Math 1ZA3
Engineering Mathematics

Fabian Bleitner

bleitnef@mcmaster.ca

Introduction

Platforms

- Resources, schedule, assignments, how to study, ... childsmath → more announcements
  - On phones the content is a bit hidden → Menu
- Lecture/tutorial times and places, self-report MSAF Mosaic
- Lecture slides, discussion chat Teams
- Lecture recordings avenue to learn → echo360

Assessments

- Assignments
  - Online on childsmath
  - See childsmath for information, deadlines, ...
- Tests:
  - In Person
  - Dates: Look at childsmath
  - Start at or after 7pm
  - If you are unable to attend
    - Early alternate on the same day: There will be a form
    - Not able the whole day: Use MSAF
  - All questions will be multiple choice
- Final Exam
  - organized via Registrar's office
  - see Mosaic
  - All questions will be multiple choice

Preparation and Help

To prepare

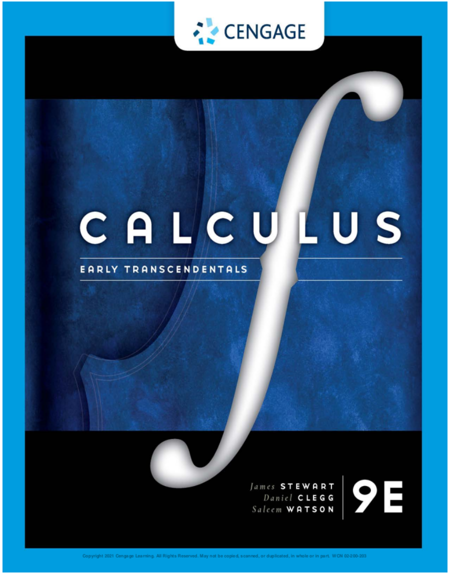
- Read everything on childsmath
- Do Assignment 0 on childsmath
- Prepare by doing the Pre-Calculus Review and Calculus Warm-Up worksheets on childsmath

Further help

- Tutorials start on Monday September 8th
- Math Help Centre starts on Monday September 8th
- My office hours: Mo, Th 2:30-3:30 in Hamilton Hall 414

Book

- Lectures will be based on this book
- Suggested problems refer to this book
- You can just read the relevant book chapters but I strongly advise you to come to class
- Try to get access to the book



Lectures

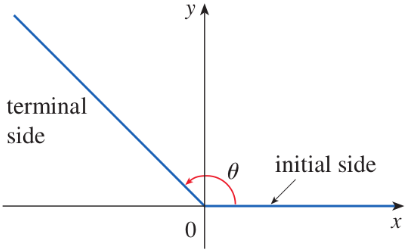
- Typical lecture
  - Introduce *new* concept
  - Calculate examples
- Before the lecture download the blank slides
- Follow along filling the gaps and calculating the examples
  - Ask questions if you don't understand it
  - Say if you can not read my handwriting or I mumble too much
- After the lecture I will upload my annotated slides

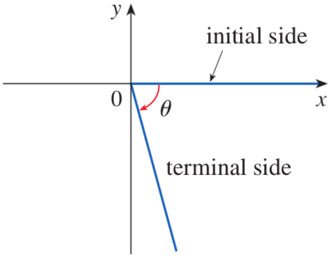
- In the beginning the content will be a repetition for most
  - adjust to pace
  - revise algebra
  - some concepts will be new
- At some point we will learn new things and the difficulty will increase
- I will try to be a bit faster in the beginning leaving time for review sessions before the midterms and to have a bit more time for the later content

Appendix D - Trigonometry

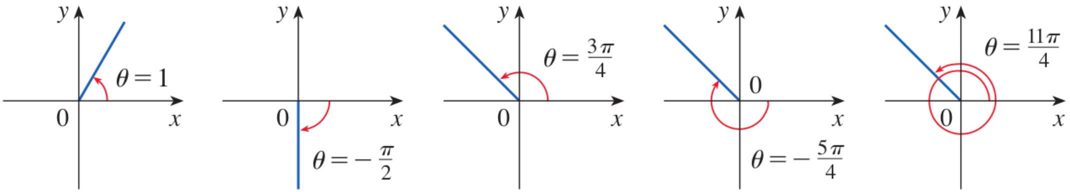
Trigonometry

- We always use radians without saying rad: $2\pi = 2\pi\,\text{rad} = 360°$
- Positive angles counterclockwise, negative angles clockwise





Examples



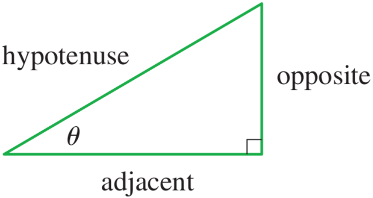
Trigonometric Functions

For **acute** angles

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

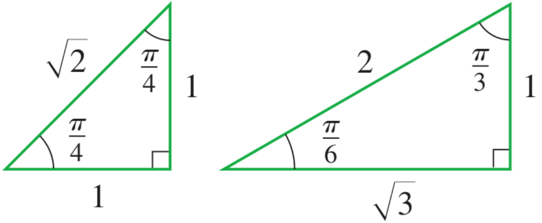


called sine, cosine, tangent, cosecant, secant and cotangent

Trigonometric Functions

Examples

- Calculate $\cos\frac{\pi}{4}$
- Calculate $\cot\frac{\pi}{3}$



Important Trigonometric Values

$$\cos\theta = \frac{\mathsf{adj}}{\mathsf{hyp}}$$

$$\cot\theta = \frac{\mathsf{adj}}{\mathsf{opp}}$$

$\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$

Trigonometric Functions

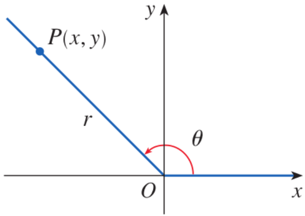
For **general** angles

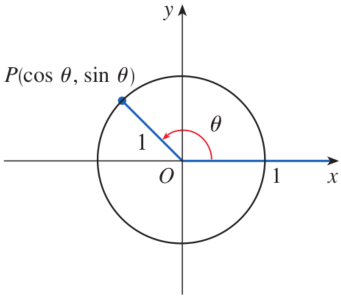
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

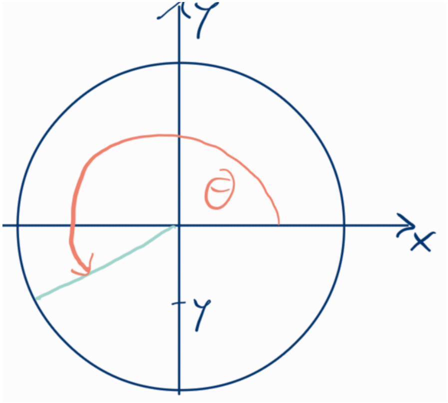
called sine, cosine, tangent, cosecant, secant and cotangent

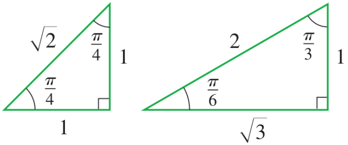




Trigonometric Functions

Example: Calculate $\sin \frac{7\pi}{6}$





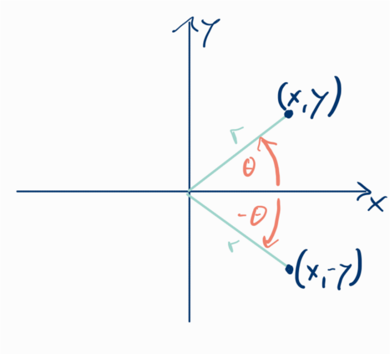
$-\frac{1}{2}$

Important Trigonometric Identities

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

since

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$



Important Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$

since

Important Trigonometric Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

More Trigonometric Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

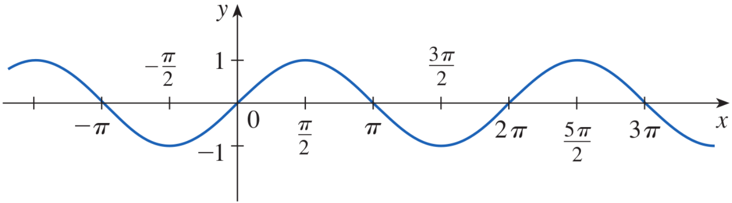
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

and many many more

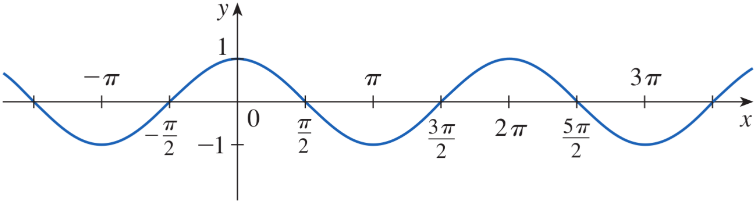
Important Trigonometric Identities

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$



(a) $f(x) = \sin x$



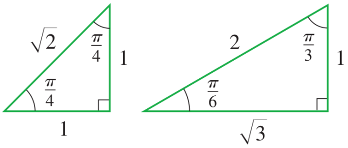
(b) $g(x) = \cos x$

Trigonometric Identities

Example

- Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$

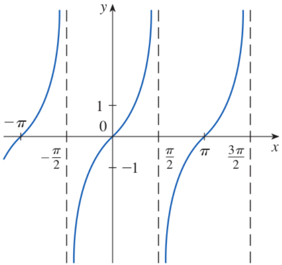
$$\sin 2x = 2 \sin x \cos x$$



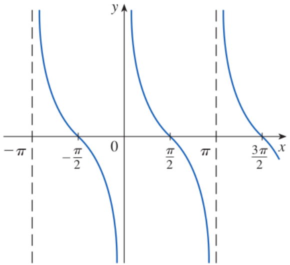
Trigonometric Identities

$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

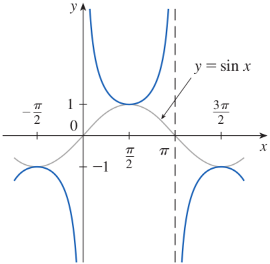
Plot of Functions



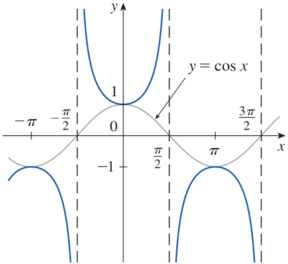
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$



(d) $y = \sec x$

Trigonometric Identities

Example: Proof $1 + \cot^2(x) = \csc^2(x)$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\cot(x) = \frac{1}{\tan(x)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

Section 1.5 - Inverse Functions

Inverse Functions - Motivation

You grow bacteria in a lab and measure the population

t (hours)	$N = f(t)$ $=$ population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

N	$t = f^{-1}(N)$ $=$ time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

Similar for the acceleration of a car.

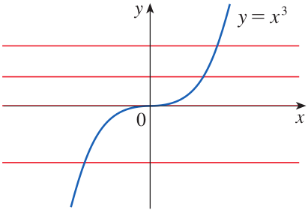
Inverse Functions - Preparation

Definition

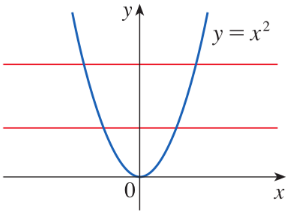
A function is *one-to-one* if it never takes on the same value twice, i.e. $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

Examples

$f(x) = x^3$ is one-to-one.



$f(x) = x^2$ is **not** one-to-one.



$$f(-1) = (-1)^2 = 1 = (1)^2 = f(1)$$

$\rightarrow$ horizontal line test

Inverse Functions - Defintion

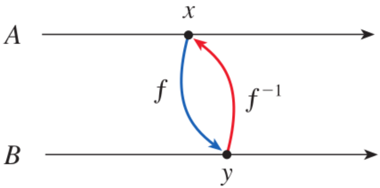
Definition

Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

The -1 is not an exponent, $f^{-1} \neq \dfrac{1}{f}$



domain of f = range of f^{-1}

range of f = domain of f^{-1}

Inverse Functions - Example

If f is a one-to-one function with

$$\begin{aligned} f(1) &= 5, \\ f(8) &= -10, \\ f(-4) &= 3, \\ f(3) &= 7 \end{aligned}$$

then

$$\begin{aligned} f^{-1}(7) &= \\ f^{-1}(5) &= \\ f^{-1}(-10) &= \end{aligned}$$

Inverse Functions

From the definition, i.e.

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

it follows that

$$f^{-1}(f(x)) = f^{-1}(y) = x.$$

Usually x is an independent variable, so we swap the variables

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

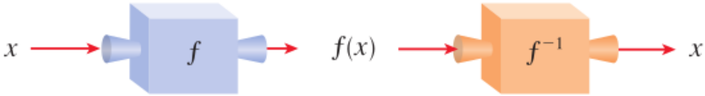
and plugging the expressions into each other as before

$$f(f^{-1}(x)) = f(y) = x$$

Inverse Functions

$$f^{-1}(f(x)) = x$$
for all x in A, the domain of f.

$$f(f^{-1}(x)) = x$$
for all x in B, the range of f.



Calculating Inverse Functions

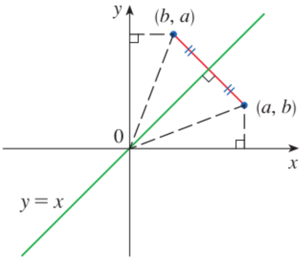
1. Write the function as $y = f(x)$
2. Solve for x
3. Swap x and y to get $f^{-1}(x)$

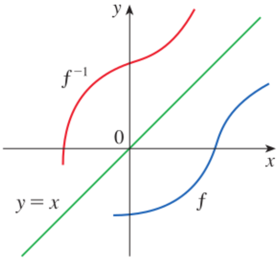
Example: Find the inverse function of $f(x) = x^3 + 2$:

$\sqrt[3]{x-2}$

Graphs of Inverse Functions

The graph of f^{-1} is obtained by reflecting the graph of f about the $y = x$ line.





Section 1.5 - Logarithms

Logarithm - Definition

For $b > 0$ and $b \neq 1$ the function $f(x) = b^x$ is either increasing or decreasing and so a one-to-one function. Therefore its inverse f^{-1} exists and it is called the logarithmic function with base b.

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$

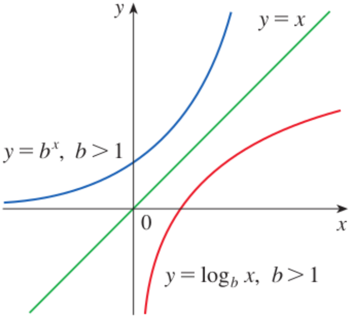
and the previous cancellation equations yield

$$b^{\log_b x} = x \quad \text{for all } x > 0$$

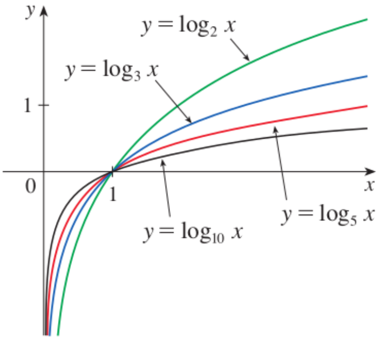
and

$$\log_b (b^x) = x \quad \text{for all } x \in \mathbb{R}$$

Logarithm - Graphs



Reflecting the exponential function.



Logarithmic functions for different bases.

Logarithm - Important Identities

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b\left(x^r\right) = r\log_b(x) \qquad \text{for all } r \in \mathbb{R}$$

Example: Calculate $\log_2(80) - \log_2(5)$

4

Natural Logarithm

Last time

$$\log_b x = y \Leftrightarrow b^y = x$$

Natural Logarithm

$$\ln x = \log_e x$$

so

$$\ln x = y \Leftrightarrow e^y = x$$

As earlier

$$e^{\ln x} = x \quad x \in \mathbb{R}$$
$$\ln(e^x) = x \quad x > 0$$

Graph of e^x and $\ln x$

Natural Logarithm - Identities

$$\ln 1 = 0$$

since

$$x^r = e^{r \ln x}$$

since

Natural Logarithm - Exercises

- Solve $\ln x = 5$ for x

- Solve $e^{5-3x} = 10$ for x

e^5, $\frac{1}{3}(5 - \ln 10)$

Natural Logarithm - Exercises

- Expand $\ln \frac{x^2\sqrt{x^2+2}}{3x+1}$ for $x > 0$

- Express $\ln a + \frac{1}{2} \ln b$ as a single logarithm for $a, b > 0$

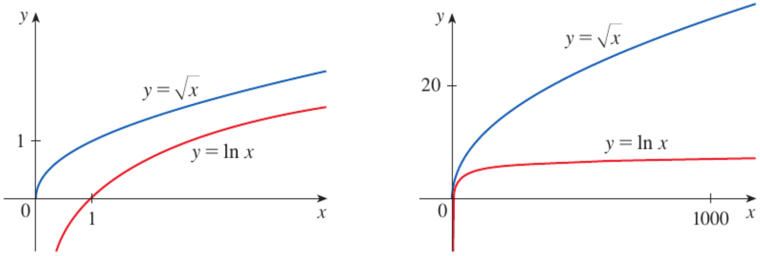
$2 \ln x + \frac{1}{2} \ln(x^2 + 2) - \ln(3x + 1)$, $\ln(a\sqrt{b})$

Change of Logarithm Formula

$$\log_b x = \frac{\ln x}{\ln b}$$

since

Logarithm - Asymptotic Behaviour



Section 1.5 - Inverse Trigonometric Functions

Definitions

Recall: For a one-to-one function f

$$f^{-1}(x) = y \qquad \Leftrightarrow \qquad f(y) = x$$

but $\sin x$ is not one-to-one. So we restrict ourselves to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

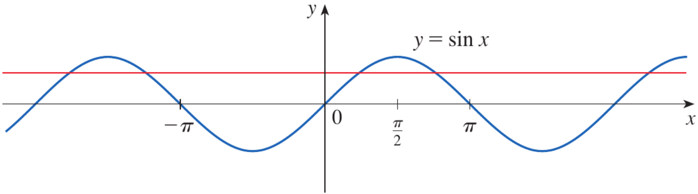


FIGURE 17

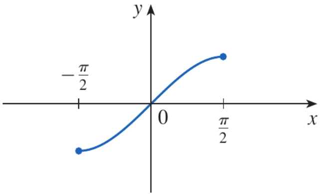


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Definition

$$\sin^{-1} x = y \qquad \Leftrightarrow \qquad \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$\sin^{-1}$ or $\arcsin$ is called the *inverse sine function* or *arcsine function*.

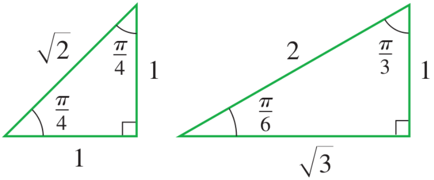
Properties

$$\sin^{-1}(\sin(x)) = x \qquad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\sin^{-1}(x)) = x \qquad \text{for } -1 \leq x \leq 1$$

- The domain of $\sin^{-1}$ is $[-1, 1]$
- The range of $\sin^{-1}$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Example

- Evaluate $\sin^{-1}(\frac{1}{2})$



$\frac{\pi}{6}$

More Definitions

Definition

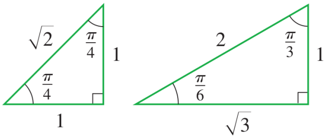
$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \text{ and } 0 \leq y \leq \pi$$
$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The following ones are not really used and there is no universal definition.

$$y = \csc^{-1} x \ (|x| \geqslant 1) \quad \Longleftrightarrow \quad \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
$$y = \sec^{-1} x \ (|x| \geqslant 1) \quad \Longleftrightarrow \quad \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1} x \ (x \in \mathbb{R}) \quad \Longleftrightarrow \quad \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

Examples

- Evaluate $\tan(\arcsin(\frac{1}{3}))$ geometrically



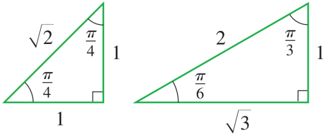
$$\sin^2 x + \cos^2 x = 1$$

- Evaluate $\tan(\arcsin(\frac{1}{3}))$ analytically

$\frac{1}{2\sqrt{2}}$

Examples

- Evaluate $\cos(\tan^{-1}(x))$ geometrically



$$1 + \tan^2 x = \sec^2 x$$

$$\sec x = \frac{1}{\cos x}$$

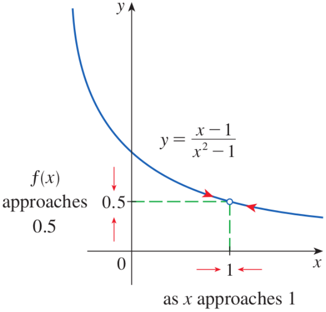
- Evaluate $\cos(\tan^{-1}(x))$ analytically

$1/(\sqrt{1+x^2})$

Section 2.5 - Limits

Limits

Consider $f(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$



$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

If $f(x)$ is defined near a we say that the *limit of $f(x)$, as x approaches a, is L* and write

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ arbitrary close to L by restricting x to be sufficiently close to a.

One Sided Limits

The *left-hand limit* of $f(x)$ as x approaches a, is L and write

$$\lim_{x \to a^-} f(x) = L$$

if we can make $f(x)$ arbitrary close to L by restricting x to be sufficiently close to a with x *less than* a.

The *right-hand limit* of $f(x)$ as x approaches a, is L and write

$$\lim_{x \to a^+} f(x) = L$$

if we can make $f(x)$ arbitrary close to L by restricting x to be sufficiently close to a with x *greater than* a.

Sometimes one-sided limits are written as

$$\lim_{x \searrow a} = \lim_{x \to a^+} \qquad \lim_{x \nearrow a} = \lim_{x \to a^-}$$

Limits and Infinity

Let f be a function defined on both sides of a, except possibly a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that f can be made arbitrarily large by taking x sufficiently close to a, but not equal a.

Let f be defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that $f(x)$ can be made arbitrary close to L by requiring x to be sufficiently large.

Limits - Examples

- $\lim_{x \to 1} x + 1$

- $\lim_{x \to 0} \frac{1}{x^2}$

- $\lim_{x \to 0} \frac{1}{x}$

Limits - Examples 2

- $\lim\limits_{x \to 0^+} \frac{1}{x}$

- $\lim\limits_{x \to 0^-} \frac{1}{x}$

- $\lim\limits_{x \to 0^+} \ln x$

Limits - Example

For $f(x) = \sin(\frac{\pi}{x})$ the $\lim\limits_{x \to 0} f(x)$ does not exist!

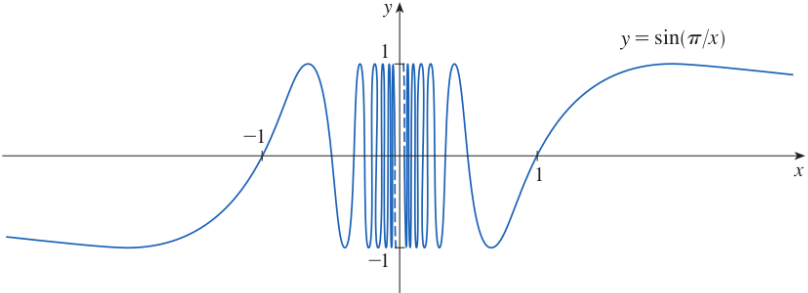
One has

$$f(1) = \sin(\pi) = 0 \qquad f\left(\frac{1}{2}\right) = \sin(2\pi) = 0 \qquad f\left(\frac{1}{3}\right) = \sin(3\pi) = 0 \qquad \ldots$$

$$f(0.1) = \sin(10\pi) = 0 \qquad f(0.01) = \sin(100\pi) = 0 \qquad f(0.001) = \sin(1000\pi) = 0 \qquad \ldots$$

But

$$f\left(\frac{2}{5}\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = 1 \qquad f\left(\frac{2}{9}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = 1 \qquad f\left(\frac{2}{17}\right) = \sin\left(8\pi + \frac{\pi}{2}\right) = 1 \qquad \ldots$$



Section 2.5 - Continuity

Continuity

Definition

A function f is continuous from the right at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

A function f is continuous from the left at a if

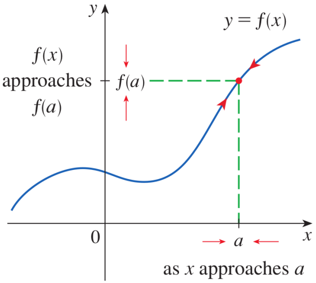
$$\lim_{x \to a^-} f(x) = f(a).$$

Continuity

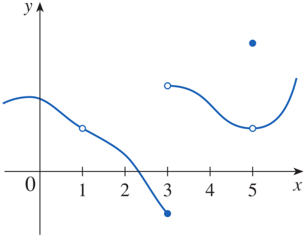
Definition

A function f is called *continuous* at a if

$$\lim_{x \to a} f(x) = f(a)$$



Continuous Function



Function with 3 Discontinuities (at $x = 1, 3, 5$)

Definition

A function f is called continuous on an interval if it is continuous at every number in the interval.

Examples 1

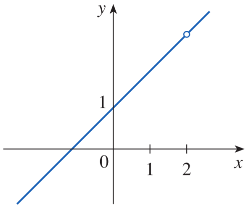
Discontinuous functions

a) $f(x) = \frac{x^2 - x - 2}{x - 2}$

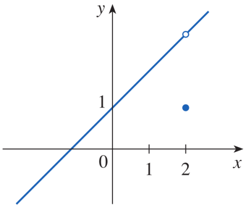
b) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases}$

Continuous functions

1) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 3 & x = 2 \end{cases} = x + 1$



(a) A removable discontinuity



(b) A removable discontinuity

Examples 2

Discontinuous functions

c) $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$

d) $f(x) = [\![x]\!]$

e) $f(x) = \tan x$ if defined on $\mathbb{R}$

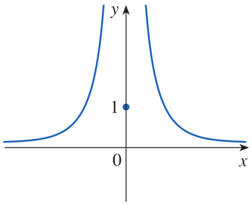
Continuous functions

2) $f(x) = e^x$

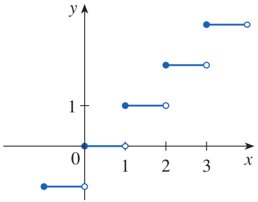
3) $f(x) = \sin x$

4) $f(x) = 3x^4 + 182x^2 + 4$

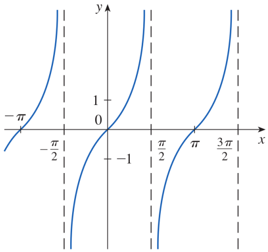
5) $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$



(c) An infinite discontinuity



(d) Jump discontinuities



Limit Laws

Theorem

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} (f(g(x))) = f\left(\lim_{x \to a} g(x)\right)$$

Limit Laws

Suppose that c is a constant, n is an integer and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \text{if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a}[(f(x))^n] = \left(\lim_{x\to a} f(x)\right)^n \qquad \text{if } n \text{ is a positive integer}$$

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)} \qquad \text{if } n \text{ is a positive integer. If } n \text{ is even then } f(a) \text{ has to be positive.}$$

$$\lim_{x\to a} c = c$$

Continuity Laws

If f and g are continuous (at a) and c is a constant then

$$f+g, \qquad f-g, \qquad cf, \qquad f\cdot g, \qquad \frac{f}{g} \text{ for } g\neq 0, \qquad g\circ f = g(f)$$

are continuous (at a).

Continuity Example

For which values is $\ln(1 + \cos(x))$ continuous

$x \neq (2k+1)\pi$

Limit Example

Calculate $\lim\limits_{x \to 0} \frac{(3+x)^2 - 9}{x}$

6

Limit Example

Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

0

Continuity Example

For what value of c is $f(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$ continuous for all $x \in \mathbb{R}$

-2

Limit - Precise Definition

The previous introduction of limits is vague.

Example $\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10000}\right)$

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

Definition ε-δ-criterion

Let f be defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\text{if} \quad |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Section 2.5 - Intermediate Value Theorem

Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

Illustration of the Intermediate Value Theorem

c is not necessarily unique.

Intermediate Value Theorem - Example

Show that there is a solution of

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

It can fail if the assumptions are not fulfilled.

1. Continuous only on $[a, b)$
2. Discontinuous between a and b
3. Also $f(a) = f(b)$ does not imply that there exists c between a and b such that $f(c) = N = f(a)$

Intermediate Value Theorem - Bonus Example

Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

Intermediate Value Theorem - Example

Let $f(x) = x + \frac{6}{x-4}$. For which of the following intervals $[a, b]$ can we use the Intermediate Value Theorem to conclude that $f(x) = 10$ for some x in $[a, b]$?

a) $[5, 6]$

b) $[1, 5]$

c) $[5, 16]$

---

a)

Intermediate Value Theorem - Example

Determine if the following statements are true or false

a) Suppose f is continuous on $[a, b]$ and let N be any number that is not between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there is no number c in (a, b) such that $f(c) = N$.

b) Suppose that f is continuous and one-to-one on $[a, b]$ and let N be any number that is not between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there is no number c in (a, b) such that $f(c) = N$.

c) Suppose that f is one-to-one on $[a, b]$ and let N be any number that is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

f,t,f

Outlook - Real World Applications

Theorem (Borsuk–Ulam Theorem)

There is always a pair of antipodal points (points opposing each other) on the Earth's surface with equal temperatures and equal barometric pressures.

Sketch of Proof

- For every arbitrary circle through the poles let $f(\theta)$ be the temperature on that circle and define $h(\theta) = f(\theta) - f(\theta + \pi)$.
- Since $h(\theta + \pi) = -h(\theta)$ by the Intermediate Value Theorem there has to be $\bar{\theta}$ with $h(\bar{\theta}) = 0$.
- Similarly let φ be the meridian (rotation around the equator) of the circle and $g(\varphi)$ be the pressure at $(\varphi, \theta(\varphi))$ and $l(\varphi) = g(\varphi) - g(\varphi + \pi)$.
- Then similar to before by the Intermediate Value Theorem there has to be $\bar{\varphi}$ with $l(\bar{\varphi}) = 0$.
- By the definitions $(\bar{\varphi}, \bar{\theta})$ is the point.

Theorem (Wobbly Table Theorem)

On every arbitrary smooth surface a 4 leg table can be turned so that it does not wobble. (see here)

Section 2.7 - Derivatives

Tangent Line

Definition

The *tangent line* to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Illustration of the secant line

Illustration of the limit

Tangent Line - Example

Find the equation for the tangent line to $y = x^2$ at the point $P(1, 1)$.

$$y = 2x - 1$$

Derivatives

If we substitute $x = a + h$ into the definition of the slop of the tangent line we get

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \Rightarrow \qquad m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition

- The *derivative of a function* f *at a number* a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  if it exists.
- f is differentiable on (b, c) if f is differentiable at every point in (b, c).

The tangent line at $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slop is equal to $f'(a)$, the derivative of f at a.

Tangent Line - Example

If the tangent line to $y = f(x)$ satisfies the equation $y = 4x - 5$ at $a = 2$. Find $f(2)$ and $f'(2)$.

$3, \ 4$

Derivative - Example

- Find the derivative of $f(x) = x^2 - 8x + 9$ at 2

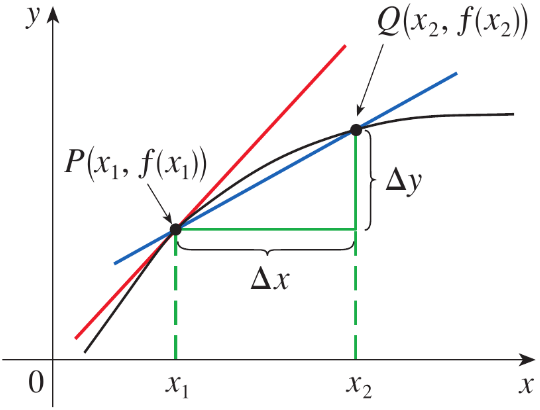
- Find the derivative of $f(x) = x^2 - 8x + 9$ at a

-4, $2a - 8$

Rate of Change

The instantaneous rate of change of f with respect to x is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$$



Derivative - Example 2

Find the derivative of $f(x) = \frac{1}{\sqrt{x}}$ at the number a for $a > 0$.

$-1/(2a^{\frac{3}{2}})$

Outlook - Importance of Derivatives

- Velocity $v = \frac{dx}{dt}$, acceleration $a = \frac{dv}{dt}$
- Newton's second law $F = \frac{dp}{dt}$, where $p = mv$ is the momentum. If m is constant $F = \frac{dp}{dt} = m\frac{dv}{dt} = ma$.
- Schrödinger Equation $\quad i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi \quad$ where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$
- Fluid dynamics $\quad \frac{\partial}{\partial t}u + u \cdot \nabla u + \nabla p - \nabla^2 u = F, \quad \nabla \cdot u = 0$
- Electromagnetism $\quad \nabla \cdot E = \frac{\rho}{\varepsilon_0}, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times B = \mu_0 \left(j + \varepsilon_0 \frac{\partial E}{\partial t}\right)$

Activate Your Accommodations

Please scan the QR code to go to **MySAS Portal** and activate your accommodations for this term.

Accommodation

Activate your Accommodations

Click Here To Proceed

Semester

Fall (2259)

Submit For All Accommodations

Review The Renewal

Book Your Tests

You can book as soon as your accommodations are active.

Test Booking

New Booking Request

1. Select a Course and Test

Select your Course and Test below. Note the start time listed in the Test title. Then click Next Step.

Course *

Test Course - Anatomy 1 (ANAT 101)

Test *

Clear Filters

Next Step

2. Select a Date

Select the bolded date from the calendar below.

3. Select a Room and Start Time

Click on "SAS Testing Centre - MUSC B101" to see a list of all available rooms and start times. Scroll until you reach the correct start time, as noted in the Test title in "1. Select a Course and Test". Then click Request. If the correct start time is not available, select the next closest start time from the options shown.

SAS Testing Centre - MUSC B101 78 Times Available

Request

Pending Booking Requests

Your test bookings may remain in the "Pending" section up until 3 calendar days before your test.

1 results

MA MUSC B101 - Large Group Room A1

Test Course - Anatomy 1

September 18, 2025 11:30 am

SAS Testing Centre - MUSC B101

Done!

Why do I need this? Can't I just use ...?

- Can we just use rules such as $\frac{d}{dt}x^k = kx^{k-1}$? Yes, when possible.
- Why do we do this then?
  - Learn how to calculate
  - Get everybody on the same level
  - What if there is no rule?
  - The rule can not give you insight
  - The definition can be applied to show that a function is not differentiable

Example from my research

$$
\begin{aligned}
&\mathcal{E}_T'(u_0, u_0') \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}(\mathcal{E}_T(u_0 + \varepsilon u_0') - \mathcal{E}_T(u_0)) \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\|\nabla \times (u(T) + \varepsilon u'(T))\|_2^2 - \|\nabla \times u(T)\|_2^2\right) \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(2\varepsilon\int_\Omega \nabla \times u(T)\cdot\nabla \times u'(T) + \varepsilon^2\|\nabla \times u'(T)\|_2^2\right) \\
&= \int_\Omega \nabla \times u(T)\cdot\nabla \times u'(T)
\end{aligned}
$$

Simulate any dynamics

- One has a grid with size h
- One checks the order of converges (p) of the approximation f_h to the real solution f

  $$\frac{|\tilde{f}_h(x) - f(x)|}{h} \leq C|h|^p$$

  for $h \to 0$
- The value of p tells you how good your simulation is (in some sense).

Section 2.8 - The Derivative as a Function

Derivative as a Function

Previous Definition

The *derivative of a function* f *at a number* a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists.

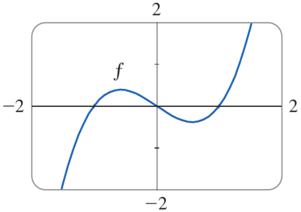
If we set $a = x$ we get a new function

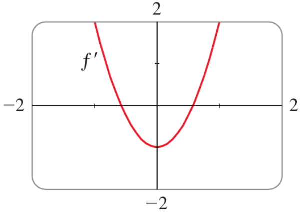
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Illustration of $f(x)$ and $f'(x)$

Derivative as a Function - Example

Find $f'(x)$ for $f(x) = x^3 - x$





$3x^2 - 1$

Derivative as a Function - Example

Where is $f(x) = |x|$ differentiable?

$\mathbb{R}\backslash 0$

Differentiablity

Is $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable at 0?

Differentiablity

Is $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable at 0?

Differentiability implies Continuity

Theorem

If f is differentiable at a, then f is continuous at a.

The converse is not always true as the example $f(x) = |x|$ illustrates.

Non-differentiability Criteria

- f is not differentiable where it has a corner or kink since the two sided limits do not coincide.
- f is not differentiable where it has discontinuities.
- f is not differentiable where it has a vertical tangent line, since then $f'(x) \to \infty$ or $f'(x) \to -\infty$ for $x \to a$

Differentiablity Criterium

- f is differentiable where by zooming its graph straightens out

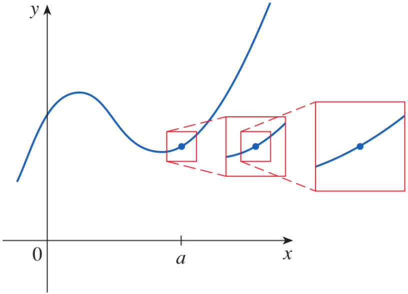


FIGURE 8
f is differentiable at a.

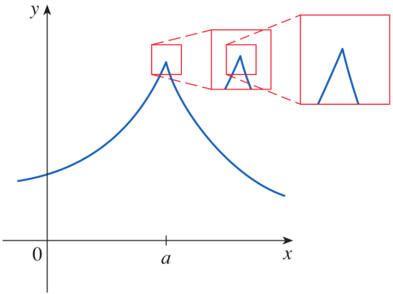


FIGURE 9
f is not differentiable at a.

Further Notation

Derivatives are also denoted as

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The evaluation of the derivative $f'(x)$ at a point a is

$$f'(a) = \left.\frac{df(x)}{dx}\right|_{x=a} \qquad \text{or} \quad \left.\frac{df(x)}{dx}\right]_{x=a}$$

Higher Order derivatives

We can also build the second derivative as

$$f'' = (f')',$$

which is the derivative of the derivative.

And similar $f''' = (f'')', \ldots$. The n-th derivative is denoted by $f^{(n)}(x)$. So $f^{(n)}(x) = f\overbrace{'\cdots'}^{n \text{ times}}$.

Higher Derivatives - Example

Find $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ for $f(x) = x^3 - x$. Previously $f'(x) = 3x^2 - 1$

6x, 6, 0

Derivatives - Example

The following is the derivative of a function f at some point a. $\lim_{h\to 0} \frac{\cos(2\pi+h)-1}{h}$ What are f and a?

Sections 3.x - Calculating Derivatives

Polynomials, Exponentials, Rules, Trigonometric Functions

Calculating Derivatives

Theorem

For any real number k one has

$$(x^k)' = kx^{k-1}.$$

- For a constant function $f(x) = c$ one has $f'(x) = 0$.
- For the function $f(x) = x$ one has $f'(x) = 1$.

Example: Calculate $(x^8)'$

Multiplying by a constant

Theorem

For a constant c

$$(cf(x))' = cf'(x)$$

Example: Calculate $(7x^2)'$

Derivatives of Exponentials

Calculate $f'(x)$ for $f(x) = b^x$ and interpret it

e can be defined by $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$f'(x) = f(x) f'(0) = b^x f'(0)$

using chain rule: $(b^x)' = (e^{\ln b^x})' = (e^{x \ln b})' = \ln b \; e^{x \ln b} = \ln b \; e^{\ln b^x} = \ln b \; b^x$, so $f'(0) = \ln b$

Exponential Function

Theorem

One has

$$(e^x)' = e^x$$

Sums and Differences of Functions

Theorem

If f and g are differentiable, then

$$(f(x)+g(x))' = f'(x)+g'(x)$$
$$(f(x)-g(x))' = f'(x)-g'(x)$$

Example: Calculate $(5x^3-x)'$

Product Rule

Theorem

If f and g are differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Example: Calculate $(x^3e^x)'$

Simplify $(fgh)'$

Quotient Rule

Theorem

If f and g are differentiable, then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Example: Calculate $\left(\frac{x^2}{e^x}\right)'$

Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $F(x) = f(g(x))$ is differentiable at x and F' is given by

$$F'(x) = f'(g(x)) \cdot g'(x) \quad \Longleftrightarrow \quad (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, i.e. $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example: Calculate $(\sqrt{x^2+1})'$

Derivatives of Trigonometric Functions

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \sec^2 x$$
$$(\csc x)' = -\csc x \cot x \qquad (\sec x)' = \sec x \tan x \qquad (\cot x)' = -\csc^2 x$$

Example: Show $(\tan x)' = \sec^2 x$

Calculate $f^{(4)}(x)$ for $f(x) = \sin x$. What is $f^{(1000)}(x)$?

Summary

$$(c)' = 0 \qquad (x^k)' = kx^{k-1} \qquad (e^x)' = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad (f(g))' = f'(g) \cdot g'$$

$$\sin' = \cos \qquad \cos' = -\sin$$

Exercise

Let $f(x) = \cos(x + g(x))$, $g(\frac{\pi}{2}) = 0$, $g'(\frac{\pi}{2}) = 5$. Calculate $f'(\frac{\pi}{2})$

-6

Section 4.8 - Newton's Method

Motivation - How to solve an equation you can't solve?

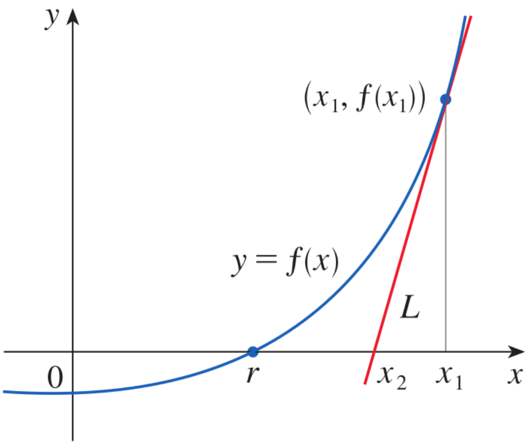
Suppose a car dealer sells you a for $\$18,000$ and you pay $\$375$ per month over five years. What is your monthly interest rate.

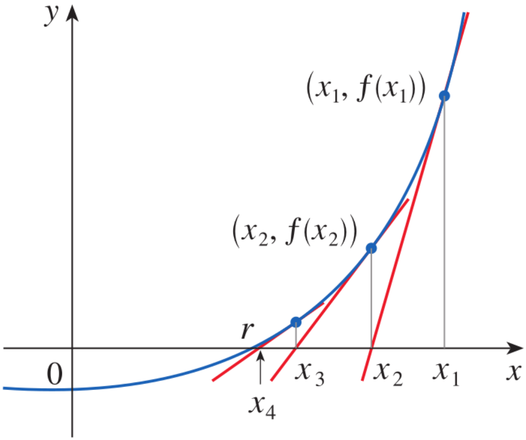
The total value V for n payments of size P with interest rate per time period x is given by

$$V = \frac{P}{x}\left(1 - (1 + x)^{-n}\right)$$

$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$

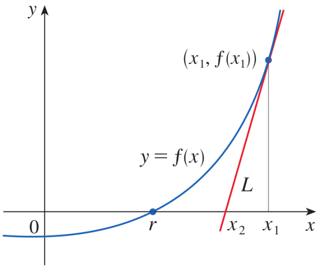
Newton's Method - Illustration





Towards Newton's Method

Start at x_1 and find x_2, x_3, . . .



$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Newton's Method

For approximating the root of a function f start with an initial guess x_1 and calculate iterative

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until the difference of iterations is sufficiently small.

Example

Starting with $x_1 = 2$ find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$

$2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.094568$, exact root $\frac{1}{3}\sqrt[3]{\frac{135}{2} - \frac{1}{2}(3\sqrt{1929})} + 3^{-\frac{2}{3}}\sqrt[3]{\frac{1}{2}(45 + \sqrt{1929})} \approx 2.094551$

Back To The Motivating Example

Find the monthly interest rate, i.e. approximate the root of $48x(1+x)^{60} - (1+x)^{60} + 1 = 0$, starting with an initial guess of 1%.

$f'(x) = 12(244x - 1)(1+x)^{59}$, $x_2 \approx 0.0082202, x_3 \approx 0.0076802, x_4 \approx 0.0076291, x_5 \approx 0.0076286, x_6 \approx 0.0076286$
yearly $\approx 9.548\%$

More than roots

Solve $\cos x = x$

$$x_1 = 1$$
$$x_2 \approx 0.7503638678402439$$
$$x_3 \approx 0.7391128909113617$$
$$x_4 \approx 0.739085133385284$$
$$x_5 \approx 0.7390851332151607$$
$$x_6 \approx 0.7390851332151607$$

Calculate $\sqrt[6]{2}$

$$x_1 = 1.5$$
$$x_2 \approx 1.293895747599451$$
$$x_3 \approx 1.170160594100723$$
$$x_4 \approx 1.127066579185062$$
$$x_5 \approx 1.122508821428220$$
$$x_6 \approx 1.122462053181502$$

Real World Implementation

Algorithm

1. Transform your equation to $f(x) = 0$
2. Calculate $f'(x)$ by hand
3. Make an initial guess x_1
4. Write code that calculates the next value

Example ($\cos x = x$)

1. $f(x) = \cos x - x = 0$
2. $f'(x) = -\sin x - 1$
3. $x_1 = 1$
4. The (Python) code

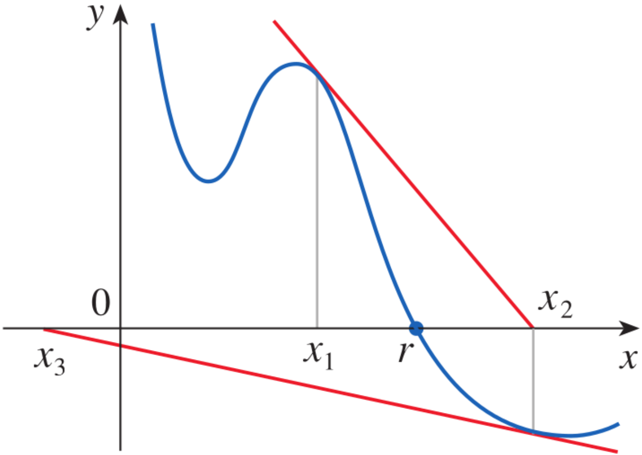
```python
import math                            # to use sin/cos
x = 1                                  # initial guess
for n in range(6):                     # repeat 6 times
    print("x",int(n+1),":\t",x) # output result
    x = x - (math.cos(x)-x)/(-math.sin(x)-1)
```

produces the output

```
$ python newton_example.py
x 1 :     1
x 2 :     0.7503638678402439
x 3 :     0.7391128909113617
x 4 :     0.739085133385284
x 5 :     0.7390851332151607
x 6 :     0.7390851332151607
```

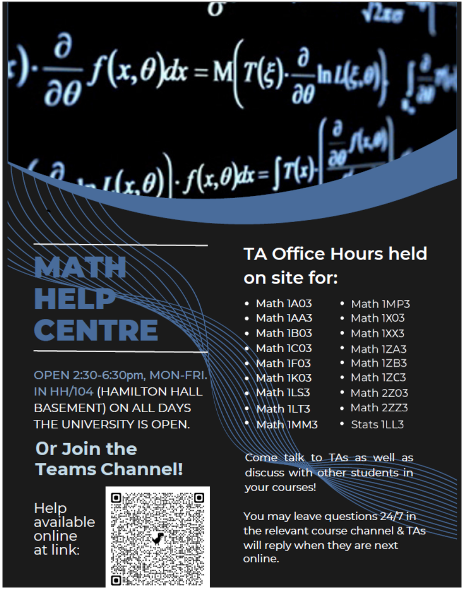
Non-Convergence

- Can fail to converge
- Usually when $f'(x_n) \approx 0$
- Make a better initial guess x_1



Further Questions

- How do you choose an initial guess?
- How do I know to stop?
- How do I know it fails?
- What if there are more roots? How to find them?
- What if there are no roots?



Teams Channel:

Direct link to Time Tables

Section 3.4 - The Chain Rule

Previously

Theorem (Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $F(x) = f(g(x))$ is differentiable at x and F' is given by

$$F'(x) = f'(g(x)) \cdot g'(x) \qquad \Longleftrightarrow (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, i.e. $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Is anyone not comfortable with it?

Chain Rule - Exercise A

Calculate $(e^{x^8-\sin(x)})'$

Chain Rule - Exercise B

Calculate $\left(\sqrt[5]{e^{\cos(x^2)}}\right)'$

$-\frac{2}{5}x\sin(x^2)\sqrt[5]{e^{\cos(x^2)}}$

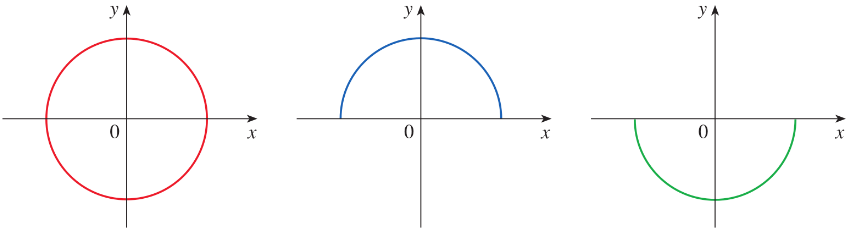
Chain Rule - Exercise C

Find an expression for $(b^x)'$ for a general base $b > 0$

$$\boxed{(b^x)' = b^x \ln b}$$

Section 3.5 - Implicit Differentiation

Implicit Functions - Circle



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Implicit Differentiation

Find the derivative of the circle $x^2 + y^2 = 25$ at the point $(3, 4)$ without solving for y.

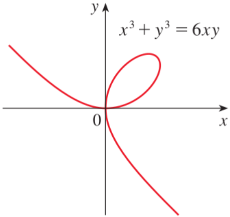
$-\frac{3}{4}$

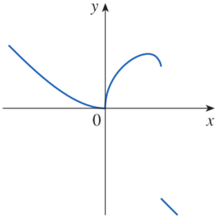
Implicit Differentiation

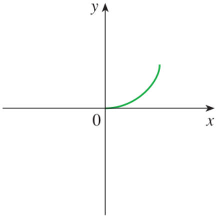
Find the derivative of the circle $x^2 + y^2 = 25$ at the point $(3, 4)$ using its explicit function $f(x) = \sqrt{25 - x^2}$ of the upper branch.

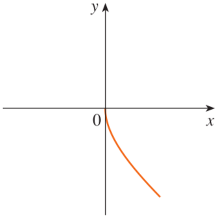
$-\frac{3}{4}$

Implicit Functions - Folium of Descartes









Folium of Descartes

Find y' for $x^3 + y^3 = 6xy$

$\frac{2y-x^2}{y^2-2x}$

Folium of Descartes

Alternatively we could have used the cubic root formulas to solve y' for $x^3 + y^3 = 6xy$ for y and get

$$y = f(x) = \sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}$$

and

$$y = \tfrac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}\right)\right]$$

and then calculated the derivative.

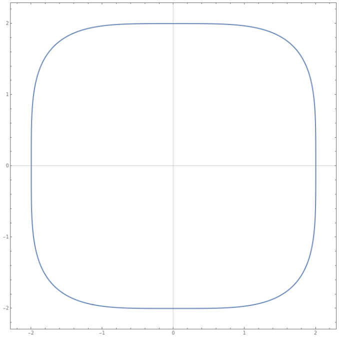
Non-Polynomial Example

Find y' for $\sin(x+y) = y^2 \cos x$

$$\frac{y^2 \sin x + \cos(x+y)}{2y \cos x - \cos(x+y)}$$

Higher Order Implicit Derivatives

Find y'' for $x^4 + y^4 = 16$



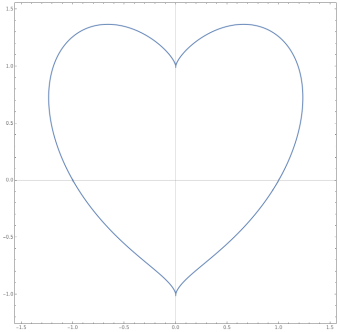
$-3x^2 \frac{y^4+x^4}{y^7}$

Example - Heart

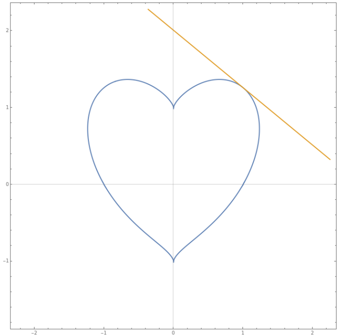
Often times functions are given implicitly.

Say you want to know the tangent line of the heart function,
$(x^2 + y^2 - 1)^3 = 2x^2y^3$ in $(x, y) = (1, \sqrt[3]{2})$

We can find the tangent line without knowing $f(x)$ explicitly.



$(x^2 + y^2 - 1)^3 = 2x^2y^3$



$$(x^2 + y^2 - 1)^3 = 2x^2y^3$$

and

$$y = \left(\frac{2\sqrt[3]{2}}{3} - 2^{\frac{2}{3}}\right)x + \frac{\sqrt[3]{2}}{3} + 2^{\frac{2}{3}}$$

$$y' = \frac{4xy^3 - 6x(x^2+y^2-1)^2}{6y(x^2+y^2-1)^2 - 6x^2y^2}, \; y = \left(\frac{2\sqrt[3]{2}}{3} - 2^{\frac{2}{3}}\right)x + \frac{\sqrt[3]{2}}{3} + 2^{\frac{2}{3}}$$

$$y' = \frac{4xy^3 - 6x(x^2+y^2-1)^2}{6y(x^2+y^2-1)^2 - 6x^2y^2},\ y = \left(\frac{2\sqrt[3]{2}}{3} - 2^{\frac{2}{3}}\right)x + \frac{\sqrt[3]{2}}{3} + 2^{\frac{2}{3}}$$

Section 3.6 - Derivatives of Logarithmic and Inverse Trig Functions

Derivative of Logarithmics

When $x > 0$ (when $x \leq 0$ the left-hand side is not defined, so we ignore it)[a]

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

[a] Also one can define logarithms of negative numbers using complex numbers. Then this formula is valid whenever $x \neq 0$.

since

Example

Differentiate $\ln(x^3 + 1)$

$$(\ln g(x))' = \frac{g'(x)}{g(x)}$$

$\frac{3x^2}{x^3+1}$

Example

Differentiate $\sqrt{\ln(\sin x)}$

$\frac{\cot x}{2\sqrt{\ln(\sin x)}}$

Example

Calculate $(\ln |x|)'$

$\frac{1}{x}$

Implicit Logarithmic Differentiate

Sometimes it is useful to first apply the logarithm

1. Take the logarithm of both sides of $y = f(x)$
2. Use implicit differentiation
3. Solve for y'

Example: Find the derivative of $x^{\sqrt{x}}$

$\frac{1}{2}(\ln x + 2)x^{\sqrt{x}-\frac{1}{2}}$

Derivative of Inverse Functions

$$\boxed{(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}}$$

since

Example

Calculate $(\sin^{-1}(x))'$

$\frac{1}{\sqrt{1-x^2}}$

Derivatives of Inverse Trigonometric Functions

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}} \qquad (\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}} \qquad (\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}} \qquad (\cot^{-1} x)' = -\frac{1}{1+x^2}$$

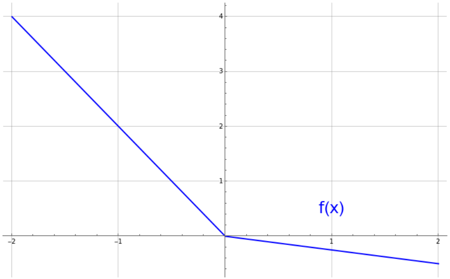
Example

Differentiate $x \tan^{-1} \sqrt{x}$

$\arctan \sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$

Example

Find the derivative of inverse of the shown function at 2, i.e. $(f^{-1})'(2)$



$-\frac{1}{2}$

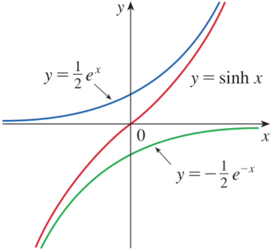
Section 3.11 - Hyperbolic Functions

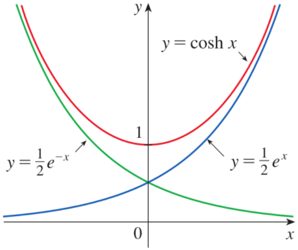
Hyperbolic Functions

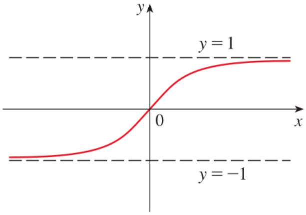
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

called hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cosecant, hyperbolic secant, hyperbolic cotangent.







Connection to Trigonometric Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

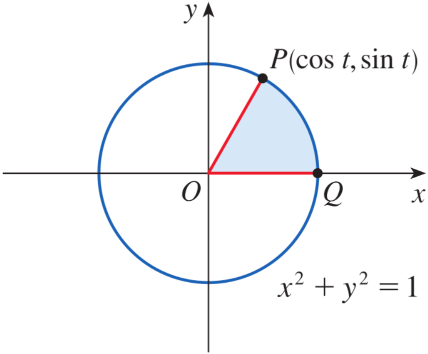
$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

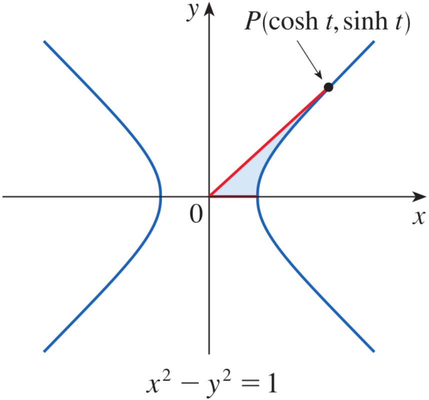
The trigonometric functions can be written as

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Connection to Trigonometric Functions



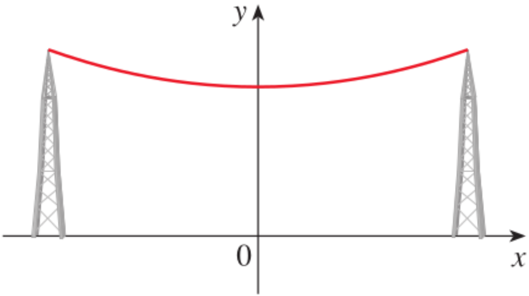


Application

The shape of a wire hanging between to points at the same height can be described by

$$y = c + a \cosh \frac{x}{a}$$

for some constants a and c.



Exercise

Simplify $\sinh(\ln x)$

$\frac{x^2-1}{2x}$

Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Exercise

Prove $\sinh(-x) = -\sinh x$

Exercise

Prove $\cosh^2 x - \sinh^2 x = 1$

Derivatives

$$(\sinh x)' = \cosh x \qquad (\cosh x)' = \sinh x \qquad (\tanh x)' = \operatorname{sech}^2 x$$

$$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x \qquad (\operatorname{sech} x)' = -\operatorname{sech} x \tanh x \qquad (\coth x)' = -\operatorname{csch}^2 x$$

Exercise

Prove $(\sinh x)' = \cosh x$

Exercise

Prove $(\tanh x)' = \operatorname{sech}^2 x$

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right) \qquad x \neq 0$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad 0 < x \leq 1$$

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1} \qquad |x| > 1$$

called inverse hyperbolic sin $\sinh^{-1}, \ldots$ (or arsinh (for area), or $\operatorname{arcsinh}$).

Exercise

Find $\sinh^{-1}(0)$ without the explicit formula

Derivatives of the inverse hyperbolic functions

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}} \qquad (\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}} \qquad (\tanh^{-1} x)' = \frac{1}{1-x^2}$$

$$(\operatorname{csch}^{-1} x)' = -\frac{1}{|x|\sqrt{1+x^2}} \qquad (\operatorname{sech}^{-1} x)' = -\frac{1}{x\sqrt{1-x^2}} \qquad (\coth^{-1} x)' = \frac{1}{1-x^2}$$

where they and their respective inverse hyperbolic function is defined

Exercise

Prove $(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$

Important Trigonometric/Hyperbolic Formulas

$$\sin x = \dots \qquad \cos x = \dots \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin' = \cos \qquad \cos' = -\sin$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh' = \cosh \qquad \cosh' = \sinh$$

Which formulas to learn for the test

- Take a sheet of paper
- Write down every formula you need to solve the assignment question/suggest problems/previous test
- think if you can derive the formula during the test
- otherwise remember it!

Section 4.1 - Maximum and Minimum Values

Definition

Definition (Absolut)

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- *absolut* (or *global*) *maximum* of f on D if $f(c) \geq f(x)$ for all x in D
- *absolut* (or *global*) *minimum* of f on D if $f(c) \leq f(x)$ for all x in D

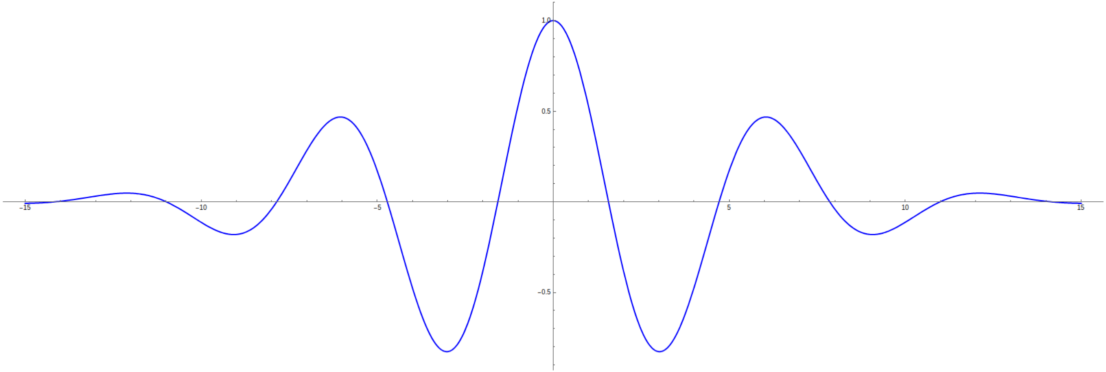
Definition (Local)

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- *local maximum* of f if $f(c) \geq f(x)$ when x is near c
- *local minimum* of f if $f(c) \leq f(x)$ when x is near c

- Maxima and Minima are called extreme values.
- Here we use the convention that when x is near c we mean the inequality has to hold on both sides of c. So end points of the domain can not be local extrema. (Some other authors (for example Wikipedia) allow endpoints to be local extrema.)

Illustration



Exercise

Sketch the graph of a function on $[1,3]$ that has

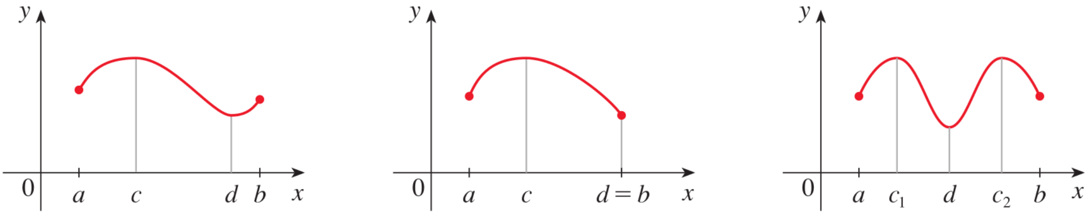
(a) an absolute maximum but no local maximum

(b) a local maximum but no absolute maximum

A Theorem About Maxima And Minima

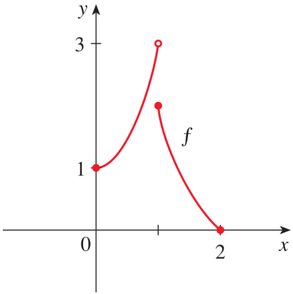
Theorem (Extrem Value Theorem)

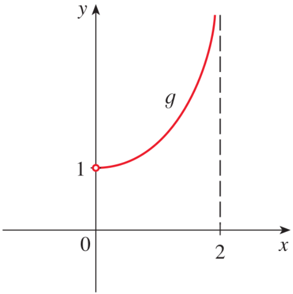
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$



Crucialness of the Assumptions

If either of the assumptions (continuity or closed interval) of the Extrem Value Theorem are not met, the function does not need to have extrem values.





Exercise

True or False:

1. If f is differentiable on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.
2. If f is continuous on an open interval (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).
3. If neither $f(a)$ nor $f(b)$ is the absolute maximum of f on the interval $[a, b]$, then f must attain an absolute maximum at some point (a, b).

T, F, F

Derivatives and Extrema

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

Careful!

This does <u>not</u> mean that if $f'(c) = 0$ the function has an extremum. See $f(x) = x^3$

Careful!

This does <u>not</u> mean that if f has a maximum in c that $f'(c) = 0$. See $f(x) = |x|$

Critical Numbers

Definition

A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example: Find critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$

0, $\frac{3}{2}$

Finding Extrema

Closed Interval Method

To find the absolute extrema of a continuous function on a closed interval:

1. Find all critical points and the function values at these points.
2. Find the values at the end points of the interval.
3. The largest value of these previous values is the absolute maximum, the smallest is the absolute minimum

Example

Find the absolute maximum and absolute minimum of $f(x) = \frac{\ln x}{x}$ on the interval $[1, 10]$.

$0, \frac{1}{e}, \frac{\ln(10)}{10} \approx 0.230259, \frac{1}{e} \approx 0.367879$

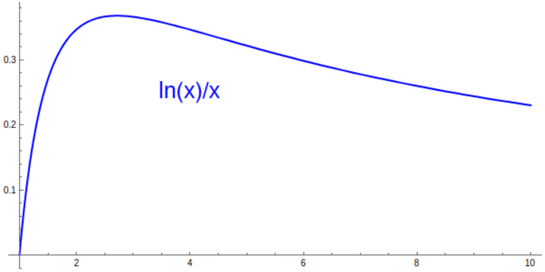
Example

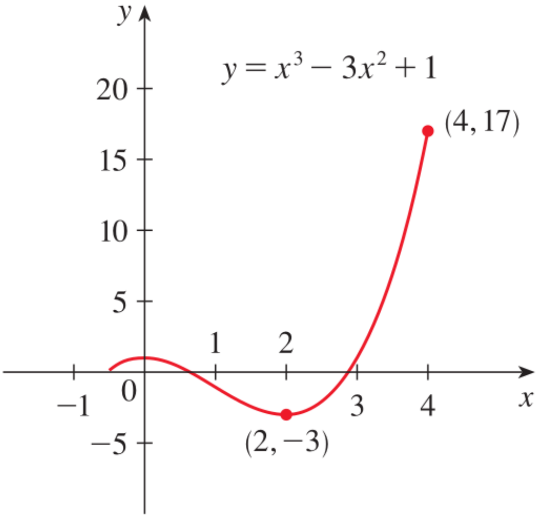
Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ in the interval $[-\frac{1}{2}, 4]$

Example

$-3, 17$

Illustration of the Examples





End of material for Test 1.

Test Information

- Coverage: Suggested Problems #1 $\rightarrow$ up until (including) 4.1 (Maximum and Minimum Values) so until here
- Look at Test #1 Information on Childsmath!
  - How to fill the solution card!
  - Which calculator
  - ...
- Previous year's test and problem sampler on are on Childsmath

Review Test 1

Read childsmath/more announcements/Test #1 Information!

Important Trigonometric/Hyperbolic Formulas

$$\sin x = \dots \qquad \cos x = \dots \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin' = \cos \qquad \cos' = -\sin$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh' = \cosh \qquad \cosh' = \sinh$$

Inverse functions

Definition

A function is *one-to-one* if it never takes on the same value twice, i.e. $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

Definition

If f is one-to-one

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

$$\text{range}(f) = \text{domain}(f^{-1}) \qquad \text{range}(f^{-1}) = \text{domain}(f)$$

$$f^{-1}(f(x)) = x \qquad\qquad f(f^{-1}(x)) = x$$

Natural Logarithm

Definition

$$\ln e^x = x \qquad\qquad e^{\ln x} = x$$

Calculation Rules

$$\ln a + \ln b = \ln(ab) \qquad\qquad \log_b x = \frac{\ln x}{\ln b}$$

$$\ln a^b = b \ln a \qquad\qquad a^b = e^{b \ln a}$$

Inverse Trigonometric Functions

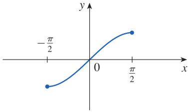


FIGURE 18
$y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

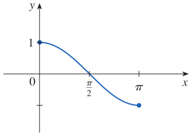


FIGURE 21
$y = \cos x, 0 \leqslant x \leqslant \pi$

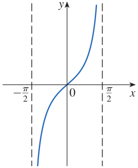


FIGURE 23
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

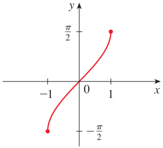


FIGURE 20
$y = \sin^{-1} x = \arcsin x$

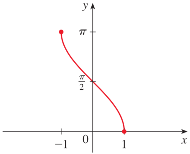


FIGURE 22
$y = \cos^{-1} x = \arccos x$

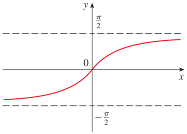
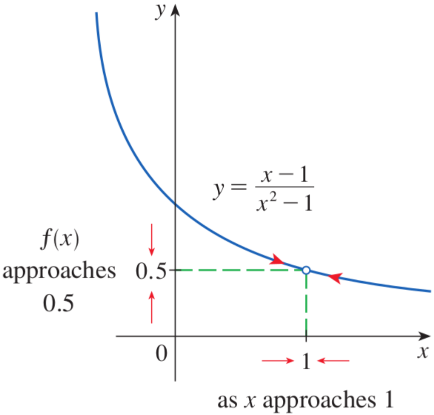


FIGURE 25
$y = \tan^{-1} x = \arctan x$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
range	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[0, \pi]$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ arbitrary close to L by restricting x to be sufficiently close to a.



Continuity

Definition

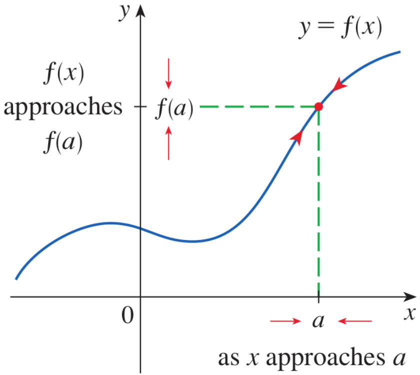
f is *continuous* at a if

$$\lim_{x \to a} f(x) = f(a)$$

Theorem

If f is continuous

$$\lim_{x \to a} (f(g(x))) = f\left(\lim_{x \to a} g(x)\right)$$



Limit Laws

Suppose that c is a constant, n is an integer and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{if } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a} [(f(x))^n] = \left(\lim_{x \to a} f(x)\right)^n \qquad \text{if } n \text{ is a positive integer}$$

$$\lim_{x \to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x \to a} f(x)} \qquad \text{if } n \text{ is a positive integer. If } n \text{ is even then } f(a) \text{ has to be positive.}$$

$$\lim_{x \to a} c = c$$

Continuity Laws

If f and g are continuous (at a) and c is a constant then

$$f+g, \qquad f-g, \qquad cf, \qquad f\cdot g, \qquad \frac{f}{g} \text{ for } g \neq 0, \qquad g\circ f = g(f)$$

are continuous (at a).

Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

Derivatives

Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists.

Tangent line at a

$$y = f'(a)x + b$$

through $(a, f(a))$.

Calculating Derivatives

$$(c)' = 0 \qquad (x^k)' = kx^{k-1} \qquad (e^x)' = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad (f(g))' = f'(g) \cdot g'$$

$$\sin' = \cos \qquad \cos' = -\sin \qquad (b^x)' = b^x \ln b$$

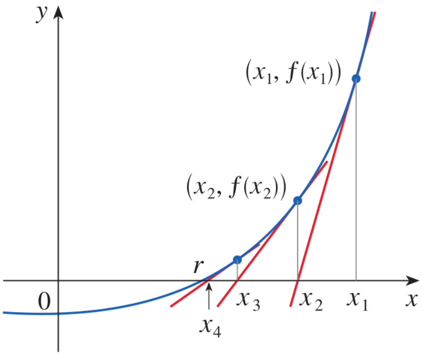
$$(\log_b x)' = \frac{1}{x \ln b} \qquad (\ln x)' = \frac{1}{x} \qquad (\ln |x|)' = \frac{1}{x}$$

$$\sinh' = \cosh \qquad \cosh' = \sinh \qquad (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Newton's Method

Make an initial guess x_1 and iteratively calculate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Implicit Differentiation

Illustrative Example

For $xy = x^3 + y^2$ consider the solution $y(x)$ and differentiate using the chain rule

$$y + xy' = 3x^2 + 2yy'$$

therefore

$$y' = \frac{3x^2 - y}{x - 2y}$$

and so the derivative at $(-2, -4)$ is

$$y' = \frac{3(-2)^2 - (-4)}{-2 - 2(-2)} = \frac{8}{3}$$

Extrema

Definition (Absolut)

$f(c)$ is the

- *absolut* (or *global*) *maximum* of f on D if $f(c) \geq f(x)$ for all x in D
- *absolut* (or *global*) *minimum* of f on D if $f(c) \leq f(x)$ for all x in D

Definition (Local)

$f(c)$ is the

- *local maximum* of f if $f(c) \geq f(x)$ when x is near c
- *local minimum* of f if $f(c) \leq f(x)$ when x is near c

Theorem (Extrem Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$

Extrema

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

Critical Number

c is a critical number if either $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method

To find the absolute extrema of a continuous function on a closed interval:

1. Find all critical points and the function values at these points.
2. Find the values at the end points of the interval.
3. The largest value of these previous values is the absolute maximum, the smallest is the absolute minimum

Section 4.2 - Mean Value Theorem

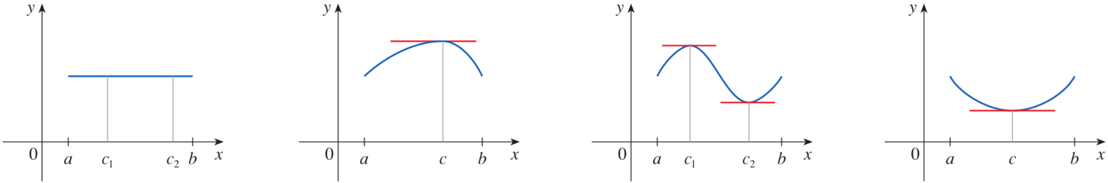
Rolle's Theorem

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following

- f is continuous on the closed interval $[a, b]$
- f is differentiable on the open interval (a, b)
- $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



Importance of the Assumptions

Rolle's theorem can fail if

1. f is only continuous on $(a, b]$ or
2. f is not everywhere differentiable

Exercise

How many roots does $x^3 + x - 1 = 0$ have?

Exercise Continued

How many roots does $x^3 + x - 1 = 0$ have?

1

Exercise

Find all values of c that satisfy the conclusion of Rolle's Theorem for $f(x) = x\sqrt{x+6}$ on the interval $[-6, 0]$.

-4

Mean Value Theorem

Theorem (Mean Value Theorem)

Let f be a function that satisfies the following

- f is continuous on the closed interval $[a, b]$
- f is differentiable on the open interval (a, b)

Then there is a number c in (a, b) such that

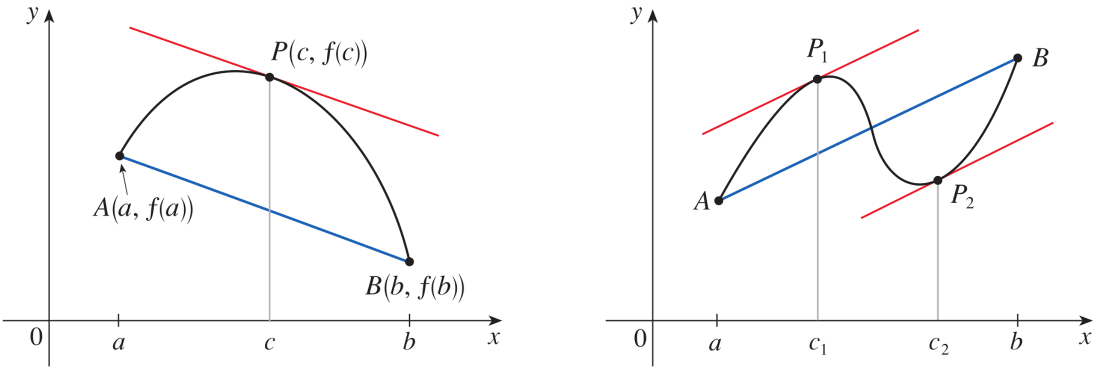
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently

$$f(b) - f(a) = f'(c)(b - a)$$

Simplified it says that a continuous differentiable function has its average slope in some point.

Illustration



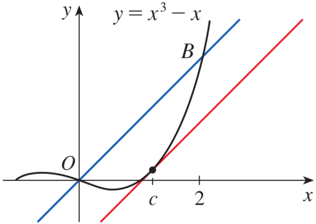
Importance of the Assumptions

The Mean Value Theorem can fail if

1. f is only continuous on $(a, b]$ or
2. f is not everywhere differentiable

Exercise

Find all values of c that satisfy the conclusion of the Mean Value Theorem for $f(x) = x^3 - x$ on the interval $[0, 2]$.



$\frac{2}{\sqrt{3}}$

Application

Theorem

If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).

Exercise

Suppose $f(2) = -3$ and $f'(x) \leq 2$ for $2 \leq x \leq 5$. According to the Mean Value Theorem, what is the largest possible value for $f(5)$?

3

Section 4.3 - How Derivatives Affect the Shape of a Graph

First Derivatives

Increasing/Decreasing

Increasing/Decreasing Test

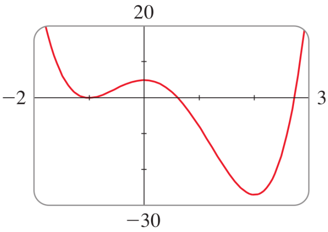
- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Since $f'(x) > 0$ we assume that f' exists on that interval, so f is differentiable and therefore continuous.

Exercise

Where is $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing and where is it decreasing?

Exercise - Continued



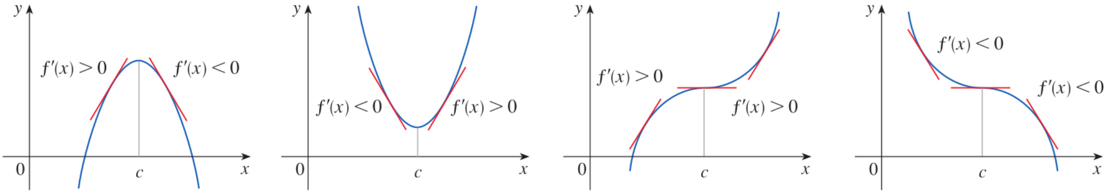
decreasing in $(-\infty, -1) \cup (0, 2)$ and increasing in $(-1, 0) \cup (2, \infty)$

Increasing/Decreasing

First Derivative Test

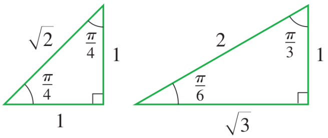
Suppose that c is a critical number of a continuous function

- If f' changes from positive to negative at c, then f has a local maximum in c
- If f' changes from negative to positive at c, then f has a local minimum in c
- If f' is positive (or negative) on both sides of c, then f does not have a local maximum or minimum in c.

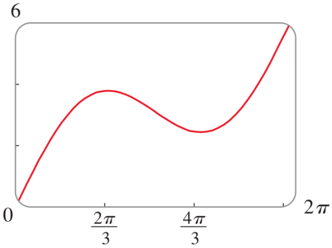


Exercise

Find all local maximum and minimum values of $f(x) = x + 2\sin x$ on $0 \le x \le 2\pi$.



Exercise - Continued



$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83,\ f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$

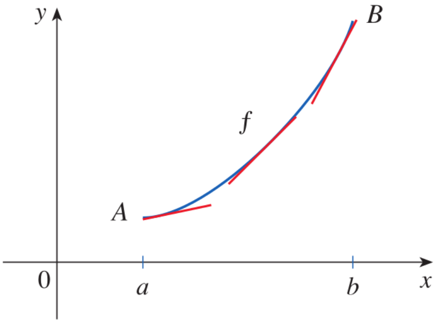
Section 4.3 - How Derivatives Affect the Shape of a Graph

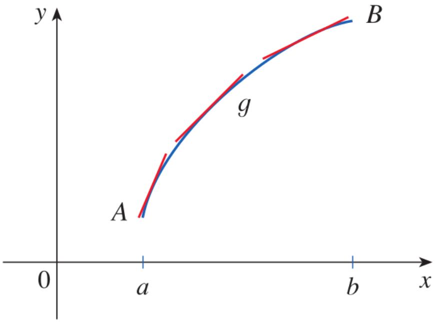
Second Derivative

Concave Upward/Downward

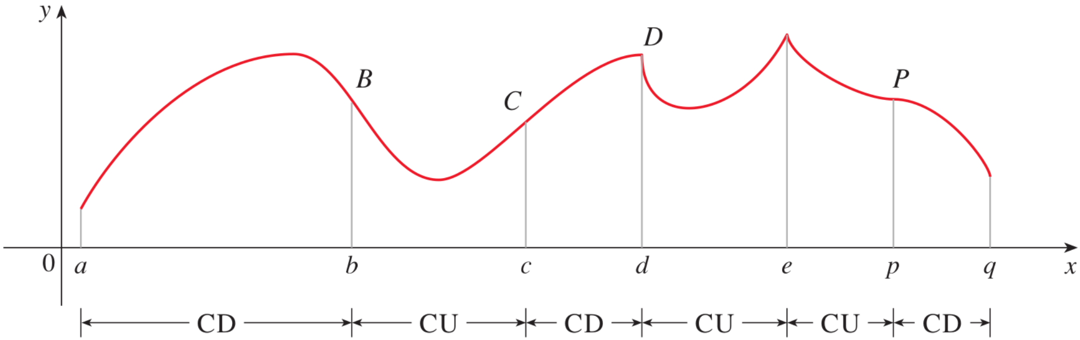
Definition

If the graph of f lies above all its tangents on an interval I, then f is called *concave upward* on I. If the graph of f lies below all of its tangents on I, then f is called *concave downward* on I.





Illustration



Relation to Second Derivatives

Concavity Test

- If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Inflection Point

Definition

A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Inflection Point and Second Derivative

$f''(c) = 0$ does not imply that there is an inflection point in c.

Example

Sketch the graph of a function f that satisfies the following conditions

- $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$
- $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$
- $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

Criteria for Extrema

Second Derivative Test

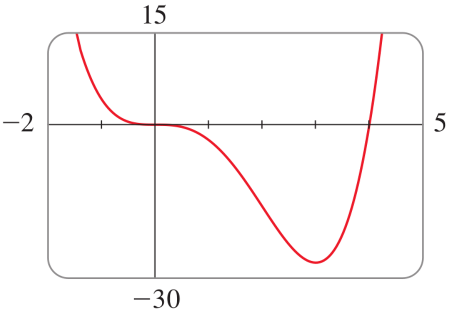
Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum in c
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum in c

Example

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima.

Example - Continued



Example

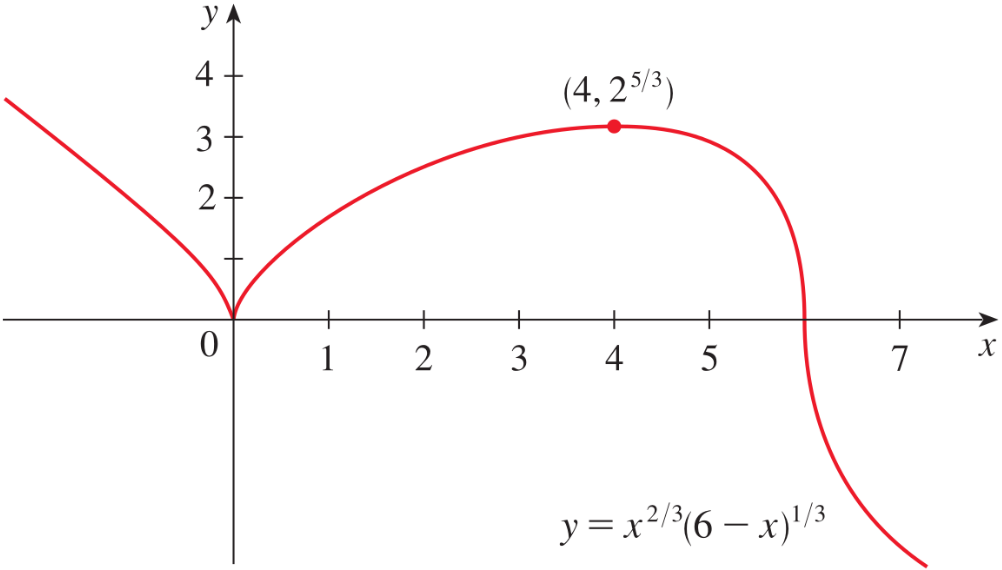
Sketch the graph of $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

Example - Continued

$f'(x) = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$

$$f''(x) = -\frac{8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

Example - Illustration



Section 4.4 - Indeterminate Forms and L'Hôpital's Rule

Motivation

Often times we have "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$", when calculating $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$.

These are called *indetermined form of type* $\frac{0}{0}$ and *indetermined form of type* $\frac{\infty}{\infty}$

- Rational functions

$$\lim_{x\to 1} \frac{x^2 - x}{x^2 - 1}$$

$$\lim_{x\to \infty} \frac{x^2 - 1}{2x^2 + 1}$$

- What do we do for

$$\lim_{x\to 0} \frac{\sin x}{x}? \qquad \lim_{x\to 1} \frac{\ln x}{x - 1}?$$

L'Hôpital's rule

L'Hôpital's rule

Suppose

- f and g are differentiable and $g'(x) \neq 0$ in an interval around a (except possibly a itself)
- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or
  $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the right-hand side limit exists or is ∞ or $-\infty$

This also works for $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$.

We do not use the chain rule here!

Example

Calculate $\lim_{x \to 0} \frac{\sin x}{x}$

1

Example

Calculate $\lim_{x \to 1} \frac{\ln x}{x-1}$

1

Example

Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$

∞

Example

Calculate $\lim\limits_{x\to 0} \frac{\tan x - x}{x^3}$

Example - Continued

$\frac{1}{3}$

Careful!

Calculate $\lim_{x \to \pi} \frac{\sin x}{1 - \cos x}$

0

More Indetermined Forms

- For $"0 \cdot \infty"$ write $\lim f(x)g(x) = \lim \frac{f(x)}{\frac{1}{g(x)}}$ and use L'Hôpital's rule
- For $"\infty - \infty"$ including fractions try to find a common denominator and use L'Hôpital's rule

Example

Calculate $\lim_{x \to 0^+} x \ln x$

0

Example

Calculate $\lim\limits_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

$\frac{1}{2}$

Even More Indetermined Forms

f^g

For $\lim_{x \to a} \left(f(x)^{g(x)} \right)$ use

$$\lim_{x \to a} \left(f(x)^{g(x)} \right) = \lim_{x \to a} \left(e^{\ln\left(f(x)^{g(x)}\right)} \right) = \lim_{x \to a} \left(e^{g(x) \ln(f(x))} \right) = e^{\lim_{x \to a} (g(x) \ln(f(x)))}$$

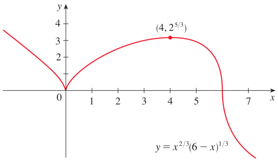
Example

Calculate $\lim_{x\to 0^+} x^x$ (This is indetermined since $0^x = 0$ for $x > 0$ and $x^0 = 1$ for $x > 0$)

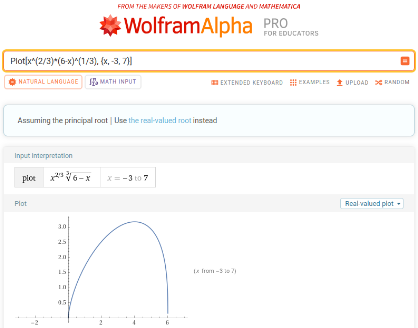
1

Section 4.5 - Summary of Curve Sketching

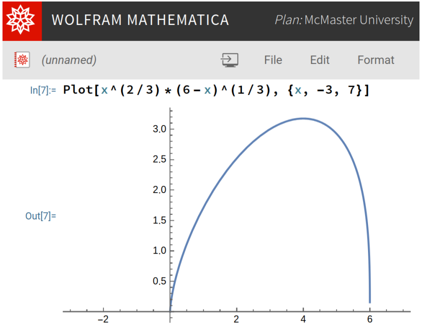
Be Careful When Using Resources



Book Plot



wolfram-alpha plot



mathematica plot

Guide for Curve Sketching

1. Domain
2. Intercepts
3. Symmetries
   - Even
   - Odd
   - Periodic
4. Asymptotes
   - Horizontal
   - Vertical
   - Slant
5. Increasing/Decreasing
6. Extrema
7. Concavity and Inflection Points
8. Sketch the curve by using the previous steps

Domain

Find where the function is not defined

- $\ln(x)$ is not defined for $x \leq 0$
- $\frac{1}{x}$ is not defined for $x = 0$
- $\sqrt{x}$ and $x^{\frac{1}{k}} = \sqrt[k]{x}$ where k is even is not defined for $x < 0$
- Compositions of these with functions

Intercepts

- y-intercept: Calculate $f(0)$
- x-intercepts: Set $f(x) = 0$ and solve for x.

Symmetries

- Even functions if $f(-x) = f(x)$. For example $f(x) = x^2$, $f(x) = \frac{1}{x^6}$, $f(x) = \cos(x)$, $f(x) = \cosh(x)$
- Odd functions if $f(-x) = -f(x)$. For example $f(x) = 3x^5 - x^3 + \frac{1}{x}$, $f(x) = \sin(x)$, $f(x) = \cot(x)$
- Periodic functions if $f(x) = f(x + p)$ for all x and some p. For example $f(x) = \sin(x)$

Asymptotes

- Horizontal Asymptotes if $\lim_{x \to \infty} = L$ or $\lim_{x \to -\infty} = L$
- Vertical Asymptotes if

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- Slant Asymptotes if

$$\lim_{x \to \infty} (f(x) - (mx + b)) = 0 \qquad \lim_{x \to -\infty} (f(x) - (mx + b)) = 0$$

Increasing/Decreasing and Extrema - Previously

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Closed Interval Method

To find the absolute extrema of a continuous function on a closed interval:

1. Find all critical points and the function values at these points.
2. Find the values at the end points of the interval.
3. The largest value of these previous values is the absolute maximum, the smallest is the absolute minimum

First Derivative Test

Suppose that c is a critical number of a continuous function

- If f' changes from positive to negative at c, then f has a local maximum in c
- If f' changes from negative to positive at c, then f has a local minimum in c
- If f' is positive (or negative) on both sides of c, then f does not have a local maximum or minimum in c.

Increasing/Decreasing and Extrema

1. Calculate f'
2. Look where
   - $f' > 0 \implies f$ is increasing
   - $f' < 0 \implies f$ is increasing
   - $f' = 0$ if
     - If f' changes from positive to negative there is a maximum at this point (alternatively if $f'' < 0$ at this point)
     - If f' changes from negative to positive there is a minimum at this point (alternatively if $f'' > 0$ at this point)
     - If f' has the same sign there is no extremum at this point
   - f' does not exist $\implies$ check further: Jump? Corner?

Concavity

1. Calculate f''
2. If
   - $f'' > 0$ it is concave upward in this point/interval
   - $f'' < 0$ it is concave downward in this point/interval
   - f'' changes sign it is an inflection point

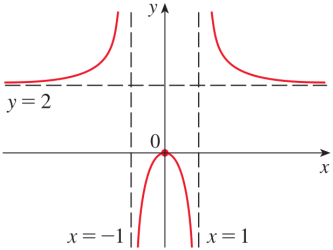
Example 1

Sketch the graph of $f(x) = \frac{2x^2}{x^2-1}$

Example 1

$f' = \frac{-4x}{(x^2-1)^2}$, $f'' = \frac{4(x^2-1)(1+3x^2)}{(x^2-1)^4}$

Example 1 - Continued

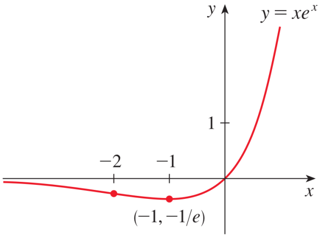


Example 2

Sketch the graph of $f(x) = xe^x$

Example 2 - Continued

Example 2 - Continued



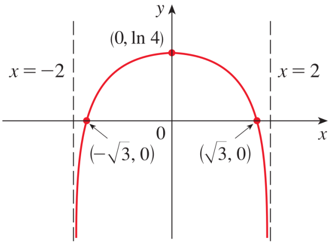
Example 3

Sketch the graph of $f(x) = \ln(4 - x^2)$

Example 3 - Continued

$f' = \frac{-2x}{4-x^2}$, $f'' = -2\frac{4+x^2}{(4-x^2)^2}$

Example 3 - Continued



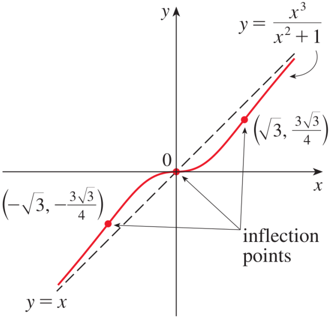
Example 4

Sketch the graph of $f(x) = \frac{x^3}{x^2+1}$

Example 4 - Continued

$f' = x^2 \frac{x^2+3}{(x^2+1)^2}$, $f'' = 2x \frac{3-x^2}{(x^2+1)^3}$

Example 4 - Continued



Example 5

Sketch the graph of $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

Example 5 - Continued

$f' = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$, $f'' = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$

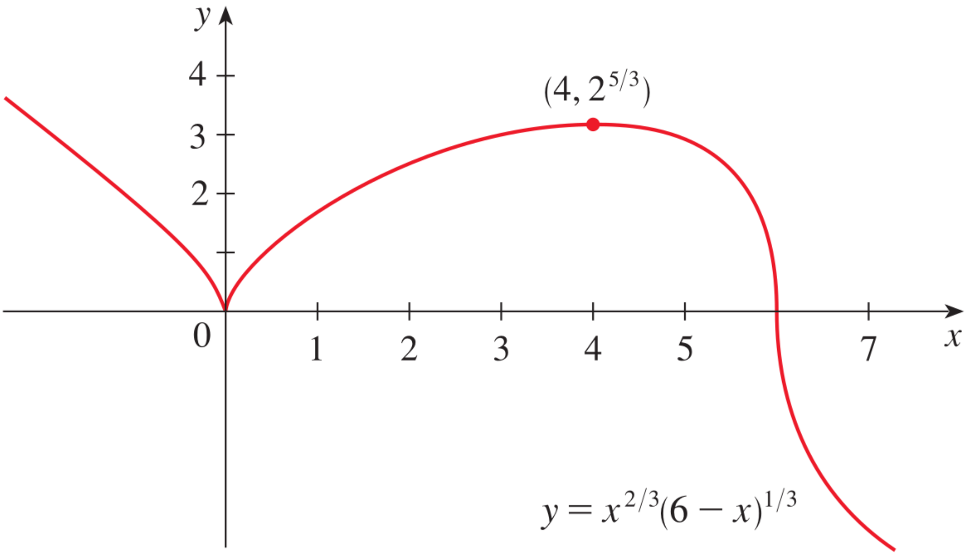
Example 5 - Continued

$f' = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$, $f'' = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$

Example 5 - Continued

Sketch the graph of $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

Illustration of Example 5



Test Feedback

- Difficulty?
- Time?
- Better preparation?

Section 4.7 - Optimization Problems

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

600ft, 1200ft

Example - Continued

600ft, 1200ft

Guide

1. Understand the problem. Unknowns? Constraints? Given Quantities?
2. Draw a sketch
3. Introduce Notation: Assign variables to the quantities of interest
4. Express the objective quantity in terms of the unknowns
5. Use the constraints to simplify the expression to one variable
6. Use the closed interval method to find the absolute maximum/minimum

Example

A cylindrical can is to be made to hold $1L$ of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Example - Continued

$r = \frac{10}{\sqrt[3]{2\pi}}$cm ≈ 5.4cm, $h = 2r \approx 10.8$cm

Useful Result

If $f > 0$ then the value that maximizes $f(x)$ is the same as the one that maximizes $(f(x))^2$

Example

Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Example - Continued

$A = r^2$

Example

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Example - Continued

$(2, 2)$

Example

A woman launches her boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. She could row her boat directly across the river to point C and then run to B, or she could row directly to B, or she could row to some point D between C and B and then run to B. If she can row $6\,\frac{\text{km}}{\text{h}}$ and run $8\,\frac{\text{km}}{\text{h}}$, where should she land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the woman rows.)

Example - Continued

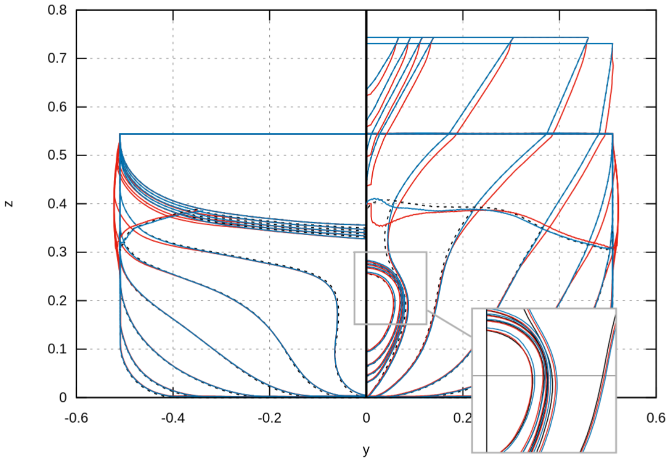
Example - Continued

$y = \frac{v_r x_{\text{max}}}{\sqrt{v_l^2 - v_r^2}} = \frac{9}{\sqrt{7}}\,\text{km}$

Lots of Applications

- Economics
  - Minimize Cost
  - Maximize Revenue
- Engineering
  - Maximize output of a wind turbine/farm
  - Optimize network coverage
  - Optimize efficiency of an engine
  - ...
- Physics
  - Least Action Principle $\rightarrow$ modern physics
  - Soap bubbles minimize the area
- ...

Connected Problems - Outlook



Shape optimization of a container ship

Sections 4.9/5.4 Antiderivatives

Antiderivative

Definition (Antiderivative)

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

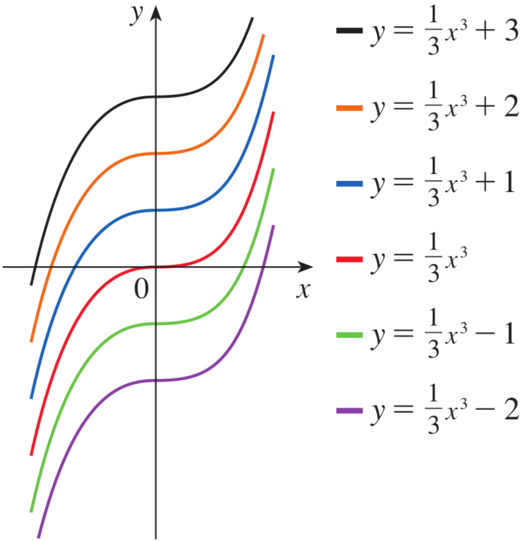
Example $f(x) = x^2$

Non-Uniqueness

If F is the antiderivative of f on I, then the most general antiderivative of f on I is

$$F(x) + C,$$

where C is an arbitrary constant.



Indefinite Integral

Definition (Indefinite Integral)

The indefinite integral

$$\int f(x)\;dx$$

is the general antiderivative $F(x)$, i.e. all $F(x)$ with

$$F'(x) = f(x).$$

Example

Calculate $\int \sin(x)\ dx$

Abuse of Notation

Find the general antiderivative of $\frac{1}{x^2}$

Nevertheless we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

and imply that it is valid on the interval $(-\infty, 0)$ and the interval $(0, \infty)$

Formula Sheet

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Exercise

Find $g(x)$ for $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$ and $g(0) = 2$

$-4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + 6$

Exercise 2

Find $f(x)$ for $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$

$x^4 + x^3 - 2x^2 - 3x + 4$

Sections 4.9/5.4 Antiderivatives

Graphing Antiderivatives

Graphing Idea

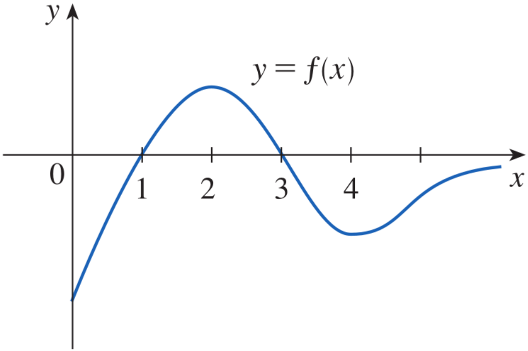
One has

$$\int f(x)\ dx = F(x) \implies F'(x) = f(x)$$

and the derivative yields the slope of the function.
So we can graph an antiderivative $F(x)$ by starting at some arbitrary point (since we can always add a constant value) and use $f(x)$ as the slope.

Example

Graph an antiderivative $F(x)$ of the function on the right



Linear Motion

The acceleration $a(t)$ is the derivative of the velocity $v(t)$ with respect to time, which again is the derivative of the position $x(t)$ with respect to time.
The earths acceleration is $\approx 9.8\frac{\text{m}}{\text{s}^2}$. Disregarding drag, what is the final velocity, when jumping from a 10m tower into a pool?

Linear Motion - Continued

$T = \sqrt{\frac{20}{9.8}}s \approx 1.43s$, $V = -14\,\frac{\mathsf{m}}{\mathsf{s}} = -50.4\,\frac{\mathsf{km}}{\mathsf{h}}$

Appendix E - Sigma Notation

Sigma Notation

Definition

If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n.$$

$\sum\limits_{i=m}^{n}$ is the summation over integer values of i from m to n.

You can also think of a_i as a function

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Examples

- $\sum_{i=1}^{4} i^2$

- $\sum_{i=1}^{3} \frac{i-1}{i^2+3}$

- $\sum_{i=1}^{n} i$

More Examples

- $\sum_{i=1}^{4} 2$

- $\sum_{i=1}^{n} 1$

- $\sum_{k=1}^{4} \frac{1}{k}$

- $\sum_{j=1}^{5} 2^j$

Other way

Write the following in sigma notation

- $2^3 + 3^3 + 4^3 + 5^3 + 6^3$

- $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^k}$

Useful Identities

Theorem

If c is a constant (it is independent of i), then

- $$\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$$

- $$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

- $$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Gauß Exercise

(You do not need to know this derivation for the test) Find a simplified expression for $\sum_{i=1}^{n} i$

$\frac{n(n+1)}{2}$, Carl Friedrich Gauß is said to have solved this when he was 10

Identites

Important identities

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example

Evaluate $\sum_{i=1}^{n} i(4i^2 - 3)$

$\frac{n(n+1)(2n^2+2n-3)}{2}$

Example

Evaluate $\sum_{i=0}^{100}\left(\frac{1}{2^i} - \frac{1}{2^{i+1}}\right)$

$1 - \frac{1}{2^{101}} \approx 0.9999999999999999999999999999996$

Example

Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\left(\frac{i}{n}\right)^2 + 1\right)$

Example - Continued

4

Example

Find the number n such that $\sum_{i=1}^{n} i = 78$

12

Outlook

Just Information, Not important for the Test:

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_{n-2} \cdot a_{n-1} \cdot a_n$$

For $|q| < 1$

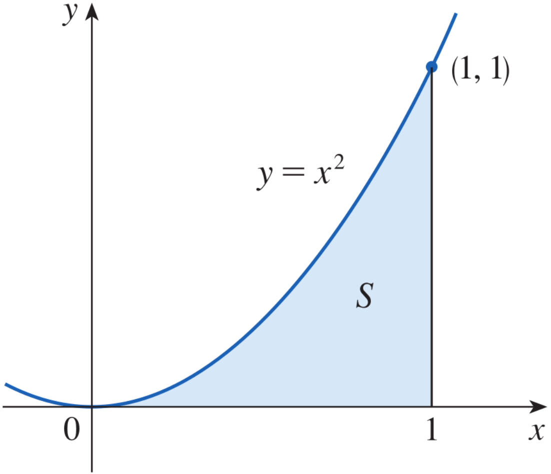
$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2$$

is called the geometric series, which is connected to the ζ-function, which again is connected to the Riemann Hypothesis. And if one could assign a value to $1+2+3+4+\ldots$, a good candidate would be $-\frac{1}{12}$. There is also a Numberphile Videos about this on Youtube.

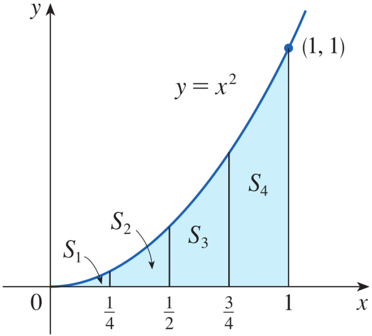
Section 5.1 - Area and Distance

Motivation

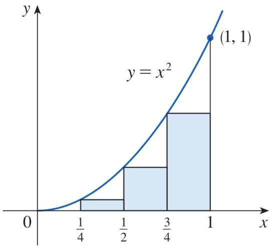
We want to find the following area S

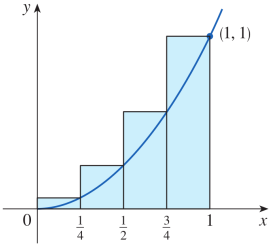


Theory - Towards a Result



Left and Right Side Areas





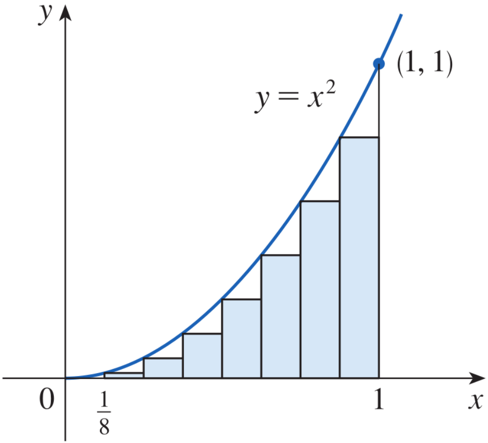
$$\begin{aligned} L_4 &= A_1 + A_2 + A_3 + A_4 \\ &= \Delta x f(0) + \Delta x f\left(\frac{1}{4}\right) + \Delta x f\left(\frac{1}{2}\right) + \Delta x f\left(\frac{3}{4}\right) \\ &= \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) \\ &= \frac{1}{4}\frac{1}{4^2} + \frac{1}{4}\frac{1}{2^2} + \frac{1}{4}\frac{3^2}{4^2} = \frac{7}{32} = 0.21875 \end{aligned}$$

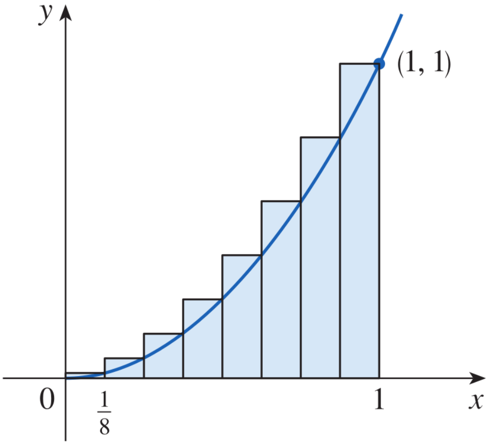
$$\begin{aligned} R_4 &= A_1 + A_2 + A_3 + A_4 \\ &= \Delta x f\left(\frac{1}{4}\right) + \Delta x f\left(\frac{1}{2}\right) + \Delta x f\left(\frac{3}{4}\right) + \Delta x f(1) \\ &= \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1) \\ &= \frac{1}{4}\frac{1}{4^2} + \frac{1}{4}\frac{1}{2^2} + \frac{1}{4}\frac{3^2}{4^2} + \frac{1}{4} = \frac{15}{32} = 0.46875 \end{aligned}$$

So $0.21875 = L_4 < S < R_4 = 0.46875$

Theory - Left and Right Side Areas

Increase the number of sections from 4 to 8

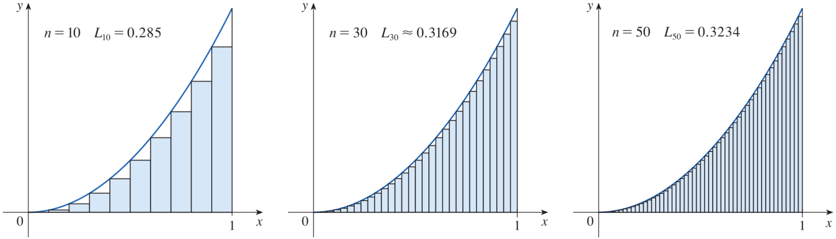




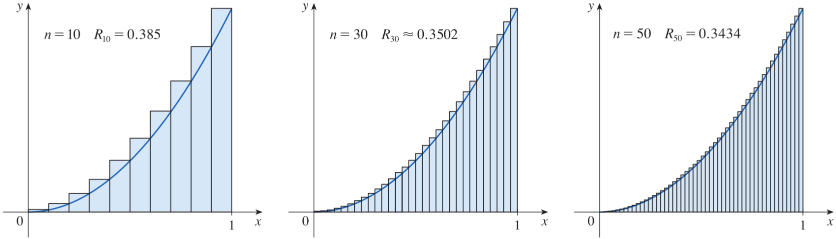
So $0.2734375 = L_8 < S < R_8 = 0.3984375$

Theory - Increase the number of sections more and more

Then the left side area will look like this



And the right side area will look like this



Theory - Convergence

Approaching infinitely many rectangles

$$
\begin{aligned}
S = \lim_{n\to\infty} R_n &= \lim_{n\to\infty} \left(\frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right) \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 \right) = \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{i=1}^{n} i^2 \right) \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{(n^2+n)(2n+1)}{6n^3} \\
&= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} \\
&= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) = \frac{1}{3}
\end{aligned}
$$

Example

Find an expression for an approximation of the area under $f(x) = e^{-x}$ on the interval $[0, 2]$ using 6 rectangles

Example

$$A_6 = \frac{1}{3}\left(e^{-\frac{1}{3}} + e^{-\frac{2}{3}} + e^{-1} + e^{-\frac{4}{3}} + e^{-\frac{5}{3}} + e^{-2}\right) \approx 0.73$$

General Area Formula

Theorem

The area A between a continuous function $f(x) \geq 0$ and the x-axis on the interval $[a, b]$ is given by

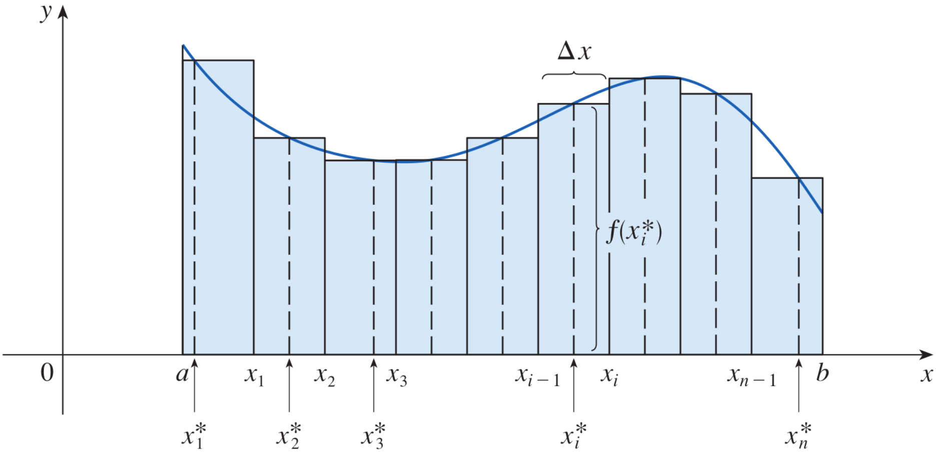
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \; f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \; f\left(a + i\frac{b-a}{n}\right),$$

where the sampling points are $x_i = a + i\Delta x$ and their distance is $\Delta x = \frac{b-a}{n}$.

Remarks

- The limit will always exist (since f is continuous and we assume a and b are finite)
- In the definition the right side area is chosen, but we could have also chosen the left side limit or any sampling point in the equally spaced intervals. We could have taken the left side area L_n or any sampling point x_i in the equally spaced portions (x_i in $[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}]$)

Sampling Points



General Sampling Points

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \; f(x_i^*), \qquad \Delta x = \frac{b-a}{n}, \quad x_i^* \in \left[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right]$$

Example

Find an expression for the area under $f(x) = e^{-x}$ on the interval $[0, 2]$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} e^{-\frac{2i}{n}} \approx 0.86$

Distance Problem

The average velocity v of an object is given by

$$v = \frac{\text{distance}}{\text{time}} = \frac{d}{t} \qquad \Longrightarrow \qquad d = vt.$$

If we know the velocity $v = f(t)$ of an object at equally spaced time intervals t_i in the interval $a \leq t \leq b$ and $f(t) \geq 0$. Then the distance traveled in the time interval is approximately

$$d_n = \Delta t \;f(t_1) + \Delta t \;f(t_2) + \Delta t \;f(t_3) + \cdots + \Delta t \;f(t_n) = \sum_{i=1}^{n} \Delta t \;f(t_i),$$

where $\Delta t = \frac{b-a}{n}$ and $t_i = a + i\frac{b-a}{n}$.

If we increase the measurement points (sampling points) the approximation gets better and in the limit we recover the actual distance

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta t \;f(t_i),$$

which is the same formula as the Area formula $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \;f(x_i)$.

Exercise

Determine a region whose area is equal to the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \frac{1}{4 + \ln\left(3 + \frac{2i}{n}\right)}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \ f(x_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \ f\left(a + i\frac{b-a}{n}\right),$$

where $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{n}$

Exercise

$f(x) = \frac{1}{4+\ln x}$ on $[3, 5]$

Section 5.2 - The Definite Integral

Previously

Definition

The area A between a continuous function $f(x) \geq 0$ and the x-axis on the interval $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \; f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \; f\left(a + i\frac{b-a}{n}\right),$$

where the sampling points are $x_i = a + i\Delta x$ and their distance is $\Delta x = \frac{b-a}{n}$.

Remarks

- The limit will always exist (since f is continuous and we assume a and b are finite)
- In the definition the right side area is chosen, but we could have also chosen the left side limit or any sampling point in the equally spaced intervals. We could have taken the left side area L_n or any sampling point x_i in the equally spaced portions (x_i in $[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}]$)

Theory - General Definition

Definition (The Definite Integral)

If f is a function defined for $a \leq x \leq b$, we set $\Delta x = \frac{b-a}{n}$, divide the interval in pieces of length Δx and let $a = x_0, x_1, \dots, x_n = b$ be the end points of these pieces. Let $x_1^\star, x_2^\star, \dots, x_n^\star$ be any sampling points in these pieces of the interval, i.e. $x_i^\star$ is in the i-th subinterval $[x_{i-1}, x_i]$. Then the *definite integral of f from a to b* is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\ \Delta x,$$

provided this limit exists and gives the same value for all sampling points.

Then f is called *integrable* on $[a, b]$.

Definition

Theorem (The Definite Integral for nice functions)

If f is continuous (or f is continuous except for a finite number of jump discontinuities) on $[a, b]$, then f is integrable on $[a, b]$, that is the definite integral

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\, \Delta x,$$

exists and is independent of the sample points in the subintervals.

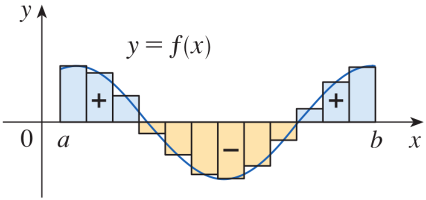
Notation/Nomenclature

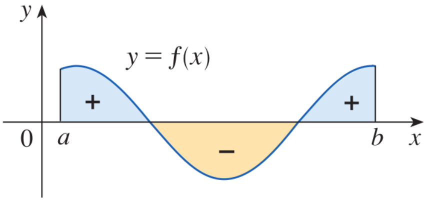
- $\int_a^b$ Integral sign with integration limits a and b
- $f(x)$ integrand
- dx differential
- $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\, \Delta x$ Riemann Sum

Note

- $\int \leftrightarrow \lim \sum$
- $f(x) \leftrightarrow f(x_i^\star)$
- $dx \leftrightarrow \Delta x$

Interpretation of the Definite Integral





The definite Integral calculates the net area between f and the x-axis $=$ area above - area under

Example (Version 1)

Find $\int_{-1}^{2} x\ dx$

$\frac{3}{2}$

Example (Version 1)

Evaluate $\int_1^3 (3 - 2x^2)\ dx$

Example (Version 1) - Continued

Previously

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$-\frac{34}{3}$

Example (Version 1)

Evaluate $\int_0^1 \sqrt{1-x^2}\ dx$

$\frac{\pi}{4}$

Properties 1

- $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

  even if $c > b$ or $c < a$

- $$\int_a^b c\ dx = (b-a)c$$

Properties 2

- $$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$
- $$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$
- $$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$
- $$\int_a^a f(x)\ dx = 0$$

Properties Overview

$$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

$$\int_a^b c\ dx = (b-a)c$$

$$\int_a^a f(x)\ dx = 0$$

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

Section 5.3 - The Fundamental Theorem of Calculus

Integrals and Derivatives

Theorem (The Fundamental Theorem Of Calculus - Part 2)

Suppose f is continuous on $[a, b]$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a) =: F(x)\Big|_a^b,$$

where F is any antiderivative of f, that is $F' = f$.

Remarks

- Part 1 later
- Since this holds for any antiderivative we can choose the one that works best for us
- $F(x)\Big|_a^b$ is just notation for $F(b) - F(a)$ and it will come in handy later
- The assumption that f is continuous on $[a, b]$ is somewhat important. It can be weakened but we need some kind of regularity.

