Artificial Intelligence

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Knowledge Representation and Inference: Logical Languages

Suggested textbook

S. Russell, P. Norvig, *Artificial Intelligence – A Modern Approach*, 4th Ed., Pearson, 2021 (or a previous edition)

Introduction

Introduction

Consider the following problems, and assume that your goal is to design **rational agents**, in the form of computer programs, capable of **autonomously** solving them.

Automatic theorem proving

Write a computer program capable to **prove** or to **refute** the following statement:

Goldbach's conjecture (1742)

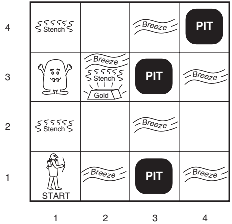
For any even number $p \geq 2$*, there exists at least one pair of prime numbers* q *and* r *(identical or not) such that*

$$q + r = p$$

Game playing

Write a computer program capable of playing the **wumpus game**, a text-based computer game (G. Yob, c. 1972) used in a modified version as an AI's toy-problem. A basic version:

- **the wumpus world**: a cave made up of connected rooms, bottomless pits, a heap of gold, and the **wumpus**, a beast that eats anyone who enter its room



- **goal**: starting from room (1,1), find the gold and go back to (1,1), without falling into a pit or hitting the wumpus
- main **rules of the game**:
  - the content of any room is known only **after** entering it
  - in rooms neighboring the wumpus and pits a **stench** and a **breeze** is perceived, respectively

Knowledge-based systems

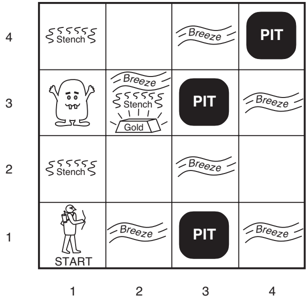
Humans usually solve problems like the ones above by combining two **high-level** capabilities: abstract **knowledge representation** and **reasoning**.

Knowledge-based systems aim at **mechanizing** the above human capabilities:

- representing knowledge about the world
- reasoning to derive new knowledge and to **guide action**

An example: playing the wumpus game

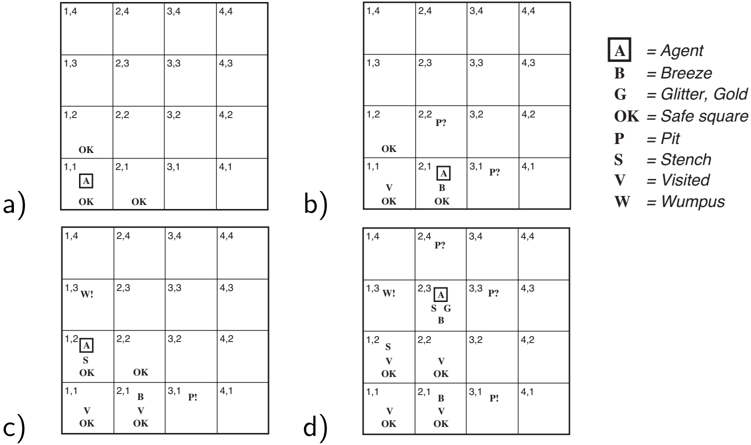
Consider the following initial configuration of the **wumpus game**, and remember that the player knows the content of any room only after entering it:



If **you** were the player, how would **you** reason to **decide** the next move to do at each game step?

An example: playing the wumpus game

Sketch of a possible **reasoning process** for deciding the next move, starting from the configuration previously shown (some moves are omitted).



Main approaches to AI system design

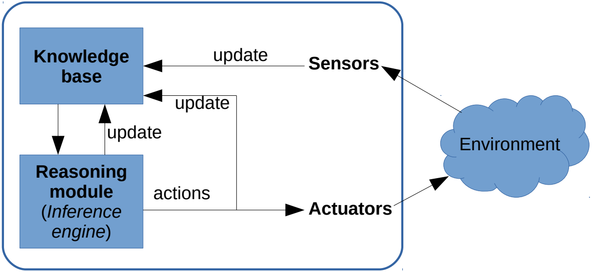
Procedural: the desired behavior (actions) are encoded directly as program code (no explicit knowledge representation and reasoning).

Declarative: explicit representation, in a **knowledge base**, of

- **background knowledge** (e.g., the rules of the wumpus game)
- knowledge about a specific **problem instance** (e.g., what the agent knows about a specific wumpus cave it is exploring)
- the agent's **goal** (e.g., finding the gold and going back to the starting square, without falling into a pit or hitting the wumpus)

Actions are then derived by **reasoning**.

Architecture of knowledge-based systems



Main feature: **separation** between **knowledge representation** and **reasoning**

- the **knowledge base** contains all the agent's knowledge about its environment, in **declarative** form
- the **inference engine** implements a reasoning process (algorithm) to derive new knowledge and to make decisions

Main applications in computer science and AI

Artificial Intelligence:

- expert systems (medicine, engineering, finance, etc.)
- automatic theorem provers

Computer science:

- logic programming languages (**Prolog**, etc.)
- databases (relational calculus, SQL)
- semantic Web
- satisfiability (SAT) problems:
  hardware/software design, planning (travel, logistics, ...), etc.

A short introduction to logic

- What is logic?
- Propositions and argumentations
- Logical (formal) languages
- Logical reasoning

Logic

Logic is one of the main tools used in AI for

- **knowledge representation**: logical languages
  - propositional logic
  - predicate (first-order) logic
- **reasoning**: inference rules and algorithms

Some of the main contributions:

- Aristotle (4th cent. BC): the “laws of thought”
- G. Boole (1815–64): Boolean algebra (propositional logic)
- G. Frege (1848–1925): predicate logic
- K. Gödel (1906–78): investigation of the limitations of logic, incompleteness theorem

Logic

A possible definition of logic:

> ***Logic*** *is the study of conditions under which an* ***argumentation*** *(reasoning) is* ***correct****.*

This definition involves the following concepts:

- **argumentation**: a set of statements consisting of some **premises** and one **conclusion**, e.g.:
  All men are mortal; Socrates is a man;
  then, Socrates is mortal
- **correctness**: an argumentation is said to be correct when its conclusion cannot be false when all its premises are true
- **proof**: a procedure to assess correctness

Propositions

Natural language is very complex and vague, and therefore difficult to formalize. Logic considers argumentations made up of only a subset of statements: **propositions** (**declarative statements**).

> *A **proposition** is a statement expressing a concept that can be either **true** or **false**.*

Example

- *Socrates is a man*
- *Two and two makes four*
- *If the Earth had been flat, then Columbus would have not reached America*

A counterexample: *Read that book!*

Simple and complex propositions

A proposition is:

- **simple**, if it does not contain simpler propositions
- **complex**, if it is made up of simpler propositions connected by **logical connectives**

Example

Simple propositions:

- *Socrates is a man*
- *Two and two makes four*

Complex propositions:

- *A tennis match can be won **or** lost*
- ***If** the Earth had been flat, **then** Columbus would have not reached America*

Argumentations

How to determine whether a proposition is true or false?
This is a **philosophical** question.
Logic does not address this question: it only analyzes the **structure** of an argumentation.

Example

All men are mortal; Socrates is a man;
then, Socrates is mortal.

Informally, the **structure** of this argumentation is:

all P are Q; x is P; then x is Q.

Its correctness depends only on its structure, **whatever** P, Q and x mean, that is, **regardless** of whether the corresponding propositions "all P are Q", "x is P" and "x is Q" are true or false.

Formal languages

Logic provides **formal languages** for representing (the structure of) propositions, in the form of **sentences**.
A formal language is defined by a **syntax** and a **semantics**:

- **syntax** (grammar): rules that define what sentences are "well-formed", i.e., sentences which a meaning can be attributed to
- **semantics**: rules that define the meaning of well-formed sentences

Examples of formal languages:

- **arithmetic**: **propositions** about numbers
- **programming languages**: **instructions** to be executed by a computer (for imperative languages like C)

Natural vs logical (formal) languages

In **natural languages**:

- syntax is not rigorously defined
- semantics defines the "content" of a statement, i.e., "what it refers to in the real world"

Example (syntax)

- *The book is on the table*: syntactically correct statement, with a clear semantics
- *Book the on is table the*: syntactically incorrect statement, no meaning can be attributed to it
- *Colorless green ideas sleep furiously*:[1] syntactically correct, but what does it mean?

[1] N. Chomsky, *Syntactic Structures*, 1957

Natural vs logical (formal) languages

Logical languages:

- syntax: formally defined
- semantics: rules that define the **truth value** of each well-formed sentence with respect to each possible **model**, i.e., a possible "world" represented by that sentence

Example (arithmetic)

- **Syntax**: $x + y = 4$ is a well-formed sentence, $x4y+=$ is not
- **Model**: the symbol '4' represents the natural number four, 'x' and 'y' any pair of natural numbers, '$+$' the sum operator, etc.
- **Semantics**: $x + y = 4$ is true for $x = 1$ and $y = 3$, for $x = 2$ and $y = 2$, etc.

Logical entailment

Logical reasoning is based on the relation of **logical entailment** between sentences, that defines when a sentence **logically follows** from another one:

> *a sentence α **entails** a sentence β, if and only if, in every model in which α is true, also β is true. In symbols:*
>
> $$\alpha \models \beta$$

Example (from arithmetic)

$$x + y = 4 \quad \models \quad x = 4 - y \ ,$$

because in every model (i.e., for any assignment of numbers to x and y) in which $x + y = 4$ is true, also $x = 4 - y$ is true.

Logical inference

Logical inference: *the process of deriving conclusions from premises*

Inference algorithm: *a procedure that derives sentences (conclusions) from other sentences (premises), in a given formal language*

Formally, the fact that an inference algorithm A derives a sentence α from a set of sentences ("knowledge base") KB is written as:

$$KB \vdash_A \alpha$$

Properties of inference algorithms

Soundness *(truth-preservation): if an inference algorithm derives* ***only*** *sentences entailed by the premises, i.e.:*

$$\text{if } KB \vdash_A \alpha, \text{then } KB \models \alpha$$

Completeness*: if an inference algorithm derives* ***all*** *the sentences entailed by the premises, i.e.:*

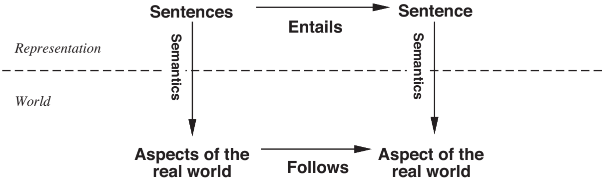
$$\text{if } KB \models \alpha, \text{then } KB \vdash_A \alpha$$

A **sound** algorithm derives conclusions that are guaranteed to be true in any world in which the premises are true.

Properties of inference algorithms

Inference algorithms operate only at the **syntactic** level:

- sentences are physical configurations of an agent (e.g., bits in registers)
- inference algorithms construct new physical configurations from previous ones
- logical reasoning should ensure that new configurations constructed by inference algorithms represent aspects of the world that actually follow from the ones represented by starting configurations

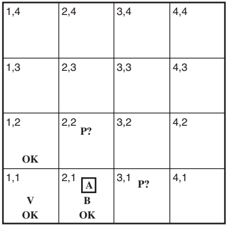


Applications of inference algorithms

In AI inference is used to answer two main kinds of questions:

- does **a given** conclusion α logically follows from the agent's knowledge KB? (i.e., $KB \models \alpha$?)
- what are **all** the conclusions that logically follow from the agent's knowledge? (i.e., find all α's such that $KB \models \alpha$)

Example (the wumpus world)



- does a breeze in room (2,1) entail the presence of a pit in room (2,2)?
- what conclusions can be derived about the presence of pits and of the wumpus in each room, from the current knowledge?

Inference algorithms: model checking

The definition of **entailment** can be directly applied to construct a simple inference algorithm, named **Model checking**:
Given a set of premises, KB, and a sentence α, **enumerate** all possible models and **check** whether α is true in **every** model in which KB is true.

Example (arithmetic)

- $KB : \{x + y = 4\}$
- $\alpha : y = 4 - x$

Is the following inference correct?

$$\{x + y = 4\} \vdash y = 4 - x$$

Model checking: enumerate all possible pairs of numbers x, y, and check whether $y = 4 - x$ is true whenever $x + y = 4$ is.

The issue of *grounding*

A knowledge base *KB* (i.e., a set of sentences that the agents considers true) is just “syntax” (a physical configuration of the agent):

- what is the connection between a KB and the real world?
- how does one know that sentences in the KB are true in the real world?

This is the same **philosophical** question met before. For humans:

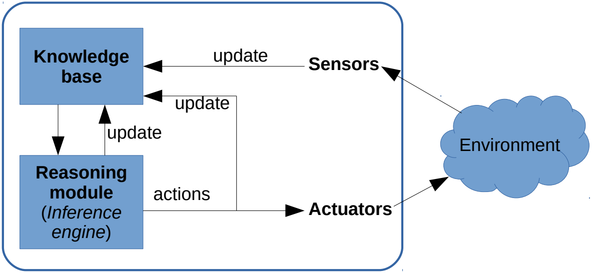
- a set of beliefs (set of statements considered true) is a physical configuration of our brain
- how do we know that our beliefs are true in the real world?

A simple answer can be given for agents (e.g., computer programs or robots): the connection is created by

- **sensors**, e.g.: perceiving a breeze in the wumpus world
- **learning**, e.g., when a breeze is perceived, there is a pit in some adjacent room

Of course, both perception and learning are **fallible**.

Architecture of knowledge-based systems revisited



If **logical languages** are used:

- **knowledge base**: a set of sentences in a given logical language
- **inference engine**: an inference algorithm for the same language

Focus of this course: propositional and predicate logic for knowledge representation, inference algorithms for propositional logic.

Logical languages

Propositional logic

- the simplest logical language
- an extension of Boolean algebra (G. Boole, 1815–64)

Predicate (or **first-order**) **logic**

- a more expressive and concise extension of propositional logic
- seminal work: G. Frege (1848–1925)

Propositional Logic

Syntax

- **Atomic sentences**
  - either a **propositional symbol** that denotes a given proposition (usually written in capitals), e.g.: P, Q, ...
  - or a propositional symbol with fixed meaning: *True* and *False*
- **Complex sentences** consist of atomic or (recursively) complex sentences connected by **logical connectives** (corresponding to natural language connectives like `and`, `or`, `not`, etc.)
- **Logical connectives** (only the commonly used ones are shown – different notations exist):

$\wedge$	(and)
$\vee$	(or)
$\neg$	(not)
$\Rightarrow$	(implication / if. . . then. . .)
$\Leftrightarrow$	(biconditional / logical equivalence – if and only if)

Syntax

A formal grammar in Backus-Naur Form (BNF):

$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{True} \mid \mathit{False} \mid \mathit{Symbol} \\
\mathit{Symbol} & \rightarrow & P \mid Q \mid R \mid \ldots \\
\mathit{ComplexSentence} & \rightarrow & \neg \mathit{Sentence} \\
 & \mid & (\ \mathit{Sentence} \wedge \mathit{Sentence}\ ) \\
 & \mid & (\ \mathit{Sentence} \vee \mathit{Sentence}\ ) \\
 & \mid & (\ \mathit{Sentence} \Rightarrow \mathit{Sentence}\ ) \\
 & \mid & (\ \mathit{Sentence} \Leftrightarrow \mathit{Sentence}\ )
\end{array}
$$

Semantics

Semantics of propositional logic:

- **model** of a sentence: a possible assignment of truth values to all propositional symbols that appear in it
- **meaning** of a sentence: its **truth value** with respect to a particular **model**

Example

The sentence $P \wedge Q \Rightarrow R$ has $2^3 = 8$ possible models.
For instance, one model is $\{P = \mathit{True}, Q = \mathit{False}, R = \mathit{True}\}$.

Note: models are abstract mathematical objects with no unique connection to the real world (e.g., P may stand for **any** proposition in natural language).

Semantics

- **Atomic sentences**:
  - *True* is true in every model
  - *False* is false in every model
  - the truth value of every propositional symbol (atomic sentence) must be specified in the model
- **Complex sentences**:
  their truth value is **recursively** determined as a function of
  - the truth value of the simpler sentences they are made up of
  - the **truth tables** of the logical connectives they contain

Truth tables of commonly used connectives

The semantics of logical connectives is **defined** by their truth tables:

P	Q	$\neg$P	P $\wedge$ Q	P $\vee$ Q	P $\Rightarrow$ Q	P $\Leftrightarrow$ Q
false	false	true	false	false	true	true
false	true	true	false	true	true	false
true	false	false	false	true	false	false
true	true	false	true	true	true	true

Note: $P \Rightarrow Q$ reads as "P implies Q, or "if P then Q".

Example

Determining the truth value of $\neg P \wedge (Q \vee R)$ in all possible models, i.e., for all possible assignment of truth values to P, Q and R:

P	Q	R	$(Q \vee R)$	$\neg P \wedge (Q \vee R)$
false	false	false	false	false
false	false	true	true	true
false	true	false	true	true
false	true	true	true	true
true	false	false	false	false
true	false	true	true	false
true	true	false	true	false
true	true	true	true	false

Logical connectives and natural language

The truth tables of the connectives $\wedge$, $\vee$ and $\neg$ agree with our intuition about the words "and", "or" and "not".
However, they do not capture all the nuances of such terms.

Example

- *He felt down **and** broke his leg*
  Here "and" includes **temporal** and **causal** relations, that are not represented by $\wedge$: whereas $P \wedge Q \equiv Q \wedge P$, the proposition *He broke his leg and felt down* has a different meaning than the one above
- *A tennis match can be won **or** lost*
  This proposition cannot be represented as $P \vee Q$ since in this case "or" has a **disjunctive** meaning (a tennis match cannot be both won and lost), corresponding to the exclusive OR operator $\oplus$ of Boolean algebra, whereas $\vee$ has a **conjunctive** meaning

Logical connectives and natural language

The truth table of the implication may not seem in agreement with the intuitive understanding of "P implies Q¡¡, or "if P then Q".

P	Q	$P \Rightarrow Q$
false	false	true
false	true	true
true	false	false
true	true	true

It can be understood as representing the concept of "**sufficient** but **not necessary** condition", i.e.:

> P *is a* ***sufficient*** *but* ***not necessary*** *condition for* Q *to be true*

In other words, the sentence $P \Rightarrow Q$ represents a proposition of the form:

> *if* P *is true, then I am claiming that also* Q *is true; otherwise,* ***I am making no claim***

In both cases, the only way for $P \Rightarrow Q$ to be false is when P is true and Q is false.

