Task 4: Vince Velocci

For any 4×4 matrix, we can write

$$H = \sum_{i,j} a_{ij}\, \sigma_i \otimes \sigma_j \qquad \text{where} \quad i,j = \{I, x, y, z\}$$

$$\sigma_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $$a_{ij} = \frac{1}{4} Tr\left[(\sigma_i \otimes \sigma_j) H\right]$$

(See the code...)

Using this, we find that with $H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$H = -\frac{1}{2}\sigma_I \otimes \sigma_I + \frac{1}{2}\left[\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right]$$

<u>Goal</u>: Find lowest eigenvalue of H

$\Rightarrow$ Find $|\psi\rangle$ that minimizes the expectation value

$\langle\psi|H|\psi\rangle$. Min value will be lowest eigenvalue

Ansatz for $|\psi\rangle$:

$$|\psi\rangle = (I \otimes \sigma_X)(CX)(R_z(\theta) \otimes I)(H \otimes I)|00\rangle$$

$$= (\sigma_I \times \sigma_x)(CX)(R_z \otimes \sigma_I)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$$

$$= (\sigma_I \otimes \sigma_x)(CX)\frac{1}{\sqrt{2}}\begin{bmatrix}e^{-i\theta/2} & 0\\0 & e^{i\theta/2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$$

$$= (\sigma_I \otimes \sigma_x)(CX)\frac{1}{\sqrt{2}}\begin{bmatrix}e^{-i\theta/2}\\e^{i\theta/2}\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$$

$$= (\sigma_I \otimes \sigma_x)\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}(e^{-i\theta/2}|0\rangle + e^{i\theta/2}|1\rangle)\otimes|0\rangle$$

$$= (\sigma_I \otimes \sigma_x)\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}(e^{-i\theta/2}|00\rangle + e^{i\theta/2}|10\rangle)$$

$$= (\sigma_I \otimes \sigma_x)\frac{1}{\sqrt{2}}(e^{-i\theta/2}|00\rangle + e^{i\theta/2}|11\rangle)$$

$$= \frac{e^{-i\theta/2}}{\sqrt{2}}|0\rangle \otimes \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \frac{e^{i\theta/2}}{\sqrt{2}}|1\rangle \otimes \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \frac{e^{-i\theta/2}}{\sqrt{2}} |01\rangle + \frac{e^{i\theta/2}}{\sqrt{2}} |10\rangle$$

$$= e^{-i\theta/2} \left[\frac{1}{\sqrt{2}} |01\rangle + \frac{e^{i\theta}}{\sqrt{2}} |10\rangle \right]$$

Global/overall phase doesn't matter when calculating $\langle\psi|H|\psi\rangle$.

So we can choose $|\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle + \frac{e^{i\theta}}{\sqrt{2}} |10\rangle$

Makes sense because $0 = H|00\rangle = H|11\rangle$

so no contributions to $\langle H\rangle_{|\psi\rangle}$ from those states anyway.

$$H|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(|01\rangle + e^{i\theta} |10\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(-|01\rangle + |10\rangle + e^{i\theta} |01\rangle - e^{i\theta} |10\rangle \right)$$

$$\Rightarrow \langle\psi|H|\psi\rangle = \frac{1}{2} \left(\langle 01| + e^{-i\theta} \langle 10| \right) \left(-|01\rangle + |10\rangle + e^{i\theta} |01\rangle - e^{i\theta} |10\rangle \right)$$

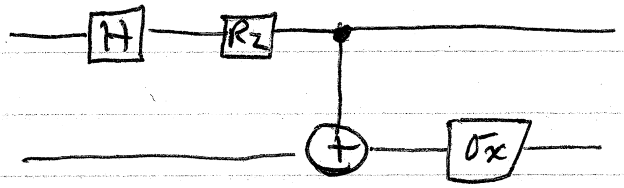
$$= \frac{1}{2}\left(-1 + e^{i\theta} + e^{-i\theta} - 1\right) = \cos\theta - 1$$

which is a minimum when $\theta = \pi$

$\therefore$ min eigenvalue should be -2

$$\Rightarrow |\Psi\rangle_{\theta=\pi} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

2-Qubit Ansatz preparation:



$$H = -\frac{1}{2}\sigma_I \otimes \sigma_I + \frac{1}{2}\left[\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right]$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{eigenvectors are } \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \equiv |+\rangle \quad (\lambda = 1)$$

$$\text{\& } \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \equiv |-\rangle \quad (\lambda = -1)$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow \text{eigenvectors are } \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \equiv |i\rangle \quad (\lambda = 1)$$

$$\text{\& } \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) \equiv |-i\rangle \quad (\lambda = -1)$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{ eigenvectors are } |0\rangle \ (\lambda = 1)$$
$$\text{\& } |1\rangle \ (\lambda = -1)$$

On the Bloch sphere:

for any $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = |\psi(\theta,\phi)\rangle$

$\therefore \ |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |\psi(\frac{\pi}{2}, 0)\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |\psi(\frac{\pi}{2}, \pi)\rangle$

