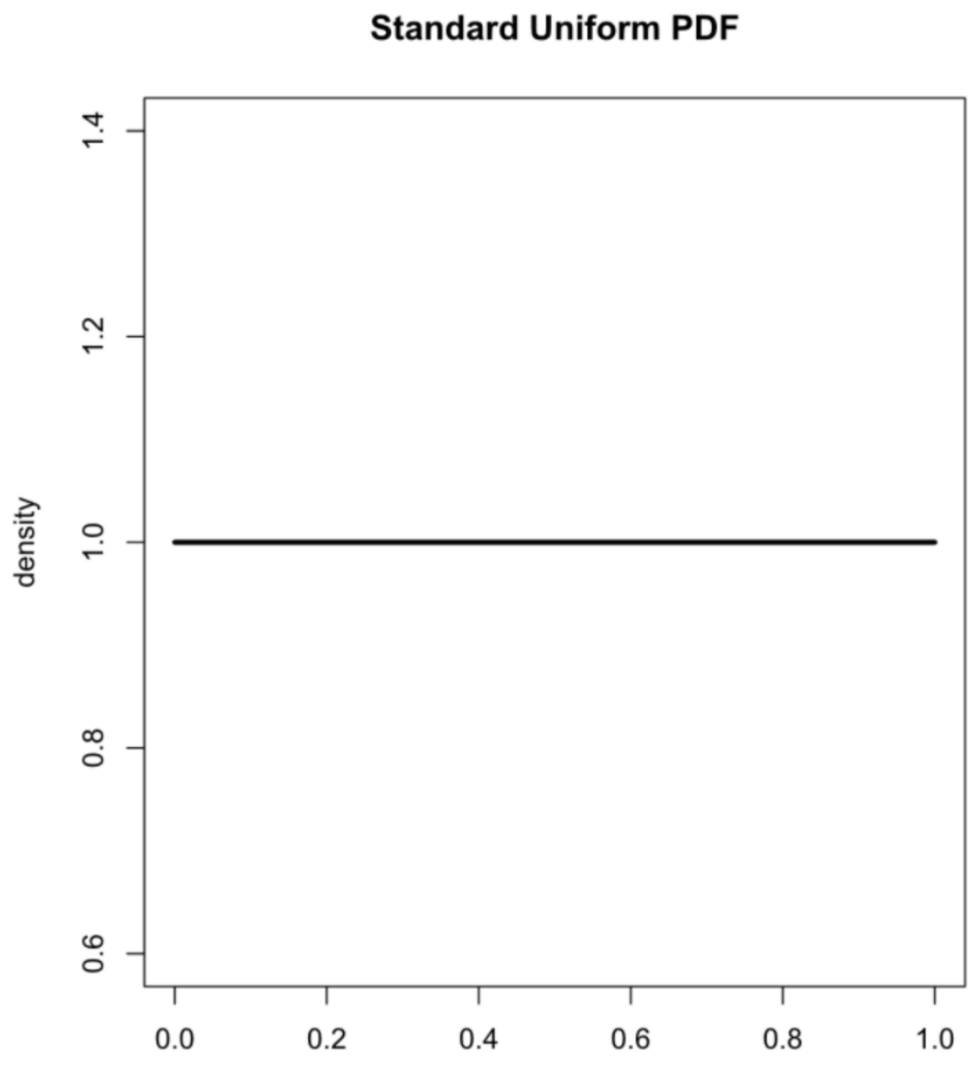


Introducing the Weibull distribution

The Weibull distribution

A continuous probability distribution that models time-to-event data very well (but originally applied to model particle size distribution).

The Weibull probability density function

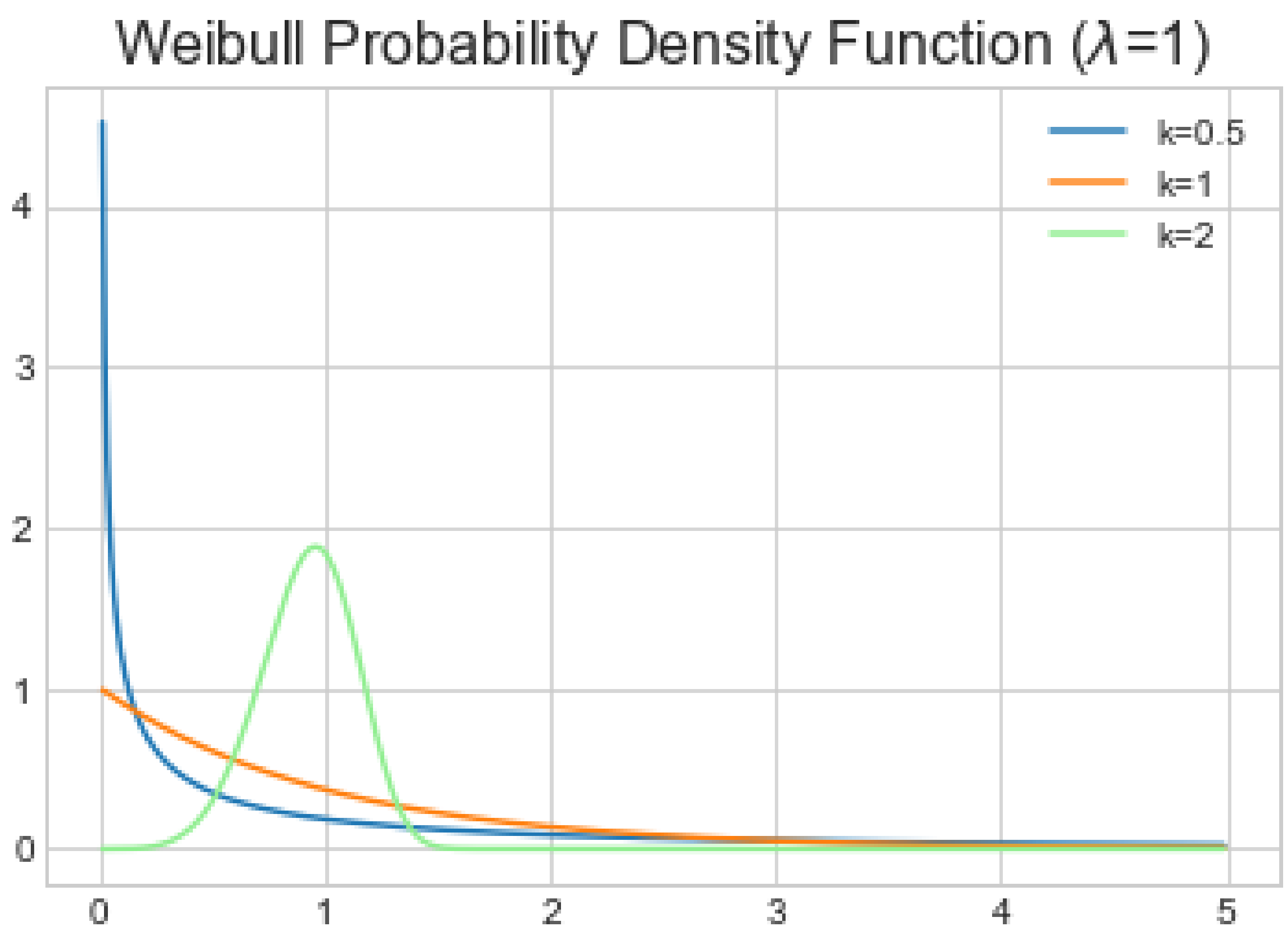
$$f(x; \lambda, k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

$$x \geq 0, k > 0, \lambda > 0$$

Introducing the Weibull distribution

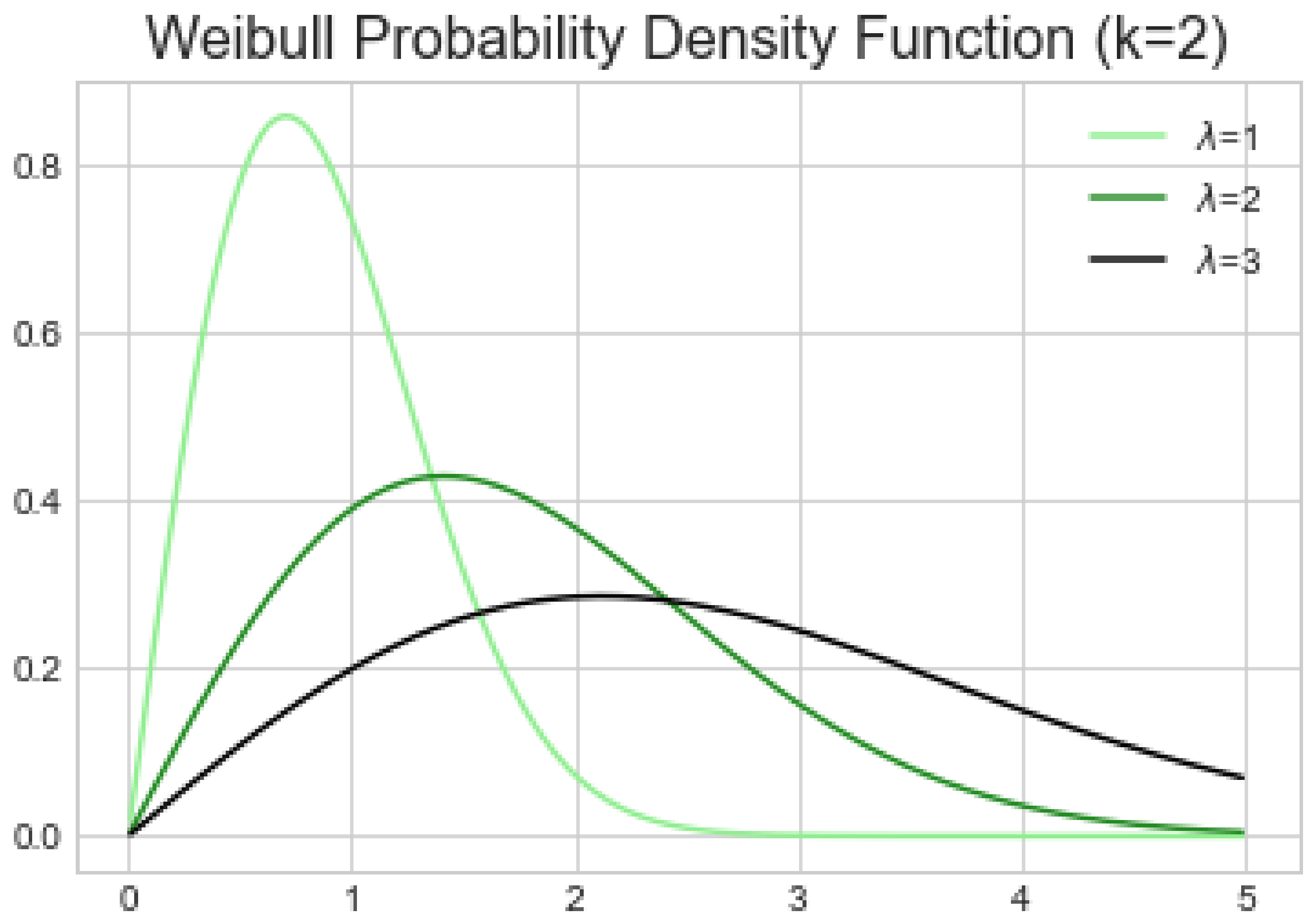
k

Determines the shape



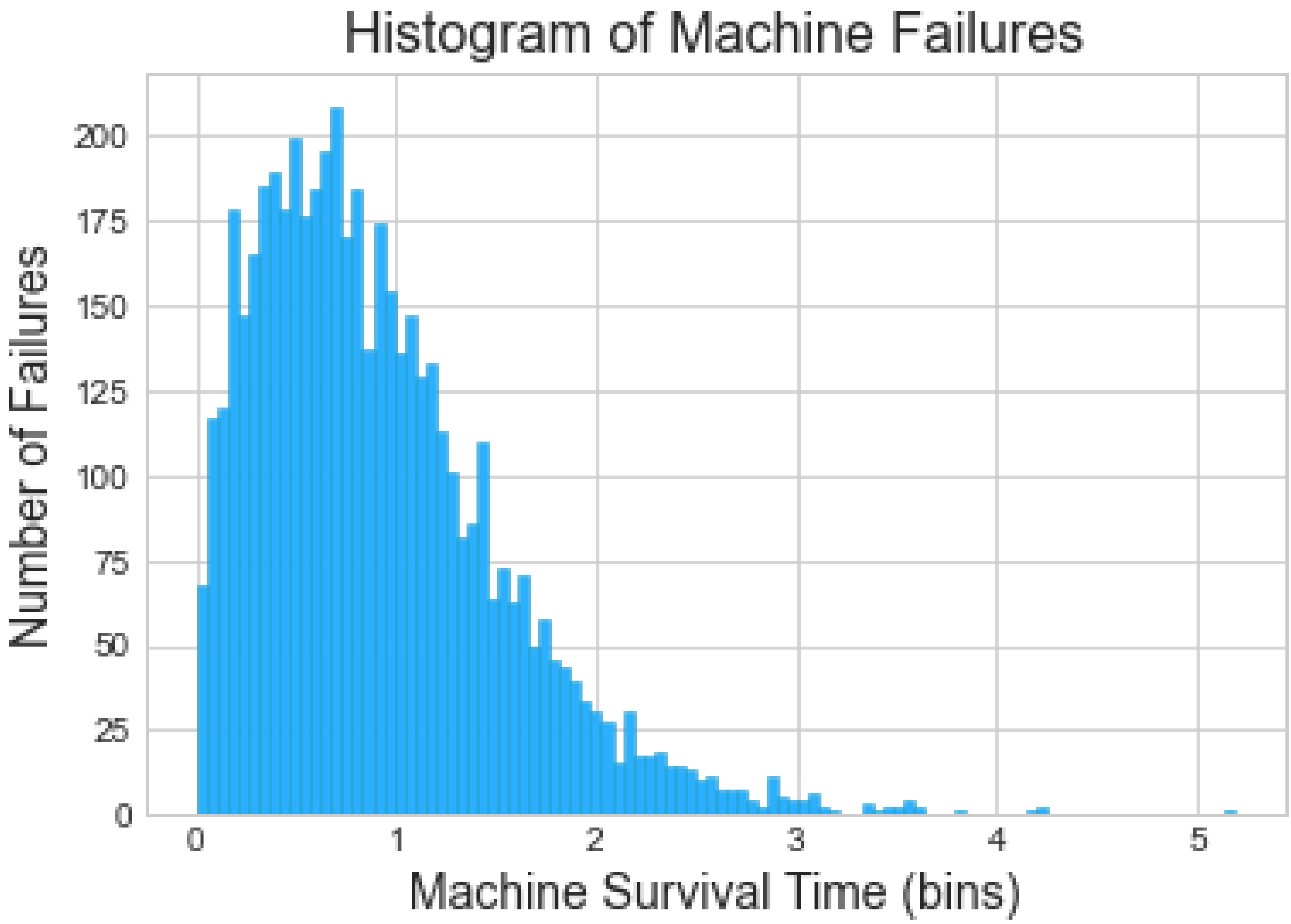
λ

Determines the scale



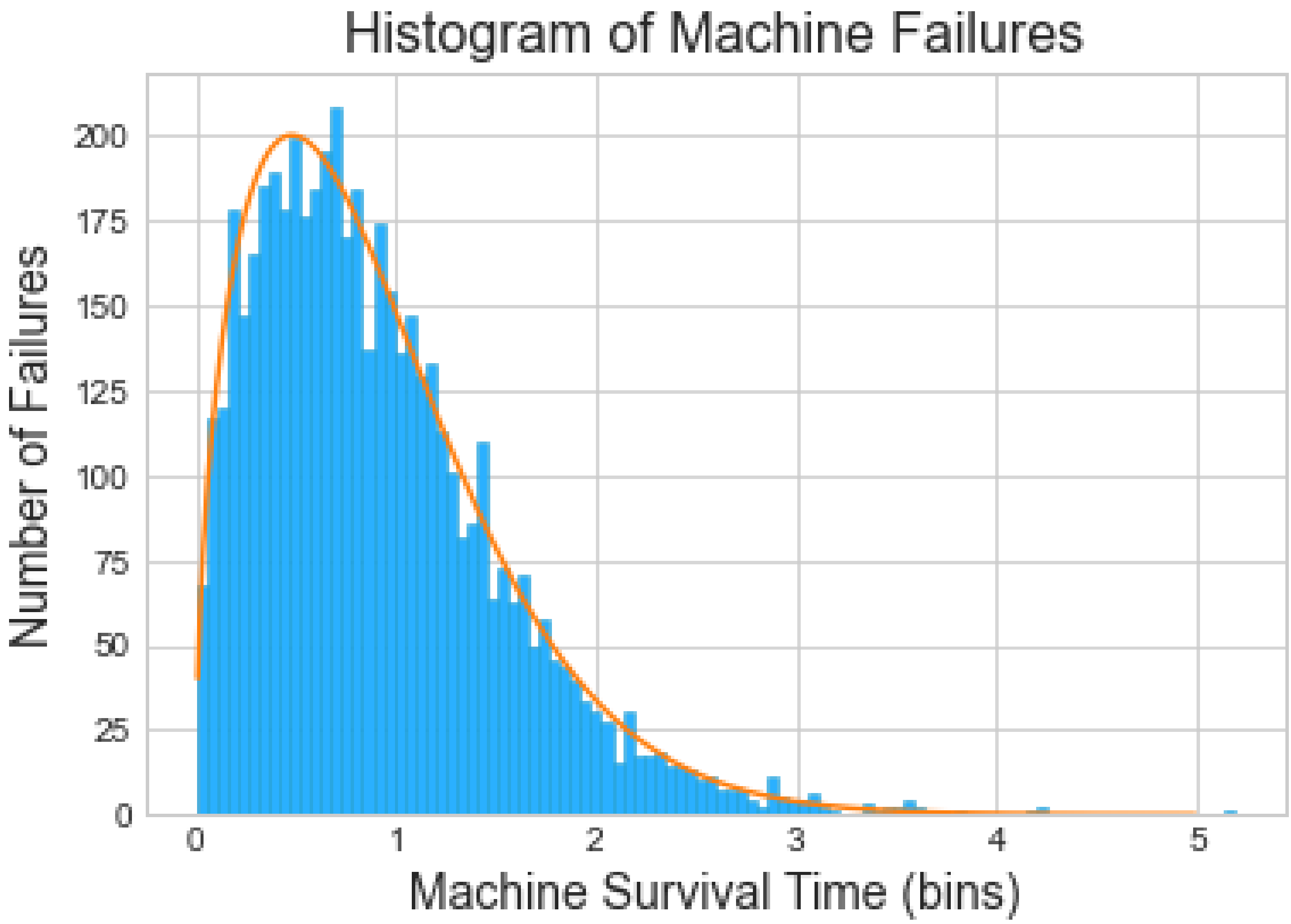
Fitting the Weibull distribution to data

A company maintains a fleet of machines that are prone to failure...



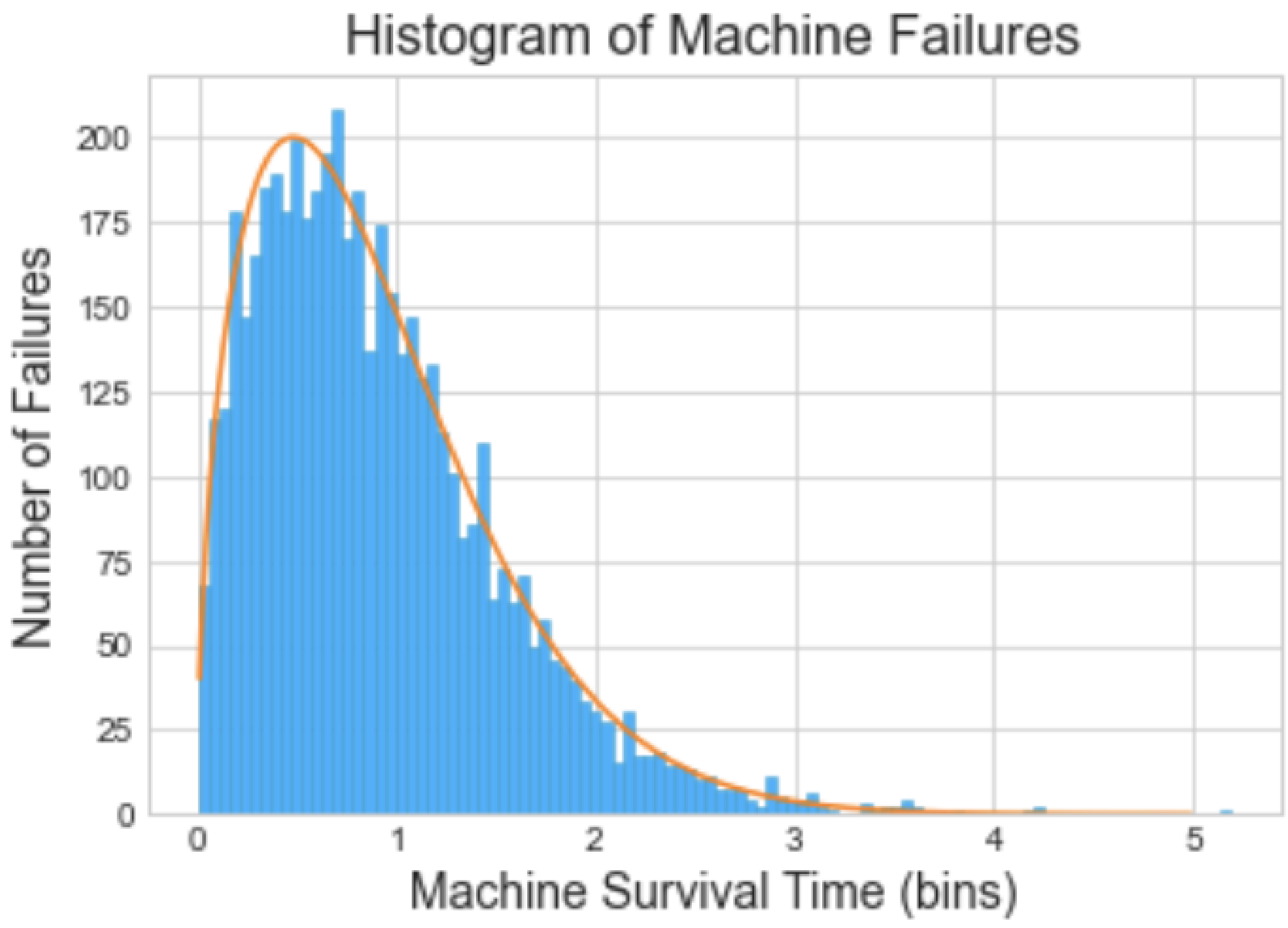
Fitting the Weibull distribution to data

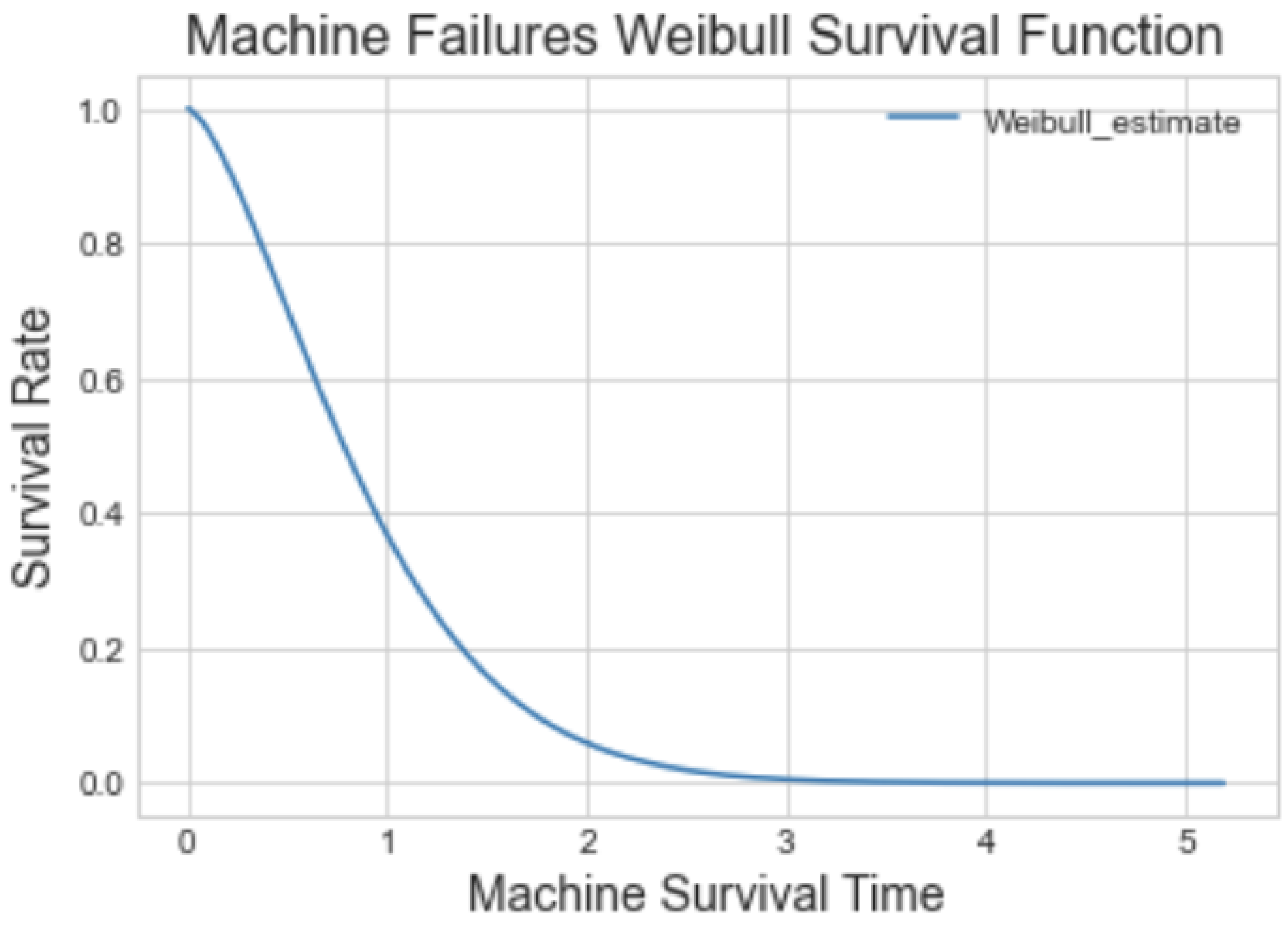
A company maintains a fleet of machines that are prone to failure...



From Weibull distribution to survival function

$$f(x;\lambda,k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} \quad \rightarrow \quad S(t) = e^{-(t/\lambda)^{\rho}}$$





The knobs: k and lambda

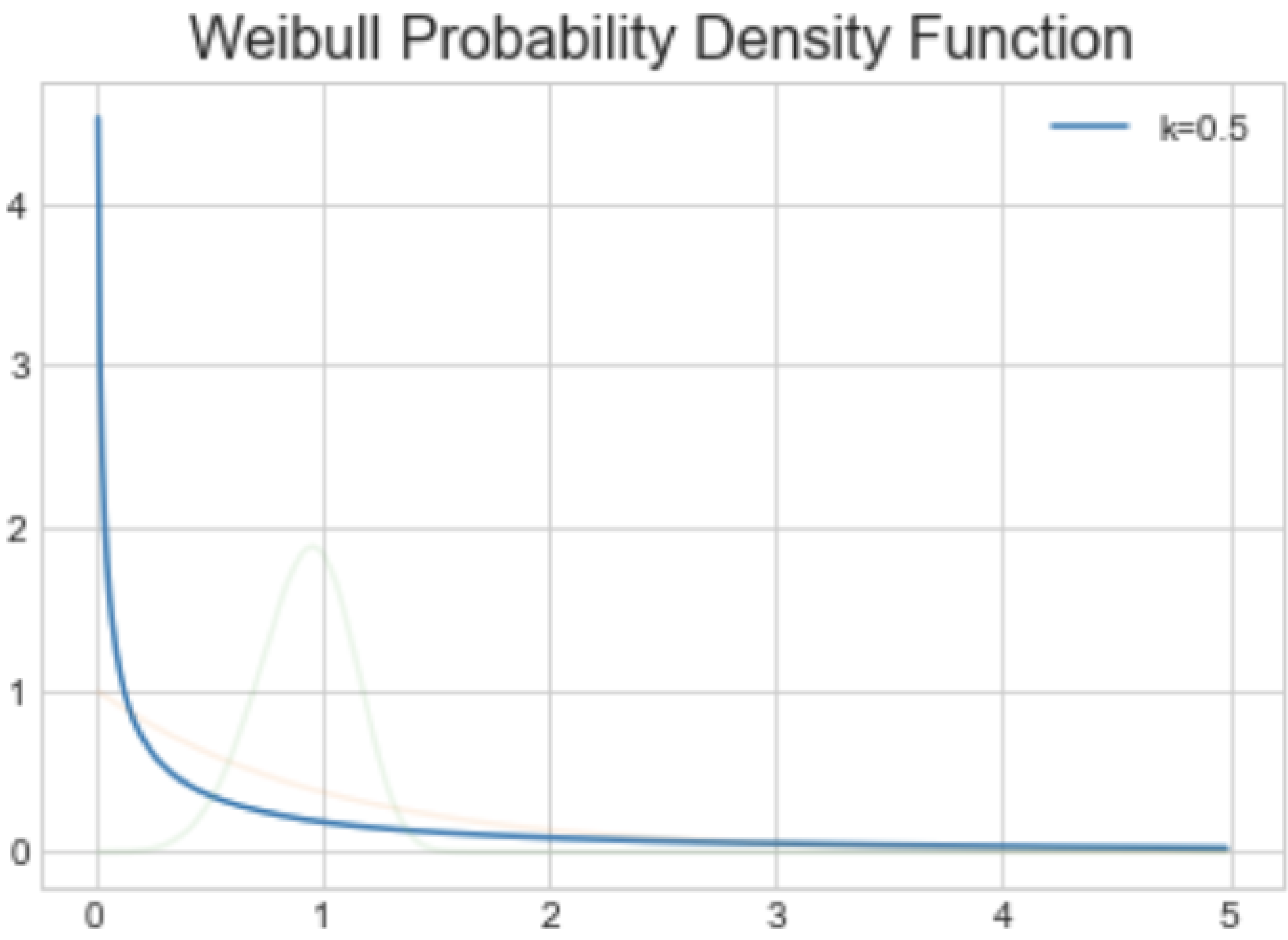
k and λ

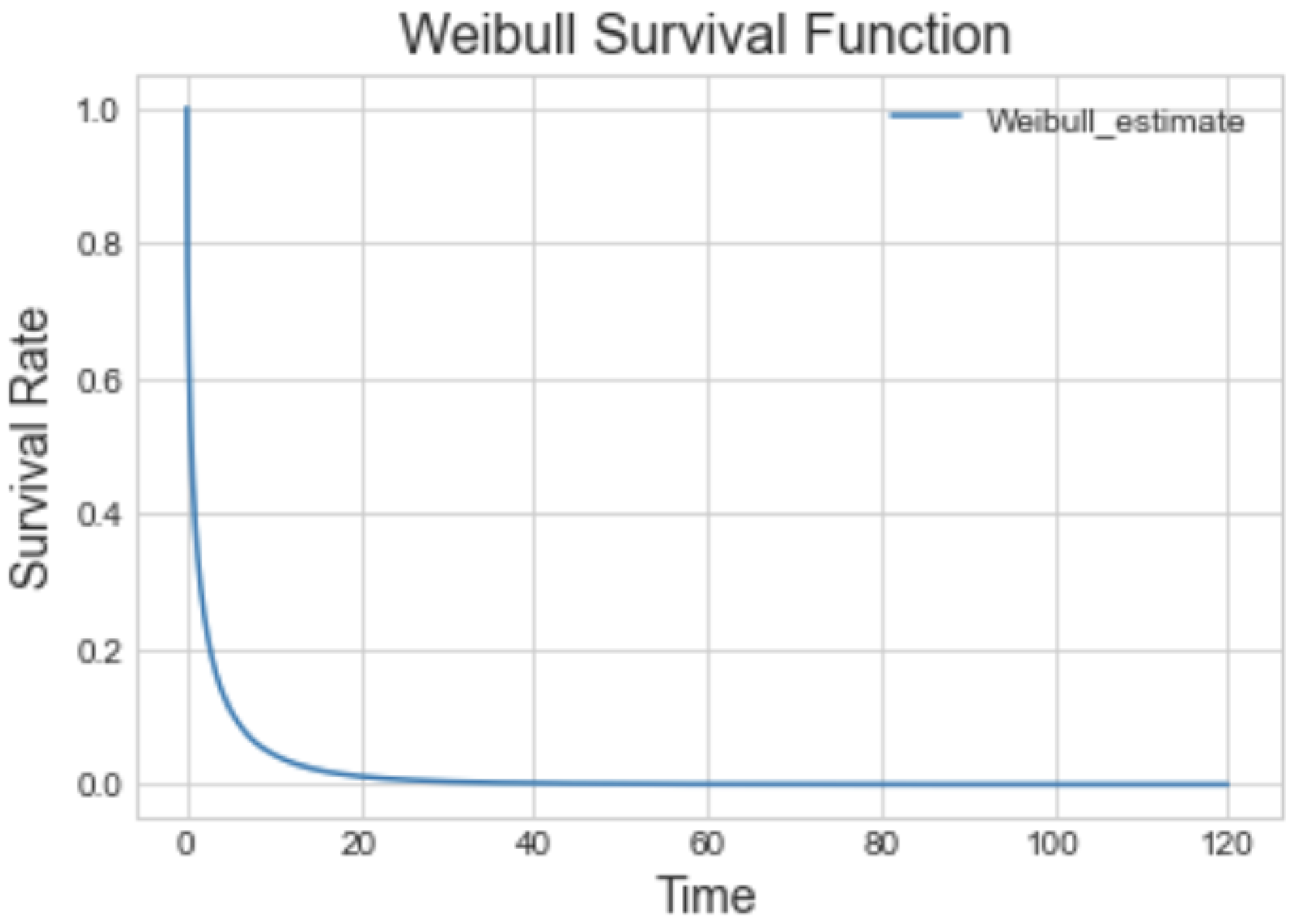
- k (or ρ): determines the shape
- λ: determines the scale (indicates when 63.2% of the population has experienced the event)

$$f(x;\lambda,k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} \quad \rightarrow \quad f(x;\lambda,k=3) = \frac{3}{\lambda}\left(\frac{x}{\lambda}\right)^{2} e^{-(x/\lambda)^3}$$

- Weibull distribution: the failure/event rate is proportional to a power of time.

