$$f_1'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = 1$$

Next we suppose that for fixed value of N, we know that for $f_N(x) = x^N$, $f_N'(x) = Nx^{N-1}$. Consider the derivative of $f_{N+1}(x) = x^{N+1}$,

$$f_{N+1}'(x) = (x \cdot x^N)' = (x)'x^N + x \cdot (x^N)' = x^N + x \cdot N \cdot x^{N-1} = (N+1)x^N.$$

We have shown that the statement $f_n'(x) = n \cdot x^{n-1}$ is true for $n = 1$ and that if this statement holds for $n = N$, then it also holds for $n = N+1$. Thus by the principle of mathematical induction, the statement must hold for $n = 1, 2, \ldots$.

13.3 Quotient rule

There is a similar rule for quotients. To prove it, we go to the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x)g(x+h)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

This leads us to the so-called "quotient rule":

Derivatives of quotients (Quotient Rule)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Which some people remember with the mnemonic "low D-high minus high D-low (over) square the low and away we go!"

13.4 Examples

The derivative of $(4x-2)/(x^2+1)$ is: