Matchmaking Strategies for Maximizing Player Engagement in Video Games

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Managing player engagement is an important problem in the video game industry, as many games generate revenue via subscription models and microtransactions. We consider a class of online video games whereby players are repeatedly matched by the game to compete against one another. Players have different skill levels, which affect the outcomes of matches, and the win-loss outcomes affect the probability of remaining engaged. The goal is to maximize overall player engagement over time by optimizing the dynamic matchmaking strategy. We propose a general but tractable framework to solve this problem, which can be formulated as an infinite linear program. We focus on a geometric losing streak model where players have a fixed probability to churn after each consecutive loss. Compared to the industry status quo that matches players within the same skill level, we show the benefit of the optimal policy can be exponential in the number of skill levels.

To provide more insights, we consider a special case where there are two skill levels and players churn only when they experience a losing streak. We explicitly solve for the optimal policy, which matches players myopically for short-term reward while balancing skill distribution for long-term reward. We then use our framework to analyze two common but controversial interventions to increase engagement: adding AI bots and a pay-to-win system. We show that optimal matchmaking may significantly reduce the number of bots needed without loss of engagement. The pay-to-win system can influence player engagement positively when the majority of players are low-skilled. Finally, we conduct a case study with real data from an online chess platform. We show that the optimal policy can improve engagement by 4-6% in the absence of new player arrival or reduce the percentage of bot players by 10% in comparison to skill-based matchmaking.

Key words: matchmaking; video games; engagement; infinite linear program

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1. Introduction

As of 2020, the global revenue of the gaming industry surged about 20% compared to 2019 and was estimated at $180 billion (Witkowski 2020). One large sector in the video game industry is competitive online games, where players play against one another in matches (one-on-one or in teams). In 2020, online competitive games *PUBG: Mobile* and *Honors of Kings* earned 2.6 and 2.5 billion dollars in revenue, respectively, which were the most among all mobile games (Verma 2020). Both games (and many others) adopt the idea of games-as-a-service (GaaS) and provide the game content on a "continuing revenue model". These games are usually free-to-play, and the revenue comes from in-game advertisements, microtransactions for virtual items such as loot boxes, and subscriptions for seasonal premium passes (sometimes referred to as "battle passes", which offer premium content for subscribers). Managing player engagement is crucial for such free-to-play games because the total revenue is directly tied to the number of active players. In the gaming industry, daily active users (DAU) or concurrent users (CCU) are used as key performance indicators, and a high DAU/CCU indicates good product performance in general.

Player engagement can be affected by many factors, such as game content, game mechanics, and user interfaces. Apart from these features, which are determined before the game's release, an important factor that influences engagement in competitive video games is how the players are matched together. The matchmaking system determines with whom a player plays in each match, which directly affects the outcome of the current match and indirectly influences the future engagement behavior of the players. Chen et al. (2017b) and Huang et al. (2019) both documented that the outcomes of the matches, either receiving wins/losses or scores, have a significant impact on engagement. Industry practitioners have also recognized the potential of matchmaking as a tool to improve engagement.[1] The status quo is called *skill-based matchmaking* (SBMM), which simply matches by trying to find an opponent with the closest skill level (Park 2021) and ignores the outcomes of previous matches. Since data is easily available, video game companies have begun exploration on how to leverage matchmaking systems to improve player engagement (thereby increasing revenue). For example, Electronics Arts have filed a patent for an engagement-oriented matchmaking system, which largely follows Chen et al. (2017b).[2] Although they recognize the dynamic nature of the problem, their patent is based on a static one-shot model. *In this paper, we study the fundamental problem of dynamic matchmaking in video games and explore the value of optimal matchmaking on total player engagement.* We also provide significant insights for industry practitioners with respect to the optimal policy and interventions such as AI bots and pay-to-win.

In addition to selecting matching partners, video game companies also influence the matchmaking system in other ways. One controversial way is to add AI-powered bots into the matchmaking

[1] Josh Menke, a former lead engagement designer for the popular game *Halo 5*, said that "Matchmaking guarantees gameplay experience as intended, and prevents disengagement when possible". (`https://www.gdcvault.com/play/1026588/Matchmaking-for-Engagement-Lessons-from`)

[2] `https://www.pcgamesn.com/ea-matchmaking-microtransactions-eomm-engagement-patent`

pool. Unlike traditional bots for tutorials or training, AI-powered bots imitate real human players in competitive games, and players are not notified when they are matched with a bot. Many competitive games are suspected of having a high bot ratio (Gelbart 2021), although only a few games, such as *Fortnite*, publicly confirmed the existence of bots in human matches.[3] Such practices are actually double-edged swords. On the one hand, the outcome of a match with bots can be easily controlled because bots can either make intentional mistakes at clutch times or outperform human players significantly as needed (especially in games that require quick reactions). Thus, AI-powered bots may influence player engagement positively by manipulating the match outcomes properly. On the other hand, bots can be clumsy or predictable at times and are often recognized by human players after repetitive encounters. Players are not happy when they intend to compete with other human players but find out that they are frequently in a match with bots. Players who have made in-game purchases and want to use (and parade) them on other players instead of bots can be particularly upset about this.

Another controversial practice is introducing a *pay-to-win system* (PTW). PTW allows players to improve their competence by paying real currency for competitive games. For example, the popular competitive game *Dota 2* offers a subscription service called “Dota Plus”. At the start of each match, ten players pick their characters one by one, and then paying players receive analytical suggestions on which character to pick based on the existing picks of others and winrate data from recent weeks. Moreover, during the match, paying players receive data-driven suggestions on items to buy and skills to learn, as well as information that is not available to regular players. Such information may not be necessary for professional players, but it provides substantial help to most amateur players. While PTW provides direct revenue, it is not hard to imagine that PTW may also influence the player engagement of payers and non-payers, influencing the overall revenue associated with engagement. The interplay between PTW and the matchmaking system also matters because the existence of PTW may change the optimal matchmaking policy. The benefits of PTW have yet to be previously studied when considering player engagement and matchmaking policies. In practice, the value of either AI-powered bots or PTW is hard to evaluate through field experiments since the impact is confounded with matchmaking policies. We provide a theoretical analysis to analyze the value of such strategies in the context of matchmaking.

In this work, we propose a novel infinite-horizon dynamic program that aims to maximize the cumulative active players, which is our metric for engagement. In each period, the game decides whom to match with each other. We assume that players can have heterogeneous skill levels, and their state depends on the win-loss outcomes of the most recent past matches. We focus on a

[3] `https://www.epicgames.com/fortnite/en-US/news/matchmaking-bots-controls-and-the-combine-update`

geometric losing streak model to derive sharp characterizations and managerial insights, where players churn (quit) with a fixed probability, starting from the second loss in a row. Below we provide a summary of our contributions and findings.

(1) We first discuss the value of optimal matchmaking over traditional skill-based matchmaking (SBMM). We find that under the geometric losing streak model, with a mild inflow of new players, the advantage of the optimal policy over SBMM can grow exponentially with the number of skill levels. We also show that the power of the optimal policy over SBMM is stronger when the players are more patient (less likely to churn after an additional loss), when more new players arrrive in each period, and when the advantage of high-skilled players over low-skilled players increases. Our proof relies on using strong duality arguments for infinite dimensional linear programming to show that an intuitive policy is optimal under an important set of initial states.

(2) To deliver more insights, we focus on a simplified model in Section 4 with only two skill levels and no new players. We fully characterize the optimal dynamic policy under the baseline model. The optimal policy always matches as many low-skilled players who are not at risk of churning to high-skilled players who are one loss away from churning in order to keep the latter in the system and increase the overall engagement. Such a matching minimizes the population loss in the current period. In some scenarios when there are too many low-skilled players, high-skilled players can also be matched to low-skilled players that are at risk of churning. This helps adjust the skill distribution to be more beneficial in the long run.

(3) Next, we investigate the value of optimizing the matchmaking system in the presence of AI-powered bots. We show that a platform using SBMM with a relatively small bot ratio can potentially not use bots at all if they transition to an optimal matchmaking policy. On the other hand, when bot usage is high (implying the bots have evolved to a very sophisticated state), the gap between the optimal and SBMM policies vanishes.

(4) We then consider the interplay between PTW strategies and the matchmaking system. We find that contrary to the conventional wisdom that PTW is simply to increase revenue, it can also be an effective lever to change the distribution of player states and increase engagement. Surprisingly, even the non-paying low-skilled players may enjoy longer lifespan in some scenarios. The potential positive externality of player engagement may transform the public perception of PTW strategies.

(5) Finally, we conduct a case study with data from Lichess (a large online chess platform) to validate our findings. After fitting a player behavior model within our framework, we show that the optimal policy may improve engagement by 4-6% over SBMM. Also, the optimal policy may reduce the bot ratio by 8-10% while maintaining the same level of engagement as SBMM when the bot ratio is less than 30%.

1.1. Literature Review

Our work contributes to the emerging literature on operations management in video games. Closest to our work, Chen et al. (2017b) and Huang et al. (2019) also investigate the problem of maximizing player engagement in video games through matchmaking. Chen et al. (2017b) proposes a model that estimates the churn risk of every pair of possible matches through logistic regression and myopically minimizes the churn risk in the next round. Their numerical study considers one-period problems and shows 0.7% improvement over SBMM in one period. On the other hand, Huang et al. (2019) estimates user engagement with a hidden Markov model and proposes a heuristic algorithm that assigns a selected player to one of the pre-specified candidate matches (assuming all the other players are fixed). In contrast, our paper investigates how to optimally solve the dynamic matchmaking problem through a fluid model, taking both the myopic reward as well as the long-term player engagement into account. We also explicitly solve for the optimal policy in special cases and provide insights on how the policy looks. Our new framework also enables us to analytically investigate the value of optimal dynamic matchmaking, AI-powered bots, as well as a pay-to-win system. Recent works also investigate player engagement empirically (Goetz and Lu (2022), and Jiao et al. (2022)), but they provide little guidance to the matchmaking policy.

Aside from matchmaking, several papers in this field consider the monetization of video games. For example, Sheng et al. (2020) considers the problem of incentivizing ad-clicking actions in freemium games. Chen et al. (2020) considers how to optimally price and design 'loot boxes', which is a popular randomized selling scheme for virtual items in video games. Mai and Hu (2022) considers the introduction and pricing of premium content in freemium games. Jiao et al. (2021) considers whether the seller should disclose an opponent's skill level when selling PTW items. Our paper also investigates PTW, but it is different in several ways: first, we assume that PTW is a subscription service that is not dependent on specific matches. Second, we focus on the player's lifetime engagement instead of the player's utility for a single match. Finally, our model enables us to check the joint value of PTW with the optimal matchmaking policy. Apart from monetization, there is increasing interest in the design of video games. Hanguir et al. (2021) recently considered an interesting problem on how to design the game loadouts to maximize the strategies' diversity. Li et al. (2022) considers how to maximize player utility by sequencing game elements in a level.

Our work also connects to the growing literature on dynamically managing user engagement and user lifetime value in a service system. In such systems, the revenue is usually proportional to the amount of cumulative active users, and users' subscription behavior is based on the service history up to date. Aflaki and Popescu (2014) considers how to maximize user engagement by dynamically adjusting the service quality. Kanoria et al. (2018) considers how a fund manager should switch between risk mode and safe mode to maximize customer lifetime value. Bernstein et al. (2022)

and Caro and Martínez-de Albéniz (2020) consider how a content provider (e.g., video streaming services) could maximize the subscription revenue by dynamically changing their content. Our work contributes to this field by considering how to manage user retention with dynamic matchmaking in video games. While the decision variable in most of the above papers is a single variable (service quality or service mode), the decision in our paper is multi-dimensional (matchmaking flows), which brings fundamentally new technical challenges and insights.

More broadly, our work relates to dynamic decision-making when players have a memory of the decision history. Research in this field has considered dynamic pricing with reference effects (Popescu and Wu 2007, Hu et al. 2016, Chen et al. 2017a), dynamic capacity allocation with customer memory effects (Adelman and Mersereau 2013), network revenue management with repeated customer interactions (Calmon et al. 2020), improving matching rates in dating markets (Rios et al. 2022), and dynamic personalized pricing with service quality variability (DeCroix et al. 2021). When customer preferences are not fully specified, Bastani et al. (2022), and Cao et al. (2019) consider how to learn customer preferences on the fly in the context of product recommendation or promotion, with the risk that customers may leave the system permanently upon consecutive bad decisions. In our work, user churn decision depends on the outcomes of the most recent previous matches.

Finally, our work contributes to the broad literature on dynamic matching. Here we only review papers where agents may only stay in the system for certain periods before leaving. Ashlagi et al. (2019) provides an approximation algorithm for a setting where customers will stay a fixed number of periods before leaving. Aouad and Saritaç (2020) propose approximate algorithms for dynamic matching over edge-weighted graphs, where the arrival and abandonment of agents are stochastic. Hu and Zhou (2022) considers dynamic matching over a bipartite graph with finite types of nodes on both sides. Unmatched supply and demand may incur waiting or holding costs and will be partially carried over to the next period. Our paper considers dynamic matching with finite types of players in a fluid model (see, e.g., Azevedo and Leshno 2016), and their churn risk evolves dynamically based on outcomes of past matches.

2. Model and Preliminaries

We now present our model of a matchmaking system for a 1-versus-1 competitive video game. We describe the player behavior in Section 2.1, then introduce the engagement maximization problem in Section 2.2.

2.1. Player Behavior

We assume that each player has a skill level that describes their relative competence in the game. There are K ordered skill levels, where level 1 is the lowest and level K is the highest. For each match, exactly one of the two players will be the winner, and the outcome of each game is either a win or a loss (no draw). The outcome of a match is a Bernoulli random variable depending on the skill levels of the two players. Let p_{kj} be the winrate of a level k player versus a level j player, implying that $p_{kj} = 1 - p_{jk}$. Players of the same skill level are equally likely to win, i.e., $p_{kk} = 0.5$. A player with a higher skill level than their opponent has a strictly larger than 0.5 probability of winning, i.e., $p_{kj} > 0.5$ if $k > j$. We assume that a player's skill level is fixed over their lifespan in the matching system. In practice, most players are casual players, and it is reasonable to assume that they cannot significantly improve their relative competence once they are familiar with the game. In Huang et al. (2019), players' skill level is an evolving metric that monotonically increases as the players play more. In our setting, skill levels reflect *relative* competence and are more stable because others are also getting more familiar with the game.

In practice, many factors may influence the players' engagement behavior. The outcome of past matches has been shown to have a significant impact on player engagement. For example, Chen et al. (2017b) documented that players' churn risk varies significantly with the outcomes of the last three matches. We assume that a player's *engagement* state is determined by the win-loss record of the last m matches and transitions according to a Markov chain when the player plays a new match. We use q to denote the 'churn state', i.e., a player quitting the game permanently. Let $\mathcal{G}$ be the set of all possible states of a player, which has cardinality at most $2^m + 1$ (history of wins/losses and the churn state).

A player is fully characterized by their skill level k and engagement state $g \in \mathcal{G}$; we shall refer to this pair as a *demographic* of players. Let $P^k_{win}, P^k_{lose} \in [0,1]^{|\mathcal{G}|\times|\mathcal{G}|}$ be the transition matrix of a level k player's engagement state, given that they win/lose the next match. Hence, if they are matched with a level j player, their aggregate transition matrix is given by $M_{kj} = p_{kj} P^k_{win} + (1 - p_{kj}) P^k_{lose}$. For ease of notation, we also define $\bar{\mathcal{G}}$ to be the set of all the active states except the churn state q and $\bar{M}_{kj}$ be the reduced aggregate transition matrix without the churn state. We define an active player as one who has not churned and is thus in one of the states in $\bar{\mathcal{G}}$.

Finally, we allow for new players to arrive. We assume that in each period, one unit of customer kg leads to $\mu_{kg,kg'}$ units of new player kg' at the start of the next period. We use $N_k = \{\mu_{kg,kg'}\} \in \mathbb{R}^{\bar{\mathcal{G}}\times\bar{\mathcal{G}}}_{\geq 0}$ to denote the new player arrival rate. Below we use a simple example to illustrate the notation.

Example 1. Suppose that players have two skill levels, either high or low (denoted by level 2 and 1, respectively). They quit with a probability 0.2 if they experience two consecutive losses and with a probability 0.5 if they experience three consecutive losses. This implies that $m = 2$, as only two previous matches plus the current match outcome affect the transition state. We further assume that a high-skilled player wins against a low-skill player with probability $p_{21} = 0.8$. Hence, for each skill level, there are 4 engagement states in $\mathcal{G}$: the player may experience 0, 1 or 2 consecutive losses or reach state q. We use 20, 21, 22, $2q$ and 10, 11, 12, $1q$ to denote the states of high- and low-skilled players, respectively. For $k=1,2$, the transition matrix P^k_{win} and P^k_{lose} is given by

$$P^k_{win} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \\ kq \end{array} \begin{array}{c} \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad P^k_{lose} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \\ kq \end{array} \begin{array}{c} \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The aggregate transition matrix is given by

$$M_{kk} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \\ kq \end{array} \begin{array}{c} \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.4 & 0.1 \\ 0.5 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad M_{21} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \\ kq \end{array} \begin{array}{c} \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.8 & 0 & 0.16 & 0.04 \\ 0.8 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad M_{12} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \\ kq \end{array} \begin{array}{c} \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.2 & 0 & 0.64 & 0.16 \\ 0.2 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

and the reduced transition matrix $\bar{M}_{kj}$ is defined as

$$\bar{M}_{kk} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \end{array} \begin{array}{c} \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.4 \\ 0.5 & 0 & 0.25 \end{pmatrix} \end{array}, \quad \bar{M}_{21} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \end{array} \begin{array}{c} \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.8 & 0 & 0.16 \\ 0.8 & 0 & 0.1 \end{pmatrix} \end{array}, \quad \bar{M}_{12} = \begin{array}{c} \\ k0 \\ k1 \\ k2 \end{array} \begin{array}{c} \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.2 & 0 & 0.64 \\ 0.2 & 0 & 0.4 \end{pmatrix} \end{array}.$$

Finally, suppose that each player would introduce μ units of same-type players in the next period. Then we have $N_k = \mu I$, where I is the identity matrix. □

2.2. Firm's Dynamic Optimization Problem

Now we describe the problem from the perspective of the firm/matchmaker. In practice, players request matches randomly, and the matchmaker reviews the matchmaking pool periodically and formulates matches based on specific constraints. For simplicity, we assume that for each time period, all the active players (i.e., players with engagement state that is not the churn state) request a match, and each player is assigned to an opponent by the matchmaker. The outcomes of all matches are then revealed simultaneously, and players update their engagement states upon completion. Then the next time period starts, and all players that have not churned request a match, and so on.

In practice, a popular competitive game usually has millions of concurrent online players. Motivated by this, we follow the literature of fluid matching (see, e.g., Azevedo and Leshno 2016) and assume that players are infinitely divisible. Let s^t_{kg} be the number of players at time t in the demographic with skill level k and engagement state g. The population of level k active players in period t is given by the vector $\mathbf{s}^t_k \in \mathbb{R}^{|\bar{\mathcal{G}}|}$, and we use $\mathbf{s}^t = [\mathbf{s}^t_1, \dots, \mathbf{s}^t_K]$ to denote the system state. Let $f^t_{kg,jg'} \geq 0$ be the amount of level k players in state g that are matched to a level j opponent in state g' in time t. Note that it is possible to use flow variables that only consider one of $f^t_{kg,jg'}$ and $f^t_{jg',kg}$.

However, the current formulation allows an easier presentation of the evolution of demographics' sizes over time. A feasible match given $\mathbf{s}^t$ is a set of matching flows $f^t_{kg,jg'}$ that satisfies:

$$\begin{aligned}
\sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f^t_{kg,jg'} &= s^t_{kg},\ k=1,\ldots,K, \forall g\in\bar{\mathcal{G}},\\
\sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f^t_{jg',kg} &= s^t_{kg},\ k=1,\ldots,K, \forall g\in\bar{\mathcal{G}},\\
f^t_{kg,jg'} &= f^t_{jg',kg},\ j=1,\ldots,K, k=1,\ldots,K, \forall g\in\bar{\mathcal{G}}, g'\in\bar{\mathcal{G}}\\
f^t_{kg,jg'} &\geq 0,\ j=1,\ldots,K, k=1,\ldots,K, \forall g\in\bar{\mathcal{G}}, g'\in\bar{\mathcal{G}}
\end{aligned} \tag{FB}$$

Namely, (FB) are *flow balance* constraints that make sure every active player is matched. The first equation ensures that every level k player in state g is matched with some opponents, and the second equation ensures that the total amount of matches against level k players in state g equals the number of such players. The third equation makes sure that for every pair of demographics, a match results in an equal effect on supply and demand.

Next, we depict the evolution of the system. Using $\mathbf{f}^t_{kj} = \{f^t_{kg,jg'}\} \in \mathbb{R}^{\bar{\mathcal{G}}\times\bar{\mathcal{G}}}_{\geq 0}$ to denote the flow matrix between level k and j, the *evolution of demographics* is given by

$$\mathbf{s}^{t+1}_k = \sum_{j=1,\ldots,K} \left(\mathbf{f}^t_{kj}\mathbf{1}\right)^\top \left(\bar{M}_{kj} + N_k\right)\ \ k=1,\ldots,K, \tag{ED}$$

where $\mathbf{1}$ is a $|\bar{\mathcal{G}}|\times 1$ unit vector. Note that in (ED), $\mathbf{f}^t_{kj}\mathbf{1}$ is the vector describing how many level k players are matched to level j players for all states in $\bar{\mathcal{G}}$, and recall that $\bar{M}_{kj}$ is the state transition matrix for level k players matched to level j players, N_k is the matrix for level k players to introduce new players (regardless they are matched with). The engagement at period t is given by $\sum_{k=1}^{K}\sum_{g\in\bar{\mathcal{G}}} s^t_{kg}$, the total amount of active players. The firm's objective is to maximize engagement, which we measure by the cumulative amount of active players across all periods, $\sum_{t=1}^{\infty}\gamma^{t-1}\sum_{k=1}^{K}\sum_{g\in\bar{\mathcal{G}}} s^t_{kg}$, where $\gamma\in(0,1]$ is the discount factor. The engagement maximization problem can be formulated as a Markov decision process, where the states are $\mathbf{s}^t$, the amount of active players in each demographic. Let V^π be the value function of a feasible policy π. The value-to-go function is given by

$$V^\pi(\mathbf{s}^t) = \sum_{k=1}^{K}\sum_{g\in\bar{\mathcal{G}}} s^{t+1}_{kg} + \gamma V^\pi(\mathbf{s}^{t+1}) \tag{1}$$
$$\text{subject to (FB), (ED).}$$

Our goal is to find the optimal policy with a value function $V^*(\cdot)$, such that $V^*(\mathbf{s}^t) \geq V^\pi(\mathbf{s}^t)$ for any feasible policy π. Note that the system dynamics are all linear, so when the initial size of each demographic is given, maximizing the engagement is equivalent to the following infinite linear program:

$$
\begin{aligned}
V^*(\mathbf{s}^0) = \max & \sum_{t=1}^{\infty} \gamma^{t-1} \sum_{k} \sum_{g \in \bar{\mathcal{G}}} s_{kg}^t \\
\text{s.t.} & \sum_{j=1}^{K} \sum_{g' \in \bar{\mathcal{G}}} f_{kg,jg'}^t = s_{kg}^t, \forall k, \forall g \in \bar{\mathcal{G}}, t = 0,1,\dots \\
& \sum_{j=1}^{K} \sum_{g' \in \bar{\mathcal{G}}} f_{jg',kg}^t = s_{kg}^t, \forall k, \forall g \in \bar{\mathcal{G}}, t = 0,1,\dots \\
& f_{kg,jg'}^t = f_{jg',kg}^t, j = 1,\dots,K, k = 1,\dots,K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}}, t = 0,1,\dots \\
& f_{kg,jg'}^t \geq 0, j = 1,\dots,K, k = 1,\dots,K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}}, t = 0,1,\dots \\
& \mathbf{s}_k^{t+1} = \sum_{j=1,\dots,K} \left(\mathbf{f}_{kj}^t \mathbf{1}\right)^\top \left(\bar{M}_{kj} + N_k\right), \forall k, t = 0,1,\dots
\end{aligned}
\tag{2}
$$

The LP formulation makes it flexible for industry practitioners to add various practical considerations into this framework.

In Appendix B, we discuss four possible extensions of our model: (1) adding a draw outcome (2) only a fraction of active players join the matchmaking pool (3) the game may last multiple (and possibly random) periods; (4) new players arrive depending on the entire history of active players. All of the above extensions can be incorporated into the LP formulation.

2.3. SBMM and Preliminary Results

As mentioned in Section 1, the status quo in the industry is SBMM, the skill-based matchmaking policy. Hence, it is important to characterize its performance and use it as a benchmark in our following analysis. In our setting, SBMM refers to any policy that lets players in the same skill level match with each other (they are all equivalent with respect to engagement). SBMM is always feasible by letting players in the same state match each other, i.e., $f_{kg,kg}^t = s_{kg}^t$. Let $V^{\text{SBMM}}(\cdot)$ be the value function of SBMM. We first characterize the value of SBMM in Theorem 1.

THEOREM 1 **(Value of Skill-based Policy)**. *Suppose the largest eigenvalue of $\gamma(\bar{M}_{kk} + N_k)$ is less than 1. Then the value function of SBMM is*

$$
V^{SBMM}(\mathbf{s}^0) = \sum_{k=1}^{K} \mathbf{v}_k^\top \mathbf{s}_k^0,
$$

where $\mathbf{v}_k$ is given by

$$
\mathbf{v}_k = \gamma^{-1} \left(\left(I - \gamma(\bar{M}_{kk} + N_k)\right)^{-1} - I \right) \mathbf{1}.
$$

If the largest eigenvalue of $\gamma(\bar{M}_{kk} + N_k)$ is greater than or equal to 1, then the total engagement is infinite.

All proofs are provided in Appendix A. Under SBMM, a level k player starts from an active state g and transitions to the next engagement state according to the Markov chain M_{kk}. Note that each unit of players generates one unit of engagement in each period. Thus, the total engagement a unit of player can generate, which we refer to as their shadow price, is the average number of periods they stay active and is described by $\mathbf{v}_k$. If there are no new players, then the eigenvalue of $\bar{M}_{kk}$ is less than 1 and the total engagement is finite even when $\gamma = 1$. When there are new players, a level k player also generate new players according to N_k. In this case, the engagement is finite only when the largest eigenvalue of $\bar{M}_{kk} + N_k$ is less than $1/\gamma$, meaning that the inflow is mild. Otherwise, the total engagement goes to infinity. The skill-based policy provides an important benchmark and is trivially the only (and optimal) policy when there is only one skill level.

To proceed, we state a mild assumption on customer behavior.

Assumption 1. *If a player and all the new players they introduce have a winrate less than or equal to 50% all the time, then their total engagement is finite.*

In the absence of new players, Assumption 1 simply assumes that a player would eventually leave if their winning probability is at most 50%. When new player arrivals are considered, Assumption 1 assumes that the new player arrival rate cannot compensate the loss of existing players. In practice, most online games have a short lifecycle and eventually lose most of the active players. Note that Assumption 1 is stronger than the assumption that the largest eigenvalue of $\gamma(\bar{M}_{kk} + N_k)$ is less than one (appeared in Theorem 1), because the latter only guarantees finite engagement when the winrate is exactly 50% (SBMM).

With Assumption 1, we can show that any policy has total finite engagement, with a mild inflow of new players.

Lemma 1 **(Finiteness of the Value Function)**. *Suppose Assumption 1 holds. Then for any policy π and initial state of demographics $\mathbf{s}^0$, $V^\pi(\mathbf{s}^0)$ is finite.*

Lemma 1 is somewhat counter-intuitive: because a myopic policy gains non-negative advantages in every period compared to SBMM, it is intuitive that the power of optimal policy should grow exponentially with time (also conjectured by Chen et al. (2017b)). However, Lemma 1 shows that the benefit of the optimal policy is finite in the absence of a large flow of new players. While the power of optimal policy is time-independent, we shall show in Section 3 that the benefit of optimal policy grows exponentially with the number of skill levels.

Finally, because (2) is an infinite LP, strong duality is not always guaranteed, although we prove it does hold in our setting.

Lemma 2 **(Strong Duality)**. *Suppose Assumption 1 holds. For any policy π initial state of demographics $\mathbf{s}^0$, strong duality and complementary slackness hold for* (2).

Although it is challenging and redundant to write out a dual problem in closed-form for the model presented in this section due to the generality of player behavior, in Appendix A.3 we provide the dual problem of the focal model presented in Section 4.

3. The Value of Optimal Matchmaking

In this section, we study the value of optimal matchmaking over SBMM in settings where players churn only after experiencing a losing streak. Specifically, we assume that for a player who just lost, another consecutive loss will result in the player staying in the matching system with probability $\rho \in [0,1)$ and churning with probability $1-\rho$. Players who just won do not churn. For example, the survival probability after 7 consecutive losses is ρ^6. Therefore, the outcome of a player's latest match fully describes their state. The set of demographics is $\{1w, 1\ell, ..., kw, k\ell, ..., Kw, K\ell\}$ and s_i^t denotes the amount of players in *demographic* $i \in \{1w, 1\ell, ..., Kw, K\ell\}$. Thus, s_{kw}^t and $s_{k\ell}^t$ represent the amount of level $k = 1, ..., K$ players at time t who just won and lost their last match, respectively. With probability ρ, players in demographic $s_{k\ell}^t$ stay in $s_{k\ell}^t$ and with probability $1-\rho$, they churn. Furthermore, we restrict $\gamma = 1$ in this section for convenience. We refer to this as the *geometric losing streak* model.

The churn behavior assumption based on losing streaks implies that people favor a winning outcome over a losing outcome in competitive games. This assumption follows the cognitive evolution theory in psychology that the *intrinsic* motivation to an activity (by the underlying need for competence and self-determination) would increase (decrease) when perceiving oneself as competent (incompetent) (Deci and Ryan 1980). Winning and losing are natural signals of competence and incompetence, and the relative benefit of winning over losing on the intrinsic motivation (measured by subsequent time spent on the game) has been widely supported in the literature (Reeve et al. 1985, Vansteenkiste and Deci 2003). Despite the outcome itself, game designers also offer various *extrinsic* incentives contingent on match outcomes, such as badges and in-game currencies, which further reinforce players' desires to win (Richter et al. 2015).

We also consider the arrival of new players in each period. We consider a word-of-mouth effect in each period where winning players can share their positive gaming experience and recruit new players to join in the next period. Mathematically, we assume that the players who just won a game, say in demographic s_{kw}^t, introduce an additional μs_{kw}^t amount of new players to the system in the next period. We assume that these μs_{kw}^t units of new players are added to demographics $s_{k\ell}^{t+1}$ in the next period, i.e., $N_k = \begin{bmatrix} 0 & \mu \\ 0 & 0 \end{bmatrix}$. This assumption reflects the reality that new players tend to be more prone to churning as they are deciding whether or not they like the game. We also remark that these assumptions have been designed to balance analytical tractability with practicality.

We shall further assume that the winrate p_{kj}, where is $k \geq j$, only depends on $k - j$. Specifically, we define $W(i)$ to be the winning probability for a player who is $i > 0$ skill levels higher than her opponent, so that $p_{kj} = W(i)$, if $k - j = i$. We shall assume that $W(i)$ is weakly increasing. The transition matrix $\bar{M}_{kj}$ is given by

$$\bar{M}_{kj} = \begin{array}{c} \\ kw \\ kl \end{array}\begin{array}{c} \begin{array}{cc} \quad kw \quad & \quad kl \quad \end{array} \\ \begin{pmatrix} W(k-j) & 1-W(k-j) \\ W(k-j) & \rho(1-W(k-j)) \end{pmatrix} \end{array}.$$

A detailed formulation Eq. ($\mathrm{P_K}$) is provided in Section A.2, and we focus on illustrating the performance of the optimal matching policy over a SBMM policy. By Theorem 1, the engagement of SBMM is finite when $\mu < (1-\rho)/2$. Intuitively, the condition $\mu < (1-\rho)/2$ requires the inflow of new players is less than the outflow of leaving players, which is consistent with the observation that the active players of all video games decay over time. Note that under a losing streak model, players may only churn after a loss. Thus, the engagement of players with less than 50 percent winrate all the time is bounded by that of players under SBMM (exactly 50% winrate). Therefore, their engagement is finite, and Assumption 1 is satisfied under the geometric losing streak model in this section.

Our goal is measure the power of the optimal policy, which we do by lower bounding the maximum ratio of the total engagement between the optimal policy and SBMM (the minimum ratio is trivially 1 when all the players are in the same skill level). Numerically, computing the maximum ratio can be done by solving the following optimization problem:

$$\begin{aligned} \max_{\mathbf{s^0}} & \sum_{t=1}^{\infty}\sum_{k}(s_{kw}^t + s_{kw}^t) \\ \text{s.t. } & V^{SBMM}(\mathbf{s^0}) = 1, \\ & \text{(FB) and (ED)}, \end{aligned} \tag{3}$$

which is still an infinite linear program because $V^{SBMM}(\cdot)$ is a linear function. To analytically quantify the maximum power of the optimal policy, we need to develop a feasible and reasonably good policy with known engagement. The performance of such a policy lower bounds the power of the optimal policy. However, besides SBMM, it is challenging to know the engagement of any policy in closed form, even a purely random one, due to the non-stationary nature of our problem. Nevertheless, we find that when the initial state belongs to a special set $\mathcal{S}$ (described in Appendix A.2) and the winrate function $W(i)$ is a constant, the feasible policy we study is *(i)* optimal and *(ii)* allows us to characterize the total engagement via the shadow price of each type of player.

LEMMA 3 **(Optimal Policy when Demographics are in $\mathcal{S}$).** *Consider $W(i) = a \in (0.5, 1]$ for all $i = 1, .., K-1$, $\rho \in [0, 1)$, and $\mu \in [0, (1-\rho)/2)$, such that $a + \mu \geq 1$. When $\mathbf{s}^0 \in \mathcal{S}$ (described in Appendix A.2), then for all $t = 0, 1, \ldots,$ the optimal matching policy induces matching flows*

$$f_{Kw,Kw}^t = s_{Kw}^t,$$

$$f^t_{(k+1)\ell,kw} = s^t_{kw}, \forall k = 1, \ldots K-1,$$
$$f^t_{k\ell,k\ell} = s^t_{k\ell} - s^t_{(k-1)w}, \forall k = 2, \ldots K,$$
$$f^t_{1l,1l} = s^t_{1l},$$

while all other matching flows are equal to zero. The total engagement is $\sum_{i\in\{1w,1l,\ldots,Kw,Kl\}} \lambda_i s^0_i$*, where* λ_i *is the shadow price given by*

$$\lambda_{Kw} = \frac{5+6\mu-\rho}{1-2\mu-\rho},$$
$$\lambda_{kw} = \frac{1+\rho+3\mu+3a(1-\rho)}{a(1-\rho)-\mu} + \frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\lambda_{(k+1)w}, \ k = 1, \ldots, K-1,$$
$$\lambda_{kl} = \frac{1+\rho}{2-\rho} + \frac{1}{2-\rho}\lambda_{kw}, \ k = 1, \ldots, K.$$

We leave the exact characterization of $\mathcal{S}$ to Appendix A.2. Note that set $\mathcal{S}$ contains some common initial demographics, such as almost uniformly distributed demographics where the numbers of players in each state are almost identical. Using strong duality, we show that as long as the initial demographics belong to set $\mathcal{S}$, the optimal matchmaking policy has a simple and intuitive form: it always matches all players in states $k\ell$ to players in states $(k-1)w$ for all $k = 2, ..., K$, while all other players conduct SBMM, i.e., matching with players from the same state. In particular, lower-skilled players who just won are matched with higher-skilled players who are on losing streaks so that with high probability, the latter's losing streak may be reset at the expense of the former. With the help of Lemma 3, Theorem 2 below characterizes the power of optimal matching policy compared to SBMM with respect to the number of total skill levels in the system.

THEOREM 2. *Suppose there exists some* $i \in \{1, 2, .., K-1\}$ *such that* $W(i) + \mu > 1$ *and* $0 < \mu < (1-\rho)/2$*. Then*

$$\max_{\mathbf{s}^0} \frac{V^*_K(\mathbf{s}^0)}{V_{SBMM}(\mathbf{s}^0)} \geq \underline{R}_K,$$

where

$$\underline{R}_K = \Theta\left(\frac{B^{\frac{K}{I(W)}}}{K}\right), \ I(W) := \min\{i \,|\, W(i) + \mu > 1\}, \ \textit{and } B := \frac{(2W(I(W))-1)(1-\rho)}{W(I(W))(1-\rho)-\mu} > 1. \quad (4)$$

We leave the closed-form expression of $\underline{R}_K$ *to Appendix A.2.*

Theorem 2 states that the maximum benefit of using the optimal matchmaking policy, when compared to SBMM, grows at least exponentially in the total number of skill levels as long as $\mu > 0$. Note that now we have generalized the winrate function $W(i)$ in Theorem 2 from a constant in Lemma 3 to a more general function.

We briefly discuss the intuition behind this result. The lower bound $\underline{R}_K$ is constructed by using our result in Lemma 3. Recall that in Lemma 3, the optimal matchmaking policy always matches all players in states $k\ell$ to players in states $(k-1)w$ for all $k=2,...,K$, while all other players conduct skilled based matchmaking. In particular, lower-skilled players who just won are matched with higher-skilled players who are on losing streaks so that with high probability, the latter's losing streak may be reset at the expense of the former. However, the "sacrificed" lower-skilled players will be matched to players with an even lower skill level in the future once they experience losing streaks. In other words, relatively low-skilled players are treated as a repairable resource that can be used to "save" players from higher skill levels. As a result, players from the lowest level can be used to save those from the second lowest level, which can further be used to save players from the third lowest level, and so on. Hence, the more skill levels there are, the more this improves engagement for all existing levels and the marginal benefit increases with K.

By analyzing the lower bound, we can gain more insights into how the power of optimal policy depends on the survival probability ρ, the new player arrival rate μ, and the winrate function W. First, note that as ρ increases, the constant B increases, leading to a greater power of optimal matchmaking compared to SBMM. Thus, a lower player churn rate in a losing streak (higher patience) enhances the power of optimal matchmaking. This result is somewhat surprising: intuitively, when the survival probability is high, we may not need to save players strategically because the players will keep playing with high probability anyway. However, in this case, the value of a saved player also becomes higher since they will stay much longer after a win. In Fig. 1a, we show how the exact ratio grows with K when we vary the value of ρ (computed by solving Eq. (3)), and it is consistent with our analysis on the lower bound.

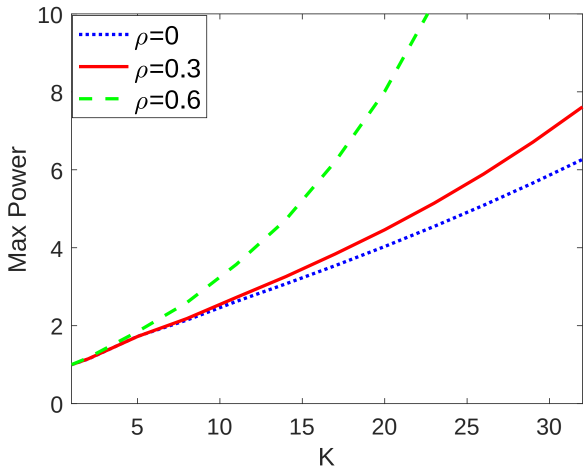
Next, we consider the impact of μ, the new player rate. As μ decreases, B also decreases, which means that the optimal policy becomes less powerful when fewer new players exist. In fact, when μ goes to 0, $W(I(W))$ goes to 1, and B goes to 1. We show that when no new players are introduced to the system, the power of optimal matchmaking is at least linear K. We formally state the result in Corollary 1 below.

COROLLARY 1. *When μ goes to 0, we have $\underline{R}_K = \Theta(K)$.*

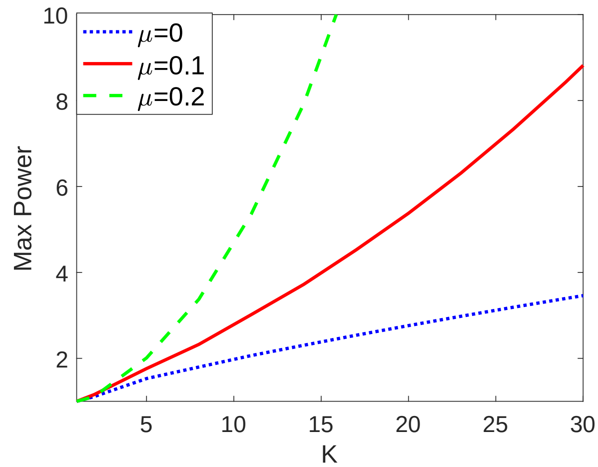
In Fig. 1b, we show the exact power of optimal policy with different μ, and the result is consistent with our analysis on the lower bound. In particular, when $\mu = 0$, the power has a linear trend when $K > 5$. Our result shows that new players reinforce the benefit of optimal matchmaking because they may be carefully matched later on.

Finally, the winrate function influences the lower bound $\underline{R}_K$ through both $I(W)$ and $W(I(W))$. Recall that $I(W)$ is the level difference such that $W(I(W)) + \mu > 1$. Thus, $I(W)$ measures how fast the winrate function $W(\cdot)$ converges to 1. Because $I(W)$ appears in the power term of $\underline{R}_K$, it significantly impacts the growth rate. The faster $W(i)$ increases, the more powerful the optimal

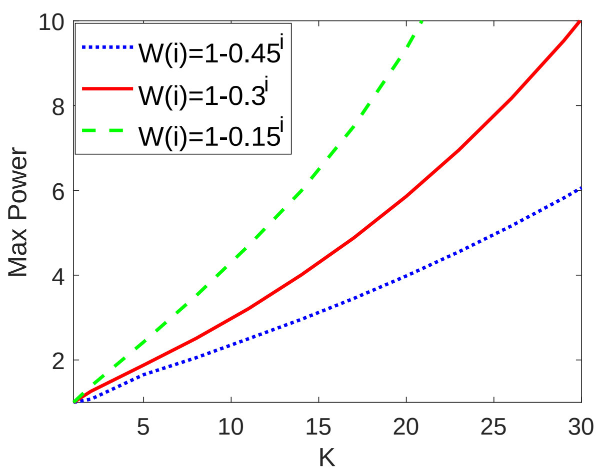
Figure 1 **The Maximum Power of Optimal Policy over SBMM**



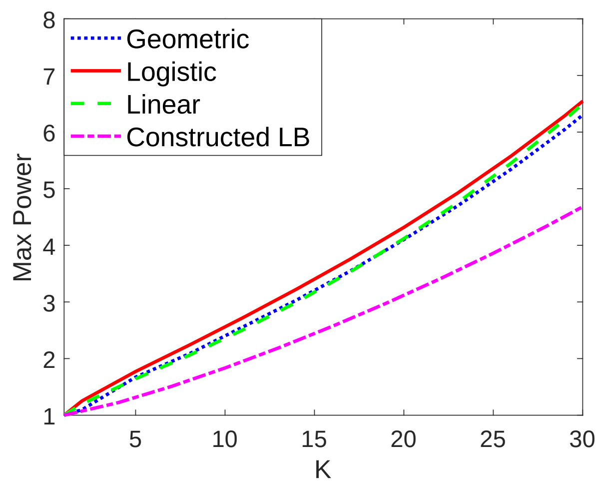
(a) Different ρ



(b) Different μ



(c) Different Convergence Rates



(d) Different $W(i)$

Note. *We plot how the maximum ratio Eq.* (3) *changes with (a)* ρ *and* K*, when* $\mu = 0.1$, $W(i) = 1 - 0.4^i$*; (b)* μ *and* K*, when* $\rho = 0.3$, $W(i) = 1 - 0.4^i$*; (c)* $W(i)$ *and* K*, when* $\mu = 0.1$, $\rho = 0.3$*; (d) different types of winrate functions, but with same* $I(W)$ *and* $W(I(W))$*. Parameter values:* $\mu = 0.1$, $\rho = 0.3$*. The three winrate functions are* $1 - 0.4365^i$, $1/(1 + \exp -0.8i)$, $\min(1, 0.5 + 0.1389i)$*, respectively.*

policy. The intuition is that optimal matching policy has more power when the matches become close to deterministic outcomes. In Fig. 1c, we show how the exact power grows with a geometric winrate function but with different convergence rates. As we expect, if the convergence rate is high, the optimal policy is more powerful. On the other hand, if we fix $I(W)$ and $W(I(W))$, the power does not change significantly. In Fig. 1d, we show an instance with geometric winrate, logistic winrate (also know as ELO), and linear winrate, respectively. In this instance, we fix $I(W) = 3$,

$W(I(W)) = 0.9168$, and the lower bound $\underline{R}_K$ is the same for all the three winrate functions. The power of the optimal policy is almost the same under these winrate functions.

4. A Stylized Baseline Model

In this section, we consider a more stylized model where players are one of two skill levels, and a high-skilled player (level 2) wins a game when matched with a low-skilled player (level 1) with probability 1 ($W(1) = 1$). Moreover, we do not consider new player arrivals ($\mu = 0$) and restrict our attention to the case where players churn immediately after two losses ($\rho = 0$). This stylized setting allows us to characterize the optimal matching policy's essence explicitly. Moreover, we are able to derive important insights related to AI bots and pay-to-win strategies.

While the model in this section is highly stylized, it does capture the essence and fundamental tradeoff of the engagement maximization problem. The assumptions of two skill levels and the 100% high-versus-low winrate are simplifications of heterogeneous skill levels. The dependency on the most recent two matches captures the players' bounded memory, which is common in the literature to deliver tractable results (e.g., Hu et al. 2016, Cohen-Hillel et al. 2019). In the context of pricing, empirical evidence also suggests that customers may only remember the current and most recent prices (see, e.g., Krishnamurthi et al. (1992) and the numerical part of Hu et al. (2016)). It is also consistent with industry practitioners' opinions that players have very limited memory capacity (Hiwiller 2015, Hodent 2020). In Section 5, we numerically show that looking at losing streaks more than two matches does not help much in estimating churn risk, and our theoretical insights hold in the case study when memory is longer than two matches. The assumption of no new players is reasonable for franchises that publish new titles regularly or those games that cannot afford continuous user acquisition campaigns. For franchises that publish new titles annually (e.g., *NBA2k* and *Call of Duty*), most new players arrive in the first few days. For example, *NBA2k21* sold 8 million copies by the first month of 2020-2021 season, but by the end of the season, the total sales were just 10 million.[4] For companies that cannot afford continuous user acquisition campaigns, their visibility on distribution channels (e.g., *Steam*) drops significantly after the initial launch, especially if the game is not a blockbuster.[5]

Under this stylized model, we can classify players into 4 *demographics*: high (level 2) and low-skilled (level 1) players who won the past game, denoted by $2w$ and $1w$, respectively; high and low-skilled players who lost the past game, denoted by class 2ℓ and 1ℓ, respectively. To avoid repetition, we present the detailed formulations and derivations of this stylized setting in Appendix A.3 and only focus on delivering meaningful insights in this section.

[4] The sales by Dec. 2020 is given by `https://www.vg247.com/nba-2k21-8-million-ps5-xbox-series-x-s-price-hike`. The sales by May 2021 are given by `https://www.gamesindustry.biz/take-two-hits-usd3-37bn-revenue-in-record-year`.

[5] https://partner.steamgames.com/doc/marketing/visibility

Before showing the optimal matchmaking policy, it is worth considering a one-period scenario, where the matchmaker only maximizes player engagement in the immediate next period. It turns out the *myopic* decision (formally stated in Appendix A.3, Lemma EC.2) that maximizes player engagement is to maximize the matching flow between demographics 2ℓ and $1w$. In other words, the matchmaker saves as many high-skilled players who are about to leave the system as possible, using low-skilled players who are not at risk of churning. Other players can simply be matched with same-level opponents or any other ways as long as $2w$ and 1ℓ are not matched. The intuition behind the optimal matching policy in this one-period setting is fairly intuitive. It preserves as many high-skilled players as possible without sacrificing low-skilled players at all. However, in an infinite-horizon matchmaking problem, we also need to balance trade-off between short-term and long-term rewards. Compared to SBMM, the myopic policy increases the number of players in demographic 1ℓ, leading to a higher churn rate in the subsequent period, and thus decreases the possible $2\ell - 1w$ matching in the future.

It turns out, however, the optimal matchmaking policy in the infinite-horizon problem resembles the aforementioned one for one-period. To be more specific, the matchmaker still maximizes the flow between 2ℓ and $1w$ demographics to maximize the overall player engagement, despite the forward-looking nature of the problem. However, the matchmaker also needs to balance the populations in each demographic at the same time, which is not a concern in the one-period matching problem. We present the formal result in the next proposition.

THEOREM 3 **(Optimal Matchmaking Policy)**. *The optimal matching policy is summarized in Table 1 with respect to different demographics. The optimal policy always maximizes matching flows between demographics 2ℓ and $1w$. The rest of the players are matched via SBMM in most scenarios (third row in Table 1), except when there are less high-skilled players compared to low-skilled players and too many 1ℓ players (first and second rows in Table 1). In this case, the matchmaker also matches players in demographic 2ℓ with players in 1ℓ.*

Table 1 **Optimal Matchmaking Policy,** $K_1 := \frac{18}{5}s^t_{2w} + \frac{9}{5}s^t_{2\ell} + \frac{3}{5}s^t_{1w}, K_2 := \frac{18}{5}s^t_{2w} + \frac{23}{5}s^t_{2\ell} - \frac{11}{5}s^t_{1w}$

States of demographics	Optimal matching policy
$s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1w}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $s^t_{1\ell} > K_2$	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$, $f^t_{2\ell,1\ell} = f^t_{1\ell,2\ell} = s^t_{2\ell} - s^t_{1w}$, and $f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}$
$s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1w}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $K_1 < s^t_{1\ell} \leq K_2$	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$, $f^t_{2\ell,2\ell} = \frac{9}{7}s^t_{2w} + \frac{23}{14}s^t_{2\ell} - \frac{11}{14}s^t_{1w} - \frac{5}{14}s^t_{1\ell}$, $f^t_{2\ell,1\ell} = f^t_{1\ell,2\ell} = \frac{5}{14}s^t_{1\ell} - \frac{9}{7}s^t_{2w} - \frac{9}{14}s^t_{2\ell} - \frac{3}{14}s^t_{1w}$, and $f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}$
Otherwise	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = s^t_{2\ell} - \min\{s^t_{2\ell}, s^t_{1w}\}$, $f^t_{1w,1w} = s^t_{1w} - \min\{s^t_{2\ell}, s^t_{1w}\}$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = \min\{s^t_{2\ell}, s^t_{1w}\}$

Table 1 summarizes the detailed matching flows by separating the four-dimensional state space into three regions. The first two regions represent special cases when the system has more high-skilled players than low-skilled players. The last region reflects that a myopic policy similar to the one in the one-period problem (Lemma EC.2) can be extended to a dynamic setting and remains optimal. In our proof, we split the four-dimensional state space into 7 scenarios (presented in Appendix A.3), and show the transition between scenarios under the proposed policy. Then we rely on strong duality and complementary slackness in Lemma 2 to prove optimality.

According to Theorem 3, the optimal matching policy always maximizes the flows between demographic 2ℓ and $1w$. Thus, matching players from different skill levels is beneficial, just like the one-period example. On the one hand, the platform should always "save" as many high-skilled players (2ℓ, who are about to leave the platform) as possible by matching them with low-skilled players ($1w$, without losing streaks). When matched with low-skilled players, high-skilled players (2ℓ) can enjoy a free win to break the losing streak and participate in future matchings. On the other hand, since low-skilled players can potentially prevent high-skilled players from leaving when they are matched, low-skilled players are very valuable to the matchmaker, so they should not be completely exhausted. Note that low-skilled players used to "save" high-skilled players are those from $1w$, so they may potentially recover from the losing streak in the next period. In most cases, the matchmaker does not match remaining players in 2ℓ to players in 1ℓ since players from the latter leave the matching process afterward, resulting in no immediate improvements and losing the potential for them to come back to demographic $1w$. One exception is when there are not enough low-skilled players in $1w$ but too many players in 1ℓ compared to the size of 2ℓ (the first two rows in Table 1). Under these scenarios, after exhausting players in $1w$, the matchmaker also matches low-skilled players in 1ℓ to high-skilled players in 2ℓ. Here, the matchmaker is essentially adjusting the overall player distribution among the demographics. Although they have the potential to increase other players' engagement, low-skilled players are only valuable if there are enough high-skilled players that need to be saved. The adjustment preserves high-skilled players in the process who may need to be saved by low-skilled players in the future.

Another way to gain insights into the effect of player distribution on the optimal policy is to look at the shadow price of each demographic, which can be loosely interpreted as how valuable a demographic is to the matchmaker. In general, the shadow prices for players who are not on losing streaks are higher than those that just lost a game. On the other hand, the shadow prices with respect to the skills ($2w$ versus $1w$) depend on the relative population of high and low players. Interestingly, when there are many more high-skilled players compared to low-skilled players, the shadow prices of low-skilled players are always bounded. However, as the ratio between the number of high- and low-skilled players goes to 0, the shadow prices for high-skilled players increases since high-skilled players can survive “forever” if injected into the system. This also explains why the matchmaker is willing to sacrifice players in 1ℓ to save high-skilled players. We leave the detailed descriptions of shadow prices in the proof of Theorem 3 in Appendix A.3.

Finally, we analyze the performance of the optimal matchmaking policy compared to the SBMM benchmark in Theorem 4 below. Since we are looking at a special case of the model in Section 3, we are able to obtain a tight analysis comapred to the more general results of Theorem 2.

THEOREM 4 **(Engagement Improvement, Stylized)**. *(a) In a single period, the myopic decision (from Lemma EC.2) that maximizes the engagement in one period, garners at most 4/3 engagement as SBMM. (b) In the infinite-horizon setting, we have that the optimal policy from Proposition 3 garners at most 3/2 engagement as SBMM. In other words,*

$$\frac{V^*(\mathbf{s}^0)}{V^{SBMM}(\mathbf{s}^0)} \leq \frac{3}{2}, \tag{5}$$

for any $\mathbf{s}^0 \geq 0$, where V^* is the value function under the optimal policy. Furthermore, the upper bound is achieved for some $s^0_{2\ell} = s^0_{1w} > 0$ and $s^0_{2w} = s^0_{1\ell} = 0$.

Theorem 4 first states that the the value of the myopic policy over SBMM is at most 4/3 in a single period. While Chen et al. (2017b) conjectured that the power of myopic matchmaking grows exponentially as the time horizon increases, we resolve this in the negative: over the whole time horizon, the benefit of optimal policy over SBMM is at most 50%. As discussed in Section 4, the optimal matching policy utilizes matches between 2ℓ and $1w$ players to improve engagement. Thus, as we can see from the second statement in Theorem 4, the upper bound of the performance ratio is attained when the initial demographics only have equal amount of 2ℓ and $1w$ players. In this scenario, the matchmaker improves the most compared to SBMM by using low-skilled players who are not in danger of churning to save high-skilled players who are about to churn.

We use the following two subsections to discuss two important interventions to increase engagement: AI-powered bots in Section 4.1 and pay-to-win strategies in Section 4.2.

4.1. AI-powered bots

Game designers routinely develop AI-powered bots to mimic human player behavior closely. Ideally, a matchmaker can use bots whenever a human player is about to churn if they experience one more loss. The bot can be designed so that it is competitive but still loses to the human player, which may result in the human breaking their losing streak and remaining in the system longer. On the other hand, due to the limitations of technology, AI-powered bots can be identified by experienced players. If a human player is frequently matched with bots, they may find out that their opponents are not human and perhaps be discouraged from playing the game. This section provides a framework to analyze the value of adding a percentage of AI-powered bots to the matching pool.

We focus on the impact on players' engagement when a certain percentage of the demographics are bots. We consider the same stylized model with the addition that each active player who just lost a game is matched to a bot with an independent and fixed probability $\alpha \in [0,1)$ in each period. With probability $1-\alpha$, the players who just lost are matched with another human player. This setup tailors toward players who are on losing streaks and focuses on using bots to improve player engagement. In practice, the value of α depends on the players' ability to detect bots and tolerance towards bots. The more developed the AI technology is, the higher α can be. Adding bots only slightly changes ($\mathrm{FB_S}$) and ($\mathrm{ED_S}$). To avoid repetition, we leave the formulation details in the Appendix (Eq. (EC.48)). With slight abuse of notation, denote $V^{\mathrm{SBMM}}(\mathbf{s},\alpha)$ and $V^*(\mathbf{s},\alpha)$ as the value functions of SBMM and the optimal matchmaking policy when the probability of matching with bots is α, respectively.

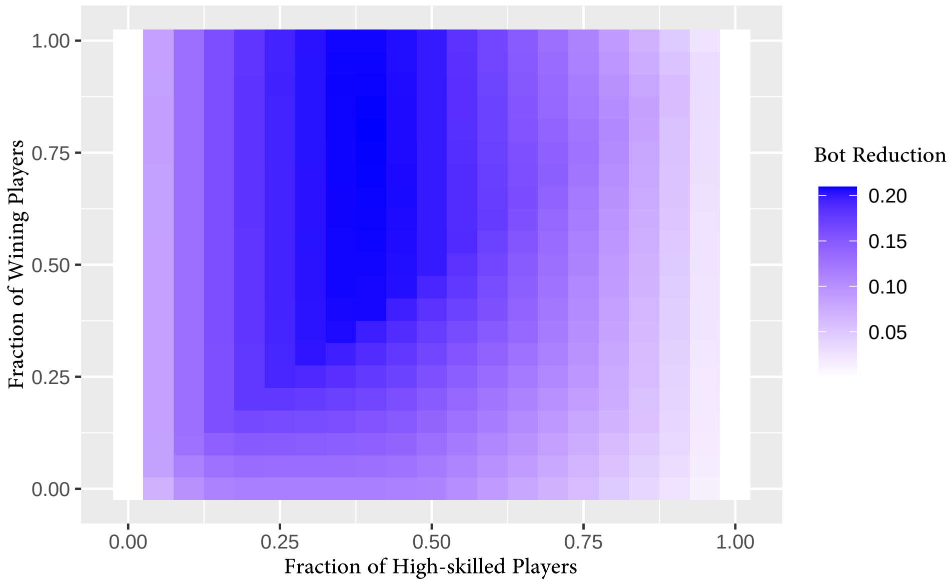
Engagement can be increased by improving the matchmaking policy from SBMM to optimal, but it can also be improved by adding bots. To measure the value of an optimal matchmaking policy, we consider the increased bot ratio needed under SBMM to achieve the same level of engagement as the optimal policy. In Figure 2, we draw a heatmap of bot reduction with different initial states. In Figure 2, we assume the initial state is $s_{2w}=xy, s_{2l}=x(1-y), s_{1w}=(1-x)y, s_{1l}=(1-x)(1-y)$, where x is the fraction of high-skilled players, and y is the fraction of winning players. Such an initial state may be achieved if the platform uses SBMM for several rounds before using the optimal policy or adding bots. As we can see, for a wide range of parameters, the optimal policy with zero bots can achieve at least the same engagement with SBMM with 10% bots. The reduction effect is strong when the fraction of high-skilled players is medium, which is consistent with our results in Section 3 that the benefit of optimal matching policy comes from cross-level matching. When there is only one skill level (the fraction of high-skilled players is either 0 or 1), the optimal policy has to use the same bot ratio as SBMM, as policies are the same. Thus, we are interested in the maximum bot reduction of the optimal policy. We summarize the maximum bot reduction of the optimal policy in the presence of bots in Theorem 5 below.

THEOREM 5. *Let α be the fractions of bots in SBMM.*

(a) For any $\alpha \le 25\%$, there exists some state $\mathbf{s}$ of demographics such that $V^*(\mathbf{s},0)=V^{SBMM}(\mathbf{s},\alpha)$, i.e., a state where the optimal policy without bots is as good as SBMM with α fraction of bots. For $\alpha > 25\%$, no such state exists.