Motivating Example (Version 2)

Find $\int_0^1 x^2 \; dx$

$\frac{1}{3}$

Example (Version 2)

Calculate $\int_{-1}^{2} x \ dx$

$\frac{3}{2}$

Example (Version 2)

Evaluate $\int_1^3 (3 - 2x^2)\ dx$

$-\frac{34}{3}$

Example (Version 2)

Evaluate $\int_0^1 \sqrt{1-x^2}\ dx$

Hint: Try

$$\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}(x)\right)$$

Previously

$$(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$$

$\frac{\pi}{4}$

Examples

Calculate $\int_0^\pi \sin(x)\ dx$

2

Examples

Calculate $\int_1^5 \frac{1}{x^3}\ dx$

$\frac{12}{25} = 0.48$

Careful

$\int_{-1}^{3} \frac{1}{x^2}\ dx$

Examples

Calculate $\int_3^1 4x^2 + \frac{1}{x} - e^x \; dx$

$-\frac{104}{3} - e + e^3 - \log(3) \approx -18.4$

Theorem (The Fundamental Theorem Of Calculus - Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ dt, \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) = f(x)$$

Remarks

- g describes the (net) area under f from a to a variable x.
- In short $\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$

Example

Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\, dt$

$\sqrt{1+x^2}$

Example

Calculate $\frac{d}{dx}\int_1^{x^4} \sec t \; dt$

$4x^3 \sec(x^4)$

Example - Substitution

Calculate $\frac{d}{dx}\int_1^{x^4} \sec t \ dt$

$4x^3 \sec(x^4)$

Overview

Theorem (The Fundamental Theorem of Calculus)

1. If f is continuous on $[a, b]$
$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$
in (a, b)
2. If f' is continuous on $[a, b]$
$$\int_a^x f'(t)\ dt = f(x) - f(a)$$
for x in $[a, b]$

- Integration is the opposite of differentiation

General Overview Of Integration

Suppose $F'(x) = f(x)$ (so F is an antiderivative of f) is continuous. Then

- the indefinite integral (or general antiderivative) is

$$\int f(x)\ dx = F(x) + C$$

- the definite integral (or net area between f and the x-axis between a and b) is

$$\text{Net Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n}\ f\left(a + i\frac{b-a}{n}\right) \overset{\text{Def}}{=} \int_a^b f(x)\ dx \overset{\text{FTC2}}{=} F(x)\Big|_a^b \overset{\text{Notation}}{=} F(b) - F(a)$$

- Fundamental Theorem of Calculus 1

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

- Fundamental Theorem of Calculus 2

$$\int_a^x f(t)\ dt = \int_a^x F'(t)\ dt = F(x) - F(a)$$

For $F(x) = \frac{1}{2}x^2$ and $f(x) = F'(x) = x$

- the indefinite integral (or general antiderivative) is

$$\int x \; dx = \frac{1}{2}x^2 + C$$

- the definite integral (or net area between $y = x$ and the x-axis between -1 and 2) is

$$\text{Net Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left(-1 + i\frac{3}{n}\right) \overset{\text{Def}}{=} \int_{-1}^{2} x \; dx \overset{\text{FTC2}}{=} \frac{1}{2}x^2\Big|_{-1}^{2} \overset{\text{Notation}}{=} \frac{1}{2}(2)^2 - \frac{1}{2}(-1)^2$$

- Fundamental Theorem of Calculus 1

$$\frac{d}{dx}\int_a^x t \; dt = x$$

- Fundamental Theorem of Calculus 2

$$\int_a^x t \; dt = \frac{1}{2}x^2 - \frac{1}{2}a^2$$

Outlook - Not Important for Test

Fundamental Theorem

$$\int_{\text{Interval}} \frac{d}{dx} F \ dx = F\Big|_{\text{Endpoints}}$$

Generalizations

- This also works in higher dimensions Gauß/Divergence Theorem

$$\int_{\text{Volume}} \nabla \cdot F \ dV = \int_{\text{Surface}} n \cdot F \ dS$$

- Or even more general domains Stokes Theorem

$$\int_{\text{Interior}} d\omega = \int_{\text{Boundary}} \omega$$

Related

- The Sobolev Embedding

$$\left(\int_{R^n} |\nabla^l f|^q \right)^{\frac{1}{q}} \leq C \left(\int_{R^n} |\nabla^k f|^p \right)^{\frac{1}{p}}$$

for $\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$ if $(k-l)p < n$ connects differentiability and integration in higher dimensions. It also says that if the function has *good* integrability properties it is differentiable/smooth.

Test 2 Information is on childsmath.


TEACHING
AWARDS
2025
Nominations open from
Oct 27 to Nov 10
Nominate
a Professor
Nominate
a TA
Find links at @msu_macademics
MSU.MACADEMICS
MSU_MACADEMICS
MSU
Macademics
McMaster Students Union
MSU

Section 5.5 - The Substitution Rule

Substitution for Indefinite Integrals

Derivation

By the chain rule

$$\big(F(g(x))\big)' = F'(g(x))g'(x)$$

and therefore

$$\int F'(g(x))g'(x)\ dx = \int \big(F(g(x))\big)'\ dx = F(g(x)) + C$$

If $u = g(x)$, then

$$\int F'(g(x))g'(x)\ dx = F(g(x)) + C = F(u) + C = \int F'(u)\ du$$

If we set $f = F'$ we have

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du$$

Substitution for Indefinite Integrals

Theorem Substitution Rule

If $u = g(x)$ is a differentiable function whose range is I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du$$

Remarks

- This helps us calculating Integrals
- Use this when a derivative of other parts of the integrant is multiplied by
- Note $g'(x)\ dx = du$
- (Strictly speaking one needs different tools to see $g'(x)\ dx = du \longleftrightarrow \frac{du}{dx} = g'(x)$ but this works)

Example

Calculate $\int x^3 \cos(x^4 + 2) \; dx$

$\frac{1}{4} \sin(x^4 + 2) + C$

Example

Calculate $\int \sqrt{2x + 1}\ dx$

$\frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

Example

Calculate $\int \frac{x}{\sqrt{1-4x^2}}\ dx$

$-\frac{1}{4}\sqrt{1-4x^2}+C$

Example - less obvious

Calculate $\int \sqrt{1+x^2}x^5 \; dx$

$$\frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

Example - less obvious

Calculate $\int \tan x \ dx$

$-\ln|\cos x| + C$

Section 5.5 - The Substitution Rule

Substitution for Definite Integrals

Theory

Previously we had for $u = g(x)$

$$\int F'(g(x))g'(x)\ dx = F(g(x)) + C = F(u) + C$$

and so with $u_a = g(a)$ and $u_b = g(b)$

$$\int_a^b F'(g(x))g'(x)\ dx = F(g(b)) - F(g(a)) = F(u_b) - F(u_a) = \int_{u_a}^{u_b} F'(u)\ du = \int_{g(a)}^{g(b)} F'(u)\ du,$$

and writing again $f = F'$

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Substitution for Definite Integrals

Theorem Substitution Rule

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Remark

- When using this we can either
  - calculate $g(a)$ and $g(b)$ first and change the integration limit in the beginning

    $$\int_{g(a)}^{g(b)} f(u)\ du = F(u)\Big|_{g(a)}^{g(b)}$$

    or
  - Plug in $u = g(x)$ before evaluating and use a and b

    $$\int_{g(a)}^{g(b)} f(u)\ du = F(u)\Big|_{g(a)}^{g(b)} = F(g(x))\Big|_a^b$$

Example

Calculate $\int_0^4 \sqrt{2x+1}\ dx$

Example - Continued

$\frac{26}{3}$

Example

Calculate $\int_1^2 \frac{dx}{(3-5x)^2}$

$\frac{1}{14}$

Example

Calculate $\int_1^e \frac{\ln x}{x}\ dx$

$\frac{1}{2}$

Section 5.5 - The Substitution Rule

Symmetries

Symmetries

Theorem

Suppose f is continuous on $[-a, a]$. If

- f is even ($f(-x) = f(x)$), then
$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$
- f is odd ($f(-x) = -f(x)$), then
$$\int_{-a}^{a} f(x)\ dx = 0$$

Example

Calculate $\int_{-2}^{2} x^6 + 1 \; dx$

$\frac{284}{7}$

Example

Calculate $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\ dx$

0

Section 6.1 - Areas Between Curves

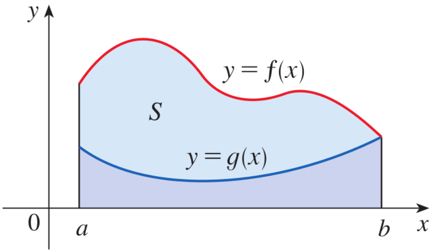
Theory

Suppose $f(x) \geq g(x)$ and we want find the area between f and g in an interval $[a, b]$

$$S = \text{Area between } f \text{ and } g$$

$$= (\text{Area under } f) - (\text{Area under } g)$$

$$= \int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b f(x) - g(x)\ dx$$



Area between "horizontal" Curves

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is

$$A = \int_a^b f(x) - g(x)\ dx$$

Note that for $g = 0$ we get the previous formula for the area between a positive function and the x-axis.

Example

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

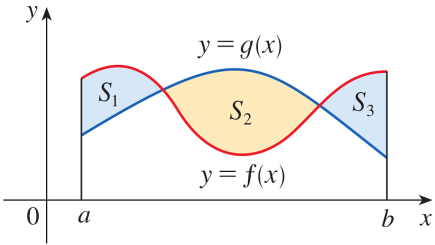
Example - Continued

$\frac{1}{3}$

General Area between "horizontal" Curves

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous is

$$A = \int_a^b |f(x) - g(x)| \, dx$$



Example

Find the area between the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$

Example - Continued

$2\sqrt{2}-2$

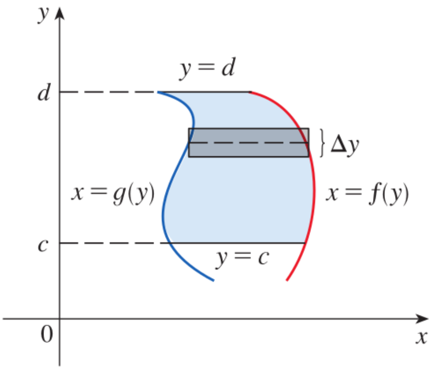
Section 6.1 - Areas Between Curves

Integrating with respect to y

Area between "vertical" Curves

The area A bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where $f(y) \geq g(y)$ and f and g are continuous is

$$A = \int_c^d f(y) - g(y)\, dy$$

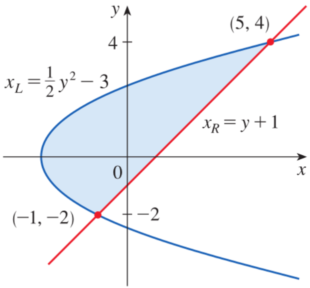


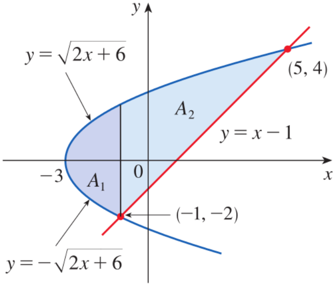
Example

Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$

18

Example - Continued





Example

Find the area between the parabola $y = x^2$, its tangent line at $x = 1$ and the x-axis

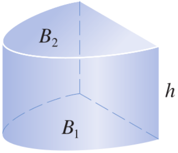
Example - Continued

$\frac{1}{12}$

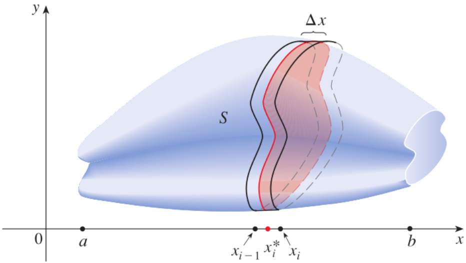
End of material for Test 2.

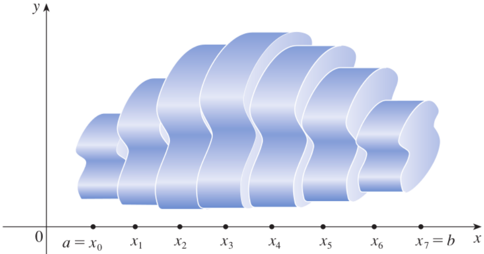
Section 6.2 - Volumes

Volumes - Theory



(a) Cylinder $V = Ah$

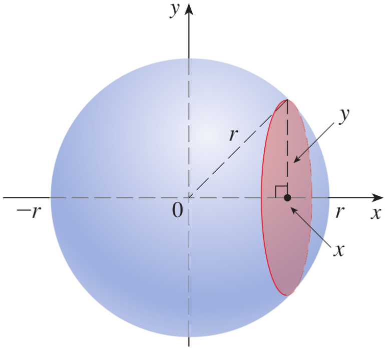




$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x A(x_i^\star) = \int_a^b A(x)\ dx$$

Example - Volume of a Sphere (Mostly Theory)

Calculate the volume of a sphere of radius r.

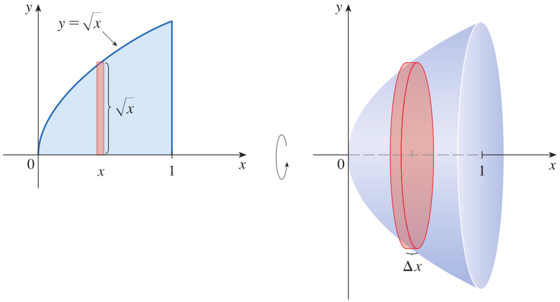


Example - Volume of a Sphere (Mostly Theory) - Continued

$V = \frac{4}{3}\pi r^3$

Example

Find the volume of the solid obtained by rotating the region under the curve $y = \sqrt{x}$ from 0 to 1 about the x-axis.

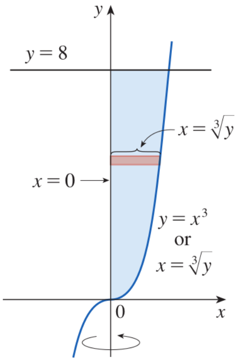


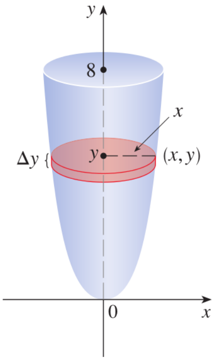
Example - Continued

$$\frac{\pi}{2}$$

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



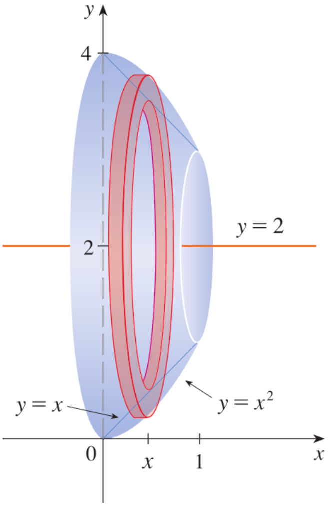


Example - Continued

$\frac{96}{5}\pi$

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the line $y = 2$.



Example - Continued

$\frac{8}{15}\pi$

Section 6.4 - Work

Definition Work

For a constant force F over a distance d the work W is given by

$$W = F\,d$$

with units

$$\begin{array}{ll} \text{metric} & [F] = \mathsf{N} = \dfrac{\mathsf{kg\,m}}{\mathsf{s}^2}, \quad [d] = \mathsf{m}, \quad [W] = \mathsf{Nm} = \mathsf{J} = \mathsf{Joule} \\ \text{imperial} & [F] = \mathsf{lb}, \quad [d] = \mathsf{ft}, \quad [W] = \mathsf{ft\text{-}lb} = \mathsf{foot\text{-}pound}, \quad 1\mathsf{ft\text{-}lb} \approx 1.36J \end{array}$$

For a non-constant force $f(x)$ over a distance from a to b the work is given by

$$W = \int_a^b f(x)\;dx.$$

For gravitational forces

- In metric units 1kg exerts a force of 9.81N

$$F = mg, \qquad g = 9.81\frac{\mathsf{m}}{\mathsf{s}^2}, \qquad\qquad F = 1\mathsf{kg}\cdot 9.81\frac{\mathsf{m}}{\mathsf{s}^2} = 9.81\frac{\mathsf{kg\,m}}{\mathsf{s}^2} = 9.81\mathsf{N}$$

- In imperial units 1lb exerts a force of 1lb

$$\text{"}F = m\cdot 1\text{"} \qquad\qquad F = 1\mathsf{lb}_{\mathsf{mass}} \cdot 1\frac{\mathsf{lb}_{\mathsf{force}}}{\mathsf{lb}_{\mathsf{mass}}} = 1\mathsf{lb}_{\mathsf{force}} = 1\mathsf{lb}$$

Example

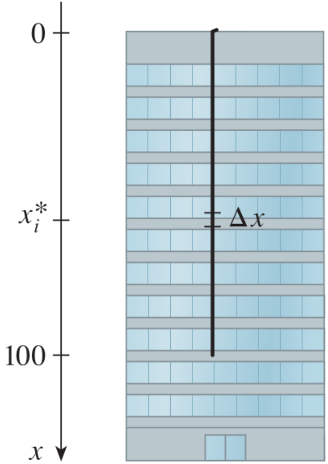
A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done in stretching the spring from 15cm to 18cm?

Example - Continued

$W = 8\,\frac{\mathsf{N}}{\mathsf{cm}} \int_{5\mathsf{cm}}^{8\mathsf{cm}} x \; dx = 1.56\mathsf{J}$

Example

A 200lb cable is 100ft long and hangs vertically from the top of a tall builds. How much work is required to lift the cable to the top of the building.

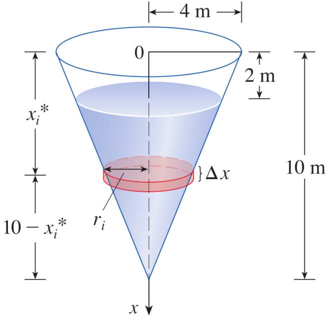


Example - Continued

$W = \frac{ML}{2} = 10000$ft-lb

Example

A tank has the shape of an inverted circular cone with height 10m and base radius of 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (Water density $1000\frac{\text{kg}}{\text{m}^3}$)



Example - Continued

Example - Continued

$W = \frac{R^2}{H^2}\pi\rho g \int_{x_0}^{H} x(H-x)^2 \; dx = 1568\pi \int_2^{10} x(10-x)^2 \; dx\text{J} \approx 3.4 \cdot 10^6\,\text{J}$

Review Test 2

Read childsmath/more announcements/Test #2 Information!

Mean Value Theorem

Theorem (Mean Value Theorem)

If f is continuous and differentiable, then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Simplified it says that a continuous differentiable function has its average slope in some point.

Suppose (everything is smooth and) $f'(x) \leq d$ then

$$f(b) \leq f(a) + (b - a)d$$

How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Concavity Test

- If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Second Derivative Test

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum in c
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum in c

Indeterminate Forms and L'Hospital's Rule

L'Hôpital's rule

Suppose

- f and g are differentiable and $g'(x) \neq 0$ in an interval around a (except possibly a itself)
- $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ or
  $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$

Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the right-hand side limit exists or is ∞ or $-\infty$

- For "$0 \cdot \infty$" write $\lim f(x)g(x) = \lim \frac{f(x)}{\frac{1}{g(x)}}$ and use L'Hôpital's rule
- For "$\infty - \infty$" including fractions try to find a common denominator and use L'Hôpital's rule

f^g

For $\lim_{x\to a} \left(f(x)^{g(x)}\right)$ use
$$\lim_{x\to a} \left(f(x)^{g(x)}\right) = \lim_{x\to a} \left(e^{\ln\left(f(x)^{g(x)}\right)}\right) = \lim_{x\to a} \left(e^{g(x)\ln(f(x))}\right) = e^{\lim_{x\to a}(g(x)\ln(f(x)))}$$

Summary of Curve Sketching

Guide for Curve Sketching

1. Domain
2. Intercepts
3. Symmetries
   - Even
   - Odd
   - Periodic
4. Asymptotes
   - Horizontal
   - Vertical
   - Slant
5. Increasing/Decreasing
6. Extrema
7. Concavity and Inflection Points
8. Sketch the curve by using the previous steps

Optimization Problems

Optimization Problems Guide

1. Understand the problem. Unknowns? Constraints? Given Quantities?
2. Draw a sketch
3. Introduce Notation: Assign variables to the quantities of interest
4. Express the objective quantity in terms of the unknowns
5. Use the constraints to simplify the expression to one variable
6. Use the closed interval method to find the absolute maximum/minimum

If $f > 0$ then the value that maximizes $f(x)$ is the same as the one that maximizes $(f(x))^2$

Antiderivatives and indefinite Integrals

F is an antiderivative of f if $F' = f$.

If F is the antiderivative of f on I, then the general antiderivative/indefinite Integral of f on I is

$$\int f(x)\;dx = F(x) + C,$$

where C is an arbitrary constant.

