Chapter 24, 25
Shortest Paths

Algorithm Analysis

School of CSEE

Shortest path

- Input : directed graph $G = (V, E)$ with weight function $w : E \to R$
- Determine the path p with minimum weight from source to destination.
- Weight $w(p)$ of path p : sum of edge weights on path p
- *shortest-path weight u to v :*

$$\delta(u,v) = \begin{cases} \min \{ w(p) : \text{path } p \text{ from } u \text{ to } v \} \\ \qquad \text{if there exists a path from } u \text{ to } v. \\ \infty, \text{ otherwise.} \end{cases}$$

Variants

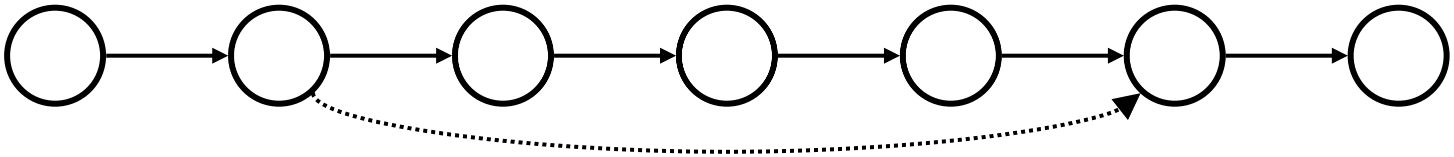
- Single-source shortest path : find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$
- Single-destinations : find shortest paths to a given destination vertex. By reversing the direction of the each edge in the graph, the problem is reduced to single-source problem.
- Single-pair : find shortest path from u to v. No easier than single-source problem.
- All-pairs shortest-paths : find shortest path from u to v for all u, $v \in V$

Variants

- Single-source shortest path
  - Bellman-Ford algorithm
  - In DAG
  - Dijkstra's algorithm
- All-pairs shortest-paths
  - Floyd-Warshall algorithm

Shortest Path Properties : Lemma 24.1

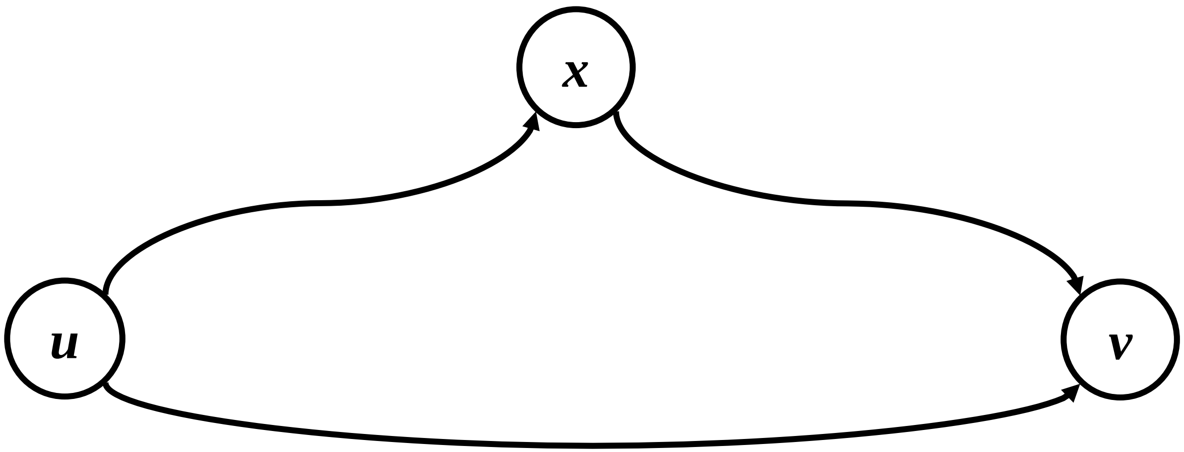
- Again, we have *optimal substructure*: the shortest path consists of shortest subpaths:



  – Proof: suppose some subpath is not a shortest path.
    - There must then exist a shorter subpath.
    - We could substitute the shorter subpath for a shortest path.
    - But then overall path is not shortest path. Contradiction!!

Shortest Path Properties

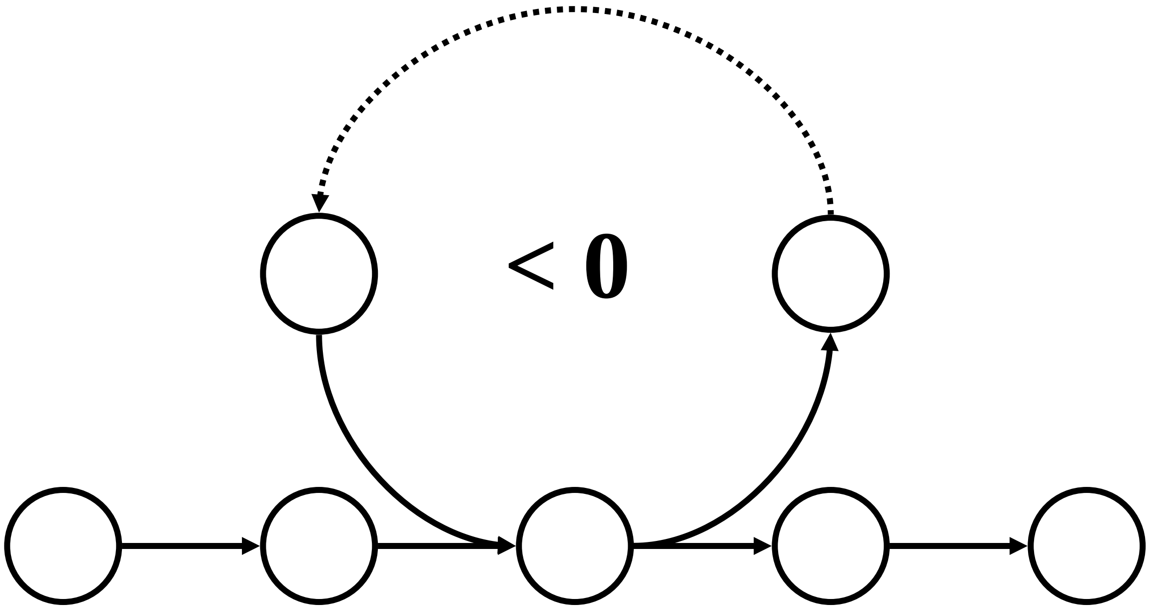
- Define $\delta(u,v)$ to be the weight of the shortest path from u to v. Shortest paths satisfy the *triangle inequality*: $\delta(u,v) \leq \delta(u,x) + \delta(x,v)$



This path is no longer than any other path

Negative-weight edges

- OK, as long as no negative-weight cycle are reachable from the source.
- If we reach a negative-weight cycle, we can just keep going around it and get $w(s, v) = -\infty$ for all v in the cycle.
- Some algorithms work only if there are no negative-weight edges.

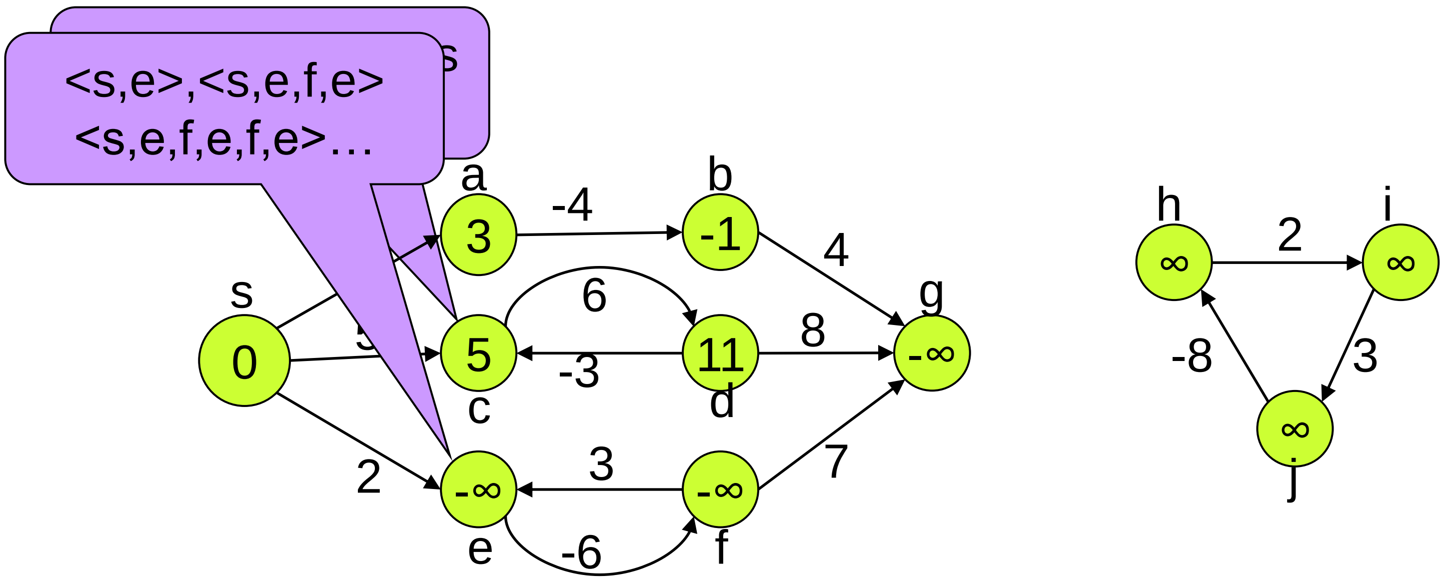


Cycles

- Shortest paths can't contain cycles.
  - We assumed that there is no negative-weight cycles.
  - Positive-weight cycle : Just omit it to get a shorter path.
  - Zero-weight cycle : There is no reason to use them. Assume that our solutions won't use them.

Cycles

- If there is a cycle reachable from s, shortest-path weights are not well defined.
- $\delta(s,v) = -\infty \ (v \in V)$



Single Source Shortest Path

SS shortest-path algorithm

- Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.
- Output for each vertex $v \in V$:
  - $d[v] = \delta(s,v)$
    - Initially, $d[v] = \infty$
    - Reduces $d[v]$ as algorithm progress.
      But always maintain $d[v] \geq \delta(s,v)$
    - $d[v]$: *shortest-path estimate*
  - $\pi[v]$ = predecessor of v on a shorteat path from s.
    - If no predecessor, $\pi[v]$ = NIL
    - π induces a tree – *shortest-path tree*

Initialization

INIT-SINGLE-SOURCE(V, s)

for each $v \in V$

do $d[v] = \infty$

$\pi[v]$ = NIL

$d[s] = 0$

Relaxation

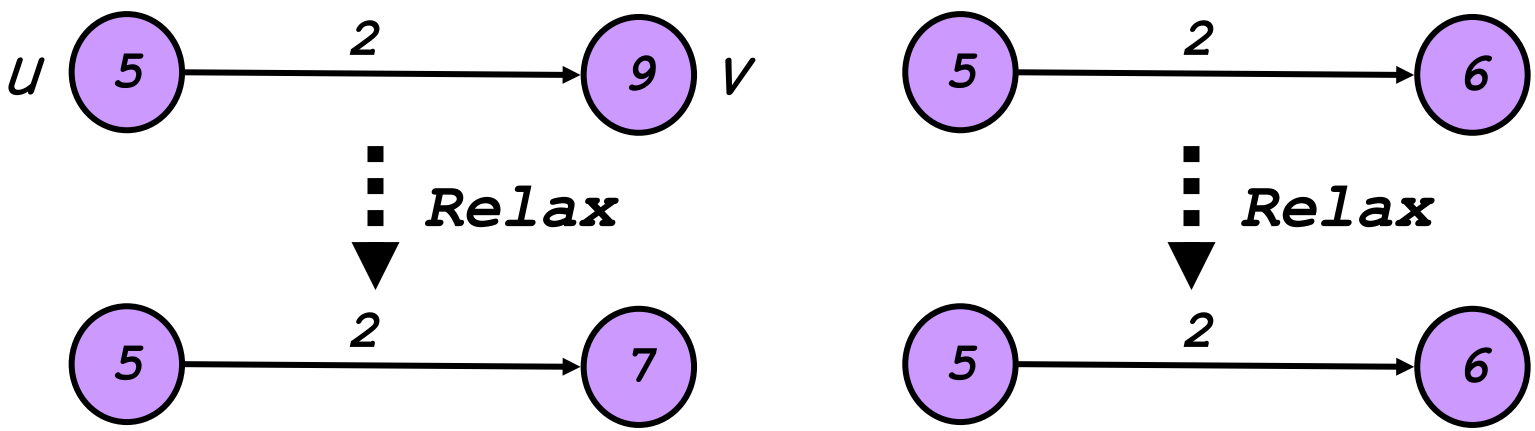
- A key technique in shortest path algorithms is *relaxation*.
  - Idea: for all *v*, maintain upper bound *d*[*v*] on δ(s,v)

Relax(*u*,*v*,*w*) {

if (*d*[*v*] > *d*[*u*]+*w*) /* *w* = *w*(*u*,*v*) */

then *d*[*v*] = *d*[*u*]+*w*;

π[*v*] = *u* }



[1] Bellman-Ford algorithm

- Solves the single-source shortest-paths problem in general case in which edge weights may be negative.

- Allow negative-weight edges and produce a correct answer as long as no negative-weight cycles are reachable from the source.

- Returns a boolean value
  - negative-weight cycle – FALSE
  - No such cycle – TRUE

Pseudocode

BELLMAN-FORD(V, E, w, s)

1. INIT-SINGLE-SOURCE(V, s)
2. for i = 1 to $|V|$ - 1
3. do for each edge $(u,v) \in E$
4. do RELAX(u, v, w)
5. for each edge $(u,v) \in E$
6. do if $d[v] > d[u] + w(u,v)$
7. then return FALSE
8. return TRUE

Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relaxing each edge

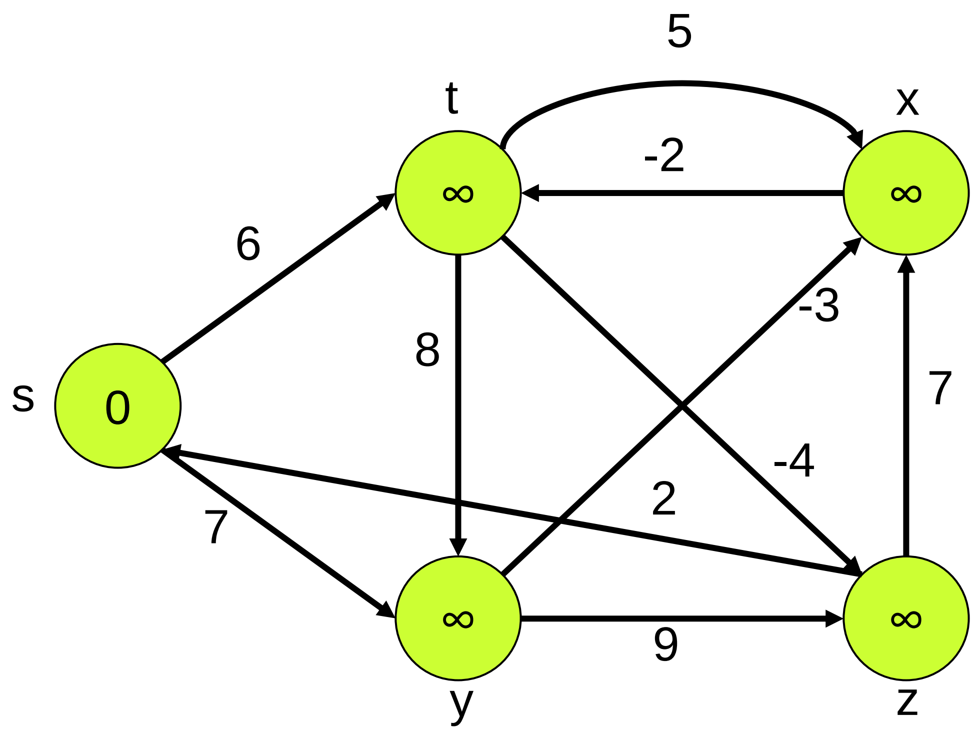
Test for solution Under what condition do we get a solution?

The first for loop relaxes all edges $|V|$ - 1 times.

Time : $\Theta(VE)$

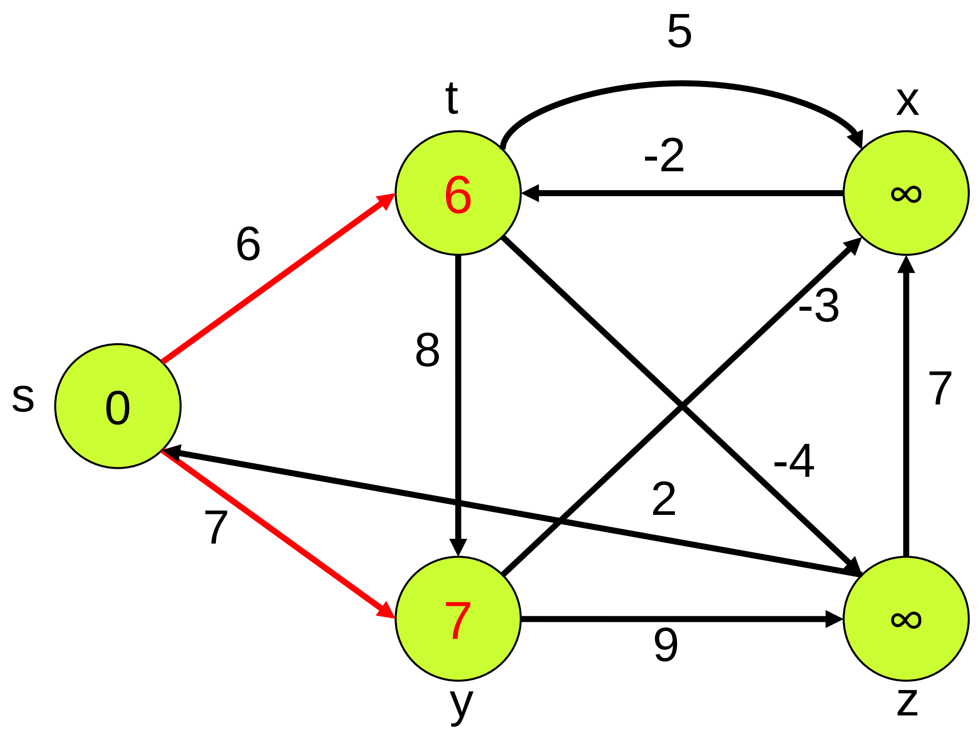
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- After INIT-SINGLE-SOURCE(V, s)



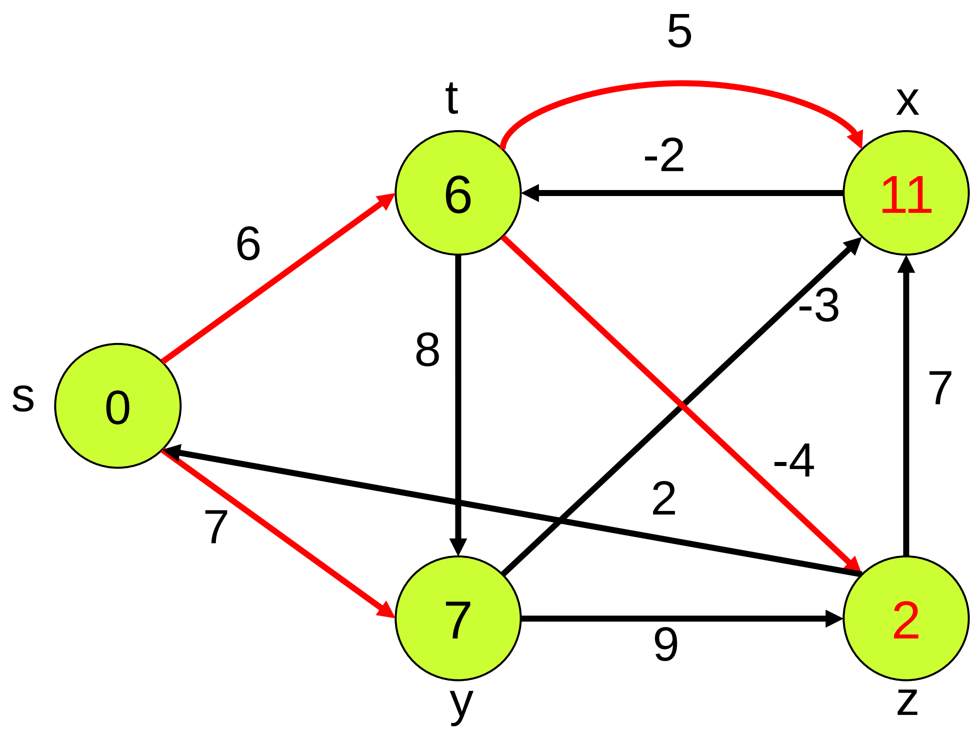
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- At first pass



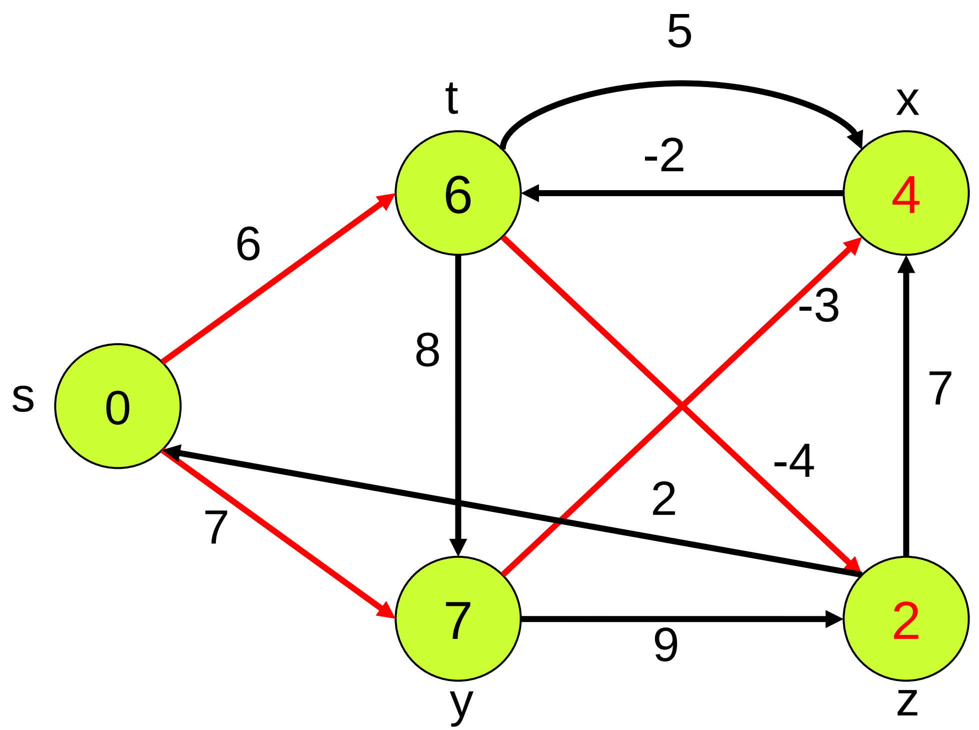
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- At second pass



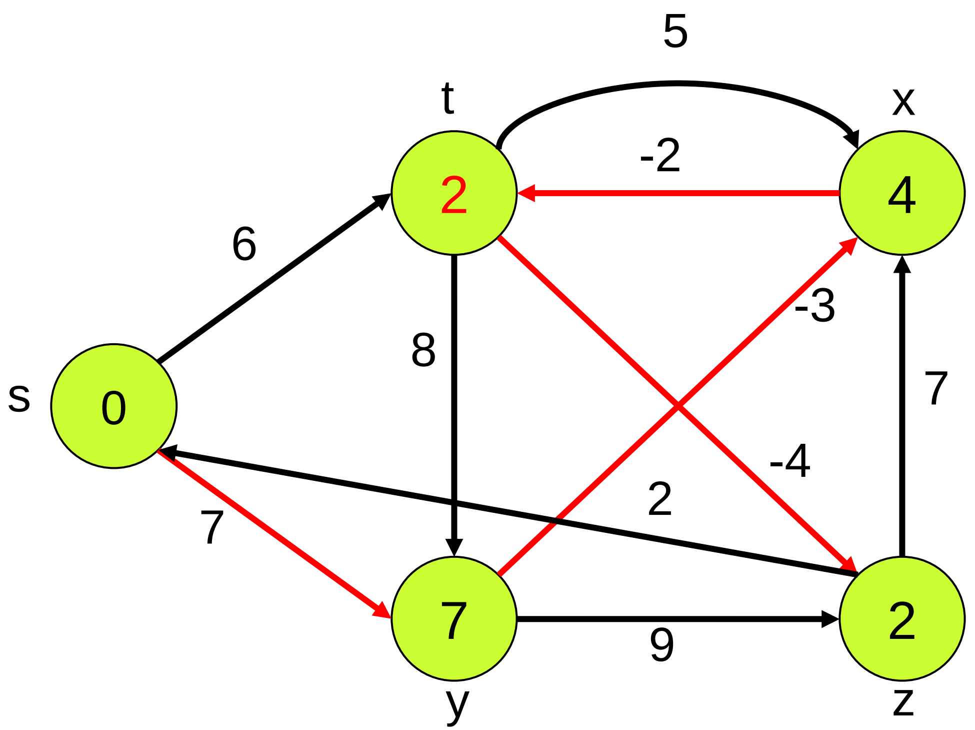
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- At second pass



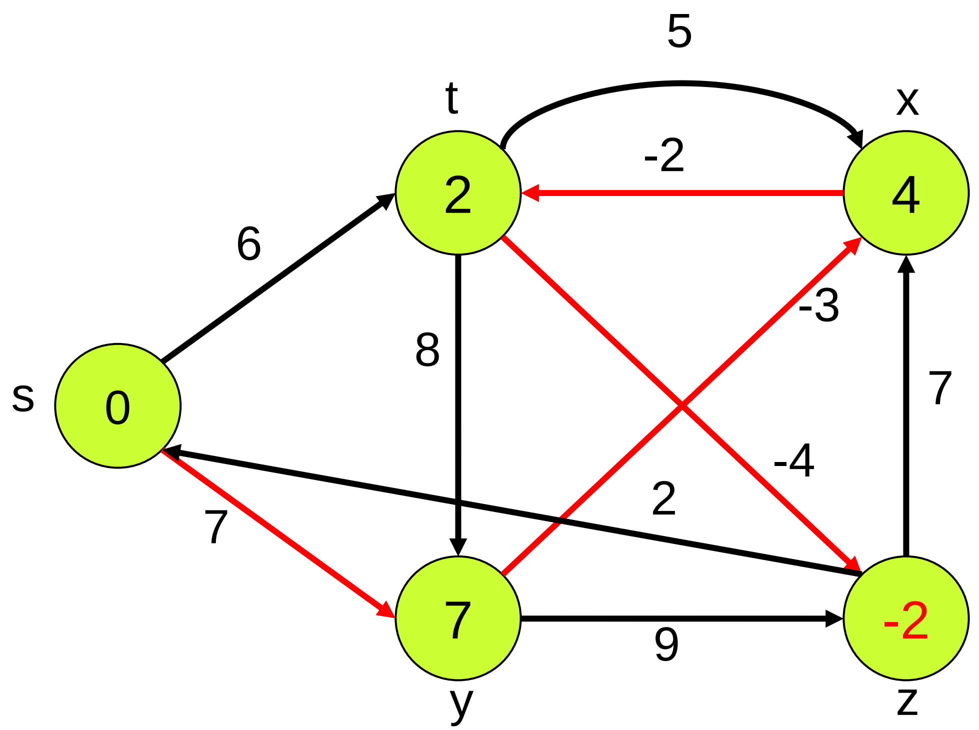
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- At third pass



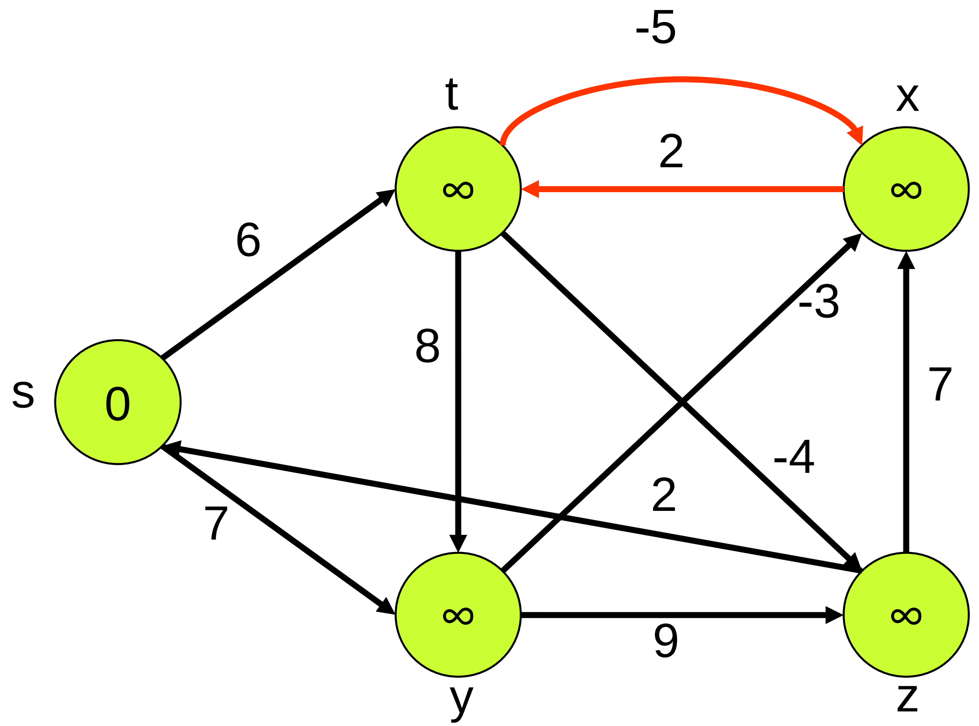
Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- At fourth pass



Neg. Weight Cycle Example

- Order : (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)
- After INIT-SINGLE-SOURCE(V, s)



Correctness

- Values you get on each pass and how quickly it converges depends on order of relaxation.
- But guaranteed to converge after |*V*|-1 passes, assuming no negative-weight cycles.

