ON ERDŐS-GINZBURG-ZIV INVERSE THEOREMS FOR DIHEDRAL AND DICYCLIC GROUPS

JUN SEOK OH AND QINGHAI ZHONG

Abstract. Let G be a finite group and $\exp(G) = \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G\}$. A finite unordered sequence of terms from G, where repetition is allowed, is a product-one sequence if its terms can be ordered such that their product equals the identity element of G. We denote by $\mathsf{s}(G)$ (or $\mathsf{E}(G)$ respectively) the smallest integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $\exp(G)$ (or $|G|$ respectively), which are called the Erdős-Ginzburg-Ziv constant. In this paper, we provide the exact value of the Erdős-Ginzburg-Ziv constant, and provide explicit characterizations of all sequences of length $\mathsf{s}(G) - 1$ (or $\mathsf{E}(G) - 1$ respectively) having no product-one subsequence of length $\exp(G)$ (or $|G|$ respectively) over Dihedral and Dicyclic groups.

1. Introduction

Let G be a finite group and $\exp(G) = \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G\}$. By a sequence S over G, we mean a finite sequence of terms from G which is unordered, repetition of terms allowed. We say that S is a product-one sequence if its terms can be ordered so that their product equals the identity element of G. The *small Davenport constant* $\mathsf{d}(G)$ is the maximal integer ℓ such that there is a sequence of length ℓ which has no non-trivial product-one subsequence. We denote by $\mathsf{s}(G)$ (or $\mathsf{E}(G)$ respectively) the smallest integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $\exp(G)$ (or $|G|$ respectively). When G is cyclic, $\exp(G) = |G|$ and $\mathsf{s}(G) = \mathsf{E}(G) = \mathsf{d}(G) + |G| = 2|G| - 1$, which is due to Erdős, Ginzburg, and Ziv in 1961. Since that time, the study of (zero-sum) sequences has begun experiencing rapid growth and development into a flourishing branch of Combinatorial Number Theory. We refer to [7, 10, 9, 13] for the surveys.

In the case when G is a non-cyclic abelian group, $\exp(G) \neq |G|$, and hence $\mathsf{s}(G)$ and $\mathsf{E}(G)$ have been studied independently. In recent years, it has been completely formulated by Gao that $\mathsf{E}(G) = \mathsf{d}(G) + |G|$ ([13, Chapter 16]) for every abelian groups G. Unlike $\mathsf{E}(G)$, the precise value of $\mathsf{s}(G)$ is quite open, and it has determined so far only for certain classes of abelian groups. We refer to [6, 10, 4, 11, 12] for the progress with respect to the Erdős-Ginzburg-Ziv constant over abelian groups.

Although the abelian setting has been the dominant for studying sequences, many of (combinatorial) problems have also been studied in the non-abelian setting. For example, Olson and White (dating back to 1977) have been given an upper bound on $\mathsf{d}(G)$ for non-cyclic group G. In this direction, the study of Erdős-Ginzburg-Ziv constant has received wide attention in the literature, and it has been studied with respect to $\mathsf{E}(G)$ for specific non-abelian groups. In the meanwhile, the study with respect to $\mathsf{s}(G)$ has been also studied to non-abelian groups, however, its precise value is unknown for every non-abelian groups. We refer to [1, 14] for recent progress with respect to the mentioned invariants for non-abelian groups.

In some earlier papers ([3, 14]), authors also studied an invariant $\mathsf{s}'(G)$ defined as the smallest integer ℓ such that every sequence over G of length at least ℓ has a product-one subsequence of length

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$\max\{\mathrm{ord}(g)\,|\,g\in G\}$. If G is nilpotent (in particular, if G is abelian), then $\exp(G)=\max\{\mathrm{ord}(g)\,|\,g\in G\}$, whence $\mathsf{s}(G)=\mathsf{s}'(G)$. Similarly, if G is a dihedral group of order $2n$ or a dicyclic group of order $4n$ for some even n, then we also have $\exp(G)=\max\{\mathrm{ord}(g)\,|\,g\in G\}$. If G is a dihedral group of order $2n$ with n odd, then there are arbitrarily long sequences over G having no product-one subsequence of length $\max\{\mathrm{ord}(g)\,|\,g\in G\}$ (see the discussion after Definition 2.1). In the present paper, we prove the following results.

Theorem 1.1.

1. *Let G be a dihedral group of order $2n$, where $n\in\mathbb{N}_{\ge 3}$. Then*

$$\mathsf{s}(G)=\begin{cases}3n & \text{if } n \text{ is odd}\\ 2n & \text{if } n \text{ is even}\end{cases}, \qquad \text{and} \qquad \mathsf{E}(G)=\mathsf{d}(G)+|G|=3n\,.$$

2. *Let G be a dicyclic group of order $4n$, where $n\in\mathbb{N}_{\ge 2}$. Then*

$$\mathsf{s}(G)=\begin{cases}6n & \text{if } n \text{ is odd}\\ 4n & \text{if } n \text{ is even}\end{cases}, \qquad \text{and} \qquad \mathsf{E}(G)=\mathsf{d}(G)+|G|=6n\,.$$

After discussing the direct problem, which asks the precise value of group invariants, we consider the associated inverse problem, which asks the structure of extremal sequences. Structural results characterizing which sequences achieve equality are rare. Even in the abelian case, little is known precisely outside of groups of rank at most 2. Among others, we refer to [2, 15] for very recent works associated the Davenport constants for specific non-abelian groups. In the present paper, we concentrate the inverse problem associated to Erdős-Ginzburg-Ziv constant, and we prove the following results.

Theorem 1.2. *Let $n\in\mathbb{N}_{\ge 3}$, G a dihedral group of order $2n$, and $S\in\mathcal{F}(G)$.*

1. *The following statements are equivalent*:
   (a) *$n\ge 4$ is even, $|S|=\mathsf{s}(G)-1$, and S has no product-one subsequence of length* $\exp(G)=n$.
   (b) *There exist $\alpha,\tau\in G$ such that $G=\langle\alpha,\tau\,|\,\alpha^n=\tau^2=1_G$ and $\tau\alpha=\alpha^{-1}\tau\rangle$ and $S=(\alpha^{r_1})^{[n-1]}\boldsymbol{\cdot}(\alpha^{r_2})^{[n-1]}\boldsymbol{\cdot}\alpha^{r_3}\tau$, where $r_1,r_2,r_3\in[0,n-1]$ with* $\gcd(r_1-r_2,n)=1$.
2. *The following statements are equivalent*:
   (a) *$|S|=\mathsf{E}(G)-1$ and S has no product-one subsequence of length $|G|$.*
   (b) *There exist $\alpha,\tau\in G$ such that $G=\langle\alpha,\tau\,|\,\alpha^n=\tau^2=1_G$ and $\tau\alpha=\alpha^{-1}\tau\rangle$ and either that both $n=3$ and $S=1_G^{[5]}\boldsymbol{\cdot}\tau\boldsymbol{\cdot}\alpha\tau\boldsymbol{\cdot}\alpha^2\tau$, or that $S=(\alpha^{r_1})^{[2n-1]}\boldsymbol{\cdot}(\alpha^{r_2})^{[n-1]}\boldsymbol{\cdot}\alpha^{r_3}\tau$, where $r_1,r_2,r_3\in[0,n-1]$ with* $\gcd(r_1-r_2,n)=1$.

Theorem 1.3. *Let $n\in\mathbb{N}_{\ge 2}$, G a dicyclic group of order $4n$, and $S\in\mathcal{F}(G)$.*

1. *The following statements are equivalent*:
   (a) *$n\ge 2$ is even, $|S|=\mathsf{s}(G)-1$, and S has no product-one subsequence of length* $\exp(G)=2n$.
   (b) *S has one of the following forms*:
      (1) *There exist $\alpha,\tau\in G$ such that $G=\langle\alpha,\tau\,|\,\alpha^{2n}=1_G,\tau^2=\alpha^n$, and $\tau\alpha=\alpha^{-1}\tau\rangle$ and $S=(\alpha^{r_1})^{[2n-1]}\boldsymbol{\cdot}(\alpha^{r_2})^{[2n-1]}\boldsymbol{\cdot}\alpha^{r_3}\tau$, where $r_1,r_2,r_3\in[0,2n-1]$ with* $\gcd(r_1-r_2,2n)=1$.
      (2) *There exist $\alpha,\tau\in G$ such that $G=\langle\alpha,\tau\,|\,\alpha^4=1_G,\tau^2=\alpha^2$, and $\tau\alpha=\alpha^{-1}\tau\rangle$ and either $S=(\alpha^{t_1})^{[3]}\boldsymbol{\cdot}(\alpha^{t_2}\tau)^{[3]}\boldsymbol{\cdot}\alpha^{t_3}$ or $S=(\alpha^{t_1})^{[3]}\boldsymbol{\cdot}(\alpha^{t_2}\tau)^{[3]}\boldsymbol{\cdot}\alpha^{t_4}\tau$, where $t_1,t_2,t_3,t_4\in[0,3]$ such that t_1 is even, t_3 is odd, and $t_4\not\equiv t_2$* $\pmod 2$.
2. *The following statements are equivalent*:
   (a) *$|S|=\mathsf{E}(G)-1$ and S has no product-one subsequence of length $|G|$.*
   (b) *S has one of the following forms*:
      (1) *There exist $\alpha,\tau\in G$ such that $G=\langle\alpha,\tau\,|\,\alpha^{2n}=1_G,\tau^2=\alpha^n$, and $\tau\alpha=\alpha^{-1}\tau\rangle$ and $S=(\alpha^{r_1})^{[4n-1]}\boldsymbol{\cdot}(\alpha^{r_2})^{[2n-1]}\boldsymbol{\cdot}\alpha^{r_3}\tau$, where $r_1,r_2,r_3\in[0,2n-1]$ with* $\gcd(r_1-r_2,2n)=1$.

(2) *There exist* $\alpha, \tau \in G$ *such that* $G = \langle \alpha, \tau \mid \alpha^4 = 1_G, \tau^2 = \alpha^2,$ *and* $\tau\alpha = \alpha^{-1}\tau\rangle$ *and either* $S = (\alpha^{t_1})^{[7]} \boldsymbol{\cdot} (\alpha^{t_2}\tau)^{[3]} \boldsymbol{\cdot} \alpha^{t_3}$ *or* $S = (\alpha^{t_1})^{[3]} \boldsymbol{\cdot} (\alpha^{t_2}\tau)^{[7]} \boldsymbol{\cdot} \alpha^{t_4}\tau$*, where* $t_1, t_2, t_3, t_4 \in [0, 3]$ *such that* t_1 *is even,* t_3 *is odd, and* $t_4 \not\equiv t_2 \pmod 2$.

2. Preliminaries

Much of the following notation can be found in [15] and is repeated here for the convenience of the reader. We denote by $\mathbb{N}$ the set of positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $k \in \mathbb{N}$, we also denote by $\mathbb{N}_{\geq k}$ the set of positive integers greater than or equal to k. For integers $a, b \in \mathbb{Z}$, $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ is the discrete interval.

Sequences over groups. Let G be a multiplicatively written finite group with identity element 1_G, and $G_0 \subset G$ a subset. For an element $g \in G$, we denote by $\mathrm{ord}(g) \in \mathbb{N}$ the order of g, by $\exp(G) = \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G\}$ the exponent of G, and by $\langle G_0 \rangle \subset G$ the subgroup generated by G_0.