(b) Let $a(\alpha)$ be the bot ratio such that $V^*(\mathbf{s},a(\alpha))=V^{SBMM}(\mathbf{s},\alpha)$. For any state $\mathbf{s}$ of demographics and bot ratio α we have that $\lim_{\alpha\to 1} a(\alpha)=1$, i.e., the optimal policy requires a bot ratio that also approaches 1 in order to achieve the same engagement.

Figure 2 **Bot Reduction with Different Initial States**



Note. When the fraction of high-skilled player is x, the fraction of winning player is y, the initial state is $s_{2w} = xy, s_{2l} = x(1-y), s_{1w} = (1-x)y, s_{1l} = (1-x)(1-y)$. The bot reduction is the value of α such that $V^*(\mathbf{s}, 0) = V^{SBMM}(\mathbf{s}, \alpha)$.

Theorem 5 delivers two important insights regarding matchmaking systems with bots. As of today, the moderate sophistication of AI-powered bots requires that α be relatively low so that players are not too frustrated. In this case, Theorem 5(a) shows that using the optimal policy with no bots can be just as good as SBMM with bots. In our baseline model, the optimal policy without bots can offset a bot ratio of up to 25% under SBMM. Thus, for companies that are criticized for using high bot ratios (Gelbart 2021) and that use SBMM, optimizing the matchmaking system may significantly alleviate such problems without loss of engagement. In Section 5, our case study based on real data also shows that the optimal policy has the power to reduce the bot ratio significantly when the value of α for SBMM is moderate.

Finally, Theorem 5(b) points out that the optimal policy's value is negligible as α goes to 1. Intuitively, when $\alpha = 1$, everyone only plays with bots, and all policies are functionally the same. Thus, developing more sophisticated AI bots can still provide value for companies regardless of the matchmaking policy.

4.2. Pay-to-win system

Building upon the baseline model in Section 4, suppose the matchmaker can offer low-skilled players a chance to purchase items or information to gain an advantage in the next match. Specifically, we follow the practice in *Dota 2* (introduced in Section 1) and consider a subscription service that provides additional information to players. Such features are largely useless for high-skilled players but may help low-skilled players substantially. Therefore, in the model of consideration, we only consider a pay-to-win (PTW) feature for low-skilled players in every period.[6] We assume that the subscription fee is r per period, and low-skilled players either opt-in at time zero and keep their subscription for all matches until they churn, or never pay for it. This assumption is a reasonable reflection of reality since players willing to pay for the subscription feature are likely those who accept the idea behind microtransactions in video games. The majority of players opposing the idea would rarely switch to a subscription in the middle of their lifespan in the system. Beyond these observations in the gaming industry, the vast amount of literature also supports a one-time decision on getting a subscription service upfront (see e.g., Cachon and Feldman 2011, Belavina et al. 2017, Wang et al. 2019). We use $\beta \in [0,1)$ to denote the proportion of the low-skilled players who pay for the subscription. In practice, only about 5% of players pay in a freemium game, so β should be small (Seufert 2013). We leave r and β exogenous and focus on the interplay between PTW and the matchmaker.

With a pay-to-win system like the subscription feature mentioned above, monetary elements have been added to the matching system. We assume that there is a conversion rate between players' engagement and the seller's revenue. For simplicity, we normalize a unit of player engagement translates to one unit of revenue generated for the seller.

Adopting the notation from Section 4, suppose the matchmaker faces an initial state of demographics $\mathbf{s}^0 = \{s^0_{1w}, s^0_{1\ell}, s^0_{2w}, s^0_{2\ell}\}$. By having the subscription, a β portion of low-skilled (level 1) players elevate their gaming skills to the high level (level 2), denoted by $\bar{2}$, so that they behave exactly like high-skilled players in Section 4. However, purchasing the subscription feature does not (and should not) reset a player's losing streak. Thus, instead of having 4 demographics, with the addition of the subscription feature, there are 6 demographics in each period, denoted by $s^t_{\bar{2}w}, s^t_{\bar{2}\ell}, s^t_{\underline{2}w}, s^t_{\underline{2}\ell}, s^t_{\underline{1}w}, s^t_{\underline{1}\ell}$. Note that $s^t_{\bar{2}w}, s^t_{\bar{2}\ell}$ represent players with subscriptions who just won and lost a game, respectively. To differentiate from the original demographics, we use $s^t_{\underline{2}w}, s^t_{\underline{2}\ell}$ to denote non-paying high-skilled players, and $s^t_{\underline{1}w}, s^t_{\underline{1}\ell}$ to denote non-paying low-skilled players. In the initial period $t = 0$, given β and $\mathbf{s}^0$, we have $s^0_{\bar{2}w} = \beta s^0_{1w}$, $s^0_{\bar{2}\ell} = \beta s^0_{1\ell}$, $s^0_{\underline{2}w} = s^0_{2w}$, $s^0_{\underline{2}\ell} = s^0_{2\ell}$, $s^0_{\underline{1}w} = (1-\beta)s^0_{1w}$, $s^0_{\underline{1}\ell} = (1-\beta)s^0_{1\ell}$. In Appendix Section A.3, we introduce new flows between demographics (Eq. ($\mathrm{FB}_{\mathrm{ptw}}$), Eq. ($\mathrm{ED}_{\mathrm{ptw}}$)) and the updated matchmaking problem (Eq. ($\mathrm{P}_{\mathrm{ptw}}$)).

Our focus is on the interplay between PTW and matchmaking, as well as their aggregate value. We provide our main findings in Theorem 6 below.

THEOREM 6. *Consider $\beta \in [0,1)$.*

[6] In practice, such membership is bundled with other perks, e.g., decorative staffs, so high-skilled players may also pay for it. However, our qualitative insights are consistent as long as high-skilled players do not improve their skill level further.

(a) With PTW under SBMM, one unit of subscribed players is as valuable as $1+r$ units of unsubscribed players. With PTW under the optimal policy, one unit of subscribed players is more valuable than $1+r$ units of unsubscribed players.

(b) Subscribed players have a higher priority compared to unsubscribed high-skilled players. That is, unsubscribed high-skilled players in $\underline{2}w$ ($\underline{2}\ell$) would only be matched with any unsubscribed low-skilled players after all the subscribed players in $\bar{2}w$ ($\bar{2}\ell$) have matched with unsubscribed low-skilled players.

(c) When there are more high-skilled players than low-skilled players, implementing PTW decreases engagement. However, in terms of the overall revenue (engagement plus subscription revenue) generated, implementing PTW is beneficial if and only of r is greater than a specific threshold.

(d) When the ratio of high- over low-skilled players is sufficiently small, implementing PTW increases engagement and subscription revenue simultaneously.

Theorem 6(a) says that under SBMM, the value of a subscribed player in a PTW system equals $r+1$ unsubscribed high-skilled players. However, under the optimal policy, the value of a subscribed player in a PTW system goes beyond replacing them with $r+1$ unsubscribed players. On the other hand, Theorem 6(b) depicts the impact of PTW on the matchmaking policy. We prove that the optimal policy gives subscribed players high priority and would save an unsubscribed high-skilled player from churning only when all the paid players within the same engagement state are saved. Theorem 6(c) and (d) describes the influence of PTW on user engagement and the total value function. When there are more high-skilled than low-skilled players, Theorem 6(c) claims that the corresponding engagement inevitably falls, and it is only worth introducing PTW when the unit profit r is high enough. This is because, in this case, low-skilled players are already scarce resources, and a PTW system further increases their scarcity. On the other hand, Theorem 6(d) claims that PTW can increase engagement while providing direct revenue. This happens when there are too few high-skilled players. Intuitively, this is because now the high-skilled players are scarce resources, and a few more high-skilled players can facilitate cross-level matching significantly and thus improve engagement. Together, Theorem 6(c) and (d) provide important managerial insights on the value of a PTW system when taking matchmaking and player engagement into account: besides its direct revenue, it also works as a lever to change the skill distribution and may influence the total engagement positively. This happens when the majority of players are low-skilled, which is reasonable for most games. On the other hand, if the majority of players are high-skilled, PTW will hurt engagement, and the seller should be careful to introduce it.

Intuitively, all the non-paying players are worse off in the presence of a PTW system, because high-skilled non-paying players are in low priority for cross-level matching, and the low-skilled non-paying players need to save more high-skilled players now. However, in Example 2 we show that it is possible that low-skilled non-paying players may also be better off due to the redistribution effect. The intuition behind such a surprising observation is that, the optimal matchmaking strategy would utilize low-skilled non-paying players in a more sustainable way as they become more scarce in the presence of PTW. Such an observation again emphasizes the potential positive externality of PTW on user engagement, in addition to the potential increase in revenue.

Example 2 (PTW May Make Low-skilled Non-paying Players Better Off). Suppose $\beta = 0.2$, $r = 0$, and the initial state is $\mathbf{s} = (2, 2, 1, 15)$. Without the PTW system, some players in 1ℓ have to be matched with 2ℓ players in the first period and leave directly, leading to an average engagement of 2.27 for all the low players over the lifespan. When the PTW system is considered, we have $s^0_{\bar{2}w} = 0.2$, $s^0_{\bar{2}\ell} = 3$, $s^0_{\underline{2}w} = 2$, $s^0_{\underline{2}\ell} = 2$, $s^0_{\underline{1}w} = 0.8$, $s^0_{\underline{1}\ell} = 12$. Because $r = 0$, such a problem is equivalent to a problem with initial state $\mathbf{s}' = (2.2, 5, 0.8, 12)$ but without the PTW system. In this case, the 1ℓ players become more scarce and would not be matched to high-skilled players in the first period, and the average engagement for the low players increases to 2.31. □

5. Case Study of Lichess

In this section, we conduct a numerical case study with the Lichess Open Database (Lichess 2021) to demonstrate the power of optimal matchmaking policies in a realistic setting.

5.1. Data Description

Lichess is a free and open-source Internet chess server, and all the match data since 2013 are available to the public. For each match, the data includes the game mode, starting time, players' IDs, the match's outcome, and the players' ratings before and after the match. Each player has a rating for each game mode. The platform adopts the Glicko-2 rating, a generalization of the Elo rating (Glickman 2012). We collected all the matches in the most popular mode "Rated Blitz Game" during 2013-2014, which includes 5.41 million matches and 135,073 unique players. To simplify our problem, we remove all the matches where the outcomes are draws, which represent 3.7% of the matches, and focus on the 5.23 million matches with unique winners. We say a player is churned after a match if s/he stops playing for 14 days after the match. Hence, a player may churn more than once.

We focus on players who played for at least 5 matches for three reasons. First, players who churned in less than 5 matches may largely churn due to other factors (e.g., user interface, internet connection). Second, new players' ratings change wildly after the first few matches, making it difficult to estimate their skill levels. Finally, we would like to test models with a range of game memory m (up to 4), which means that the players need to play at least $m+1$ matches. In our dataset, there are $N = 60,334$ qualified players. For each player, we take at most 500 matches on their record (92% of the 60,334 players play less than 500 matches). This is done to avoid underestimating the churn probability since a few highly dedicated players never quit under any circumstance.

Because the ratings oscillate at all times, we use the average of the ratings after the last 3 matches to estimate the rating, which is then used to determine their skill level. A summary of the players' engagement and ratings is in Table 2. Compared to the median rating (1518), the fluctuation is not large in general: for the focal players, the average of their lifetime standard deviation is 57, and in a calendar month the average standard deviation is 39. Thus, it is reasonable to assume that the skill levels are stationary. Based on the estimated ratings, we divide players into 13 skill levels: for the 96% of the considered players whose rating is between 1000 and 2100, we separate them into 11 levels (level 2 to 12) based on intervals of 100; for players with a rating less than 1000 and greater than 2100, we group them into two new levels, level 1 and level 13, respectively. To compute the winning probability between different skill levels, we use the realized winrate when the level difference is less than or equal to 8. When the level difference is greater than 8, we have too little data for accurate estimation and simply assume that the player with a higher skill level wins with probability 1. Table 3 describes the winrate we used in the case study.

Table 2 Summary of Considered Players ($N = 60334$)

	Mean	Median	Min	Max
Number of Matches	170.6	21	5	18764
Estimated Ratings	1525	1518	764	2663

Table 3 Winrate $W(i)$ Used in Section 5

Level Difference	0	1	2	3	4	5	6	7	8	9-12
Winrate	0.500	0.596	0.690	0.776	0.844	0.893	0.924	0.936	0.948	1.000

5.2. Players' Churn Behavior Estimation

Recall that in Section 2, we define a Markovian engagement model where the state g is uniquely determined by the win-loss record of the last m matches. We now illustrate how to estimate P^k_{win} and P^k_{lose} through maximum log-likelihood estimation. To simplify our problem and reduce the number of parameters, we assume that the players' churn behavior is independent of skill level and drop the superscript k. In the state transition matrix, the row that represents state g in P_{win} (P_{lose}) only has two positive entries: the probability of churning after winning (losing) the next match and the probability of moving to a certain non-churn state. Our goal is then to estimate ρ_g^{win} (ρ_g^{lose}), the churn probability of a player in state $g \in \mathcal{G}$ after winning (losing) the next match (Note that $\rho_q^{win} = \rho_q^{lose} = 1$ since we assume that a churned player will stay churned.) Let g_t^i be the state player i reaches before playing t-th match, $\mathbb{W}_t^i$ be the indicator of whether player i wins the t-th match, and $\mathbb{I}_t^i$ be the indicator of whether the player i churns right after reaching g_t^i. For a player who played T^i total matches, we record his/her states and churn decisions starting from the 5th match because they only have a valid state right before the 5th match if $m = 4$. If the state sequence is given by $(g_5^i, g_6^i, \ldots, g_{T^i}^i)$, the outcome sequence is $(\mathbb{W}_5^i, \mathbb{W}_6^i, \ldots, \mathbb{W}_{T^i}^i)$, and churn decision sequence is $(\mathbb{I}_5^i, \mathbb{I}_6^i, \ldots, \mathbb{I}_{T^i}^i)$, then the log-likelihood (LLH) function is given by

$$\sum_{i=1}^{N}\sum_{t=5}^{T^i} \log\left(\rho_{g_t^i}^{win}\mathbb{W}_t^i\mathbb{I}_t + (1-\rho_{g_t^i}^{win})\mathbb{W}_t^i(1-\mathbb{I}_t) + \rho_{g_t^i}^{lose}(1-\mathbb{W}_t^i)\mathbb{I}_t + (1-\rho_{g_t^i}^{lose})(1-\mathbb{W}_t^i)(1-\mathbb{I}_t)\right).$$

Because the state transition is exogenous, the estimation is computationally simple and usually has a closed form. Below we propose two reasonable models that use $m+2$ parameters and show the closed form of the parameters.

Losing Streak Model: The losing streak model assumes that the players' churn probability depends on the length of the losing streak they experience (after the next match). For a given m, players have $m+1$ possible engagement states $0, \ldots, m$, which represents the length of the losing streak before the next match. In this model, ρ_g^{win} is the same for all $g = 0, \ldots, m$, because all such players have a 0 match losing streak upon winning the next game. On the other hand, ρ_g^{lose} represents the churn probability upon losing the next game, given that the players have already lost g consecutive games (thus, they experience a $(g+1)$-match losing streak). Our baseline model in Section 4 can be viewed as a special case of the losing streak model, with $m = 1, \rho_g^{win} = 0$, $\rho_0^{lose} = 0$, and $\rho_1^{lose} = 1$. For the losing streak model, the solution for the MLE is given by

$$\rho_g^{win} = \frac{\text{Number of churn decisions after a win}}{\text{Number of wins}}, \ g = 0, \ldots, m$$

$$\rho_g^{lose} = \frac{\text{Number of churn decisions after } g+1 \text{ consecutive losses}}{\text{Number of times that players experience } g+1 \text{ consecutive losses}}, \ g = 0, \ldots, m.$$

Winrate Model: The winrate model assumes that the players' churn probability depends on the number of wins over the last m matches plus the next match (total m+1 matches). Hence, the model can be described by $m+2$ parameters, the churn probabilities when the player has 0 to $m+1$ wins. Note that although we only need to estimate $m+2$ parameters, we still need $2^m + 1$ states. This is because, for the LP formulation, the customer transition matrix needs the full win-loss record. Hence, a state g is a length-m binary sequence denoting the win-loss record, and we use $|g|$ to denote the wins in g. The solution for the MLE is

$$\rho_g^{win} = \frac{\text{Number of churn decision when players won } |g+1| \text{ matches over last } m+1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over last } m+1 \text{ matches}},$$

$$\rho_g^{lose} = \frac{\text{Number of churn decision when players won } |g| \text{ matches over the } m+1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over the } m+1 \text{ matches}}.$$

5.2.1. Estimation Results We estimate the models for the players who played at least 5 matches. We randomly sample 70% of the players to train the model and use the rest of them to validate the model. We test both the losing streak model and the winrate model, ranging m from 0 to 4. When $m=0$, the churn probability only depends on the outcome of the current game, and the losing streak model coincides with the winrate model. In addition, we also estimate a null model as a benchmark, which assumes a uniform churn probability regardless of the states.

Table 4 Out-of-sample Negative Log-likelihood for the Candidate Models

Null Model			
144424.4			
Losing Streak Model		Winrate Model	
$m=0$	140019.4	$m=0$	140019.4
$m=1$	139641.5	$m=1$	139713.1
$m=2$	139488.8	$m=2$	139566.7
$m=3$	139362.9	$m=3$	139375.9
$m=4$	139281.6	$m=4$	139255.2

In Table 4, we summarize the out-of-sample negative LLH of the candidate models. Notably, compared with the null model, even when m equals 0 or 1, both the losing streak model and the winrate model improve the LLH by more than 4000, while increasing m from 1 to 4 only improves the LLH further by less than 500. Thus, assuming $m=1$ does not result in much loss in the goodness-of-fit of the models. When $m=1$, the parameter of the losing streak model is given by $\rho_g^{win} = 1.32\%$, $\rho_0^{lose} = 1.63\%$, and $\rho_1^{lose} = 2.54\%$. The surge in churn probability from one loss to two consecutive losses reflects the players' negative sentiment toward a losing streak. Our stylized model in Section 4 reasonably captures the essence of this behavioral pattern.

5.3. Power of Optimal Matchmaking on Lichess

In this section, we test the power of optimal matchmaking, and validate the insights in Section 3. We consider a scenario where in each period, all the active players join the matchmaking pool and the outcome is revealed immediately. We use the realized skill levels (13 total) and the realized engagement states after the fifth match as the initial input.

We compare four policies: SBMM, the optimal policy, the random policy where a player has a uniform chance to be matched with any other active player, and a one-step-look-ahead policy that maximizes engagement in the next round. We include the one-step-look-ahead policy because it is easy to compute, and Theorem 3 shows that the optimal policy belongs to this class under the stylized baseline model. To compute the total engagement under the optimal policy, we simply solve the LP formulation Eq. (2) with a large enough T. In our case, we set $T = 1000$. The discount factor is 1. We use Gurobi to solve the formulation. The one-step-look-ahead policy is computed by setting $T = 1$, and we use the policy output by Gurobi. In Table 5, we show the power of various policies compared to SBMM. SBMM is better than the random policy, which shows that the status quo of SBMM is better than doing nothing. Notably, the optimal policy may improve the total engagement by 4.17-6.62%, depending on the choice of churn model and the memory parameter m. The one-step heuristic is comparable to the optimal policy in all scenarios, providing a computationally more efficient alternative in practice.

Table 5 Relative Power of Candidate Policies Compared to SBMM

	Random	One-step	Optimal
Losing Streak, $m = 1$	-2.05%	4.04%	4.17%
Losing Streak, $m = 2$	-3.80%	5.56%	5.81%
Losing Streak, $m = 3$	-5.04%	5.77%	6.00%
Losing Streak, $m = 4$	-5.83%	5.55%	5.73%
Winrate, $m = 1$	-2.14%	3.94%	4.32%
Winrate, $m = 2$	-4.14%	5.04%	5.87%
Winrate, $m = 3$	-6.17%	5.60%	6.62%

We also consider the case when there exist new customer arrivals. Following our setting in Section 3, we consider the losing streak model with $m = 1$, and assume that each winning player may introduce μ units of losing players in the same skill level at the start of the next period. We compute how the power of the optimal policy changes when we vary μ from 0% to 1%. Consistent with our theoretical results in Theorem 2, the optimal policy provides more benefits when new customer arrivals exist. We see the improvement from the optimal policy over SBMM increase from 4.17% to 5.93%.

Next, we check the power of optimal matchmaking with a varying number of skill levels. Recall that in Theorem 2, the maximum power of optimal matchmaking over SBMM increases with the number of skill levels. To verify this insight in a realistic setting, we now compute the relative power of optimal matchmaking when the top or bottom K skill levels are considered for $K = 2, \ldots, 13$. In Fig. 3, we show a representative example when $m = 1$. It turns out that the power of the optimal policy monotonically increases with K regardless of how we add the levels (bottom-up or top-down), which is consistent with the insight we derive on the maximum power.

Finally, we investigate scenarios with AI-powered bots. We assume that every player who just experienced a loss has an α chance to be matched with a bot and compare the performance of the optimal policy and SBMM. In Fig. 4, we show how the total engagement changes with α when $m = 3$. Notably, when α is reasonably small ($\leq 30\%$), the optimal policy can achieve the

Figure 3 **Power of Optimal Matchmaking with K Skill Levels ($m = 1$)**

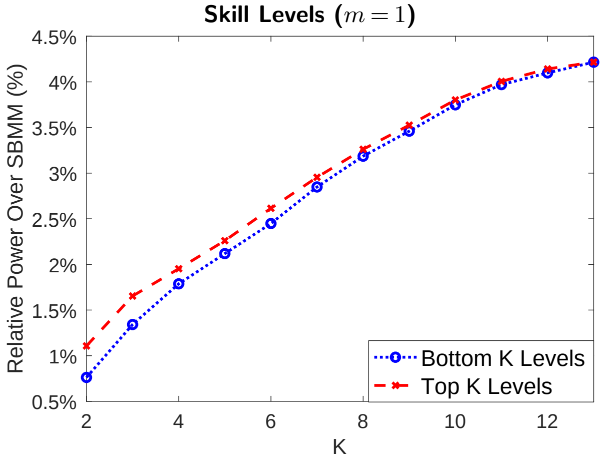
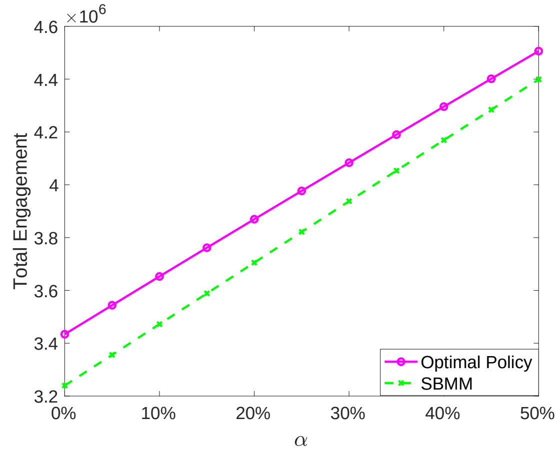


Figure 4 **Total Engagement with Different Fractions of Bot Matches ($m = 3$)**



same engagement level with SBMM but with $8-10\%$ fewer bot matches. Thus, optimizing the matchmaking system can significantly reduce the bots needed and avoid the perils of a high bot ratio. From the players' perspective, the reduction effect could be significant: if they meet a bot every 5 matches ($\alpha = 20\%$) under SBMM, then they would meet a bot every 10 matches ($\alpha = 10\%$) without loss of engagement under the optimal policy. On the other hand, as α keeps increasing, the difference between SBMM and the optimal policy decreases. This is because when a high bot ratio can be tolerated, most players play with a bot and transition to an ideal state, making all policies similar.

6. Conclusions

This paper presents a modeling framework for matchmaking in competitive video games. We provide several novel insights to industry practitioners through sharp characterizations of a baseline model and a numerical case study based on real data. The current standard is SBMM, which matches players with similar skill levels. Although SBMM is intuitive and easy to implement, it does not maximize long-term engagement. For the engagement-optimal matchmaking rule, we show that in the special case, the optimal policy myopically maximizes the short-term reward in the next period but also adjusts player distribution for the long-term reward. We also highlight the significant improvement in player engagement that can be achieved using an optimal matchmaking policy over SBMM. Surprisingly, the benefit increases exponentially with the number of skill levels.

In addition, we provide new perspectives on controversial topics such as AI bots and pay-to-win systems in competitive video games. Our results show that using an optimal matchmaking policy instead of SBMM may reduce the required bot ratio significantly while maintaining the same level of engagement. By investigating the interplay between pay-to-win strategies and the optimal matchmaking policy, we provide a novel viewpoint of PTW as a lever to control the distribution over the demographics of players. Importantly, we show that PTW may even increase engagement, sometimes even for low-skilled non-paying players. This potential positive externality on user engagement contrasts with the negative public image of PTW strategies. Finally, using real data from an online chess platform, we show that players do indeed churn based on recent match outcomes, and the optimal policy can significantly improve over SBMM.

We note some limitations of the present research and suggest possible directions for future work. First, our model focuses on players' skill levels when conducting matchmaking. Future works can consider other features, such as players' preference for a specific playing style or strategy during matchmaking. Second, we do not consider games where players' skill levels can change quickly over time (on the same time scale as matches being played), which can be of interest for games with quick learning curves. Third, one can consider more strategic gamers that can decide strategically when to buy a pay-to-win benefit, which is a one-shot decision in our current model.

Finally, we remark that video games have positive benefits for training memory (Blacker et al. 2014) and improving cognitive skills (Bader et al. 2022). At the same time, we must also consider the social responsibility issues related to video games and other forms of digital addiction (Allcott et al. 2022). Our work suggests that designing better matchmaking algorithms increases engagement for the video game at hand (offering a competitive advantage), but empirical work is needed to see whether or not it would increase the total time dedicated to video games. More generally, the problem we study is a customer retention problem. In particular, our modeling framework and insights can potentially be extended to other settings where keep users engaged with the platform is critical, such as online dating, education, and gig economy.

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Appendix A: Omitted Proofs

A.1. Omitted Proofs from Section 2

Proof of Theorem 1. Denote $\mathbf{v}_k = \{v_{kg}\}_{g\in\bar{\mathcal{G}}}$ as the vector of value (engagement) that 1 unit of players in level k can create. Note that v_{kg} is the average active time starting from state g in the absorbing Markov chain M_{kk} before absorption. By Theorem 3.2.4 in Kemeny et al. (1960), we have

$$\begin{aligned}\mathbf{v}_k &= \left((\bar{M}_{kk}+N_k)+\gamma(\bar{M}_{kk}+N_k)^2+\gamma^2(\bar{M}_{kk}+N_k)^3+\cdots\right)\mathbf{1}\\ &= \left(\frac{1}{\gamma}\left(I+\gamma(\bar{M}_{kk}+N_k)+\gamma^2(\bar{M}_{kk}+N_k)^2+\cdots\right)-\frac{1}{\gamma}I\right)\mathbf{1}\\ &= \gamma^{-1}\left(\left(I-\gamma(\bar{M}_{kk}+N_k)\right)^{-1}-I\right)\mathbf{1},\end{aligned}$$

where the last line requires that the largest eigenvalue of $\gamma(\bar{M}_{kk}+N_k)$ is less than 1. Otherwise, $I-\gamma(\bar{M}_{kk}+N_k)$ is not invertable, which indicates that the summation goes to infinity. Summing up players at all levels, we have

$$V^{\mathrm{SBMM}}(\mathbf{s}^0)=\sum_{k=1}^{K}\mathbf{v}_k^\top\mathbf{s}_k^0.$$

□

Proof of Lemma 1. We prove this lemma by induction. For the base case, consider the engagement of level 1 players. Because their winrate is at most 0.5, by Assumption 1, their engagement is finite.

The induction hypothesis is that the engagement is finite for players of level 1 to level k. We then show that the engagement is also finite for players of level $k+1$. From the induction hypothesis, the engagement with players from levels 1 to k must be finite. Thus, we only need to consider engagement generated from matches with players of level $k+1$ to K, and show it is also finite. Note that level $k+1$ players' win rate is at most 0.5 since they are matched to players with at least the same skill level. Thus, Assumption 1 implies that the engagement from matches with levels $k+1$ to K is also finite. □

Proof of Lemma 2. Our proof relies on Theorem 2.3 in Ghate (2015), which requires showing that our LP formulation (2) satisfies the following five hypotheses. Let $X\subseteq\mathcal{R}^{\mathbb{N}}$ be a linear subspace. For an infinite primal LP

$$V(P)=\sup\sum_{j=1}^{\infty}c_jx_j \tag{EC.1}$$

$$\sum_{j=1}^{\infty}a_{ij}x_j=b_i,\ i=1,2,\cdots \tag{EC.2}$$

$$x_j\geq 0,\ j=1,2,\cdots \tag{EC.3}$$

$$x\in X, \tag{EC.4}$$

we assume for any $x\in X$

H1. $\sum_{j=1}^{\infty}c_jx_j<\infty$.

H2. $\sum_{j=1}^{\infty}a_{ij}x_j<\infty,\ j=1,2,\cdots$.

Further, let $Y\subseteq\mathcal{R}^{\mathbb{N}}$ be the subset of all $y\in R^{\mathbb{N}}$ such that

H3. $\sum_{i=1}^{\infty} b_i y_i < \infty$.

H4. For every $x \in X$, $\sum_{j=1}^{\infty} |a_{ij} x_j y_i|$ converges to some limit $L_i(x, y_i)$ for $i = 1, 2, \cdots$, and

H5. The above limits $L_i(x, y_i)$ have the property that $\sum_{i=1}^{\infty} L_i(x, y_i) < \infty$.

Then consider the dual problem

$$V(D) = \inf \sum_{i=1}^{\infty} b_i y_i \tag{EC.5}$$

$$\sum_{i=1}^{\infty} a_{ij} y_i \geq c_j, \ \ j = 1, 2, \cdots \tag{EC.6}$$

$$y \in Y. \tag{EC.7}$$

By Theorem 2.3 in Ghate (2015), suppose $x \in X$ and $y \in Y$ are feasible to the primal and dual problems and are complementary ($x_j(c_j - \sum_{i=1}^{\infty} a_{ij} y_i) = 0$ for all j). Then x and y are optimal solutions to the primal and dual problems, and $V(P) = V(D)$.

For our problem in (2), let X to be the l_1 sequence space. Because of Lemma 1, letting X be the l_1 sequence space is without of generality. We check hypotheses H1 to H5, respectively. For dual variables, we only consider y from l_∞ space. As we will show, any y from l_∞ space satisfy H3 to H5, and we only use such y in the following proofs.

Hypothesis H1 is satisfied because X is the l_1 space. Hypothesis H2 and H4 are satisfied since we have finitely many primal variables in each constraint of problem 2 and X is the l_1 space. Hypothesis H3 is satisfied since only the constraints associate with the initial period ($t = 0$) leads to nonzero values (thus finite), and y is in l_∞ space. Finally, for hypothesis H5, let A_i be the set of nonzero columns for row i. Note that we have

$$\sum_{i=1}^{\infty} L_i(x, y_i) = \sum_{i=1}^{\infty} \sum_{j \in A_i} |a_{ij} x_j y_i| \leq \sum_{i=1}^{\infty} \sup_{i \in \mathbb{N}} |y_i| \sum_{j \in A_i} x_j = \sup_{i \in \mathbb{N}} |y_i| \sum_{i=1}^{\infty} \sum_{j \in A_i} x_j \leq 2|\bar{\mathcal{G}}| \sup_{i \in \mathbb{N}} |y_i| \sum_{j \in \mathbb{N}} |x_j| < \infty,$$

where the first inequality follows the fact $a_{ij} \in [0, 1]$ and Y is l_∞ space, the second inequality follows the fact that each primal variable x_j appears in two periods, so it shows up in at most $2|\bar{\mathcal{G}}|$ constraints in problem (2), and the last inequality follows from the fact that X is the l_1 space. □

A.2. Omitted Proofs from Section 3

We first show the primal problem in this section. We denote $f_{i,j}^t$ as the amount of players in demographic i that are assigned to match with players in demographic j in period t where $i, j \in \{1w, 1\ell, ..., Kw, K\ell\}$. The flow balancing constraints (FB) now become

$$\begin{aligned} \sum_j f_{i,j}^t &= s_i^t, \forall i \in \{1w, 1\ell, ..., Kw, K\ell\}, \\ \sum_i f_{i,j}^t &= s_j^t, \forall j \in \{1w, 1\ell, ..., Kw, K\ell\}, \\ f_{i,j}^t &= f_{j,i}^t, \forall i \neq j, \, i, j \in \{1w, 1\ell, ..., Kw, K\ell\}, \\ f_{i,j}^t &\geq 0, \forall i, j \in \{1w, 1\ell, ..., Kw, K\ell\}. \end{aligned} \tag{FB$_\text{K}$}$$

Similarly, the evolution of demographics in (ED) now becomes

$$
\begin{aligned}
s_{kw}^{t+1} &= \frac{1}{2}(f_{kw,kw}^t + f_{kw,k\ell}^t + f_{k\ell,kw}^t + f_{k\ell,k\ell}^t) + \sum_{j<k} W(k-j)(f_{kw,jw}^t + f_{kw,j\ell}^t + f_{k\ell,jw}^t + f_{k\ell,j\ell}^t),\\
s_{k\ell}^{t+1} &= \left(\frac{1}{2}+\mu\right)(f_{kw,kw}^t + f_{kw,k\ell}^t) + \frac{\rho}{2}(f_{k\ell,kw}^t + f_{k\ell,k\ell}^t)\\
&\quad + \sum_{j>k}\left[(W(j-k)+\mu)(f_{kw,jw}^t + f_{kw,j\ell}^t) + W(j-k)\rho(f_{k\ell,jw}^t + f_{k\ell,j\ell}^t)\right]\\
&\quad + \sum_{j<k}\left[(1-W(k-j)+\mu)(f_{kw,jw} + f_{kw,j\ell}) + (1-W(k-j))\rho(f_{k\ell,jw}^t + f_{k\ell,j\ell}^t)\right].
\end{aligned}
\tag{ED$_\mathrm{K}$}
$$

The evolution of demographics in (ED$_\mathrm{K}$) reflects that a player in skill level k wins a game with probability $W(k-j)$ when matched with another player who has lower skill level, i.e., $j<k$, and only players who just lost a game are subjected to churn with probability $1-\rho$.

With an initial state of demographics $\mathbf{s}^0 = (s_{1w}^0, s_{1\ell}^0, \ldots, s_{Kw}^0, s_{K\ell}^0)$, we define the matchmaker's problem as

$$
\begin{aligned}
V_K^*(\mathbf{s}^0) = \max_{\{f_{i,j}^t\}_{t=1}^\infty} \sum_{t=1}^{\infty} \sum_{i\in\{1w,1\ell,\ldots,Kw,K\ell\}} s_i^t \\
\text{s.t. (FB}_\mathrm{K}\text{) and (ED}_\mathrm{K}\text{)} \ \forall t=0,1,\ldots, \text{ and } i,j\in\{1w,1\ell,...,Kw,K\ell\},
\end{aligned}
\tag{P$_\mathrm{K}$}
$$

which is a infinite dimensional linear program.

Before proving Lemma 3 and Theorem 2, we first show the set $\mathcal{S}$, where the optimal policy remains simple and informative. Fix $W(i)=a\in(0.5,1]$, and the set $\mathcal{S}$ consists of states $\mathbf{s}^t$ that satisfy

$$s_{k\ell}^t \geq s_{(k-1)w}^t, \quad \forall k=2,...,K, \tag{a}$$

$$(a+\mu)s_{kw}^t \geq (1-a)s_{(k-1)w}^t + \frac{1}{2}s_{(k-1)\ell}^t + \left(a-\frac{1}{2}\right)s_{(k-2)w}^t, \quad \forall k=3,...,K-1, \tag{b}$$

$$s_{Kw}^t \geq 2as_{(K-1)w}^t, \tag{c}$$

$$s_{Kw}^t + s_{K\ell}^t \geq s_{(K-1)w}^t + 2as_{(K-1)\ell}^t, \tag{d}$$

$$\left(\frac{1}{2}+\mu\right)s_{Kw}^t \geq (1-a)s_{(K-1)w}^t + \frac{1}{2}s_{(K-1)\ell}^t + \left(a-\frac{1}{2}\right)s_{(K-2)w}^t, \tag{e}$$

$$(a+\mu)\,s_{2w}^t \geq (1-a)s_{1w}^t + \frac{1}{2}s_{1\ell}^t \tag{f}$$

$$s_{K\ell}^t \geq s_{(K-1)\ell}^t \geq ... \geq s_{2\ell}^t \geq s_{1\ell}^t, \tag{g}$$

$$s_{Kw}^t \geq s_{(K-1)w}^t \geq ... \geq s_{2w}^t \geq s_{1w}^t. \tag{h}$$

Proof of Lemma 3 We break down this proof into two steps. In *Step 1*, we check the primal feasibility under the proposed policy. To be more specific, we show that based on the proposed policy and the evolution in (ED$_\mathrm{K}$), then $\mathbf{s}^t$ always belong to set $\mathcal{S}$ if $\mathbf{s}^0\in\mathcal{S}$. In *Step 2*, we provide a set of dual variables that satisfy complementary slackness, and check dual feasibility. Then by strong duality (Lemma 2), the proposed policy is optimal. We also characterize the properties of dual variables.

Step 1. Based on the matching flows in Proposition 3 and the demographic evolution in ($\mathrm{ED_K}$), for any time period t, we have

$$\begin{aligned}
s_{Kw}^{t+1} &= \frac{1}{2}s_{Kw}^t + \frac{1}{2}s_{K\ell}^t + \left(a - \frac{1}{2}\right)s_{(K-1)w}^t,\\
s_{K\ell}^{t+1} &= \left(\frac{1}{2}+\mu\right)s_{Kw}^t + \frac{\rho}{2}s_{K\ell}^t + \left(\frac{1}{2}-a\right)\rho s_{(K-1)w}^t,\\
s_{kw}^{t+1} &= (1-a)s_{kw}^t + \left(a-\frac{1}{2}\right)s_{(k-1)w}^t + \frac{1}{2}s_{k\ell}^t, \quad \forall k = 2,..,K-1,\\
s_{k\ell}^{t+1} &= (\mu+a)s_{kw}^t + \frac{\rho}{2}s_{k\ell}^t + \left(\frac{1}{2}-a\right)\rho s_{(k-1)w}^t, \quad \forall k=2,..,K-1,\\
s_{1w}^{t+1} &= \frac{1}{2}s_{1\ell}^t + (1-a)s_{1w}^t,\\
s_{1\ell}^{t+1} &= (\mu+a)s_{1w}^t + \frac{\rho}{2}s_{1\ell}^t,
\end{aligned} \tag{EC.8}$$

as the one step state transition. We use an induction argument to show that condition (a), (b), (c), (d),(e), and (f) are satisfied over time. We conclude this step by checking the two monotonicity conditions (g) and (h), respectively.

Suppose condition (a) to (h) hold for period t. Now, consider period $t+1$.

To show condition (a), we first consider skill level K. We have that

$$\begin{aligned}
s_{K\ell}^{t+1} &= \left(\frac{1}{2}+\mu\right)s_{Kw}^t + \frac{\rho}{2}s_{K\ell}^t + \left(\frac{1}{2}-a\right)\rho s_{(K-1)w}^t\\
&= \left(\frac{1}{2}+\mu\right)s_{Kw}^t + \frac{1}{2}\rho(s_{K\ell}^t - s_{(K-1)w}^t) + \rho(1-a)s_{(K-1)w}\\
&\geq \left(\frac{1}{2}+\mu\right)s_{Kw}^t\\
&\geq (1-a)s_{(K-1)w}^t + \frac{1}{2}s_{(K-1)\ell}^t + \left(a-\frac{1}{2}\right)s_{(K-2)w}^t = s_{(K-1)}^{t+1},
\end{aligned}$$

where the first equality follows from (EC.8); the second equality follows simple algebra; the first inequality follows condition (a) in period t and the fact $a \leq 1$; the second inequality follows condition (e); and the final equality follows from (EC.8). For skill levels $k = 2, ..., K-1$, we have that

$$\begin{aligned}
s_{k\ell}^{t+1} &= (a+\mu)\, s_{kw}^t + \frac{\rho}{2}s_{k\ell}^t + \left(\frac{1}{2}-a\right)\rho s_{(k-1)w}^t\\
&= (a+\mu)\, s_{kw}^t + \frac{1}{2}\rho(s_{k\ell}^t - s_{(k-1)w}^t) + \rho(1-a)s_{(k-1)w}\\
&\geq (a+\mu)\, s_{kw}^t\\
&\geq (1-a)s_{(k-1)w}^t + \frac{1}{2}s_{(k-1)\ell}^t + \left(a-\frac{1}{2}\right)s_{(k-2)w}^t = s_{(k-1)}^{t+1},
\end{aligned}$$

where the first equality follows from (EC.8) and the last inequality follows from (b). Thus, condition (a) holds in period $t+1$.

Next, consider (b). For skill levels $k = 3, .., K-1$, we have

$$\begin{aligned}
(a+\mu)\, s_{kw}^{t+1} &= (a+\mu)\left[(1-a)s_{kw}^t + \frac{1}{2}s_{k\ell}^t + \left(a-\frac{1}{2}\right)s_{(k-1)w}^t\right]\\
&\geq (a+\mu)\left[(1-a)s_{kw}^t + a s_{(k-1)w}^t\right]
\end{aligned}$$

$$\begin{aligned}
&= (1-a)\,(a+\mu)\, s^t_{kw} + \left(a - \frac{1}{2}\right)(a+\mu) s^t_{(k-1)w} + \frac{1}{2}(a+\mu) s^t_{(k-1)w} \\
&\geq (1-a)\left[(1-a) s^t_{(k-1)w} + \frac{1}{2} s^t_{(k-1)\ell} + \left(a - \frac{1}{2}\right) s^t_{(k-2)w}\right] \\
&\quad + \left(a - \frac{1}{2}\right)\left[(1-a) s^t_{(k-2)w} + \frac{1}{2} s^t_{(k-2)\ell} + \left(a - \frac{1}{2}\right) s^t_{(k-3)w}\right] + \frac{1}{2}(a+\mu) s^t_{(k-1)w} \\
&= (1-a) s^{t+1}_{(k-1)w} + \frac{1}{2} s^t_{(k-1)\ell} + \left(a - \frac{1}{2}\right) s^t_{(k-2)w},
\end{aligned}$$

where the first equality follows (EC.8); the first inequality follows from condition (a); the second equality follows from simple algebra; the second inequality follows from (b) and (e) in period t; and the last equality follows from (EC.8). Thus, condition (b) holds in period $t+1$.

To show condition (c), consider skill level K at period $t+1$:

$$\begin{aligned}
s^{t+1}_{Kw} &= \frac{1}{2} s^t_{Kw} + \frac{1}{2} s^t_{K\ell} + \left(a - \frac{1}{2}\right) s^t_{(K-1)w}, \\
&\geq \frac{1}{2} s^t_{(K-1)w} + a s^t_{(K-1)\ell} + \left(a - \frac{1}{2}\right) s^t_{(K-1)w}, \\
&= 2a\left(\frac{1}{2} s^t_{(K-1)w} + \frac{1}{2} s_{(K-1)\ell}\right) \\
&\geq 2a\left((1-a) s^t_{(K-1)w} + \left(a - \frac{1}{2}\right) s^t_{(K-2)w} + \frac{1}{2} s_{(K-1)\ell}\right) \\
&= 2a s_{(K-1)w},
\end{aligned}$$

where the first equality follows from (EC.8); the first inequality follows from (d); the second equality follows from simple algebra; the second inequality follows from $s^t_{(K-2)w} \leq s^t_{(K-1)w}$, implied by condition (h); and the last equality follows from (EC.8).

Before showing condition (d) holds in period $t+1$, we first show the following inequality holds in period t:

$$(\mu+1) s^t_{Kw} + \frac{1+\rho}{2} s^t_{K\ell} \geq \left(1 + 2a\mu + \frac{2a-1}{2}\rho\right) s^t_{(K-1)w} + \left(\frac{1}{2} + a\rho\right) s^t_{(K-1)\ell}. \tag{EC.9}$$

Consider

$$\begin{aligned}
(\mu+1) s^t_{Kw} + \frac{1+\rho}{2} s^t_{K\ell} &= (\mu+1) s^t_{Kw} + \frac{1+\rho}{2} s^t_{K\ell} - \frac{\frac{1}{2} + a\rho}{2a}(s^t_{Kw} + s_{K\ell}) + \frac{\frac{1}{2} + a\rho}{2a}(s^t_{Kw} + s_{K\ell}) \\
&= \frac{2a(1+\mu) - a\rho - \frac{1}{2}}{2a} s^t_{Kw} + \frac{2a-1}{4a} s_{K\ell} + \frac{\frac{1}{2} + a\rho}{2a}(s^t_{Kw} + s_{K\ell}) \\
&\geq \frac{2a(1+\mu) - a\rho - \frac{1}{2}}{2a} s^t_{Kw} + \frac{2a-1}{4a} s_{K\ell} + \frac{\frac{1}{2} + a\rho}{2a} s^t_{(K-1)w} + \left(\frac{1}{2} + a\rho\right) s^t_{(K-1)\ell} \\
&\geq \left(2a(1+\mu) - a\rho - \frac{1}{4a}\right) s^t_{(K-1)w} + \frac{\frac{1}{2} + a\rho}{2a} s^t_{(K-1)w} + \left(\frac{1}{2} + a\rho\right) s^t_{(K-1)\ell} \\
&\geq \left(2a\mu - \frac{1}{4a} + 1 - \rho + a\rho\right) s^t_{(K-1)w} + \frac{\frac{1}{2} + a\rho}{2a} s^t_{(K-1)w} + \left(\frac{1}{2} + a\rho\right) s^t_{(K-1)\ell} \\
&= \left(1 + 2a\mu + \frac{2a-1}{2}\rho\right) s^t_{(K-1)w} + \left(\frac{1}{2} + a\rho\right) s^t_{(K-1)\ell},
\end{aligned}$$

where the first two equalities follow from simple algebra; the first inequality follows from condition (d) in period t; the second inequality follows from conditions (a) and (c); the third inequality follows from

$2a-a\rho \geq 1-\rho+a\rho$ since $(2a-1)(1-\rho)\geq 0$; and the final equality follows from simple algebra. Now, we consider condition (d) in period $t+1$:

$$\begin{aligned}
s^{t+1}_{Kw}+s^{t+1}_{K\ell} &= \frac{1}{2}s^t_{Kw}+\frac{1}{2}s^t_{K\ell}+\left(a-\frac{1}{2}\right)s_{(K-1)w}+\left(\mu+\frac{1}{2}\right)s^t_{Kw}+\frac{1}{2}\rho s^t_{K\ell}+\left(\frac{1}{2}-a\right)\rho s^t_{(K-1)w}\\
&=(\mu+1)s^t_{Kw}+\frac{1+\rho}{2}s^t_{K\ell}+\left(a-\frac{1}{2}\right)(1-\rho)s^t_{(K-1)w}\\
&\geq\left(2a\mu+a+\frac{1}{2}\right)s^t_{(K-1)w}+\left(\frac{1}{2}+a\rho\right)s^t_{(K-1)\ell}\\
&=(1-a)s^t_{(K-1)w}+\left(a-\frac{1}{2}\right)s^t_{(K-1)w}+\frac{1}{2}s^t_{(K-1)\ell}+2a\left[(\mu+a)s^t_{(K-1)w}+\frac{1}{2}\rho s^t_{(K-1)\ell}+\left(\frac{1}{2}-a\right)\rho s^t_{(K-1)w}\right]\\
&\geq(1-a)s^t_{(K-1)w}+\left(a-\frac{1}{2}\right)s^t_{(K-2)w}+\frac{1}{2}s^t_{(K-1)\ell}+2a\left[(\mu+a)s^t_{(K-1)w}+\frac{1}{2}\rho s^t_{(K-1)\ell}+\left(\frac{1}{2}-a\right)\rho s^t_{(K-2)w}\right]\\
&=s^{t+1}_{(K-1)w}+2as^{t+1}_{(K-1)\ell},
\end{aligned}$$

where the first and second equalities follows from (EC.8) and simple algebra, respectively; the first inequality follows from (EC.9); the third equality follows from simple algebra; the second inequality follows condition (h); and the last equality follows from (EC.8). Thus, condition (d) holds in period $t+1$.

Next, we consider condition (e). For skill level K, we have

$$\begin{aligned}
\left(\frac{1}{2}+\mu\right)s^{t+1}_{Kw} &= \left(\frac{1}{2}+\mu\right)\left[\frac{1}{2}s^t_{Kw}+\frac{1}{2}s^t_{K\ell}+\left(a-\frac{1}{2}\right)s^t_{(K-1)w}\right]\\
&\geq\left(\frac{1}{2}+\mu\right)\left[\frac{1}{2}s^t_{Kw}+as^t_{(K-1)w}\right]\\
&\geq(1-a)\left(\frac{1}{2}+\mu\right)s^t_{Kw}+a(a+\mu)s^t_{(K-1)w}\\
&\geq(1-a)\left[(1-a)s^t_{(K-1)w}+\frac{1}{2}s^t_{(K-1)\ell}+\left(a-\frac{1}{2}\right)s^t_{(K-2)w}\right]\\
&\quad+\left(a-\frac{1}{2}\right)\left[(1-a)s^t_{(K-2)w}+\frac{1}{2}s^t_{(K-2)\ell}+\left(a-\frac{1}{2}\right)s^t_{(K-3)w}\right]+\frac{1}{2}(a+\mu)s^t_{(K-1)w}\\
&=(1-a)s^{t+1}_{(K-1)w}+\frac{1}{2}s^t_{(K-1)\ell}+\left(a-\frac{1}{2}\right)s^t_{(K-2)w},
\end{aligned}$$

where the first equality follows from (EC.8); the first inequality follows from (a); the second inequality follows (c), which implies $(1/2+\mu)s^t_{Kw}\geq as^t_{(K-1)w}$; the third inequality follows from (b) and (e) in period t; and the last equality follows from (EC.8). Thus, condition (e) holds in period $t+1$.

Next, we show condition (f) holds in period $t+1$. We have

$$\begin{aligned}
(a+\mu)s^t_{2w} &= (a+\mu)\left[(1-a)s^t_{2w}+\frac{1}{2}s^t_{2\ell}+\left(a-\frac{1}{2}\right)s^t_{1w}\right]\\
&\geq(a+\mu)\left[(1-a)s^t_{2w}+\frac{2\mu+\rho}{2}s^t_{2\ell}+\left(a-\frac{1}{2}\right)s^t_{1w}\right]\\
&\geq(a+\mu)\left[(1-a)s^t_{2w}+\frac{\rho}{2}s^t_{1\ell}+\mu s^t_{2\ell}+\left(a-\frac{1}{2}\right)s^t_{1w}\right]\\
&\geq(a+\mu)\left[(1-a)s^t_{2w}+\mu s^t_{2\ell}+\left(a-\frac{1}{2}\right)s^t_{1w}\right]+\frac{\rho}{4}s^t_{1\ell}\\
&\geq(a+\mu)\left[(1-a)s^t_{2w}+\left(\mu+a-\frac{1}{2}\right)s^t_{1w}\right]+\frac{\rho}{4}s^t_{1\ell}
\end{aligned}$$

$$\begin{aligned}
&\geq (a+\mu)\left[(1-a)s_{2w}^t + \frac{1}{2}s_{1w}^t\right] + \frac{\rho}{4}s_{1\ell}^t \\
&\geq (1-a)\left[\frac{1}{2}s_{1\ell}^t + (1-a)s_{1w}^t\right] + \frac{1}{2}\left[(\mu+a)s_{1w}^t + \frac{\rho}{2}s_{1\ell}^t\right] = (1-a)s_{1w}^{t+1} + \frac{1}{2}s_{1\ell}^{t+1},
\end{aligned}$$

where the first equality follows from (EC.8); the first inequality follows from $\mu < (1-\rho)/2$; the second inequality follows from condition (g); the third inequality follows from $\mu + a \geq 1$; the fourth inequality follows from condition (a); the fifth inequality follows from $\mu + a \geq 1$; the last inequality follows from condition (f) in period t; and the last equality follows from (EC.8).

Finally, we show the two monotonicity conditions (g) and (h) hold in period $t+1$. For skill level K, we have

$$\begin{aligned}
s_{Kw}^{t+1} &= \frac{1}{2}s_{Kw}^t + \frac{1}{2}s_{K\ell}^t + \left(a - \frac{1}{2}\right)s_{(K-1)w}^t \\
&\geq \frac{1}{2}s_{(K-1)w}^t + \frac{1}{2}s_{(K-1)\ell}^t + \left(a - \frac{1}{2}\right)s_{(K-2)w}^t = s_{(K-1)w}^{t+1},
\end{aligned}$$

and

$$\begin{aligned}
s_{K\ell}^{t+1} &= \left(\frac{1}{2}+\mu\right)s_{Kw}^t + \frac{\rho}{2}s_{K\ell}^t + \left(\frac{1}{2} - a\right)\rho s_{(k-1)w}^t \\
&= \left(\frac{1}{2}+\mu\right)s_{Kw}^t + \frac{\rho}{2}\left(s_{Kw}^t - s_{(K-1)w}^t\right) + \rho(1-a)s_{(K-1)w}^t \\
&\geq \left(\frac{1}{2}+\mu\right)s_{(K-1)w}^t + \frac{\rho}{2}\left(s_{(K-1)w}^t - s_{(K-2)w}^t\right) + \rho(1-a)s_{(K-2)w}^t = s_{(K-1)\ell}^{t+1},
\end{aligned}$$

where the equalities follow from (EC.8) and inequalities follow from conditions (g) and (h), respectively. We can follow the same steps to show monotonicity for general skill levels $k = 3, \ldots, K-1$. That is, we have

$$\begin{aligned}
s_{kw}^{t+1} &= (1-a)s_{kw}^t + \frac{1}{2}s_{k\ell}^t + \left(a - \frac{1}{2}\right)s_{(k-1)w}^t \\
&\geq (1-a)s_{(k-1)w}^t + \frac{1}{2}s_{(k-1)\ell}^t + \left(a - \frac{1}{2}\right)s_{(k-2)w}^t = s_{(k-1)w}^{t+1},
\end{aligned}$$

and

$$\begin{aligned}
s_{k\ell}^{t+1} &= (a+\mu)\,s_{kw}^t + \frac{\rho}{2}s_{k\ell}^t + \left(\frac{1}{2} - a\right)\rho s_{(k-1)w}^t \\
&= (a+\mu)\,s_{kw}^t + \frac{\rho}{2}\left(s_{kw}^t - s_{(k-1)w}^t\right) + \rho(1-a)s_{(k-1)w}^t \\
&\geq (a+\mu)\,s_{(k-1)w}^t + \frac{\rho}{2}\left(s_{(k-1)w}^t - s_{(k-2)w}^t\right) + \rho(1-a)s_{(k-2)w}^t = s_{(k-1)\ell}^{t+1}.
\end{aligned}$$

Finally, consider skill levels 1 and 2. We have

$$s_{2w}^{t+1} = (1-a)s_{2w}^t + \frac{1}{2}s_{2\ell}^t + \left(a - \frac{1}{2}\right)s_{1w}^t \geq (1-a)s_{1w}^t + \frac{1}{2}s_{1\ell}^t = s_{1w}^{t+1},$$

and

$$s_{2\ell}^{t+1} = (a+\mu)\,s_{2w}^t + \frac{\rho}{2}s_{2\ell}^t + \left(\frac{1}{2} - a\right)\rho s_{1w}^t \geq (a+\mu)\,s_{1w}^t + \frac{\rho}{2}s_{1\ell}^t = s_{1\ell}^{t+1}.$$

This completes the proof of *Step 1*, primal feasibility, which shows that conditions (a) to (h) hold for all subsequent periods under the proposed policy.

Step 2. Next, we check the dual feasibility. Here, we propose a set of dual variables that that remain constant over time. By complementary slackness, the constraints corresponding to $f^t_{Kw,Kw}, f^t_{(k+1)l,kw}, f^t_{kl,kl}$ should be tight. That is,

$$\lambda^t_{Kw} - \frac{1}{2}\lambda^{t+1}_{Kw} - (\frac{1}{2}+\mu)\lambda^{t+1}_{Kl} = 1+\mu$$
$$\lambda^t_{kl} - \frac{1}{2}\lambda^{t+1}_{kw} - \frac{1}{2}\rho\lambda^{t+1}_{kl} = \frac{1}{2}+\frac{1}{2}\rho,\ k=1,\ldots,K$$
$$\lambda^t_{kl} + \lambda^t_{(k-1)w} - a\lambda^{t+1}_{kw} - (1-a)\rho\lambda^{t+1}_{kl} - (1-a)\lambda^{t+1}_{(k-1)w} - (a+\mu)\lambda^{t+1}_{(k-1)l} = 1+a+\mu+(1-a)\rho,\ k=2,\ldots,K.$$

Then, one can easily verify the following dual variables satisfy the complementary slackness constraints. For all t, we propose a set of dual variables that are constant over time, and they satisfy the complementary slackness constraints:

$$\lambda_{Kw} = \frac{5+6\mu-\rho}{1-2\mu-\rho}, \tag{EC.10}$$
$$\lambda_{kw} = \frac{1+\rho+3\mu+3a(1-\rho)}{a(1-\rho)-\mu} + \frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\lambda_{(k+1)w},\ k=1,\ldots,K-1, \tag{EC.11}$$
$$\lambda_{kl} = \frac{1+\rho}{2-\rho} + \frac{1}{2-\rho}\lambda_{kw},\ k=1,\ldots,K. \tag{EC.12}$$

Here, we identify a key property of the dual variables: when $a+\mu>1$, λ_{kw} grows to infinity in an exponential rate as k decreases (players with lower skills are more valuable). To see this, note that the iterative equation (EC.11) has a constant term and a linear term. The denominator $a(1-\rho)-\mu$ is positive, because $\mu < \frac{1}{2}(1-\rho) < a(1-\rho)$. The numerator of the constant term is obviously positive. For the linear term, the coefficient is greater than 1, because

$$\frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu} - 1 = \frac{a+\mu-1+\rho(1-a)}{a(1-\rho)-\mu} \geq \frac{\rho(1-a)}{a(1-\rho)-\mu} \geq 0,$$

where the first inequality is from $a+\mu\geq 1$. Hence, λ^t_{kw} is an linear function of $\lambda^t_{(k+1)w}$ with positive constant term, and the linear coefficient is greater than 1. So we have λ^t_{kw} goes exponentially as k decreases. Because λ^t_{kl} is a linear function of λ^t_{kw}, it also grows to infinity in an exponential rate. A special case is when $a+\mu=1$, $\rho=0$, in that case, the dual variables grow in a linear rate.

Next, we show that the proposed dual solution is indeed feasible. That is, the dual constraints that correspond to $f^t_{kw,kw}$ $(k<K)$, $f^t_{kw,jw}$ $(k>j)$, $f^t_{kw,jl}$ $(k>j)$, $f^t_{kl,jw}$ $(k>j+1)$, and $f^t_{kl,jl}$ $(k>j)$ are feasible. For brevity, we will omit the superscript t since the dual variable remain constant over time.

First, we check $f^t_{kw,kw}$, for $k<K$. The corresponding constraint is

$$\lambda_{kw} - \frac{1}{2}\lambda_{kw} - (\frac{1}{2}+\mu)\lambda_{kl} \geq 1+\mu$$
$$\iff \frac{5+6\mu-\rho}{2(-2+\rho)} + \frac{-1+2\mu+\rho}{2(-2+\rho)}\lambda_{kw} \geq 0,$$

where the second line follows from plugging in (EC.12). Note that the coefficient of λ_{kw} is greater than 0, because $\rho-2<0$, and $2\mu+\rho-1<0$ when $\mu<(1-\rho)/2$ by our assumption. Then we have

$$\frac{5+6\mu-\rho}{2(-2+\rho)} + \frac{-1+2\mu+\rho}{2(-2+\rho)}\lambda_{kw} \geq \frac{5+6\mu-\rho}{2(-2+\rho)} + \frac{-1+2\mu+\rho}{2(-2+\rho)}\lambda_{Kw} = 0,$$

where the inequality is because $\lambda_{kw} > \lambda_{Kw}$ for any $k<K$, and the equality is by plugging in (EC.10).