$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |\psi(\frac{\pi}{2}, \frac{\pi}{2})\rangle$

$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |\psi(\frac{\pi}{2}, \frac{3\pi}{2})\rangle$

$|0\rangle = |\psi(0,0)\rangle \qquad |1\rangle = |\psi(\pi, 0)\rangle$

$\therefore$ σ_x has eigenvectors $|\psi(\frac{\pi}{2}, 0)\rangle$ \& $|\psi(\frac{\pi}{2}, \pi)\rangle$

σ_y has eigenvectors $|\psi(\frac{\pi}{2}, \frac{\pi}{2})\rangle$ \& $|\psi(\frac{\pi}{2}, \frac{3\pi}{2})\rangle$

σ_z has eigenvectors $|\psi(0, 0)\rangle$ \& $|\psi(\pi, 0)\rangle$

$\therefore \ |0\rangle \xrightarrow{R_Y(\frac{\pi}{2})} |+\rangle \ , \ |1\rangle \xrightarrow{R_Y(\frac{\pi}{2})} |-\rangle$

$|0\rangle \xrightarrow{R_x(-\frac{\pi}{2})} |i\rangle \ , \ |1\rangle \xrightarrow{R_x(-\frac{\pi}{2})} |-i\rangle$

$\langle\psi|\sigma_I|\psi\rangle = \langle\psi|\psi\rangle = 1$

$|\psi\rangle$ is a linear combination of $|01\rangle$ & $|10\rangle$

So we can transform $|01\rangle$ into $|+-\rangle$ & $|i\,{-i}\rangle$

and can transform $|10\rangle$ into $|-+\rangle$ & $|{-i}\,i\rangle$,

so that when we measure the $\sigma_x, \sigma_y, \sigma_z$

i.e. "measure $|\psi\rangle$"

observables, we will just get ± 1 and can get

the weighted average

Transforming $|0\rangle, |1\rangle$ with R_x & R_Y will put

$|\psi\rangle$ into a linear combination of eigenvectors of

σ_Y & σ_x. So measuring $|\psi\rangle$ in those bases

will return ± 1

Notice that $R_y(\theta) R_y^T(\theta) =$

$$\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R_y^T(\theta) R_y(\theta)$$

where T denotes conjugate transpose

$$R_x(\theta) R_x^T(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R_x^T(\theta) R_x(\theta)$$

$\therefore$ R_y & R_x are unitary

$\therefore$ inner products will be preserved

i.e. If $|\psi'\rangle = R_y|\psi\rangle$ and $|\rho'\rangle = R_y|\rho\rangle$

then $\langle \rho'|\psi'\rangle = \langle \rho | R_y^T R_y | \psi\rangle = \langle \rho | I | \psi \rangle$

$= \langle \rho | \psi \rangle$

Also we can write R_y & R_x as $e^{i\theta A}$ where

A is a matrix:

$R_x(\theta) = e^{-i\theta\sigma_x/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_x$.

and $R_y(\theta) = e^{-i\theta\sigma_y/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_y$

$\therefore$ we can regard $R_x|\psi\rangle$ and $R_y|\psi\rangle$ as just multiplying $|\psi\rangle$ by a phase factor. we're just changing bases, not changing the state or its probability predictions!

So we can transform $|\psi\rangle$ using R_x & R_y, then prepare and measure $|\psi\rangle$ many times to find expectation values. If $|\psi\rangle$ is in terms of $|\pm\rangle$ or $|{\pm i}\rangle$, then measuring $|\psi\rangle$ in those bases and averaging the results as a weighted average weighted by the eigenvalues will give us $\langle\psi|\sigma_x|\psi\rangle$ and $\langle\psi|\sigma_y|\psi\rangle$. No transformation needed to find $\langle\psi|\sigma_z|\psi\rangle$ since $|\psi\rangle$ is first in the basis of eigenvectors of σ_z

So consider $\langle \Psi | H | \Psi \rangle$

There 4 terms:

Term ①: $-\frac{1}{2} \langle \Psi | \sigma_I \otimes \sigma_I | \Psi \rangle = -\frac{1}{2}$

Term ②: $\frac{1}{2} \langle \Psi | \sigma_x \otimes \sigma_x | \Psi \rangle$

$\Rightarrow$ transform to basis of eigenvectors of σ_x by applying $R_Y(\frac{\pi}{2})$ $\Rightarrow$ measure $\Rightarrow$ average $\times \frac{1}{2}$

Term ③: $\frac{1}{2} \langle \Psi | \sigma_y \otimes \sigma_y | \Psi \rangle$

$\Rightarrow$ transform to basis of eigenvectors of σ_y by applying $R_X(-\frac{\pi}{2})$ $\Rightarrow$ measure $\Rightarrow$ average $\times \frac{1}{2}$

Term ④: $\frac{1}{2} \langle \Psi | \sigma_z \otimes \sigma_z | \Psi \rangle$

$\Rightarrow$ already in basis of eigenvectors of σ_z

$\Rightarrow$ measure $\Rightarrow$ average $\times \frac{1}{2}$

SEE CODE...