Logical connectives and natural language

Also the implication connective does not represent all the nuances of the word “implies”.

Example

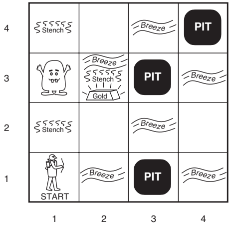
- *The number four is even implies Tokyo is the capital of Japan*
  One may consider this proposition as false, although both the antecedent and consequent are true, since in this case “implies” includes a relation of **causation** or **relevance**. However, if it is translated as $P \Rightarrow Q$, it is **true** according to the truth table of the implication connective, since this connective does not represent causation or relevance
- *The number five is even implies Sam is smart*
  Also this proposition may be considered as false since there is no relation of causation or relevance between antecedent and consequent, **regardless** of whether Sam is smart or not. However, if represented as $P \Rightarrow Q$, since P is false, also this proposition is **true**, **regardless** of the truth value of Q

Exercise

1. Define a set of propositional symbols to represent the wumpus world: the position of the agent, the wumpus, pits, etc.
2. Define the model corresponding to the configuration below
3. Define the part of the initial agent's KB corresponding to its knowledge about the cave configuration in the figure below
4. Write a sentence for the propositions:

(a) *If no breeze is perceived in room (1,1), then there is no pit in room (1,2)*

(b) *If the wumpus is in room (3,1), there is a stench in rooms (2,1), (4,1) and (3,2)*



Solution of exercise 1

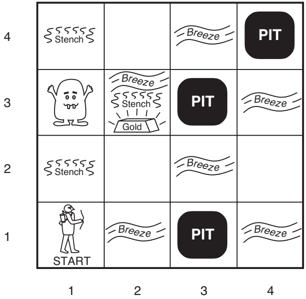
A possible choice of propositional symbols:

- $A_{1,1}$ ("the agent is in room (1,1)"), $A_{1,2}$, ..., $A_{4,4}$
- $W_{1,1}$ ("the wumpus is in room (1,1)"), $W_{1,2}$, ..., $W_{4,4}$
- $P_{1,1}$ ("there is a pit in room (1,1)"), $P_{1,2}$, ..., $P_{4,4}$
- $G_{1,1}$ ("the gold is in room (1,1)"), $G_{1,2}$, ..., $G_{4,4}$
- $B_{1,1}$ ("there is a breeze in room (1,1)"), $B_{1,2}$, ..., $B_{4,4}$
- $S_{1,1}$ ("there is stench in room (1,1)"), $S_{1,2}$, ..., $S_{4,4}$

Solution of exercise 2

Model corresponding to the configuration on the right:

- $A_{1,1}$ is true;
  $A_{1,2}$, $A_{1,3}$,... are false
- $W_{3,1}$ is true;
  $W_{1,1}$, $W_{1,2}$, ... are false
- $P_{1,3}$, $P_{3,3}$, $P_{4,4}$ are true;
  $P_{1,1}$, $P_{1,2}$, ... are false
- $G_{3,2}$ is true;
  $G_{1,1}$, $G_{1,2}$, ... are false
- $B_{1,2}$, $B_{1,4}$, ... are true;
  $B_{1,1}$, $B_{1,3}$, ... are false
- $S_{2,1}$, $S_{3,2}$, $B_{4,1}$ are true;
  $S_{1,1}$, $S_{1,2}$, ... are false



Solution of exercise 3

What the agent knows in the starting configuration:

- I am in room (1,1) (starting position of the game)
- there is no pit nor the wumpus in room (1,1)
- there is no gold in room (1,1)
- I do not perceive a breeze nor a stench in room (1,1)

The corresponding agent's KB in propositional logic (the set of sentences the agent **believes** to be true):

- $A_{1,1}$, $\neg A_{1,2}$, $\neg A_{1,3}$, $\ldots$, $\neg A_{4,4}$ (16 sentences)
- $\neg W_{1,1}$, $\neg P_{1,1}$
- $\neg G_{1,1}$
- $\neg B_{1,1}$, $\neg S_{1,1}$

Solution of exercise 4(a)

If no breeze is perceived in room (1,1), then there is no pit in room (1,2)

It is easy to see that the absence of breeze in a room is a sufficient condition for the absence of a pit in any adjacent room. It is however not a necessary condition, since, even if a breeze is present, e.g., in room (1,1), it may be due to a pit in room (2,1), whereas room (1,2) can still contain no pit.

The above proposition can therefore be correctly represented by an implication:

$$\neg B_{1,1} \Rightarrow \neg P_{1,2}$$

Solution of exercise 4(b)

If the wumpus is in room (3,1) then there is a stench in rooms (2,1), (4,1) and (3,2)

One may be tempted to use an implication:

$$W_{3,1} \Rightarrow (S_{2,1} \wedge S_{4,1} \wedge S_{3,2})$$

However the above sentence is true in the wumpus world, but **incomplete**. Since there is only one wumpus, its presence in room (3,1) is also a **necessary** condition for a stench to be present in the neighboring rooms. In other words, the previous sentence does not rule out models in which $W_{3,1}$ is false and $S_{2,1} \wedge S_{4,1} \wedge S_{3,2}$ is true, which would violate the rules of the wumpus world. Indeed, the opposite is also true:

$$(S_{2,1} \wedge S_{4,1} \wedge S_{3,2}) \Rightarrow W_{3,1}$$

The correct representation of the above statement is therefore:

$$W_{3,1} \Leftrightarrow (S_{2,1} \wedge S_{4,1} \wedge S_{3,2})$$

Inference in Propositional Logic

The Model checking inference algorithm

Goal of logical inference: given a KB and a sentence α, deciding whether $KB \models \alpha$.

A simple inference algorithm: **model checking** (see above).

Application to propositional logic:

- enumerate all possible models for the sentences $KB \cup \{\alpha\}$
- check whether α is true in every model in which KB is true

Implementation: **truth tables**.

Model checking: an example

Determine whether $\{P \vee Q, P \Rightarrow R, Q \Rightarrow R\} \models P \vee R$, using model checking.

Propositional symbols			Premises			Conclusion
P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$P \vee R$
false	false	false	false	true	true	false
false	false	true	false	true	true	true
false	true	false	true	true	false	false
false	true	true	true	true	true	true
true	false	false	true	false	true	true
true	false	true	true	true	true	true
true	true	false	true	false	false	true
true	true	true	true	true	true	true

The answer is **yes**, because the conclusion is true in every model in which the premises are true (grey rows).

The Model checking inference algorithm

Properties of Model checking:

- **soundness**: **yes**, since it directly implements the definition of entailment
- **completeness**: **yes**, since it works for any (finite) KB and any α, and the corresponding set of models is finite
- **computational complexity**: $O(2^n)$, where n is the number of propositional symbols appearing in KB and α

The drawback of model checking is its **exponential** computational complexity, which makes it infeasible when the number of propositional symbols is high.

Example

In the exercise about the wumpus world, 96 propositional symbols have been introduced: the corresponding truth table is made up of $2^{96} \approx 10^{28}$ rows.

Inference rules

To avoid the exponential computational complexity of model checking, **practical** inference algorithms based on **inference rules** have been devised.

An inference rule represents a **standard pattern of inference**: it implements a **simple reasoning step** whose soundness can be easily proven, that can be applied to a set of premises having a **specific** structure to derive a conclusion.

Inference rules are represented as follows:

$$\frac{\text{premises}}{\text{conclusion}}$$

Examples of inference rules

In the following, α and β denote any propositional **sentences**.

And Elimination	$\frac{\alpha_1 \wedge \alpha_2}{\alpha_i}$, $i = 1, 2$
And Introduction	$\frac{\alpha_1, \alpha_2}{\alpha_1 \wedge \alpha_2}$
Or Introduction	$\frac{\alpha_1}{\alpha_1 \vee \alpha_2}$ (α_2 can be **any** sentence)
First De Morgan's law	$\frac{\neg(\alpha_1 \wedge \alpha_2)}{\neg\alpha_1 \vee \neg\alpha_2}$
Second De Morgan's law	$\frac{\neg(\alpha_1 \vee \alpha_2)}{\neg\alpha_1 \ \wedge \ \neg\alpha_2}$
Double Negation	$\frac{\neg(\neg\alpha)}{\alpha}$
Modus Ponens	$\frac{\alpha \Rightarrow \beta, \ \alpha}{\beta}$

The first five rules above easily generalize to any set of sentences $\alpha_1, \ldots, \alpha_n$.

Soundness of inference rules

Since inference rules usually involve a few sentences, their soundness can be easily proven using model checking.

An example: **Modus Ponens**

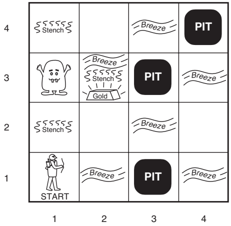
premise α	conclusion β	premise $\alpha \Rightarrow \beta$
false	false	true
false	true	true
true	false	false
true	true	true

An example

In the initial configuration of the wumpus game shown in the figure below, the agent's KB includes:

(a) $\neg B_{1,1}$ (current percept)

(b) $\neg B_{1,1} \Rightarrow \neg P_{1,2} \wedge \neg P_{2,1}$
(one of the rules of the game)



The agent can be interested in knowing whether room $(1, 2)$ does not contains any pit, i.e., whether $KB \models \neg P_{1,2}$.
The proof can be carried out using the above inference rules, e.g.:

- applying *Modus Ponens* to (a) and (b): $\neg P_{1,2} \wedge \neg P_{2,1}$
- applying *And elimination* to the new sentence above: $\neg P_{1,2}$

The agent can **conclude** that room $(1, 2)$ does not contain any pit.

Inference algorithms based on inference rules

Given a set of premises *KB* and a conclusion of interest α, the goal of an inference algorithm A is to find a **proof** $KB \vdash_A \alpha$ (if any), i.e., a **sequence** of applications of inference rules that leads from *KB* to α.

A basic inference algorithm based on a set of inference rules $\mathcal{I}$:

repeat
  apply in all possible ways the rules in $\mathcal{I}$ to the sentences of *KB*
  add to *KB* each derived sentence, if not already present in it
until some sentences not yet present in *KB* have been derived,
  and α is not present in *KB*

If α is present in the final *KB*, then $KB \vdash_A \alpha$, and if A is sound, one can conclude that $KB \models \alpha$.
What about the case when α is **not** present in the final *KB*? Can one conclude that $KB \not\models \alpha$? In other words, is A **complete**?

Properties of inference algorithms

Three main issues:

- is a given inference algorithm **sound** (correct)?
- is it **complete**?
- what is its **computational complexity**?

It is not difficult to see that, if the considered inference rules are **sound**, so is an inference algorithm based on them.

Completeness is more difficult to prove: it depends on the set of available inference rules, and on the ways in which they are applied.

Properties of inference algorithms

What about **computational complexity**?

There can be a huge number of ways to apply the inference rules at hand to sentences of KB, then to sentences of the **updated** KB, and so on, leading to a very high computational complexity.

Efficiency can be improved in different ways:

- ignoring sentences of KB **irrelevant** to the conclusion α
- using few inference rules (even only one), without compromising completeness

Complete inference algorithms for propositional logic

Forward and Backward Chaining are

- complete inference algorithms, **limited** to
  - premises (KB's) in the form of Horn clauses
  - conclusions in the form of atomic, non-negated sentences
- based on a **single** inference rule (Modus Ponens)

A family of complete inference algorithms, named **Resolution**, exists for the **whole** propositional logic (i.e., applicable to premises and conclusions in **any** form), which is also based on a **single**, homonymous inference rule.

Resolution algorithms require that the premises and conclusion are written in **conjunctive normal form**, which is possible for **any** propositional sentence.

Conjunctive normal form

A **literal** is a propositional symbol, possibly negated; e.g., P and $\neg Q$ are literals.

A **clause** is a sentence made up of either a single literal or a **disjunction** of two or more literals, e.g.:

- $\neg Q$
- $P \vee Q \vee \neg R$

A sentence is said to be in **conjunctive normal form** (CNF) if it is made up of either a single clause, or a **conjunction** of two or more clauses, e.g.:

- $\neg Q$
- $P \vee Q \vee \neg R$
- $(\neg R \vee S) \wedge P \wedge (Q \vee T \vee \neg Z)$

Conversion into conjunctive normal form

Any propositional sentence can be rewritten in CNF through the following steps:

1. eliminating biconditionals, through the equivalence:
   $(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$
2. eliminating implications, through the equivalence:
   $(P \Rightarrow Q) \equiv (\neg P \vee Q)$
3. "moving negations inwards" up to individual propositional symbols, using the following equivalences:
   - De Morgan's laws:
     $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$
     $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
   - double-negation elimination: $(\neg(\neg P)) \equiv P$
4. distributing $\vee$ over $\wedge$, using the equivalence:
   $((P \wedge Q) \vee R) \equiv ((P \vee R) \wedge (Q \vee R))$

The correctness of the above equivalences is easy to prove using truth tables. Note that each step produces a **clause**.

Conversion into conjunctive normal form: an example

Consider the following sentence encoding one of the rules of the Wumpus game:

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

It can be rewritten in CNF as follows:

1. eliminating the biconditional:
   $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$
2. eliminating the implication:
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$
3. moving negations inwards (using one of De Morgan's laws):
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$
4. distributing $\vee$ over $\wedge$:
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

The resolution inference rule

Consider two clauses (α_i's and α_j's denote literals):

$$\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_m$$
$$\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$$

with $n, m \geq 1$, and assume they contain a pair of **complementary** literals α_p and α_q, say, $\alpha_p = P$ and $\beta_q = \neg P$.

The **resolution** inference rule derives a new **clause** made up of the disjunction of all the literals of the premises, **except** for α_p and α_q:

$$\frac{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_m, \qquad \beta_1 \vee \beta_2 \vee \ldots \vee \beta_n}{\alpha_1 \vee \ldots \vee \alpha_{p-1} \vee \alpha_{p+1} \vee \ldots \vee \alpha_m \vee \beta_1 \vee \ldots \vee \beta_{q-1} \vee \beta_{q+1} \vee \ldots \vee \beta_n}$$

Its soundness can be formally proven using truth tables.
Intuitively, since **one** among α_p and α_q must be **false**, the disjunction of the **remaining** literals of the corresponding clause must be true.

The resolution inference rule

An example:

$$\frac{P \vee \neg Q \vee \neg \mathbf{R}, \quad \mathbf{R} \vee \neg S}{P \vee \neg Q \vee \neg S}$$

As a particular case, two clauses made up of **single**, **complementary** literals, such as P and $\neg P$, are **contradictory**, i.e., cannot be simultaneously true. Applying the resolution rule to such clauses leads to an "empty" clause, denoted by [ ]:

$$\frac{P, \quad \neg P}{[\,]}$$

Note that two contradictory clauses cannot be part of the **premises** (or KB) of any **correct** argumentation.

Proof by refutation

A sentence is **satisfiable** if it is true an at least one model. For instance, in propositional logic both P and $Q \vee \neg R$ are satisfiable, whereas $P \wedge \neg P$ is unsatisfiable. Unsatisfiable sentences are said to be **contradictory**.

Remember that a sentence α is said to logically follow from a set of premises KB ($KB \models \alpha$), if it cannot be false when KB is true. Viewing KB as a single, equivalent sentence made up of the **conjunction** of all its components, this can be restated as:

$KB \models \alpha$ if and only if $KB \wedge \neg\alpha$ is unsatisfiable (i.e., contradictory).

Proving that $KB \models \alpha$ by showing that $KB \wedge \neg\alpha$ is unsatisfiable (contradictory) corresponds to the mathematical technique of **proof by refutation** or **contradiction** (*reductio ad absurdum*).

Inference algorithms based on the resolution rule

The resolution rule, coupled with the proof by refutation technique, leads to **refutation-complete** algorithms for propositional logic:

- for any **given** KB and α, they can determine in a **finite** number of steps whether $KB \models \alpha$ or $KB \not\models \alpha$
- however, they **cannot** derive **all** the sentences that logically follow from a given set of premises

To this aim, it is necessary to use **complete** search strategies to carry out the proof.

A resolution inference algorithm

function RESOLUTION (KB, α) **returns** *true* or *false*
  clauses $\leftarrow$ the set of clauses in CNF form of $KB \cup \{\neg\alpha\}$
  new $\leftarrow \emptyset$
  for each distinct α_i, α_j **in** *clauses* **do**
    resolvents $\leftarrow$ `resolve`(α_i, α_j)
    if *resolvents* contains the empty clause **then return** *true*
    new $\leftarrow$ *new* $\cup$ *resolvents*
  if *new* $\subseteq$ *clauses* **then return** *false*
  clauses $\leftarrow$ *clauses* $\cup$ *new*

`resolve` denotes a function that applies the resolution rule to all pairs of complementary literals (if any) of a pair of clauses.

Resolution inference algorithms: computational complexity

Inference algorithms based on the resolution rule and on the proof by refutation technique check whether a set of sentences is **satisfiable**.

Checking satisfiability is known to be a **NP-complete** problem – informally, its computational complexity is very high: it requires up to an **exponential** number of steps in the size of the KB.

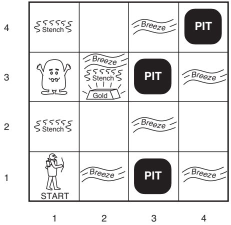
Efficiency can be improved through several strategies, such as discarding derived clauses containing complementary literals, which are true by definition, e.g.: $P \vee \neg R \vee Q \vee \neg Q$.

Resolution inference algorithms: an example

Consider the same sentence as in the previous example, encoding one of the rules of the Wumpus game:

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

Assuming the agent is in the starting square, the fact that it does not perceive any breeze (as in the figure on the right) can be expressed as $\neg B_{1,1}$.



To choose the next action, among other things the agent can be interested in knowing whether room $(1, 2)$ contains no pit, i.e., whether $\neg P_{1,2}$ is true.

