( )F520 ( )MS550 - Primeira Prova - 25/04/2011

RA: ________________ **Nome:** ____

(1) (i) Sejam as coordenadas esféricas (r, θ, ϕ) dadas por

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

onde $0 \le r < \infty$, $0 \le \theta \le \phi$ e $0 \le \phi < 2\pi$. Mostre que os vetores tangentes unitários $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ são dados por

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k},\\
\mathbf{e}_\theta &= \cos\theta\cos\phi\boldsymbol{i} + \cos\theta\sin\phi\boldsymbol{j} - \sin\theta\boldsymbol{k},\\
\mathbf{e}_\phi &= -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j},
\end{aligned}$$

onde $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ são os vetores unitários cartesianos tais que $\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

(ii) Sejam ∇ e $\mathbf{V}$ dados por

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \qquad \mathbf{V} = V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\phi\mathbf{e}_\phi,$$

e $\mathbf{V}\cdot\nabla$ o operador dado por

$$\mathbf{V}\cdot\nabla = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

Mostre que

$$(\mathbf{V}\cdot\nabla)\mathbf{r} = \mathbf{V}.$$

(2) Seja a equação diferencial

$$x(1+x)y'' + y' = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que a equação indicial correspondente apresenta raízes iguais $r_1 = r_2 = 0$. Podemos também notar que uma solução dessa equação diferencial é $y_1(x) = 1$. Utilize o método de Frobenius para encontrar uma segunda solução $y_2(x)$ linearmente independente.

(3) Seja a equação diferencial

$$xy'' - y' + 4x^3y = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que as raízes da equação indicial são $r_1 = 2$ e $r_2 = 0$. Mostre que nesse caso a menor das raízes não representa um problema para determinar a relação de recorrência das séries, e utilize essa menor raíz para encontrar as duas soluções linearmente independentes dessa equação.

(4) Mostre que a única equação diferencial linear de segunda ordem que possui apenas dois pontos singulares regulares e localizados em $z = 0$ e $z = \infty$ no plano complexo estendido é a equação diferencial de Euler

$$z^2y'' + \alpha zy' + \beta y = 0.$$

ℹ Valor das questões: (1) i - 1,0; ii - 2,0 (2) 3,0 (3) 3,0 (4) 2,0.

① $\vec{r} = r\sin\theta\cos\phi\,\vec{i} + r\sin\theta\sin\phi\,\vec{j} + r\cos\theta\,\vec{k}$

(i) $\dfrac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k}$

$$\left|\frac{\partial\vec{r}}{\partial r}\right|^2 = \underbrace{\sin^2\theta\cos^2\phi + \sin^2\phi\sin^2\phi}_{\sin^2\theta} + \cos^2\theta = 1$$

$\therefore\ \vec{e}_r = \sin\theta\cos\theta\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k}$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\vec{i} + r\cos\theta\sin\phi\,\vec{j} - r\sin\theta\,\vec{k}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta}\right|^2 = \underbrace{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi}_{r^2\cos^2\theta} + r^2\sin^2\theta = r^2$$

$\therefore\ \vec{e}_\theta = \cos\theta\cos\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j} - \sin\theta\,\vec{k}$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{i} + r\sin\theta\cos\phi\,\vec{j}$$

$$\left|\frac{\partial\vec{r}}{\partial\phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\phi$$

$\therefore\ \vec{e}_\phi = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}$

+1,0

(ii) $\vec{r} = r\vec{e}_r$

$$(\vec{V}\cdot\nabla)\vec{r} = V_r \frac{\partial}{\partial r}(r\vec{e}_r) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}(r\vec{e}_r) + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}(r\vec{e}_r)$$

mas:

$$\frac{\partial}{\partial r}(r\vec{e}_r) = \underbrace{\left(\frac{\partial r}{\partial r}\right)}_{1}\vec{e}_r + r\underbrace{\left(\frac{\partial \vec{e}_r}{\partial r}\right)}_{0} = \vec{e}_r$$

$$\frac{\partial}{\partial\theta}(r\vec{e}_r) = \underbrace{\left(\frac{\partial r}{\partial\theta}\right)}_{=0}\vec{e}_r + r\frac{\partial\vec{e}_r}{\partial\theta}$$

$$= r(\cos\theta\cos\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j} - \sin\theta\,\vec{k}) = r\vec{e}_\theta$$

$$\frac{\partial}{\partial\phi}(r\vec{e}_r) = \underbrace{\left(\frac{\partial r}{\partial\phi}\right)}_{=0}\vec{e}_r + r\frac{\partial\vec{e}_r}{\partial\phi}$$

