University of Cagliari

MSc programs in: Computer Engineering, Cybersecurity and Artificial Intelligence – Electronic Engineering – Internet Engineering

Artificial Intelligence

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Instructor: Giorgio Fumera

Exercises on logical languages for knowledge representation and reasoning

Propositional logic

1. Use the Backward chaining inference algorithm to prove that the propositional sentence R logically follows from the knowledge base shown below, which is expressed in the form of Horn clauses:

$$KB = \{C \Rightarrow A, \quad P \wedge Q \Rightarrow R, \quad A \wedge B \Rightarrow R, \quad P, \quad S \Rightarrow Q, \quad P \wedge A \wedge C \Rightarrow B, \quad S \Rightarrow B, \quad C\}$$

2. Consider a robot, powered by a battery, capable of moving objects that are liftable. Its knowledge base includes the fact that, if its battery is charged, and it tries to move a liftable object, then that object will move. It also has sensors that tell it whether its battery is charged or not, and whether an object it is trying to lift moves or does not.

   Assume that, after the robot encounters an object and tries to lift it, its sensors indicate that that object does not move, and that its battery is charged. Intuitively, this implies that the object is not liftable.

   (a) Represent the robot's knowledge (both the knowledge base and the sensory information) in propositional logic.

   (b) Prove that the object is not liftable, using the resolution inference rule.

Predicate logic

1. Give a representation in predicate logic of the following propositions related to search trees, starting from the definition of the domain of discourse and of the predicate, function and constant symbols (they have to be the same for all propositions).

   (a) The root node of a search tree has zero depth

   (b) A node that has not yet been expanded has no successors

   (c) A node that has already been expanded has at least one successor

   (d) Each node has exactly one parent node, except for the root node

2. Give a representation in predicate logic of the following groups of propositions. First of all, separately for each group, *clearly* define the domain of discourse and the predicate, function and constant symbols (they have to be the same for all propositions in the same group).

   (a) Not all students take both the History and the Biology courses. The best mark in History is higher than the best mark in Biology. One only student failed in History. One only student failed both in History and in Biology.

   (b) Politicians can fool some of the people all of the time, and they can fool all the people some of the time, but they can't fool all of the people all of the time.
   (*From a phrase attributed to Abraham Lincoln.*)

(c) Every person who dislikes all vegetarians is smart. Nobody likes a vegetarian who is intelligent. There is a woman who likes all men who are not vegetarians.

(d) Some barber shaves every man who does not shave himself.
(*From an informal version of the famous paradox of set theory by the British philosopher and mathematician Bertrand Russell, about the barber who shaves all and only those who do not shave themselves: who shaves that barber, assuming it is a male?*)

(e) No person likes a professor unless the professor is smart.

(f) All Germans speak the same languages.

3. Give a representation in predicate logic of the following propositions, starting from the definition of the domain of discourse and of the predicate, function and constant symbols (they have to be the same for all propositions).

   (a) Cows, pigs and horses are mammals

   (b) The child of a horse is a horse

   (c) Bluebeard is a horse

   (d) Bluebeard is Charlie's father

   (e) Child and father are inverse relations

   (f) Every mammal has a father

4. Give a representation in predicate logic of the following propositions, including the definition of the domain of discourse and of the predicate, function and constant symbols (they have to be the same for all propositions).

   (a) Tony, Mike and John are members of the Alpine Club

   (b) Every member of the Alpine Club is a skier or a climber

   (c) No climber likes rain

   (d) Every skier likes snow

   (e) Mike does not like everything that Tony likes

   (f) Mike likes everything Tony does not like

   (g) Tony likes both rain and snow

5. Give a representation in predicate logic of the following propositions related to the *block's world* domain, including the definition of the domain of discourse and of the predicate, function and constant symbols (they have to be the same for all propositions):