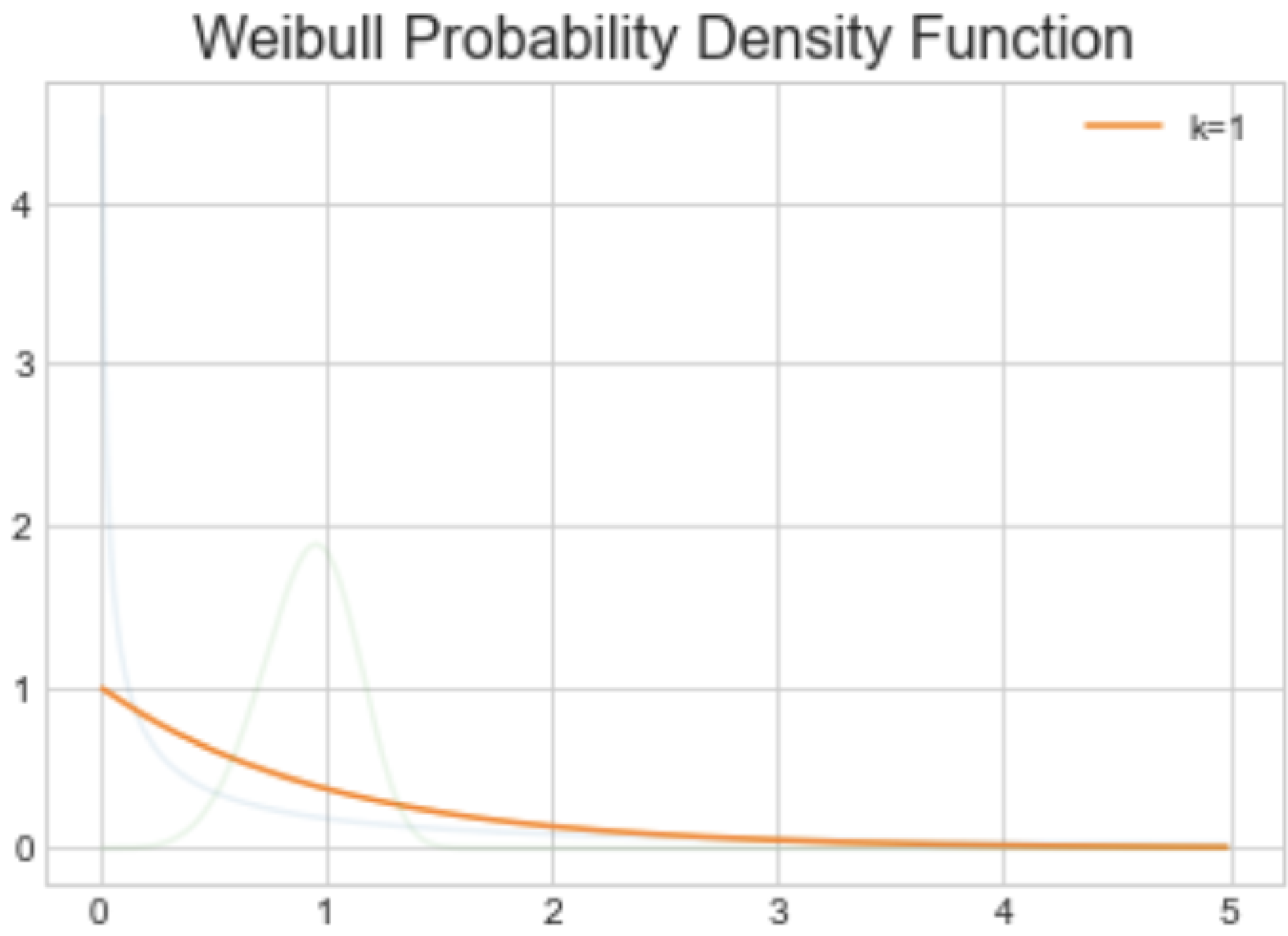
Interpreting k (or ρ)

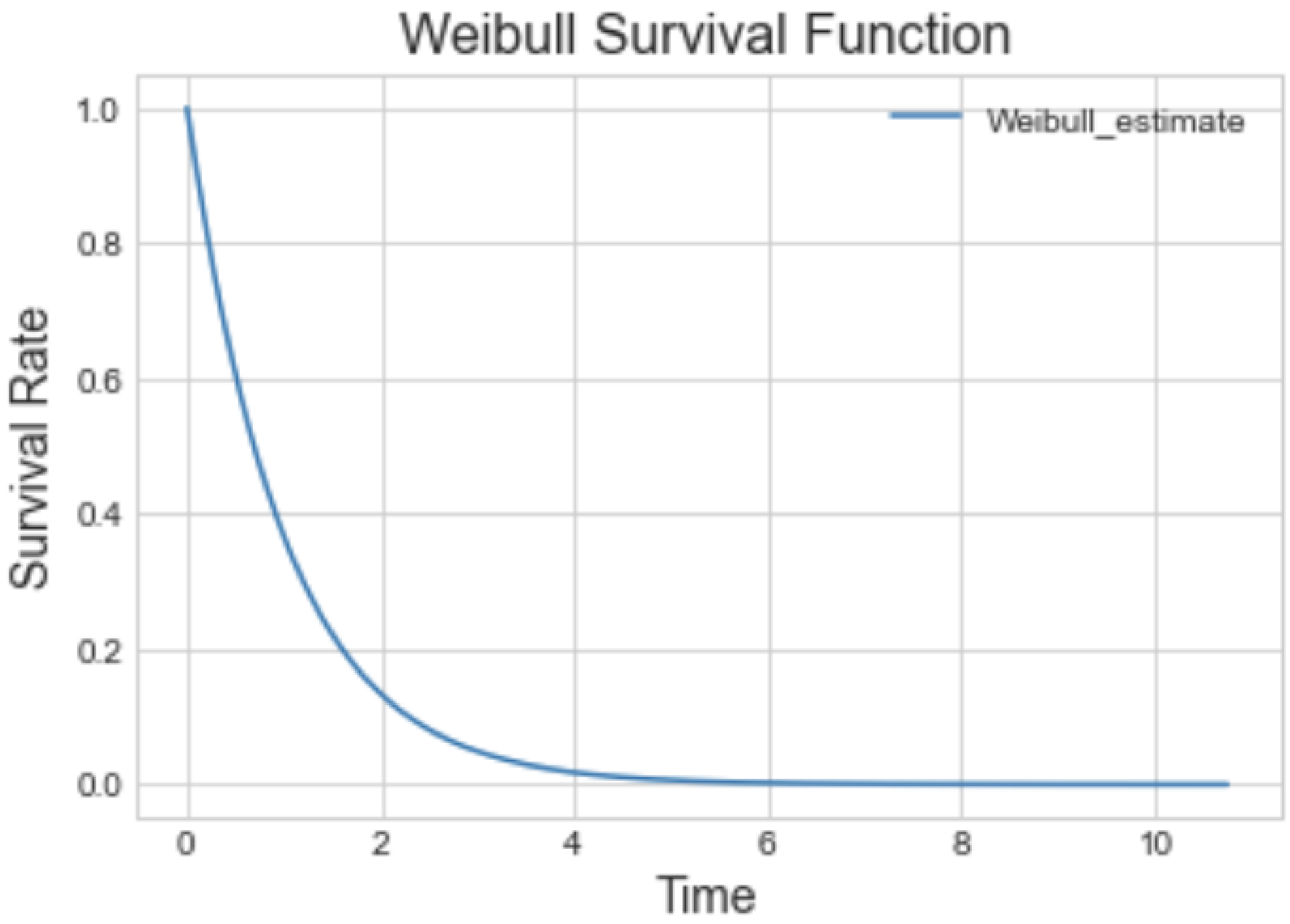




- When $k < 1$, the failure/event rate decreases over time.

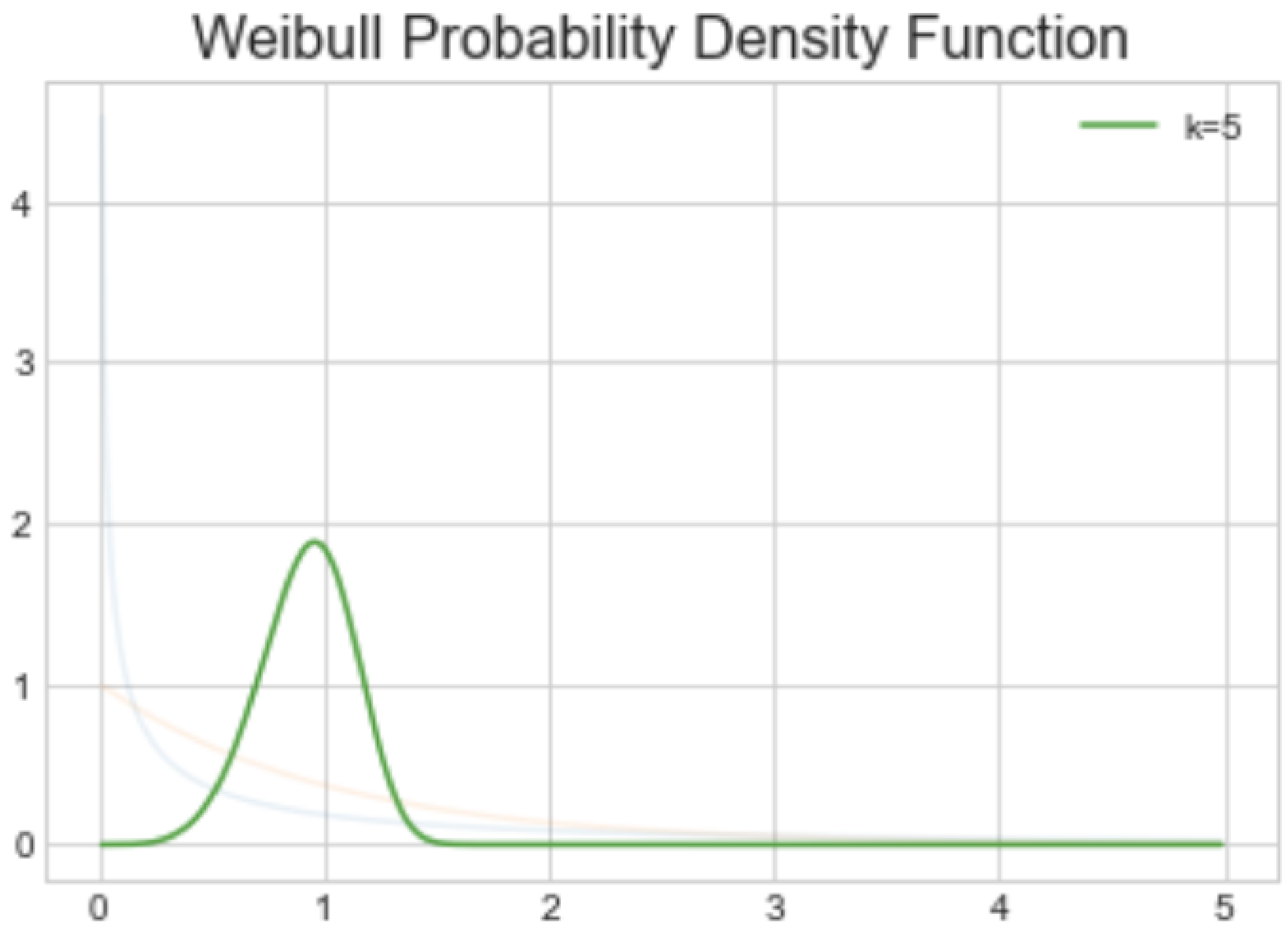
Interpreting k (or ρ)