Second, we check $f^t_{kw,jw}(k>j)$. The corresponding constraint is

$$\begin{aligned} &\lambda_{kw}+\lambda_{jw}-a\lambda_{kw}-(1-a+\mu)\lambda_{kl}-(1-a)\lambda_{jw}-(a+\mu)\lambda_{jl}\geq 2+2\mu \\ \iff &\frac{1-\mu-\rho-a(1-\rho)}{2-\rho}\lambda_{kw}+\frac{a(1-\rho)-\mu}{2-\rho}\lambda_{jw}-\frac{5+6\mu-\rho}{2-\rho}\geq 0, \end{aligned} \tag{EC.13}$$

where the second line is by replacing λ_{kl}, λ_{jl} with (EC.12). When $j=k-1$, we can plug (EC.11) into (EC.13), which further reduce (EC.13) to

$$\frac{a(1-\rho)-\mu}{2-\rho}\lambda_{kw}-\frac{4+3\mu+3a(-1+\rho)-2\rho}{2-\rho}\geq\frac{a(1-\rho)-\mu}{2-\rho}\lambda_{Kw}-\frac{4+3\mu+3a(-1+\rho)-2\rho}{2-\rho}=\frac{2(2a-1)(1-\rho)}{1-2\mu-\rho}\geq 0.$$

The first inequality is because the left-hand-side is an increasing function of λ_{kw}. To see this, note that $a(1-\rho)>\frac{1}{2}(1-\rho)>\mu$ and $2-\rho>0$. Thus, we only need to check the smallest λ_{kw} when $k=K$. The equality is from plugging in (EC.10). The last inequality is because $2a-1>0$, $1-\rho>0$, and $1-2\mu-\rho>0$. Thus, for any $j=k-1$, we have verified the dual feasibility. For $j<k-1$, note that the left-hand-side of (EC.13) is an increasing function of λ_{jw} because $a(1-\rho)-\mu>\frac{1}{2}(1-\rho)-\mu>0$. Hence, when we decrease j, λ_{jw} increases, the left-hand-side of (EC.13) increases, so the inequality still holds.

Third, we check $f^t_{kw,jl}$, for $k>j$. The corresponding constraints is

$$\begin{aligned} &\lambda_{kw}+\lambda_{jl}-a\lambda_{kw}-(1-a+\mu)\lambda_{kl}-(1-a)\lambda_{jw}-a\rho\lambda_{jl}\geq 2+\mu+a\rho-a \\ \iff &\frac{1-\mu-\rho-a(1-\rho)}{2-\rho}\lambda_{kw}+\frac{(2a-1)(1-\rho)}{2-\rho}\lambda_{jw}-\frac{4+3\mu-3a(1-\rho)-2\rho}{2-\rho}\geq 0., \end{aligned} \tag{EC.14}$$

where the second line is by replacing λ_{kl}, λ_{jl} with (EC.12). When $j=k-1$, we can plug (EC.11) into (EC.14), which further reduce (EC.14) to

$$\begin{aligned} &\frac{\mu^2-\mu(1-\rho)+(1-\rho)^2(3a^2-3a+1)}{(2-\rho)(a(1-\rho)-\mu)}\lambda_{kw}+\frac{-1+\mu+3\mu^2+5a(-1+\rho)+9a^2(-1+\rho)^2+\mu\rho-7a(-1+\rho)\rho+\rho^2}{(2-\rho)(a(1-\rho)-\mu)} \\ \geq &\frac{\mu^2-\mu(1-\rho)+(1-\rho)^2(3a^2-3a+1)}{(2-\rho)(a(1-\rho)-\mu)}\lambda_{Kw}+\frac{-1+\mu+3\mu^2+5a(-1+\rho)+9a^2(-1+\rho)^2+\mu\rho-7a(-1+\rho)\rho+\rho^2}{(2-\rho)(a(1-\rho)-\mu)} \\ = &\frac{2(2a-1)(1-\rho)((3a-1)(1-\rho)-\mu)}{(1-2\mu-\rho)(a(1-\rho)-\mu)}\geq 0. \end{aligned}$$

The first inequality is because the first line is an increasing function of λ_{kw}. To see this, we show that the coefficient of λ_{kw} is non-negative. First, the denominator is positive, because $2-\rho>0$ and $\mu<(1-\rho)/2<a(1-\rho)$. Second, the numerator is positive, because $\mu^2-\mu(1-\rho)>-0.25(1-\rho)^2$ for $\rho\in(0,(1-\rho)/2)$, and $(1-\rho)^2(3a^2-3a+1)>0.25(1-\rho)^2$ for $a\in(0.5,1]$. Thus, we only need to check the smallest λ_{kw}, which is λ_{Kw}. The third line is by plugging in (EC.10). To see that it is indeed non-negative, note that $1-2\mu-\rho$, $a(1-\rho)-\mu$, $2a-1$, $1-\rho$ are obviously positive by our assumption, and $(3a-1)(1-\rho)-\mu>(3a-1)(1-\rho)-(1-\rho)/2=(3a-\frac{3}{2})(1-\rho)>0$ because $\mu<(1-\rho)/2$. Thus, for any $j=k-1$, (EC.14) holds. For $j<k-1$, note that the left-hand-side of (EC.14) is an increasing function of λ_{jw}. As j decreases, λ_{jw} increases, and the inequality still holds.

Fourth, we check $f^t_{kl,jw}$, for $k>j+1$. The corresponding constraint is

$$\begin{aligned} &\lambda^t_{kl}+\lambda^t_{(k-1)w}-a\lambda^{t+1}_{kw}-(1-a)\rho\lambda^{t+1}_{kl}-(1-a)\lambda^{t+1}_{(k-1)w}-(a+\mu)\lambda^{t+1}_{(k-1)l}\geq 1+a+\mu+(1-a)\rho \\ \iff &\frac{1-\rho-2a(1-\rho)}{2-\rho}\lambda_{kw}+\frac{a(1-\rho)-\mu}{2-\rho}\lambda_{jw}-\frac{1+3\mu-3a(-1+\rho)+\rho}{2-\rho}\geq 0, \end{aligned} \tag{EC.15}$$

where (EC.15) is by replacing λ_{kl}, λ_{jl} with (EC.12). Note that (EC.15) equals to 0 when $j = k-1$. When $j < k-1$, λ_{jw} increases and j decreases, and the left-hand-side of (EC.15) increases because the coefficient of λ_{jw} is positive. Thus, (EC.15) is always positive.

Finally, we check $f^t_{kl,jl}$, for $k > j$. The corresponding constraint is

$$\begin{aligned} &\lambda_{kl} + \lambda_{jl} - a\lambda_{kw} - (1-a)\rho\lambda_{kl} - (1-a)\lambda_{kw} - a\rho\lambda_{jl} \geq 1-\rho \\ \iff &\frac{1-a\rho}{2-\rho}(\lambda_{jw} - \lambda_{kw}) \geq 0, \end{aligned}$$

where the second line is by replacing λ_{kl} and λ_{jl} with (EC.12), and it is naturally positive because $\lambda_{jw} > \lambda_{kw}$ for any $k > j$.

We have prove that the proposed primal and dual solutions are indeed feasible and satisfy complementary slackness, and are hence optimal. Given (EC.10)-(EC.12), the general formula for the shadow price is

$$\lambda_{Kw} = \frac{5+6\mu-\rho}{1-2\mu-\rho},$$
$$\lambda_{Kl} = \frac{3+2\mu+\rho}{1-2\mu-\rho},$$

$$\lambda_{kw} = \frac{1+\rho+3\mu+3a(1-\rho)}{a(1-\rho)-\mu}\sum_{i=1}^{K-k-1}\left(\frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\right)^i + \left(\frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\right)^{K-k}\frac{5+6\mu-\rho}{1-2\mu-\rho},\ k = 1,\ldots,K-1. \tag{EC.16}$$

$$\lambda_{kl} = \frac{1+\rho}{2-\rho} + \frac{1}{2-\rho}\left(\frac{1+\rho+3\mu+3a(1-\rho)}{a(1-\rho)-\mu}\sum_{i=1}^{K-k-1}\left(\frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\right)^i + \left(\frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}\right)^{K-k}\frac{5+6\mu-\rho}{1-2\mu-\rho}\right),$$
$$k = 1,\ldots,K-1.$$

Proof of Theorem 2. In *Step 1*, we use a model that satisfies the assumptions in Lemma 3, and then generalize in *Step 2* and *Step 3*.

Step 1. Suppose the winrate function is $W(i) = a$, $a+\mu > 1$, $\mu < (1-\rho)/2$, which satisfies the assumptions in Lemma 3. Then we know the optimal engagement for any initial state within $\mathcal{S}$, because we already know the shadow price of each type of players. Now consider an initial state $\underline{\mathbf{s}}$ such that $\underline{s}_{Kw} = \underline{s}_{Kl} = 2$, and $\underline{s}_{kw} = \underline{s}_{kl} = 1$ for $k = 1,\ldots,K-1$. It is easy to check that $\underline{\mathbf{s}}$ belongs to $\mathcal{S}$. The maximum of $V^*(\mathbf{s})/V^{SBMM}(\mathbf{s})$ is at least $V^*(\underline{\mathbf{s}})/V^{SBMM}(\underline{\mathbf{s}}) := R^1_K$. The shadow price of SBMM is λ_{Kw} for any kw and λ_{Kl} for any kl (computed from Theorem 1). For ease of notation, let $C = \frac{1+\rho+3\mu+3a(1-\rho)}{a(1-\rho)-\mu}$, and $B = \frac{(2a-1)(1-\rho)}{a(1-\rho)-\mu}$.

Then R^1_K can be expressed as

$$\begin{aligned} R^1_K &= \frac{\sum_{i=\{1w,1l,\ldots,Kw,Kl\}}\lambda_i s_i}{(K+1)(\lambda_{Kw}+\lambda_{Kl})} \\ &= \frac{\lambda_{Kw}+\lambda_{Kl}+\sum_{k=1}^K(\lambda_{kw}+\lambda_{kl})}{\lambda_{Kw}+\lambda_{Kl}+K(\lambda_{Kw}+\lambda_{Kl})} \\ &= \frac{\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw}+\sum_{k=1}^K(\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{kw})}{\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw}+K(\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw})} \\ &= \frac{\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw}+K\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\sum_{k=1}^K(C\sum_{i=0}^{K-k-1}B^i+B^{K-k}\lambda_{Kw})}{\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw}+K(\frac{1+\rho}{2-\rho}+\frac{3-\rho}{2-\rho}\lambda_{Kw})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(K+1)\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw} + \frac{3-\rho}{2-\rho}\sum_{k=1}^{K}(C\frac{B^{K-k}-1}{B-1} + B^{K-k}\lambda_{Kw})}{(K+1)(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw})} \\
&= \frac{(K+1)\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw} + \frac{3-\rho}{2-\rho}(\frac{C(B^K-1)}{(B-1)^2} - K\frac{C}{B-1} + \frac{B^K-1}{B-1}\lambda_{Kw})}{(K+1)(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw})} \\
&= \frac{(K+1)\frac{1+\rho}{2-\rho} - \frac{3-\rho}{2-\rho}K\frac{C}{B-1} + \frac{3-\rho}{2-\rho}\lambda_{Kw} + \frac{3-\rho}{2-\rho}(\frac{C(B^K-1)}{(B-1)^2} + \frac{B^K-1}{B-1}\lambda_{Kw})}{(K+1)(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw})},
\end{aligned}
$$

where the first equality follows from writing the objective values using shadow prices, the second line is by $\lambda_{kl} = \frac{1+\rho}{2-\rho} + \frac{1}{2-\rho}\lambda_{kw}$, and the third line is by (EC.16).

Step 2. Now consider a model with $W(1) = a$, but there is no restriction for $W(i)$ when $i > 1$. Still, we assume that $a + \mu > 1$, $\mu < (1-\rho)/2$. Then, for any $\mathbf{s} \in \mathcal{S}$, we can follow the proposed policy in Lemma 3 to achieve exactly the same engagement as in Step 1, because we only use matching pairs between neighbor levels, and such a policy is still feasible. The engagement from SBMM and the propose policy is also the same. Thus, if we let $R_K^2 = \max V^*(\mathbf{s})/V^{SBMM}(\mathbf{s})$, then we have $R_K^2 \geq R_K^1$.

Step 3. Finally consider a general winrate function $W(i)$, and the only assumption is that $W(i) > 1-\mu$ for some $i < K$. Let $I(W) = \min\{i : W(i) + \mu > 1\}$. Suppose $K = JI(W)$ for some positive number J. We can construct the following initial state: Let $s_{kw} = s_{kl} = 1$ for $k = I(W)j$, $j = 1, \ldots, J-1$, and $s_{Kw} = s_{Kl} = 2$. That is, we only have players every $I(W)$ levels, so that the winrate between every skill levels satisfies the condition in Step 2. The ratio V^*/V^{SBMM} is at least R_J^2, and we have $R_K^3 \geq R_J^2 \geq R_J^1$.

Hence, for any model in Step 3, the lower bound is given by

$$
\underline{R}_K = \frac{\left(\lfloor\frac{K}{I(W)}\rfloor + 1\right)\frac{1+\rho}{2-\rho} - \frac{3-\rho}{2-\rho}\lfloor\frac{K}{I(W)}\rfloor\frac{C}{B-1} + \frac{3-\rho}{2-\rho}\lambda_{Kw} + \frac{3-\rho}{2-\rho}\left(\frac{C(B^{\lfloor\frac{K}{I(W)}\rfloor}-1)}{(B-1)^2} + \frac{B^{\lfloor\frac{K}{I(W)}\rfloor}-1}{B-1}\lambda_{Kw}\right)}{\left(\lfloor\frac{K}{I(W)}\rfloor + 1\right)\left(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw}\right)},
$$

where $B = \frac{(2W(I(W))-1)(1-\rho)}{W(I(W))(1-\rho)-\mu}$. It is straightforward to see that for any W, μ, ρ, $\underline{R}_K = \Theta(\frac{B^{K/I(W)}}{K})$.

Proof of Corollary 1. When μ approaches 0, we have $W(I(W))$, B go to 1, and

$$
\begin{aligned}
\lim_{\mu\to 0}\underline{R}_K &= \frac{\left(\lfloor\frac{K}{I(W)}\rfloor + 1\right)\frac{1+\rho}{2-\rho} - \frac{3-\rho}{2-\rho}\lfloor\frac{K}{I(W)}\rfloor\frac{C}{B-1} + \frac{3-\rho}{2-\rho}\lambda_{Kw} + \frac{3-\rho}{2-\rho}\left(\frac{C(B^{\lfloor\frac{K}{I(W)}\rfloor}-1)}{(B-1)^2} + \frac{B^{\lfloor\frac{K}{I(W)}\rfloor}-1}{B-1}\lambda_{Kw}\right)}{\left(\lfloor\frac{K}{I(W)}\rfloor + 1\right)\left(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw}\right)} \\
&= \frac{(K+1)\left(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw}\right) + \frac{3-\rho}{2(2-\rho)}K(K-1)}{(K+1)\left(\frac{1+\rho}{2-\rho} + \frac{3-\rho}{2-\rho}\lambda_{Kw}\right)},
\end{aligned}
$$

where the equality follows

$$
\lim_{\mu\to 0}\frac{3-\rho}{2-\rho}\left(-\frac{KC}{B-1} + \frac{C(B^K-1)}{(B-1)^2}\right) = \frac{3-\rho}{2-\rho}\frac{CK(K-1)}{2}, \text{ and } \lim_{\mu\to 0}\frac{3-\rho}{2-\rho}\frac{C(B^K-1)}{(B-1)} = \frac{3-\rho}{2-\rho}CK,
$$

by L'Hôpital's rule. Thus, we have $\underline{R}_K = \Theta(K)$ when μ goes to 0.

A.3. Omitted Proofs from Section 4

Before proving results in Section 4, we first present the formulation of the stylized model being considered.

For convenience, we define $\mathcal{P} := \{1w, 1\ell, 2w, 2\ell\}$. Let $\mathbf{s}^t = (s_{1w}^t, s_{1\ell}^t, s_{2w}^t, s_{2\ell}^t)$ denote the size of the four demographics in period $t = 0, 1, 2, \ldots$. The aggregate transition matrix is given by

$$\bar{M}_{kk}=\begin{array}{c c} & \begin{array}{cc} kw & k\ell \end{array} \\ \begin{array}{c} kw \\ k\ell \end{array} & \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0 \end{pmatrix} \end{array}, \quad \bar{M}_{21}=\begin{array}{c c} & \begin{array}{cc} kw & k\ell \end{array} \\ \begin{array}{c} kw \\ k\ell \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \bar{M}_{12}=\begin{array}{c c} & \begin{array}{cc} kw & k\ell \end{array} \\ \begin{array}{c} kw \\ k\ell \end{array} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}.$$

Furthermore, denote $f^t_{i,j}$, as the amount of players in demographic i that are assigned to match with players in demographic j in period t, where $i,j\in\mathcal{P}$. The flow balance constraints (FB) now become

$$\begin{aligned} \sum_{j\in\mathcal{P}} f^t_{i,j} &= s^t_i,\ \forall i\in\mathcal{P},\\ \sum_{i\in\mathcal{P}} f^t_{i,j} &= s^t_j,\ \forall j\in\mathcal{P},\\ f^t_{i,j} &= f^t_{j,i},\ \forall i\neq j,\ i,j\in\mathcal{P}\\ f^t_{i,j} &\geq 0,\ \forall i\neq j,\ i,j\in\mathcal{P}. \end{aligned} \tag{FB$_\mathrm{S}$}$$

In any period $t=1,2,...$, the evolution of demographics (ED) now becomes

$$\begin{aligned} s^{t+1}_{2w} &= \frac{1}{2}(f^t_{2w,2w}+f^t_{2w,2\ell}+f^t_{2\ell,2w}+f^t_{2\ell,2\ell})+f^t_{2w,1w}+f^t_{2w,1\ell}+f^t_{2\ell,1w}+f^t_{2\ell,1\ell},\\ s^{t+1}_{2\ell} &= \frac{1}{2}(f^t_{2w,2w}+f^t_{2w,2\ell}),\\ s^{t+1}_{1w} &= \frac{1}{2}(f^t_{1w,1w}+f^t_{1w,1\ell}+f^t_{1\ell,1w}+f^t_{1\ell,1\ell}),\\ s^{t+1}_{1\ell} &= \frac{1}{2}(f^t_{1w,1w}+f^t_{1w,1\ell})+f^t_{1w,2w}+f^t_{1w,2\ell}. \end{aligned} \tag{ED$_\mathrm{S}$}$$

The evolution in (ED$_\mathrm{S}$) reflects that high-skilled players always beat low-skilled players and players from the same skill level have equal chances to win a match. Furthermore, since churn is guaranteed after two losses, only players in demographic 2ℓ and 1ℓ may potentially leave the system in the next period.

Consistent with our original model in Section 2, the matchmaker is interested in maximizing players' engagement by designing matching flows satisfying both (FB) and (ED) constraints. We shall focus on the case where there is no discount factor, i.e., $\gamma=1$. Thus, given the initial demographics $\mathbf{s}^0=(s^0_{1w},s^0_{1\ell},s^0_{2w},s^0_{2\ell})$, we define the matchmaker's problem as

$$\begin{aligned} V^*(\mathbf{s}^0) = &\max_{\{f^t_{i,j}\}^\infty_{t=0},\{s^t_i\}^\infty_{t=1}} \sum_{t=1}^\infty\sum_{i\in\mathcal{P}} s^t_i \\ &\text{s.t. (FB}_\mathrm{S}\text{) and (ED}_\mathrm{S}\text{)}\ \forall t=0,1,2,\dots,\ \text{and}\ i,j\in\mathcal{P}, \end{aligned} \tag{P}$$

which is an infinite dimensional linear program. Although the objective is an infinite sum, we note that Lemma 1 guarantees the finiteness of (P).

Next, we present a simplified formulation to the matchmaker's problem in (P). Recall that $\mathcal{P}:=\{2w,2\ell,1w,1\ell\}$. To begin, in any period $t=0,1,2,\dots$, matching flows $f^t_{2w,2\ell}$, $f^t_{1w,1\ell}$, $f^t_{2\ell,2w}$, and $f^t_{1\ell,1w}$ can be set to zero without loss of generality. Taking flows $f^t_{2w,2\ell}=f^t_{2\ell,2w}=a\in(0,\min\{s^t_{2w},s^t_{2\ell}\}]$ as an example, it can be represented by $f^t_{2w,2w}=a$ and $f^t_{2\ell,2\ell}=a$, since they induce the same evolution to players' demographics $\{s^{t+1}_{2w},s^{t+1}_{2\ell}\}$ in the next period. We also use the fact that $f^t_{i,j}=f^t_{j,i},\forall i,j\in\mathcal{P}$ to reduce the problem to 8 flow variables for each period, which is half of the original described in (P). Thus, we can rewrite the matchmaker's problem in (P) as

$$\max_{\{\mathbf{f}^t\}^\infty_{t=0}} \sum_{t=1}^\infty \left(s^t_{2w}+s^t_{2l}+s^t_{1w}+s^t_{1l}\right)$$

$$
\begin{aligned}
\text{s.t. } & s_{2w}^0 = f_{2w,2w}^0 + f_{2w,1w}^0 + f_{2w,1\ell}^0, \\
& s_{2\ell}^0 = f_{2\ell,2\ell}^0 + f_{2\ell,1w}^0 + f_{2\ell,1\ell}^0, \\
& s_{1w}^0 = f_{1w,1w}^0 + f_{2\ell,1w}^0 + f_{2w,1w}^0, \\
& s_{1\ell}^0 = f_{1\ell,1\ell}^0 + f_{2w,1\ell}^0 + f_{2\ell,1\ell}^0, \\
& s_{2w}^t = f_{2w,2w}^t + f_{2w,1w}^t + f_{2w,1\ell}^t, \ t = 1,2,..., \qquad \text{(FB)} \\
& s_{2\ell}^t = f_{2\ell,2\ell}^t + f_{2\ell,1w}^t + f_{2\ell,1\ell}^t, \ t = 1,2,..., \\
& s_{1w}^t = f_{1w,1w}^t + f_{2\ell,1w}^t + f_{2w,1w}^t, \ t = 1,2,..., \\
& s_{1\ell}^t = f_{1\ell,1\ell}^t + f_{2w,1\ell}^t + f_{2\ell,1\ell}^t, \ t = 1,2,..., \\
& s_{2w}^t = \frac{1}{2}\left(f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1}\right) + f_{2w,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2\ell,1\ell}^{t-1}, \ t = 1,2,..., \qquad \text{(ED)} \\
& s_{2\ell}^t = \frac{1}{2} f_{2w,2w}^{t-1}, \ t = 1,2,..., \\
& s_{1w}^t = \frac{1}{2}\left(f_{1w,1w}^{t-1} + f_{1\ell,1\ell}^{t-1}\right), \ t = 1,2,..., \\
& s_{1\ell}^t = \frac{1}{2} f_{1w,1w}^{t-1} + f_{2w,1w}^{t-1} + f_{2\ell,1w}^{t-1}, \ t = 1,2,..., \\
& f_{i,j}^t \geq 0, \ \forall i,j,t \in \mathcal{P}.
\end{aligned}
$$

By merging the flow balance and evolution of demographic constraints, we can further remove s_i^t for $t > 1$, with only 8 decision variables per period:

$$
\max_{\{\mathbf{f}^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(f_{2w,2w}^t + \frac{1}{2} f_{2\ell,2\ell}^t + f_{1w,1w}^t + \frac{1}{2} f_{1\ell,1\ell}^t + 2 f_{2w,1w}^t + f_{2\ell,1\ell}^t + f_{2w,1\ell}^t + 2 f_{2\ell,1w}^t \right) \qquad \text{(P')}
$$

s.t.

$$
\begin{aligned}
& s_{2w}^0 = f_{2w,2w}^0 + f_{2w,1w}^0 + f_{2w,1\ell}^0, \\
& s_{2\ell}^0 = f_{2\ell,2\ell}^0 + f_{2\ell,1w}^0 + f_{2\ell,1\ell}^0, \\
& s_{1w}^0 = f_{1w,1w}^0 + f_{2\ell,1w}^0 + f_{2w,1w}^0, \\
& s_{1\ell}^0 = f_{1\ell,1\ell}^0 + f_{2w,1\ell}^0 + f_{2\ell,1\ell}^0, \\
& f_{2w,2w}^t + f_{2w,1w}^t + f_{2w,1\ell}^t = \frac{1}{2}\left(f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1}\right) + f_{2w,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2\ell,1\ell}^{t-1}, \ t = 1,2,..., \\
& f_{2\ell,2\ell}^t + f_{2\ell,1w}^t + f_{2\ell,1\ell}^t = \frac{1}{2} f_{2w,2w}^{t-1}, \ t = 1,2,..., \\
& f_{1w,1w}^t + f_{2\ell,1w}^t + f_{2w,1w}^t = \frac{1}{2}\left(f_{1w,1w}^{t-1} + f_{1\ell,1\ell}^{t-1}\right), \ t = 1,2,..., \\
& f_{1\ell,1\ell}^t + f_{2w,1\ell}^t + f_{2\ell,1\ell}^t = \frac{1}{2} f_{1w,1w}^{t-1} + f_{2w,1w}^{t-1} + f_{2\ell,1w}^{t-1}, \ t = 1,2,..., \\
& f_{i,j}^t \geq 0, \ \forall i,j,t \in \mathcal{P}.
\end{aligned}
$$

Although we have already stripped the matching model to its simplest form, it still possesses intricate dynamics. While traditional literature on Markov decision processes usually considers steady-state solutions, our problem (P) cannot have a steady-state solution since the total engagement is finite. We thus consider a generalization of steady-state referred to as *decaying steady-state*. A policy admits a decaying steady state $\mathbf{s}^t$, if there exists some $c \in (0,1]$ such that under the given policy, we have $\mathbf{s}^{t+1} = c\mathbf{s}^t$. Unfortunately, Lemma EC.1 shows that no matching policy, besides SBMM, results in its demographics reaching a decaying steady state.

LEMMA EC.1 **(No Steady State Exists)**. *Consider a fixed time period t.*

(i) SBMM can induce a non-zero decaying steady-state, but only for $c=(1+\sqrt{5})/4$ and $\mathbf{s}^t$ a positive multiple of the vector $((1+\sqrt{5})/2,1,(1+\sqrt{5})/2,1)$.

(ii) For any matching policy that involves matching flows between high-skilled and low-skilled players, there is no non-zero decaying steady-state for any $c \in (0,1]$.

Proof of Lemma EC.1. We work with the alternative formulation in (P') which has 8 flow variables each period.

(i) Let $f^t=(f^t_{2w,2w},f^t_{2\ell,2\ell},f^t_{1w,1w},f^t_{1\ell,1\ell},f^t_{2w,1w},f^t_{2w,1\ell},f^t_{2\ell,1w},f^t_{2\ell,1\ell})$ be the flow vector at time t. With (FB_S) conditions, the state $\mathbf{s}^t$ can then be expressed as $B\mathbf{f}^t$, where

$$B=\begin{array}{c} \\ s^t_{2w} \\ s^t_{2\ell} \\ s^t_{1w} \\ s^t_{1\ell} \end{array}\begin{array}{c} \begin{array}{cccccccc} f^t_{2w,2w} & f^t_{2\ell,2\ell} & f^t_{1w,1w} & f^t_{1\ell,1\ell} & f^t_{2w,1w} & f^t_{2w,1\ell} & f^t_{2\ell,1w} & f^t_{2\ell,1\ell} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Similarly, using the (ED_S) conditions, the state $\mathbf{s}^{t+1}$ can be expressed as $A\mathbf{f}^t$, where

$$A=\begin{array}{c} \\ s^t_{2w} \\ s^t_{2\ell} \\ s^t_{1w} \\ s^t_{1\ell} \end{array}\begin{array}{c} \begin{array}{cccccccc} f^t_{2w,2w} & f^t_{2\ell,2\ell} & f^t_{1w,1w} & f^t_{1\ell,1\ell} & f^t_{2w,1w} & f^t_{2w,1\ell} & f^t_{2\ell,1w} & f^t_{2\ell,1\ell} \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

If there exists a decaying steady state for some $c \in [0,1]$, then there exists vector $\mathbf{f}^t \geq 0$ such that

$$A\mathbf{f}^t = cB\mathbf{f}^t \iff (A-cB)\mathbf{f}^t = 0.$$

Now we provide the null space of $A-cB$. When $c \neq (1+\sqrt{5})/4$, the null space is given by

$$\begin{pmatrix} 0.5c/g(c) & (-c^2+0.5c+0.5)/g(c) & -0.5c/g(c) & (-c^2+0.5c)/g(c) & 0 & 0 & 0 & 1 \\ 0.5c/g(c) & (-c^2+0.5c+0.5)/g(c) & (-c^2+0.5)/g(c) & (0.5c-0.5)/g(c) & 0 & 0 & 1 & 0 \\ (-c^2+c)/g(c) & (-0.5c+0.5)/g(c) & -0.5c/g(c) & (-c^2+0.5c)/g(c) & 0 & 1 & 0 & 0 \\ (-c^2+c)/g(c) & (-0.5c+0.5)/g(c) & (-c^2+0.5)/g(c) & (0.5c-0.5)/g(c) & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{EC.17}$$

where $g(c)=c^2-0.5c-0.25$. Note that $g(c)=0$ if $c=(1+\sqrt{5})/4$. In that case, the null space of $A-cB$ is given by

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{2}{-3+\sqrt{5}} & 0 & 0 & 1 \\ \frac{1+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{-5+\sqrt{5}}{3-\sqrt{5}} & 1 & 1 & 0 \\ 0 & 0 & \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If there exists a linear combination of rows in the null space resulting in all non-negative elements and at least one non-zero element, then we have found a valid flow vector $\mathbf{f}^t$ and thus a valid demographic $\mathbf{s}^t$ that decays steadily at a rate of c.