The resolution inference algorithm: an example

For the sake of simplicity, consider a KB made up only of the two following sentences:

$$KB = \{B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1}), \ \ \neg B_{1,1}\}$$

The above resolution algorithm can now be applied to prove whether $KB \models \neg P_{1,2}$. Preliminary steps:

- converting into CNF the first sentence of KB (as shown in a previous example):

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$$

- adding to KB the negation of the sentence to be proven, and separating the resulting conjuncts, which leads to the clauses:

$$\{\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}, \ \ \neg P_{1,2} \vee B_{1,1}, \ \ \neg P_{2,1} \vee B_{1,1}, \ \ \neg B_{1,1}, \ \ P_{1,2}\}$$

The resolution inference algorithm: an example

What follows is a possible sequence of steps of the resolution algorithm, chosen ad hoc to complete the proof in a small number of steps.

Clauses 1–5 are the initial ones (i.e., coming from $KB \cup P_{1,2}$).
Subsequent clauses are obtained by the application of the resolution rule to the pair or clauses whose numbers are shown on the right.

1. $\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}$
2. $\neg P_{1,2} \vee B_{1,1}$
3. $\neg P_{2,1} \vee B_{1,1}$
4. $\neg B_{1,1}$
5. $P_{1,2}$
6. $\neg B_{1,1} \vee P_{1,2} \vee B_{1,1}$ (1, 3)
7. $P_{1,2} \vee P_{2,1} \vee \neg P_{2,1}$ (1, 3)
8. $\neg B_{1,1} \vee P_{2,1} \vee \vee B_{1,1}$ (1, 2)
9. $P_{1,2} \vee P_{2,1} \vee \neg P_{1,2}$ (1, 2)
10. $\neg P_{2,1}$ (3, 4)
11. $\neg P_{1,2}$ (2, 4)
12. [ ] (5, 11)

A contradiction has been derived in step 12, which proves that room $(1, 2)$ does not contain any pit.

Horn clauses

In many domains of practical interest, the whole KB can be expressed in the form of "if... then..." propositions that can be encoded as **Horn clauses**, i.e., implications where:

- the **antecedent** is a conjunction ($\wedge$) of atomic sentences (non-negated propositional symbols)
- the **consequent** is a single atomic sentence

$$P_1 \wedge \ldots \wedge P_n \Rightarrow Q$$

For instance, $S_{2,1} \wedge S_{4,1} \wedge S_{3,2} \Rightarrow W_{3,1}$ is a Horn clause.

As particular cases, also **atomic sentences** (i.e., propositional symbols) and their negation can be rewritten as Horn clauses. Indeed, since $(P \Rightarrow Q) \equiv (\neg P \vee Q)$:

$$\begin{array}{rclcl} P & \equiv & \neg \mathit{True} \vee P & \equiv & \mathit{True} \Rightarrow P \\ \neg P & \equiv & \neg P \vee \mathit{False} & \equiv & P \equiv \mathit{False} \end{array}$$

Forward and Backward Chaining inference algorithms

In the particular case when:

- the KB can be expressed as a set of Horn clauses
- the conclusion is an atomic and non-negated sentence (CIOÈ un caso particolare di clausola di Horn)

si dimostra che esistono inference algorithms, named **Forward** and **Backward Chaining**, exhibit the following characteristics:

- they use a **single** inference rule, as Resolution algorithms, in this case **Modus Ponens**
- they are **complete** – E IN PARTICOLARE FC consente di derivare TUTTE le conseguenze di una KB (limited to), cosa che Resolution algorithms cannot do
- they exhibit a computational complexity **linear** in the size of the KB, thus **lower** (?) than Resolution algorithms

Forward chaining

Given a KB made up by Horn clauses, **Forward Chaining** (FC) derives **all** the entailed atomic (non-negated) sentences:

function FORWARD-CHAINING (*KB*)
    repeat
        apply MP in all possible ways to sentences in *KB*
        add to *KB* the derived sentences not already present (if any)
    until some sentences not yet present in *KB* have been derived
    return *KB*

Forward Chaining

FC is an example of **data-driven** reasoning: it starts from known data, and derives their consequences.

For instance, in the wumpus game FC could be used to **update** the agent's knowledge about the environment (the presence of pits in each room, etc.), based on the new percepts after each move.

The inference engine of **expert systems** is inspired by the FC inference algorithm.

Forward Chaining: an example

Consider the KB shown below, made up of Horn clauses:

1. $P \Rightarrow Q$
2. $L \wedge M \Rightarrow P$
3. $B \wedge L \Rightarrow M$
4. $A \wedge P \Rightarrow L$
5. $A \wedge B \Rightarrow L$
6. A
7. B

(cont.)

Forward Chaining: an example

By applying FC one obtains:

8. the only implication whose premises (individual propositional symbols) are in the KB is 5: MP derives **L** and adds it to the current KB
9. now the premises of 3 are all true: MP derives **M** and adds it to the KB
10. the premises of 2 have become all true: MP derives **P** and adds it to the KB
11. the premises of 1 and 4 are now all true: MP derives **Q** form 1 and adds it to the KB, but disregards 4 since its consequent (L) is already present in the KB
12. no **new** sentences can be derived from 1–11: FC ends and returns the updated KB containing the original sentences 1–7 and the ones derived in the above steps: $\{L, M, P, Q\}$

Backward Chaining

For a given KB made up of Horn clauses, and a given atomic, non-negated sentence α, FC can be used to prove whether or not $KB \models \alpha$. To this aim, one has to check whether α is present or not among the derived sentences.

However, **Backward Chaining** (BC) is more effective for this task. BC **recursively** applies MP "backwards". It exploits the fact that $KB \models \alpha$, if and only if:

- either $\alpha \in KB$ (this terminates recursion)
- or KB contains some implication $\beta_1, \ldots, \beta_n \Rightarrow \alpha$, and (recursively) $KB \models \beta_1$, ..., $KB \models \beta_n$

The sentence α to be proven is also called **query**.

Backward Chaining

function BACKWARD-CHAINING (KB, α)
    if $\alpha \in KB$ **then return** True
    let B be the set of sentences of KB having α as the consequent
    for each $\beta \in B$
        let $\beta_1, \beta_2, \ldots$ be the propositional symbols in the antecedent of β
        if BACKWARD-CHAINING (KB, β_i) = True for all β_i's
        then return True
    return False

Backward Chaining

BC is a form of **goal-directed** reasoning.

For instance, in the wumpus game it could be used to answer queries like: given the current agent's knowledge, is *moving upward* the best action?

The computational complexity of BC is even **lower** than that of FC, since BC focuses only on relevant sentences.

The **Prolog** inductive logic programming language is based on the predicate logic version of the BC inference algorithm.

Backward Chaining: an example

Consider a KB representing the rules followed by a financial institution for deciding whether to grant a loan to an individual. The following propositional symbols are used:

- OK: the loan should be approved
- $COLLAT$: the collateral for the loan is satisfactory
- $PYMT$: the applicant is able to repay the loan
- REP: the applicant has a good financial reputation
- APP: the appraisal on the collateral is sufficiently greater than the loan amount
- $RATING$: the applicant has a good credit rating
- INC: the applicant has a good, steady income
- BAL: the applicant has an excellent balance sheet

Backward Chaining: an example

The KB is made up of the five rules (implications) on the left, and of the data about a specific applicant encoded by the four sentences on the right (all of them are Horn clauses):

1. $COLLAT \wedge PYMT \wedge REP \Rightarrow OK$
2. $APP \Rightarrow COLLAT$
3. $RATING \Rightarrow REP$
4. $INC \Rightarrow PYMT$
5. $BAL \wedge REP \Rightarrow OK$
6. APP
7. $RATING$
8. INC
9. $\neg BAL$

Should the loan be approved for this specific applicant?

This amounts to prove whether OK is entailed by the KB, i.e., whether:

$$KB \models OK$$

Backward Chaining: an example

The BC **recursive** proof $KB \vdash_{\mathrm{BC}} OK$ can be conveniently represented as an **AND-OR graph**, a tree-like graph in which:

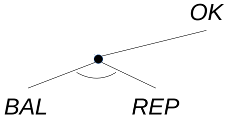
- multiple links joined by an arc indicate a **conjunction**: **every** linked proposition must be proven to prove the proposition in the parent node
- multiple links without an arc indicate a **disjunction**: **any** linked proposition can be proven to prove the proposition in the parent node

Backward Chaining: an example

The first call Backward-chaining(KB, OK) is represented by the tree root, corresponding to the sentence to be proven:

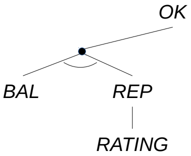
$$OK$$

Since $OK \notin KB$, implications having OK as the consequent are searched for. There are two such sentences: 1 and 5. The BC procedure tries to prove **all** the antecedents of **at least one** of them. Considering first 5, a recursive call to Backward-chaining is made for each of its two antecedents, represented by an AND-link:

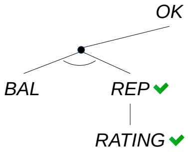


Backward Chaining: an example

Consider the call Backward-chaining(*KB*, *REP*): since $REP \notin KB$, and the only implication having *REP* as the consequent is 3, another recursive call is made for the antecedent of 3:

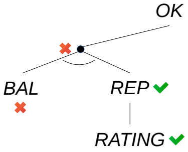


The call Backward-chaining(*KB*, *RATING*) returns *True*, since $RATING \in KB$, and thus also the call Backward-Chaining(*KB*, *REP*) returns *True*:

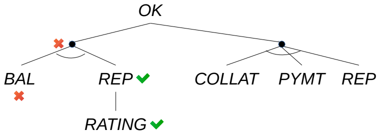


Backward Chaining: an example

However, the call BACKWARD-CHAINING(KB, BAL) returns *False*, since $BAL \notin KB$ and there are no implications having BAL as the consequent. Therefore, the first call BACKWARD-CHAINING(KB, OK) is not able to prove OK through this AND-link:

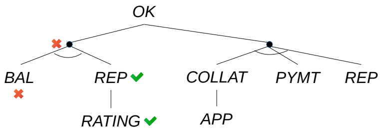


The other sentence in the KB having OK as the consequent, 1, is now considered, and another AND-link is generated with three recursive calls for each of the antecedents of 1:

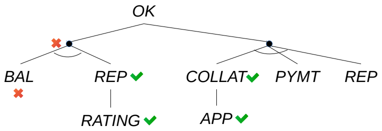


Backward Chaining: an example

The call BACKWARD-CHAINING(*KB*, *COLLAT*) generates in turn another recursive call to prove the antecedent of the only implication having *COLLAT* as the consequent, 2:



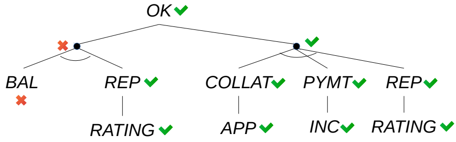
The call BACKWARD-CHAINING(*KB*, *APP*) returns *True*, since *APP* $\in$ *KB*, and thus also BACKWARD-CHAINING(*KB*, *COLLAT*) returns *True*



Backward Chaining: an example

Similarly, the calls $\textsc{Backward-chaining}(KB, PYMT)$ and $\textsc{Backward-Chaining}(KB, REP)$ return *True*.

The corresponding AND-link is then proven, which finally allows the first call $\textsc{Backward-chaining}(KB, OK)$ to return *True*:



The proof $KB \vdash_{BC} OK$ is then **successfully completed**.

Exercise 1

Construct the agent's initial KB for the wumpus game.

The KB should contain:

- the rules of the game: the agent starts in room (1,1); there is a breeze in rooms adjacent to pits, etc.
- rules to decide the agent's move at each step of the game

Note that the KB must be **updated** at each step of the game:

1. adding percepts in the current room (from **sensors**)
2. **reasoning** to derive new knowledge about the position of pits and wumpus
3. **reasoning** to decide the next move
4. updating the agent's position after a move

Exercise 1

Rules of the wumpus game:

- *the agent starts in room (1,1)*:
  $A_{1,1} \wedge \neg A_{1,2} \wedge \ldots \wedge \neg A_{4,4}$
- *there is a breeze in rooms adjacent to pits*:
  $P_{1,1} \Rightarrow (B_{2,1} \wedge B_{1,2})$,
  $P_{1,2} \Rightarrow (B_{1,1} \wedge B_{2,2} \wedge B_{1,3})$, ...
  (**one** proposition in natural language, **sixteen** sentences in propositional logic – one for each room)
- *there is only one wumpus*:
  $(W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{1,3} \wedge \ldots \wedge \neg W_{4,4}) \vee$
  $(\neg W_{1,1} \wedge W_{1,2} \wedge \neg W_{1,3} \wedge \ldots \wedge \neg W_{4,4}) \vee \ldots$
  (**one** proposition in natural language, **sixteen** sentences in propositional logic – one for each room)
- ...

Often, **one** concise proposition in natural language needs to be represented by **many** complex sentences in propositional logic.

Exercise 1

How to **update** the KB to account for the **change** of the agent's position after each move? E.g., $A_{1,1}$ is true in the starting position, and becomes false after the first move:

- adding $\neg A_{1,1}$ makes the KB contradictory, since $A_{1,1}$ is still present ...
- ... but inference rules do not allow **removing** sentences

Solution: using a **different** propositional symbol for each time step, e.g., $A^t_{i,j}$, $t = 1, 2, \ldots$

- initial KB: $A^1_{1,1}$, $\neg A^1_{1,2}$, $\ldots \neg A^1_{4,4}$
- if the agent moves to (1,2), the following sentences must be added to the KB: $\neg A^2_{1,1}$, $A^2_{1,2}$, $\neg A^2_{1,3}$ $\ldots$, $\neg A^2_{4,4}$; and so on

Things get complicated ...

Exercise 2

The following argumentation (an example of **syllogism**) is intuitively correct; prove its correctness using propositional logic:
All men are mortal; Socrates is a man; then, Socrates is mortal.

Three **distinct** propositional symbols must be used:
P (*All men are mortal*), Q (*Socrates is a man*), R (*Socrates is mortal*)

Therefore:

- premises: $\{P, Q\}$
- conclusion: R

Do the premises **entail** the conclusion, i.e., $\{P, Q\} \models R$?

Model checking easily allows one to prove that the answer is **no**: in the model $\{P = True, Q = True, R = False\}$, the premises are **true** but the conclusion is **false**.

What's wrong?

Limitations of propositional logic

Main problems:

- **limited expressive power**
- **lack of conciseness**

Example (wumpus world)

Even small knowledge bases (in natural language) require a large number of propositional symbols and sentences.

Example (syllogisms)

Inferences involving the **structure** of **atomic** sentences (e.g., *All men are mortal, . . .*) cannot be made.

From propositional to predicate logic

The description of many domains of interest for real world applications (e.g., mathematics, philosophy, AI) involve the following elements of natural language:

- **nouns** denoting **individuals** or **objects**, e.g.: the wumpus, pits, Socrates, Plato, the number four, etc.
- **predicates** denoting **properties** of individuals (or objects) or **relations** between them, e.g.: Socrates **is a man**, five **is prime**, four **is lower than** five; the sum of two and two **equals** four
- **functions** that indirectly denote an individual (or object) between objects in terms of other ones, e.g.: the **father of Mary**, **one plus three** is even
- facts involving **some** or **all** individuals or objects of a given set, e.g.: **all** squares neighboring the wumpus are smelly; **some** numbers are prime

These elements cannot be represented in propositional logic, and require the more expressive **predicate logic**.

Predicate Logic

Models

In predicate logic a **model** consists of:

- a **domain of discourse**, i.e., the set of all objects or individuals mentioned in the propositions, e.g.:
  - the set of natural numbers
  - a set of individuals: Socrates, Plato, ...
- **relations** between domain elements, **explicitly** represented as the set of tuples among which a relation holds, e.g.:
  - being a prime number (unary relation): $\{1, 2, 3, 5, 7, 11, \dots\}$
  - being greater than (binary relation): $\{(2,1), (3,1), \dots\}$
  - being equal to (binary relation): $\{(1,1), (2,2), \dots\}$

  (unary relations are also called **properties**)
- **functions** mapping tuples of domain elements to a single one, e.g.:
  - plus: $(1,1) \rightarrow 2$, $(1,2) \rightarrow 3$, ...
  - father of: John $\rightarrow$ Mary, ...

Note that relations and functions are defined **extensionally**, i.e., by explicitly enumerating the corresponding tuples.

Syntax

The basic elements are symbols that are used to represent domain elements, relations and functions:

- **constant symbols** denote domain elements (objects or individuals), e.g.: *One*, *Two*, *Three*, *John*, *Mary*
- **predicate symbols** denote relations, e.g.: *GreaterThan*, *Prime*, *Sum*, *Father*
- **function symbols** denote functions, e.g.: *Plus*, *FatherOf*

Syntax

A formal grammar in Backus-Naur Form (BNF):

$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \\
& | & (\mathit{Sentence}\ \mathit{Connective}\ \mathit{Sentence}) \\
& | & \mathit{Quantifier}\ \mathit{Variable}, \ldots \mathit{Sentence} \\
& | & \neg\ \mathit{Sentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{Predicate}(\mathit{Term}, \ldots) \\
\mathit{Term} & \rightarrow & \mathit{Function}(\mathit{Term}, \ldots) \mid \mathit{Constant} \mid \mathit{Variable} \\
\mathit{Connective} & \rightarrow & \Rightarrow \mid \wedge \mid \vee \mid \Leftrightarrow \\
\mathit{Quantifier} & \rightarrow & \forall \mid \exists \\
\mathit{Constant} & \rightarrow & \mathit{John} \mid \mathit{Mary} \mid \mathit{One} \mid \mathit{Two} \mid \ldots \\
\mathit{Variable} & \rightarrow & a \mid x \mid s \mid \ldots \\
\mathit{Predicate} & \rightarrow & \mathit{GreaterThan} \mid \mathit{Father} \mid \ldots \\
\mathit{Function} & \rightarrow & \mathit{Plus} \mid \mathit{FatherOf} \mid \ldots
\end{array}
$$

Semantics: interpretations

Remember that semantics defines the truth of well-formed sentences, related to a particular model.
In predicate logic this requires an **interpretation**: defining which domain elements, relations and functions are referred to by symbols.

Examples:

- *One*, *Two* and *Three* denote the natural numbers 1, 2, 3;
  John and *Mary* denote the individuals John and Mary
- *GreaterThan* denotes the binary relation “to be greater than” ($>$) between numbers;
  Father denotes the fatherhood relation between individuals
- *Plus* denotes the function mapping a pair of numbers to the number corresponding to their sum

Semantics: terms

Terms are logical expressions denoting domain elements.
A term can be:

- **simple**: a constant symbol, e.g.: *One*, *Two*, *John*
- **complex**: a function symbol applied (possibly, **recursively**) to other terms, e.g.:
  FatherOf(*Mary*)
  Plus(*One*, *Two*)
  Plus(*One*, *Plus*(*One*, *One*))

Note:

- assigning a constant symbol to every domain element is not required (domains can be even infinite): only elements **explicitly** mentioned in propositions (e.g., *Socrates*) should be assigned a constant symbol
- a domain element can be denoted by more than one symbol

Semantics: atomic sentences

Atomic sentences are the simplest kind of proposition: a predicate symbol applied to a list of terms.

Examples:

- $GreaterThan(Two, One)$
- $Prime(Two)$,
- $Prime(Plus(Two, Two))$
- $Sum(One, One, Two)$
- $Father(John, Mary)$
- $Father(FatherOf(John), FatherOf(Mary))$

Semantics: atomic sentences

An atomic sentence is true, in a given model and under a given interpretation, if the relation referred to by its predicate symbol holds between the objects referred to by its arguments (terms)

Example

According to the above model and interpretation:

- $GreaterThan(One, Two)$ is false
- $Prime(Two)$ is true
- $Prime(Plus(One, One))$ is true
- $Sum(One, One, Two)$ is true
- $Father(John, Mary)$ is true

Semantics: complex sentences

Complex sentences are obtained as in propositional logic, using logical connectives.

Examples:

- $\mathit{Prime}(\mathit{Two}) \wedge \mathit{Prime}(\mathit{Three})$
- $\neg \mathit{Sum}(\mathit{One}, \mathit{One}, \mathit{Two})$
- $\mathit{GreaterThan}(\mathit{One}, \mathit{Two}) \Rightarrow (\neg \mathit{GreaterThan}(\mathit{Two}, \mathit{One}))$
- $\mathit{Father}(\mathit{John}, \mathit{Mary}) \vee \mathit{Father}(\mathit{Mary}, \mathit{John})$

Semantics (truth value) is determined as in propositional logic.
Examples: the second sentence above is false, the others are true.

A note on predicates and functions

Although the syntax of predicate and function symbols is identical, their role is different:

- functions are **terms** (i.e., denote domain elements), and therefore can only appear as **arguments** of predicates and (recursively) functions
- predicates denote propositions, and **cannot** appear as arguments of predicates or functions

Semantics: quantifiers

Quantifiers allow one to express propositions involving **collections** of domain elements, **without** enumerating them **explicitly**.

Two main quantifiers are used in predicate logic:

- **universal** quantifier, e.g.:
  All *men are mortal*
  All *rooms neighboring the wumpus are smelly*
  All *even numbers that are greater than two are not prime*
- **existential** quantifier, e.g.:
  Some *numbers are prime*
  Some *rooms contain pits*
  Some *men are philosophers*

Quantifiers require a new kind of term: **variable symbols**, usually denoted with lowercase letters.

Semantics: universal quantifier

Example

Assume that the **domain** is the set of natural numbers.
All natural numbers are greater than or equal to one

$$\forall x \; GreaterOrEqual(x, One)$$

Semantics: universal quantifier

The semantics of a sentence $\forall x\ \alpha(x)$, where $\alpha(x)$ is a sentence containing the variable x, is:

$\alpha(x)$ is true for **each** domain element in place of x

Example

If the domain is the set of natural numbers,

$$\forall x\ \mathit{GreaterOrEqual}(x, \mathit{One})$$

states that the following (infinite) sentences are **all** true:
$\mathit{GreaterOrEqual}(\mathbf{One}, \mathit{One})$
$\mathit{GreaterOrEqual}(\mathbf{Two}, \mathit{One})$
...

Semantics: universal quantifier

Consider the proposition:

all even numbers greater than two are not prime

A common mistake is to represent it as:

$$\forall x \; \mathit{Even}(x) \wedge \mathit{GreaterThan}(x, \mathit{Two}) \wedge (\neg \mathit{Prime}(x))$$

The above sentence actually states that:

all numbers are even, greater than two, and are not prime

which is different from the intended meaning.

Semantics: universal quantifier

The correct sentence can be obtained by restating the original proposition as:

> ***for all*** *x*, ***if*** *x is even and greater than two,*
> ***then*** *x is not prime*

This proposition states a sufficient condition, and therefore can be represented as an **implication**:

$$\forall x \; (Even(x) \wedge GreaterThan(x, Two)) \Rightarrow (\neg Prime(x))$$

In general, propositions where "all" refers to all the domain elements **that satisfy some condition** should be represented using an **implication**.

Semantics: universal quantifier

Consider again this sentence:

$$\forall x \; (Even(x) \wedge GreaterThan(x, Two)) \Rightarrow (\neg Prime(x))$$

Claiming that it is true corresponds to claim that also sentences like the following ones are true:

$$(Even(One) \wedge GreaterThan(One, Two)) \Rightarrow (\neg Prime(One))$$

Note that the antecedent of the implication is false (the number one is not even, nor it is greater than the number two). This is not contradictory, since implications with false antecedents are true **by definition** (see again the truth table of $\Rightarrow$).

Semantics: existential quantifier

Example

Assume that the domain is the set of natural numbers.

- *Some numbers are prime*

$$\exists x \; Prime(x)$$

  This is read as: *there exists some x such that x is prime*

- *Some numbers are not greater than three, and are even*

$$\exists x \; \neg GreaterThan(x, Three) \wedge Even(x)$$

Semantics: existential quantifier

Consider a proposition like the following:

some odd numbers are prime

A common mistake is to represent it using an implication:

$$\exists x \; Odd(x) \Rightarrow Prime(x)$$

The above sentence actually states that:

for some natural numbers being odd is a necessary condition to be prime

which does not correspond to the original proposition (and is also false).

Semantics: existential quantifier

The sentence corresponding to the original proposition can be obtained by restating the latter as:

there exists *some x such that x is odd* ***and*** *x is prime*

The above proposition can be represented using a **conjunction**:

$$\exists x \; Odd(x) \wedge Prime(x)$$

In general, propositions stating several properties about "some" domain element should be represented using a **conjunction**.

Semantics: nested quantifiers

A sentence can contain more than one quantified variable.
If the quantifier is the same for all variables, e.g.:

$$\forall x (\forall y (\forall z \ldots \alpha[x, y, z, \ldots] \ldots))$$

then the sentence can be rewritten more concisely as:

$$\forall x, y, z \ldots \alpha[x, y, z, \ldots]$$

For instance, in the domain of natural numbers, the proposition:

> *If a number is greater than another number, then also the successor of the former is greater than the latter*

can be represented by the following sentence (using the function *Successor*):

$$\forall x, y \ \mathit{GreaterThan}(x, y) \Rightarrow \mathit{GreaterThan}(\mathit{Successor}(x), y)$$

Semantics: nested quantifiers

If a sentence contains both universally and existentially quantified variables, its meaning depends on the **order** of quantification. In particular, $\forall x(\exists y\ \alpha[x, y])$ and $\exists y(\forall x\ \alpha[x, y])$ are **not** equivalent, i.e., they are not true under the **same** models.

For instance,

$$\forall x\ \exists y\ Loves(x, y)$$

states that (i.e., is true under any model in which) *everybody loves somebody*. Note that the domain element denoted by y can be different for different x's.

Instead,

$$\exists y\ \forall x\ Loves(x, y)$$

states that *there is someone who is loved by everyone* (now the domain element denoted by y must be the same for **all** the x's). Therefore the meaning of the above sentences is different, i.e., they can be true under **different** sets of models.

Semantics: connections between quantifiers

The quantifiers $\forall$ and $\exists$ are connected with each other through **negation**, just like in natural language.

For instance, asserting that *Every natural number is greater than or equal to zero* is the same as asserting that *There does not exist any natural number which is not greater than or equal to zero*.

The two propositions above can be respectively translated into the following, **equivalent** sentences, whose domain is assumed to be the set of natural numbers:

$$\forall x \ \mathit{GreaterOrEqual}(x, \mathit{Zero})$$

$$\neg\,(\exists x \ \neg \mathit{GreaterOrEqual}(x, \mathit{Zero}))$$

Semantics: connections between quantifiers

In general, since $\forall$ is equivalent to a conjunction over all domain elements, and $\exists$ is equivalent to a disjunct, they obey De Morgan's rules (shown below on the left, in the usual form involving two propositional variables):

$$
\begin{array}{rcl|rcl}
\neg P \wedge \neg Q & \Leftrightarrow & \neg(P \vee Q) & \forall x(\neg\alpha[x]) & \Leftrightarrow & \neg(\exists x\ \alpha[x]) \\
\neg(P \wedge Q) & \Leftrightarrow & (\neg P) \vee (\neg Q) & \neg(\forall x\ \alpha[x]) & \Leftrightarrow & \exists x(\neg\alpha[x]) \\
P \wedge Q & \Leftrightarrow & \neg(\neg P \vee \neg Q) & \forall x\ \alpha[x] & \Leftrightarrow & \neg(\exists x(\neg\alpha[x])) \\
P \vee Q & \Leftrightarrow & \neg(\neg P \vee \wedge Q) & \exists x\ \alpha[x] & \Leftrightarrow & \neg(\forall x\ (\neg\alpha[x]))
\end{array}
$$

Exercises

Represent the following propositions using sentences in predicate logic, including the definition of the domain

1. All men are mortal; Socrates is a man; Socrates is mortal
2. All rooms neighboring a pit are breezy (wumpus game)
3. Peano-Russell's axioms of arithmetic, that define natural numbers (nonnegative integers):
   - P1 zero is a natural number
   - P2 the successor of any natural number is a natural number
   - P3 zero is not the successor of any natural number
   - P4 no two natural numbers have the same successor
   - P5 any property which belongs to zero, and to the successor of every natural number which has the property, belongs to all natural numbers

Exercises

4. Assume that the goal is to prove that West is a criminal (using suitable inference algorithms):

   The law says that it is a crime for an American to sell weapons to hostile countries. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

   Note that in a knowledge-based system the first proposition above encodes the **general** knowledge about the problem at hand (analogously to the rules of chess and of the wumpus game), whereas the second proposition encodes a **specific** problem instance (analogously to a specific configuration of a chessboard or of the wumpus maze).