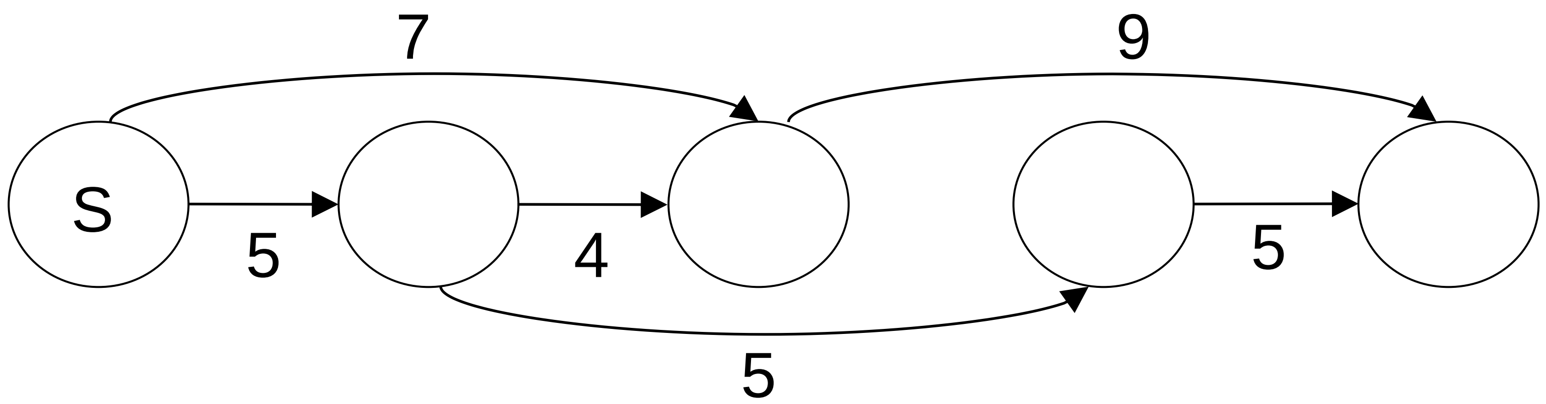
- Proof : use path-relaxation property.

  Theorem 24.4

[2] SSP in DAG

- Problem: finding shortest paths in DAG
  - Bellman-Ford takes Θ(VE) time.
  - *How can we do better?*
  - Idea: use topological sort
    - Since it is a DAG, there are no cycles.
    - Every path in a DAG is subsequence of topologically sorted, so processes vertices on each shortest path from left to right, then it would be done in one pass.
    - *What will be the running time?*

Example



SSP in DAG

DAG-SHORTEST-PATHS(V, E, w, s)

topologically sort the vertices

INIT-SINGLE-SOURCE(V, s)

for each vertex u, take in topologically sorted order

do for each vertex $v \in Adj[u]$

do RELAX(u, v, w)

Time : $\Theta(V+E)$

[3] Dijkstra's Algorithm

- If there are no negative weight edge, we can beat Bellman-Ford.
- Similar to breadth-first search : weighted version of breadth-first search.
  - Grow a tree gradually, advancing from vertices taken from a queue.
  - Instead of a FIFO queue, uses a priority queue.
  - Keys are shortest-path weights ($d[v]$).

Basic Idea

- Have two sets of vertices :
  - S = vertices whose final shortest-path weights are determined.
  - Q = priority queue = $V - S$.
- For the graph $G=(V,E)$, maintains a set S of vertices for which the shortest paths are known.
- Repeatedly selects the vertex u ($u \in V-S$), with the minimum shortest-path estimated, adds u to S, and relaxes all edges leaving u.

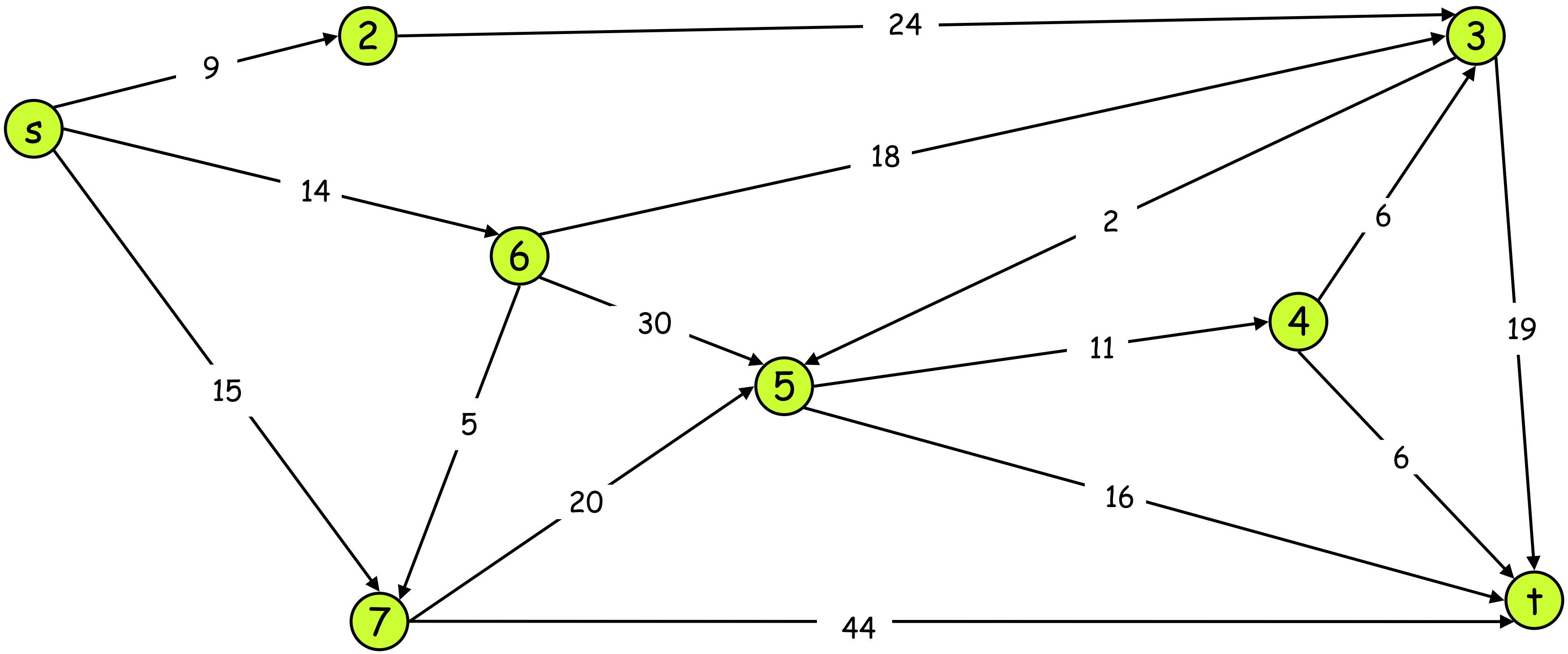
Pseudo-code

```
DIJKSTRA(G,w,s)

INITIALIZE-SINGLE-SOURCE(G, s)
S ← Ø
Q ← V[G]
while Q ≠ Ø
    do u ← EXTRACT-MIN(Q)
       S ← S U {u}
       for each vertex v ∈ Adj[u]
          do RELAX(u, v, w)
```

Example

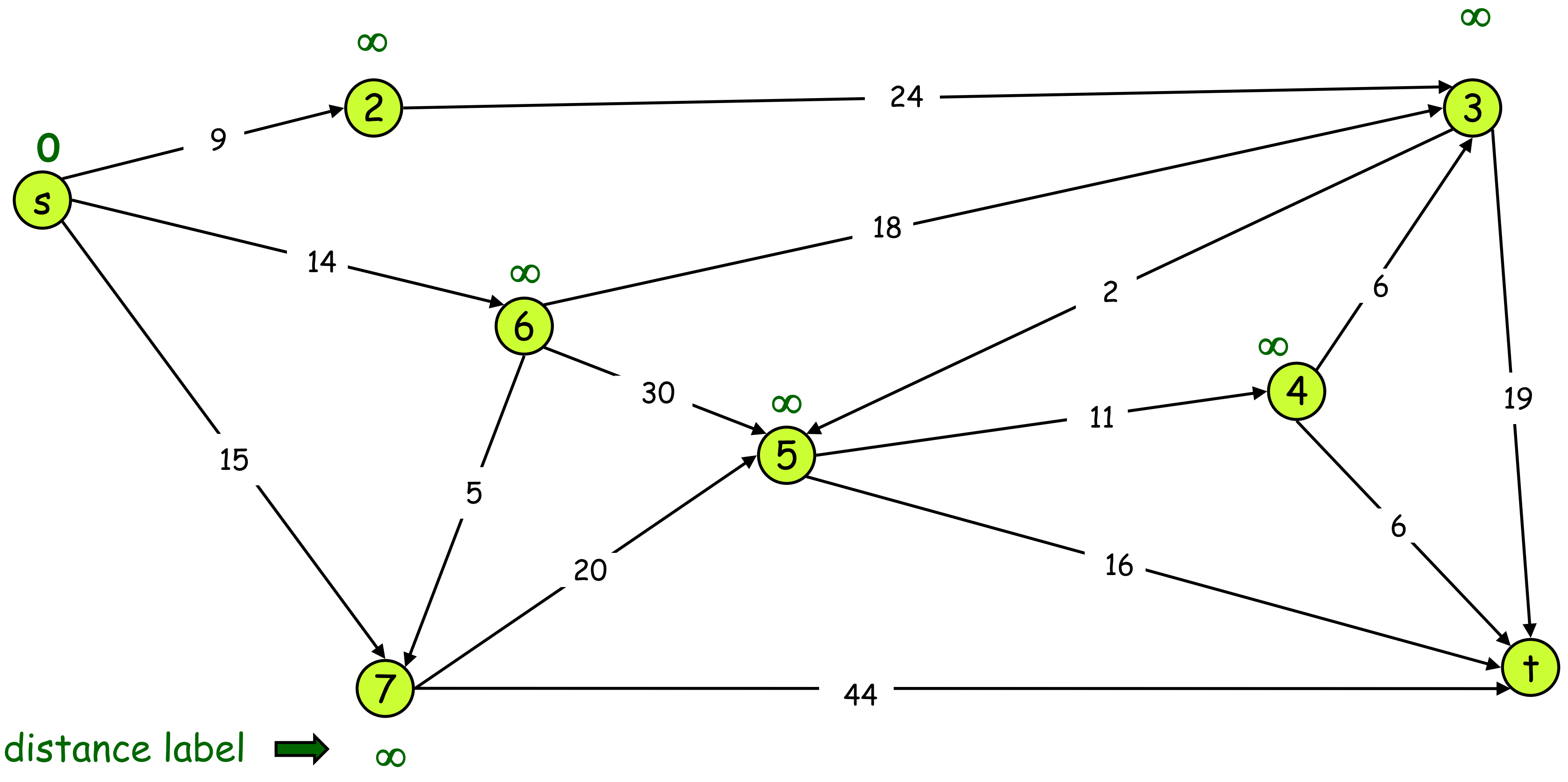
- Find shortest path from s to t.



Example

S = { }
Q = { s, 2, 3, 4, 5, 6, 7, t }

INITIALIZE-SINGLE-SOURCE(G, s)
S ← ∅ , Q ← V[G]



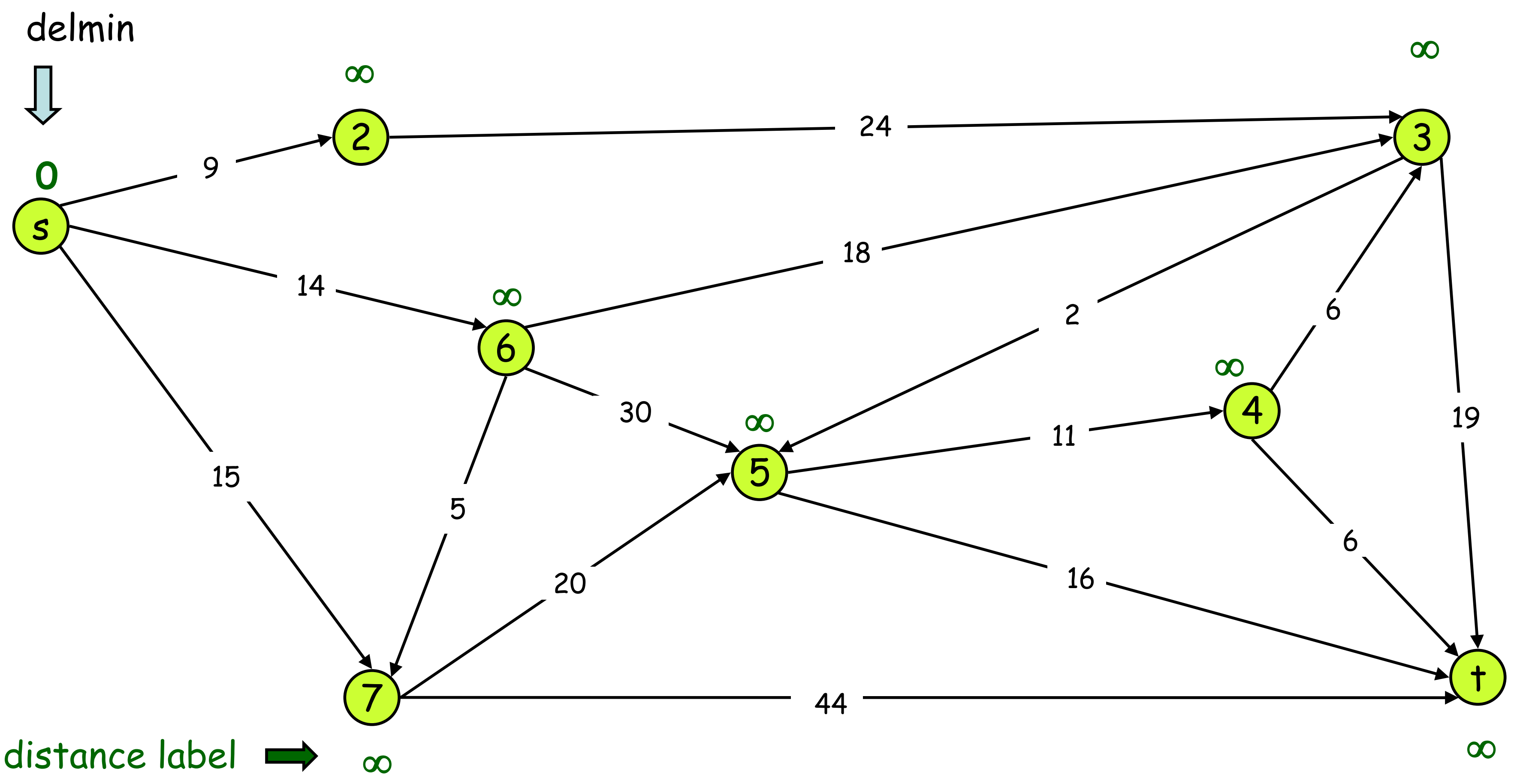
Example

S = { }

Q = { s, 2, 3, 4, 5, 6, 7, t }

While Q ≠ ∅

do u ← EXTRACT-MIN(Q)