The elements of the free abelian monoid $\mathcal{F}(G_0)$ will be called *sequences* over G_0. This terminology goes back to Combinatorial Number Theory. Indeed, a sequence over G_0 can be viewed as a finite unordered sequence of terms from G_0, where the repetition of elements is allowed. We briefly discuss our notation which follows the monograph [13, Chapter 10.1]. In order to avoid confusion between multiplication in G and multiplication in $\mathcal{F}(G_0)$, we denote multiplication in $\mathcal{F}(G_0)$ by the boldsymbol $\boldsymbol{\cdot}$ and we use brackets for all exponentiation in $\mathcal{F}(G_0)$. In particular, a sequence $S \in \mathcal{F}(G_0)$ has the form

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell = {\prod}^{\bullet}_{i \in [1,\ell]} g_i \in \mathcal{F}(G_0), \tag{2.1}$$

where $g_1, \ldots, g_\ell \in G_0$ are the terms of S. For $g \in G_0$,

- $\mathsf{v}_g(S) = |\{i \in [1, \ell] \mid g_i = g\}|$ denotes the *multiplicity* of g in S,
- $\mathrm{supp}(S) = \{g \in G_0 \mid \mathsf{v}_g(S) > 0\}$ denotes the *support* of S, and
- $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G_0\}$ denotes the *maximal multiplicity* of S.

A *subsequence* T of S is a divisor of S in $\mathcal{F}(G_0)$ and we write $T \mid S$. For a subset $H \subset G_0$, we denote by S_H the subsequence of S consisting of all terms from H. Furthermore, $T \mid S$ if and only if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G_0$, and in such case, $S \boldsymbol{\cdot} T^{[-1]}$ denotes the subsequence of S obtained by removing the terms of T from S so that $\mathsf{v}_g\big(S \boldsymbol{\cdot} T^{[-1]}\big) = \mathsf{v}_g(S) - \mathsf{v}_g(T)$ for all $g \in G_0$. On the other hand, we set $S^{-1} = g_1^{-1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell^{-1}$ to be the sequence obtained by taking elementwise inverse from S.

Moreover, if $S_1, S_2 \in \mathcal{F}(G_0)$ and $g_1, g_2 \in G_0$, then $S_1 \boldsymbol{\cdot} S_2 \in \mathcal{F}(G_0)$ has length $|S_1| + |S_2|$, $S_1 \boldsymbol{\cdot} g_1 \in \mathcal{F}(G_0)$ has length $|S_1| + 1$, $g_1 g_2 \in G$ is an element of G, but $g_1 \boldsymbol{\cdot} g_2 \in \mathcal{F}(G_0)$ is a sequence of length 2. If $g \in G_0$, $T \in \mathcal{F}(G_0)$, and $k \in \mathbb{N}_0$, then

$$g^{[k]} = \underbrace{g \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g}_{k} \in \mathcal{F}(G_0) \quad \text{and} \quad T^{[k]} = \underbrace{T \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T}_{k} \in \mathcal{F}(G_0)\,.$$

Let $S \in \mathcal{F}(G_0)$ be a sequence as in (2.1). Then we denote by

$$\pi(S) = \{g_{\tau(1)} \ldots g_{\tau(\ell)} \in G \mid \tau \text{ a permutation of } [1, \ell]\} \subset G \quad \text{and} \quad \Pi_n(S) = \bigcup_{\substack{T \mid S \\ |T| = n}} \pi(T) \subset G\,,$$

the *set of products* and n*-products* of S, and more generally, the *subsequence products* of S is denoted by

$$\Pi(S) = \bigcup_{n \geq 1} \Pi_n(S) \subset G\,.$$

It can easily be seen that $\pi(S)$ is contained in a G'-coset, where G' is the commutator subgroup of G. Note that $|S| = 0$ if and only if $S = 1_{\mathcal{F}(G)}$, and in that case we use the convention that $\pi(S) = \{1_G\}$. The sequence S is called

- a *product-one sequence* if $1_G \in \pi(S)$,
- *product-one free* if $1_G \notin \Pi(S)$, and
- *square-free* if $\mathsf h(S) \le 1$.

If $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell \in \mathcal B(G)$ is a product-one sequence with $1_G = g_1 \ldots g_\ell$, then $1_G = g_i \ldots g_\ell g_1 \ldots g_{i-1}$ for every $i \in [1, \ell]$. Every map of groups $\theta : G \to H$ extends to a monoid homomorphism $\theta : \mathcal F(G) \to \mathcal F(H)$, where $\theta(S) = \theta(g_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \theta(g_\ell)$. If θ is a group homomorphism, then $\theta(S)$ is a product-one sequence if and only if $\pi(S) \cap \ker(\theta) \ne \emptyset$. We denote by

$$\mathcal B(G_0) \;=\; \big\{ S \in \mathcal F(G_0) \mid 1_G \in \pi(S) \big\}$$

the set of all product-one sequences over G_0, and clearly $\mathcal B(G_0) \subset \mathcal F(G_0)$ is a submonoid. We denote by $\mathcal A(G_0)$ the set of irreducible elements of $\mathcal B(G_0)$ which, in other words, is the set of minimal product-one sequences over G_0. Moreover,

$$\mathsf D(G_0) \;=\; \sup \big\{ |S| \mid S \in \mathcal A(G_0) \big\} \;\in\; \mathbb N \cup \{\infty\}$$

is the *large Davenport constant* of G_0, and

$$\mathsf d(G_0) \;=\; \sup \big\{ |S| \mid S \in \mathcal F(G_0) \text{ is product-one free } \big\} \;\in\; \mathbb N_0 \cup \{\infty\}$$

is the *small Davenport constant* of G_0.

Ordered sequences over groups. These are an important tool used to study (unordered) sequences over non-abelian groups. Indeed, it is quite useful to have related notation for sequences in which the order of terms matters. Thus, for a subset $G_0 \subset G$, we denote by $\mathcal F^*(G_0) = \big(\mathcal F^*(G_0), \boldsymbol{\cdot}\big)$ the free (non-abelian) monoid with basis G_0, whose elements will be called the *ordered sequences* over G_0.

Taking an ordered sequence in $\mathcal F^*(G_0)$ and considering all possible permutations of its terms gives rise to a natural equivalence class in $\mathcal F^*(G_0)$, yielding a natural map

$$[\boldsymbol{\cdot}] \,:\, \mathcal F^*(G_0) \quad \to \quad \mathcal F(G_0)$$

given by abelianizing the sequence product in $\mathcal F^*(G_0)$. For any sequence $S \in \mathcal F(G_0)$, we say that an ordered sequence $S^* \in \mathcal F^*(G_0)$ with $[S^*] = S$ is an *ordering* of the sequence $S \in \mathcal F(G_0)$.

All notation and conventions for sequences extend naturally to ordered sequences. We sometimes associate an (unordered) sequence S with a fixed (ordered) sequence having the same terms, also denoted by S. While somewhat informal, this does not five rise to confusion, and will improve the readability of some of the arguments.

For an ordered sequence $S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell \in \mathcal F^*(G)$, we denote by $\pi^* : \mathcal F^*(G) \to G$ the unique homomorphism that maps an ordered sequence onto its product in G, so

$$\pi^*(S) \;=\; g_1 \ldots g_\ell \;\in\; G\,.$$

If G is a multiplicatively written abelian group, then for every sequence $S \in \mathcal F(G)$, we always use $\pi^*(S) \in G$ to be the unique product, and $\Pi(S) = \bigcup \big\{ \pi^*(T) \mid T \text{ divides } S \;\text{ and }\; |T| \ge 1 \big\} \subset G$.

Definition 2.1. We denote by

- $\mathsf s(G)$ the smallest integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $\exp(G)$, and
- $\mathsf E(G)$ the smallest integer ℓ such that every sequence of length at least ℓ has a product-one subsequence of length $|G|$.

Note that $\max\{\operatorname{ord}(g) \mid g \in G\} \le \exp(G)$, and equality holds for nilpotent groups G. If G is a dihedral group of order $2n$ with n even, then the equality holds true, whence our definition for $\mathsf s(G)$ coincides with one defined in the papers [3, 14]. However, when n is odd, such two invariants have different flavor. To discuss short argument, let $n \in \mathbb N_{\ge 3}$ be odd, and $G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \text{ and } \tau\alpha = \alpha^{-1}\tau \}$ a dihedral group of order $2n$. Then $\max\{\operatorname{ord}(g) \mid g \in G\} = n$ is odd, and it follows by the relation between α and τ

that every sequence over $G \setminus \langle\alpha\rangle$ cannot have a product-one subsequence of any odd length, in particular of length n.

Lemma 2.2. *Let G be a finite group. Then $\mathsf{s}(G) \geq \mathsf{d}(G) + \exp(G)$ and $\mathsf{E}(G) \geq \mathsf{d}(G) + |G|$.*

Proof. We need to find a sequence of length $\mathsf{d}(G) + \exp(G) - 1$ (or $\mathsf{d}(G) + |G| - 1$ respectively) which has no product-one subsequence of length $\exp(G)$ (or $|G|$ respectively). Take a product-one free sequence S over G of length $|S| = \mathsf{d}(G)$. Then $S \cdot 1_G^{[\exp(G)-1]}$ (or $S \cdot 1_G^{[|G|-1]}$ respectively) is the desired sequence. $\square$

For the rest of this section, we list the following preliminary results.

Lemma 2.3. [8, Lemma 7] *Let G be a finite abelian group of order n, $r \in \mathbb{N}_{\geq 2}$, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = n+r-2$. If S has no product-one subsequence of length n, then $|\Pi_{n-2}(S)| = |\Pi_r(S)| \geq r-1$.*

Lemma 2.4. [5, Theorem 1] *Let G be a cyclic group of order $n \geq 2$, $2 \leq k \leq \lfloor \frac{n}{4} \rfloor + 2$, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = 2n - k$. If S has no product-one subsequence of length n, then there exist $g, h \in G$ with $\mathrm{ord}(gh^{-1}) = n$ such that g and h each appear u and v times in S, where $u \geq n - 2k + 3$, $v \geq n - 2k + 3$, and $u + v \geq 2n - 2k + 2$.*

Finally, we conclude the following direct result from the previous lemma.

Lemma 2.5. *Let G be a cyclic group of order $n \geq 2$, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = 3n - 2$. Then the following statements are equivalent:*

(a) *S has no product-one subsequence of length $2n$.*
(b) *$S = g^{[2n-1]} \cdot h^{[n-1]}$, where $g, h \in G$ with $\mathrm{ord}(gh^{-1}) = n$.*

Proof. Since every sequence of the form $g^{[n-1]} \cdot h^{[n-1]}$, where $g, h \in G$ with $\mathrm{ord}(gh^{-1}) = n$, has no product-one subsequence of length n, it suffices to show (a) $\Rightarrow$ (b). Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 3n - 2$. Suppose that S has no product-one subsequence of length $2n$. Then it follows by $\mathsf{s}(G) = 2n - 1$ that S has a product-one subsequence S_1 of length $|S_1| = n$, whence $|S \cdot S_1^{[-1]}| = 2n - 2$. Then Lemma 2.4 ensures that there exist $g, h \in G$ with $\mathrm{ord}(gh^{-1}) = n$ such that $S = S_1 \cdot g^{[n-1]} \cdot h^{[n-1]}$. Assume to the contrary that there exists $x \in \mathrm{supp}(S_1)$ such that $x \neq g, h$. Then it follows again by Lemma 2.4 that $x \cdot g^{[n-2]} \cdot h^{[n-1]}$ has a product-one subsequence of length n, say $x \cdot g^{[r_1]} \cdot h^{[r_2]}$, where $r_1, r_2 \in [1, n-3]$ such that $r_1 + r_2 = n - 1$. Then $(S_1 \cdot x^{[-1]}) \cdot g^{[n-1-r_1]} \cdot h^{[n-1-r_2]}$ is a sequence of length $2n - 2$. Since S has no product-one subsequence of length $2n$, it follows again by Lemma 2.4 that $S_1 \cdot x^{[-1]} = g^{[r_1]} \cdot h^{[r_2]}$, whence $g^{[n]} \cdot h^{[n]}$ is a product-one subsequence of S of length $2n$, a contradiction. Thus every elements in $\mathrm{supp}(S_1)$ is either g or h. If $g, h \in \mathrm{supp}(S_1)$, then $g^{[n]} \cdot h^{[n]}$ is a product-one subsequence of S of length $2n$, a contradiction. Therefore we obtain either $S_1 = g^{[n]}$ or $S_1 = h^{[n]}$. $\square$

3. On Dihedral groups

In this section, we set $n \in \mathbb{N}_{\geq 3}$, and use the following notation for a dihedral group of order $2n$:

$$G = \langle \alpha, \tau \mid \alpha^n = \tau^2 = 1_G \text{ and } \tau\alpha = \alpha^{-1}\tau \rangle .$$

Then we set $H = \langle\alpha\rangle$, and $G_0 = G \setminus H$. Note that $\mathsf{d}(G) = n$ and $\exp(G) = \mathrm{lcm}(2, n)$.