$$= r(-\sin\theta\sin\phi\,\vec{i} + \sin\theta\cos\phi\,\vec{j}) = r\sin\theta\,\vec{e}_\phi$$

$$\therefore (\vec{V}\cdot\nabla)\vec{r} = V_r\vec{e}_r + \frac{V_\theta}{\cancel{r}}\cancel{r}\vec{e}_\theta + \frac{V_\phi}{\cancel{r\sin\theta}}\cancel{r\sin\theta}\,\vec{e}_\phi$$

$$= V_r\vec{e}_r + V_\theta\vec{e}_\theta + V_\phi\vec{e}_\phi = \vec{V}$$

+2,0

② $x(1+x)y'' + y' = xy'' + x^2y'' + y' = 0$ (*)

$y_1 = 1$, $r_1 = r_2 = 0$

$\therefore y_2(x) = y_1(x)\ln x + x^{r_2}\sum_{n=1}^{\infty} a_n x^n$

$\therefore y_2(x) = \ln x + \sum_{n=1}^{\infty} a_n x^n$

+1,0

$y_2' = \frac{1}{x} + \sum_{n=1}^{\infty} a_n n x^{n-1}$

$y_2'' = -\frac{1}{x^2} + \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

$(*) \Rightarrow -\cancel{\frac{1}{x}} + \sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} + (-1) + \sum_{n=2}^{\infty} a_n n(n-1) x^n$

$+ \cancel{\frac{1}{x}} + \sum_{n=1}^{\infty} a_n n x^{n-1} = 0$

$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} + (-1) + \sum_{n=3}^{\infty} a_{n-1}(n-1)(n-2) x^{n-1}$

$+ a_1 \cdot 1 + \sum_{n=2}^{\infty} a_n n x^{n-1} = 0$

$(a_1 - 1) + a_2 \cdot 2^2 x + \sum_{n=3}^{\infty} [a_n n^2 + a_{n-1}(n-1)(n-2)] x^{n-1} = 0$

$$\therefore \begin{cases} a_1 - 1 = 0 \Rightarrow \boxed{a_1 = 1} \\ 4a_2 = 0 \Rightarrow \boxed{a_2 = 0} \\ a_n n^2 + a_{n-1}(n-1)(n-2) = 0 \ , \ n = 3, 4, 5, \dots \quad (RR) \end{cases}$$

+1,0

$$a_2 = 0 + (RR) \Rightarrow a_3 = a_4 = \dots = 0$$

$$y_2 = \ln x + \underbrace{a_1}_{1} x + \sum_{n=2}^{\infty} \underbrace{a_n}_{=0} x^n$$

$$\therefore y_2(x) = \ln x + x$$

+1,0

(3) $xy'' - y' + 4x^3 y = 0 \quad ; \quad y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} 4a_n x^{n+r+3} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-2)x^{n+r-1} + \sum_{n=0}^{\infty} 4a_n x^{n+r+3} = 0$$

$$a_0 r(r-2)x^{r-1} + a_1(r+1)(r-1)x^r + a_2(r+2)r x^{r+1}$$
$$+ a_3(r+3)(r-1)x^{r+2} + \sum_{n=0}^{\infty}\left[a_{n+4}(n+r+4)(n+r+2) + 4a_n\right]x^{n+r+3} = 0$$

$$\therefore \begin{cases} a_0 r(r-2) = 0 & (1) \\ a_1(r+1)(r-1) = 0 & (2) \\ a_2(r+2)r = 0 & (3) \\ a_3(r+3)(r-1) = 0 & (4) \\ a_{n+4}(n+r+4)(n+r+2) + 4a_n = 0, \quad n = 0,1,2\ldots & (5) \end{cases}$$

$$(1) \xRightarrow{a_0 \neq 0} \begin{cases} r_1 = 2 \\ r_2 = 0 \end{cases}$$

Tomando $\boxed{r = r_2 = 0}$

$(2) \Rightarrow a_1 = 0$

$(3) \Rightarrow a_2$ indeterminado!