Example

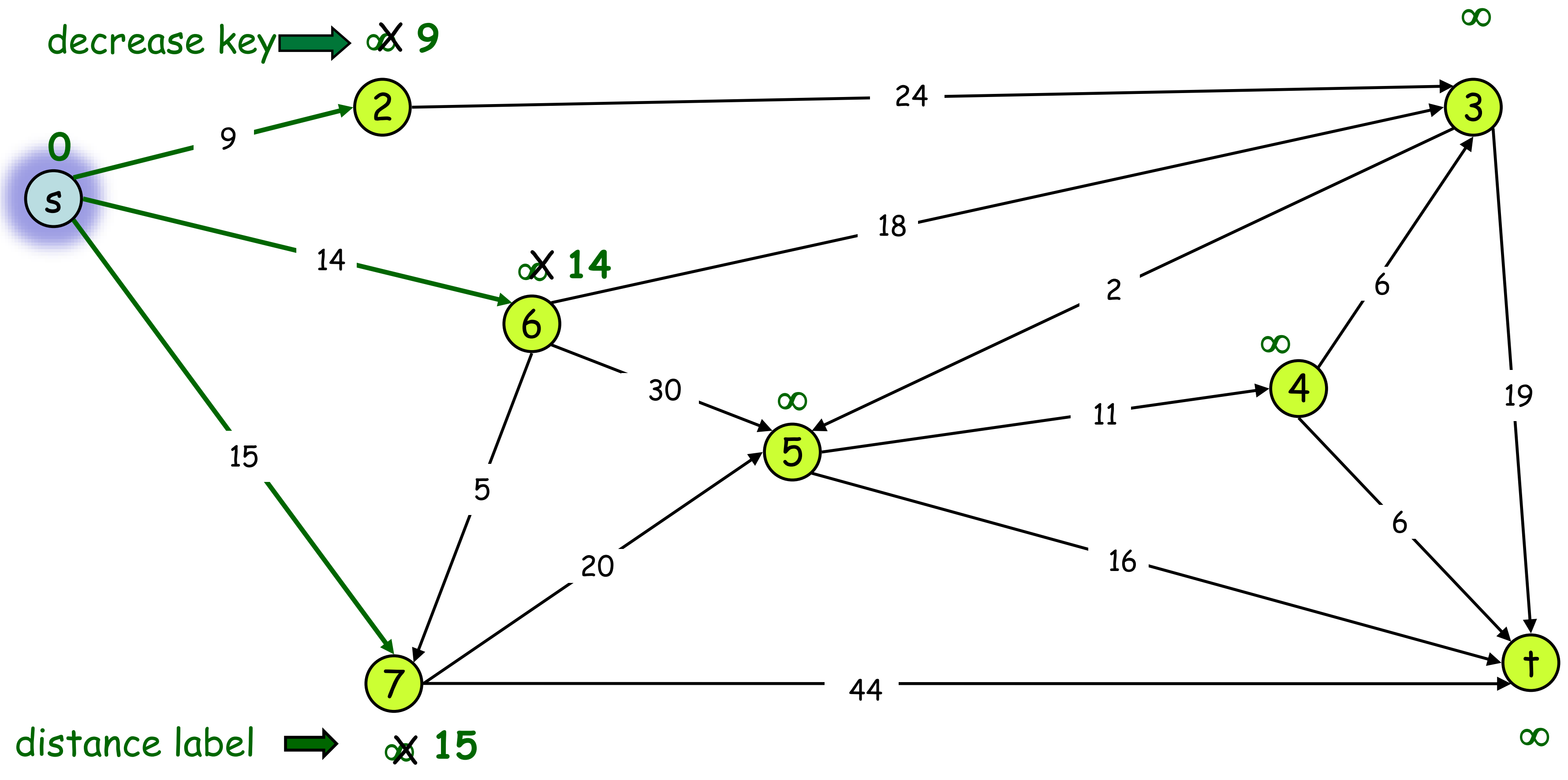
S = { s }

Q = { 2, 6, 7, 3, 4, 5, t }

S ← S ∪ {u}

for each vertex v ∈ Adj[u]

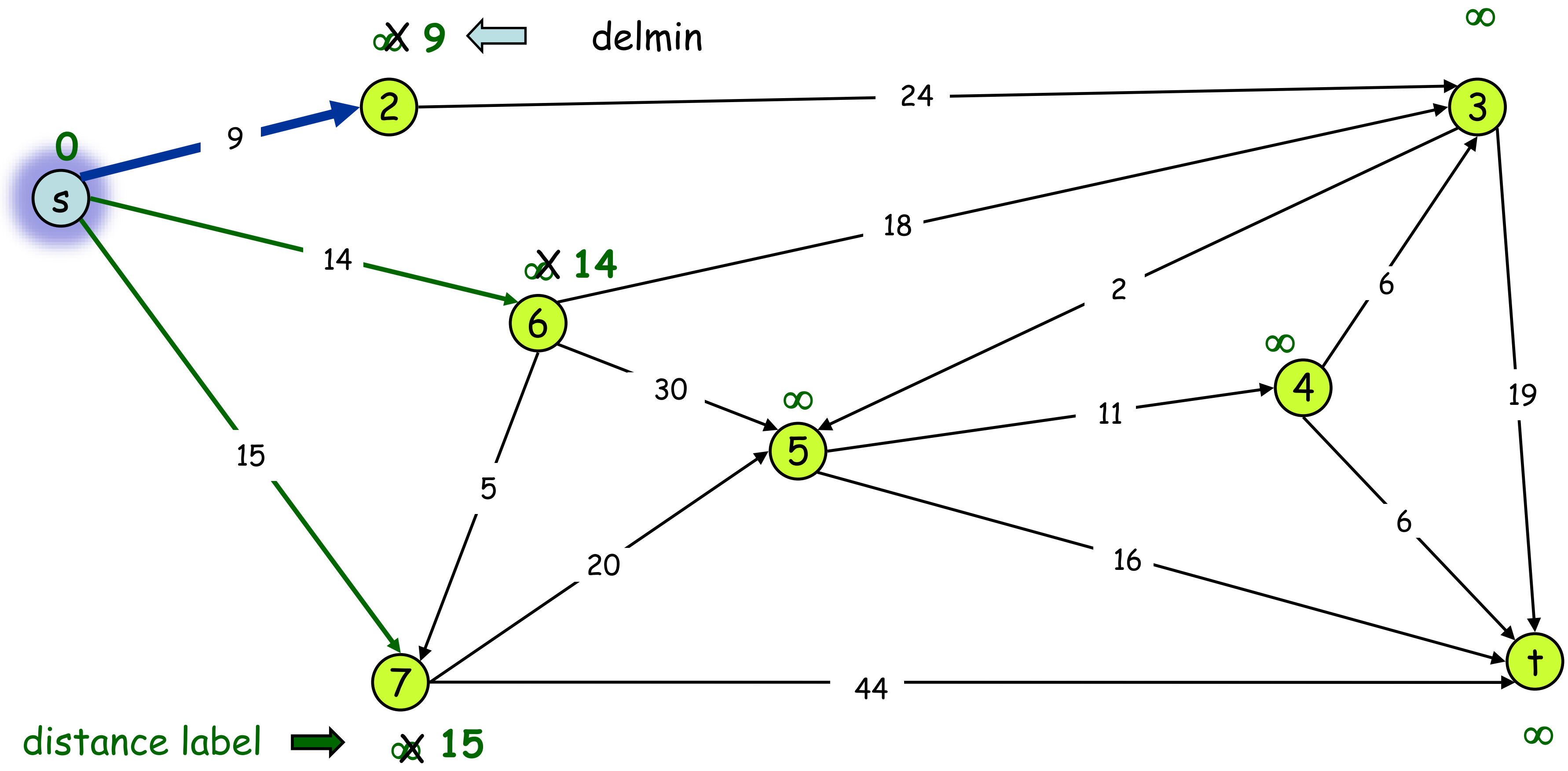
do RELAX(u, v, w)



Example

S = { s }
Q = { 2, 6, 7, 3, 4, 5, t }

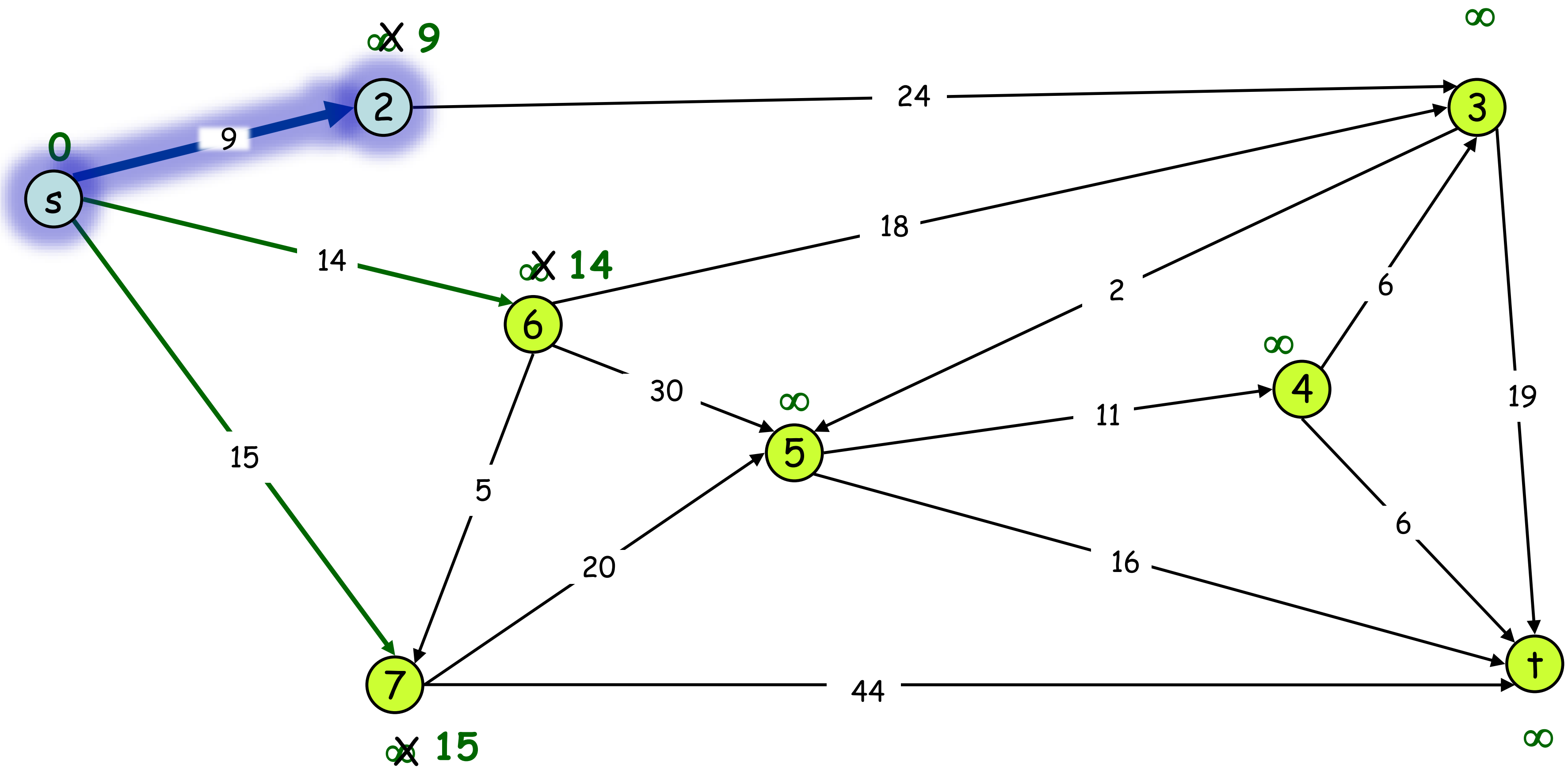
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2 }
Q = { 6, 7, 3, 4, 5, t }

S ← S U {u}



Example

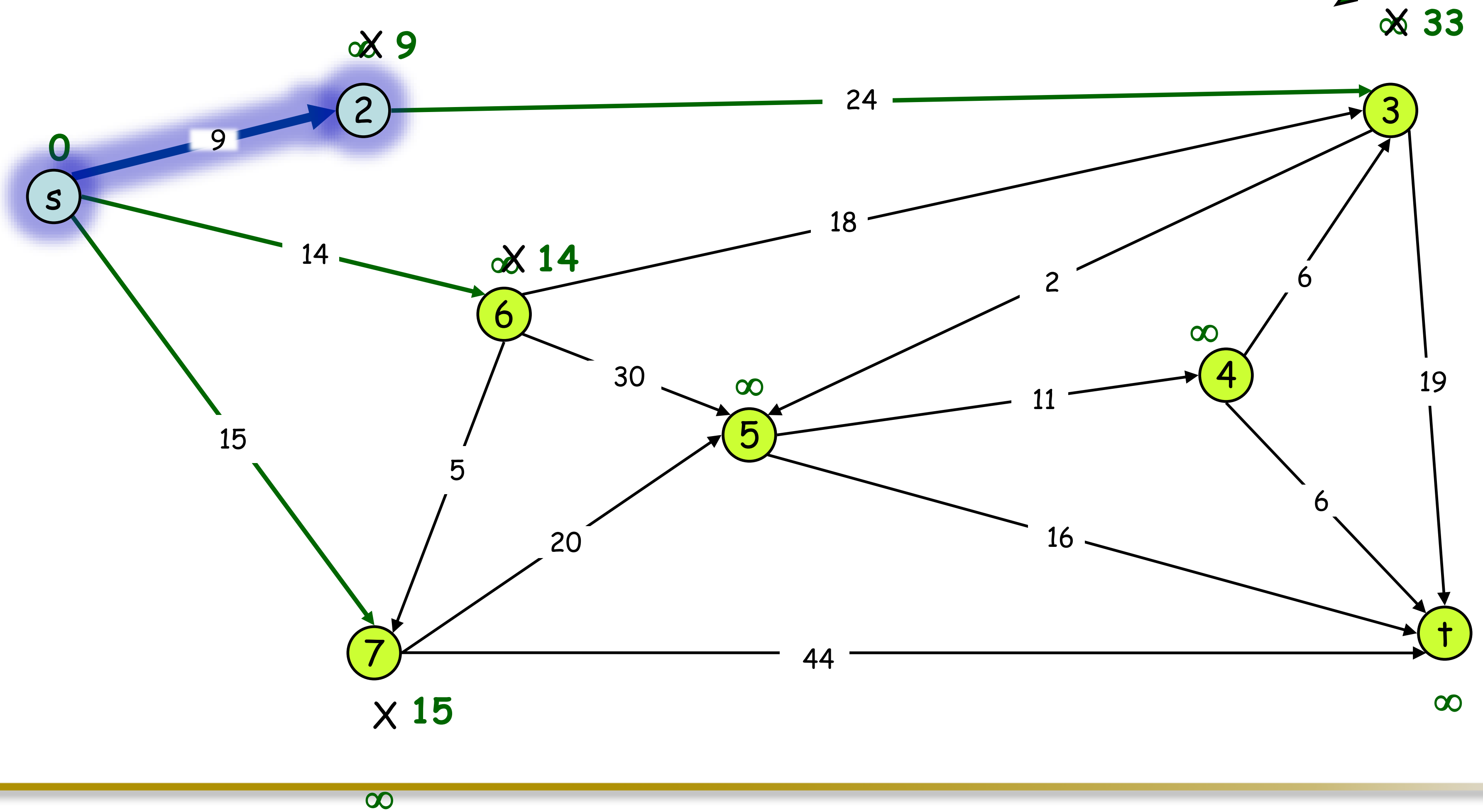
$S = \{ s, 2 \}$

$Q = \{ 6, 7, 3, 4, 5, t \}$

for each vertex $v \in Adj[u]$

do RELAX(u, v, w)