Antiderivatives and indefinite Integrals

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Σ-Notation

Definition

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n \qquad \text{for } n > m$$

Important identities

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Telescope sum $\Leftrightarrow$ stuff cancels with other terms

Area, Distance, Definite Integral

Theorem

The net area A between a continuous function and the x-axis on the interval $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \ f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \ f\left(a + i\frac{b-a}{n}\right) = \int_a^b f(x) \ dx,$$

where the sampling points are $x_i = a + i\Delta x$ and their distance is $\Delta x = \frac{b-a}{n}$.

$$\int_a^c f(x) \ dx + \int_c^b f(x) \ dx = \int_a^b f(x) \ dx \qquad \int_a^b f(x) \ dx = -\int_b^a f(x) \ dx$$

$$\int_a^b c \ dx = (b-a)c \qquad \int_a^a f(x) \ dx = 0$$

$$\int_a^b f(x) + g(x) \ dx = \int_a^b f(x) \ dx + \int_a^b g(x) \ dx \qquad \int_a^b cf(x) \ dx = c\int_a^b f(x) \ dx$$

The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus - Overview)

1. If f is continuous on $[a, b]$

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

   in (a, b)

2. If f' is continuous on $[a, b]$

$$\int_a^x f'(t)\ dt = f(x) - f(a)$$

   for x in $[a, b]$

Integration - Overview

Suppose $F'(x) = f(x)$ (so F is an antiderivative of f) is continuous. Then

- the indefinite integral (or general antiderivative) is

$$\int f(x)\ dx = F(x) + C$$

- the definite integral (or net area between f and the x-axis between a and b) is

$$\text{Net Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n}\ f\left(a + i\frac{b-a}{n}\right) \overset{\text{Def}}{=} \int_a^b f(x)\ dx \overset{\text{FTC2}}{=} F(x)\Big|_a^b \overset{\text{Notation}}{=} F(b) - F(a)$$

- Fundamental Theorem of Calculus 1

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x) \qquad \Longrightarrow \qquad \frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f(g(x))g'(x)$$

- Fundamental Theorem of Calculus 2

$$\int_a^x f(t)\ dt = \int_a^x F'(t)\ dt = F(x) - F(a)$$

Substitution

Theorem (Substitution Rule for Indefinite Integrals)

If $u = g(x)$ is a differentiable function whose range is I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du$$

Theorem (Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Area between Curves

Theorem (horizontal, $f > g$)

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous and $f(x) \geq g(x)$ is

$$A = \int_a^b f(x) - g(x)\, dx$$

Theorem (horizontal, general)

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Theorem (vertical, $f > g$)

The area A bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where $f(y) \geq g(y)$ and f and g are continuous is

$$A = \int_c^d f(y) - g(y)\, dy$$

Section 6.5 - Average of Functions

Average of Functions

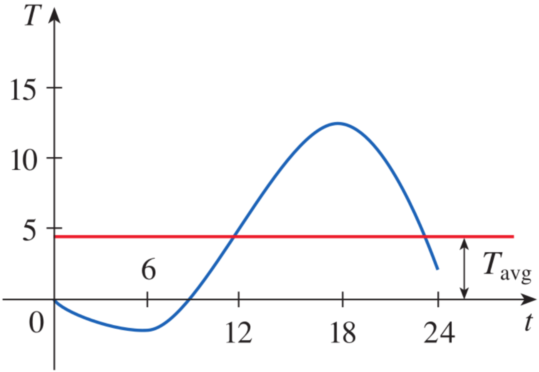
The Area below a function is its average height f_{avg} (or T_{avg}) times the width w

$$f_{\text{avg}}(b-a) = f_{\text{avg}} w = A = \int_a^b f(x)\ dx$$

and therefore

Average of a function

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\ dx$$



Alternatively splitting the interval into equally spaced cuts leads to the same as

$$f_{\text{avg}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{b-a}{n} f(a + i\frac{b-a}{n})}{b-a} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Example

Calculate the average of $f(x) = 1 + x^2$ in the interval $[-1, 2]$

2

Mean Value Theorem for Integrals

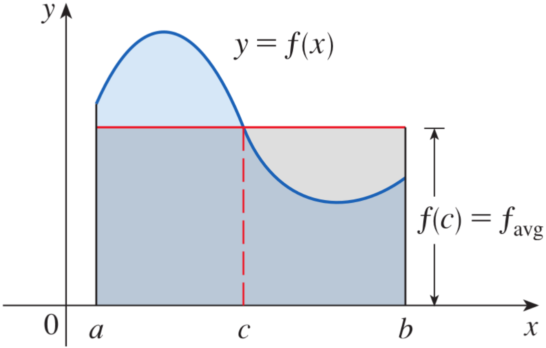
Theorem (Mean Value Theorem for Integrals)

If f is continuous in $[a, b]$, there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or equivalently

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

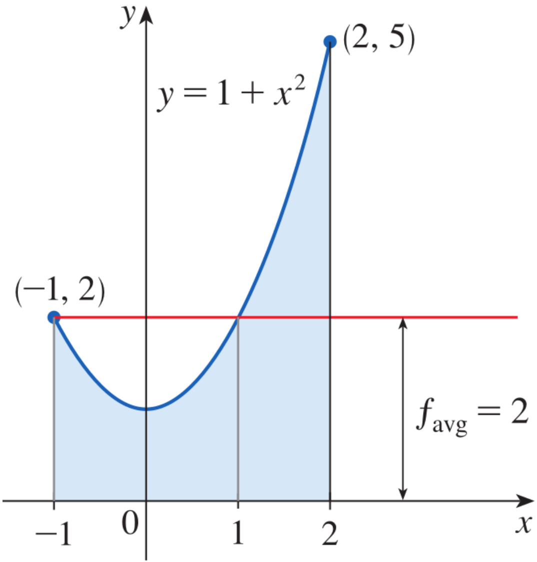


Example

What is c in the conclusion of the Mean Value Theorem for Integrals for $f(x) = 1 + x^2$ in the interval $[-1, 2]$

± 1

Example - Illustration



Section 7.1 - Integration by Parts

Theory for Indefinite Integrals

Integration by Parts for Indefinite Integrals

Integration by Parts Formula 1

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

or

$$\int u\ dv = uv - \int v\ du$$

Example

Calculate $\int x \sin x \; dx$

$-x\cos x + \sin x + c$

Example

Calculate $\int \ln x \; dx$

$x \ln x - x + c$

Example

Calculate $\int x^2 e^x \ dx$

Example - Continued

$(t^2 - 2t + 2)e^t + c$

Example

Calculate $\int e^x \sin x \; dx$

Example - Continued

$\frac{1}{2} e^x(\sin x - \cos x) + c$

Integration by Parts for Definite Integrals

Integration by Parts Formula 2

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx$$

Example

Calculate $\int_0^1 \tan^{-1} x \; dx$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$

Example - Continued

$\frac{\pi}{4} - \frac{\ln 2}{2} + c$

Example

Calculate $\int_1^e x^4(\ln x)^2 \; dx$

Example - Continued

$\frac{17}{125}e^5 - \frac{2}{125} + c$

Example - Reduction Formula

Proof the Reduction Formula: For an integer $n \geq 2$

$$\int \sin^n x \; dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \; dx$$

Example - Continued

Section 7.2 - Trigonometric Integrals

$\int \sin^m x \cos^n x \; dx$

Motivating example

Calculate $\int \sin x \cos^8 x \; dx$

This works since we started with power 1 of one trigonometric function and any other power for the other trig function and they are derivatives of each other

---

$-\frac{1}{9}\cos^9 x + c$

Strategy for $\int \sin^m x \cos^n x \; dx$ - Part 1

- If either m or n is 1 use substitution as before, e.g.

  For $\int \sin x \cos^8 x \; dx$, substitute $u = \cos x$

- If either m or n is odd use $\sin^2 x + \cos^2 x = 1$ to get to the previous case, e.g.

$$\begin{aligned}\int \sin^5 x \cos^8 x \; dx &= \int \sin x (1 - \cos^2 x)^2 \cos^8 x \; dx \\ &= \int \sin x \cos^8 x \; dx - 2 \int \sin x \cos^{10} x \; dx + \int \sin x \cos^{12} x \; dx\end{aligned}$$

  then $u = \cos x$

Strategy for $\int \sin^m x \cos^n x \ dx$ - Part 2

- For even m and $n = 0$ or even n and $m = 0$ use the half angle formulae

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

repeated to reduce to simple $\sin / \cos$, e.g.

$$\begin{aligned}\int \sin^4 x \ dx &= \frac{1}{4}\int (1 - \cos 2x)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx = \frac{3}{8}\int 1 \ dx - \frac{1}{2}\int \cos 2x \ dx + \frac{1}{8}\int \cos 4x \ dx \\ &= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c\end{aligned}$$

- If both $m > 0$ and $n > 0$ are even use $\sin^2 x + \cos^2 x = 1$ to reduce to the previous case, e.g.

$$\int \sin^4 x \cos^2 x \ dx = \int \sin^4 x (1 - \sin^2 x) \ dx = \int \sin^4 x \ dx - \int \sin^6 x \ dx$$

- Sometimes the identity following identity helps

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

Strategy Short

Short Summary

Use $\sin^2 + \cos^2 = 1$ to reduce until the power of $\sin$ or $\cos$ is 1 or 0. Then

1 use substitution

0 use half angle formula $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Exercise

Calculate $\int \sin^5 x \cos^2 x \; dx$

$$-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$$

Exercise

Calculate $\int \sin^2 x \cos^2 x \; dx$

$\frac{1}{8}x - \frac{1}{32}\sin(4x) + c$

Section 7.2 - Trigonometric Integrals

$\int \tan^m x \sec^n x \; dx$

Strategy for $\int \tan^m x \sec^n x \; dx$

- If n is even, use $\sec^2 x = 1 + \tan^2 x$ until one factor of $\sec^2 x$ is left and use $u = \tan x$ ($\tan' = \sec^2$), e.g.

$$\begin{aligned}\int \tan^4 x \sec^6 x \; dx &= \int \tan^4 x \sec^4 x \sec^2 x \; dx = \int \tan^4 x (1+\tan^2 x)^2 \sec^2 x \; dx \\ &= \int \tan^4 x \sec^2 x \; dx + 2\int \tan^6 x \sec^2 x \; dx + \int \tan^8 x \sec^2 x \; dx \\ &= \int u^4 \; du + 2\int u^6 \; du + \int u^8 \; du = \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 + c \\ &= \frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + c\end{aligned}$$

- If m is odd, use $\tan^2 x = \sec^2 x - 1$ until one factor of $\sec x \tan x$ is left and use $u = \sec x$ ($\sec' = \sec\tan$), e.g.

$$\begin{aligned}\int \tan^3 x \sec^5 x \; dx &= \int \tan x (\sec^2 x - 1)\sec^5 x \; dx = \int (\sec^6 x - \sec^4 x) \tan x \sec x \; dx \\ &= \int (u^6 - u^4) \; du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + c = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c\end{aligned}$$

Strategy Short

Short Summary

Use $\sec^2 = 1 + \tan^2$ until $\sec^2 x$ or $\sec x \tan x$ is left

$\sec^2 x$ substitute $u = \tan x$ since $\frac{du}{dx} = \sec^2 x$

$\sec x \tan x$ substitute $u = \sec x$ since $\frac{du}{dx} = \sec x \tan x$

Also

Helpful Identities

$$\int \tan x \; dx = \ln|\sec x| + c$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + c$$

Example

Calculate $\int \tan^3 x \; dx$

$\frac{1}{2}\sec^2 x - \ln|\sec x| + c$

Section 7.2 - Trigonometric Integrals

$\sin nx \cos mx$

Strategy for $\int \sin nx \cos mx \ dx$

Use

$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$
$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$

Example

Calculate $\int \sin 4x \cos 5x \; dx$

$\frac{1}{2}\cos x - \frac{1}{18}\cos 9x + c$

Section 7.3 - Trigonometric Substitution

Motivation

For integrals of type

$$\int \sqrt{a^2 - x^2}\, dx, \qquad a^2 \geq x^2 \Leftrightarrow |a| \geq |x|$$

which arise as the area of a circle or ellipse, we can do the following.
Set $\theta = \sin^{-1} \frac{x}{a}$ for $-1 \leq \frac{x}{a} \leq 1$, implying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (we need a one-to-one function to find an inverse), then $x = a \sin\theta$ and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = |a\cos\theta| = |a|\cos\theta$$

Therefore for $a > 0$ and $-a \leq x \leq a$

$$\frac{dx}{d\theta} = a\cos\theta \implies dx = a\cos\theta\; d\theta$$

$$\int \sqrt{a^2 - x^2}\; dx = \int a\cos\theta a\cos\theta\; d\theta = a^2 \int \cos^2\theta d\theta = a^2 \int \frac{1}{2}(1 + \cos(2\theta))\; d\theta$$

$$= \frac{a^2}{4}(2\theta + \sin(2\theta)) + c = \frac{a^2}{4}(2\theta + 2\sin\theta\cos\theta) + c = \frac{a^2}{2}\left(\theta + \sin\theta\sqrt{1 - \sin^2(\theta)}\right) + c$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \sin\left(\sin^{-1}\frac{x}{a}\right)\sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}\right) + c$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right) + c = \frac{1}{2}\left(a^2\sin^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2 - x^2}\right) + c$$

Trigonometric Substitution Overview

Expression	Substitution	Domain	Identity	sign
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	$\cos\theta \geq 0$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	$\sec\theta \geq 0$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3}{2}\pi$	$\sec^2\theta - 1 = \tan^2\theta$	$\tan\theta \geq 0$

Example

Calculate $\int \frac{\sqrt{9-x^2}}{x^2}\ dx$

$\cot' = -\csc^2$

$$-\cot\theta - \theta + c = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$$

Example

Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\ dx$

$\tan' = \sec^2$
$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$
$\cos\frac{\pi}{3} = \frac{1}{2}$

$\frac{3}{16}\int_1^{\frac{1}{2}} (1 - u^{-2})\ du = \frac{3}{32}$

Example - Easier

Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\ dx$

$\frac{1}{32}\int_9^{36} u^{-\frac{1}{2}} - 9u^{-\frac{3}{2}}\ du = \frac{3}{32}$

Section 7.4 - Integration of Rational Functions by Partial Fractions

Motivation

Calculate

$$\int f(x)\ dx = \int \frac{P(x)}{Q(x)}\ dx$$

as for example in

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\ dx$$

Proper and Improper Fractions

For $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials, if $\deg(P) \geq \deg(Q)$ one calls f improper and we can bring it (via long division) in the form

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

where $R(x)$, with $\deg(R) < \deg(Q)$, is the remainder and $S(x)$ is a polynomial. If $\deg(P) < \deg(Q)$ one calls f proper.

Example

$$f(x) = \frac{x^3 + x}{x - 1} = x^2 + x + 2 + \frac{2}{x - 1}$$

Here $P(x) = x^3 + x$, $Q(x) = x - 1$, $S(x) = x^2 + x + 2$ and $R(x) = 2$.

Example

Calculate $\int \frac{x^3+x}{x-1}\ dx$

$\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + c$

Section 7.4 - Integration of Rational Functions by Partial Fractions

Partial Fraction Decomposition

Explanatory Example

Calculate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}$

$A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$

$\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + c$

Goal

We can solve these integrals similar to the following

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int x+1+\frac{4x}{x^3-x^2-x+1}\,dx\\ &= \int x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\,dx\\ &= \frac{1}{2}x^2+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+c\\ &= \frac{1}{2}x^2+x-\frac{2}{x-1}+\ln\left|\frac{x-1}{x+1}\right|+c\end{aligned}$$

where the denominators in the second line are the factors of the denominator in the first line.

Overview

$f(x) = \frac{P(x)}{Q(x)}$

1. Make it a proper fraction
$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$
2. Try to factorize the denominator (see later)
3. Find the correct enumerators/denominators of the partial fraction decomposition (see later)
4. Multiply the partial fraction decomposition by $Q(x)$
5. Either
   - Plug in the roots

   or
   - Arrange the result by powers of x
6. Solve for the enumerators
7. Write $f(x) = S(x) + \frac{R(x)}{Q(x)}$ as the partial fraction decomposition and solve the integral

Overview - Factorization - 1

If the denominator Q is neither a linear function or an irreducable quadratic factorize it:
Either guess or use the quadratic formula
Example $Q(x) = x^2 - x - 2$

Example $Q(x) = 2x^3 + 3x^2 - 2x$

Overview - Factorization - 2

Example $Q(x) = 2x^3 + 3x^2 - 2x$

Example $Q(x) = x^4 - 16$

Not possible, called irreducable, if $Q(x) = (\dots)(ax^2 + bx + c)$, where $b^2 - 4ac < 0$

Overview - The Correct Choice of enumerators/denominators

Factorize as far as possible

- write linear factors as

$$\frac{A}{a_1x-b_1}+\frac{B}{a_2x-b_2}+\cdots+\frac{E}{a_kx-b_k}$$

- write irreducible quadratic factors as

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c}+\frac{A_2x+B_2}{a_2x^2+b_2x+c}+\cdots+\frac{A_kx+B_k}{a_kx^2+b_kx+c}$$

- For repeated linear or quadratic factors add terms with denominators up to this order

To summarize in one example

$$\begin{aligned}&\frac{\cdots}{x^2(3x+2)^2(x-5)^3(x^2+8x+1)(10x^2+9x+13)^3}\\ &\qquad=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{3x+2}+\frac{D}{(3x+2)^2}+\frac{E}{x-5}+\frac{F}{(x-5)^2}+\frac{G}{(x-5)^3}\\ &\qquad\quad+\frac{Hx+I}{x^2+8x+1}+\frac{Jx+K}{10x^2+9x+13}+\frac{Lx+M}{(10x^2+9x+13)^2}+\frac{Nx+O}{(10x^2+9x+13)^3}\end{aligned}$$

Overview - Solving the Integral

- Single linear terms $\displaystyle \int \frac{A}{ax-b}\,dx = \frac{A}{a}\ln|ax-b|$ by substitution or guessing

- Repeated linear terms $\displaystyle \int \frac{A}{(ax-b)^p}\,dx = \frac{A}{a(-p+1)}(ax-b)^{-p+1}$ for $p \neq 1$ by substitution or guessing

- Single quadratic terms

$$\int \frac{A}{ax^2+bx+c}\,dx = \frac{A}{D}\int \frac{1}{(x-E)^2+F^2}\,dx$$

$$= \frac{A}{D}\int \frac{1}{u^2+F^2}\,du = \frac{A}{D}\left(\frac{1}{F}\tan^{-1}\left(\frac{u}{F}\right)\right) + C$$

$$= \frac{A}{D}\left(\frac{1}{F}\tan^{-1}\left(\frac{x-E}{F}\right)\right) + C$$

by completing the square (for some D, E depending on a, b, c), substitution and the tangent formula

Completing The Square

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

Useful Identity

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

- Repeated quadratic terms are difficult but can be done by a combination of substitution, completing the squares, trigonometric substitution, ...

$$\int \frac{A}{(ax^2+bx+c)^r}\,dx = \dots$$

Example

Calculate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\ dx$

$$\int x + 1 + \tfrac{1}{x-1} + \tfrac{2}{(x-1)^2} - \tfrac{1}{x+1}\ dx = \tfrac{1}{2}x^2 + x - \tfrac{2}{x-1} + \ln\left|\tfrac{x-1}{x+1}\right| + c$$

Example

Calculate $\int \frac{2x^2-x+4}{x^3+4x}\ dx$

$$\ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right) + c$$

Example

Calculate $\int \frac{4x^2-3x+2}{4x^2-4x+3}\ dx$

$$x + \frac{1}{4} \int \frac{u}{u^2+2}\ du - \frac{1}{4} \int \frac{1}{u^2+2}\ du = x + \frac{1}{8} \ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

Example

Calculate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\ dx$

$\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x - \frac{1}{2(x^2+1)} + c$

McMaster University | BRIGHTER WORLD

Now Open! Student Course Experience Surveys

It's time to share your feedback on your learning experience.

Here are a few quick tips to make the most impact with your comments.

- Think constructively about your course(s).
- Be respectful.
- Be specific and provide a reasonable amount of information.
- Consider what's working and what's not working.

Get started on your course surveys: **mcmaster.bluera.com/mcmaster**

Scan to get started on your course surveys.

Section 8.1 - Arc Length

Theory

The length of one of the red line segments is

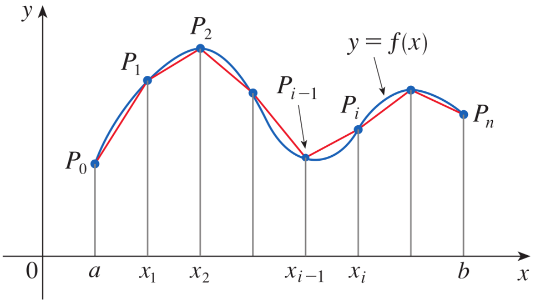
$$
\begin{aligned}
l_i^2 &= (x_{P_i} - x_{P_{i-1}})^2 + (y_{P_i} - y_{P_{i-1}})^2 \\
&= (\Delta y_i)^2 + (\Delta x_i)^2
\end{aligned}
$$

Remember that by the mean value theorem for some $x_i^\star$ in $[x_{i-1}, x_i]$

$$
\begin{aligned}
\Delta y_i = y_{P_i} - y_{P_{i-1}} &= f(x_{P_i}) - f(x_{P_{i-1}}) \\
&= f'(x_i^\star)(x_{P_i} - x_{P_{i-1}}) = f'(x_i^\star)\Delta x_i
\end{aligned}
$$

So the total length for n cuts is

$$
\begin{aligned}
L_n = \sum_{i=1}^{n} l_i &= \sum_{i=1}^{n} \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2} \\
&= \sum_{i=1}^{n} \sqrt{(f'(x_i^\star)\Delta x_i)^2 + (\Delta x_i)^2} \\
&= \sum_{i=1}^{n} \sqrt{(f'(x_i^\star))^2 + 1}\ \Delta x_i
\end{aligned}
$$



$$
\Longrightarrow \qquad
\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(x_i^\star))^2 + 1}\ \Delta x_i \\
&= \int_a^b \sqrt{(f'(x))^2 + 1}\ dx
\end{aligned}
$$

Arc Length

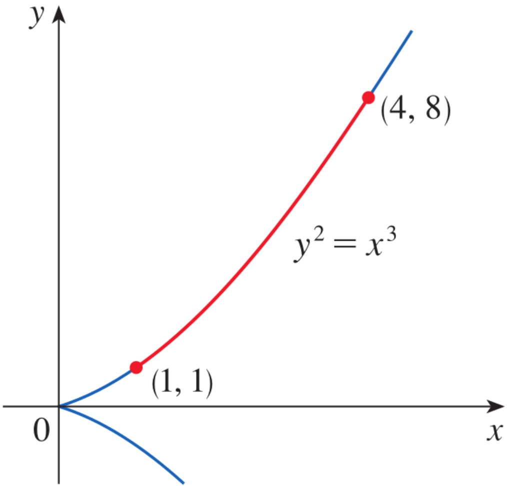
The Arc Length Formula

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example

Find the length of the arc of the semicubical parabola $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$



$\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$

Arc Length of Vertical Curves

The Arc Length Formula

If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$, $c \leq y \leq d$ is

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Example

Find the length of the curve $x = y^2 - \frac{1}{8}\ln y$ on the interval $1 \leq y \leq e$

$e^2 - \frac{7}{8}$

Arc Length Function

Arc Length Function

For a smooth curve satisfying the equation $y = f(x)$ for $a \leq x \leq b$. Then $s(x)$ is the distance along C from $P_0 = (a, f(a))$ to $Q = (x, f(x))$. s is called the arc length function and given by

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt$$

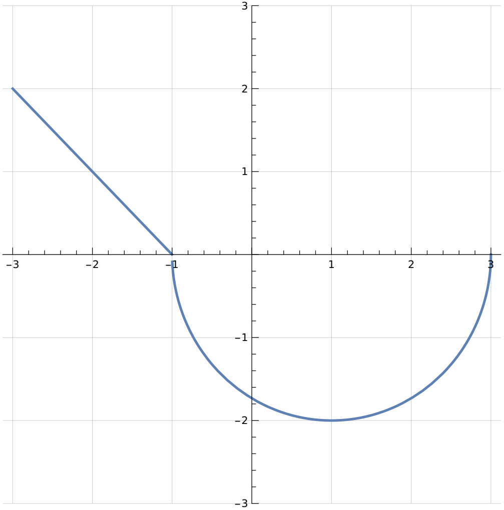
Example

Calculate the arc length function of $f(x) = x^2 - \frac{1}{8}\ln x$ for x in $(1, \infty)$

$x^2 + \frac{1}{8}\ln x - 1$

Example

Find the arc length of the function shown on the right in the interval $(-3, 3)$



$2\sqrt{2} + 2\pi$

Section 7.5 - Integration Strategy

Important Integrals

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$

14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$

16. $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

*19. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Integration Strategy - Part 1

1. Simplify the Integrand

$$\int \sqrt{x}(1+\sqrt{x})\ dx = \int \sqrt{x} + x\ dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\ d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\ d\theta = \int \sin\theta\cos\theta\ d\theta = \int \frac{1}{2}\sin 2\theta\ d\theta$$

$$\int (\sin x + \cos x)^2\ dx = \int \sin^2 x + 2\sin x\cos x + \cos^2 x\ dx = \int 1 + 2\sin x\cos x\ dx = \int 1 + \sin(2x)\ dx$$

2. Is there a memorized solution?
3. Look for *obvious* Substitutions

$$\int \frac{x}{x^2+1}\ dx = \frac{1}{2}\int \frac{1}{u}\ du$$

$$\int \sin x\ \cos^4 x\ dx = -\int u^4\ du$$

- Try to substitute the argument of ugly functions

Integration Strategy - Part 2

4. Classify the integrand and use the learned technique

   4.1 Powers of trigonometric functions → carefully use trigonometric identities (Part 10/Section 7.2)

   $$\int \sin^4 x \cos^6 x \; dx$$

   4.2 Rational functions → partial fraction decomposition (Part 10/Section 7.4)

   $$\int \frac{x^3 + 4x - 5}{x^2 + x - 8} \; dx$$

   4.3 If a factor has a nice derivative/integral → integration by parts (Part 9/Section 7.1)

   $$\int x^3 \cosh x \; dx \qquad \int e^x \sin x \; dx$$

   4.4 Trigonometric square roots → use trigonometric substitution (Part 10/Section 7.3)

   $$\int \sqrt{x^2 - a^2} \; dx \qquad \int \frac{\sqrt{9 - x^2}}{x^2}$$

Theory - Advanced Integration Strategy

5. If this does not work
   5.1 Try non-obvious substitutions
   5.2 Still try integration by parts
   5.3 Try to manipulate the integrand
   5.4 Try combining previous techniques

Example

$$\int \frac{\tan^3 x}{\cos^3 x}\ dx$$

$$\int u^{-4} - u^{-6}\ du = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + c$$