Now we prove the following technical lemmas by improving the argument from [1].

Lemma 3.1. *Let $n \in \mathbb{N}_{\geq 4}$ be even, and $S \in \mathcal{F}(G)$.*

1. *If $|S| = 2n - 1$, and S has no product-one subsequence of length n, then $|S_{G_0}| = 1$.*
2. *If $|S| = 3n - 1$, and S has no product-one subsequence of length $2n$, then $|S_{G_0}| = 1$.*

Proof. 1. Let $n \ge 4$ be even, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = 2n-1$ such that S has no product-one subsequence of length n. Note that $|S_{G_0}| \ge 1$. Assume to the contrary that $|S_{G_0}| \ge 2$.

Let $H_1 = \langle \alpha^2 \rangle$, $H_2 = H \setminus H_1$, $G_1 = \big\{ \alpha^{2k}\tau \mid k \in [0, \frac{n}{2}-1] \big\}$, and $G_2 = G_0 \setminus G_1$. Suppose that $S_{H_1} = a_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_r$, $S_{H_2} = b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_s$, $S_{G_1} = c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_t$, and $S_{G_2} = d_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} d_\ell$, where $r,s,t,\ell \in \mathbb{N}_0$. Then $r+s+t+\ell = 2n-1$ is odd. We set

$$\begin{aligned} T_1 \ = \ & (a_1a_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (a_{2\lfloor \frac{r}{2} \rfloor -1} a_{2\lfloor \frac{r}{2} \rfloor}) \boldsymbol{\cdot} (b_1b_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (b_{2\lfloor \frac{s}{2} \rfloor -1} b_{2\lfloor \frac{s}{2} \rfloor}) \boldsymbol{\cdot} \\ & (c_1c_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (c_{2\lfloor \frac{t}{2} \rfloor -1} c_{2\lfloor \frac{t}{2} \rfloor}) \boldsymbol{\cdot} (d_1d_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (d_{2\lfloor \frac{\ell}{2} \rfloor -1} d_{2\lfloor \frac{\ell}{2} \rfloor}) \,. \end{aligned}$$

Then $T_1 \in \mathcal{F}(H_1)$ of length $|T_1| = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor \ge n-2 = 2\big(\frac{n}{2}\big) - 2$. Since S has no product-one subsequence of length n, we have that T_1 has no product-one subsequence of length $\frac{n}{2}$. It follows by $\mathsf{s}(H_1) = 2\big(\frac{n}{2}\big) - 1$ that $|T_1| = n-2$ and three elements of $\{r,s,t,\ell\}$ are odd. By Lemma 2.4, there exist $g,h \in H_1$ with $\operatorname{ord}(gh^{-1}) = \frac{n}{2}$ such that

$$T_1 \ = \ g^{[\frac{n}{2}-1]} \boldsymbol{\cdot} h^{[\frac{n}{2}-1]} \,.$$

CASE 1. $n = 4$.

If $\{r,s,t,\ell\} \subset \mathbb{N}$, then $a_1b_1, c_1d_1 \in H_2$. Since $|H_2| = 2$, we obtain that either $a_1b_1 = c_1d_1$ or $a_1b_1 = (c_1d_1)^{-1} = d_1c_1$. It follows that either $c_1 \boldsymbol{\cdot} a_1 \boldsymbol{\cdot} b_1 \boldsymbol{\cdot} d_1$ or $a_1 \boldsymbol{\cdot} b_1 \boldsymbol{\cdot} c_1 \boldsymbol{\cdot} d_1$ is a product-one subsequence of S of length 4, a contradiction. Thus one element of $\{r,s,t,\ell\}$ must be zero.

Suppose that $r = 0$, and the case when $s = 0$ follows by the same argument. Then all s,t,ℓ must be odd with $s+t+\ell = 7$. Since $|H_1| = |H_2| = |G_1| = |G_2| = 2$, we obtain that there exist subsequences W_1, W_2 of S such that $W_1 = x^{[2]}$ and $W_2 = y^{[2]}$ for some $x,y \in G$. If $x,y \in G_0$, then $W_1 \boldsymbol{\cdot} W_2$ is a product-one subsequence of S of length 4, a contradiction. If $x \in H_2$ and $y \in G_0$ (or else $x \in G_0$ and $y \in H_2$ respectively), then $xyxy = 1_G$ (or $yxyx = 1_G$ respectively), which implies that $W_1 \boldsymbol{\cdot} W_2$ is a product-one subsequence of S of length 4, a contradiction. If $x,y \in H_2$, then $x = y$ or $x = y^{-1}$, which implies that $W_1 \boldsymbol{\cdot} W_2$ is a product-one subsequence of S of length 4, a contradiction.

Suppose that $t = 0$, and the case when $\ell = 0$ follows by the same argument. Then all r,s,ℓ must be odd with $\ell \ge 3$ and $r+s+\ell = 7$. Since $|H_1| = |H_2| = |G_1| = |G_2| = 2$, we obtain that there exist subsequences W_1, W_2 of S such that $W_1 = x^{[2]}$ and $W_2 = y^{[2]}$, where $x \in G$ and $y \in G_2$. If $x \in G_2$, then $W_1 \boldsymbol{\cdot} W_2$ is a product-one subsequence of S of length 4, a contradiction. If $x \in H$, then $xyxy = 1_G$, which implies that $W_1 \boldsymbol{\cdot} W_2$ is a product-one subsequence of S of length 4, a contradiction.

CASE 2. $n \ge 6$ and $t + \ell \ge 3$.

Then $t \ge 2$ or $\ell \ge 2$. If there exist distinct $t_1, t_2 \in [1,t]$ (or $\ell_1, \ell_2 \in [1,\ell]$ respectively) such that $c_{t_1} = c_{t_2}$ (or $d_{\ell_1} = d_{\ell_2}$ respectively), then after renumbering if necessary, we may assume that $c_1 = c_2$ (or $d_1 = d_2$ respectively). By symmetry, we may suppose that $c_1c_2 = g$ (or $d_1d_2 = g$ respectively). Then $h^{[\frac{n-2}{4}]} \boldsymbol{\cdot} c_1 \boldsymbol{\cdot} h^{[\frac{n-2}{4}]} \boldsymbol{\cdot} c_2$ $\big($or $h^{[\frac{n-2}{4}]} \boldsymbol{\cdot} d_1 \boldsymbol{\cdot} h^{[\frac{n-2}{4}]} \boldsymbol{\cdot} d_2$ respectively$\big)$ is a product-one sequence, and by splitting elements equal to h to be subsequence of S of length 2, we infer that S has a product-one subsequence of length n, a contradiction. Therefore

$$\mathsf{h}(S_{G_0}) \ = \ 1 \,, \quad \text{ and hence } \quad t + \ell \ \le \ n \,.$$

We only prove the case when $t \ge 2$, and the case when $\ell \ge 2$ follows by the same argument. Now we assume that $t \ge 2$, and without loss of generality that $c_1c_2 = g$.

Suppose that $\frac{n}{2}$ is even. Then $\frac{n}{2} - 1 \ge 3$, and $h^{[\frac{n-4}{4}]} \boldsymbol{\cdot} c_1 \boldsymbol{\cdot} h^{[\frac{n-4}{4}]} \boldsymbol{\cdot} g \boldsymbol{\cdot} c_2$ is a product-one sequence. It follows by splitting the elements equals h and g to be subsequences of S length 2 that there exists a product-one subsequence of S of length n, a contradiction.

Suppose that $\frac{n}{2}$ is odd, and both t and ℓ are odd. Then we set $T_2 = T_1 \boldsymbol{\cdot} (c_1c_2)^{[-1]} \boldsymbol{\cdot} (c_1c_t)$ which still has no product-one subsequence of length $\frac{n}{2}$. Comparing with T_1, we have $c_1c_2 = c_1c_t$, and hence $c_2 = c_t$, contradicting that $\mathsf{h}(S_{G_0}) = 1$.

Suppose that $\frac{n}{2}$ is odd, and both r and s are odd. Then we set $T_3 = T_1 \boldsymbol{\cdot} (a_1a_2)^{[-1]} \boldsymbol{\cdot} (a_1a_r)$ which still has no product-one subsequence of length $\frac{n}{2}$. Comparing with T_1, we have $a_1a_2 = a_1a_r$, and we obtain by proceeding the same argument that $S_{H_1}=a_1^{[r]}$ and $S_{H_2}=b_1^{[s]}$. Since $r+s=2n-1-(t+\ell)\ge n-1\ge 5$, there exist $x,y\in H$ such that $x^{[2]}\boldsymbol{\cdot} y^{[2]}\mid S_H$. If $n\ge 10$, then $h^{[\frac{n-6}{4}]}\boldsymbol{\cdot} c_1\boldsymbol{\cdot} h^{[\frac{n-6}{4}]}\boldsymbol{\cdot} g\boldsymbol{\cdot} x\boldsymbol{\cdot} c_2\boldsymbol{\cdot} x$ is a product-one sequence, and by splitting the elements equal to h and g to be subsequences of S of length 2, we infer that S has a product-one subsequence of length n, a contradiction. If $n=6$, then $x^2\ne y^2$, and hence either $c_1\boldsymbol{\cdot} x^{[2]}\boldsymbol{\cdot} y\boldsymbol{\cdot} c_2\boldsymbol{\cdot} y$ or $c_1\boldsymbol{\cdot} y^{[2]}\boldsymbol{\cdot} x\boldsymbol{\cdot} c_2\boldsymbol{\cdot} x$ is a product-one subsequence of S of length 6, a contradiction.

CASE 3. $n\ge 6$ and $t+\ell=2$.

Then $|S_H|=2n-3$. Since t or ℓ must be odd, we have $t=\ell=1$. Suppose that $S_{G_1}=\alpha^i\tau$ and $S_{G_2}=\alpha^j\tau$, where $i,j\in[0,n-1]$. Since S_H has no product-one subsequence of length n, it follows by lemma 2.4 that there exist $r_1,r_2\in[0,n-1]$ such that r_1 is odd, r_2 is even, and $\gcd(r_1-r_2,n)=1$ such that $(\alpha^{r_1})^{[u_1]}\boldsymbol{\cdot}(\alpha^{r_2})^{[u_2]}\mid S_H$ for some $u_1,u_2\ge n-3\ge\frac{n}{2}$ with $u_1+u_2\ge 2n-4\ge n+2$. Then there exist $v,w\in[0,n-1]$ such that

$$i\equiv v(r_1-r_2)\pmod n\qquad\text{and}\qquad j\equiv w(r_1-r_2)\pmod n\,,$$

and we set x to be $|w-v|$ if $|w-v|\le\frac{n}{2}$, and $n-|w-v|$ if $|w-v|>\frac{n}{2}$. Then $x\in[1,\frac{n}{2}]$ is odd, and $V=(\alpha^{r_1})^{[x]}\boldsymbol{\cdot}(\alpha^{r_2})^{[x]}\boldsymbol{\cdot}\alpha^i\tau\boldsymbol{\cdot}\alpha^j\tau$ is a product-one subsequence of S having even length. If $x\le\frac{n}{2}-1$, then by adding even terms of either α^{r_1} or α^{r_2} to V, we obtain a product-one subsequence of S of length n, a contradiction. Therefore $x=\frac{n}{2}$ is odd and $j\equiv\frac{n}{2}+i\pmod n$.

Suppose that $\gcd(r_1,n)>1$. Since r_1 is odd, we have $\gcd(r_1,n)\mid\frac{n}{2}$, and hence there exists $x_0\in[1,\frac{n}{2}-1]$ such that $r_1x_0\equiv\frac{n}{2}\pmod n$. Since $\frac{n}{2}$ is odd, we obtain that x_0 is odd. Hence $n-2x_0-4\ge 0$ and

$$(\alpha^{r_1})^{[x_0+\frac{n-2x_0-4}{4}]}\boldsymbol{\cdot}(\alpha^i\tau)\boldsymbol{\cdot}(\alpha^{r_1})^{[\frac{n-2x_0-4}{4}]}\boldsymbol{\cdot}(\alpha^j\tau)\boldsymbol{\cdot}(\alpha^{r_2})^{[\frac{n}{2}]}$$

is a product-one subsequence of S of length n, a contradiction.

Suppose that $\gcd(r_1,n)=1$. If $r_2=0$, then since $u_1,u_2\ge\frac{n}{2}$, it follows that $(\alpha^{r_2})^{[\frac{n}{2}-2]}\boldsymbol{\cdot}(\alpha^{r_1})^{[\frac{n}{2}]}\boldsymbol{\cdot}\alpha^i\tau\boldsymbol{\cdot}\alpha^j\tau$ is a product-one subsequence of S of length n, a contradiction. Thus we assume that $r_2\ne 0$, and then there exists an odd $x_0\in[1,n]\setminus\{\frac{n}{2}\}$ such that $r_1x_0\equiv\frac{n}{2}+r_2\pmod n$. Thus $|\frac{n}{2}-x_0|\ge 2$. If $x_0\in[1,\frac{n}{2}-1]$, then $n-2x_0-4\ge 0$ and

$$(\alpha^{r_1})^{[x_0+\frac{n-2x_0-4}{4}]}\boldsymbol{\cdot}(\alpha^i\tau)\boldsymbol{\cdot}(\alpha^{r_1})^{[\frac{n-2x_0-4}{4}]}\boldsymbol{\cdot}(\alpha^{r_2})^{[1+\frac{n-2}{4}]}\boldsymbol{\cdot}(\alpha^j\tau)\boldsymbol{\cdot}(\alpha^{r_2})^{[\frac{n-2}{4}]}$$

is a product-one subsequence of S of length n, a contradiction. If $x_0\in[\frac{n}{2}+1,n]$, then $2x_0-n-4\ge 0$ and

$$(\alpha^{r_1})^{[n-x_0+\frac{2x_0-n-4}{4}]}\boldsymbol{\cdot}(\alpha^i\tau)\boldsymbol{\cdot}(\alpha^{r_1})^{[\frac{2x_0-n-4}{4}]}\boldsymbol{\cdot}(\alpha^{r_2})^{[1+\frac{n-2}{4}]}\boldsymbol{\cdot}(\alpha^j\tau)\boldsymbol{\cdot}(\alpha^{r_2})^{[\frac{n-2}{4}]}$$

is a product-one subsequence of S of length n, a contradiction.

2. Let $S\in\mathcal{F}(G)$ be a sequence of length $|S|=3n-1$ such that S has no product-one subsequence of length $2n$. Assume to the contrary that $|S_{G_0}|\ge 2$.

If $|S_H|\ge 2n-1$, then there exits a product-one subsequence T_1 of S_H of length $|T_1|=n$. Thus $|S\boldsymbol{\cdot} T_1^{[-1]}|=2n-1$ and $|(S\boldsymbol{\cdot} T_1^{[-1]})_{G_0}|\ge 2$. Thus 1. implies that $S\boldsymbol{\cdot} T_1^{[-1]}$ has a product-one subsequence T_2 of length n, whence $T_1\boldsymbol{\cdot} T_2$ is a product-one subsequence of S of length $2n$, a contradiction.

If $|S_H|\le 2n-2$, then $|S_{G_0}|\ge n+1\ge 5$. Let $S=S_1\boldsymbol{\cdot} S_2$ be such that $|S_1|=2n-1$ with $|(S_1)_{G_0}|\ge 2$ and $|S_2|=n$ with $|(S_2)_{G_0}|\ge 2$, where $S_1,S_2\in\mathcal{F}(G)$. Thus 1. implies that S_1 has a product-one subsequence T_1 of length n, and that $S\boldsymbol{\cdot} T_1^{[-1]}$ has a product-one subsequence T_2 of length n, whence $T_1\boldsymbol{\cdot} T_2$ is a product-one subsequence of S of length $2n$, a contradiction. □

Lemma 3.2. *Let $n\in\mathbb{N}_{\ge 3}$ be odd, and $S\in\mathcal{F}(G)$ a sequence of length $|S|=3n-1$. If S has no product-one subsequence of length $2n$, then either $|S_{G_0}|=1$, or that both $n=3$ and $S=1_G^{[5]}\boldsymbol{\cdot}\tau\boldsymbol{\cdot}\alpha\tau\boldsymbol{\cdot}\alpha^2\tau$.*

Proof. Let $S\in\mathcal{F}(G)$ be a sequence of length $|S|=3n-1$ such that S has no product-one subsequence of length $2n$. Note that $|S_{G_0}|\ge 1$. By way of contradiction, we may suppose that $|S_{G_0}|\ge 2$, and further suppose that $S\ne\tau\boldsymbol{\cdot}\alpha\tau\boldsymbol{\cdot}\alpha^2\tau\boldsymbol{\cdot} 1_G^{[5]}$ when $n=3$.

By renumbering if necessary, we can assume that

$$S_{G_0} \;=\; \big(a_1^{[2]} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_r^{[2]}\big) \boldsymbol{\cdot} (c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u)\,,$$

where $c_i \neq c_j$ for all distinct $i, j \in [1,u]$. Then $u \le n$. Similarly we set

$$S_H \;=\; \big(b_1^{[2]} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_t^{[2]}\big) \boldsymbol{\cdot} (e_1 \boldsymbol{\cdot} e_1^{-1}) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (e_s \boldsymbol{\cdot} e_s^{-1}) \boldsymbol{\cdot} (d_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} d_v)\,,$$

where $d_i \notin \{d_j, d_j^{-1}\}$ for all distinct $i, j \in [1,v]$, and $\operatorname{ord}(b_k) \neq 2$ for all $k \in [1,v]$. Then $v \le \frac{n+1}{2}$.

CASE 1. $r \ge 1$.

Let W_1 be the maximal product-one subsequence of $c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u \boldsymbol{\cdot} d_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} d_v$ of even length. Then $|W_1| \le 2n$ and $W = c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u \boldsymbol{\cdot} d_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} d_v \boldsymbol{\cdot} W_1^{[-1]}$ has no product-one subsequence of even length. If $|W| \le n-1$, then $2t + 2r + 2s + |W_1| \ge 2n$, and hence there exist $t_1 \in [0,t]$, $r_1 \in [0,r]$, and $s_1 \in [0,s]$ with $2t_1 + 2r_1 + 2s_1 + |W| = 2n$ such that

$$W_1 \boldsymbol{\cdot} (e_1 \boldsymbol{\cdot} e_1^{-1}) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (e_{s_1} \boldsymbol{\cdot} e_{s_1}^{-1}) \boldsymbol{\cdot} (b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_{t_1}) \boldsymbol{\cdot} a_1 \boldsymbol{\cdot} (b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_{t_1}) \boldsymbol{\cdot} a_1 \boldsymbol{\cdot} a_2^{[2]} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_{r_1}^{[2]}$$

is a product-one subsequence of S of length $2n$, a contradiction. Since n is odd, we have $|W| \ge n+1$.

Suppose that $|W_H| \ge 2$. Let $\Omega = \big\{x \in H \mid x \in \Pi_2(W_H) \text{ or } x^{-1} \in \Pi_2(W_H)\big\}$. Since $\Pi_2(W_H) \subset H \setminus \{1_G\}$, we have $|\Omega| \ge |W_H|$, and hence $|\Pi_2(W_{G_0})| + |\Omega| \ge \big(|W_{G_0}| - 1\big) + |W_H| \ge n = |H|$. It follows by $\Pi_2(W_{G_0}) \cup \Omega \subset H \setminus \{1_G\}$ that there exist ordered sequences $T_1 \mid W_{G_0}$ and $T_2 \mid W_H$ with $|T_1| = |T_2| = 2$ such that $\pi^*(T_1) = \pi^*(T_2) \in \Pi_2(W_{G_0}) \cap \Omega$, which implies that $T_1 \boldsymbol{\cdot} T_2$ is a product-one subsequence of W of length 4, a contradiction to the maximality of W_1.

Suppose that $|W_H| = 1$. Then $|W_{G_0}| = n$. If $n \ge 5$, then W_{G_0} has a product-one subsequence of length 4, a contradiction to the maximality of W_1. If $n = 3$, then it is easy to verify that there are $g_1, g_2 \in \operatorname{supp}(W_{G_0})$ such that $T = a_1^{[2]} \boldsymbol{\cdot} g_1 \boldsymbol{\cdot} g_2$ is a product-one sequence. Note that $4 = 2r + 2t + 2s + |W_1|$. If W_1 is non-trivial, then since $r \ge 1$, we must have $|W_1| = 2$, whence $W_1 \boldsymbol{\cdot} T$ is a product-one subsequence of S of length 6, a contradiction. Thus W_1 is a trivial sequence, and then $r + t + s = 2$. Hence we infer that S has a product-one subsequence of length 6, a contradiction.

CASE 2. $r = 0$.

Then $2 \le |S_{G_0}| = u \le n$, and we proceed by the following assertion.

A. *For every non-trivial product-one subsequence W of S_{G_0}, we have $u - |W| \ge \frac{n+1}{2}$.*

Proof of **A**. Assume to the contrary that $S_{G_0} = c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u$ has a non-trivial product-one subsequence W such that $u - |W| \le \frac{n-1}{2}$. Since $|W|$ is even and

$$\big|\, \big(b_1^{[2]} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_t^{[2]}\big) \boldsymbol{\cdot} (e_1 \boldsymbol{\cdot} e_1^{-1}) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (e_s \boldsymbol{\cdot} e_s^{-1}) \boldsymbol{\cdot} W \,\big| \;\ge\; (3n-1) - \frac{n+1}{2} - \frac{n-1}{2} \;=\; 2n - 1\,,$$

it follows that $\big(b_1^{[2]} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_t^{[2]}\big) \boldsymbol{\cdot} (e_1 \boldsymbol{\cdot} e_1^{-1}) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (e_s \boldsymbol{\cdot} e_s^{-1}) \boldsymbol{\cdot} W$ has a product-one subsequence of length $2n$, a contradiction. $\square$

SUBCASE 2.1. $u = n$.

Since all c_i are distinct, we may assume by renumbering if necessary that $c_i = \alpha^i \tau$ for every $i \in [1, n-1]$ and $c_u = c_n = \tau$. If $n = 3$, then $S_{G_0} = \tau \boldsymbol{\cdot} \alpha\tau \boldsymbol{\cdot} \alpha^2\tau$ and $|S_H| = 5$. If $\alpha^i \in \Pi_4(S_H)$ for some $i \in [1,2]$, then S has a product-one subsequence of length 6, a contradiction. Thus $\Pi_4(S_H) = \{1_G\}$, which implies that $S_H = 1_G^{[5]}$ and $S = \tau \boldsymbol{\cdot} \alpha\tau \boldsymbol{\cdot} \alpha^2\tau \boldsymbol{\cdot} 1_G^{[5]}$, a contradiction to our assumption.

Suppose that $n \ge 5$. If $\frac{n-1}{2}$ is even, then $\alpha\tau \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{n-1}\tau$ is a product-one subsequence of S_{G_0}, a contradiction to **A**. If $\frac{n-1}{2}$ is odd, then $n \ge 7$ and $\tau \boldsymbol{\cdot} \alpha^2\tau \boldsymbol{\cdot} \alpha^3\tau \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \alpha^{n-1}\tau$ is a product-one subsequence of S_{G_0}, again a contradiction to **A**.