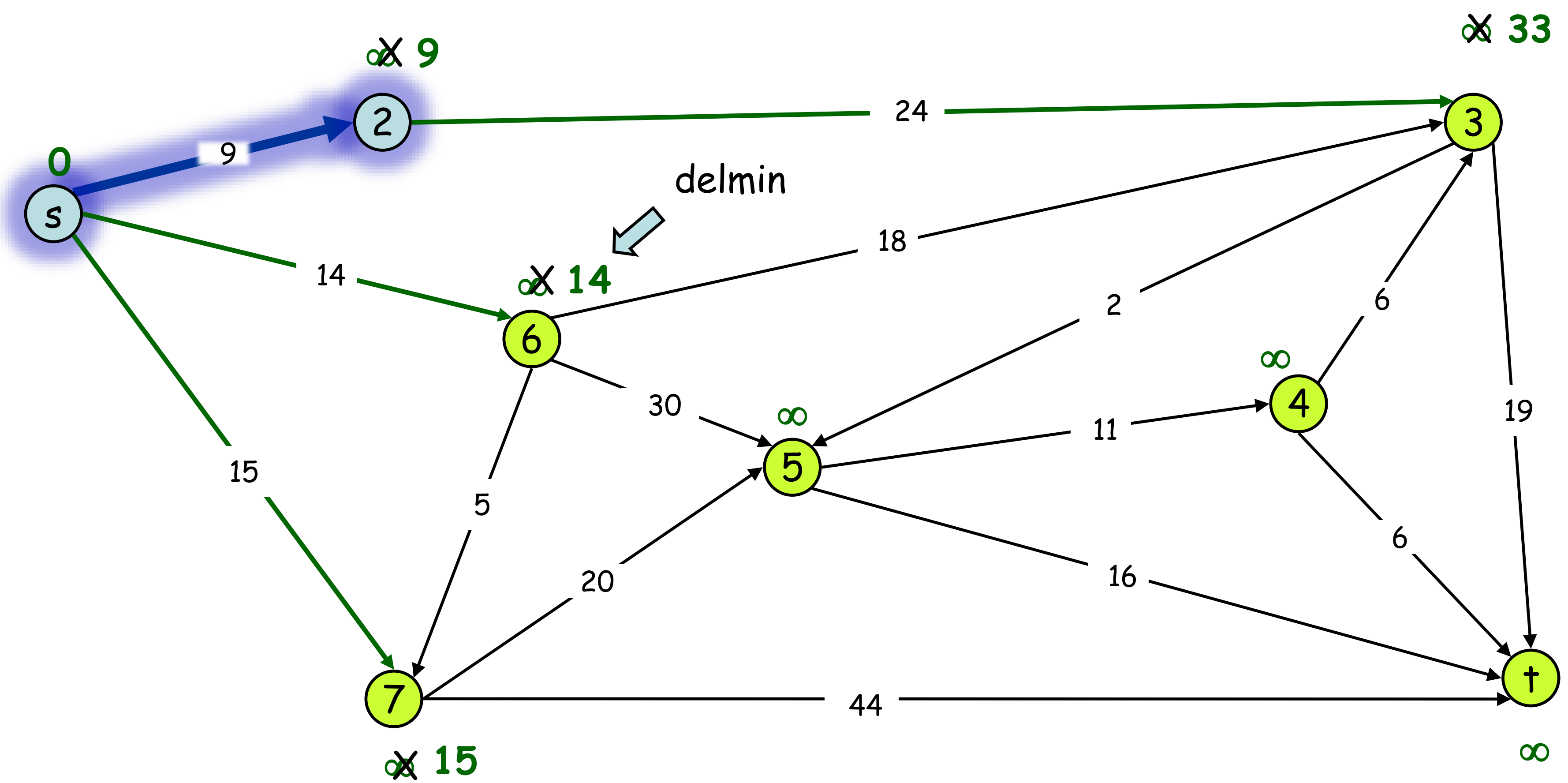
decrease key



Example

S = { s, 2 }
Q = { 6, 7, 3, 4, 5, t }

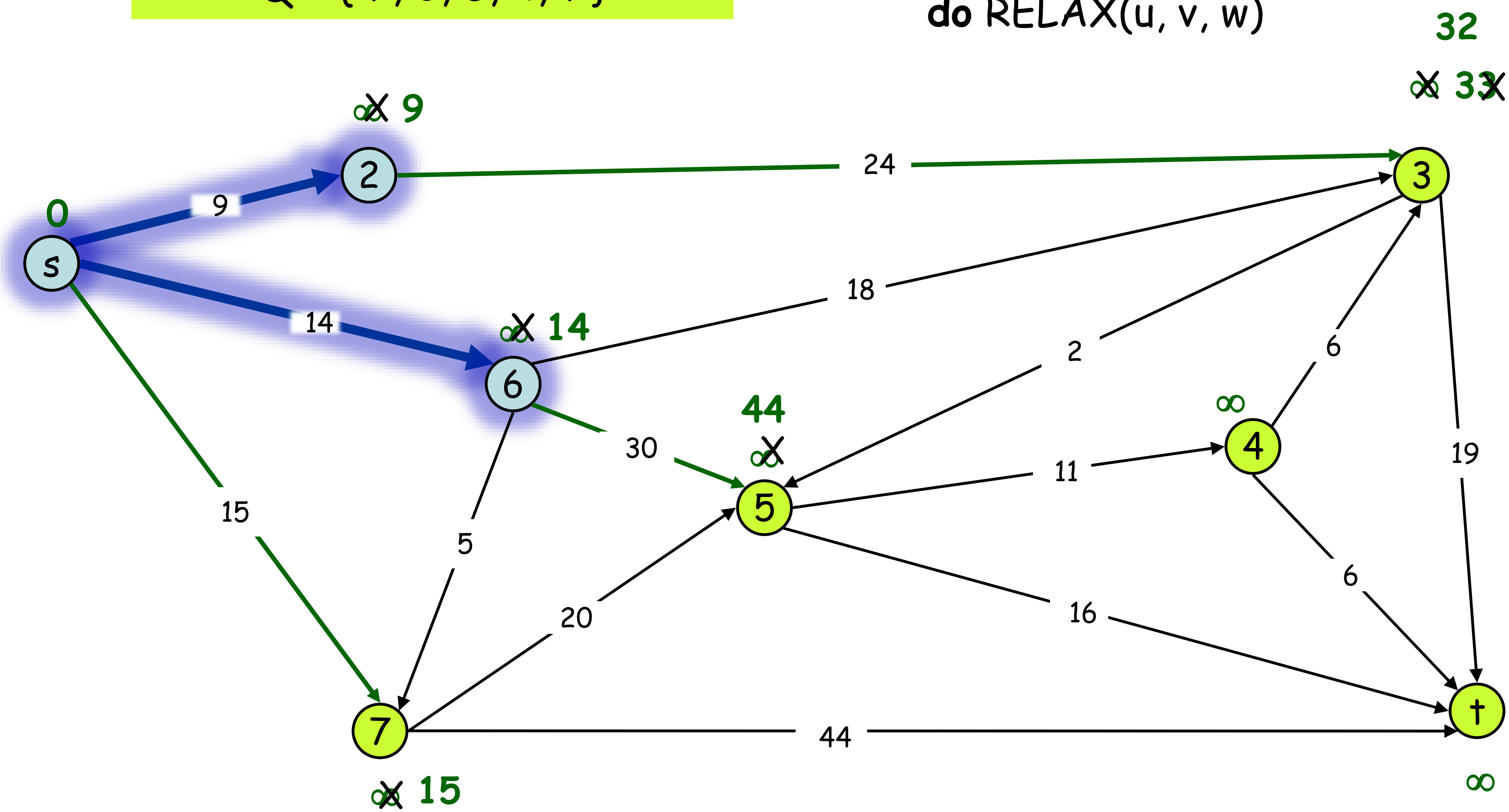
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 6 }
Q = { 7, 3, 5, 4, t }

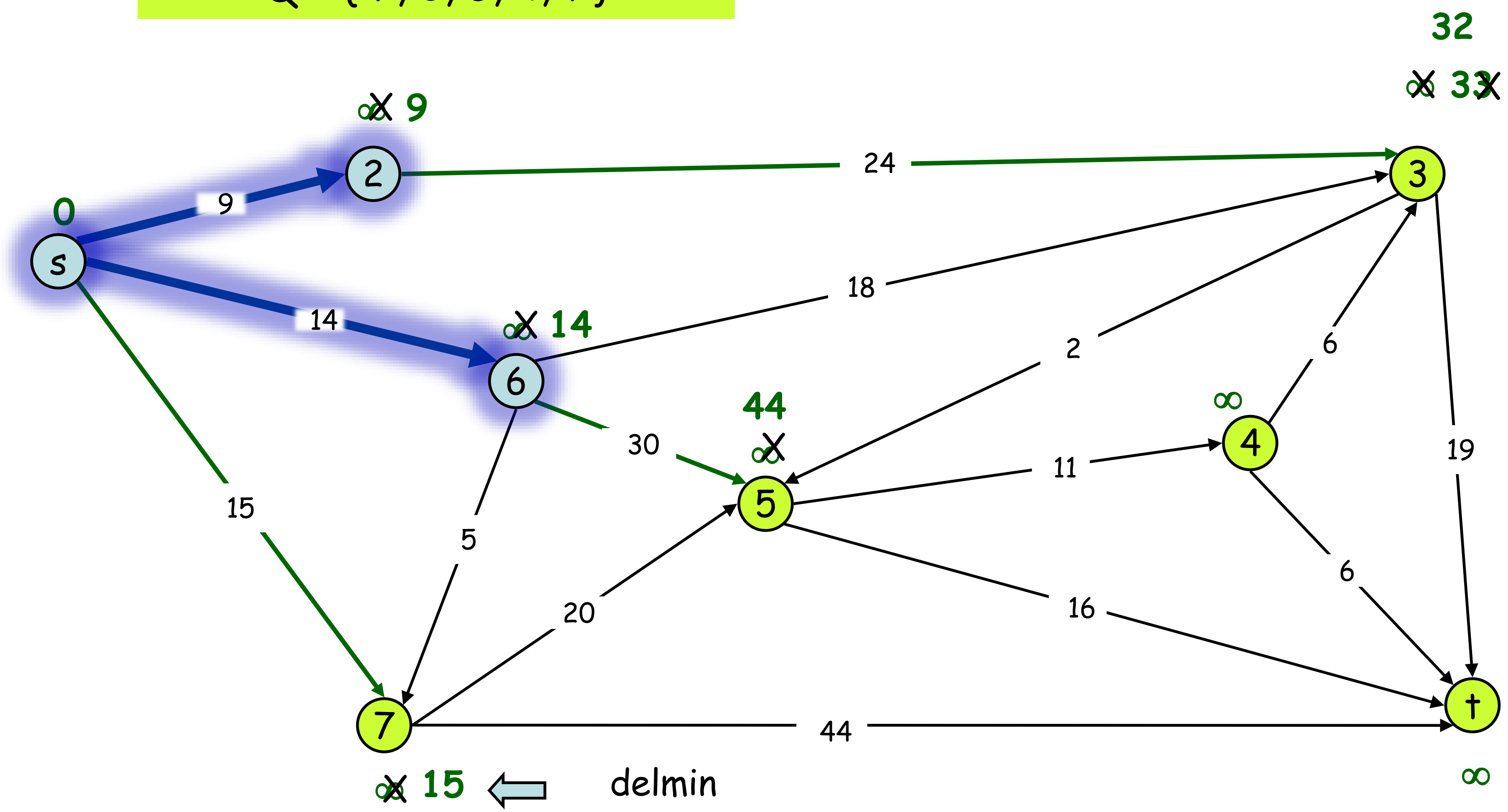
S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Example

$S = \{ s, 2, 6 \}$
$Q = \{ 7, 3, 5, 4, t \}$

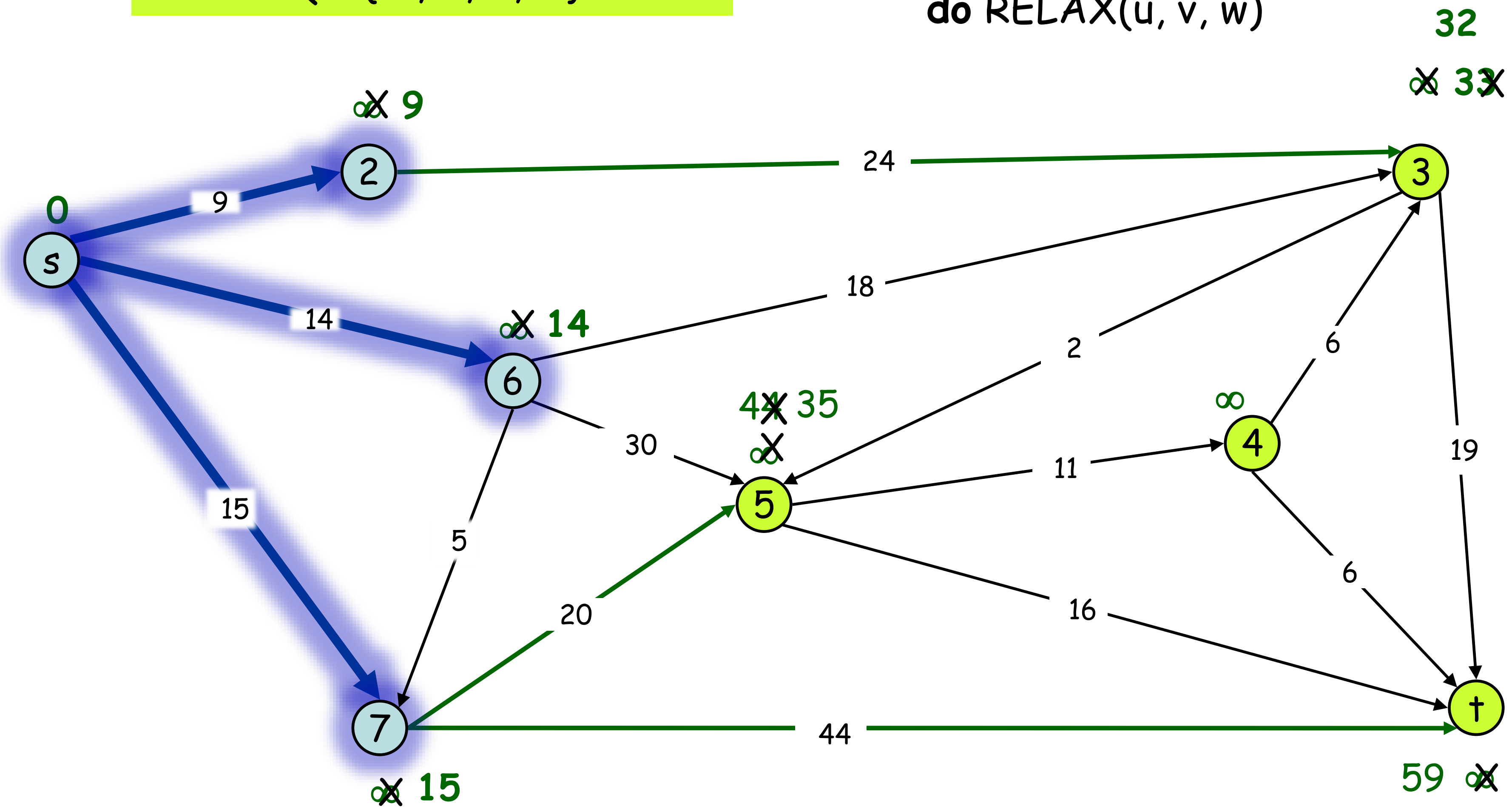
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 6, 7 }
Q = { 3, 5, 4, t }

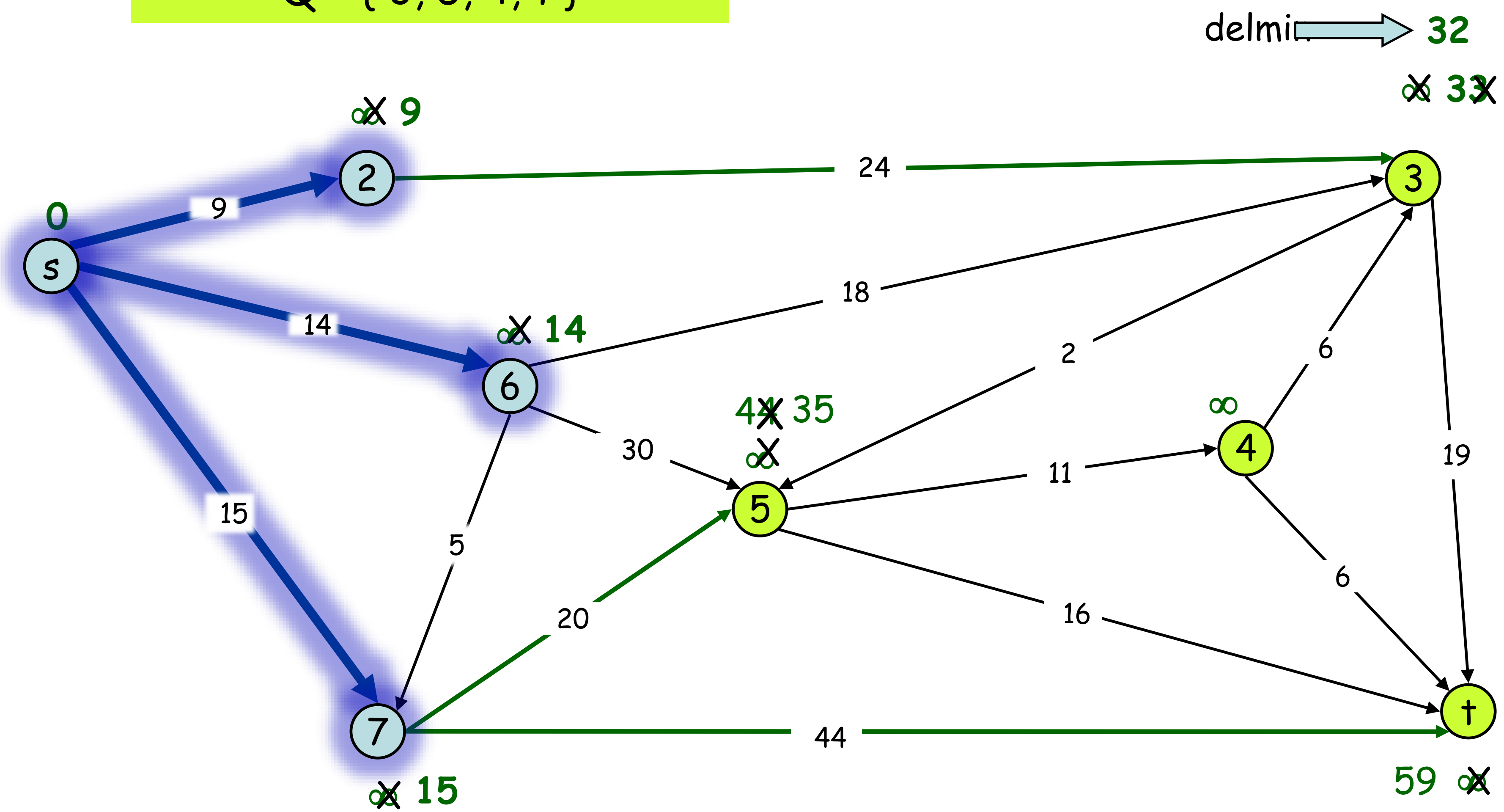
S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Example

S = { s, 2, 6, 7 }
Q = { 3, 5, 4, t }

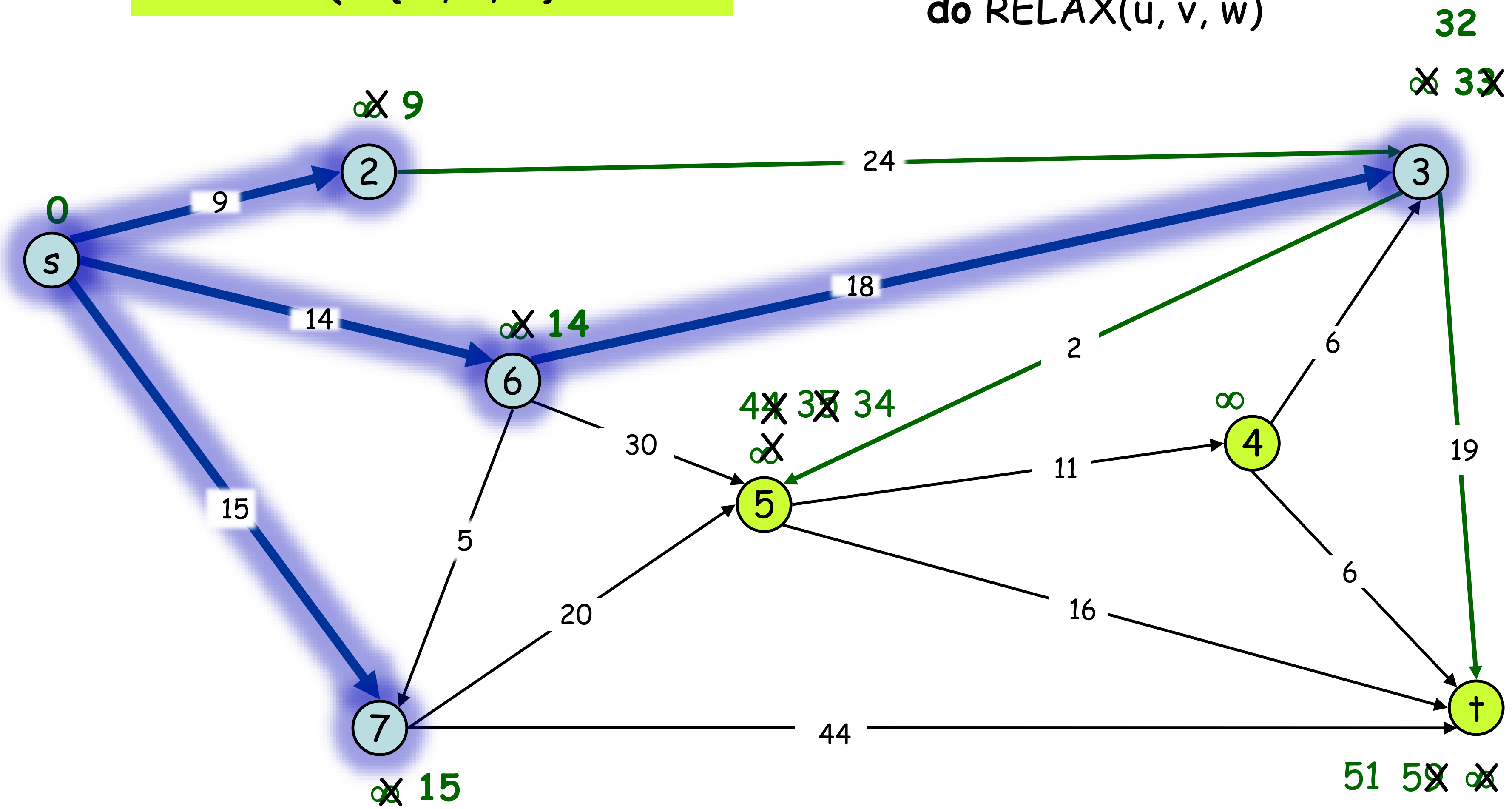
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 3, 6, 7 }
Q = { 5, 4, t }

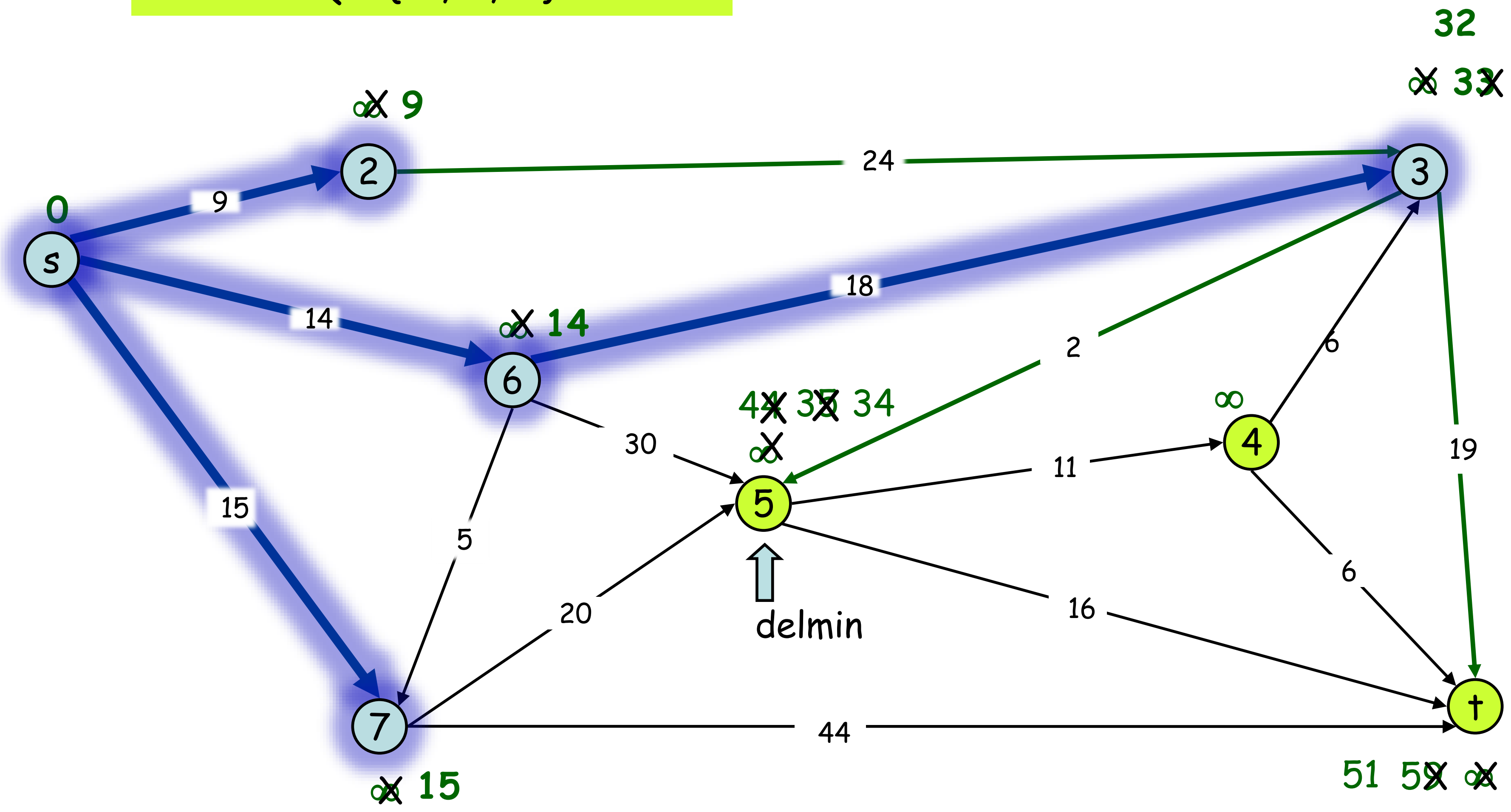
S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Example