First, consider $c \in (\frac{1+\sqrt{5}}{4},1]$, which implies $g(c)>0$. Then observe that elements in third column of Eq. (EC.17), representing flow $f^t_{1w,1w}$ are all negative. Hence, for any linear combination of rows in Eq. (EC.17), as long as the flow $f_{1w,1w}$ is positive, at least one of the element representing flows $f^t_{2w,1w},f^t_{2w,1\ell},f^t_{2\ell,1w},f^t_{2\ell,1\ell}$ is negative. Thus, no steady state exists when $c \in (\frac{1+\sqrt{5}}{4},1]$.

Next, consider $c \in (0, \frac{1+\sqrt{5}}{4})$, which implies $g(c) < 0$. Note that elements in the second column of Eq. (EC.17), representing the flow $f^t_{2\ell,2\ell}$ are all positive. Hence, for any linear combination of rows in Eq. (EC.17), as long as the flow $f_{2\ell,2\ell}$ is positive, then at least one of the element representing flows $f^t_{2w,1w}, f^t_{2w,1\ell}, f^t_{2\ell,1w}, f^t_{2\ell,1\ell}$ is negative. Thus, no steady state exists when $c \in (0, \frac{1+\sqrt{5}}{4})$.

Finally, consider $c = (1+\sqrt{5})/4$. In this case, we can easily find a non-negative flow vector $\mathbf{f}^t$ by summing up the third and fourth row, which gives $\big((1+\sqrt{5})/2, 1, (1+\sqrt{5})/2, 1, 0, 0, 0, 0\big)$, representing SBMM, and any positive multiple of $\mathbf{s} = ((1+\sqrt{5})/2, 1, (1+\sqrt{5})/2, 1)$ can induce such flows. To see that no other policy can reach a steady state when $c = (1+\sqrt{5})/4$, note that we have to find a non-zero non-negative flow vector $\mathbf{f}^t$ with the use of the first two rows. For the first row, the sign of fifth entry is opposite with the eighth entry; for the second row, the sign of fifth entry is opposite with the sixth and seventh entry. Hence, there is no way to construct a feasible flow vector $\mathbf{f}^t$ with non-negative elements. Thus, no other policy besides SBMM can induce a decaying steady state when $c = (1+\sqrt{5})/4$. □

As we show later, SBMM is actually sub-optimal as long as there are positive amounts of players in both levels. On the other hand, searching and conducting analysis for steady states is doomed to fail. Hence, even under this simplest setting, finding the optimal matching policy is highly non-trivial and needs to take the dynamics of player demographics across time into consideration.

Before presenting the optimal matching policy that maximizes the matchmaker's problem in (2), we first consider a one-period matching which provides insights on the structure of the optimal matching flows. Suppose the matchmaker only needs to design the optimal matching flows for the initial period. That is, we can modify the matchmaker's problem in (2) to

$$\max_{f^0_{i,j}} \sum_i s^1_i, \quad \text{s.t. (FB}_\text{S}\text{) and (ED}_\text{S}\text{) when } t = 0. \tag{P$_1$}$$

In other words, by designing the matching flows in the initial period $t = 0$, the matchmaker wants to retain as many players in the next period $t = 1$ as possible. Lemma EC.2 summarizes the optimal matching flows in the one-period matching problem (P$_1$).

Lemma EC.2 **(Myopic Policy)**. *Consider the one-period matching problem in* (P$_1$) *with demographics state* $\mathbf{s}^t = (s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell})$. *The optimal matching policy maximizes flow between demographics* 2ℓ *and* $1w$. *The rest of the players can be matched arbitrarily as long as* $2w$ *and* 1ℓ *are not matched to each other.*

Proof of Lemma EC.2. For each flow that involves both high-skilled and low-skilled player, we can compare the outcome of one unit of such flow with SBMM flow that uses the same amount of players. A unit of $f_{2\ell,1w}$ uses the same amount of players as one unit of $f_{2\ell,2\ell}$ and $f_{1w,1w}$, but it leads to zero loss in the next period while SBMM loses a half unit of $2l$ players. One unit of $f_{2\ell,1\ell}$ or $f_{2w,1w}$ leads to the same losses in the next period compared to the SBMM flow of one unit of $f_{2\ell,2\ell}, f_{1\ell,1\ell}, f_{2w,2w}$, and $f_{1w,1w}$. Finally, one unit of $f_{2w,1l}$ leads to a one unit loss of 1ℓ players in the next period, while the SBMM flow of one unit of $f_{1\ell,1\ell}$ and $f_{2w,2w}$ only leads only leads to a half unit loss of 1ℓ players. Hence, to maximize the population in the next period, we should always maximize $f_{2\ell,1w}$ and set $f_{2w,1\ell} = 0$. For the rest of players, they can be matched arbitrarily as the outcome in the next period is the same as SBMM. □

Finally, denote λ_i^t for $i \in \mathcal{P}$ as the dual variables (shadow price) for each demographic in period $t = 0, 1, 2, ...$ Then we can write the dual problem of (P') as

$$\min_{\{\lambda^t\}} \sum_{i \in \mathcal{P}} s_i^0 \lambda_i^0 \tag{D'}$$

s.t.

$$\begin{aligned}
&1 \leq \lambda_{2w}^t - \frac{1}{2}\lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{2\ell}^{t+1},\ t = 0, 1, 2, ...,\\
&\frac{1}{2} \leq \lambda_{2\ell}^t - \frac{1}{2}\lambda_{2w}^{t+1},\ t = 0, 1, 2, ...,\\
&2 \leq \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1},\ t = 0, 1, 2, ...,\\
&\frac{1}{2} \leq \lambda_{1\ell}^t - \frac{1}{2}\lambda_{1w}^{t+1},\ t = 0, 1, 2, ...,\\
&2 \leq \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1},\ t = 0, 1, 2, ...,\\
&1 \leq \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1},\ t = 0, 1, 2, ...,\\
&1 \leq \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1},\ t = 0, 1, 2, ...,\\
&1 \leq \lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1},\ t = 0, 1, 2, ...
\end{aligned}$$

A.3.1. Proof of Theorem 3

Proof of Theorem 3. We solve the problem in (P') by considering its dual problem (D'). In each period, there are four constraints in the primal problem (P'). Thus, we assign dual variable λ_i^t, where $i \in \{1w, 1\ell, 2w, 2\ell\}$, to each constraint representing the evolution of a players' demographics group in the primal problem. We will fully characterize the transition of primal and dual sequences, and show optimality by checking primal/dual feasibility and complementary slackness.

We break down the rest of this proof into 7 steps. In each step, we analyze a scenario corresponding to a parameter regime, which is mutually exclusive to parameter regimes in other scenarios and collectively exhaustive. That is, in any period t, we have

- Scenario 1: $s_{2w}^t + s_{2\ell}^t \geq s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, and $s_{2w}^t \geq s_{1\ell}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,2\ell}^t = s_{2\ell}^t - s_{1w}^t$, $f_{1w,1w}^t = 0$, $f_{1\ell,1\ell}^t = s_{1\ell}^t$, and $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$;
- Scenario 2: $s_{2w}^t + s_{2\ell}^t \geq s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t < s_{1w}^t$, and $s_{2w}^t \geq s_{1\ell}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,2\ell}^t = 0$, $f_{1w,1w}^t = s_{1w}^t - s_{2\ell}^t$, $f_{1\ell,1\ell}^t = s_{1\ell}^t$, and $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{2\ell}^t$;
- Scenario 3: $s_{2w}^t + s_{2\ell}^t \geq s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, and $s_{2w}^t < s_{1\ell}^t$; The optimal matching flows are the same as those in Scenario 1;
- Scenario 4: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$ and $s_{2\ell}^t < s_{1w}^t$; The optimal matching flows are the same as those in Scenario 2;
- Scenario 5: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $s_{1\ell}^t \leq K_1 = \frac{18}{5}s_{2w}^t + \frac{9}{5}s_{2\ell}^t + \frac{3}{5}s_{1w}^t$; The optimal matching flows are the same as those in Scenario 1;
- Scenario 6: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $s_{1\ell}^t > K_2 = \frac{18}{5}s_{2w}^t + \frac{23}{5}s_{2\ell}^t - \frac{11}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$, $f_{2\ell,1\ell}^t = f_{1\ell,2\ell}^t = s_{2\ell}^t - s_{1w}^t$, and $f_{1\ell,1\ell}^t = s_{1\ell}^t - f_{2\ell,1\ell}^t$;

- Scenario 7: $s^t_{2w}+s^t_{2\ell}<s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, $s^t_{2w}<s^t_{1\ell}$, and $K_1<s^t_{1\ell}\le K_2$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$, $f^t_{2\ell,2\ell}=\frac{9}{7}s^t_{2w}+\frac{23}{14}s^t_{2\ell}-\frac{11}{14}s^t_{1w}-\frac{5}{14}s^t_{1\ell}$, $f^t_{2\ell,1\ell}=f^t_{1\ell,2\ell}=\frac{5}{14}s^t_{1\ell}-\frac{9}{7}s^t_{2w}-\frac{9}{14}s^t_{2\ell}-\frac{3}{14}s^t_{1w}$, and $f^t_{1\ell,1\ell}=s^t_{1\ell}-f^t_{2\ell,1\ell}$.

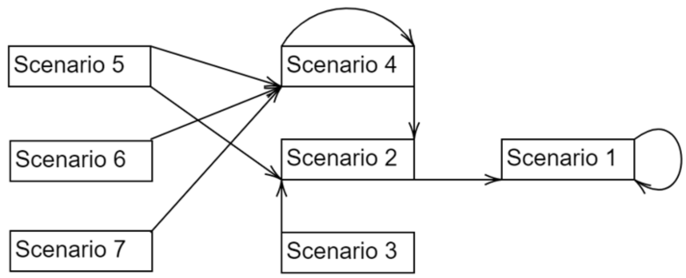

Figure EC.1 Evolution of Players' Demographics

Note that Scenarios 1 to 5 correspond to the third row of Table 1. In such cases, the optimal policy simply maximize the matching between 2ℓ and $1w$, and do skill-based matching for the remaining players. Scenario 6 to 7 corresponds to the first two rows in Table 1. The reason we classify the aforementioned scenarios in this way is because of how the scenarios evolve over time, as shown in Figure EC.1. For example, at any period t, if we are in Scenario 1, then in the next period $t+1$, one can verify that we always stay in the parameter regime of Scenario 1 under the proposed matching policy. The dual variables for any state in Scenario 1 is constant over time. Similarly, once we reach Scenario 2 at time t, we always transfer to Scenario 1 at $t+1$.

For the rest of this proof, we show that the state evolves as Fig. EC.1, and our proposed matching policy is optimal in each scenario and its subsequent scenarios as the state demographics evolve. To be more specific, our proof goes through each scenario and shows the corresponding optimal matching flows in Table 1 are optimal. Note that by construction, the proposed matching policy in Table 1 is primal feasible, i.e., satisfying all constraints in ($\mathrm{FB_S}$) and ($\mathrm{ED_S}$). We establish optimality by constructing dual variables for each scenario and show that both complementary slackness and dual feasibility conditions hold (the primal feasibility can be easily verified for our proposed policy). When possible, we write the corresponding dual variables with closed-form expressions. Our proof starts with Scenario 1 as it is at the end of all transitions (does not transition to other scenarios), then works backwards to parental scenarios. For each parental scenarios, we readily construct the dual variables for all subsequent scenarios, representing all the following time periods. For example, when the demographic state is in Scenario 2, we show it will transition to Scenario 1 in one period under the proposed matching policy. Then we only need to construct the dual variables for the current period and use the dual variables constructed in Scenario 1 for all subsequent periods, which we already have, to establish optimality.

Scenario 1: $s^t_{2w}+s^t_{2\ell}\ge s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, and $s^t_{2w}\ge s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=s^t_{2\ell}-s^t_{1w}$, $f^t_{1w,1w}=0$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$.

We first consider Scenario 1, and prove that it is the "end" of all scenarios in Figure EC.1. In other words, we shall show that once the state of players' demographics falls in Scenario 1, it will remain in Scenario 1.

To see this, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell},\}$ in Scenario 1, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w} = \frac{1}{2}s^t_{1l}$, $s^{t+1}_{1\ell} = s^t_{1w}$, $s^{t+1}_{2w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} + s^t_{1w})$, and $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w}$. Then $s^{t+1}_{2l} \geq s^{t+1}_{1w}$ because $s^t_{2w} \geq s^t_{1l}$, and $s^{t+1}_{2w} \geq s^{t+1}_{1l}$ because $s^{t+1}_{2w} - s^{t+1}_{1l} = \frac{1}{2}(s^t_{2w} + s^t_{2l} - s^t_{1w})$ and $s^t_{2l} - s^t_{1w} \geq 0$. Hence, $\mathbf{s}^{t+1}$ still belongs to Scenario 1.

Next, we show that the proposed policy in Theorem 3 is optimal for all subsequent periods once players' demographics satisfy Scenario 1. The optimal solution in Theorem 3 suggests that in Scenario 1, only 4 variables are non-zero while all other flows are zero in each period. Therefore, we have

$$\begin{aligned}
1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell}, \\
\frac{1}{2} &= \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w}, \\
2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},
\end{aligned} \tag{CS$_1$}$$

as the complementary conditions corresponding to primal non-zero variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$, respectively, and have

$$\begin{aligned}
2 &\leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\
1 &\leq \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\
1 &\leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\
1 &\leq \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}\lambda^{t+1}_{1\ell},
\end{aligned} \tag{DF$_1$}$$

as the dual feasibility conditions corresponding to variables, $f^t_{2w,1w}$, $f^t_{2w,1\ell}$, $f^t_{2\ell,1\ell}$, and $f^t_{1w,1w}$, that are zero in the primal problem, respectively.

The following dual solutions:

$$\lambda^t_{2w} = 5,\ \lambda^t_{2\ell} = 3,\ \lambda^t_{1w} = 9, \text{ and } \lambda^t_{1\ell} = 5, \quad \forall t, \tag{EC.18}$$

satisfies complementary slackness in (CS$_1$) and feasibility conditions in (DF$_1$). Thus, the proposed policy in Theorem 3 is optimal once the players' demographics fall in scenario 1.

Scenario 2: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} < s^t_{1w}$, and $s^t_{2w} \geq s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = 0$, $f^t_{1w,1w} = s^t_{1w} - s^t_{2\ell}$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{2\ell}$;

In the second step, we consider Scenario 2, which will transit to Scenario 1 after matching under the proposed policy as we have stated in Figure EC.1. To see that, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2l}\}$ in Scenario 2, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w} = \frac{1}{2}(s^t_{1w} + s^t_{1\ell} - s^t_{2\ell})$, $s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2w} = \frac{1}{2}s^t_{2w} + s^t_{2\ell}$, and $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w},$. Then $s^{t+1}_{2\ell} \geq s^{t+1}_{1w}$ because $s^{t+1}_{2\ell} - s^{t+1}_{1w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) \geq 0$. Also, $s^{t+1}_{2w} \geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w}) \geq s^t_{1\ell} \geq 0$. Hence, $\mathbf{s}^{t+1}$ belongs to Scenario 1.

Therefore, we only need to show that in any period t such that players' demographics satisfy Scenario 2, we can find solutions to dual variables, induced by the proposed policy in Theorem 3, which satisfy the complementary slackness conditions and dual feasibility conditions. Note that in period t under Scenario 2, the non-zero primal variables are $f^t_{2w,2w}$, $f^t_{2\ell,1w}$, $f^t_{1w,1w}$ and $f^t_{1\ell,1\ell}$. Therefore, by taking out the condition for $f^t_{2\ell,1\ell}$ and replacing it with the one for $f^t_{1w,1w}$, the complementary slackness conditions in (CS$_1$) change to

$$\begin{aligned}
1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t-1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\\
2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell},\\
1 &= \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}\lambda^{t+1}_{1\ell},\\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},
\end{aligned} \tag{CS$_2$}$$

in period t. Similarly, by taking out the condition for $f^t_{1w,1w}$ and replacing with the one for $f^t_{2\ell,1\ell}$, the dual feasibility conditions in (DF$_1$) turn into

$$\begin{aligned}
2 &\le \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell},\\
1 &\le \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w},\\
\frac{1}{2} &\le \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w},\\
1 &\le \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w},
\end{aligned} \tag{DF$_2$}$$

in period t. Note that the complementary slackness conditions and the dual feasibility conditions switch back to those in (CS$_1$) and (DF$_1$), starting period $t+1$, as player's demographics transit into Scenario 1. Therefore, we have

$$\lambda^s_{2w} = 5,\ \lambda^s_{2\ell} = 3,\ \lambda^s_{1w} = 9, \text{ and } \lambda^s_{1\ell} = 5, \quad \forall s = t+1, ..., T-1,$$

from (EC.18) and only need to find λ^t_i, where $i \in \{2w, 2\ell, 1w, 1\ell\}$, satisfy conditions in (CS$_2$) and (DF$_2$). Indeed, such solutions exist and one can verify that

$$\lambda^t_{2w} = 5,\ \lambda^t_{2\ell} = 4,\ \lambda^t_{1w} = 8, \text{ and } \lambda^t_{1\ell} = 5, \tag{EC.19}$$

are the desired solution. Therefore, the proof for Scenario 2 is completed.

Scenario 3: $s^t_{2w} + s^t_{2\ell} \ge s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \ge s^t_{1w}$, and $s^t_{2w} < s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1w}$, $f^t_{1w,1w} = 0$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$;

In the third step, we consider Scenario 3, which shall transit to Scenario 2 after matching is done under the proposed policy. To see this, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 1, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{2w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w}$, $s^{t+1}_{1w} = \frac{1}{2}s^t_{1\ell}$, and $s^{t+1}_{1\ell} = s^t_{1w}$. Then $s^{t+1}_{2\ell} < s^{t+1}_{1w}$ because $s^t_{2w} < s^t_{1\ell}$, and $s^{t+1}_{2w} \ge s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w})$ and $s^t_{2\ell} - s^t_{1w} \ge 0$. Finally, $s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell} = \frac{1}{2}(2s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) \ge 0$. Hence, $\mathbf{s}^{t+1}$ belongs to Scenario 2.

Thus, we only need to check we can find solutions to dual variables, which satisfy the complementary slackness conditions and dual feasibility conditions. According to the policy in Theorem 3, in period t under Scenario 3, primal variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$ are non-zero. Thus, in period t, the complementary slackness and dual feasibility conditions are

$$\begin{aligned}
1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\\
\frac{1}{2} &= \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w},\\
2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell},\\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},
\end{aligned} \tag{CS$_3$}$$

and

$$\begin{aligned}
&2 \le \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1},\\
&1 \le \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1},\\
&1 \le \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1},\\
&1 \le \lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1},
\end{aligned} \tag{DF$_3$}$$

respectively, where

$$\lambda_{2w}^{t+1} = 5,\ \lambda_{2\ell}^{t+1} = 4,\ \lambda_{1w}^{t+1} = 8, \text{ and } \lambda_{1\ell}^{t+1} = 5,$$

are from (EC.19), since players' demographic shall transit to Scenario 2 in the next period.

Finally, one can verify that

$$\lambda_{2w}^t = 5.5,\ \lambda_{2\ell}^t = 3,\ \lambda_{1w}^t = 9, \text{ and } \lambda_{1\ell}^t = 4.5, \tag{EC.20}$$

are the desired solutions we are looking for, satisfying (CS$_3$) and (DF$_3$). This completes the proof for Scenario 3.

Scenario 4: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$ and $s_{2\ell}^t < s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,2\ell}^t = 0$, $f_{1w,1w}^t = s_{1w}^t - s_{2\ell}^t$, $f_{1\ell,1\ell}^t = s_{1\ell}^t$, and $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{2\ell}^t$.

In the fourth step, we consider Scenario 4. For some $\mathbf{s}^t = \{s_{1w}^t, s_{1\ell}^t, s_{2w}^t, s_{2l}^t\}$ in Scenario 4, following the proposed policy, the state at $t+1$ is given by $s_{1w}^{t+1} = \frac{1}{2}(s_{1w}^t + s_{1l}^t - s_{2l}^t), s_{1\ell}^{t+1} = \frac{1}{2}(s_{2l}^t + s_{1w}^t)$, $s_{2w}^{t+1} = \frac{1}{2}s_{2w}^t + s_{2l}^t$, and $s_{2\ell}^{t+1} = \frac{1}{2}s_{2w}^t,$. Then $s_{2l}^{t+1} < s_{1w}^{t+1}$ because $s_{1w}^{t+1} - s_{2l}^{t+1} = \frac{1}{2}(s_{1w}^t + s_{1l}^t - s_{2w}^t - s_{2l}^t) > 0$. Hence, the state shall either transit to Scenario 2 or stay in Scenario 4 after a match. Suppose at time $t = s$, players' demographic is in Scenario 4. Denote $\tau := \min\{t \ge s \,|\, s_{2w}^t + s_{2\ell}^t \ge s_{1w}^t + s_{1\ell}^t\}$, representing the time period players' demographic transit to Scenario 2. First, we argue that $\tau < \infty$. Suppose otherwise, we have $\tau \to \infty$, which implies that the players' demographic stays in Scenario 4 forever. Note that as long as players' demographic belongs to Scenario 4, no high-skilled players shall depart from the matching system since there are always enough low-skilled players to be matched with. Thus, we have

$$\sum_{t=s}^{\infty} (s_{2w}^t + s_{2\ell}^t) = \sum_{t=s}^{\infty} (s_{2w}^s + s_{2\ell}^s) \to \infty,$$

which contradicts Lemma 1. Therefore, we have $\tau < \infty$.

Next, under Scenario 2, according to the optimal policy in Theorem 3, we have $\{f_{2w,2w}^t\}_{t=s}^{\tau-1}$, $\{f_{2\ell,1w}^t\}_{t=s}^{\tau-1}$, $\{f_{1w,1w}^t\}_{t=s}^{\tau-1}$, and $\{f_{1\ell,1\ell}^t\}_{t=s}^{\tau-1}$ as the sequences that contain non-zero primal variables, whereas each elements in sequences $\{f_{2w,1w}^t\}_{t=s}^{\tau-1}$, $\{f_{2w,1\ell}^t\}_{t=s}^{\tau-1}$, $\{f_{2\ell,2\ell}^t\}_{t=s}^{\tau-1}$, and $\{f_{2\ell,1\ell}^t\}_{t=s}^{\tau-1}$ are zero.

To proof the proposed policy is optimal in Scenario 4, we verify that there exists sequences $\{\lambda_{2w}^t\}_{t=s}^{\tau}$, $\{\lambda_{2\ell}^t\}_{t=s}^{\tau}$, $\{\lambda_{1w}^t\}_{t=s}^{\tau}$, and $\{\lambda_{1\ell}^t\}_{t=s}^{\tau}$, such that

$$\lambda_{2w}^{\tau} = 5,\ \lambda_{2\ell}^{\tau} = 4,\ \lambda_{1w}^{\tau} = 8, \text{ and } \lambda_{1\ell}^{\tau} = 5, \tag{EC.21}$$

and for all $s \le t \le \tau - 1$, we have

$$\begin{aligned} 1 &= \lambda_{2w}^t - \frac{1}{2}\lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{2\ell}^{t+1}, \\ 2 &= \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, \\ 1 &= \lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1}, \\ \frac{1}{2} &= \lambda_{1\ell}^t - \frac{1}{2}\lambda_{1w}^{t+1}, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \lambda_{2w}^t &= 1 + \frac{1}{2}\lambda_{2w}^{t+1} + \frac{1}{2}\lambda_{2\ell}^{t+1} \\ \lambda_{2\ell}^t &= 1 + \lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{1w}^{t+1} + \frac{1}{2}\lambda_{1\ell}^{t+1} \\ \lambda_{1w}^t &= 1 + \frac{1}{2}\lambda_{1w}^{t+1} + \frac{1}{2}\lambda_{1\ell}^{t+1} \\ \lambda_{1\ell}^t &= \frac{1}{2} + \frac{1}{2}\lambda_{1w}^{t+1} \end{aligned} \tag{CS$_4$}$$

as complementary slackness conditions and

$$\begin{aligned} 2 &\le \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1}, \\ 1 &\le \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \\ \frac{1}{2} &\le \lambda_{2\ell}^t - \frac{1}{2}\lambda_{2w}^{t+1}, \\ 1 &\le \lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1}, \end{aligned} \tag{DF$_4$}$$

as dual feasibility conditions.

For convenience, we also rearrange (CS$_4$) forwardly:

$$\lambda_{2w}^{t+1} = \lambda_{2\ell}^t - \lambda_{1w}^t + 2\lambda_{1\ell}^t - 1, \tag{EC.22}$$

$$\lambda_{2\ell}^{t+1} = 2\lambda_{2w}^t - \lambda_{2\ell}^t + \lambda_{1w}^t - 2\lambda_{1\ell}^t - 1, \tag{EC.23}$$

$$\lambda_{1w}^{t+1} = 2\lambda_{1\ell}^t - 1, \tag{EC.24}$$

$$\lambda_{1\ell}^{t+1} = 2\lambda_{1w}^t - 2\lambda_{1\ell}^t - 1. \tag{EC.25}$$

Equation (EC.24) follows the third equation of (CS$_4$). Equation (EC.25) follows the last equation in (CS$_4$) and (EC.24). Equation (EC.22) follows the second equation in (CS$_4$), (EC.24), and (EC.25). Finally, (EC.23) follows the first equation in (CS$_4$) and (EC.23).

Taking (EC.22)-(EC.25) into (DF$_4$), we can rewrite the dual feasibility conditions as

$$0 \le \lambda_{2w}^t - \lambda_{2\ell}^t, \tag{EC.26}$$

$$0 \le \lambda_{2w}^t - \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{1\ell}^t, \tag{EC.27}$$

$$0 \le \lambda_{2\ell}^t + \lambda_{1w}^t - 2\lambda_{1\ell}^t, \tag{EC.28}$$

$$0 \le \lambda_{1w}^t - \lambda_{1\ell}^t, \tag{EC.29}$$

Note that (EC.27) is implied by summing up (EC.26) and (EC.29). For the rest of this step, we show that the updated dual feasibility conditions (EC.26), (EC.28),(EC.29) are satisfied for all $s \le t \le \tau$. We will prove a stronger result: (EC.26), (EC.28),(EC.29) together with the following (EC.30) are satisfied for all $s \le t \le \tau$.

$$0 \le -\lambda_{2w}^t + \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{1\ell}^t, \text{ and } -1 \le \lambda_{2w}^t - \lambda_{1w}^t + \lambda_{1\ell}^t. \tag{EC.30}$$

We prove this result by backwards induction with the help of the following lemma.

Lemma EC.3. *Consider the dual sequences $\{\lambda_{2w}^t\}_{t=s}^{\tau}$, $\{\lambda_{2\ell}^t\}_{t=s}^{\tau}$, $\{\lambda_{1w}^t\}_{t=s}^{\tau}$, and $\{\lambda_{1\ell}^t\}_{t=s}^{\tau}$ that is defined by Eq.* (EC.21) *and Eq.* (CS_4)*. Then we have:*

(1) $\lambda_{1w}^t \geq 5$, $\lambda_{1l}^t \geq 3$, *for* $t = s, ..., \tau$.

(2) $\lambda_{1w}^t \leq \lambda_{1w}^{t+1}$, $\lambda_{1l}^t \leq \lambda_{1l}^{t+1}$, *for* $t = s, \ldots, \tau - 1$.

(3) $\lambda_{2w}^t \geq \lambda_{2w}^{t+1}$, $\lambda_{2l}^t \geq \lambda_{2l}^{t+1}$, *for* $t = s, \ldots, \tau - 1$.

For the base step, we first verify that the conditions are satisfied for both τ and $\tau - 1$. The solution for τ is given by $\lambda_{2w}^{\tau} = 5,\ \lambda_{2\ell}^{\tau} = 4,\ \lambda_{1w}^{\tau} = 8,$ and $\lambda_{1\ell}^{\tau} = 5$, and the solution for $\tau - 1$ is given by $\lambda_{2w}^{\tau-1} = 5.5,\ \lambda_{2\ell}^{\tau-1} = 4.5,\ \lambda_{1w}^{\tau-1} = 7.5,$ and $\lambda_{1\ell}^{\tau-1} = 4.5$, and one can easily verify that all the conditions are satisfied. As for the induction step, suppose inequalities in (EC.26), (EC.28), (EC.29) and (EC.30) hold for all periods $t = k+1, ..., \tau$. Consider period $k \leq \tau - 2$. For (EC.29), we have

$$\lambda_{1w}^k - \lambda_{1\ell}^k = (1 + \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) - (\frac{1}{2} + \frac{1}{2}\lambda_{1w}^{k+1}) = \frac{1}{2}(1 + \lambda_{1\ell}^{k+1}) \geq 0,$$

where the first equality uses (CS_4) to expand λ_{1w}^k and λ_{1l}^k, and the inequality follows Lemma EC.3(1).

Next, we check the inequalities in (EC.26) and (EC.28). Using (CS_4) to expand the two conditions, we have

$$\begin{aligned}
&0 \leq \lambda_{2w}^k - \lambda_{2\ell}^k \\
\Longleftrightarrow &0 \leq (1 + \frac{1}{2}\lambda_{2w}^{k+1} + \frac{1}{2}\lambda_{2\ell}^{k+1}) - (1 + \lambda_{2w}^{k+1} - \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) \\
\Longleftrightarrow &0 \leq -\lambda_{2w}^{k+1} + \lambda_{2\ell}^{k+1} + \lambda_{1w}^{k+1} - \lambda_{1\ell}^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
&0 \leq \lambda_{2\ell}^k + \lambda_{1w}^k - 2\lambda_{1\ell}^k \\
\Longleftrightarrow &0 \leq (1 + \lambda_{2w}^{k+1} - \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) + (1 + \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) - 2(\frac{1}{2} + \frac{1}{2}\lambda_{1w}^{k+1}) \\
\Longleftrightarrow &-1 \leq \lambda_{2w}^{k+1} - \lambda_{1w}^{k+1} + \lambda_{1\ell}^{k+1}.
\end{aligned}$$

Note that these two inequalities are exactly (EC.30) in period $k+1$, which hold by our assumption.