SUBCASE 2.2. $u = 2$.

We set $S_{G_0} = \alpha^i\tau \boldsymbol{\cdot} \alpha^j\tau$ for distinct $i, j \in [0, n-1]$. If $n = 3$, then $|S_H| = 6$ and $\pi(S_{G_0}) = \{\alpha^{i-j},\, \alpha^{j-i}\} = \{\alpha, \alpha^2\}$. If $\alpha^k \in \Pi_4(S_H)$ for some $k \in [1,2]$, then S has a product-one subsequence of length 6, a contradiction. Thus $\Pi_4(S_H)=\{1_G\}$, and hence $S_H=1_G^{[6]}$ is a product-one subsequence of length 6, a contradiction.

Suppose that $n\ge 5$. Since $|S_H|=3n-3$, it follows by $\mathsf{s}(H)=2n-1$ that there exists a product-one subsequence T_1 of S_H of length n, whence $|S_H\cdot T_1^{[-1]}|=2n-3$. Since S has no product-one subsequence of length $2n$, it follows by Lemma 2.4 that there exist $r_1,r_2\in[0,n-1]$ with $\gcd(r_1-r_2,n)=1$ such that

$$(\alpha^{r_1})^{[\gamma_1]}\cdot(\alpha^{r_2})^{[\gamma_2]}\mid S_H\,,\quad\text{where}\quad \gamma_1,\gamma_2\ \ge\ n-3\ \ge\ \frac{n-1}{2}\quad\text{and}\quad \gamma_1+\gamma_2\ \ge\ 2n-4\ \ge\ n+1\,.$$

Since $\gcd(r_1-r_2,n)=1$, there exists $x\in[0,n-1]$ such that $(r_1-r_2)x\equiv j-i+r_1\pmod n$. If $x\le\frac{n-1}{2}$, then $V=(\alpha^{r_1})^{[x-1]}\cdot\alpha^i\tau\cdot(\alpha^{r_2})^{[x]}\cdot\alpha^j\tau$ is a product-one subsequence of S. Since $\gamma_1+\gamma_2\ge n+1$, by adding even terms of either α^{r_1} or α^{r_2} to V, we obtain that $S\cdot T_1^{[-1]}$ has a product-one subsequence of length n, a contradiction. If $x\ge\frac{n+3}{2}$, then $n-x+1\le\frac{n-1}{2}$ and $V=(\alpha^{r_1})^{[n-x+1]}\cdot\alpha^j\tau\cdot(\alpha^{r_2})^{[n-x]}\cdot\alpha^i\tau$ is a product-one subsequence of S. Since $\gamma_1+\gamma_2\ge n+1$, by adding even terms of either α^{r_1} or α^{r_2} to V, we obtain that $S\cdot T_1^{[-1]}$ has a product-one subsequence of length n, a contradiction. Thus $x=\frac{n+1}{2}$, and again by $\gcd(r_1-r_2,n)=1$, there exists $y\in[0,n-1]$ such that $(r_1-r_2)y\equiv i-j+r_1\pmod n$. A similar argument shows that $y=\frac{n+1}{2}=x$, and hence $i-j+r_1\equiv j-i+r_1\pmod n$. Since n is odd, it follows that $i=j$, a contradiction.

SUBCASE 2.3. $n-1\ge u\ge 3$.

Then $|S_H|=3n-u-1\ge\mathsf{s}(H)=2n-1$ ensures that S_H has a product-one subsequence T_1 of length n. Then $S_H\cdot T_1^{[-1]}$ is a sequence over H of length $2n-u-1$ which has no product-one subsequence of length n, and thus Lemma 2.3 ensures that $\big|\Pi_{n-2}\big(S_H\cdot T_1^{[-1]}\big)\big|\ge n-u$. Since all c_i are distinct, we have that all $c_1c_2,c_1c_3,\ldots,c_1c_u$ are distinct, and hence $\big|\Pi_2(c_1\cdot\ldots\cdot c_u)\big|\ge u-1$.

SUBCASE 2.3.1 $\big|\Pi_2(c_1\cdot\ldots\cdot c_u)\big|\ge u$.

If $\Pi_{n-2}\big(S_H\cdot T_1^{[-1]}\big)\cap\Pi_2(c_1\cdot\ldots\cdot c_u)\ne\emptyset$, then there exists a product-one subsequence T_2 of $S\cdot T_1^{[-1]}$ of length n, whence S has a product-one subsequence $T_1\cdot T_2$ of length $2n$, a contradiction. Thus $\Pi_{n-2}\big(S_H\cdot T_1^{[-1]}\big)\cap\Pi_2(c_1\cdot\ldots\cdot c_u)=\emptyset$, which implies that $\big|\Pi_2(c_1\cdot\ldots\cdot c_u)\big|=u$ and $\big|\Pi_{n-2}\big(S_H\cdot T_1^{[-1]}\big)\big|=n-u$. Note that if $h\in\Pi_2(c_1\cdot\ldots\cdot c_u)$, then $h^{-1}\in\Pi_2(c_1\cdot\ldots\cdot c_u)$, for otherwise, $S\cdot T_1^{[-1]}$ must have a product-one subsequence of length n, again a contradiction. Since n is odd and $\Pi_2(c_1\cdot\ldots\cdot c_u)\subset H\setminus\{1_G\}$, we obtain that u must be even. Since all $c_1c_2,c_1c_3,\ldots,c_1c_u$ are distinct, after renumbering if necessary, we may assume that $c_1c_{2k}=(c_1c_{2k+1})^{-1}$ for all $k\in[1,\frac{u}{2}-1]$, and suppose $c_i=\alpha^{r_i}\tau$ for all $i\in[1,u]$. Then $2r_1\equiv r_2+r_3\equiv\ldots\equiv r_{u-2}+r_{u-1}\pmod n$, and hence for every $k\in\mathbb{N}_0$ such that $(2+4k)+3\le u-1$, we have

$$c_{2+4k}\,c_{(2+4k)+2}\,c_{(2+4k)+1}\,c_{(2+4k)+3}\ =\ \alpha^{r_{2+4k}+r_{(2+4k)+1}-r_{(2+4k)+2}-r_{(2+4k)+3}}\ =\ 1_G\,.$$

Suppose that $u\ge 6$ with $4\mid u-2$. Then $W=c_2\cdot\ldots\cdot c_{u-1}$ is a product-one sequence of length $u-2$. Since $n\ge u\ge 6$, we have $u-|W|=2\le\frac{n-1}{2}$, a contradiction to **A**.

Suppose that $u\ge 6$ with $4\nmid u-2$. Then $4\mid u-4$ and $W=c_2\cdot\ldots\cdot c_{u-3}$ is a product-one sequence of length $u-4$. Since $u\ge 8$, we obtain $n\ge 9$, and hence $u-|W|=4\le\frac{n-1}{2}$, again a contradiction to **A**.

Suppose that $u=4$. Then $c_3c_4\in\Pi_2(c_1\cdot\ldots\cdot c_4)=\{c_1c_2,\,c_1c_3,\,c_1c_4,\,c_4c_1\}$. Since all c_i are distinct, we have $c_3c_4\ne c_1c_4$. If $c_3c_4=c_1c_2$ or $c_3c_4=c_1c_3=c_2c_1$, then $c_1\cdot\ldots\cdot c_4$ is a product-one sequence, a contradiction to **A**, whence $c_3c_4=c_4c_1$. Similarly we can prove that $c_2c_4=c_4c_1$, which implies that $c_2c_4=c_3c_4$, and thus $c_2=c_3$, a contradiction.

SUBCASE 2.3.2 $\big|\Pi_2(c_1\cdot\ldots\cdot c_u)\big|=u-1$.

For every $i\in[1,u]$, we set $c_i=\alpha^{r_i}\tau$. Since $\big|\Pi_2(c_1\cdot\ldots\cdot c_u)\big|=u-1$, it follows that

$$\Sigma_2(c_1\cdot\ldots\cdot c_u)\ =\ \{c_1c_2,\ldots,c_1c_u\}\ =\ \big\{\alpha^{r_1-r_2},\alpha^{r_1-r_3},\ldots,\alpha^{r_1-r_u}\big\}\ =\ \big\{\alpha^{r_2-r_1},\alpha^{r_2-r_3},\ldots,\alpha^{r_2-r_u}\big\}\,.$$

By multiplying all the elements of each set, we have the modulo equation

$$(u-1)r_1 - (r_2 + r_3 + \ldots + r_u) \equiv (u-1)r_2 - (r_1 + r_3 + \ldots + r_u) \pmod{n},$$

which implies that $ur_1 \equiv ur_2 \pmod{n}$. Similarly, $ur_i \equiv ur_j \pmod{n}$ for all distinct $i, j \in [1, u]$. Thus all elements $\alpha^{r_i - r_j}$ are in the subgroup of H having order $\gcd(u, n)$. But $u = \big|\{1_G, \alpha^{r_1 - r_2}, \ldots, \alpha^{r_1 - r_u}\}\big| \le \gcd(u, n) \le u$, whence $\gcd(u, n) = u$ and $u \mid n$. Since n is odd, u is also odd, and $\Pi_2(c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u) = \big\{\alpha^{\frac{n}{u}}, \ldots, \alpha^{(u-1)\frac{n}{u}}\big\}$. By renumbering if necessary, we can assume that

$$c_1 = \alpha^{r_1}\tau, \quad c_2 = \alpha^{r_1 + \frac{n}{u}}\tau, \quad \ldots, \quad c_u = \alpha^{r_1 + (u-1)\frac{n}{u}}\tau.$$

Suppose that $u \ge 5$ and $4 \mid u-1$. Then $c_1 \boldsymbol{\cdot} c_2 \boldsymbol{\cdot} c_4 \boldsymbol{\cdot} c_3$ is a product-one sequence, and thus $c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_{u-1}$ is a product-one subsequence of S_{G_0} of length $u-1$, a contradiction to **A**.

Suppose that $u \ge 5$ and $4 \nmid u-1$. Then $u \ge 7$, and $c_1 \boldsymbol{\cdot} c_3 \boldsymbol{\cdot} c_5 \boldsymbol{\cdot} c_4 \boldsymbol{\cdot} c_7 \boldsymbol{\cdot} c_6$ is a product-one sequence. Thus $c_1 \boldsymbol{\cdot} c_3 \boldsymbol{\cdot} c_4 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_{u-1}$ is a product-one subsequence of S_{G_0} of length $u-1$, again a contradiction to **A**.