S = { s, 2, 3, 6, 7 }
Q = { 5, 4, t }

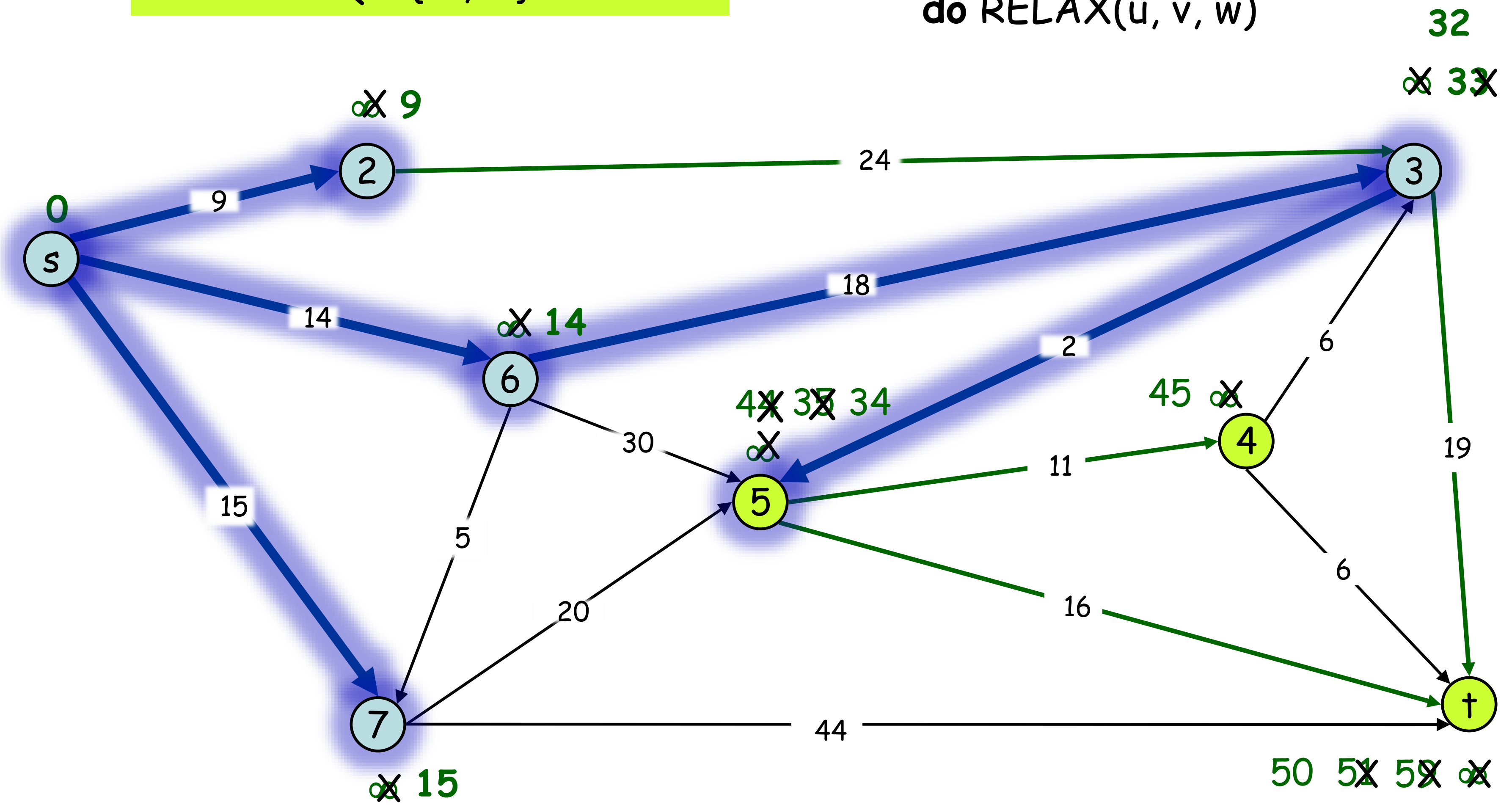
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 3, 5, 6, 7 }
Q = { 4, t }

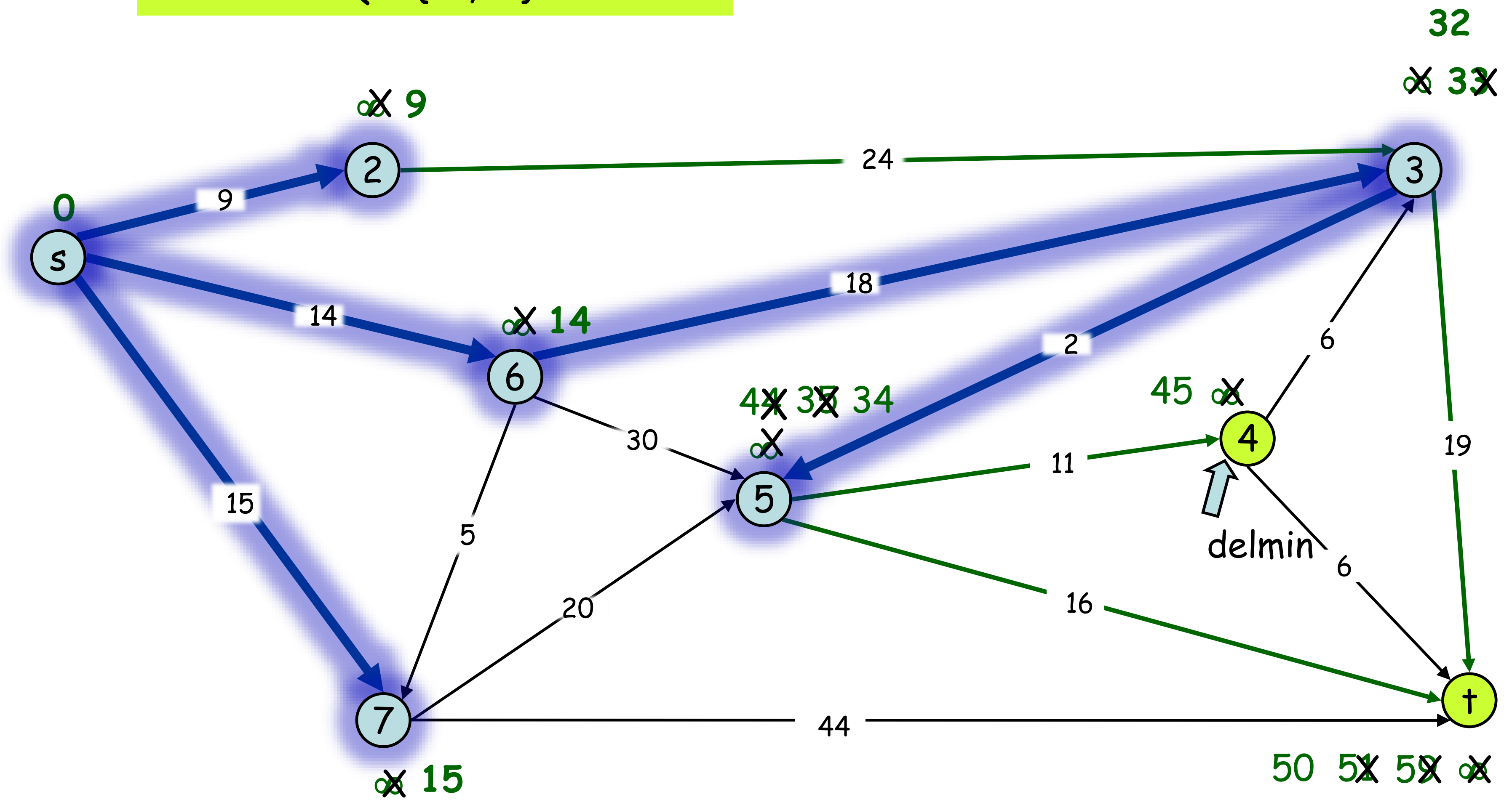
S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Example

S = { s, 2, 3, 5, 6, 7 }
Q = { 4, t }

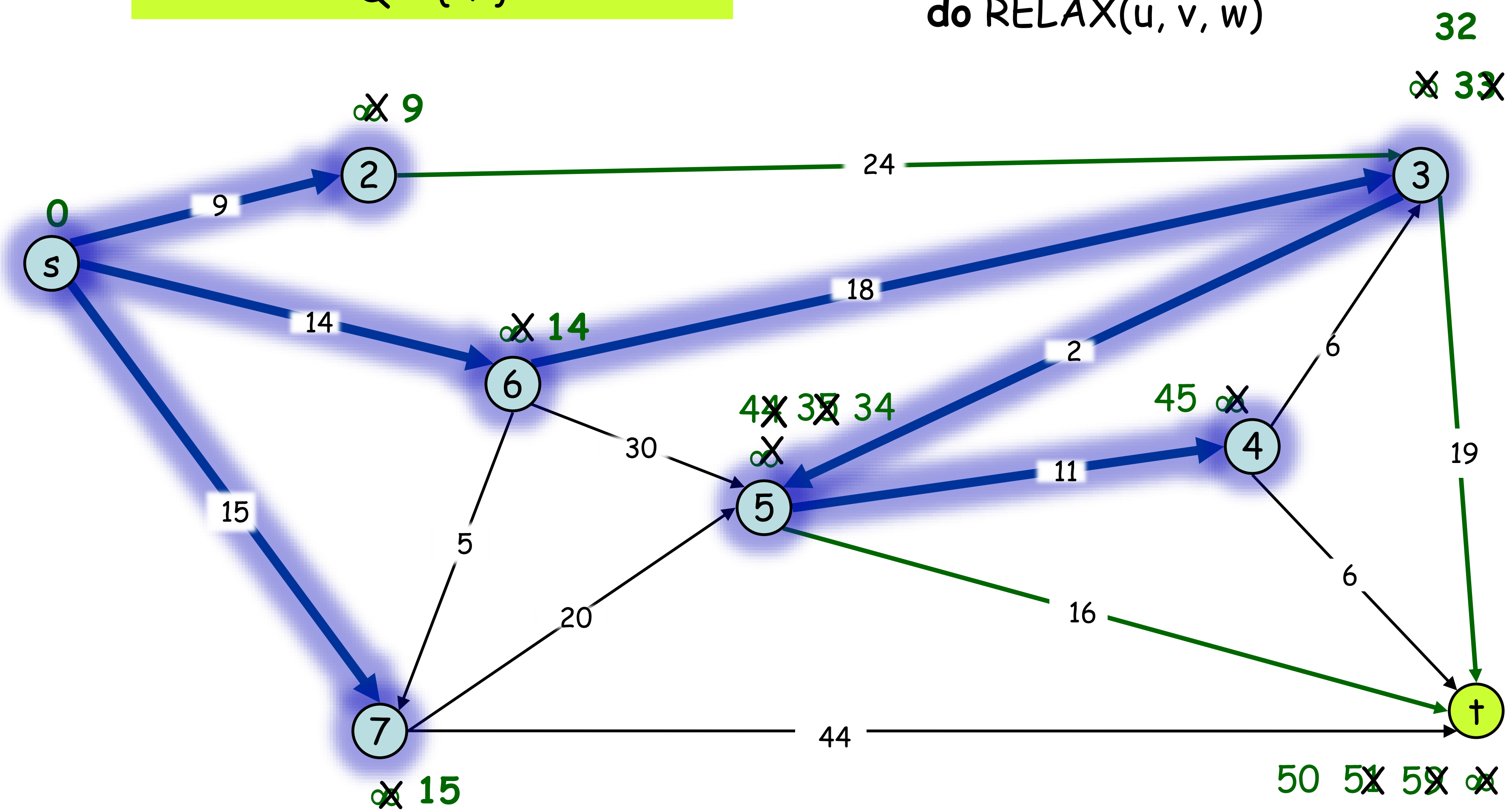
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 3, 4, 5, 6, 7 }
Q = { t }

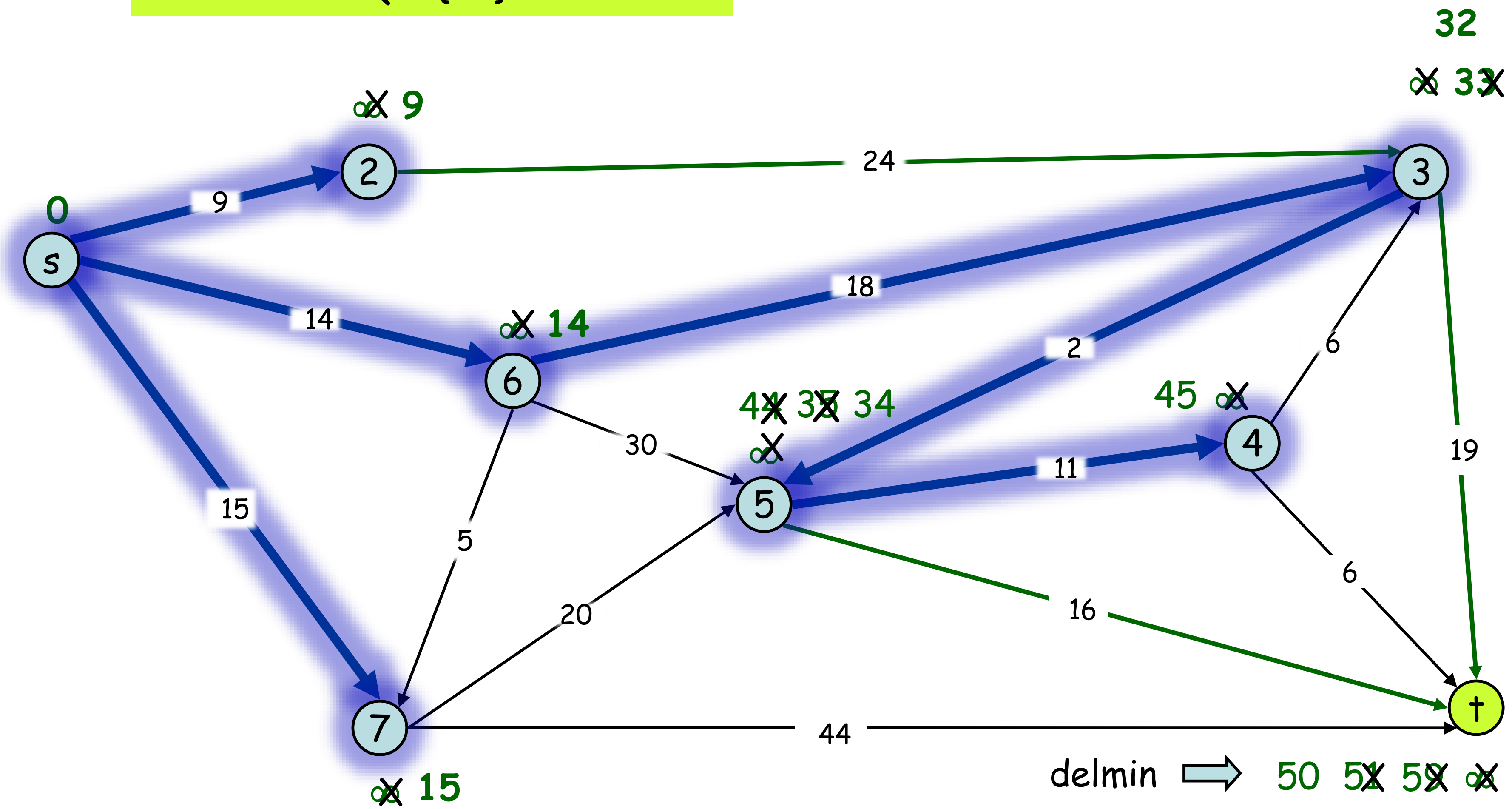
S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Example