   (a) If all heavy objects are blue, then all non-heavy objects are green

   (b) Every object is either blue or green, but not both

   (c) If there is some non-heavy object, then all heavy objects are blue

Solution

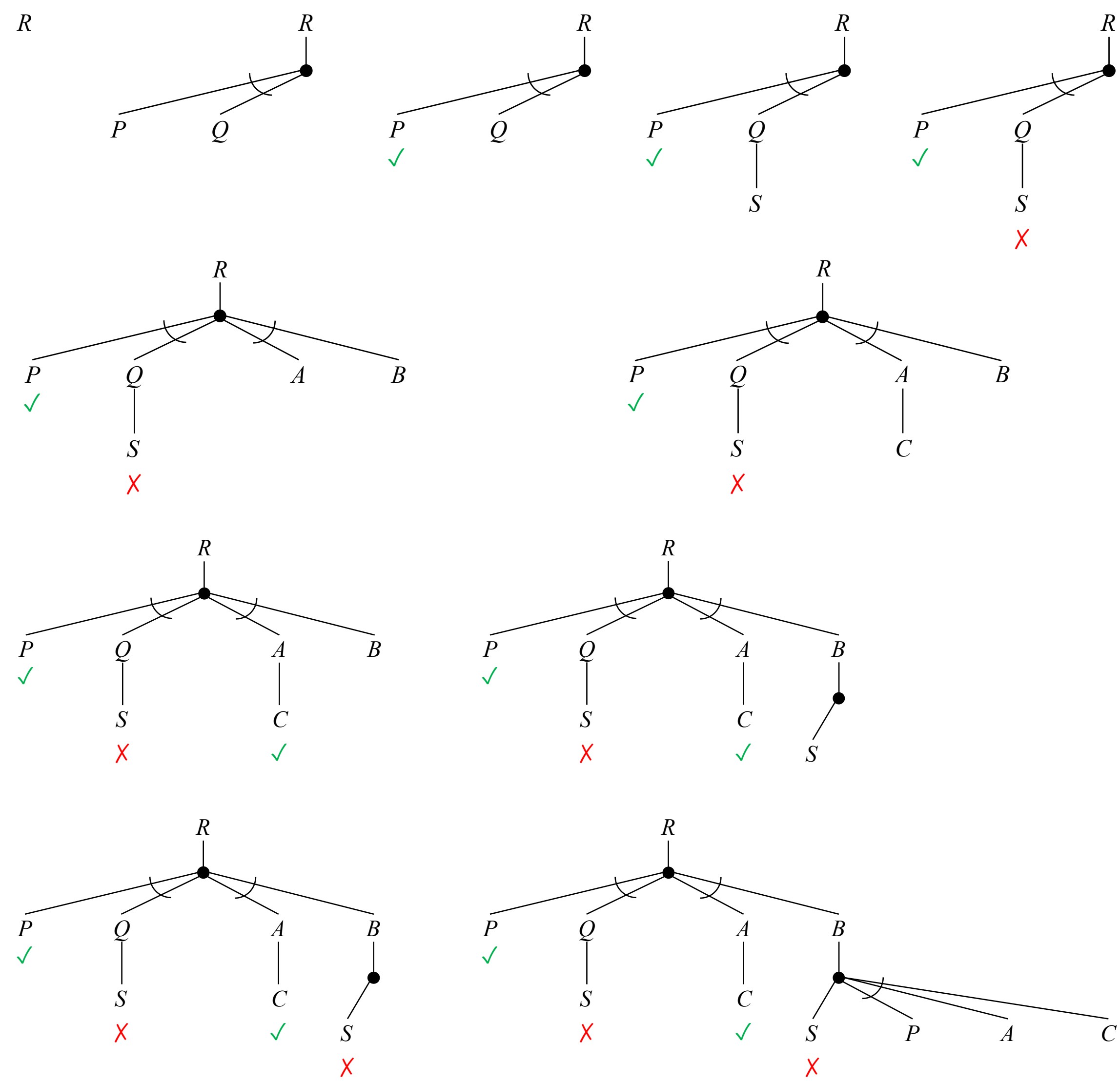
Propositional logic

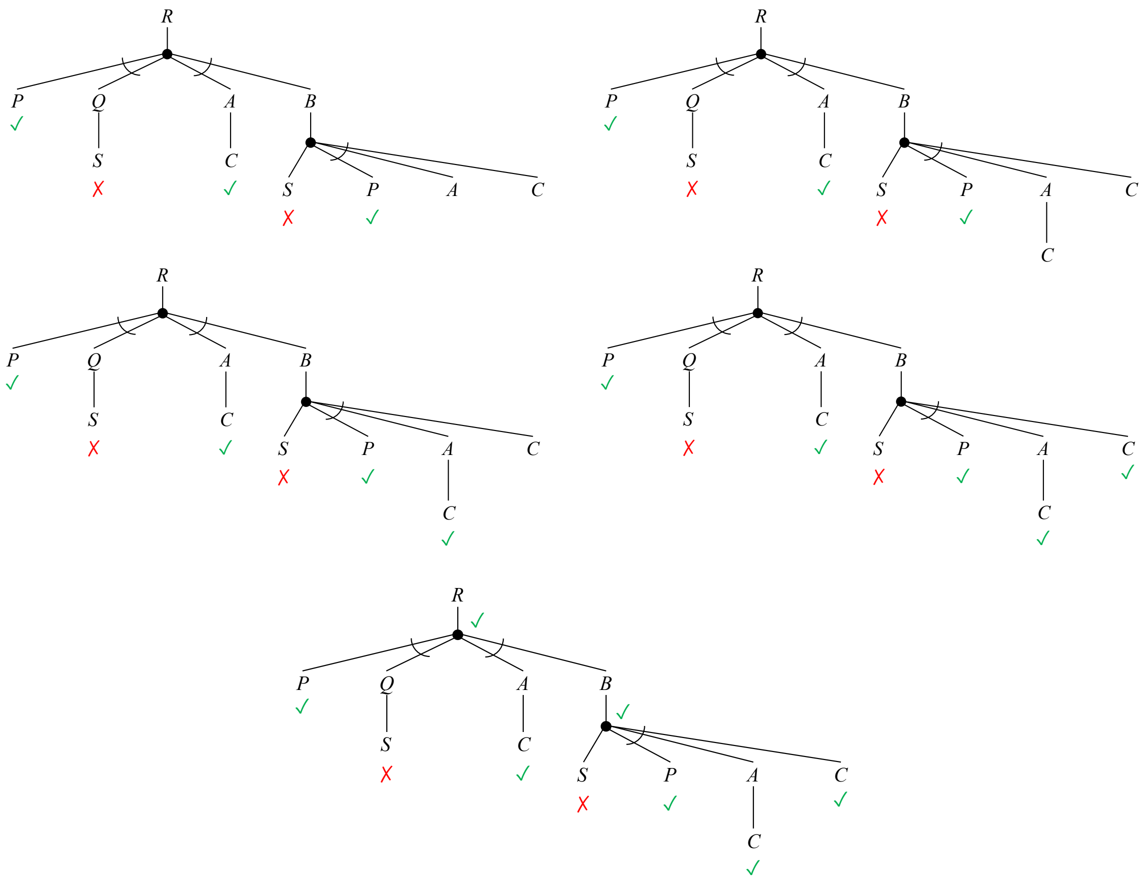
1. The sentence to prove, R, is not part of KB. It is nevertheless the consequent of two distinct implications in KB: $P \land Q \Rightarrow R$, and $A \land B \Rightarrow R$.

   The step-by-step proof by Backward chaining is shown below, in the form of an AND-OR graph, assuming it starts by attempting to prove the implication $P \land Q \Rightarrow R$. Since this first branch of the proof fails, a second one is attempted, this time successfully, using the implication $A \land B \Rightarrow R$.

   Remember that edges joined by an arc correspond to the antecedent of a *same* implication (i.e., to a conjunction of propositional symbols), and that *all* the corresponding proof branches need to succeed for the propositional symbol in their parent node to be proven; distinct groups of edges (including individual edges) not joined by an arc denote *alternative* proofs of the propositional symbol in their parent node, instead.

   Leaf nodes with a green check mark denote successful proof branches, corresponding to propositional symbols which are part of KB (and therefore are true). Leaf nodes with a red cross denote failed proof attempts, instead, i.e., propositional symbols that are not part of KB, nor consequent of implications present in KB (and therefore their truth cannot be proven by Backward chaining).





2. (a) The state of the robot can be represented by three propositional symbols: $BATTERY$, denoting whether its battery is charged or not, according to the corresponding sensor; $LIFTABLE$, denoting whether the object the robot is handling is liftable or not; and $MOVES$, denoting whether the other sensor indicates that the object the robot is handling is moving or not.

   The robot's knowledge base can therefore be represented by the following sentences:

   1. $BATTERY \wedge LIFTABLE \Rightarrow MOVES$
   2. $BATTERY$
   3. $\neg MOVES$

   (b) To apply the resolution algorithm all the sentences must be in conjunctive normal form (CNF), whereas the first one above is not. It can be converted into CNF in two steps:

   - eliminating the implication: $\neg(BATTERY \wedge LIFTABLE) \vee MOVES$
   - moving negation inwards: $\neg BATTERY \vee \neg LIFTABLE \vee MOVES$

   The proposition to prove is that the object is not liftable, i.e., $\neg LIFTABLE$. Its negation is $LIFTABLE$, which is already in CNF.

   Adding to the robot's knowledge base the negation of the sentence to be proven one gets:

   1. $\neg BATTERY \vee \neg LIFTABLE \vee MOVES$
   2. $BATTERY$
   3. $\neg MOVES$
   4. $LIFTABLE$

A possible proof by resolution is the following (the two numbers next to each newly derived clause denote the two previous clauses that have been resolved):

5. $\neg LIFTABLE \vee \neg MOVES$ (1, 2)
6. $\neg LIFTABLE$ (5, 3)
7. [ ] (6, 4)

An empty clause has been derived at step 7, by resolving the two contradictory literals 6 and 4 (respectively, $\neg LIFTABLE$ and $LIFTABLE$): this means that the sentence to be proven, $\neg LIFTABLE$, logically follows from the original knowledge base.

Predicate logic

The following ones are *possible* solutions to each exercise. Different solutions can be devised by making different choices on the domain, constants, predicates and functions.

1. Domain: a set of nodes and of natural numbers (to represent the depth of nodes).

   Predicate symbols: $Node(\cdot)$ (being a node), $Root(\cdot)$ (being a root node), $Expanded(\cdot)$ (being a node which has already been expanded), $Successor(\langle\text{child node}\rangle, \langle\text{parent node}\rangle)$ (being the successor of a node), $Equal(\cdot, \cdot)$ (being the same entity). Note that a predicate representing the relationship "being the parent node of" is not necessary, since the predicate *Successor* can be used to this aim.

   Function symbols: $Depth(\cdot)$ (the depth of a node).

   Constant symbols: Z (the number zero).

   (a) $\forall x\ Root(x) \Rightarrow Equal(Depth(x), Z)$
   (b) $\forall x\ Node(x) \wedge \neg Expanded(x) \Rightarrow \neg\exists y\ Node(y) \wedge Successor(y, x)$
   (c) $\forall x\ Node(x) \wedge Expanded(x) \Rightarrow \exists y\ Node(x) \wedge Successor(y, x)$
   (d) $\forall x\ Node(x) \wedge \neg Root(x) \Rightarrow (\exists y\ Node(y) \wedge Successor(x, y) \wedge (\forall z\ Node(z) \wedge Successor(x, z) \Rightarrow Equal(z, y)))$

2. A distinct definition of domain, constants, predicates and functions is given for each group of propositions.

   (a) Domain: a set including students, courses and scores.
   Constant symbols: *Biology* and *History* (denoting two courses).
   Predicate symbols: $Student(\cdot)$, $Takes(\langle\text{student}\rangle, \langle\text{course}\rangle)$, $Fail(\langle\text{student}\rangle, \langle\text{course}\rangle)$, $Equal(\cdot, \cdot)$, $Greater(\langle\text{score}_1\rangle, \langle\text{score}_2\rangle)$ (which means: $\text{score}_1 > \text{score}_2$).
   Function symbols: $Score(\langle\text{student}\rangle, \langle\text{course}\rangle)$ (denotes the score got by a student in the exam of a particular course).

   "Not all students take both the History and the Biology courses:"
   $\neg(\forall x\ Student(x) \Rightarrow Takes(x, History) \wedge Takes(x, Biology))$
   Equivalently, "some student did not take either History or Biology:"
   $\exists x\ Student(x) \wedge (\neg Takes(x, History) \vee \neg Takes(x, Biology))$

   "The best score in History is higher than the best score in Biology" can be restated as: "some student got a score in History which is higher than the scores got in Biology by all students:"
   $\exists y\ Student(y) \wedge\ (\forall x\ Student(x) \Rightarrow Greater(Score(y, History), Score(x, Biology)))$

   "One only student failed in History." One may think that the sentence:
   $\exists x\ Student(x) \wedge Fail(x, History)$
   is the correct representation of the above proposition. However it is not, since the existential quantifier $\exists$ means "there is *at least* one domain element that...," not "there is *exactly* one domain element that..." To correctly translate the above proposition, it can be restated as: "some student failed in History, and every student who failed in History is equal to the former:"

   $\exists x\ Student(x) \wedge Fail(x, History) \wedge (\forall y\ Student(y) \wedge Fail(y, History) \Rightarrow Equal(x, y))$

   "One only student failed both in History and in Biology:"
   $\exists x\ Student(x) \wedge Fail(x, History) \wedge Fail(x, Biology) \wedge$
   $(\forall y\ (Student(y) \wedge Fail(y, History) \wedge Fail(y, Biology) \Rightarrow Equal(x, y))$

(b) Domain: a set including people, some of which are politicians, and *discrete* time instants (the meaning of "all of the time" and "some of the time" will be expressed with reference to discrete time instants).

Predicate symbols: $Person(\cdot)$, $Politician(\cdot)$, $Instant(\cdot)$, $Fools(\langle\text{who}\rangle, \langle\text{whom}\rangle, \langle\text{when (time instant)}\rangle)$.

$$\begin{array}{ll}
\forall x\ Politician(x) \Rightarrow & \text{(Politicians...)} \\
\quad (\exists y\ Person(y) \wedge (\forall t\ Instant(t) \Rightarrow Fools(x,y,t))) \wedge & \text{(can fool some of the people all of the time)} \\
\quad (\exists t\ Instant(t) \wedge (\forall y\ Person(y) \Rightarrow Fools(x,y,t))) \wedge & \text{(can fool all the people some of the time)} \\
\quad \neg(\forall y,t\ Person(y) \wedge Instant(t) \Rightarrow Fools(x,y,t)) & \text{(can't fool all of the people all of the time)}
\end{array}$$

(c) Domain: a set of people, some of which are vegetarians.

Predicate symbols: $Likes(\langle\text{who}\rangle, \langle\text{whom}\rangle)$, $Man(\cdot)$, $Woman(\cdot)$, $Vegetarian(\cdot)$, $Smart(\cdot)$.

"Every person who dislikes all vegetarians is smart:"
$\forall x\ (\forall y\ Vegetarian(y) \Rightarrow \neg Likes(x,y)) \Rightarrow Smart(x)$

"No person likes a smart vegetarian:"
$\forall x,y\ (Smart(y) \wedge Vegetarian(y)) \Rightarrow \neg Likes(x,y)$

"There is a woman who likes all men who are not vegetarians:"
$\exists x\ Woman(x) \wedge (\forall y\ Man(y) \wedge \neg Vegetarian(y) \Rightarrow Likes(x,y))$

(d) Domain: a set of people, some of which are barbers.

Predicate symbols: $Man(\cdot)$, $Barber(\cdot)$, $Shaves(\langle\text{who}\rangle, \langle\text{whom}\rangle)$.

$\exists x\ Barber(x) \wedge (\forall y\ Man(y) \wedge \neg Shaves(y,y) \Rightarrow Shaves(x,y))$

(e) Domain: a set of people, some of which are professors.