Finally, we check (EC.30) for period k. Plugging in (CS_4) twice for period k and $k+1$, we can express $-\lambda_{2w}^k + \lambda_{2\ell}^k + \lambda_{1w}^k - \lambda_{1\ell}^k$ with variables from $k+2$:

$$\begin{aligned}
-\lambda_{2w}^k + \lambda_{2\ell}^k + \lambda_{1w}^k - \lambda_{1\ell}^k &= \frac{1}{2}(1 + \lambda_{2w}^{k+1} - \lambda_{2\ell}^{k+1} - \lambda_{1w}^{k+1} + 2\lambda_{1\ell}^{k+1}) \\
&= \frac{1}{2} - \frac{1}{4}\lambda_{2w}^{k+2} + \frac{1}{4}\lambda_{2l}^{k+2} + \frac{1}{2}\lambda_{1w}^{k+2} - \frac{1}{2}\lambda_{1\ell}^{k+2} \\
&= \frac{1}{2} + \frac{1}{4}(-\lambda_{2w}^{k+2} + \lambda_{2\ell}^{k+2} + \lambda_{1w}^{k+2} - \lambda_{1\ell}^{k+2}) + \frac{1}{4}(\lambda_{1w}^{k+2} - \lambda_{1\ell}^{k+2}) \geq 0
\end{aligned}$$

where the the inequality follows (EC.30) and (EC.29) in period $k+2$. Similarly, we also have

$$\begin{aligned}
1 + \lambda_{2w}^k - \lambda_{1w}^k + \lambda_{1\ell}^k &= \frac{1}{2}\left(\lambda_{2w}^{k+1} + \lambda_{2\ell}^{k+1} - \lambda_{1\ell}^{k+1}\right) + \frac{3}{2} \\
&\geq \frac{1}{2}\left(\lambda_{2w}^{\tau-1} + \lambda_{2\ell}^{\tau-1} - \lambda_{1\ell}^{\tau-1}\right) + \frac{3}{2} = 4.25 > 0,
\end{aligned}$$

where the inequality uses the decreasing property of λ_{2w}^t, $\lambda_{2\ell}^t$ and the increasing property of $\lambda_{1\ell}^t$ from Lemma EC.3. Thus, all dual feasible conditions in (EC.26), (EC.28), (EC.29), and (EC.30) are satisfied, which imply that conditions in (DF$_4$) also hold for all $s \le t \le \tau$. This completes the proof for Scenario 4.

Scenario 5: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \ge s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $s_{1\ell}^t \le K_1 = \frac{18}{5}s_{2w}^t + \frac{9}{5}s_{2\ell}^t + \frac{3}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,2\ell}^t = s_{2\ell}^t - s_{1w}^t$, $f_{1\ell,1\ell}^t = s_{1\ell}^t$, and $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$;

According to the proposed matching flows above, the state of demographics either goes to Scenario 2 or Scenario 4 in period $t+1$. If it goes to Scenario 4, following the proved optimal solution in Scenario 4, players' demographic eventually evolves to Scenario 2 at period $\tau \le t+4$. That is, we summarize players' demographics in period $k = t+1, ..., \tau$ when $\tau = t+4$ in the next table. Note that in period $t+4$, we have

Table EC.1 Players' Demographics for Scenario 5

k	s_{2w}^k	$s_{2\ell}^k$	s_{1w}^k	$s_{1\ell}^k$
$t+1$	$\frac{1}{2}(s_{2w}^t + s_{2\ell}^t + s_{1w}^t)$	$\frac{1}{2}s_{2w}^t$	$\frac{1}{2}s_{1\ell}^t$	s_{1w}^t
$t+2$	$\frac{1}{4}(3s_{2w}^t + s_{2\ell}^t + s_{1w}^t)$	$\frac{1}{4}(s_{2w}^t + s_{2\ell}^t + s_{1w}^t)$	$\frac{1}{4}(-s_{2w}^t + 2s_{1w}^t + s_{1\ell}^t)$	$\frac{1}{4}(s_{1\ell}^t + s_{2w}^t)$
$t+3$	$\frac{1}{8}(5s_{2w}^t + 3s_{2\ell}^t + 3s_{1w}^t)$	$\frac{1}{8}(3s_{2w}^t + s_{2\ell}^t + s_{1w}^t)$	$\frac{1}{8}(-s_{2w}^t - s_{2\ell}^t + s_{1w}^t + 2s_{1\ell}^t)$	$\frac{1}{8}(s_{2\ell}^t + 3s_{1w}^t + s_{1\ell}^t)$
$t+4$	$\frac{1}{16}(11s_{2w}^t + 5s_{2\ell}^t + 5s_{1w}^t)$	$\frac{1}{16}(5s_{2w}^t + 3s_{2\ell}^t + 3s_{1w}^t)$	$\frac{1}{16}(-4s_{2w}^t - s_{2\ell}^t + 3s_{1w}^t + 3s_{1\ell}^t)$	$\frac{1}{8}(s_{2w}^t + s_{1w}^t + s_{1\ell}^t)$

$$s_{2w}^{t+4} + s_{2\ell}^{t+4} - s_{1w}^{t+4} - s_{1\ell}^{t+4} = \frac{1}{16}(18s_{2w}^t + 9s_{2\ell}^t + 3s_{1w}^t - 5s_{1\ell}^t) \ge 0,$$

where the inequality follows $s_{1\ell}^t \le \dfrac{18}{5}s_{2w}^t + \dfrac{9}{5}s_{2\ell}^t + \dfrac{3}{5}s_{1w}^t$. Therefore, we have $\tau \le t+4$.

In Table EC.2, we list the dual variables when $\tau = t+1, ..., t+4$ respectively, and one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasible conditions.

Scenario 6: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \ge s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $s_{1\ell}^t > K_2 = \frac{18}{5}s_{2w}^t + \frac{23}{5}s_{2\ell}^t - \frac{11}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$, $f_{2\ell,1\ell}^t = f_{1\ell,2\ell}^t = s_{2\ell}^t - s_{1w}^t$, and $f_{1\ell,1\ell}^t = s_{1\ell}^t - f_{2\ell,1\ell}^t$.

Following the policy above, the state of demographics in the next period is given by $s_{1w}^{t+1} = \frac{1}{2}(s_{1l}^t - (s_{2l}^t - s_{1w}^t))$, $s_{1l}^{t+1} = s_{1w}^t$, $s_{2w}^{t+1} = \frac{1}{2}s_{2w}^t + s_{2l}^t$, $s_{2l}^{t+1} = \frac{1}{2}s_{2w}^t$. This corresponds to Scenario 4 since

$$\begin{aligned} s_{2w}^{t+1} + s_{2\ell}^{t+1} - s_{1w}^{t+1} - s_{1\ell}^{t+1} &= s_{2w}^t + \frac{3}{2}s_{2\ell}^t - \frac{3}{2}s_{1w}^t - \frac{1}{2}s_{1\ell}^t \\ &< s_{2w}^t + \frac{3}{2}s_{2\ell}^t - \frac{3}{2}s_{1w}^t - \frac{1}{2}K_2 \\ &= -\frac{4}{5}s_{2w}^t - \frac{4}{5}s_{2l}^t - \frac{2}{5}s_{1w}^t < 0. \end{aligned}$$

Also, $s_{1w}^{t+1} - s_{2l}^{t+1} = \frac{1}{2}(s_{1w}^t + s_{1l}^t - s_{2w}^t - s_{2l}^t) < 0$.

Since $t+1$, we follow the proposed policy in Scenario 4 until we reach Scenario 2 at τ. One can verify that $\tau > t+5$, because following the proposed solution in Scenario 4, we have $s_{1w}^{t+5} = -\frac{1}{4}s_{2w}^t - \frac{1}{16}s_{2l}^t + \frac{3}{16}s_{1w}^t + \frac{3}{16}s_{1l}^t$, $s_{1l}^{t+5} = \frac{1}{8}s_{2w}^t + \frac{1}{8}s_{1w}^t + \frac{1}{8}s_{1l}^t$, $s_{2w}^{t+5} = \frac{11}{16}s_{2w}^t + \frac{5}{8}s_{2l}^t$, $s_{2l}^{t+5} = \frac{5}{16}s_{2w}^t + \frac{3}{8}s_{2l}^t$. To check it still belongs to Scenario 4,

$$\begin{aligned} s_{2w}^{t+5} + s_{2\ell}^{t+5} - s_{1w}^{t+5} - s_{1\ell}^{t+5} &= \frac{9}{8}s_{2w}^t + \frac{23}{16}s_{2l}^t - \frac{11}{16}s_{1w}^t - \frac{5}{16}s_{1l}^t \\ &< \frac{9}{8}s_{2w}^t + \frac{23}{16}s_{2l}^t - \frac{11}{16}s_{1w}^t - \frac{5}{16}K_2 = 0. \end{aligned}$$

Table EC.2 Dual Variables for Scenario 5

$\tau=t+1$	$k=t$	$k=t+1$			
λ^k_{2w}	5.5	5			
$\lambda^k_{2\ell}$	3	4			
λ^k_{1w}	9	8			
$\lambda^k_{1\ell}$	4.5	5			
$\tau=t+2$	$k=t$	$k=t+1$	$k=t+2$		
λ^k_{2w}	6	5.5	5		
$\lambda^k_{2\ell}$	3.25	4.5	4		
λ^k_{1w}	8.75	7.5	8		
$\lambda^k_{1\ell}$	4.25	4.5	5		
$\tau=t+3$	$k=t$	$k=t+1$	$k=t+2$	$k=t+3$	
λ^k_{2w}	6.5	6	5.5	5	
$\lambda^k_{2\ell}$	3.5	5	4.5	4	
λ^k_{1w}	8.75	7	7.5	8	
$\lambda^k_{1\ell}$	4	4.25	4.5	5	
$\tau=t+4$	$k=t$	$k=t+1$	$k=t+2$	$k=t+3$	$k=t+4$
λ^k_{2w}	7.0625	6.5	6	5.5	5
$\lambda^k_{2\ell}$	3.75	5.625	5	4.5	4
λ^k_{1w}	8.75	6.625	7	7.5	8
$\lambda^k_{1\ell}$	3.8125	4	4.25	4.5	5

For period $t+1$ and forward, we use the dual variables proposed in Scenario 4, 2, and 1. Hence, we only need to show that the proposed policy as well as the corresponding dual variables at period t satisfies complementary slackness and dual feasibility in order to establish optimality. By complementary slackness, we have

$$\lambda^t_{2w} = 1 + \frac{1}{2}\lambda^{t+1}_{2w} + \frac{1}{2}\lambda^{t+1}_{2\ell} \tag{EC.31}$$

$$\lambda^t_{2\ell} = \frac{1}{2} + \lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{1w}, \tag{EC.32}$$

$$\lambda^t_{1w} = \frac{3}{2} + \lambda^{t+1}_{1\ell} + \frac{1}{2}\lambda^{t+1}_{1w}, \tag{EC.33}$$

$$\lambda^t_{1\ell} = \frac{1}{2} + \frac{1}{2}\lambda^{t+1}_{1w}. \tag{EC.34}$$

Then, we need to validate the dual feasibility condition corresponding to $f^t_{2\ell,2\ell}, f^t_{1w,1w}, f^t_{2w,1w}, f^t_{2w,1\ell}$, which are given by:

$$\lambda^t_{2\ell} - 0.5\lambda^{t+1}_{2w} \geq 0.5, \tag{EC.35}$$

$$\lambda^t_{1w} - 0.5\lambda^{t+1}_{1w} - 0.5\lambda^{t+1}_{1\ell} \geq 1, \tag{EC.36}$$

$$\lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell} \geq 2, \tag{EC.37}$$

$$\lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w} \geq 1. \tag{EC.38}$$

Taking (EC.31)-(EC.34) into the above inequalities, it is equivalent to validate

$$\lambda_{2w}^{t+1} \geq \lambda_{1w}^{t+1}, \tag{EC.39}$$

$$1.5 + \lambda_{1\ell}^{t+1} \geq 1, \tag{EC.40}$$

$$0.5 + 0.5\lambda_{2\ell}^{t+1} + 0.5\lambda_{1w}^{t+1} \geq 0.5\lambda_{2w}^{t+1}, \tag{EC.41}$$

$$1.5 + 0.5\lambda_{2\ell}^{t+1} + 0.5\lambda_{1w}^{t+1} \geq 0.5\lambda_{2w}^{t+1}. \tag{EC.42}$$

Among them, (EC.40) is trivially true, because λ_{1l}^{t+1} is in Scenario 4 and is greater than 5 by Lemma EC.3. (EC.41) and (EC.42) are directly from (EC.30). Thus, we only need to validate (EC.39). Note that in Scenario 6, we have $\tau \geq t+5$. At $k = \tau - 4 \geq t+1$, we have

$$\lambda_{2w}^{\tau-4} = 7.0625,\ \lambda_{2\ell}^{\tau-4} = 6.1875,\ \lambda_{1w}^{\tau-4} = 6.3125, \text{ and } \lambda_{1\ell}^{\tau-4} = 3.8125, \tag{EC.43}$$

by using the complementary slackness conditions recursively. By Lemma EC.3, we know that $\lambda_{2w}^{t+1} \geq \lambda_{2w}^{\tau-4} \geq \lambda_{1w}^{\tau-4} \geq \lambda_{1w}^{t+1}$, which completes the proof.

Scenario 7: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $K_1 < s_{1\ell}^t \leq K_2$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$, $f_{2\ell,2\ell}^t = \frac{9}{7}s_{2w}^t + \frac{23}{14}s_{2\ell}^t - \frac{11}{14}s_{1w}^t - \frac{5}{14}s_{1\ell}^t$, $f_{2\ell,1\ell}^t = f_{1\ell,2\ell}^t = \frac{5}{14}s_{1\ell}^t - \frac{9}{7}s_{2w}^t - \frac{9}{14}s_{2\ell}^t - \frac{3}{14}s_{1w}^t$, and $f_{1\ell,1\ell}^t = s_{1\ell}^t - f_{2\ell,1\ell}^t$.

In this scenario, the state of demographics will transit to Scenario 4 in the second period. Further, one can verify that the system reaches Scenario 2 in period $k = t+4$, with $s_{2w}^{t+4} + s_{2\ell}^{t+4} = s_{1w}^{t+4} + s_{1\ell}^{t+4}$. Further, in period $k = t+5$, the system goes to Scenario 1 with $s_{2\ell}^{t+5} = s_{1w}^{t+5}$. Hence, in period $k = t+5$, we reach a degenerate case with $f_{2\ell,2\ell}^{t+5} = 0$. From the view of the simplex method, under the solution of Scenario 5.1, the reduced cost of $f_{2\ell,1\ell}^t$ is positive, so we take it in to the basic feasible solution, and move $f_{2\ell,2\ell}^{t+5}$ out of the basis. The positiveness of all the other flows remains. To see the solution is optimal, we list out the dual variables for the first $t+6$ periods in Table EC.3, and for periods $k > t+6$, we always have

$$\lambda_{2w}^k = 5,\ \lambda_{2\ell}^k = 3,\ \lambda_{1w}^k = 9, \text{ and } \lambda_{1\ell}^k = 5.$$

Then one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasi-

Table EC.3 Dual Variables for Scenario 7

k	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$
λ_{2w}^k	50/7	737/112	85/14	39/7	71/14	5	5
$\lambda_{2\ell}^k$	849/224	639/112	285/56	32/7	57/14	22/7	4
λ_{1w}^k	1963/224	737/112	389/56	52/7	111/14	62/7	8
$\lambda_{1\ell}^k$	849/224	445/112	59/14	125/28	69/14	5	5
Primal	Scenario 7	Scenario 4	Scenario 4	Scenario 4	Scenario 2	Scenario 1	Scenario 1

bility. □

Proof of Lemma EC.3 (1) We prove the statements by backwards induction from period τ.

First, we show that $\lambda_{1w}^t \geq 5$, $\lambda_{1l}^t \geq 3$ for all $t=s,...,\tau$. This is true for τ according to (EC.21), which completes the base step.

The induction hypothesis is that $\lambda_{1w}^k \geq 5$, $\lambda_{1l}^k \geq 3$ for all $k=t,...,\tau$. Then for period $t-1$, from Eq. (CS$_4$) we have

$$\lambda_{1w}^{t-1} = 1 + 0.5\lambda_{1w}^t + 0.5\lambda_{1\ell}^t \geq 1 + 0.5\cdot 5 + 0.5\cdot 3 = 5,$$

$$\lambda_{1\ell}^{t-1} = 0.5 + 0.5\lambda_{1w}^t \geq 0.5 + 0.5\cdot 5 = 3,$$

and the induction step is completed.

(2) We show that $\lambda_{1w}^t \leq \lambda_{1w}^{t+1}$, $\lambda_{1\ell}^t \leq \lambda_{1\ell}^{t+1}$, for $t=s,\ldots,\tau-1$. For the induction base step, we have $\lambda_{1w}^\tau = 7.5 \leq 8 = \lambda_{1w}^\tau$ and $\lambda_{1\ell}^t = 4.5 \leq 5 = \lambda_{1\ell}^\tau$. Our induction hypothesis it that for all $k=t,...,\tau-1$, we have $\lambda_{1w}^k \leq \lambda_{1w}^{k+1}$ and $\lambda_{1\ell}^k \leq \lambda_{1\ell}^{k+1}$. Now, consider period $t-1$. Take the difference between λ_{1w}^{t-1} and λ_{1w}^t, we have

$$\begin{aligned}
&\lambda_{1w}^{t-1} - \lambda_{1w}^t \\
=&1 + 0.5\lambda_{1w}^t + 0.5\lambda_{1\ell}^t - \lambda_{1w}^t \\
=&1 + 0.5\lambda_{1\ell}^k - 0.5\lambda_{1w}^k \\
=&0.75 - 0.25\lambda_{1\ell}^{k+1} \\
\leq&0.75 - 0.25\cdot 3 = 0,
\end{aligned} \tag{EC.44}$$

where the first equality follows Eq. (CS$_4$), the third equality follows (EC.25), and the last inequality follows the fact that $\lambda_{1\ell}^t \geq 3$. Next, we show that $\lambda_{1\ell}^{t-1} \leq \lambda_{1\ell}^t$. Note that from the last equation in (CS$_4$), we have $\lambda_{1\ell}^{t-1} = 0.5 + 0.5\lambda_{1w}^t$. Since $\lambda_{1w}^t \leq \lambda_{1w}^{t+1}$, we have $\lambda_{1\ell}^{t-1} \leq 0.5 + 0.5\lambda_{1w}^{t+1} = \lambda_{1\ell}^t$.

(3) Finally, we show that $\lambda_{2w}^t \geq \lambda_{2w}^{t+1}$, $\lambda_{2l}^t \geq \lambda_{2l}^{t+1}$. For the base step, we have $\lambda_{2w}^{\tau-1} = 5.5 \geq 5 = \lambda_{2w}^\tau$ and $\lambda_{2\ell}^{\tau-1} = 4.5 \geq 4 = \lambda_{2\ell}^\tau$. The induction hypothesis is that for all $k=t,...,\tau-1$, we have $\lambda_{2w}^k \geq \lambda_{2w}^{k+1}$ and $\lambda_{2\ell}^k \geq \lambda_{2\ell}^{k+1}$. Now, consider period $t-1$. We have

$$\lambda_{2w}^{t-1} = 1 + 0.5\lambda_{2w}^t + 0.5\lambda_{2\ell}^t \geq 1 + 0.5\lambda_{2w}^{t+1} + 0.5\lambda_{2\ell}^{t+1} = \lambda_{2w}^t,$$

where the equality on the two sides follows the first equation of (CS$_4$). Next, consider λ_{2l}^{t-1}. We have

$$\begin{aligned}
\lambda_{2\ell}^{t-1} &= 1 + \lambda_{2w}^t - 0.5\lambda_{1w}^t + 0.5\lambda_{1\ell}^t \\
&= 1 + \lambda_{2w}^t - 0.5(1 + 0.5\lambda_{1w}^{t+1} + 0.5\lambda_{1\ell}^{t+1}) + 0.5(0.5 + 0.5\lambda_{1w}^{t+1}) \\
&= 1 + \lambda_{2w}^t - 0.25 - 0.25\lambda_{1\ell}^{t+1} \\
&\geq 1 + \lambda_{2w}^{t+1} - 0.25 - 0.25\lambda_{1\ell}^{t+2} \\
&= \lambda_{2\ell}^t,
\end{aligned}$$

where the first equality follows the second equation of (CS$_4$), the second equality follows the third and fourth equations of (CS$_4$), the first inequality follows the fact that $\lambda_{1\ell}^{t+1} \leq \lambda_{1\ell}^{t+2}$ and $\lambda_{2w}^t \geq \lambda_{2w}^{t+1}$. Thus, the induction step is completed and this completes the proof. □

Proof of Theorem 4. (a) In order to prove the first statement, we consider the the linear program for the one-shot matching problem in (P_1) and use the same simplification tricks we used in Appendix A.3. Without loss of generality, we will set the engagement level of SBMM in the next period to be 1, which is equivalent to the constraint $1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}$. Thus, the following optimization problem selects the initial state of the demographics $\mathbf{s}$ to maximize the ratio of the optimal policy to SBMM for the one-period problem (we drop all the superscripts, representing time periods, since it is a one-shot problem):

$$\max_{\mathbf{f},\mathbf{s}} f_{2w,2w} + \frac{1}{2}f_{2\ell,2\ell} + f_{1w,1w} + \frac{1}{2}f_{1\ell,1\ell} + 2f_{2w,1w} + f_{2\ell,1\ell} + f_{2w,1\ell} + 2f_{2\ell,1w}$$

s.t.

$$\begin{aligned}
1 &= s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}\\
s_{2w} &= f_{2w,2w} + f_{2w,1w} + f_{2w,1\ell},\\
s_{2\ell} &= f_{2\ell,2\ell} + f_{2\ell,1w} + f_{2\ell,1\ell},\\
s_{1w} &= f_{1w,1w} + f_{2\ell,1w} + f_{2w,1w},\\
s_{1\ell} &= f_{1\ell,1\ell} + f_{2w,1\ell} + f_{2\ell,1\ell}.
\end{aligned}$$

which is the linear program for the one-shot matching problem in (P_1), maximizing over the initial demographics and matching flows, with an additional constraint. Without loss of generality, the additional constraint $1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}$, normalizes the value of one-shot SBMM to 1.

We verify that the optimal solution to the above optimization problem is $s_{2\ell} = s_{1w} = 2/3$, $f_{2\ell,1w} = 2/3$, and all other matching flows are 0. The objective value is 4/3, which is the desired ratio. Denote λ_0 as the dual variable correspond to the constraint normalizing the engagement for SBMM to be 1. We verify the proposed solution using complementary slackness conditions:

$$\begin{aligned}
0 &= \frac{1}{2}\lambda_0 - \lambda_{2\ell},\\
0 &= \lambda_0 - \lambda_{1w},\\
2 &= \lambda_{2\ell} + \lambda_{1w},
\end{aligned}$$

where the conditions correspond to primal non-zero variables $s_{2\ell}$, s_{1w}, and $f_{2\ell,1w}$, respectively. There is a unique solution of dual variables solving the complementary slackness conditions: $\lambda_0 = 4/3$, $\lambda_{2\ell} = 2/3$, and $\lambda_{1w} = 4/3$. To complete the proof, we need to check dual feasibility conditions:

$$\begin{aligned}
&0 \le \lambda_0 - \lambda_{2w},\ 0 \le \frac{1}{2}\lambda_0 - \lambda_{1\ell},\ 1 \le \lambda_{2w},\ \frac{1}{2} \le \lambda_{2\ell},\ 1 \le \lambda_{1w},\ \frac{1}{2} \le \lambda_{1\ell},\\
&2 \le \lambda_{2w} + \lambda_{1w},\ 1 \le \lambda_{2\ell} + \lambda_{1\ell},\ 1 \le \lambda_{2w} + \lambda_{1\ell},\ 1 \le \lambda_{2\ell} + \lambda_{1w},
\end{aligned}$$

representing zero state variables (s_{2w}, $s_{1\ell}$) and zero matching flows ($f_{2w,2w}$, $f_{2\ell,2\ell}$, $f_{1w,1w}$, $f_{1\ell,1\ell}$, $f_{2w,1w}$, $f_{2\ell,1\ell}$, $f_{2w,1\ell}$), respectively.

(b) Next, we turn our attention to the infinite horizon problem in (P'). Using a similar idea, we can solve an optimization problem to find the maximum ratio between the optimal matching policy and SBMM. Using Proposition 1, the value function of the baseline model under SBMM is

$$V^{\mathrm{SBMM}}(\mathbf{s}^0) = 5(s^t_{2w} + s^t_{1w}) + 3(s^t_{2\ell} + s^t_{1\ell}), \quad t = 1, 2, \dots, \tag{EC.45}$$

which we normalize to 1 without loss of generality. Thus, the following optimization problem selects the initial state of the demographics $\mathbf{s^0}$ to maximize the ratio of the optimal policy to SBMM for the infinite-horizon problem (we set $t=0$ without loss of generality):

$$\max_{\{\mathbf{f}^t\},\{\mathbf{s}^0\}} \sum_{t=0}^{\infty} \left(f^t_{2w,2w} + \frac{1}{2} f^t_{2\ell,2\ell} + f^t_{1w,1w} + \frac{1}{2} f^t_{1\ell,1\ell} + 2 f^t_{2w,1w} + f^t_{2\ell,1\ell} + f^t_{2w,1\ell} + 2 f^t_{2\ell,1w} \right) \tag{EC.46}$$

s.t.

$$\begin{aligned}
1 &= 5(s^0_{2w} + s^0_{1w}) + 3(s^0_{2\ell} + s^0_{1\ell}), \qquad \text{(EC.47)}\\
0 &= f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell} - s^0_{2w},\\
0 &= f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell} - s^0_{2\ell},\\
0 &= f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w} - s^0_{1w},\\
0 &= f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell} - s^0_{1\ell},
\end{aligned}$$

and for all $t=1,2,...,$

$$\begin{aligned}
& f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell} = \frac{1}{2}\left(f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell}\right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell},\\
& f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{2w,2w},\\
& f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} = \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right),\\
& f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w},\\
& f^t_{i,j} \geq 0,\ \forall i,j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{aligned}$$

Again, we use dual complementary slackness and feasibility conditions verifying the initial state $\mathbf{s}^0 = \{1/8, 0, 0, 1/8\}$, along with the optimal matching flows in Proposition 3, is the optimal solution to the above maximization problem. The objective value is $3/2$, which is the desired ratio.

With slight abuse of notation, denote the dual variable to the new constraint (EC.47) as λ_0. We show that under the optimal initial state where $s_{2\ell} = s_{1w} = 1/8$, we have $\lambda_0 = 3/2$, representing the maximum ratio. The proposed initial state is in Scenario 1. Based on the the transition in Fig. EC.1, we shall always stay in Scenario 1. We list the states in period 0, 1, and 2 in Table EC.4.

Note that in period 0, we reach a degenerate period with only one positive flow $f^0_{2l,1w}$, and in period 1 we reach a degenerate case with only two positive flows $f^1_{2w,2w}$ and $f^1_{1l,1l}$. Thus, for these two periods, we can use only part of the equations in (CS$_1$). To be specific, given our proposed primal solution, the complementary slackness equations are given by

$$\begin{aligned}
0 &= 3\lambda_0 - \lambda^0_{2\ell},\\
0 &= 5\lambda_0 - \lambda^0_{1w},\\
2 &= \lambda^0_{2\ell} + \lambda^0_{1w} - \lambda^1_{2w} - \lambda^1_{1\ell},\\
1 &= \lambda^1_{2w} - \frac{1}{2}\lambda^2_{2w} - \frac{1}{2}\lambda^2_{2\ell},\\
\frac{1}{2} &= \lambda^1_{1\ell} - \frac{1}{2}\lambda^2_{1w},\\
& (\text{CS}_1),\ t=2,\dots
\end{aligned}$$

and the dual feasibility constraints we need to check is

$$\begin{aligned}&0 \le 5\lambda_0 - \lambda_{2w}^0, \quad 0 \le 3\lambda_0 - \lambda_{1w}^0, \quad 1 \le \lambda_{2w}^0 - \frac{1}{2}\lambda_{2w}^1 - \frac{1}{2}\lambda_{2\ell}^1,\\ &\frac{1}{2} \le \lambda_{2\ell}^0 - \frac{1}{2}\lambda_{2w}^1, \quad \frac{1}{2} \le \lambda_{1\ell}^0 - \frac{1}{2}\lambda_{1w}^1, \quad \frac{1}{2} \le \lambda_{1\ell}^1 - \frac{1}{2}\lambda_{1w}^2\\ &2 \le \lambda_{2\ell}^1 + \lambda_{1w}^1 - \lambda_{2w}^2 - \lambda_{1\ell}^2, \quad (\mathrm{DF}_1) \text{ for } t = 0, \ldots\end{aligned}$$

We then list out the dual variables in period 0, 1, 2 in Table EC.4. For $t > 2$, we use $\lambda_{2w}^0 = 5$ $\lambda_{2\ell}^0 = 3$, $\lambda_{1w} = 9$, $\lambda_{1l} = 5$. Together with $\lambda_0 = 3/2$, one can verify that the proposed dual solution satisfies complementary slackness and dual feasibility constraints.