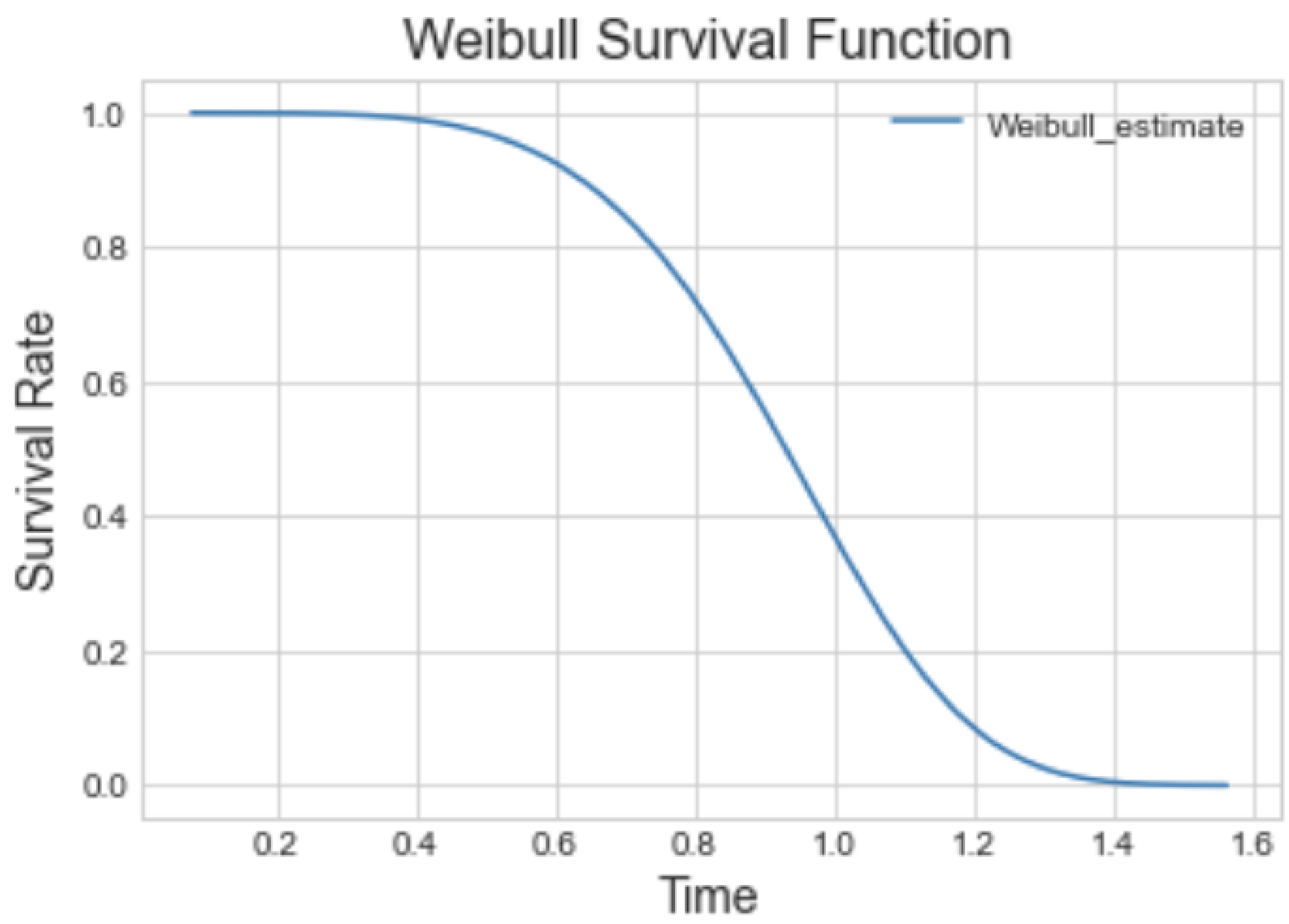




- When $k = 1$, the failure/event rate is constant over time.

Interpreting k (or ρ)





- When $k > 1$, the failure/event rate increases over time.

Survival analysis with Weibull distribution

1. Import the `WeibullFitter` class

```
from lifelines import WeibullFitter
```

2. Instantiate a `WeibullFitter` class

```
wb = WeibullFitter()
```

3. Call `.fit()` to fit the estimator to the data

```
wb.fit(durations, event_observed)
```

4. Access `.survival_function_`, `.lambda_`, `.rho_`, `.summary`, `.predict()`

Example Weibull model

DataFrame name: `mortgage_df`

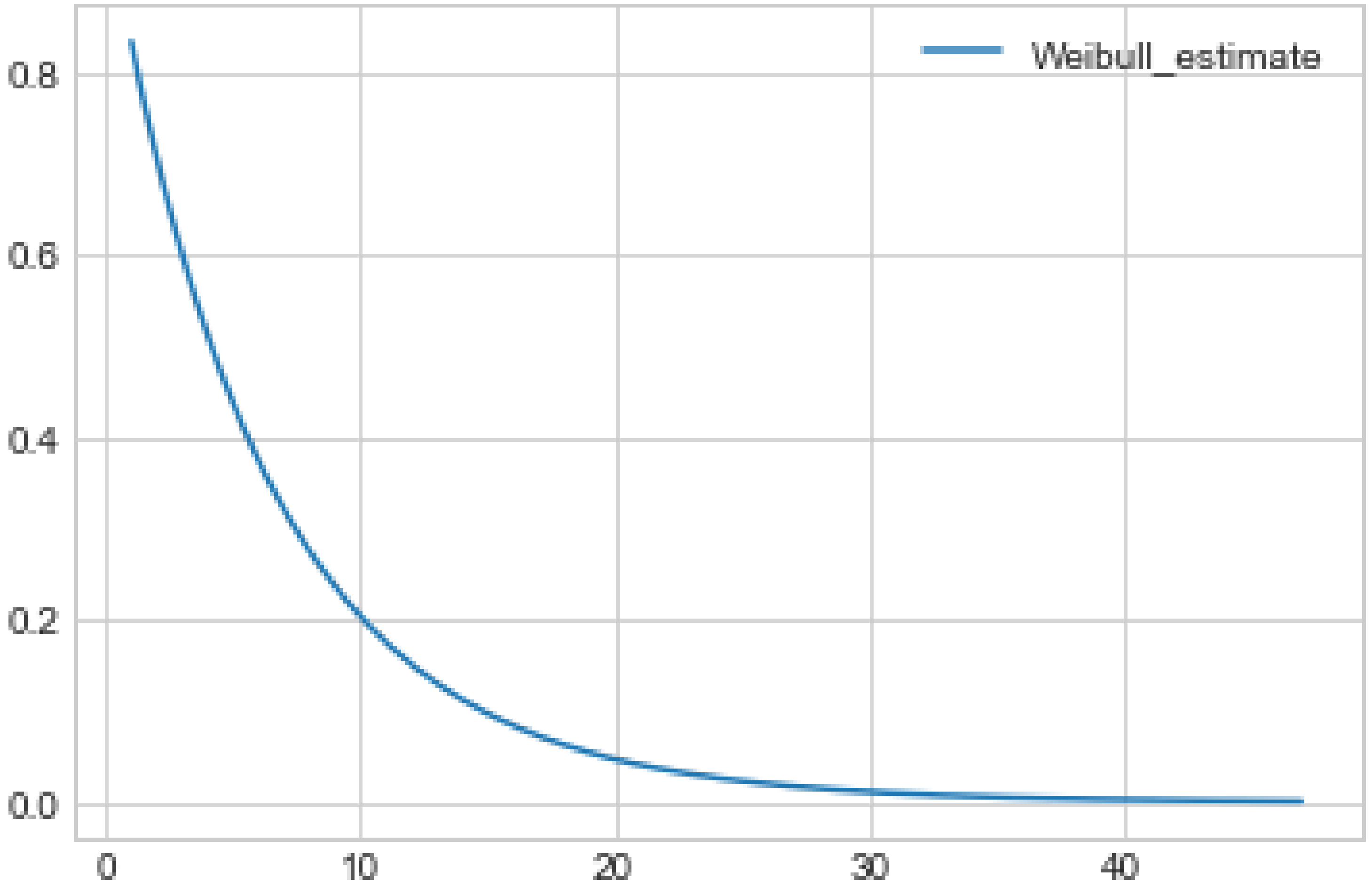
id	duration	paid_off
1	25	0
2	17	1
3	5	0
...	...	...
1000	30	1

```
from lifelines import WeibullFitter
wb = WeibullFitter()
```

```
wb.fit(durations=mortgage_df["duration"],
       event_observed=mortgage_df["paid_off"])
```

Example Weibull model

```
wb.survival_function_.plot()
plt.show()
```




```
print(wb.lambda_, wb.rho_)
```

```
6.11  0.94
```

```
print(wb.predict(20))
```

```
0.05
```

Let's practice!

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Weibull model with covariates

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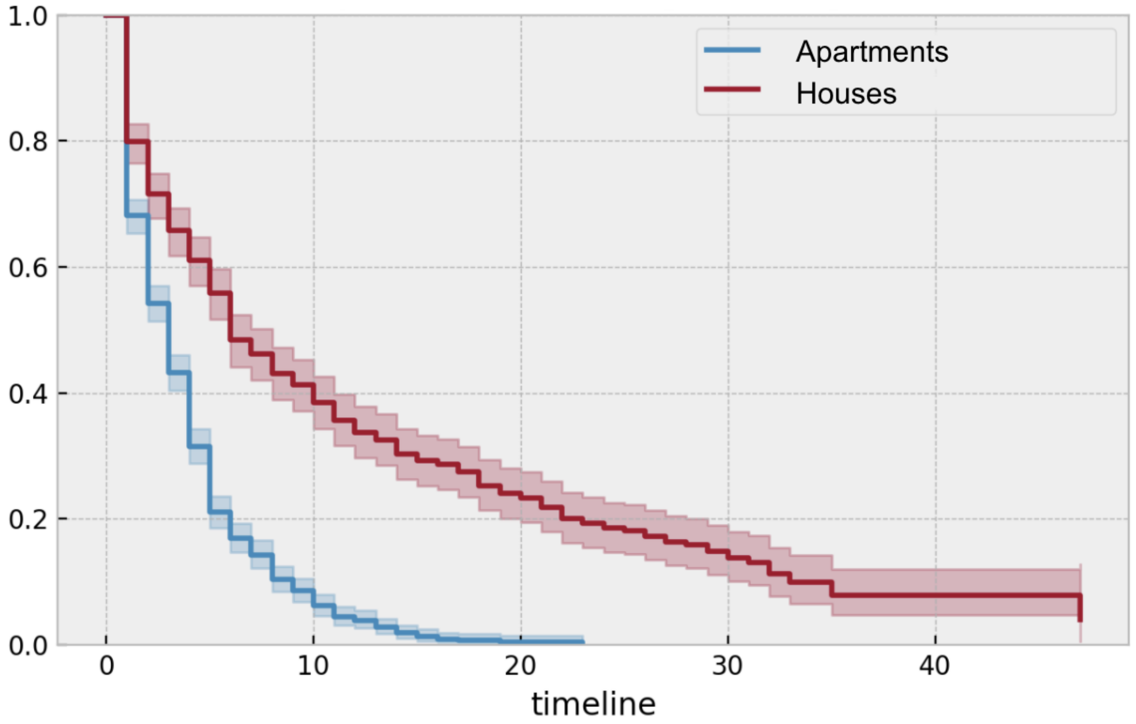
Senior Data Scientist

Comparing survival functions

Compare groups using the Kaplan-Meier estimator:



Compare groups using the log-rank test:

```
<lifelines.StatisticalResult: logrank_test>
 null_distribution = chi squared
degrees_of_freedom = 1
         test_name = logrank_test
 test_statistic    p  -log2(p)
           0.09 0.77      0.38
```

Q: How do we assess if/how one or multiple continuous variables affect the survival function?