Predicate symbols: $Likes(\langle\text{who}\rangle, \langle\text{whom}\rangle)$, $Smart(\cdot)$, $Professor(\cdot)$.

This proposition can be restated as: "no person likes professors who are not smart:"
$\forall x,y\ Professor(y) \wedge \neg Smart(y) \Rightarrow \neg Likes(x,y)$

(f) Domain: a set of people, some of which are Germans, and of languages.

Predicate symbols: $Language(\cdot)$, $German(\cdot)$, $Speaks(\langle\text{who}\rangle, \langle\text{language }\rangle)$.

$\forall x,y,z\ German(x) \wedge German(y) \wedge Language(z) \wedge Speaks(x,z) \Rightarrow Speaks(y,z)$

3. Domain: a set of animals including cows, horses and pigs.

Predicate symbols: $Mammal(\cdot)$, $Horse(\cdot)$, $Cow(\cdot)$, $Pig(\cdot)$, $Child(\langle\text{who}\rangle, \langle\text{of whom}\rangle)$, $Father(\langle\text{who}\rangle, \langle\text{of whom}\rangle)$.

Constant symbols: Bluebeard, Charlie.

(a) $\forall x\ Cow(x) \vee Pig(x) \vee Horse(x) \Rightarrow Mammal(x)$
This proposition can also be represented by three distinct sententes:
$\forall x\ Cow(x) \Rightarrow Mammal(x)$
$\forall x\ Pig(x) \Rightarrow Mammal(x)$
$\forall x\ Horse(x) \Rightarrow Mammal(x)$

(b) $\forall x,y\ Horse(x) \wedge Child(y,x) \Rightarrow Horse(y)$

(c) $Horse(Bluebeard)$

(d) $Father(Bluebeard, Charlie)$

(e) $\forall x,y\ Child(x,y) \Leftrightarrow Father(y,x)$

(f) $\forall x\ Mammal(x) \Rightarrow \exists y\ Father(y,x)$

4. Domain: a set made up of people, clubs, and entities including snow and rain.

   Constant symbols: $Tony$, $John$, $Mike$, $AlpineClub$, $Rain$, $Snow$.

   Predicate symbols: $Member(\langle \text{who} \rangle, \langle \text{club} \rangle)$, $Skier(\cdot)$, $Climber(\cdot)$, $Likes(\langle \text{who} \rangle, \langle \text{what} \rangle)$.

   (a) $Member(Tony, AlpineClub) \wedge Member(John, , AlpineClub) \wedge Member(Mike, AlpineClub)$
   This proposition can also be represented by three distinct sententes:
   $Member(Tony, AlpineClub)$
   $Member(John, AlpineClub)$
   $Member(Mike, AlpineClub)$

   (b) $\forall x \ Member(x, AlpineClub) \Rightarrow Skier(x) \vee Climber(x)$

   (c) $\forall x \ Climber(x) \Rightarrow \neg Likes(x, Rain)$

   (d) $\forall x \ Skier(x) \Rightarrow Likes(x, Snow)$

   (e) $\forall x \ Likes(Tony, x) \Rightarrow \neg Likes(Mike, x)$

   (f) $\forall x \ \neg Likes(Tony, x) \Rightarrow Likes(Mike, x)$

   (g) $Likes(Tony, Rain) \wedge Likes(Tony, Snow)$
   (this proposition can be represented by two distinct sentences as well)

5. Domain: a set of objects. Predicate symbols: $Heavy(\cdot)$, $Blue(\cdot)$, $Green(\cdot)$.

   (a) $(\forall x \ Heavy(x) \Rightarrow Blue(x)) \Rightarrow (\forall y \ \neg Heavy(y) \Rightarrow Green(y))$

   (b) $\forall x \ \ (Blue(x) \vee Green(x)) \ \wedge \ \neg (Blue(x) \wedge Green(x))$

   (c) $(\exists x \ \neg Heavy(x)) \Rightarrow (\forall y \ Heavy(y) \Rightarrow Blue(y))$