Example

$$\int \sin \sqrt{x}\ dx$$

$$2\int u\sin u\ du = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + c$$

Example

$$\int \frac{3x^2+1}{x^3+x^2+x+1}\ dx$$

$$- \tan^{-1} x + 2 \ln |1 + x| + \tfrac{1}{2} \ln |1 + x^2|$$

Example

$$\int \frac{1}{x\sqrt{\ln x}}\ dx$$

$2\sqrt{\ln x} + c$

Example

$$\int \sqrt{\frac{1-x}{1+x}}\ dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx - \int \frac{x}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + \sqrt{1-x^2} + c$$

Competitions

Now that you are experts in integrating

- MIT Integration Bee Finals 2025
- McMaster - William Lowell Putnam Mathematical Competition
- McMaster Math Outreach: Erin Clements and Miroslav Lovric

Final Exam Information

From Childsmath

- For locations and time check on Mosaic $\rightarrow$ Student Centre $\rightarrow$ Academics Section $\rightarrow$ Exam Schedule (in the drop-down)
- Duration: 2.5 hours
- Format: 32 multiple choice questions worth 1 mark each
- Coverage: All sections in all 3 sets of suggested problems
  - 7 of the questions come from the material from the sections in Suggested Problems #1
  - 9 of the questions come from the material from the sections in Suggested Problems #2
  - 16 of the questions come from the material from the sections in Suggested Problems #3
- Only the McMaster standard calculator Casio fx-991 MS or MS Plus is allowed
- You must bring your student I.D. card to the exam
- My Pre-Exam Office Hours
  - Friday, Dec 5, 9:30-10:30, HH414
  - Tuesday, Dec 9, 10:30-11:30, HH414

Teaching Awards Nomination - Kyle Sung

- Kyle is among the top nominees
- Please fill the survey
- Deadline: December 10

Review

Start $\rightarrow$ Test 1

Important Trigonometric/Hyperbolic Formulas

$$\sin x = \dots \qquad \cos x = \dots \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin' = \cos \qquad \cos' = -\sin$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh' = \cosh \qquad \cosh' = \sinh$$

Inverse functions

Definition

A function is *one-to-one* if it never takes on the same value twice, i.e. $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

Definition

If f is one-to-one

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

$$\text{range}(f) = \text{domain}(f^{-1}) \qquad \text{range}(f^{-1}) = \text{domain}(f)$$

$$f^{-1}(f(x)) = x \qquad\qquad f(f^{-1}(x)) = x$$

Natural Logarithm

Definition

$$\ln e^x = x \qquad\qquad e^{\ln x} = x$$

Calculation Rules

$$\ln a + \ln b = \ln(ab) \qquad\qquad \log_b x = \frac{\ln x}{\ln b}$$

$$\ln a^b = b \ln a \qquad\qquad a^b = e^{b \ln a}$$

Inverse Trigonometric Functions

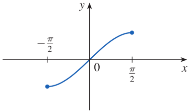


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

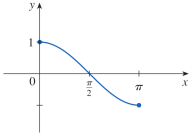


FIGURE 21
$y = \cos x,\ 0 \leqslant x \leqslant \pi$

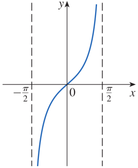


FIGURE 23
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

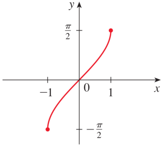


FIGURE 20
$y = \sin^{-1} x = \arcsin x$

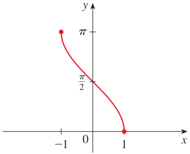


FIGURE 22
$y = \cos^{-1} x = \arccos x$

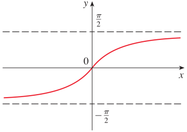


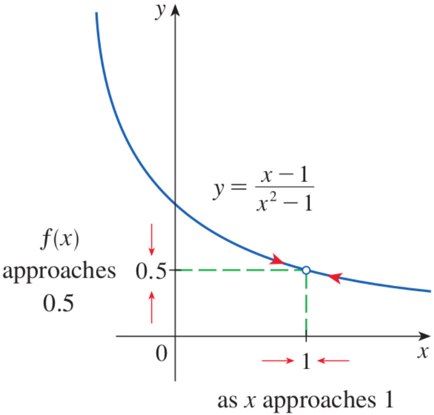
FIGURE 25
$y = \tan^{-1} x = \arctan x$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
range	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[0, \pi]$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Limit

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ arbitrary close to L by restricting x to be sufficiently close to a.



Continuity

Definition

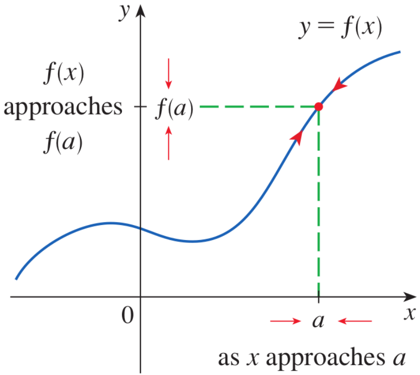
f is *continuous* at a if

$$\lim_{x \to a} f(x) = f(a)$$

Theorem

If f is continuous

$$\lim_{x \to a} (f(g(x))) = f\left(\lim_{x \to a} g(x)\right)$$



Limit Laws

Suppose that c is a constant, n is an integer and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \text{if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a} [(f(x))^n] = \left(\lim_{x\to a} f(x)\right)^n \qquad \text{if } n \text{ is a positive integer}$$

$$\lim_{x\to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)} \qquad \text{if } n \text{ is a positive integer. If } n \text{ is even then } f(a) \text{ has to be positive.}$$

$$\lim_{x\to a} c = c$$

Continuity Laws

If f and g are continuous (at a) and c is a constant then

$$f+g, \qquad f-g, \qquad cf, \qquad f\cdot g, \qquad \frac{f}{g} \text{ for } g\neq 0, \qquad g\circ f = g(f)$$

are continuous (at a).

Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

Derivatives

Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists.

Tangent line at a

$$y = f'(a)x + b$$

through $(a, f(a))$.

Calculating Derivatives

$$(c)' = 0 \qquad (x^k)' = kx^{k-1} \qquad (e^x)' = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad (f(g))' = f'(g) \cdot g'$$

$$\sin' = \cos \qquad \cos' = -\sin \qquad (b^x)' = b^x \ln b$$

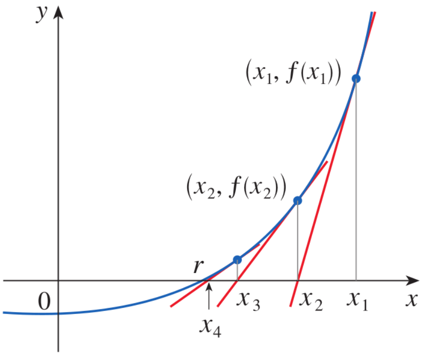
$$(\log_b x)' = \frac{1}{x \ln b} \qquad (\ln x)' = \frac{1}{x} \qquad (\ln |x|)' = \frac{1}{x}$$

$$\sinh' = \cosh \qquad \cosh' = \sinh \qquad (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Newton's Method

Make an initial guess x_1 and iteratively calculate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Implicit Differentiation

Illustrative Example

For $xy = x^3 + y^2$ consider the solution $y(x)$ and differentiate using the chain rule

$$y + xy' = 3x^2 + 2yy'$$

therefore

$$y' = \frac{3x^2 - y}{x - 2y}$$

and so the derivative at $(-2, -4)$ is

$$y' = \frac{3(-2)^2 - (-4)}{-2 - 2(-2)} = \frac{8}{3}$$

Extrema

Definition (Absolut)

$f(c)$ is the

- *absolut* (or *global*) *maximum* of f on D if $f(c) \geq f(x)$ for all x in D
- *absolut* (or *global*) *minimum* of f on D if $f(c) \leq f(x)$ for all x in D

Definition (Local)

$f(c)$ is the

- *local maximum* of f if $f(c) \geq f(x)$ when x is near c
- *local minimum* of f if $f(c) \leq f(x)$ when x is near c

Theorem (Extrem Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$

Extrema

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

Critical Number

c is a critical number if either $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method

To find the absolute extrema of a continuous function on a closed interval:

1. Find all critical points and the function values at these points.
2. Find the values at the end points of the interval.
3. The largest value of these previous values is the absolute maximum, the smallest is the absolute minimum

Review

Test 1 $\rightarrow$ Test 2

Mean Value Theorem

Theorem (Mean Value Theorem)

If f is continuous and differentiable, then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Simplified it says that a continuous differentiable function has its average slope in some point.

Suppose (everything is smooth and) $f'(x) \leq d$ then

$$f(b) \leq f(a) + (b - a)d$$

How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Concavity Test

- If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Second Derivative Test

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum in c
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum in c

Indeterminate Forms and L'Hospital's Rule

L'Hôpital's rule

Suppose

- f and g are differentiable and $g'(x) \neq 0$ in an interval around a (except possibly a itself)
- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or
  $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the right-hand side limit exists or is ∞ or $-\infty$

- For "$0 \cdot \infty$" write $\lim f(x)g(x) = \lim \frac{f(x)}{\frac{1}{g(x)}}$ and use L'Hôpital's rule
- For "$\infty - \infty$" including fractions try to find a common denominator and use L'Hôpital's rule

f^g

For $\lim_{x \to a} \left(f(x)^{g(x)} \right)$ use

$$\lim_{x \to a} \left(f(x)^{g(x)} \right) = \lim_{x \to a} \left(e^{\ln\left(f(x)^{g(x)}\right)} \right) = \lim_{x \to a} \left(e^{g(x) \ln(f(x))} \right) = e^{\lim_{x \to a} (g(x) \ln(f(x)))}$$

Indeterminate Forms and L'Hospital's Rule

L'Hôpital's rule

Suppose

- f and g are differentiable and $g'(x) \neq 0$ in an interval around a (except possibly a itself)
- $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ or
  $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the right-hand side limit exists or is ∞ or $-\infty$

- For "$0 \cdot \infty$" write $\lim f(x)g(x) = \lim \frac{f(x)}{\frac{1}{g(x)}}$ and use L'Hôpital's rule
- For "$\infty - \infty$" including fractions try to find a common denominator and use L'Hôpital's rule

f^g

For $\lim_{x \to a} \left(f(x)^{g(x)} \right)$ use
$$\lim_{x \to a} \left(f(x)^{g(x)} \right) = \lim_{x \to a} \left(e^{\ln\left(f(x)^{g(x)}\right)} \right) = \lim_{x \to a} \left(e^{g(x) \ln(f(x))} \right) = e^{\lim_{x \to a} (g(x) \ln(f(x)))}$$

Summary of Curve Sketching

Guide for Curve Sketching

1. Domain
2. Intercepts
3. Symmetries
   - Even
   - Odd
   - Periodic
4. Asymptotes
   - Horizontal
   - Vertical
   - Slant
5. Increasing/Decreasing
6. Extrema
7. Concavity and Inflection Points
8. Sketch the curve by using the previous steps

Optimization Problems

Optimization Problems Guide

1. Understand the problem. Unknowns? Constraints? Given Quantities?
2. Draw a sketch
3. Introduce Notation: Assign variables to the quantities of interest
4. Express the objective quantity in terms of the unknowns
5. Use the constraints to simplify the expression to one variable
6. Use the closed interval method to find the absolute maximum/minimum

If $f > 0$ then the value that maximizes $f(x)$ is the same as the one that maximizes $(f(x))^2$

Antiderivatives and indefinite Integrals

F is an antiderivative of f if $F' = f$.

If F is the antiderivative of f on I, then the general antiderivative/indefinite Integral of f on I is

$$\int f(x)\, dx = F(x) + C,$$

where C is an arbitrary constant.

Antiderivatives and indefinite Integrals

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Σ-Notation

Definition

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n \qquad \text{for } n > m$$

Important identities

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Telescope sum $\Leftrightarrow$ stuff cancels with other terms

Area, Distance, Definite Integral

Theorem

The net area A between a continuous function and the x-axis on the interval $[a,b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \; f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \; f\left(a + i\frac{b-a}{n}\right) = \int_a^b f(x)\; dx,$$

where the sampling points are $x_i = a + i\Delta x$ and their distance is $\Delta x = \frac{b-a}{n}$.

$$\int_a^c f(x)\; dx + \int_c^b f(x)\; dx = \int_a^b f(x)\; dx \qquad \int_a^b f(x)\; dx = -\int_b^a f(x)\; dx$$

$$\int_a^b c\; dx = (b-a)c \qquad \int_a^a f(x)\; dx = 0$$

$$\int_a^b f(x) + g(x)\; dx = \int_a^b f(x)\; dx + \int_a^b g(x)\; dx \qquad \int_a^b cf(x)\; dx = c\int_a^b f(x)\; dx$$

The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus - Overview)

1. If f is continuous on $[a, b]$
$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$
in (a, b)
2. If f' is continuous on $[a, b]$
$$\int_a^x f'(t)\ dt = f(x) - f(a)$$
for x in $[a, b]$

Integration - Overview

Suppose $F'(x) = f(x)$ (so F is an antiderivative of f) is continuous. Then

- the indefinite integral (or general antiderivative) is

$$\int f(x)\ dx = F(x) + C$$

- the definite integral (or net area between f and the x-axis between a and b) is

$$\text{Net Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n}\ f\left(a + i\frac{b-a}{n}\right) \overset{\text{Def}}{=} \int_a^b f(x)\ dx \overset{\text{FTC2}}{=} F(x)\Big|_a^b \overset{\text{Notation}}{=} F(b) - F(a)$$

- Fundamental Theorem of Calculus 1

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x) \qquad \Longrightarrow \qquad \frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f(g(x))g'(x)$$

- Fundamental Theorem of Calculus 2

$$\int_a^x f(t)\ dt = \int_a^x F'(t)\ dt = F(x) - F(a)$$

Substitution

Theorem (Substitution Rule for Indefinite Integrals)

If $u = g(x)$ is a differentiable function whose range is I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du$$

Theorem (Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Area between Curves

Theorem (horizontal, $f > g$)

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous and $f(x) \geq g(x)$ is

$$A = \int_a^b f(x) - g(x)\ dx$$

Theorem (horizontal, general)

The area A of a region between the curves $y = f(x)$ and $y = g(x)$ from a to b, where f and g are continuous is

$$A = \int_a^b |f(x) - g(x)|\ dx$$

Theorem (vertical, $f > g$)

The area A bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where $f(y) \geq g(y)$ and f and g are continuous is

$$A = \int_c^d f(y) - g(y)\ dy$$

Review

Test 2 $\rightarrow$ End

Volume

- Carefully read the question
- Sketch the problem
- Find the end points
- Find cross sectional area
- Integrate orthogonal to the cross sectional area

Work

- constant force F over distance d

$$W = F\,d$$

- non-constant force $f(x)$ over a distance from a to b

$$W = \int_a^b f(x)\ dx.$$

Gravitational forces in

- Metric units

$$F = mg, \qquad g = 9.81\,\frac{\mathsf{m}}{\mathsf{s}^2}, \qquad\qquad F = 1\mathsf{kg}\cdot 9.81\,\frac{\mathsf{m}}{\mathsf{s}^2} = 9.81\,\frac{\mathsf{kg\,m}}{\mathsf{s}^2} = 9.81\mathsf{N}$$

- Imperial units

$$1\text{lb (mass)} \implies 1\text{lb (force)}$$

Usual strategy

- consider a small piece
- calculate the work required for the small piece
- sum up all the pieces
- decrease the size of the pieces by making more cuts $\implies \lim\limits_{n\to\infty}$ which results in an integral

$$\lim_{n\to\infty}\sum_{i=1}^{n} \to \int_{\text{start}}^{\text{end}} \qquad\qquad \Delta x \to dx$$

- solve the integral

Average of Functions/Mean Value Theorem for Integrals

Average of a function

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

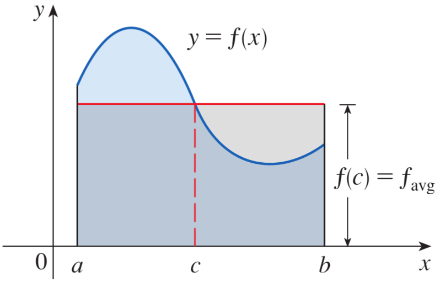
Theorem (Mean Value Theorem for Integrals)

If f is continuous in $[a,b]$, there exists a number c in $[a,b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or equivalently

$$\int_a^b f(x)\,dx = f(c)(b-a)$$



Integration by Parts

Integration by Parts - Formula 1 - Indefinite Integrals

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

Integration by Parts - Formula 2 - Indefinite Integrals

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx$$

Short Summary

For

$$\int \sin^m x \cos^n x \; dx$$

use $\sin^2 + \cos^2 = 1$ to reduce until the power of $\sin$ or $\cos$ is 1 or 0. Then

1: use substitution

0: use half angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Trigonometric Integrals - $\tan^m x \sec^n$

Short Summary

For

$$\int \tan^m x \sec^n x \, dx$$

use $\sec^2 = 1 + \tan^2$ until $\sec^2 x$ or $\sec x \tan x$ is left

$\sec^2 x$: substitute $u = \tan x$ since $\frac{du}{dx} = \sec^2 x$

$\sec x \tan x$: substitute $u = \sec x$ since $\frac{du}{dx} = \sec x \tan x$

Helpful Identities

$$\int \tan x \, dx = \ln|\sec x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

Multiple angles

For

$$\int \sin nx \cos mx \; dx$$

use

$$\sin A \cos B = \frac{1}{2}\left(\sin(A - B) + \sin(A + B)\right)$$
$$\sin A \sin B = \frac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$$
$$\cos A \cos B = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right)$$

Trigonometric Substitution

Expression	Substitution	Domain	Identity	sign
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	$\cos\theta \geq 0$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	$\sec\theta \geq 0$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3}{2}\pi$	$\sec^2\theta - 1 = \tan^2\theta$	$\tan\theta \geq 0$

Partial Fraction Decomposition - Overview

To solve

$$\int f(x)\ dx = \int \frac{P(x)}{Q(x)} dx,$$

where P, Q are polynomials

1. Make it a proper fraction
$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$
2. Factorize the denominator until only linear terms and irreducable quadratic terms are left
3. Find the correct enumerators/denominators of the partial fraction decomposition (see later)
4. Multiply the partial fraction decomposition by $Q(x)$
5. Either
    - Plug in the roots

    or
    - Arrange the result by powers of x
6. Solve for the enumerators
7. Write $f(x) = S(x) + \frac{R(x)}{Q(x)}$ as the partial fraction decomposition and solve the integral

Partial Fraction Decomposition - The Correct Choice of enumerators/denominators

$$\frac{\ldots}{x^2(3x+2)^2(x-5)^3\underbrace{(x^2+8x+1)}_{\text{irreducable}}\underbrace{(10x^2+9x+13)}_{\text{irreducable}}^3}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3x+2} + \frac{D}{(3x+2)^2} + \frac{E}{x-5} + \frac{F}{(x-5)^2} + \frac{G}{(x-5)^3}$$

$$+ \frac{Hx+I}{x^2+8x+1} + \frac{Jx+K}{10x^2+9x+13} + \frac{Lx+M}{(10x^2+9x+13)^2} + \frac{Nx+O}{(10x^2+9x+13)^3}$$

Partial Fraction Decomposition - Solving the Integral

- Single linear terms $\int \frac{A}{ax-b}\,dx = \frac{A}{a}\ln|ax-b|$ by substitution or guessing
- Repeated linear terms $\int \frac{A}{(ax-b)^p}\,dx = \frac{A}{a(-p+1)}(ax-b)^{-p+1}$ for $p \neq 1$ by substitution or guessing
- Single quadratic terms

$$\int \frac{A}{ax^2+bx+c}\,dx \overset{(200)}{=} \frac{A}{D}\int \frac{1}{(x+E)^2+F^2}\,dx$$

$$= \frac{A}{D}\int \frac{1}{u^2+F^2}\,du \overset{(201)}{=} \frac{A}{D}\left(\frac{1}{F}\tan^{-1}\left(\frac{u}{F}\right)\right)+C$$

$$= \frac{A}{D}\left(\frac{1}{F}\tan^{-1}\left(\frac{x+E}{F}\right)\right)+C$$

Completing The Square

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c \quad (200)$$

Useful Identity

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c \quad (201)$$

- Repeated quadratic terms are difficult but can be done by a combination of substitution, completing the squares, trigonometric substitution, ...

$$\int \frac{A}{(ax^2+bx+c)^r}\,dx = \dots$$

Arc Length

Arc Length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

Arc Length - Vertical Functions

If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$, $c \leq y \leq d$ is

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\ dy$$

Arc Length Function

For a smooth curve satisfying the equation $y = f(x)$ for $a \leq x \leq b$. Then $s(x)$ is the distance along C from $P_0 = (a, f(a))$ to $Q = (x, f(x))$. s is called the arc length function and given by

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt$$

Integration Strategy - Important Integrals

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$

14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$

16. $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

*19. $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Integration Strategy - First Approach

1. Simplify the Integrand

$$\int \sqrt{x}(1+\sqrt{x})\ dx = \int \sqrt{x} + x\ dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\ d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\ d\theta = \int \sin\theta\cos\theta\ d\theta = \int \frac{1}{2}\sin 2\theta\ d\theta$$

$$\int (\sin x + \cos x)^2\ dx = \int \sin^2 x + 2\sin x\cos x + \cos^2 x\ dx = \int 1 + 2\sin x\cos x\ dx = \int 1 + \sin(2x)\ dx$$

2. Is there a memorized solution?
3. Look for *obvious* Substitutions

$$\int \frac{x}{x^2+1}\ dx = \frac{1}{2}\int \frac{1}{u}\ du$$

$$\int \sin x\ \cos^4 x\ dx = -\int u^4\ du$$

- Try to substitute the argument of ugly functions

Integration Strategy - Second Approach

4. Classify the integrand and use the learned technique

   4.1 Powers of trigonometric functions → carefully use trigonometric identities (Part 10/Section 7.2)

   $$\int \sin^4 x \cos^6 x \; dx$$

   4.2 Rational functions → partial fraction decomposition (Part 10/Section 7.4)

   $$\int \frac{x^3 + 4x - 5}{x^2 + x - 8} \; dx$$

   4.3 If a factor has a nice derivative/integral → integration by parts (Part 9/Section 7.1)

   $$\int x^3 \cosh x \; dx \qquad \int e^x \sin x \; dx$$

   4.4 Trigonometric square roots → use trigonometric substitution (Part 10/Section 7.3)

   $$\int \sqrt{x^2 - a^2} \; dx \qquad \int \frac{\sqrt{9 - x^2}}{x^2}$$

Integration Strategy - Third Approach

5. If this does not work
   - 5.1 Try non-obvious substitutions
   - 5.2 Still try integration by parts
   - 5.3 Try to manipulate the integrand
   - 5.4 Try combining previous techniques

Review

Review Exercises

Section 6.4 - Exercise 15

A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

650 000 ft-lb

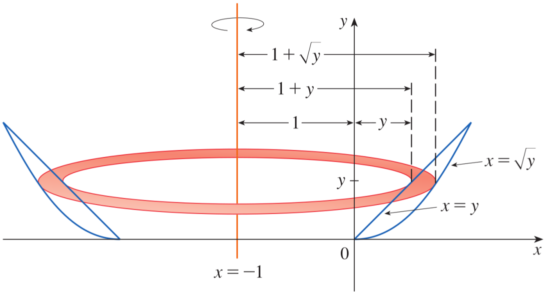
Section 6.4 - Exercise 17

A 10 ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it is level with the upper end.

62.5 ft-lb

Section 6.2 - Example 6

Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the line $x = -1$.



$\frac{\pi}{2}$

Section 7.2 - Exercise 29

Calculate $\int \tan^3 x \sec^6 x \; dx$

$\frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 + c$

Section 7.2 - Exercise 29

Calculate $\int \tan^3 x \sec^6 x \; dx$

$\frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 + c$

Section 7.3 - Exercise 35

Calculate $\int x\sqrt{1-x^4}\ dx$

$\frac{1}{4}\arcsin(x^2) + \frac{1}{4}x^2\sqrt{1-x^4} + c$