Survival regression

- A method that models survival functions with covariates
- Quantifies how each covariate affects the survival function

$$Y_i = f(X_i, \beta)$$

$$Y_i : \text{durations}, X_i : \text{covariates}$$

- Example covariates: age, weight, country

The Accelerated Failure Time (AFT) model

$$\text{Population A} : S_A(t)$$

$$\text{Population B} : S_B(t)$$

$$S_A(t) = S_B(t * \lambda)$$

- $S_B(t)$ is speeding up (accelerating) or slowing down (decelerating) along $S_A(t)$ by a factor of λ.
- AFT models this acceleration/deceleration relationship based on model covariates.
- When a covariate changes from a to b, time-to-event speeds up or slows down by the accelerated failure rate λ.
- Example: $S_{dog}(t) = S_{human}(t * 7)$

Data for survival regression

DataFrame example: `mortgage_df`

id	property_type	principal	interest	property_tax	credit_score	duration	paid_off
1	house	1275	0.035	0.019	780	25	0
2	apartment	756	0.028	0.020	695	17	1
3	apartment	968	0.029	0.017	810	5	0
...	...	...	...	...	...	...	...
1000	house	1505	0.041	0.023	750	30	1

Combining Weibull with AFT: the Weibull AFT model

- DataFrame: `mortgage_df`
- Covariates:
  - `property_type` is **replaced** with a dummy variable:
    - `house` : 1 if "house", 0 if "apartment"
  - `principal`
  - `interest`
  - `property_tax`
  - `credit_score`

1. Import and instantiate the `WeibullAFTFitter` class

```
from lifelines import WeibullAFTFitter
aft = WeibullAFTFitter()
```

2. Call `.fit()` to fit the estimator to the data

```
aft.fit(df=mortgage_df,
        duration_col="duration",
        event_col="paid_off")
```

Interpreting model output

```
print(aft.summary)
```

```
<lifelines.WeibullAFTFitter: fitted with 1808 observations, 340 censored>
                      coef  exp(coef)  se(coef)      z        p
lambda_ house         0.04       1.04      0.01   0.99     0.32
        principal    -0.03       0.97      0.22  -1.04     0.30
        interest      0.11       1.11      0.15   1.96     0.05
        property_tax  0.31       1.36      0.27   1.15     0.25
        credit_score -0.16       0.85      0.14  -2.33     0.02
        Intercept     3.99      54.06      0.41   9.52 <0.0005
rho_    Intercept     0.34       1.40      0.08   3.80 <0.0005
```

WeibullAFTFitter with custom formula

Using `formula` to handle custom model covariates:

```
aft.fit(df=mortgage_df,
        duration_col="duration",
        event_col="paid_off",
        formula="principal + interest * house")
```

Analogous to the linear model with interaction term:

$$\beta_1 \text{principal} + \beta_2 \text{interest} + \beta_3 \text{house} + \beta_4 \text{interest} \cdot \text{house}$$

Interpreting model output

```
print(aft.summary)
```

```
<lifelines.WeibullAFTFitter: fitted with 1808 observations, 340 censored>
                             coef  exp(coef)  se(coef)      z        p
lambda_ principal           -0.03       0.97      0.22  -1.04     0.30
        interest             0.11       1.11      0.15   1.96     0.05
        house                0.04       1.04      0.01   0.99     0.32
        interest:house       0.06       1.06      0.14   0.42     0.64
        Intercept            3.99      54.06      0.41   9.52 <0.0005
rho_    Intercept            0.34       1.40      0.08   3.80 <0.0005
```

Let's practice!

SURVIVAL ANALYSIS IN PYTHON

Visualization and prediction with Weibull model

SURVIVAL ANALYSIS IN PYTHON

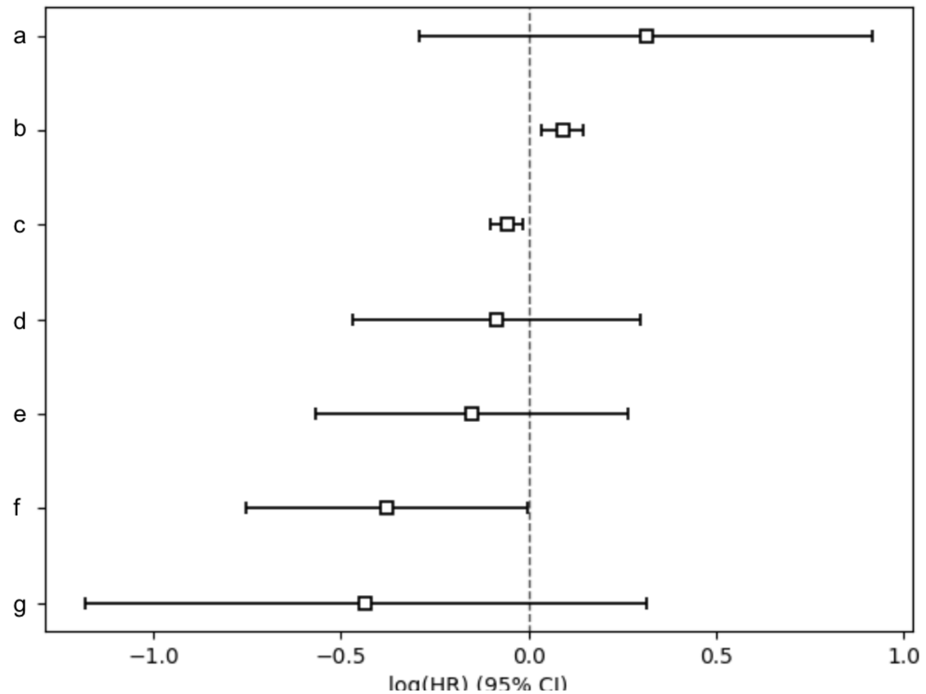
Shae Wang
Senior Data Scientist

.plot()

Returns a plot of the coefficients and their ranges from the 95% confidence intervals.

```
aft.plot()
plt.show()
```

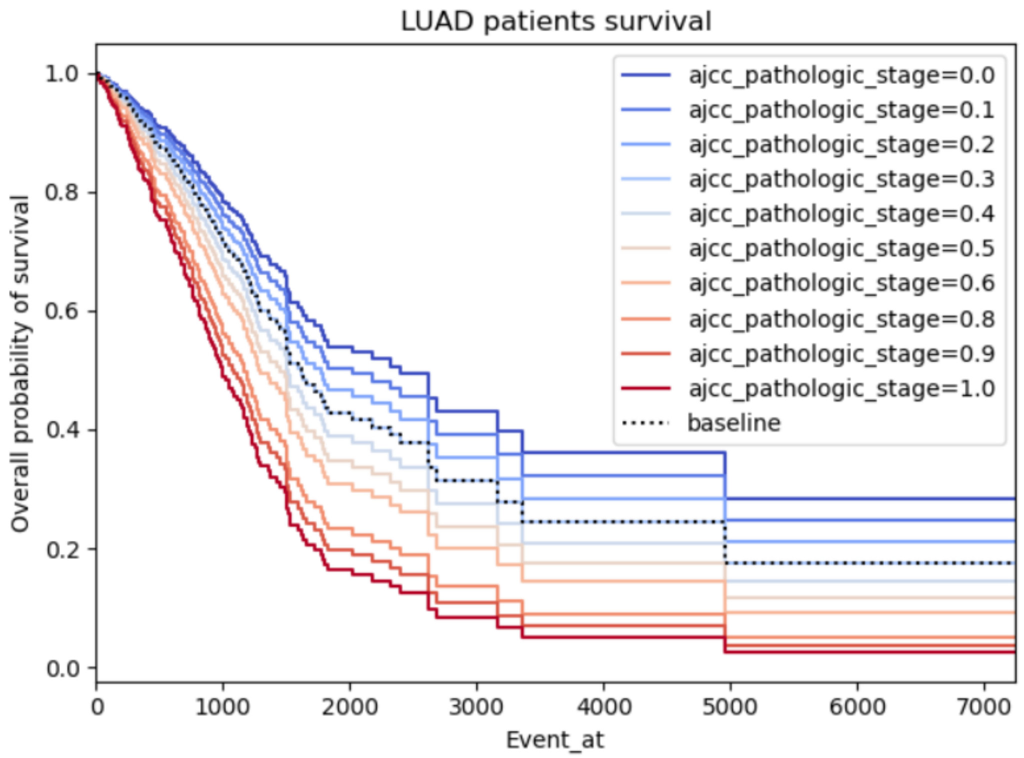
Sample plot:



.plot_partial_effects_on_outcome()

Returns a plot comparing the baseline survival curve versus what happens when covariates are varied over values.

```
aft.plot_partial_effects_on_outcome(covariates, values)
plt.show()
```

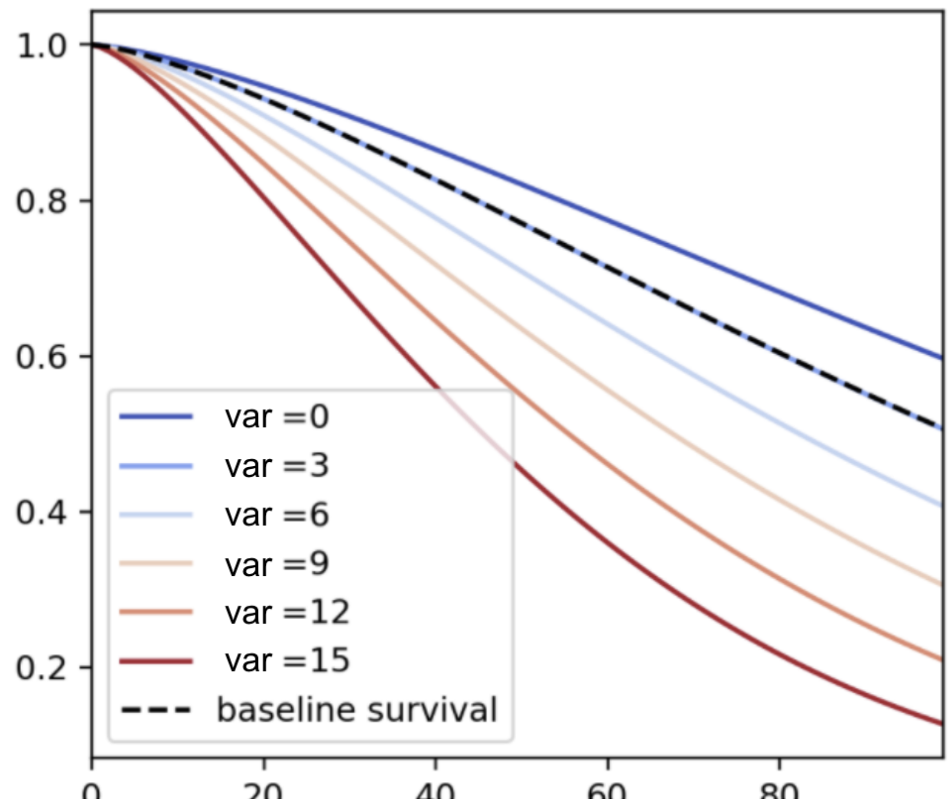


How to plot partial effects?

`.plot_partial_effects_on_outcome()`

- `covariates` (string or list): covariate(s) in the original dataset that we wish to vary.
- `values` (1d or 2d iterable): values we wish the covariate to take on.
- **Baseline survival curve**: predicted survival curve at all average values in the original dataset.

```
aft.plot_partial_effects_on_outcome(
  covariates='var',
  values=[0, 3, 6, 9, 12, 15]
)
plt.show()
```



How to plot partial effects?

- Hard-code values:

```
aft.plot_partial_effects_on_outcome(
  covariates='a',
  values=[0, 3, 6]
)
```

- Use a range function:

```
aft.plot_partial_effects_on_outcome(
  covariates='a',
  values=np.arange(10)
)
```

- Multiple covariates:

```
aft.plot_partial_effects_on_outcome(
  covariates=['a','b'],
  values=[[1,2],[1,3],[2,3]]
)
```

- Custom formula:
  - Necessary transformations (interactions, one-hot encoding, etc.) will be made internally and automatically.

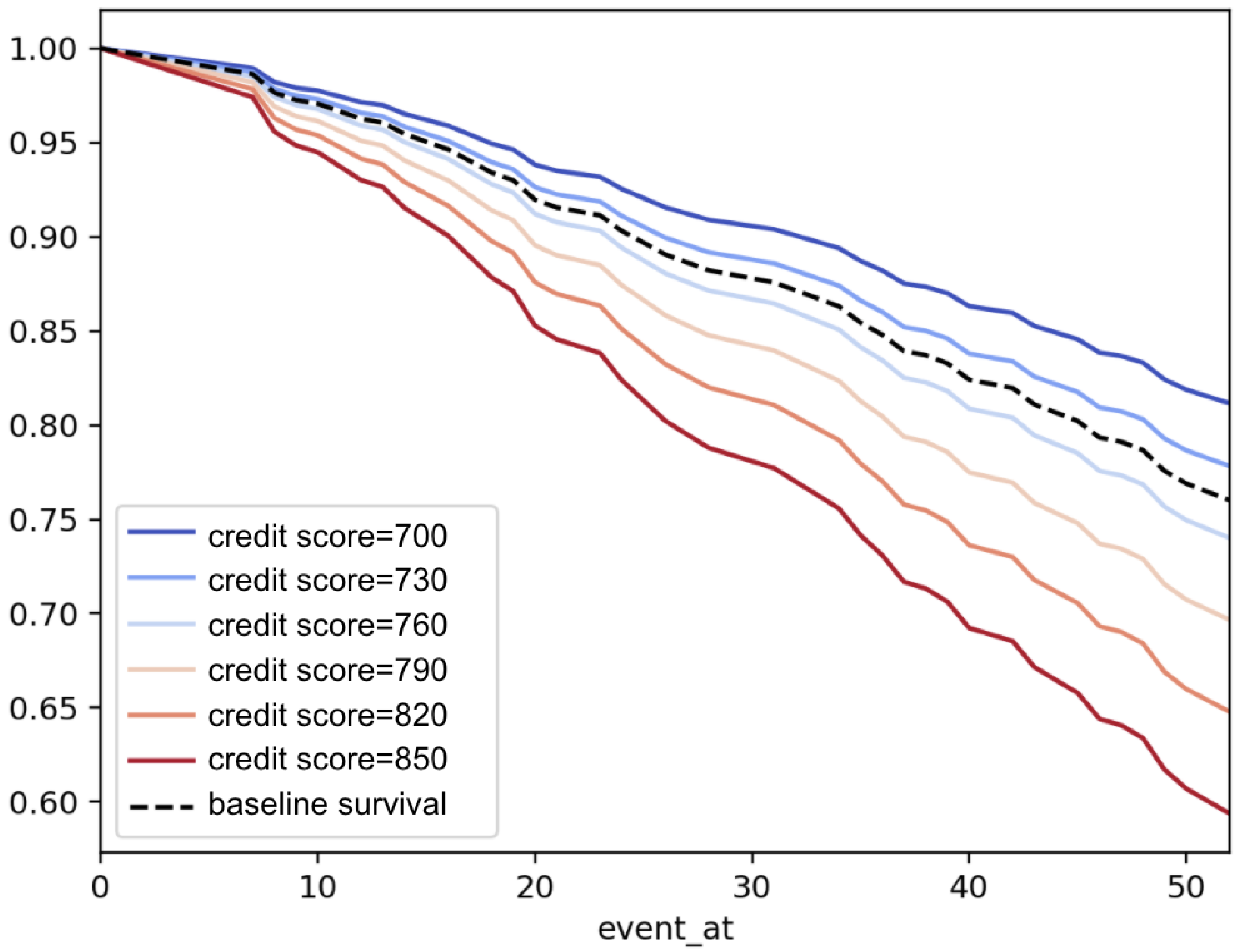
Mortgage example

DataFrame example: `mortgage_df`

id	house	principal	interest	property_tax	credit score	duration	paid_off
1	1	1275	0.035	0.019	780	25	0
2	0	756	0.028	0.020	695	17	1
3	0	968	0.029	0.017	810	5	0
...	...	...	...	...	...	...	...
1000	1	1505	0.041	0.023	750	30	1

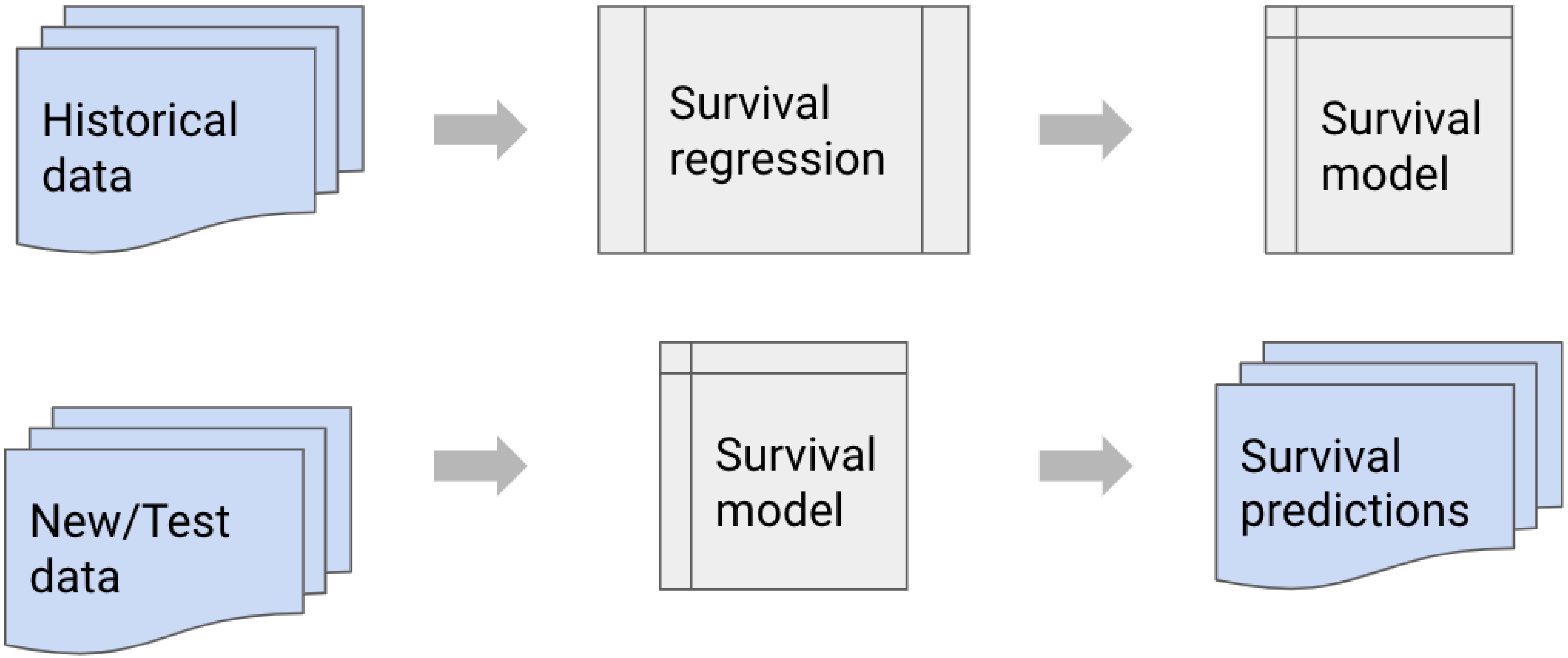
Mortgage example

```
aft.plot_partial_effects_on_outcome(
    covariates='credit score',
    values=np.arange(700, 860, 30)
)
plt.show()
```



Predict survival functions

- Survival curves vary based on their covariates' values.



Predict survival functions

Predict survival functions of individuals based on covariate values.

`.predict_survival_function()`

Arguments:

- `X` (`np` array or DataFrame): covariates. If a DataFrame, columns can be in any order.

Predict median survival durations of individuals based on covariate values.

`.predict_median()`

Arguments:

- `df` (`np` array or DataFrame): covariates. If a DataFrame, columns can be in any order.

Conditional after current durations

Predict survival function or median survival duration conditional after current duration.

- `.predict_survival_function(X, conditional_after)`
- `.predict_median(df, conditional_after)`

Example:

```
aft.predict_median(new_subject)
```

```
4.0
```

```
aft.predict_median(new_subject, conditional_after=[2])
```

```
2.0
```

Let's practice!

Other distributions and model selection

SURVIVAL ANALYSIS IN PYTHON

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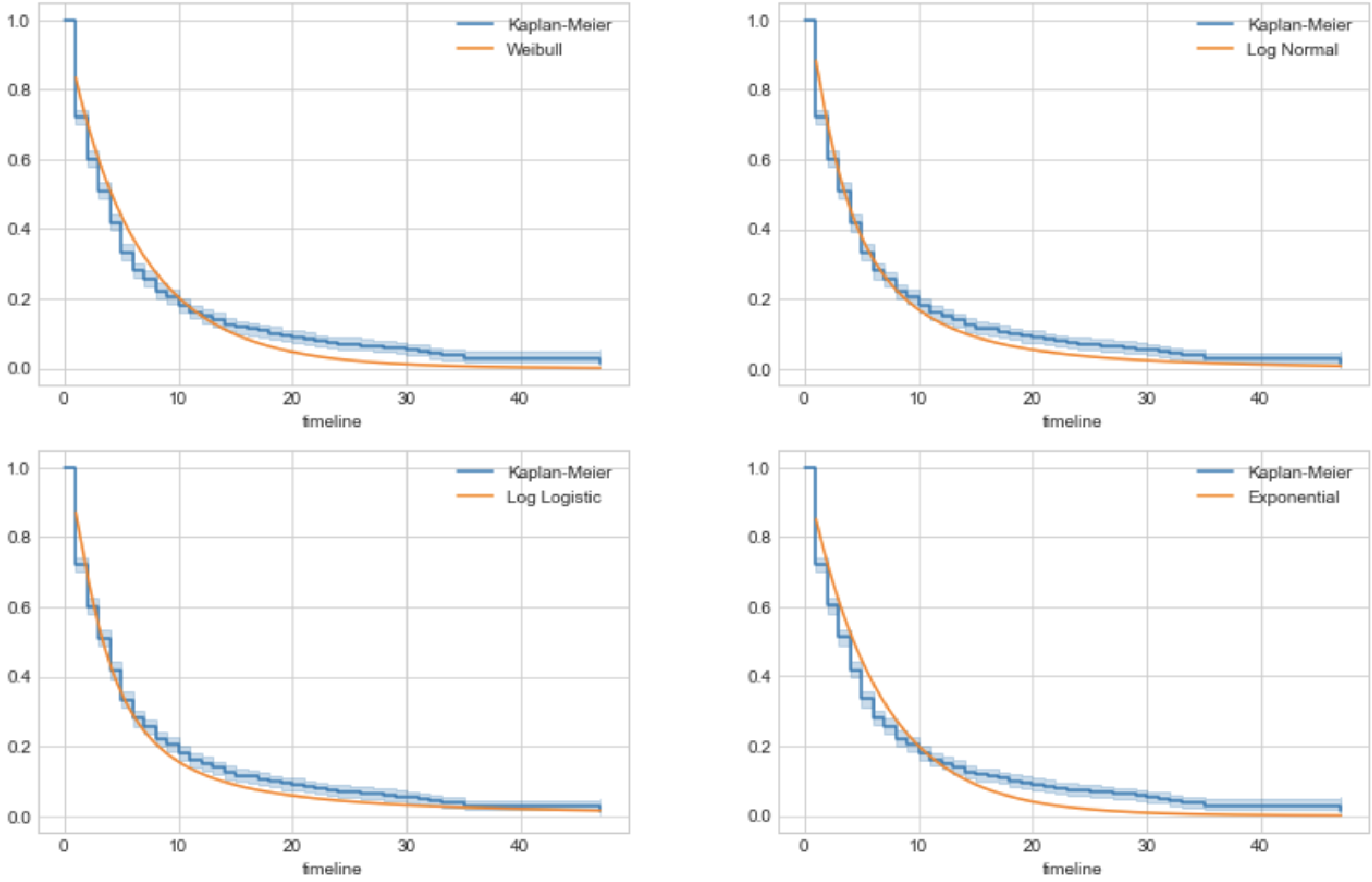
Which model fits the data the best?



Choosing parametric models

- **Non-parametric modeling** (i.e. the Kaplan-Meier model)
  - Describes the data accurately because it's distribution-free
  - Is not smooth/continuous/differentiable
- **Parametric modeling** (i.e. the Weibull model)
  - Parametric statistics will usually give us more information
  - When the wrong model is used, they lead to significantly biased conclusions

Common parametric survival models

- The Weibull model

```
from lifelines import WeibullFitter
```

- The Exponential model

```
from lifelines import ExponentialFitter
```

- The Gamma model

```
from lifelines import GeneralizedGammaFitter
```

- The Log Normal model

```
from lifelines import LogNormalFitter
```

- The Log Logistic model

```
from lifelines import LogLogisticFitter
```

The Akaike Information Criterion (AIC)

- AIC: An estimator of **prediction error** and **relative quality of statistical models** for a given set of data.
- Estimates the relative amount of information lost by a given model and penalizes large number of estimated parameters.
  - The less information a model loses, the higher the quality of that model.
  - The fewer parameters (less complex) a model is, the higher the quality of that model.
- Given a set of candidate models for the data, the one with **the minimum AIC value** is the preferred model.

Using the AIC for model selection

Step 1) Fit parametric models in `lifelines`

Step 2) Print and compare each model's `AIC_` property

Step 3) The lowest AIC value is preferred

```
from lifelines import WeibullFitter,
                     ExponentialFitter,
                     LogNormalFitter
```

```
wb = WeibullFitter().fit(D, E)
exp = ExponentialFitter().fit(D, E)
log = LogNormalFitter().fit(D, E)
```

```
print(wb.AIC_, exp.AIC_, log.AIC_)
```

```
215.9091   216.1183   202.3498
```

find_best_parametric_model()

- `find_best_parametric_model()` : a built-in `lifelines` function to automate AIC comparisons between parametric models.
- Iterates through each parametric model available in `lifelines` .

How to use it?

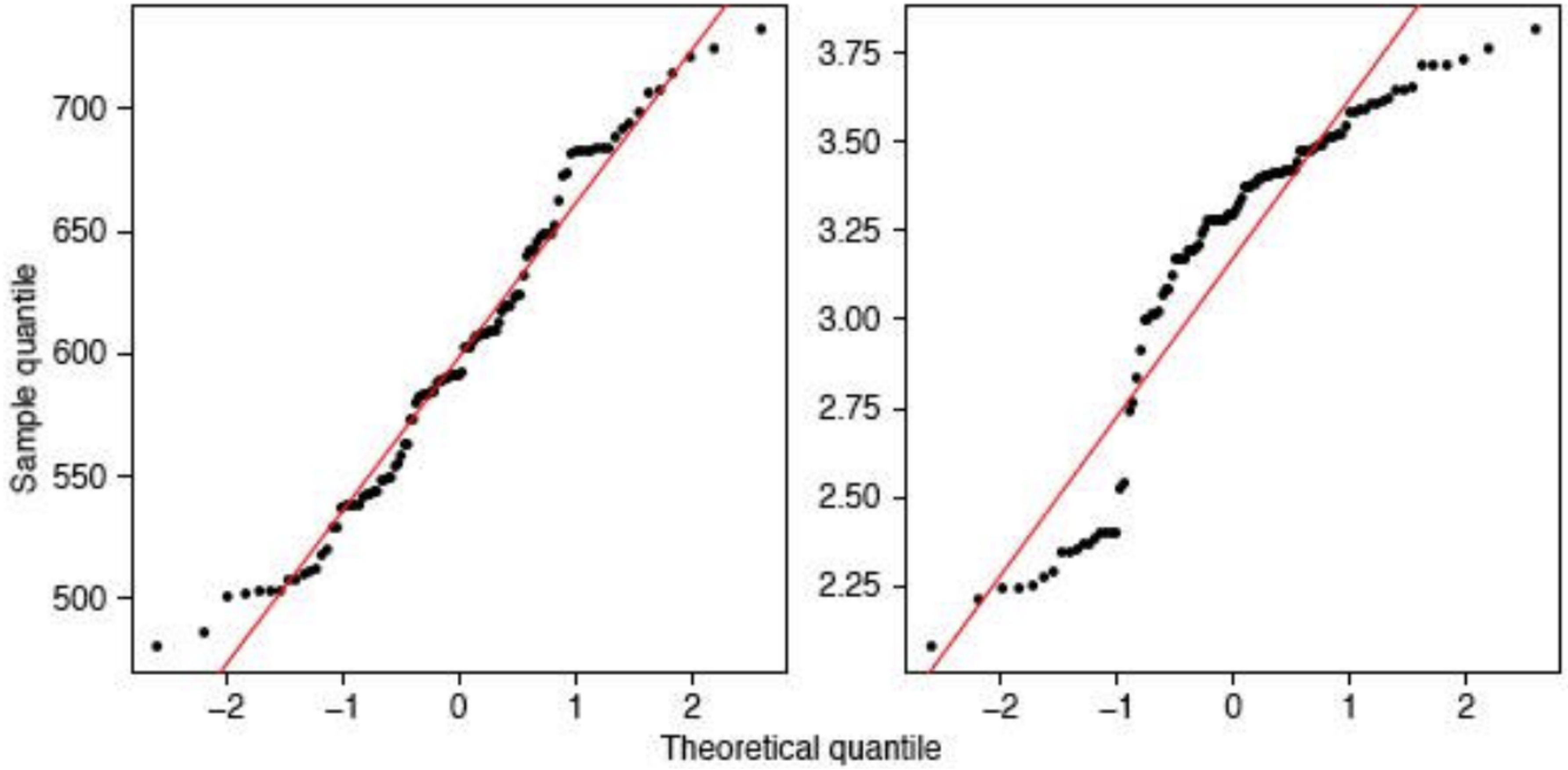
- `T` : durations, `E` : censorship

```
best_model, best_aic_ = find_best_parametric_model(event_times=T,
                                                    event_observed=E,
                                                    scoring_method="AIC")
print(best_model)
```

```
<lifelines.WeibullFitter:"Weibull_estimate",
fitted with 686 total observations, 387 right-censored observations>
```

The QQ plot

- QQ plot: Compares two probability distributions by plotting their quantiles against each other.
- If the two distributions being compared are similar, the points in the QQ plot will approximately lie on the line `y = x` .



Using QQ plots for model selection

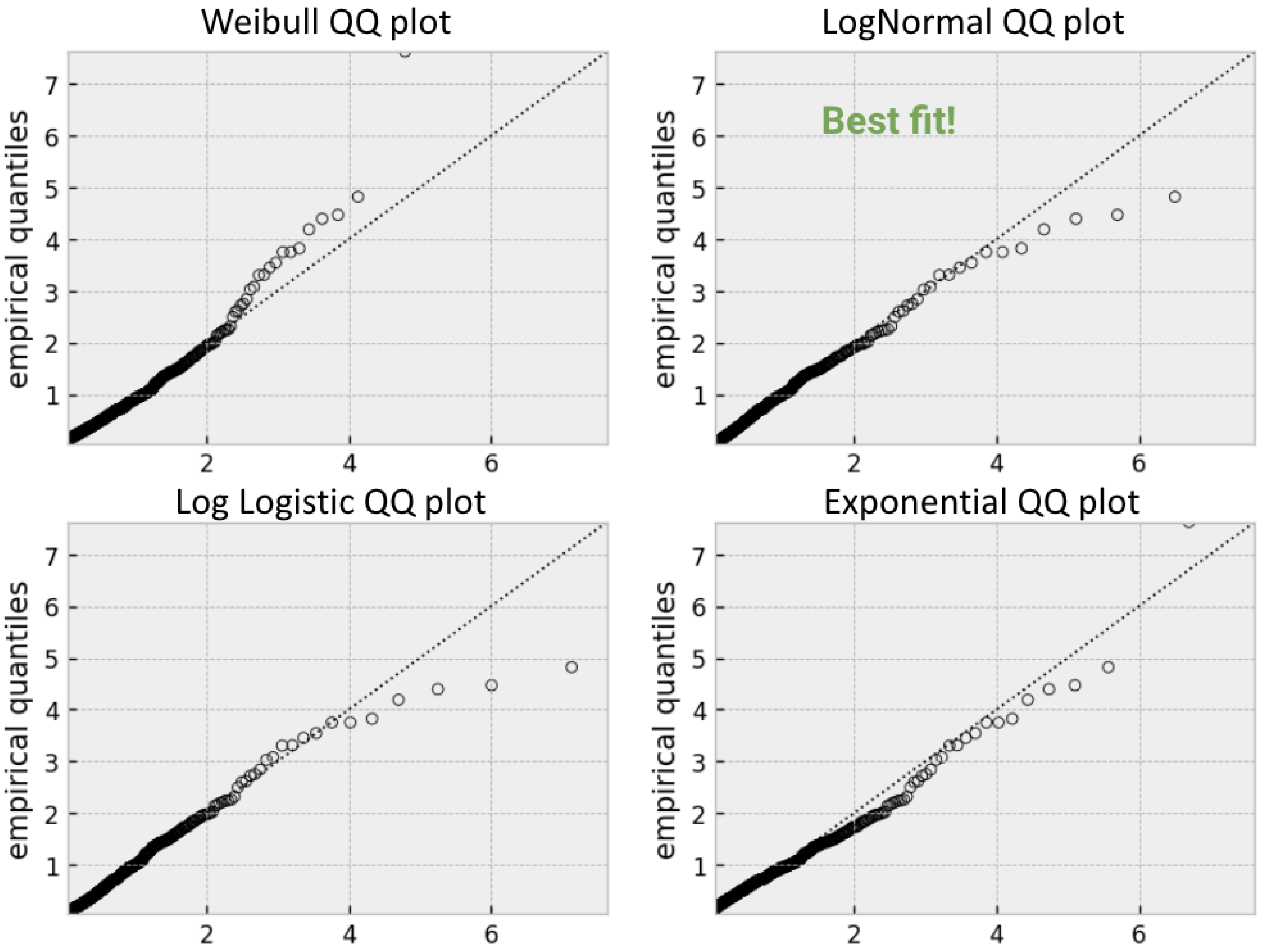
Step 1) Fit parametric models in `lifelines`.

Step 2) Plot the QQ plot of each model.

Step 3) The QQ plot closest to `y = x` is preferred.

```
from lifelines.plotting import qq_plot
for model in [WeibullFitter(),
              LogNormalFitter(),
              LogLogisticFitter(),
              ExponentialFitter()]:
  model.fit(T, E)
  qq_plot(model)
plt.show()
```

Using QQ plots for model selection



Let's practice!

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