Suppose that $u = 3$. Since $u \mid n$ and $u \le n-1$, we obtain $n \ge 9$, and since $|S_H| = 3n-4 \ge \mathsf{s}(H) = 2n-1$, we obtain that S_H has a product-one subsequence T_1 of length n. Then $|S_H \boldsymbol{\cdot} T_1^{[-1]}| = 2n-4$ and $S_H \boldsymbol{\cdot} T_1^{[-1]}$ has no product-one subsequence of length n. It follows by Lemma 2.4 that there exist $r_1, r_2 \in [0, n-1]$ with $\gcd(r_1 - r_2, n) = 1$ such that

$$(\alpha^{r_1})^{[\gamma_1]} \boldsymbol{\cdot} (\alpha^{r_2})^{[\gamma_2]} \,\big|\, S_H \boldsymbol{\cdot} T_1^{[-1]} \quad \text{for some } \gamma_1, \gamma_2 \ge n-5 \ge \frac{n-1}{2} \text{ and } \gamma_1 + \gamma_2 \ge 2n-6 \ge n+3.$$

Since $\gcd(r_1 - r_2, n) = 1$, there exists $x \in [0, n-1]$ such that $(r_1 - r_2)x \equiv \frac{n}{3} + r_1 \pmod{n}$. If $x \le \frac{n-1}{2}$, then $V = (\alpha^{r_1})^{[x-1]} \boldsymbol{\cdot} c_1 \boldsymbol{\cdot} (\alpha^{r_2})^{[x]} \boldsymbol{\cdot} c_2$ is a product-one subsequence of S. Since $\gamma_1 + \gamma_2 \ge n+3$, by adding even terms of either α^{r_1} or α^{r_2} to V, we obtain that $S \boldsymbol{\cdot} T_1^{[-1]}$ has a product-one subsequence of length n, a contradiction. If $x \ge \frac{n+3}{2}$, then $n - x + 1 \le \frac{n-1}{2}$ and $V = (\alpha^{r_1})^{[n-x+1]} \boldsymbol{\cdot} c_2 \boldsymbol{\cdot} (\alpha^{r_2})^{[n-x]} \boldsymbol{\cdot} c_1$ is a product-one subsequence of S. Since $\gamma_1 + \gamma_2 \ge n+3$, by adding even terms of either α^{r_1} or α^{r_2} to V, we obtain that $S \boldsymbol{\cdot} T_1^{[-1]}$ has a product-one subsequence of length n, a contradiction. Thus $x = \frac{n+1}{2}$, and again by $\gcd(r_1 - r_2, n) = 1$, there exists $y \in [0, n-1]$ such that $(r_1 - r_2)y \equiv \frac{2n}{3} + r_1 \pmod{n}$. A similar argument shows that $y = \frac{n+1}{2} = x$, and hence $\frac{n}{3} + r_1 \equiv \frac{2n}{3} + r_1 \pmod{n}$, a contradiction. □

Proof of Theorem 1.1.1. We only prove $\mathsf{s}(G) = 2n$ for $n \ge 4$ being even. The remains can be proved by similar argument, or can be found in [1]. Let $n \ge 4$ be even. By Lemma 2.2, it suffices to show that every sequence of length $2n$ has a product-one subsequence of length n. Let $S \in \mathcal{F}(G)$ be a sequence of length $2n$, and assume to the contrary that S has no product-one subsequence of length n.

Let $T \mid S$ be a subsequence of length $2n-1$. Then T has no product-one subsequence of length n. By Lemma 3.2.1, we have $|T_{G_0}| = 1$, and hence $|S \boldsymbol{\cdot} T_{G_0}^{[-1]}| = 2n-1$. Again by Lemma 3.2.1, we have $\big|(S \boldsymbol{\cdot} T_{G_0}^{[-1]})_{G_0}\big| = 1$, which implies that $|S_{G_0}| = 2$. Let $W \mid S$ be a subsequence of length $2n-1$ with $S_{G_0} \mid W$. Then Lemma 3.2.1 ensures that W, and hence S, has a product-one subsequence of length n, a contradiction. □

Proof of Theorem 1.2. 1. (b) ⇒ (a) This is obvious.

(a) ⇒ (b) Let $n \ge 4$ be even. Then $\mathsf{s}(G) = 2n$ by Theorem 1.1.1. Let $S \in \mathcal{F}(G)$ be a sequence of length $2n-1$ such that it has no product-one subsequence of length n. It follows by Lemma 3.1.1 that $|S_{G_0}| = 1$ and $|S_H| = 2n-2$. Therefore Lemma 2.4 ensures that there exist $r_1, r_2, r_3 \in [0, n-1]$ with $\gcd(r_1 - r_2, n) = 1$ such that $S_{G_0} = \alpha^{r_3}\tau$ and $S_H = (\alpha^{r_1})^{[n-1]} \boldsymbol{\cdot} (\alpha^{r_2})^{[n-1]}$.

2. (b) ⇒ (a) This is obvious by Lemma 2.5 and 3.2.

(a)⇒(b) Let $n \ge 3$. Note $\mathsf{E}(G) = 3n$. Let $S \in \mathcal{F}(G)$ be a sequence of length $3n-1$ such that it has no product-one subsequence of length $2n$. Suppose in addition that $S \ne 1_G^{[5]} \boldsymbol{\cdot} \tau \boldsymbol{\cdot} \alpha\tau \boldsymbol{\cdot} \alpha^2\tau$ when $n = 3$. It follows by Lemmas 3.1.2 and 3.2 that $|S_{G_0}| = 1$ and $|S_H| = 3n-2$. Therefore Lemma 2.5 ensures that there exist $r_1, r_2, r_3 \in [0, n-1]$ with $\gcd(r_1 - r_2, n) = 1$ such that $S_{G_0} = \alpha^{r_3}\tau$ and $S_H = (\alpha^{r_1})^{[2n-1]} \boldsymbol{\cdot} (\alpha^{r-2})^{[n-1]}$. $\square$

4. On Dicyclic groups

In this section, we set $n \in \mathbb{N}_{\ge 2}$, and use the following notation for a dicyclic group of order $4n$:

$$G = \langle \alpha, \tau \mid \alpha^{2n} = 1_G,\, \tau^2 = \alpha^n, \text{ and } \tau\alpha = \alpha^{-1}\tau \rangle .$$

Then we set $H = \langle \alpha \rangle$, and $G_0 = G \setminus H$. Note that $\mathsf{d}(G) = 2n$ and $\exp(G) = \operatorname{lcm}(4, 2n)$. Since $G/\{1_G, \alpha^n\}$ is a dihedral group of order $2n$, we denote by

$$\psi : G \to \overline{G} = G/\{1_G, \alpha^n\} .$$

the natural epimorphism, by $\overline{H}$ the cyclic subgroup of $\overline{G}$, and by $\overline{G}_0 = \overline{G} \setminus \overline{H}$.

Likewise, we prove the following technical lemmas by improving the argument from [1], and we will use here results from the dihedral case.

Lemma 4.1. *Let $n \in \mathbb{N}_{\ge 2}$ be even, and $S \in \mathcal{F}(G)$.*

1. *If $|S| = 4n-1$, and S has no product-one subsequence of length $2n$, then either $|S_{G_0}| = 1$, or that both $n = 2$ and S has one of the following forms:*
$$S = (\alpha^{r_1})^{[3]} \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[3]} \boldsymbol{\cdot} \alpha^{r_3} \quad \text{or} \quad S = (\alpha^{r_1})^{[3]} \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[3]} \boldsymbol{\cdot} \alpha^{r_4}\tau ,$$
where $r_1, r_2, r_3, r_4 \in [0, 3]$ such that r_1 is even, r_3 is odd, and $r_3 \not\equiv r_2 \pmod 2$.
2. *If $|S| = 6n-1$, and S has no product-one subsequence of length $4n$, then either $|S_{G_0}| = 1$, or that both $n = 2$ and S has one of the following forms:*
$$S = (\alpha^{r_1})^{[7]} \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[3]} \boldsymbol{\cdot} \alpha^{r_3} \quad \text{or} \quad S = (\alpha^{r_1})^{[3]} \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[7]} \boldsymbol{\cdot} \alpha^{r_4}\tau ,$$
where $r_1, r_2, r_3, r_4 \in [0, 3]$ such that r_1 is even, r_3 is odd, and $r_3 \not\equiv r_2 \pmod 2$.

Proof. 1. It is easy to check that any sequence having asserted structure has no product-one subsequence of length 4. Let $n \ge 2$ be even, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = 4n-1$ such that S has no product-one subsequence of length $2n$. Note that $|S_{G_0}| \ge 1$. By way of contradiction, we may suppose that $|S_{G_0}| \ge 2$, and further suppose that S fail to satisfy the given structure when $n = 2$.

CASE 1. $n \ge 4$.

Then $\psi(S)$ is a sequence of length $4n-1$ over $\overline{G}$ and $|\psi(S)_{\overline{G}_0}| = |S_{G_0}| \ge 2$. Let $g_1, g_2 \in G_0$ such that $g_1 \boldsymbol{\cdot} g_2 \mid S_{G_0}$. Since $|\psi(S \boldsymbol{\cdot} (g_1 \boldsymbol{\cdot} g_2)^{[-1]})| = 4n-3 \ge 2n$, it follows by Theorem 1.1.1 that there exits $T_1 \mid S \boldsymbol{\cdot} (g_1 \boldsymbol{\cdot} g_2)^{[-1]}$ of length n such that $\psi(T_1)$ is a product-one sequence. Since $|\psi(S \boldsymbol{\cdot} (g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} T_1)^{[-1]})| = 3n-3 \ge 2n$, it follows again by Theorem 1.1.1 that there exits $T_2 \mid S \boldsymbol{\cdot} (g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} T_1)^{[-1]}$ of length n such that $\psi(T_2)$ is a product-one sequence. Therefore $|\psi(S \boldsymbol{\cdot} (T_1 \boldsymbol{\cdot} T_2)^{[-1]})| = 2n-1$ and $|\psi(S \boldsymbol{\cdot} (T_1 \boldsymbol{\cdot} T_2)^{[-1]})_{\overline{G}_0}| \ge |\psi(g_1 \boldsymbol{\cdot} g_2)| = 2$. We deduce by Lemma 3.1.1 that there exits $T_3 \mid S \boldsymbol{\cdot} (T_1 \boldsymbol{\cdot} T_2)^{[-1]}$ of length n such that $\psi(T_3)$ is a product-one sequence. Since $\pi(T_i) \cap \{1_G, \alpha^n\} \ne \emptyset$ for all $i \in [1, 3]$, we obtain that there exist distinct $i, j \in [1, 3]$ such that $T_i \boldsymbol{\cdot} T_j$ is a product-one subsequence of S of length $2n$, a contradiction.

CASE 2. $n = 2$.

Let $H_1 = \{1_G, \alpha^2\}$, $H_2 = \{\alpha, \alpha^3\}$, $G_1 = \{\tau, \alpha^2\tau\}$, and $G_2 = \{\alpha\tau, \alpha^3\tau\}$. Suppose that $S_{H_1} = a_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_r$, $S_{H_2} = b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_s$, $S_{G_1} = c_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_t$, and $S_{G_2} = d_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} d_\ell$, where $r, s, t, \ell \in \mathbb{N}_0$. Then $r + s + t + \ell = 7$. We set

$$\begin{aligned} T_1 \;=\; & (a_1a_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (a_{2\lfloor \frac{r}{2} \rfloor -1} a_{2\lfloor \frac{r}{2} \rfloor}) \boldsymbol{\cdot} (b_1b_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (b_{2\lfloor \frac{s}{2} \rfloor -1} b_{2\lfloor \frac{s}{2} \rfloor}) \boldsymbol{\cdot} \\ & (c_1c_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (c_{2\lfloor \frac{t}{2} \rfloor -1} c_{2\lfloor \frac{t}{2} \rfloor}) \boldsymbol{\cdot} (d_1d_2) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} (d_{2\lfloor \frac{\ell}{2} \rfloor -1} d_{2\lfloor \frac{\ell}{2} \rfloor}) . \end{aligned}$$

Then $T_1 \in \mathcal{F}(H_1)$ of length $|T_1| = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor \ge 2$. Since S has no product-one subsequence of length 4, we obtain that T_1 has no product-one subsequence of length 2. Since $|H_1| = 2$, we obtain that $|T_1| = 2$, and hence three elements of $\{r, s, t, \ell\}$ are odd.

If $\{r, s, t, \ell\} \subset \mathbb{N}$, then $a_1b_1,\, c_1d_1 \in H_2$. Since $|H_2| = 2$, we obtain that either $a_1b_1 = c_1d_1$ or $a_1b_1 = (c_1d_1)^{-1}$. It follows that either $c_1 \cdot b_1 \cdot a_1 \cdot d_1$ or $a_1 \cdot b_1 \cdot c_1 \cdot d_1$ is a product-one subsequence of S of length 4, a contradiction. Thus one element of $\{r, s, t, \ell\}$ must be zero, and we distinguish four cases.

Suppose that $r = 0$. Then all s, t, ℓ must be odd with $s+t+\ell = 7$. Since $|H_1| = |H_2| = |G_1| = |G_2| = 2$, we obtain that there exist subsequences W_1, W_2 of S such that $W_1 = x^{[2]}$ and $W_2 = y^{[2]}$ for some $x, y \in G$. If $x \in G_0$ or $y \in G_0$, then $W_1 \cdot W_2$ is a product-one subsequence of S of length 4, a contradiction. If $x, y \in H_2$, then $x = y$ or $x = y^{-1}$, which implies that $W_1 \cdot W_2$ is a product-one subsequence of S of length 4, a contradiction.

Suppose that $s = 0$. Then all r, t, ℓ must be odd with $r+t+\ell = 7$. Since $|H_1| = |H_2| = |G_1| = |G_2| = 2$, we obtain that there exist subsequences W_1, W_2 of S such that $W_1 = x^{[2]}$ and $W_2 = y^{[2]}$ for some $x, y \in G$. If $x, y \in H_1$ or $x, y \in G_0$, then $W_1 \cdot W_2$ is a product-one subsequence of S of length 4, a contradiction. Thus we may assume that $x \in H_1$ and $y \in G_0$, whence $r = 3$ and $t + \ell = 4$. Thus either $S = \big(a_1^{[2]} \cdot a_2\big) \cdot \big(c_1^{[2]} \cdot c_2\big) \cdot d_1$ or $S = \big(a_1^{[2]} \cdot a_2\big) \cdot c_1 \cdot \big(d_1^{[2]} \cdot d_2\big)$. In the former case, if $a_1 \ne a_2$ (or $c_1 \ne c_2$ respectively), then $a_1 \cdot a_2 \cdot c_1^{[2]}$ (or $a_1^{[2]} \cdot c_1 \cdot c_2$ respectively) is a product-one subsequence of S of length 4, a contradiction, whence $S = a_1^{[3]} \cdot c_1^{[3]} \cdot d_1$, a contradiction to our assumption. In the latter case, we similarly obtain $S = a_1^{[3]} \cdot c_1 \cdot d_1^{[3]}$, again a contradiction.

Suppose that $t = 0$. Then all r, s, ℓ must be odd with $\ell \ge 3$ and $r + s + \ell = 7$. Since $|H_1| = |H_2| = |G_1| = |G_2| = 2$, we obtain that there exist subsequences W_1, W_2 of S such that $W_1 = x^{[2]}$ and $W_2 = y^{[2]}$, where $x \in G$ and $y \in G_2$. If $x \in G_2$, then $W_1 \cdot W_2$ is a product-one subsequence of S of length 4, a contradiction. Thus we must have $\ell = 3$ and $r + s = 4$. If $s = 3$, then $x \in H_2$, and hence $W_1 \cdot W_2$ is a product-one subsequence of S of length 4, a contradiction. Hence we must have $r = 3$ and $s = 1$, and thus after renumbering if necessary, we have $S = \big(a_1^{[2]} \cdot a_2\big) \cdot b_1 \cdot \big(d_1^{[2]} \cdot d_2\big)$. If $a_1 \ne a_2$ (or $d_1 \ne d_2$ respectively), then $a_1 \cdot a_2 \cdot d_1 \cdot d_1$ (or $a_1 \cdot a_1 \cdot d_1 \cdot d_2$ respectively) is a product-one subsequence of S of length 4, a contradiction. Therefore $S = a_1^{[3]} \cdot b_1 \cdot d_1^{[3]}$, a contradiction to our assumption.

Suppose that $\ell = 0$. A similar argument as used in the case $t = 0$ shows $S = a_1^{[3]} \cdot b_1 \cdot c_1^{[3]}$, a contradiction.

2. It is easy to check that any sequence having asserted structure has no product-one subsequence of length 8. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 6n - 1$ such that S has no product-one subsequence of length $4n$. By way of contradiction, we may suppose that $|S_{G_0}| \ge 2$, and further suppose that S fail to satisfy the given structure when $n = 2$.

Let $g_1, g_2 \in G_0$ such that $g_1 \cdot g_2 \,|\, S_{G_0}$. Since $|S \cdot (g_1 \cdot g_2)^{[-1]}| = 6n - 3 \ge 4n$, it follows by Theorem 1.1.2 that $S \cdot (g_1 \cdot g_2)^{[-1]}$ has a product-one subsequence T_1 of length $2n$. Then $|S \cdot T_1^{[-1]}| = 4n - 1$ and $|(S \cdot T_1^{[-1]})_{G_0}| \ge |g_1 \cdot g_2| = 2$.

If $n \ge 4$, then we infer by 1. that $S \cdot T_1^{[-1]}$ has a product-one subsequence T_2 of length $2n$. Therefore $T_1 \cdot T_2$ is a product-one subsequnce of S of length $4n$, a contradiction. Thus $n = 2$, and it follows again by 1. that

$$S \cdot T_1^{[-1]} = (\alpha^{r_1})^{[3]} \cdot (\alpha^{r_2}\tau)^{[3]} \cdot x\,,$$

where $r_1, r_2 \in [0, 3]$ such that r_1 is even and $x = \alpha$, or α^3, or $\alpha^{r_3}\tau$ with $r_3 \not\equiv r_2 \pmod 2$. If $\alpha^{r_1} \,|\, T_1$, then $(\alpha^{r_1})^{[4]} \,|\, S$, and we obtain by 1. that $S \cdot \big((\alpha^{r_1})^{[4]}\big)^{[-1]} = x \cdot (\alpha^{r'})^{[3]} \cdot (\alpha^{r_2}\tau)^{[3]}$, where $r' \in [0, 3]$ is even. Thus $T_1 = \alpha^{r_1} \cdot (\alpha^{r'})^{[3]}$ is a product-one sequence, which implies that $r' = r_1$ and $S = x \cdot (\alpha^{r_1})^{[7]} \cdot (\alpha^{r_2}\tau)^{[3]}$, a contradiction. If $\alpha^{r_2}\tau \,|\, T_1$, then a similar argument shows that $S = x \cdot (\alpha^{r_1})^{[3]} \cdot (\alpha^{r_2}\tau)^{[7]}$, a contradiction. Then $\mathrm{supp}(T_1) \cap \{\alpha^{r_1}, \alpha^{r_2}\tau\} = \emptyset$. Let $g \,|\, T_1$. Then $S \cdot (\alpha^{r_2}\tau \cdot x \cdot g)^{[-1]}$ has length 8, and by Theorem 1.1.2, it has a product-one subsequence T_2 of length 4. Since $T_2 \ne (\alpha^{r_1})^{[4]}$ and $S \cdot T_2^{[-1]}$ has no product-one subsequence of length 4, it follows by 1. that $\alpha^{r_1} \cdot \alpha^{r_2}\tau \cdot x \cdot g \mid S \cdot T_2^{[-1]}$, and hence

$$S \cdot T_2^{[-1]} = (\alpha^{r_1})^{[3]} \cdot \alpha^{r_2}\tau \cdot x^{[3]}, \text{ where } g = x = \alpha^{r_3}\tau \text{ for some } r_3 \in [0,3] \text{ with } r_3 \not\equiv r_2 \pmod 2.$$

Then $T_1 = x^{[4]}$, and thus $S = (\alpha^{r_1})^{[3]} \cdot (\alpha^{r_2}\tau)^{[3]} \cdot x^{[5]}$. Since $(\alpha^{r_1} \cdot x \cdot \alpha^{r_1} \cdot x) \cdot (x \cdot \alpha^{r_2}\tau \cdot x \cdot \alpha^{r_2}\tau)$ is a product-one sequence of length 8, we get a contradiction. $\square$

Lemma 4.2. *Let $n \in \mathbb{N}_{\ge 3}$ be odd, and $S \in \mathcal{F}(G)$ a sequence of length $|S| = 6n-1$. If S has no product-one subsequence of length $4n$, then $|S_{G_0}| = 1$.*

Proof. Let $S \in \mathcal{F}(G)$ be a sequence of length $6n-1$ such that it has no product-one subsequence of length $4n$. Note that $|S_{G_0}| \ge 1$. Assume to the contrary that $|S_{G_0}| \ge 2$.

CASE 1. $|S_{G_0}| \le n$.

Then $|S_H| \ge 5n-1$, and by mapping under ψ, we obtain a sequence $\psi(S_H) \in \mathcal{F}(\overline{H})$ of length at least $5n-1$. Since $5n-1 > 2n-1 = \mathsf{s}(\overline{H})$, we obtain that there exists $T_1 \mid S_H$ of length n such that $\psi(T_1)$ is a product-one sequence over $\overline{H}$, whence $\pi^*(T_1) \in \{1_G, \alpha^n\}$. Continuing this process, we obtain that there exist subsequences T_2, T_3, T_4 of S_H such that $|T_2| = |T_3| = |T_4| = n$ and $\pi^*(T_i) \in \{1_G, \alpha^n\}$ for all $i \in [2,4]$. Since S has no product-one subsequence of length $4n$, we may assume that $\pi^*(T_1) \ne \pi^*(T_2) = \pi^*(T_3) = \pi^*(T_4)$. Consider the sequence $\psi\big(S \cdot (T_1 \cdot T_2 \cdot T_3)^{[-1]}\big) \in \mathcal{F}(\overline{G})$ of length $3n-1$.

Suppose that $n \ge 5$. Since $|S_{G_0}| \ge 2$, it follows by Lemma 3.2 that there exists an ordered subsequence $T_5 \mid S \cdot (T_1 \cdot T_2 \cdot T_3)^{[-1]}$ of length $2n$ such that $\pi^*(T_5) \in \{1_G, \alpha^n\}$. It follows that either $T_1 \cdot T_3 \cdot T_5$ or $T_2 \cdot T_3 \cdot T_5$ is a product-one subsequence of S of length $4n$, a contradiction.

Suppose that $n = 3$. Since $|S_{G_0}| \ge 2$, Lemma 3.2 ensures that

$$S = T_1 \cdot T_2 \cdot T_3 \cdot \big(\alpha^{r_1} \cdot \alpha^{r_2} \cdot \alpha^{r_3} \cdot \alpha^{r_4} \cdot \alpha^{r_5}\big) \cdot \Big(\alpha^{r_6}\tau \cdot \alpha^{r_7}\tau \cdot \alpha^{r_8}\tau\Big),$$

where $r_1, \ldots, r_6 \in \{0,3\}$, $r_7 \in \{1,4\}$, and $r_8 \in \{2,5\}$. Since there exit distinct $i,j \in [6,8]$ such that $r_i \equiv r_j \pmod 2$, we obtain $\{\alpha, \alpha^5\} = \big\{\alpha^{r_i}\tau\alpha^{r_j}\tau,\, \alpha^{r_j}\tau\alpha^{r_i}\tau\big\} \subset \Pi_2(\alpha^{r_6}\tau \cdot \alpha^{r_7}\tau \cdot \alpha^{r_8}\tau)$, and assert that $S \cdot (T_2 \cdot T_3)^{[-1]}$ has a product-one subsequenc of length 6, leading a contradiction to the fact that S has no product-one subsequence of length 12.

If $\pi^*(T_1) \in \Pi_3(\alpha^{r_1} \cdot \ldots \cdot \alpha^{r_5})$, then $T_1 \cdot \alpha^{r_1} \cdot \ldots \cdot \alpha^{r_5}$ has a product-one subsequence of length 6. Suppose that $|\Pi_3(\alpha^{r_1} \cdot \ldots \cdot \alpha^{r_5})| = 1$ and $\pi^*(T_1) \notin \Pi_3(\alpha^{r_1} \cdot \ldots \cdot \alpha^{r_5})$. Then $r_1 = \ldots = r_5$ and $\pi^*(T_1) \ne \alpha^{r_1}$, whence $\pi^*(T_1 \cdot \alpha^{r_1}) = \alpha^3$. If $\{1_G, \alpha^3\} \subset \Pi_2(T_1 \cdot \alpha^{r_1})$, then together with $\alpha^{r_2} \cdot \ldots \cdot \alpha^{r_5}$ we obtain a product-one subsequence of length 6. If $\{\alpha, \alpha^2\} \subset \Pi_2(T_1 \cdot \alpha^{r_1})$ or $\{\alpha^4, \alpha^5\} \subset \Pi_2(T_1 \cdot \alpha^{r_1})$, then $(T_1 \cdot \alpha^{r_1}) \cdot \alpha^{r_6}\tau \cdot \alpha^{r_7}\tau \cdot \alpha^{r_8}\tau$ has a product-one subsequence of length 4. Together with $\alpha^{r_2} \cdot \alpha^{r_3}$ we obtain a product-one subsequence of length 6.

CASE 2. $|S_{G_0}| \ge n+1$.

By renumbering if necessary, we can assume that

$$S_{G_0} = (a_1 \cdot a_1^{-1}) \cdot \ldots \cdot (a_r \cdot a_r^{-1}) \cdot \big(W_1 \cdot \ldots W_x\big) \cdot (c_1 \cdot \ldots \cdot c_u),$$

where W_i is a product-one sequence of length 4 for all $i \in [1,x]$, and $c_1 \cdot \ldots \cdot c_u$ has no product-one subsequence of length 2 and 4. Then $\mathsf{h}(c_1 \cdot \ldots \cdot c_u) \le 3$, and thus $u - 2 \le |\operatorname{supp}(c_1 \cdot \ldots \cdot c_u)| \le n$, whence $u \le n+2$. Similarly we set

$$S_H = (e_1 \cdot e_1^{-1}) \cdot \ldots \cdot (e_s \cdot e_s^{-1}) \cdot \big(b_1^{[2]} \cdot \ldots \cdot b_t^{[2]}\big) \cdot (d_1 \cdot \ldots \cdot d_v),$$

where $d_i \notin \{d_j, d_j^{-1}\}$ for all distinct $i,j \in [1,v]$, and $\operatorname{ord}(b_k) \ne 2$ for all $k \in [1,t]$. Then $d_1 \cdot \ldots \cdot d_v$ is square-free, and hence $v \le n+1$. Let $R = c_1 \cdot \ldots \cdot c_u \cdot d_1 \cdot \ldots \cdot d_v$.

SUBCASE 2.1. $r + s + t = 0$.

If $\mathsf{h}(R_{G_0}) = 1$, then $u \le n$, and hence $4x \ge (6n-1) - u - v \ge 4n-2$. It follows that $x \ge n$, whence $W_1 \cdot \ldots W_n$ is a product-one subsequence of S of length $4n$, a contradiction. Thus we must have $\mathsf{h}(R_{G_0}) \ge 2$, and by renumbering if necessary, we may assume that $c_1 = c_2$. Note that $4x \ge 4n-4$. If

$v \ge \frac{n+1}{2} + 1$, then by mapping under ψ, we infer that there exist $i, j \in [1, v]$ such that $\pi^*(d_i d_j) = \alpha^n$, whence $c_1^{[2]} \boldsymbol{\cdot} d_i \boldsymbol{\cdot} d_j$ is a product-one sequence of length 4. Thus $W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{n-1} \boldsymbol{\cdot} c_1^{[2]} \boldsymbol{\cdot} d_i \boldsymbol{\cdot} d_j$ is a product-one subsequence of S of length $4n$, a contradiction. Thus we must have $v \le \frac{n+1}{2}$.

Suppose that $n \ge 5$. Then $u + v \le \frac{3n+5}{2} \le 2n$, and since $u + v$ is odd, it follows that $u + v \le 2n - 1$. Hence $4x \ge 4n$, and therefore $W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_n$ is a product-one subsequence of S of length $4n$, a contradiction.

Suppose that $n = 3$. Then $u \le 5$ and $v \le 2$. If $u + v \le 5$, then $4x \ge 12$, whence S has a product-one subsequence of length 12, a contradiction. Thus we must have that $u = 5$ and $v = 2$, and then $4x = (6n - 1) - u - v = 10$, a contradiction.

SUBCASE 2.2. $r + s + t \ge 1$.

Suppose that $u \le n$. If $2r + 4x + 2s + 2t \ge 4n$, then S must have a product-one subsequence of length $4n$. We assume that $2r + 4x + 2s + 2t \le 4n - 2$. Then $2n + 1 \ge u + v \ge (6n - 1) - (4n - 2) = 2n + 1$, and hence $u = n$ and $v = n + 1$. Since R_H is square-free, $|\Pi_2(R_H)| \ge v - 1 = n \ge 3$, and by renumbering if necessary, we may assume that $d_1 d_2 = \alpha^n$ and $d_3 d_4 \notin \{1_G, \alpha^n\}$. Since $|R_{G_0} \boldsymbol{\cdot} (d_3 d_4)| = n + 1$, it follows by mapping under ψ that there exists an ordered subsequence T of $R_{G_0} \boldsymbol{\cdot} d_3 \boldsymbol{\cdot} d_4$ having even length such that $\pi^*(T) \in \{1_G, \alpha^n\}$. Since R_{G_0} has no product-one subsequence of length 2, we obtain that either T or $T \boldsymbol{\cdot} d_1 \boldsymbol{\cdot} d_2$ is a product-one subsequence of S having even length at least 4. Since $2r + 4x + 2s + 2t = 4n - 2$ and $r + s + t \ge 1$, it follows that S has a product-one subsequence of length $4n$, a contradiction.

Suppose that $u \ge n + 1$. Then $\mathsf{h}(R_{G_0}) \ge 2$, and by renumbering if necessary, we may assume that $c_1 = c_2$. If $t = 0$ and $v \le 1$, then $2r + 2s + 4x \ge 5n - 3 \ge 4n - 1$, and hence S must have a product-one subsequence of length $4n$, a contradiction. Thus either $t \ge 1$ or $v \ge 2$. By mapping under ψ, [2, Theorem 1.3] ensures that there exists an ordered sequence T of $c_3 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u \boldsymbol{\cdot} d_1 \boldsymbol{\cdot} d_2$ (or $c_3 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} c_u \boldsymbol{\cdot} b_1^{[2]}$ if $v \le 1$) having even length such that $\pi^*(T) \in \{1_G, \alpha^n\}$. Since R_{G_0} has no product-one subsequence of length 2, we obtain that either T or $T \boldsymbol{\cdot} c_1^{[2]}$ is a product-one subsequence of S having even length at least 4. Since $2r + 4x + 2s + 2t \ge 4n - 4$ (or $2r + 4x + 2s + 2(t - 1) \ge 5n - 6 \ge 4n - 4$ if $v \le 1$) and $r + s + t \ge 1$, it follows that S has a product-one subsequence of length $4n$, a contradiction. $\square$

Proof of Theorem 1.1.2. we only prove $\mathsf{s}(G) = 4n$ for $n \ge 2$ being even. The remains can be proved by similar argument, or can be found in [1]. Let $n \ge 2$ be even. By Lemma 2.2, it suffices to show that every sequence of length $4n$ has a product-one subsequence of length $2n$. Let $S \in \mathcal{F}(G)$ be a sequence of length $4n$, and assume to the contrary that S has no product-one subsequence of length $2n$.

Let $T \mid S$ be a subsequence of length $4n - 1$. Then T has no product-one subsequence of length $2n$. By Lemma 4.1.1, we have that either $|T_{G_0}| = 1$, or else $n = 2$ and T satisfy the stated structure.

Suppose that $n \ge 4$, and then $|T_{G_0}| = 1$. Hence $|S \boldsymbol{\cdot} T_{G_0}^{[-1]}| = 4n - 1$, and again by Lemma 4.1.1, we have $\big|(S \boldsymbol{\cdot} T_{G_0}^{[-1]})_{G_0}\big| = 1$, which implies that $|S_{G_0}| = 2$. Let $W \mid S$ be a subsequence of length $4n - 1$ with $S_{G_0} \mid W$. Then Lemma 4.1.1 ensures that W, and hence S, has a product-one subsequence of length $2n$, a contradiction.

Suppose that $n = 2$. If $|T_{G_0}| = 1$, then either $|S_{G_0}| = 2$ or $|S_{G_0}| = 1$. In the former case, the argument from the case $n \ge 4$ leads to a contradiction. In the letter case, we have $|S_H| = 4n - 1 = 7$, and it follows by $\mathsf{s}(H) = 4n - 1 = 7$ that S_H has a product-one subsequence of length 4, a contradiction. Thus T must have the form given in Lemma 4.1.1, and hence

$$S \,=\, g \boldsymbol{\cdot} (\alpha^{r_1})^{[3]} \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[3]} \boldsymbol{\cdot} x\,,$$

where $g \in G$, $r_1, r_2 \in [0, 3]$ such that r_1 is even, and $x \in \{\alpha, \alpha^3, \alpha^{r_3}\tau\}$ with $r_3 \not\equiv r_2 \pmod 2$. Proceeding the same argument on $S \boldsymbol{\cdot} x^{[-1]}$, we obtain that $g \in \{\alpha, \alpha^3, \alpha^{r_4}\tau\}$ with $r_4 \not\equiv r_2 \pmod 2$. If $g, x \in H$ or $g, x \in G_0$, then either $g \boldsymbol{\cdot} x \boldsymbol{\cdot} (\alpha^{r_1})^{[2]}$ or $g \boldsymbol{\cdot} x \boldsymbol{\cdot} (\alpha^{r_2}\tau)^{[2]}$ is a product-one subsequence of S of length 4, a contradiction. Thus we may assume without loss of generality that $g = \alpha^{r_5}$ and $x = \alpha^{r_3}\tau$, where $r_3, r_5 \in [0, 3]$ such that $r_3 \not\equiv r_2 \pmod 2$ and r_5 is odd. Then $r_5 + r_3 \equiv r_2 \pmod 2$, and hence $\alpha^{r_1} \boldsymbol{\cdot} g \boldsymbol{\cdot} \alpha^{r_2}\tau \boldsymbol{\cdot} x$ or $\alpha^{r_1} \boldsymbol{\cdot} \alpha^{r_2}\tau \boldsymbol{\cdot} x \boldsymbol{\cdot} g$ is a product-one subsequence of length 4, a contradiction. $\square$

Proof of Theorem 1.3. 1. (b) $\Rightarrow$ (a) This is obvious with Lemma 4.1.1.

(a) $\Rightarrow$ (b) Let $n \ge 2$ be even. Then $\mathsf{s}(G) = 4n$ by Theorem 1.1.2. Let $S \in \mathcal{F}(G)$ be a sequence of length $4n-1$ such that it has no product-one subsequence of length n. Suppose in addition that S fail to satisfy the structure (2). It follows by Lemma 4.1.1 that $|S_{G_0}| = 1$ and $|S_H| = 4n-2$. Therefore Lemma 2.5 ensures that there exist $r_1, r_2, r_3 \in [0, 2n-1]$ with $\gcd(r_1 - r_2, 2n) = 1$ such that $S_{G_0} = \alpha^{r_3}\tau$ and $S_H = (\alpha^{r_1})^{[2n-1]} \boldsymbol{\cdot} (\alpha^{r_2})^{[2n-1]}$.

2. (b) $\Rightarrow$ (a) This is obvious by Lemma 2.5 and 4.1.2.

(a) $\Rightarrow$ (b) Let $n \ge 2$. Note $\mathsf{E}(G) = 6n$. Let $S \in \mathcal{F}(G)$ be a sequence of length $6n-1$ such that it has no product-one subsequence of length $4n$. Suppose in addition that S fail to satisfy the structure (2). It follows by Lemma 4.1.2 and 4.2 that $|S_{G_0}| = 1$ and $|S_H| = 6n-2$. Therefore Lemma 2.5 ensures that there exist $r_1, r_2, r_3 \in [0, 2n-1]$ with $\gcd(r_1 - r_2, 2n) = 1$ such that $S_{G_0} = \alpha^{r_3}\tau$ and $S_H = (\alpha^{r_1})^{[4n-1]} \boldsymbol{\cdot} (\alpha^{r_2})^{[2n-1]}$. $\square$

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Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstrasse 36, 8010 Graz, Austria

E-mail address: `junseok.oh@uni-graz.at, qinghai.zhong@uni-graz.at`

URL: `https://imsc.uni-graz.at/zhong/`