$(4) \Rightarrow a_3 = 0$

+1,0

$\therefore$ 2 ctes. arbitrárias (a_0 e a_2) $\Rightarrow$ 2 soluções!

$$RR\ (5) \Longrightarrow a_{n+4}(n+4)(n+2) = -4a_n \Longrightarrow a_{n+4} = \frac{-4a_n}{(n+4)(n+2)} \quad RR$$

$$a_1 = 0 \xRightarrow{RR} a_5 = a_9 = \cdots = a_{1+4K} = \cdots = 0$$

$$\therefore\ a_{1+4K} = 0\ ,\ K = 0,1,2,\ldots$$

$$a_3 = 0 \xRightarrow{RR} a_7 = a_{11} = \cdots = a_{3+4K} = \cdots = 0$$

$$\therefore\ a_{3+4K} = 0,\ K = 0,1,2\ldots$$

+0,5

$\underline{n=0}$ $\quad a_4 = -\dfrac{4a_0}{4\cdot 2} = -\dfrac{a_0}{2!}$

$\underline{n=4}$ $\quad a_8 = -\dfrac{4a_4}{8\cdot 6} = -\dfrac{a_4}{4\cdot 3} = \dfrac{a_0}{4!}$

$\underline{n=8}$ $\quad a_{12} = \dfrac{-4a_8}{12\cdot 10} = \dfrac{-a_8}{6\cdot 5} = \dfrac{-a_0}{6!}$

$$\therefore\ a_{4K} = \frac{(-1)^K a_0}{(2K)!} \qquad K = 0,1,2\ldots$$

$\underline{n=2}$ $\quad a_6 = -\dfrac{4a_2}{6\cdot 4} = -\dfrac{a_2}{3\cdot 2} = -\dfrac{a_2}{3!}$

$\underline{n=6}$ $\quad a_{10} = -\dfrac{4a_6}{10\cdot 8} = \dfrac{-a_6}{5\cdot 4} = \dfrac{a_2}{5!}$

$$\therefore\ a_{2+4K} = \frac{(-1)^K a_2}{(2K+1)!} \qquad K = 0,1,2,\ldots$$

+1,0

$$\therefore y = \sum_{K=0}^{\infty} a_{4K}\, x^{0+4K} + \sum_{K=0}^{\infty} \underbrace{a_{1+4K}}_{=0} x^{0+1+4K} + \sum_{K=0}^{\infty} a_{2+2K}\, x^{0+2+2K} + \sum_{K=0}^{\infty} \underbrace{a_{3+4K}}_{=0} x^{0+3+4K}$$

$$= a_0 \sum_{K=0}^{\infty} \frac{(-1)^K x^{4K}}{(2K)!} + a_2\, x^2 \sum_{K=0}^{\infty} \frac{(-1)^K x^{4K}}{(2K+1)!}$$

+0,5

$$y_1(x) = \sum_{K=0}^{\infty} \frac{(-1)^K x^{4K}}{(2K)!} \quad ;\ y_2(x) = x^2 \sum_{K=0}^{\infty} \frac{(-1)^K x^{4K}}{(2K+1)!}$$

(4) $y'' + p(z)y' + q(z)y = 0 \quad (*)$

ponto singular regular em $z=0$ $\iff \begin{cases} p(z) = \dfrac{\alpha}{z} + \displaystyle\sum_{n=0}^{\infty} a_n z^n \\ q(x) = \dfrac{\beta}{z^2} + \dfrac{\beta'}{z} + \displaystyle\sum_{n=0}^{\infty} b_n z^n \end{cases}$

+0,5

$z \to \infty \iff t = \frac{1}{z} \to 0$

$$\frac{dy}{dz} = -t^2\frac{dy}{dt} \;;\; \frac{d^2y}{dz^2} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

$$(*) \Rightarrow t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} + p(1/t)(-t^2)\frac{dy}{dt} + q(1/t)y = 0$$

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = 0$$

onde: $P(t) = \frac{2}{t} - \frac{1}{t^2}p(1/t) = \frac{2}{t^2} - \frac{1}{t^2}\left[\alpha t + \sum_{n=0}^{\infty} a_n \frac{1}{t^n}\right]$

$Q(t) = \frac{1}{t^4}q(1/t) = \frac{\beta}{t^2} + \frac{\beta'}{t^3} + \sum_{n=0}^{\infty} b_n \frac{1}{t^{n+4}}$

+0,5

$z=\infty$ é ponto singular regular $\Rightarrow$ $t=0$ é ponto singular regular

$\therefore$ $P(t)$ tem no máximo um pólo simples em $t=0$ $\iff a_n = 0$

$Q(t)$ tem no máximo um pólo de ordem 2 em $t=0$ $\iff \beta' = 0,\; b_n = 0$

$\therefore \; p(z) = \frac{\alpha}{z}, \quad q(z) = \frac{\beta}{z^2}$

+1,0

$\therefore (*) \Rightarrow y'' + \frac{\alpha}{z}y' + \frac{\beta}{z^2}y = 0$, que é uma eq. de Euler.