Solution of exercise 1

Domain and symbols:

- domain: any set including all men
- constant symbols: *Socrates*
- predicate symbols: *Man* and *Mortal*, unary predicates; e.g., $Man(Socrates)$ means that Socrates is a man.

The sentences are:

$$\forall x \ Man(x) \Rightarrow Mortal(x)$$
$$Man(Socrates)$$
$$Mortal(Socrates)$$

Solution of exercise 2

A **possible** choice of domain and symbols:

- domain: row and column coordinates
- constant symbols: 1, 2, 3, 4
- predicate symbols:
  - *Pit*, binary predicate; e.g., $P(1,2)$ means that there is a pit in room (1,2)
  - *Adjacent*, predicate with four terms; e.g., $Adjacent(1,1,1,2)$ means that room (1,1) is adjacent to room (1,2)
  - *Breezy*, binary predicate; e.g., $Breezy(2,2)$ means that there is a breeze in room (2,2)

Solution of exercise 2

The required sentence can be obtained by restating the considered proposition as:

> *if a room contains a pit, then all the adjacent rooms are breezy*

Note that the latter proposition represents a **sufficient** condition for all the rooms adjacent to given one to be breezy, but not a necessary one, since the same rooms can be breezy also due to the presence of pits in other rooms. The latter proposition can therefore be represented as an **implication**:

$$\forall x, y \; Pit(x, y) \Rightarrow (\forall p, q \; Adjacent(x, y, p, q) \Rightarrow Breezy(p, q))$$

Solution of exercise 2

One could however argue that the original proposition does not **completely** represent the corresponding rule of the wumpus game, that can be stated as follows:

> *a room is breezy, if and only if at least one of the adjacent rooms contains a pit*

It is easy to see that the above proposition states a **necessary** and **sufficient** condition, which can be represented as an **equivalence**:

$$\forall x, y \ Breezy(x, y) \Leftrightarrow (\exists p, q \ Adjacent(x, y, p, q) \land Pit(p, q))$$

Solution of exercise 3

A **possible** choice of domain and symbols:

- domain: any set including all natural numbers (e.g., the set of real numbers)
- constant symbols: Z, denoting the number zero
- predicate symbols:
  - N, unary predicate denoting the fact of being a natural number; e.g., $N(Z)$ means that zero is a natural number
  - Eq, binary predicate denoting equality; e.g., $Eq(Z, Z)$ means that zero equals zero
  - P denoting any **given** property
- function symbols: S, mapping a natural number to its successor; e.g., $S(Z)$ denotes one, $S(S(Z))$ denotes two

Solution of exercise 3

P1 $N(Z)$

P2 $\forall x \; N(x) \Rightarrow N(S(x))$

P3 $\neg(\exists x \; Eq(Z, S(x)))$

P4 $\forall x, y \; Eq(S(x), S(y)) \Rightarrow Eq(x, y)$

P5 $(P(Z) \wedge \forall x((N(x) \wedge P(x)) \Rightarrow P(S(x)))) \Rightarrow (\forall x \; (N(x) \Rightarrow P(x)))$

Solution of exercise 4

A **possible** choice of domain and symbols:

- domain: a set including different individuals (among which Colonel West), nations (among which America and Nono), and missiles
- constant symbols: *West*, *America* and *Nono*
- predicate symbols:
  - $Country(\cdot)$, $American(\cdot)$, $Missile(\cdot)$, $Weapon(\cdot)$, $Hostile(\cdot)$: respectively, being: a country, an American citizen, a missile, a weapon, hostile
  - $Enemy(< who >, < to\ whom >)$: being enemies
  - $Owns(< who >, < what >)$: owning something
  - $Sells(< who >, < what >, < to\ whom >)$: selling something to someone
- no function symbols are necessary

Solution of exercise 4

The law says that it is a crime for an American to sell weapons to hostile nations:
$\forall x, y, z\ (American(x) \wedge Country(y) \wedge Hostile(y) \wedge Weapon(z) \wedge$
$\quad Sells(x, y, z)) \Rightarrow Criminal(x)$

The second proposition can be conveniently split into simpler ones:

Nono is a country...:
$Country(Nono)$

...Nono is an enemy of America (which is also a country)...:
$Enemy(Nono, America)$
$Country(America)$

...Nono has some missiles...:
$\exists x\ Missile(x) \wedge Owns(Nono, x)$

...all Nono's missiles were sold to it by Colonel West:
$\forall x\ (Missile(x) \wedge Owns(Nono, x)) \Rightarrow Sells(West, Nono, x)$

Solution of exercise 4

A human would intuitively say that the above propositions in natural language imply that West is a criminal.
However, it is not difficult to see that the above sentences in predicate logic are **not** sufficient to prove this.
The reason is that humans exploit **background knowledge** (or **common sense**) that is **not** represented **explicitly** in the above propositions. In particular, there are two "missing links":

- an enemy nation is hostile
- a missile is a weapon

To use such additional knowledge, it must be **explicitly** represented by sentences in predicate logic:

- $\forall x, y \; (Country(x) \land Enemy(x, America)) \Rightarrow Hostile(x)$
- $\forall x \; Missile(x) \Rightarrow Weapon(x)$

Knowledge engineering

Knowledge engineering is the process of constructing a KB.

It consists of investigating a specific domain, identifying the relevant concepts (**knowledge acquisition**), and formally representing them.

This requires the interaction between

- a **domain expert** (DE)
- a **knowledge engineer** (KE), who is expert in knowledge representation and inference, but usually **not** in the domain of interest

A possible approach, suitable for **special-purpose** KBs (in predicate logic), is the following.

Knowledge engineering

1. Identify the task:
   - what range of queries will the KB support?
   - what kind of facts will be available for each problem instance?
2. Knowledge acquisition: eliciting from the domain expert the **general** knowledge about the domain (e.g., the rules of chess)
3. Choice of a **vocabulary**: what concepts have to be represented as objects, predicates, functions?
   The result is the domain's **ontology**, which affects the complexity of the representation and the inferences that can be made.
   E.g., in the wumpus game pits can be represented either as objects, or as unary predicates on squares.

(cont.)

Knowledge engineering

4. Encoding the domain's general knowledge acquired in step 2 (this may require to revise the vocabulary of step 3)
5. Encoding a specific problem instance (e.g., a specific chess game)
6. Posing queries to the inference procedure and getting answers
7. Debugging the KB, based on the results of step 6

Applications of predicate logic and inference algorithms

Logic programming languages, in particular **Prolog**, used for:

- rapid prototyping
- symbol processing applications (compilers, natural language parsers, etc.)
- developing expert systems

Example of a Prolog **clause**:

```
criminal(X) :- american(X), weapon(Y), sells(X,Y,Z),
               hostile(Z).
```

Running a program consists of proving a sentence (**query**) using a specific inference algorithm, e.g.:

- `?- criminal(west)`
  produces `Yes`
- `?- criminal(A)`
  produces `A=west, Yes`

Applications of predicate logic and inference algorithms

Theorem provers, used for:

- assisting (not replacing) mathematicians
- proof checking
- verification and synthesis of hardware and software
  - hardware design (e.g., entire CPUs)
  - programming languages (syntax)
  - software engineering (verifying program specifications, e.g., RSA public key encryption algorithm)

Other applications

Encoding condition-action rules to recommend actions, based on a data-driven approach: **expert systems** and **production systems** (production: condition-action rule).

Expert systems

- encoding human experts' problem-solving knowledge in the form of `IF...THEN...` rules, in **specific** application domains for which **no algorithmic solutions exist** (e.g., medical diagnosis)
- used as **decision support systems**, to support (not to replace) experts' decisions
- popular in the 1980s, used in niche/restricted domains since the 1990s (medical diagnosis, geology, finance, military strategies, software engineering, help desk)
- examples of free Expert System shells
  - CLIPS (C Language Integrated Production System) `https://www.clipsrules.net/`
  - Drools (Business Rules Management System) `http://www.drools.org`

Beyond classical logic

Classical logic is based on two principles:

- **bivalence**: there exist only two truth values, *true* and *false*
- **determinateness**: each proposition has only one truth value

But: how to deal with propositions like the following ones?

- *Tomorrow will be a sunny day*: is this true or false, **today**?
- *John is tall*: is this "completely" true (or false)?
  This kind of problem is addressed by **fuzzy logic**
- Goldbach's conjecture: *Every even number is the sum of a pair of prime numbers*
  No proof has been found yet: can we still say this is either true or false?