S = { s, 2, 3, 4, 5, 6, 7 }
Q = { t }

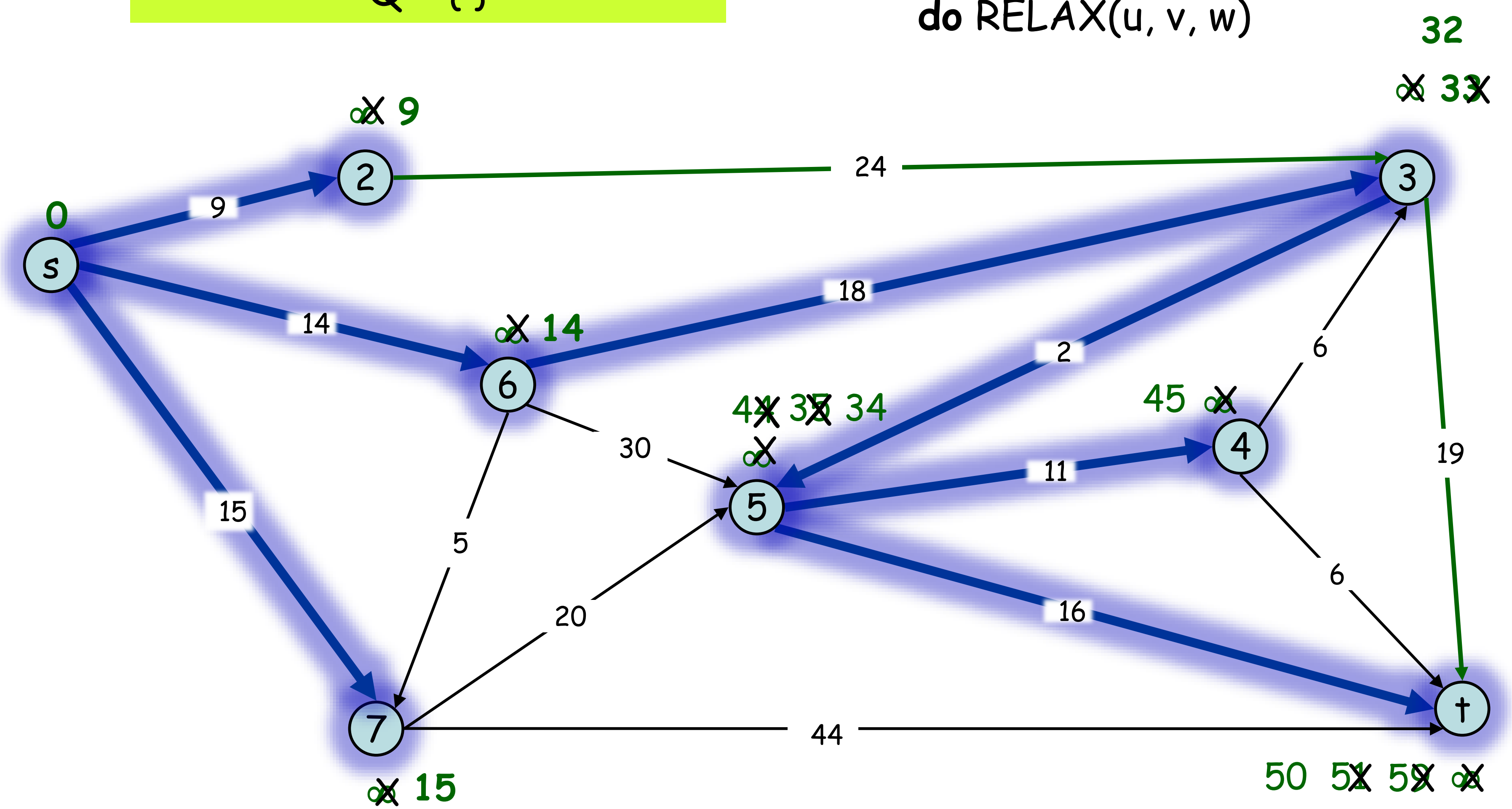
do u ← EXTRACT-MIN(Q)



Example

S = { s, 2, 3, 4, 5, 6, 7, t }
Q = { }

S ← S U {u}
for each vertex v ∈ Adj[u]
do RELAX(u, v, w)



Correctness

Theorem 24.6

loop invariant : At the start of each iteration of the while loop of lines 4-8, $d[v] = \delta(s,v)$ for each vertex $v \in S$.

Proof at page 660 ~ 661

Time Analysis

- Like Prim's algorithm, performance depends on implementation of priority queue.
  - Binary heap :
    - Each operation takes $O(\lg V)$ time

      $\rightarrow O(E \lg V)$
  - Fibonacci heap :
    - $O(V \lg V + E)$ time.

All Pairs Shortest Path

Floyd-Warshall Algorithm

- The easiest way!
  - Iterate Dijkstra's and Bellman-Ford $|V|$ times!

- Dijkstra:
  - $O(V\lg V + E)$ -> $\boxed{O(V^2\lg V + VE)}$
- Bellman-Ford:
  - $O(VE)$ -> $\boxed{O(V^2E)}$

On dense graph → $\boxed{O(V^3)}$, $\boxed{O(V^4)}$

- Faster-All-Pairs-Shortest-Paths (Ch 25.1):
  - $O(V^3\lg V)$ -> better than Dijkstra and Bellman-Ford ?
- Any other faster algorithms?
  - Floyd-Warshall Algorithm

Floyd-Warshall Algorithm

- Negative edges are allowed
- Assume that no negative-weight cycle
- Dynamic Programming Solution
  - Optimal substructure

The structure of a shortest path

- Intermediate vertex
  - In simple path $p = \langle v_1,\ldots,v_L \rangle$, any vertex of p other than v_1 and v_L, i.e., any vertex in the set $\{v_2,\ldots,v_{L-1}\}$.
- Key Observation
  - For any pair of vertices i, j in V.
  - Let p be a minimum-weight path of all paths from i to j whose intermediate vertices are all from $\{1,2,\ldots,k\}$.
  - Assume that we have all shortest paths from i to j whose intermediate vertices are from $\{1,2,\ldots,k-1\}$.
  - Observe relationship between path p and above shortest paths.

Key Observation

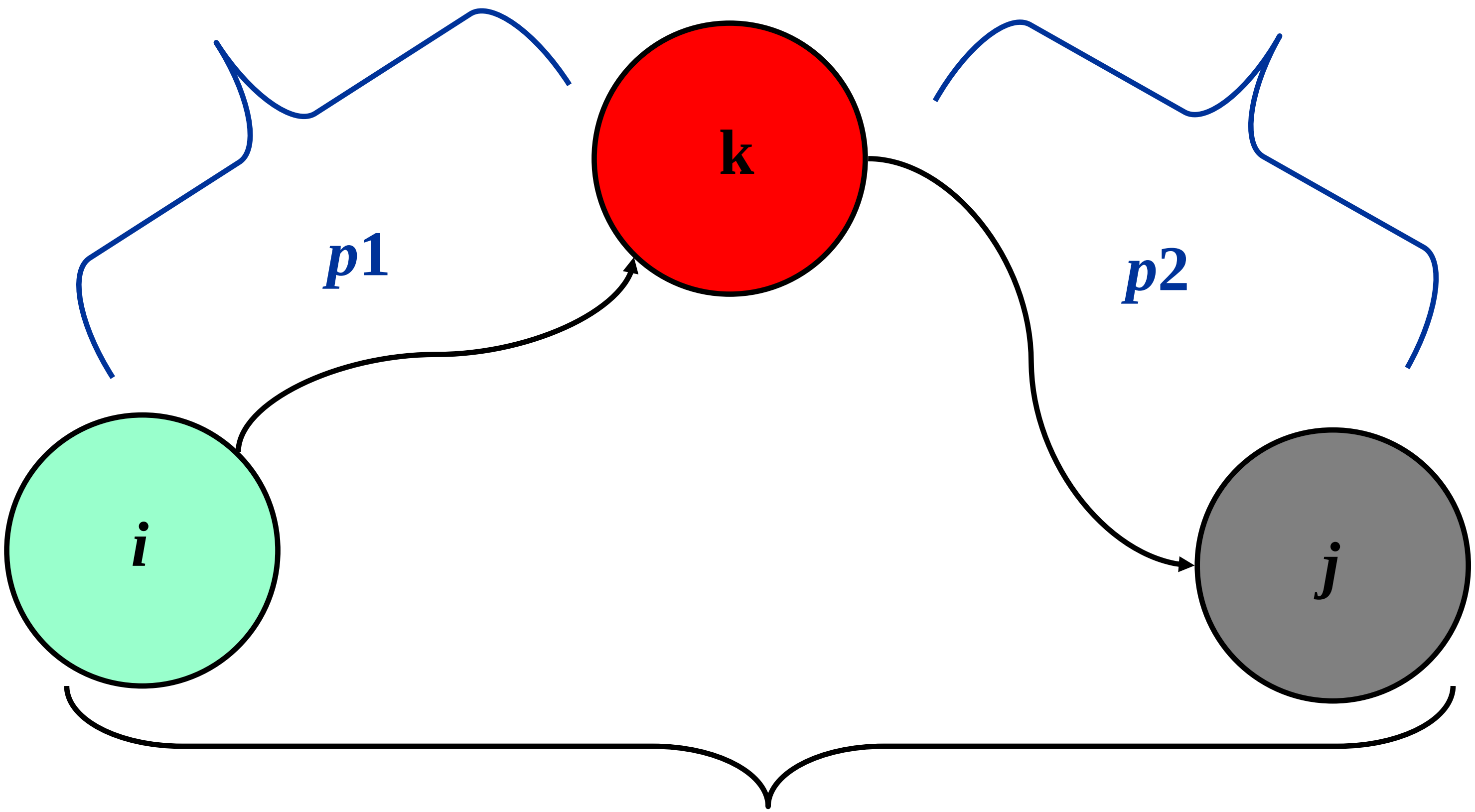
- A shortest path does not contain the same vertex twice.
  - Proof: A path containing the same vertex twice contains a cycle. Removing cycle give a shorter path.

Key Observation

- p is determined by the shortest paths whose intermediate vertices from {1,…,k-1}.
- Case1: If k is not an intermediate vertex of p.
  - Path p is the shortest path from i to j with intermediates from {1,…k-1}.
- Case2: If k is an intermediate vertex of path p.
  - Path p can be broken down into $i \xrightarrow{p1} k \xrightarrow{p2} j$.
  - $p1$ is the shortest path from i to k with all intermediate vertices in the set {1,2,…,k-1}.
  - $p2$ is the shortest path from k to j with {1,2,…,k-1}.

Key Observation

***p*1**: **All intermediate vertices in {1,2,..,*k*-1}** ***p*2**: **All intermediate vertices in {1,2,..,*k*-1}**



p: **All intermediate vertices in {1,2,..,*k*}**

A recursive solution

- Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that all intermediate vertices on the path are in set $\{1,2,\ldots,k\}$.
- Let $D^{(k)}$ be the n X n matrix $[d_{ij}^{(k)}]$.
- $d_{ij}^{(0)}$ is set to be w_{ij} (no intermediate vertex).
- $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$ $(k \geq 1)$
- $D^{(n)} = (d_{ij}^{(n)})$ gives the final answer, for all intermediate vertices are in the set $\{1,2,\ldots,n\}$.

A recursive solution

- $$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{(if } k=0\text{)} \\ \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) & \text{(if } k\geq 1\text{)} \end{cases}$$
- The Matrix $D^{(n)} = (d_{ij}^{(n)})$ gives the final answer: $d_{ij}^{(n)} = \delta(i,j)$ for all $i,j \in V$.

Extracting the Shortest Paths

- The predecessor pointers *pred*[*i*,*j*] can be used.
- Initially all *pred*[*i*,*j*] = nil
- Whenever the shortest path from *i* to *j* passing through an intermediate vertex *k* is discovered, we set *pred*[*i*,*j*] = *k*

Extracting the Shortest Paths

- Observation:
  - If *pred*[*i*,*j*] = nil, shortest path does not exist.
  - If there exists shortest path and the shortest path does not pass through any intermediate vertex, then *pred*[*i*,*j*] = *i*.
  - If *pred*[*i*,*j*] = *k*, vertex k is an intermediate vertex on shortest path form *i* to *j*

Extracting the Shortest Paths

- How to find?
  - If $pred[i,j] = i$, the shortest path is edge (i,j)
  - Otherwise, recursively compute

    $(i\ ,\ pred[i,j])$ and $(pred[i,j]\ ,\ j)$

Computing the weights bottom up

- The Floyd-Warshall Algorithm: Version 1

Floyd-Warshall(*w,n*)

{ for *i =1* to *n* do **initialize**

for *j=1* to *n* do

{ $D^0[i,j] = w[i,j]$;

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$

for k = 1 to n do **dynamic programming**

for i = 1 to n do

for j = 1 to n do

$\text{if}(d^{(k-1)}[i,k] + d^{(k-1)}[k,j] < d^{(k-1)}[i,j])$

$\{ d^{(k)}[i,j] = d^{(k-1)}[i,k] + d^{(k-1)}[k,j];$

$pred[i,j] = k;$

$\text{else } d^{(k)}[i,j] = d^{(k-1)}[i,j];$

return d(n)[1..n, 1..n];

}

case2

case1

See figure 25.4.

Can you find a shortest path from vertex 1 to vertex 2 from the π matrix?

Analysis

- Running time is clearly $\Theta(?)$
- $\Theta(n^3)$ -> $\Theta(|V|^3)$
- Faster than previous algorithms. $O(|V|^4)$,$O(|V|^3\lg|V|)$
- Problem: Space Complexity $\Theta(|V|^3)$.
- It is possible to reduce this down to $\Theta(|V|^2)$by keeping only one matrix instead of n.

Transitive Closure

- Given directed graph $G = (V, E)$
- Compute $G^* = (V, E^*)$
- $E^* = \{(i,j) : \text{there is path from } i \text{ to } j \text{ in } G\}$
- Could assign weight of 1 to each edge, then run FLOYD-WARSHALL
- If $d_{ij} < n$, then there is a path from i to j.
- Otherwise, $d_{ij} = \infty$ and there is no path.

Transitive Closure – Warshall

- Using logical operations $\vee$ (OR), $\wedge$ (AND)
- Assign weight of 1 to each edge, then run FLOYD-WARSHALL with this weights.
- Instead of $D^{(k)}$, we have $T^{(k)} = (t_{ij}^{(k)})$
  - $t_{ij}^{(0)} = \begin{cases} 0 & (\text{if } i \neq j \text{ and } (i, j) \notin E) \\ 1 & (\text{if } i = j \text{ or } (i, j) \in E) \end{cases}$
  - $t_{ij}^{(k)} = \begin{cases} 1 & (\text{ if there is a path from } i \text{ to } j \text{ with all intermediate vertices in } \{1, 2, \ldots, k\}) \\ & (t_{ij}^{(k-1)} \text{ is 1) or } (t_{ik}^{(k-1)} \text{ is 1 and } t_{kj}^{(k-1)} \text{ is 1}) \\ 0 & (\text{otherwise}) \end{cases}$

Transitive Closure

TRANSITIVE-CLOSURE(E, n)

for i = 1 to n

    do for j = 1 to n

        do if $i=j$ or $(i, j) \in E$

            then $t_{ij}^{(0)} = 1$

            else $t_{ij}^{(0)} = 0$

for k = 1 to n

    do for i = 1 to n

        do for j = 1 to n

            do $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$

return $T^{(n)}$