Table EC.4 Primal States and Dual Variables in Period 0, 1, and 2.

	s_{2w}^t	s_{2l}^t	s_{1w}^t	s_{1l}^t	λ_{2w}^t	λ_{2l}^t	λ_{1w}^t	λ_{1l}^t
$t=0$	0	1/8	1/8	0	11/2	9/2	15/2	9/2
$t=1$	1/8	0	0	1/8	5	4	8	5
$t=2$	1/16	1/16	1/16	0	5	3	9	5

□

A.3.2. Proof of Theorem 5 Before proving Theorem 5, we first provide the primal problem in the presence of bots. The primal problem is slightly modified from Eq. (P'). Since we consider a scenario where bots can only be matched with players who is on a losing streak, we only match bots with players in demographics 2ℓ and 1ℓ. In particular, only α fraction of players in states ℓ shall automatically go to state w in the next period. To help facilitate the analysis, we define $f_{1\ell,b}^t := \alpha s_{1\ell}^t$ and $f_{2\ell,b}^t := \alpha s_{1\ell}^t$ representing the matching flows dedicated for bot matching in each period t for each skill level of losing players. The rest of flow variables only contain human matching. Thus, we have

$$\begin{aligned}&\max_{\mathbf{f}^t} \sum_{t=0}^{\infty} \left(f_{2w,2w}^t + \frac{1}{2} f_{2\ell,2\ell}^t + f_{1w,1w}^t + \frac{1}{2} f_{1\ell,1\ell}^t + 2 f_{2w,1w}^t + f_{2\ell,1\ell}^t + f_{2w,1\ell}^t + 2 f_{2\ell,1w}^t + f_{2\ell,b}^t + f_{1\ell,b}^t \right)\\ &= \max_{\mathbf{f}^t} \sum_{t=0}^{\infty} \left(f_{2w,2w}^t + \frac{1+\alpha}{2(1-\alpha)} f_{2\ell,2\ell}^t + f_{1w,1w}^t + \frac{1+\alpha}{2(1-\alpha)} f_{1\ell,1\ell}^t + 2 f_{2w,1w}^t + \frac{1}{1-\alpha} f_{2\ell,1\ell}^t + \frac{1}{1-\alpha} f_{2w,1\ell}^t + \frac{2-\alpha}{1-\alpha} f_{2\ell,1w}^t \right)\end{aligned} \tag{EC.48}$$

s.t.

$$\begin{aligned}&s_{2w}^0 = f_{2w,2w}^0 + f_{2w,1w}^0 + f_{2w,1\ell}^0,\\ &(1-\alpha) s_{2\ell}^0 = f_{2\ell,2\ell}^0 + f_{2\ell,1w}^0 + f_{2\ell,1\ell}^0,\\ &s_{1w}^0 = f_{1w,1w}^0 + f_{2\ell,1w}^0 + f_{2w,1w}^0,\\ &(1-\alpha) s_{1\ell}^0 = f_{1\ell,1\ell}^0 + f_{2w,1\ell}^0 + f_{2\ell,1\ell}^0,\end{aligned}$$

and for all $t = 1, 2, \ldots$,

$$\begin{aligned}s_{2w}^t = f_{2w,2w}^t + f_{2w,1w}^t + f_{2w,1\ell}^t &= f_{2\ell,b}^{t-1} + \frac{1}{2}\left(f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1}\right) + f_{2w,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2\ell,1\ell}^{t-1}\\ &= \frac{\alpha}{1-\alpha}\left(f_{2\ell,2\ell}^t + f_{2\ell,1w}^t + f_{2\ell,1\ell}^t\right) + \frac{1}{2}\left(f_{2w,2w}^{t-1} + f_{2\ell,2\ell}^{t-1}\right) + f_{2w,1w}^{t-1} + f_{2w,1\ell}^{t-1} + f_{2\ell,1w}^{t-1} + f_{2\ell,1\ell}^{t-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}f^{t-1}_{2w,2w} + \frac{1+\alpha}{2(1-\alpha)}f^{t-1}_{2\ell,2\ell} + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + \frac{1}{1-\alpha}\left(f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell}\right),\\
(1-\alpha)s^t_{2\ell} = f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} &= \frac{1}{2}(1-\alpha)f^{t-1}_{2w,2w},\\
s^t_{1\ell} = f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} &= f^{t-1}_{1\ell,b} + \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right)\\
&= \frac{\alpha}{1-\alpha}\left(f^t_{2w,1\ell} + f^t_{2\ell,1\ell}\right) + \frac{1}{2}f^{t-1}_{1w,1w} + \frac{1+\alpha}{2(1-\alpha)}f^{t-1}_{1\ell,1\ell},\\
(1-\alpha)s^t_{1\ell} = f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} &= (1-\alpha)\left(\frac{1}{2}f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w}\right),\\
f^t_{i,j} &\geq 0,\ \forall i,j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{aligned}
$$

Proof of Theorem 5. *(a)* First, we derive the shadow price for each demographics under SBMM, denoted by μ_w and μ_ℓ for players who just won and lost a game, respectively. We do not need to specify time or skill level due to the nature of SBMM. That is, we have

$$
\begin{aligned}
\mu_w &= 1 + \frac{1}{2}\mu_w + \frac{1}{2}\mu_\ell,\\
\mu_\ell &= \alpha(1+\mu_w) + \frac{1-\alpha}{2}(1+\mu_w),
\end{aligned}
$$

which gives

$$
\mu_w = \frac{5+\alpha}{1-\alpha},\ \text{and}\ \mu_\ell = \frac{3(1+\alpha)}{1-\alpha}. \tag{EC.49}
$$

For a given α, to check whether there exists $\mathbf{s}$ such that $V^*(\mathbf{s},0) = V^{\text{SBMM}}(\mathbf{s},\alpha)$, one can solve

$$
\max_{\{\mathbf{f}^t\},\{\mathbf{s}^0\}} \sum_{t=0}^{\infty}\left(f^t_{2w,2w} + \frac{1}{2}f^t_{2\ell,2\ell} + f^t_{1w,1w} + \frac{1}{2}f^t_{1\ell,1\ell} + 2f^t_{2w,1w} + f^t_{2\ell,1\ell} + f^t_{2w,1\ell} + 2f^t_{2\ell,1w}\right) \tag{EC.50}
$$

s.t.

$$
\begin{aligned}
1 &\geq \frac{5+\alpha}{1-\alpha}\left(s^t_{2w} + s^t_{1w}\right) + \frac{3(1+\alpha)}{1-\alpha}\left(s^t_{2\ell} + s^t_{1\ell}\right), &\text{(EC.51)}\\
0 &= f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell} - s^0_{2w},\\
0 &= f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell} - s^0_{2\ell},\\
0 &= f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w} - s^0_{1w},\\
0 &= f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell} - s^0_{1\ell},
\end{aligned}
$$

and for all $t = 1, 2, \ldots,$

$$
\begin{aligned}
&f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell} = \frac{1}{2}\left(f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell}\right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell},\\
&f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} = \frac{1}{2}f^{t-1}_{2w,2w},\\
&f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} = \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right),\\
&f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} = \frac{1}{2}f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w},\\
&f^t_{i,j} \geq 0,\ \forall i,j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{aligned}
$$

and see if the objective is greater than 1. Furthermore, as α increases, both μ_w and μ_ℓ increase, making the optimization problem's feasible region smaller. Thus, using binary search, we numerically find that for $\alpha \leq 0.25$, there exists $\mathbf{s}$ such that $V^*(\mathbf{s},0) = V^{\text{SBMM}}(\mathbf{s},\alpha)$.

(b) We write down the dual problem for the new primal problem in (EC.48):

$$\min_{\{\lambda^t\}} \sum_{i\in\mathcal{P}} s_i^0 \lambda_i^0 \tag{EC.52}$$

s.t.

for all $t = 0, 1, 2, ...,$

$$\begin{aligned}
1 &\le \lambda_{2w}^t - \frac{1}{2}(1+\alpha)\lambda_{2w}^{t+1} - \frac{1}{2}(1-\alpha)\lambda_{2\ell}^{t+1},\\
\frac{1+\alpha}{2(1-\alpha)} &\le \lambda_{2\ell}^t - \frac{1}{2}(1+\alpha)\lambda_{2w}^{t+1},\\
\frac{2-\alpha}{1-\alpha} &\le \lambda_{2\ell}^t + \lambda_{1w}^t - \frac{1}{1-\alpha}\lambda_{2w}^{t+1} - (1-\alpha)\lambda_{1\ell}^{t+1},\\
\frac{1+\alpha}{2(1-\alpha)} &\le \lambda_{1\ell}^t - \frac{1+\alpha}{2(1-\alpha)}\lambda_{1w}^{t+1},\\
2 &\le \lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1-\alpha)\lambda_{1\ell}^{t+1},\\
\frac{1}{1-\alpha} &\le \lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} - \frac{\alpha}{1-\alpha}\lambda_{1w}^{t+1},\\
\frac{1}{1-\alpha} &\le \lambda_{2\ell}^t + \lambda_{1\ell}^t - \frac{1}{1-\alpha}\lambda_{2w}^{t+1} - \frac{\alpha}{1-\alpha}\lambda_{1w}^{t+1},\\
1 &\le \lambda_{1w}^t - \frac{1}{2}(1+\alpha)\lambda_{1w}^{t+1} - \frac{1}{2}(1-\alpha)\lambda_{1\ell}^{t+1}.
\end{aligned}$$

One can easily verify that the following solutions

$$\lambda_{2w}^t = \frac{5+\alpha}{1-\alpha}, \lambda_{2l}^t = \frac{3(1+\alpha)}{(1-\alpha)^2}, \lambda_{1w}^t = \frac{9-2\alpha-\alpha^2}{(1-\alpha)^2}, \lambda_{1l}^t = \frac{5-3\alpha+2\alpha^2}{(1-\alpha)^3}, \quad \forall t = 0, 1, 2, ...,$$

are feasible for the updated dual problem. As we can see, the proposed solutions are increasing in α. Thus, we have

$$V^*(\mathbf{s},\alpha) \le U(\mathbf{s},\alpha) := \frac{5+\alpha}{1-\alpha}s_{2w} + \frac{3(1+\alpha)}{(1-\alpha)^2}s_{2l} + \frac{9-2\alpha-\alpha^2}{(1-\alpha)^2}s_{1w} + \frac{5-3\alpha+2\alpha^2}{(1-\alpha)^3}s_{1l}.$$

Let $a = \underline{a}(\alpha)$ be solution of $U(\mathbf{s},a) = V^{\text{SBMM}}(\mathbf{s},\alpha)$, and $a = a(\alpha)$ be solution of $V^*(\mathbf{s},a) = V^{\text{SBMM}}(\mathbf{s},\alpha)$. Then $a(\alpha) \ge \underline{a}(\alpha)$ because $V^*(\mathbf{s},\alpha) \le U(\mathbf{s},\alpha)$ for any $\alpha \in [0,1]$ and function $V^*(\mathbf{s},\alpha)$ is increasing in α. Since $V^{\text{SBMM}}(\mathbf{s},\alpha)$ goes to infinity as α goes to 1, we must have $\underline{a}(\alpha)$ as well. Hence, taking the limit on both sides of $\underline{a}(\alpha) \le a(\alpha) \le \alpha$, we have $\lim_{\alpha\to 1} a(\alpha) = 1$. □

A.3.3. Proof of Theorem 6 Before showing the results of the interplay between PTW strategies and matchmaking, we update the flow balancing and demographic evolution constraints, then formally state the matchmaker's problem when facing PTW.

In any period t, the flow balance constraints are now

$$\begin{aligned}
&\sum_j f_{i,j}^t = s_i^t, \forall i \in \bar{\mathcal{P}},\\
&\sum_i f_{i,j}^t = s_j^t, \forall j \in \bar{\mathcal{P}},\\
&f_{i,j}^t = f_{j,i}^t, \forall i \ne j, i,j \in \bar{\mathcal{P}},\\
&f_{i,j}^t \ge 0, \forall i \ne j, i,j \in \bar{\mathcal{P}},
\end{aligned} \tag{FB$_{\text{ptw}}$}$$

where $\bar{\mathcal{P}} := \{\bar{2}w, \bar{2}\ell, \underline{2}w, \underline{2}\ell, \underline{1}w, \underline{1}\ell\}$, and evolution of demographics are now

$$\begin{aligned}
s_{\bar{2}w}^{t+1} &= \frac{1}{2}(f_{\bar{2}w,\bar{2}w}^t + f_{\bar{2}w,\bar{2}\ell}^t + f_{\bar{2}\ell,\bar{2}w}^t + f_{\bar{2}\ell,\bar{2}\ell}^t + f_{\bar{2}w,\underline{2}w}^t + f_{\bar{2}w,\underline{2}\ell}^t + f_{\bar{2}\ell,\underline{2}w}^t + f_{\bar{2}\ell,\underline{2}\ell}^t) + f_{\bar{2}w,\underline{1}w}^t + f_{\bar{2}w,\underline{1}\ell}^t + f_{\bar{2}\ell,\bar{1}w}^t + f_{\bar{2}\ell,\underline{1}\ell}^t,\\
s_{\bar{2}\ell}^{t+1} &= \frac{1}{2}(f_{\bar{2}w,\bar{2}w}^t + f_{\bar{2}w,\bar{2}\ell}^t + f_{\bar{2}w,\underline{2}w}^t + f_{\bar{2}w,\underline{2}\ell}^t),\\
s_{\underline{2}w}^{t+1} &= \frac{1}{2}(f_{\underline{2}w,\underline{2}w}^t + f_{\underline{2}w,\underline{2}\ell}^t + f_{\underline{2}\ell,\underline{2}w}^t + f_{\underline{2}\ell,\underline{2}\ell}^t + f_{\underline{2}w,\bar{2}w}^t + f_{\underline{2}w,\bar{2}\ell}^t + f_{\underline{2}\ell,\bar{2}w}^t + f_{\underline{2}\ell,\bar{2}\ell}^t) + f_{\underline{2}w,\underline{1}w}^t + f_{\underline{2}w,\underline{1}\ell}^t + f_{\underline{2}\ell,\underline{1}w}^t + f_{\underline{2}\ell,\underline{1}\ell}^t,\\
s_{\underline{2}\ell}^{t+1} &= \frac{1}{2}(f_{\underline{2}w,\underline{2}w}^t + f_{\underline{2}w,\underline{2}\ell}^t + f_{\underline{2}w,\bar{2}w}^t + f_{\underline{2}w,\bar{2}\ell}^t),\\
s_{\underline{1}w}^{t+1} &= \frac{1}{2}(f_{\underline{1}w,\underline{1}w}^t + f_{\underline{1}w,\underline{1}\ell}^t + f_{\underline{1}\ell,\underline{1}w}^t + f_{\underline{1}\ell,\underline{1}\ell}^t),\\
s_{\underline{1}\ell}^{t+1} &= \frac{1}{2}(f_{\underline{1}w,\underline{1}w}^t + f_{\underline{1}w,\underline{1}\ell}^t) + f_{\underline{1}w,\underline{2}w}^t + f_{\underline{1}w,\underline{2}\ell}^t + f_{\underline{1}w,\bar{2}w}^t + f_{\underline{1}w,\bar{2}\ell}^t.
\end{aligned} \tag{ED$_{\text{ptw}}$}$$

From (FB_{ptw}) and (ED_{ptw}) conditions, we can see that players in the new demographics with skill level $\bar{2}$ behave the same as those with skill level h. However, the matchmaker's objective is largely different from the one in (P). The matchmaker also needs to consider the revenue generated by paid subscriptions. Given β, r, and the initial demographics $\mathbf{s}^0 = \{s_{2w}^0, s_{2\ell}^0, s_{1w}^0, s_{1\ell}^0\}$, the matchmaker's problem is

$$\begin{aligned}
V^*(\beta, r, \mathbf{s}^0) := \max_{\{f_{i,j}^t\}_{t=1}^\infty} \sum_{t=1}^\infty \sum_i s_i^t + \sum_{t=1}^\infty r(s_{\bar{2}w}^t + s_{\bar{2}\ell}^t) &= \max_{\{f_{i,j}^t\}_{t=1}^\infty} ENG(\beta, r, \mathbf{s}^0) + REV(\beta, r, \mathbf{s}^0) \qquad (\text{P}_{\text{ptw}})\\
\text{s.t. } s_{\bar{2}w}^0 &= \beta s_{1w}^0,\\
s_{\bar{2}\ell}^0 &= \beta s_{1\ell}^0,\\
s_{\underline{2}w}^0 &= s_{2w}^0,\\
s_{\underline{2}l}^0 &= s_{2l}^0,\\
s_{\underline{1}w}^0 &= (1-\beta) s_{1w}^0,\\
s_{\underline{1}\ell}^0 &= (1-\beta) s_{1\ell}^0,\\
&(\text{FB}_{\text{ptw}}) \text{ and } (\text{ED}_{\text{ptw}})\ \forall\, t = 0, 1, 2, \ldots, \text{ and } i, j \in \bar{\mathcal{P}},
\end{aligned}$$

where the objective function is to maximize the sum of player engagement $ENG(\beta, r, \mathbf{s}^0)$ and revenue from paid subscription $REV(\beta, r, \mathbf{s}^0)$.

Proof of Theorem 6. We prove the four parts separately.

(a) Consider two initial states, $\mathbf{s}_1 = (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$ and $\mathbf{s}_2 = (s_{2w} + (r+1)\beta s_{1w}, s_{2\ell} + (r+1)\beta s_{1\ell}, (1-\beta)s_{1w}, (1-\beta)s_{1\ell})$. We show that $V^{SBMM}(\beta, r, \mathbf{s}_1) = V^{SBMM}(0, 0, \mathbf{s}_2)$ and $V^*(\beta, r, \mathbf{s}_1) \geq V^*(0, 0, \mathbf{s}_2)$.

Note that one can easily verify $V^{SBMM}(\beta, r, \mathbf{s}_1) = V^{SBMM}(0, 0, \mathbf{s}_2)$ using the closed-form expression of V^{SBMM} in Theorem 1. Thus, we omit the details and only focused on the value functions under the optimal matching policy.

Consider the optimal trajectory when the initial state of demographics is $\mathbf{s}_2 = (s_{2w} + (r+1)\beta s_{1w}, s_{2\ell} + (r+1)\beta s_{1\ell}, (1-\beta)s_{1w}, (1-\beta)s_{1\ell})$ and there is no PTW system (referred as non-PTW problem). Let $f_{i,j}^t$ be the optimal flow between $i, j \in \{1w, 1\ell, 2w, 2\ell\}$ at time t, and s_i^t be the population at time t. From Theorem 3, we know that it may involve cross-level matching between 2ℓ and $1w$, as well as 2ℓ and 1ℓ (only happens when $t = 0$ in Scenario 6 and 7, after which the states transit to Scenario 1-4). We now show that we can collect at least the same value with demographic $\mathbf{s}_1 = (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$ and PTW (referred as PTW problem). Note that with PTW, the reward of 1 unit flow between $\bar{2}\ell$ and level 1 players are the same as $r+1$ units of flow between 2ℓ and level 1 players without PTW, due to the extra r unit of revenue. However, such a flow use less amount of low players, which makes the level 1 players better off. This observation provides a natural way to construct a feasible flow for the PTW problem. To distinguish from the state of demographics s_i^t and the matching flows $f_{i,g}^t$ in the system without PTW, where $i, j \in (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$, we denote σ_i^t and $g_{i,j}^t$ as the state of demographics and matching flow in the PTW problem, respectively, where $i, j \in (s_{\bar{2}w}^t, s_{\bar{2}\ell}^t, s_{\underline{2}w}^t, s_{\underline{2}\ell}^t, s_{\underline{1}w}^t, s_{\underline{1}\ell}^t)$.

Next, we construct a specific set of matching flows in the PTW problem. In each period, consider the following proposed flows: $g_{\bar{2}\ell,\underline{1}w}^t = \min\{f_{2\ell,1w}^t/(1+r), \sigma_{\bar{2}\ell}^t\}$ and $g_{\underline{2}\ell,\underline{1}w}^t = f_{2\ell,1w}^t - (1+r)g_{\bar{2}\ell,\underline{1}w}^t$. Furthermore, at $t=0$, if the matching system without PTW has non-zero flows between players in 2ℓ and 1ℓ, i.e., $f_{2\ell,1\ell}^0 > 0$, then we set $g_{\bar{2}\ell,\underline{1}\ell}^0 = \min\{\sigma_{\bar{2}\ell}^0 - g_{\bar{2}\ell,\underline{1}w}^0, f_{2\ell,1\ell}^0/(1+r)\}$ and $g_{\underline{2}\ell,\underline{1}\ell}^0 = f_{2\ell,1\ell}^0 - (1+r)g_{\bar{2}\ell,\underline{1}\ell}^0$. In other words, we prioritize cross-level matching flows between $\bar{2}\ell$ players and level 1 players, and only after which is exhausted, we then use players in $\underline{2}\ell$ to match with level 1 players. For all the remaining players, we simply match them with other players in the same state.

Next, we show that with the initial state of demographics $\mathbf{s}_1$ the proposed flows $g_{i,j}^t$, where $i, j \in (s_{\bar{2}w}^t, s_{\bar{2}\ell}^t, s_{\underline{2}w}^t, s_{\underline{2}\ell}^t, s_{\underline{1}w}^t, s_{\underline{1}\ell}^t)$ is not only feasible, but also collect at least the same rewards as the non-PTW problem with initial state of demographics $\mathbf{s}_2$ and optimal flows $f_{i,g}^t$, where $i, j \in (s_{2w}, s_{2\ell}, s_{1w}, s_{1\ell})$.

To show the statement above, we prove that in the PTW problem, we must have $\sigma_{\underline{2}w}^t + (1+r)\sigma_{\bar{2}w}^t = s_{2w}^t$, $\sigma_{\underline{2}\ell}^t + (1+r)\sigma_{\bar{2}\ell}^t = s_{2\ell}^t$, $\sigma_{\underline{1}w}^t \geq s_{1w}^t$, and $\sigma_{\underline{1}w}^t + \sigma_{\underline{1}\ell}^t \geq s_{1w}^t + s_{1\ell}^t$ for any $t \geq 0$.

It is trivially true for $t=0$ by construction. Now consider $t=1$. First, recall that one unit of subscribed (level $\bar{2}$) player in the PTW problem correspond to $1+r$ units of high-skilled (level 2) players in the non-PTW problem, due to the extra unit of revenue r generated by subscription fee. Thus, we must have $\sigma_{\underline{2}w}^1 + (1+r)\sigma_{\bar{2}w}^1 = s_{2w}^1$ and $\sigma_{\underline{2}\ell}^1 + (1+r)\sigma_{\bar{2}\ell}^1 = s_{2\ell}^1$. Second, in the non-PTW problem, if $f_{2\ell,1\ell}^0 > 0$, then $f_{2\ell,1\ell}^0$ unit of level 1 players will lose in the upcoming game and leave the system permanently. However, in a corresponding PTW problem, we have $g_{\bar{2}\ell,\underline{1}\ell}^0 \leq f_{2w,1\ell}^0/(1+r)$ according to the proposed policy. Thus, instead of matched with high-skill players and leaving the system permanently in a non-PTW problem, we have $f_{2\ell,1\ell}^0 - g_{\bar{2}\ell,\underline{1}\ell}^0$ unit of players in state $\underline{1}\ell$ are matched to other players in the same state in the PTW problem and half of them can survive to period $t=1$. Thus, we must have $\sigma_{\underline{1}w}^1 \geq s_{1w}^1$, and $\sigma_{\underline{1}w}^1 + \sigma_{\underline{1}\ell}^1 \geq s_{1w}^1 + s_{1\ell}^1$. Third, in a non-PTW problem, if we have $f_{2\ell,1w}^0 > 0$, then we have $f_{2\ell,1w}^0$ unit of players in state $1w$ transfer to state 1ℓ just prior to period $t=1$. However, in a corresponding PTW problem, we have $g_{\bar{2}\ell,\underline{1}w}^0 \leq f_{2\ell,1w}^0/(1+r)$ according to the proposed policy. Thus, instead of matching with high-skilled players, we have $f_{2\ell,1w}^0 - g_{\bar{2}\ell,\underline{1}w}^0$ unit of $\underline{1}w$ players matched with other players in the same state. As a result, just prior to period $t=1$, we have half of the players in the aforementioned flow with size $f_{2\ell,1w}^0 - g_{\bar{2}\ell,\underline{1}w}^0$ remain in $\underline{1}w$ and half of them transfer to $\underline{1}\ell$. Hence, we still have $\sigma_{\underline{1}w}^1 \geq s_{1w}^1$, and $\sigma_{\underline{1}w}^1 + \sigma_{\underline{1}\ell}^1 \geq s_{1w}^1 + s_{1\ell}^1$.

For $t \geq 1$, we only need to consider the matching flow between $1w$ and 2ℓ (since Scenario 5-7 in the proof of Theorem 3, which involves other cross-skill matching, only occurs in period $t=0$). At $t+1$, $\sigma_{\underline{2}w}^1 + (1+r)\sigma_{\bar{2}w}^1 = s_{2w}^1$ and $\sigma_{\underline{2}\ell}^1 + (1+r)\sigma_{\bar{2}\ell}^1 = s_{2\ell}^1$ still hold for the same reason as in $t=1$. If in a non-PTW problem, we have

$f^{t-1}_{2\ell,1w} > 0$. Then using the exact same third argument in the proof for period $t = 1$, we can show that we still have $\sigma^1_{\underline{1}w} \geq s^1_{1w}$, and $\sigma^1_{\underline{1}w} + \sigma^1_{\underline{1}\ell} \geq s^1_{1w} + s^1_{1\ell}$. The reason is the same as before: some of the low-skill players who are matched with high-skilled players in a non-PTW problem shall be matched to other low-skill players in a PTW problem, which leads to less players transferring to state $\underline{1}\ell$ and more to $\underline{1}w$ instead.

Since we have $\sigma^t_{\underline{2}l} + (1+r)\sigma^t_{\bar{2}l} = s^t_{2l}$ and $\sigma^t_{\underline{1}w} \geq s^t_{1w}$, the proposed flow is feasible because for any $t \geq 1$ all matching flows are either skill-based or between $\bar{2}\ell$, $\underline{2}\ell$, and $\underline{1}w$ players, which are determined by $f^t_{2l,1w}$ in a corresponding non-PTW problem. Also, in every period we collect reward no less than the non-PTW problem because the reward we collect from high-skilled players is $\sum_{i=w,l}(\sigma^t_{\underline{2}i} + (1+r)\sigma^t_{\bar{2}i}) = s^t_{2w} + s^t_{2l}$, and the reward from low-skilled problem is $\sigma^t_{\underline{1}w} + \sigma^t_{\underline{1}\ell}$ which we have shown is no less than $s^t_{1w} + s^t_{1\ell}$.

(b) We show that unsubscribed high-skilled players in $\underline{2}w$ ($\underline{2}\ell$) would only be matched with any unsubscribed low-skilled players after all the subscribed players in $\bar{2}w$ ($\bar{2}\ell$) have matched with unsubscribed low-skilled players.

We prove the statement above by contradiction. Suppose on the optimal trajectory, the flow between high-skilled non-paying players in $\underline{2}i$ and low-skilled unsubscribed players $\underline{1}j$ are positive for some $i, j = w, \ell$, while there exists subscribed players $\bar{2}i$ who are matched by skill levels. Then by matching $\bar{2}i$ with $\underline{1}j$, we can collect strictly more rewards in the current period, and a player in $\bar{2}i$ would replace a player $\underline{2}i$ in all the subsequent periods. Hence, the solution cannot be optimal.

(c) We show that if $s^0_{2w} + s^0_{2\ell} \geq s^0_{1w} + s^0_{1\ell}$, then $ENG(\beta, r, \mathbf{s}^0) < V^*(0, 0, \mathbf{s}^0)$. Furthermore, there exists a threshold $\bar{r}$ such that $V^*(\beta, r, \mathbf{s}^0) \geq V^*(0, 0, \mathbf{s}^0)$ if and only if $r \geq \bar{r}$.

We first show that $ENG(\beta, r, \mathbf{s}^0) \leq V^*(0, 0, \mathbf{s}^0)$. The engagement $ENG(\beta, r, \mathbf{s}^0)$ is at most $V^*(\beta, 0, \mathbf{s}^0)$, which is the optimal engagement with the same demographic but without revenue. We now show that $V^*(\beta, 0, \mathbf{s}^0) \leq V^*(0, 0, \mathbf{s}^0)$, and the equality only holds when we do not have low players at all.

When the high-skilled players are more than low-skilled players in a non-PTW system, we are in Scenario 1-3 in the proof of Theorem 3. Note that for these scenarios we have explicit shadow prices for each type of players. We now discuss the three scenarios separately.

Consider Scenario 1: $s^0_{2w} + s^0_{2\ell} \geq s^0_{1w} + s^0_{1\ell}$, $s^0_{2\ell} \geq s^0_{1w}$, and $s^0_{2w} \geq s^0_{1\ell}$. The shadow price in this case is 5,3,9,5, respectively. The PTW system shift $1w$ (1ℓ) player to $2w$ (2ℓ), so with PTW the initial demographic is still in scenario 1. The total value change will be $(5-9)\beta s^0_{1w} + (3-5)s^0_{1\ell} \leq 0$, where the equality only holds when $s^0_{1w} = s^0_{1\ell} = 0$.

Consider Scenario 2: $s^0_{2w} + s^0_{2\ell} \geq s^0_{1w} + s^0_{1\ell}$, $s^0_{2\ell} < s^0_{1w}$, and $s^0_{2w} \geq s^0_{1\ell}$; The shadow price in this case is 5,4,8,5, respectively. The PTW system shift $1w$ (1ℓ) player to $2w$ (2ℓ), so with PTW the initial demographic is either in Scenario 1 or Scenario 2. If it remains in Scenario 2, then the total value change will be $(5-8)\beta s^0_{1w} + (4-5)s^0_{1\ell} < 0$ because now we have $s^0_{1w} > s^0_{2\ell} \geq 0$. If the state of demographics transits to Scenario 1, the total value change would be

$$
\begin{aligned}
& 5(s^0_{2w} + \beta s^0_{1w}) + 3(s^0_{2\ell} + \beta s^0_{1\ell}) + 9(1-\beta)s^0_{1w} + 5(1-\beta)s^0_{1\ell} - 5s^0_{2w} - 4s^0_{2\ell} - 8s^0_{1w} - 5s^0_{1\ell} \\
= & -s^0_{2\ell} + (1-4\beta)s^0_{1w} - 2\beta s^0_{1\ell} \\
\leq & -s^0_{2\ell} + (1-4\beta)\frac{s^0_{2\ell} + \beta s^0_{1\ell}}{1-\beta} - 2\beta s^0_{1\ell} \qquad \text{(EC.53)} \\
= & \left(-1 + \frac{1-4\beta}{1-\beta}\right)s^0_{2\ell} + \left(-2\beta + \frac{1-4\beta}{1-\beta}\right)s^0_{1\ell} < 0, \qquad \text{(EC.54)}
\end{aligned}
$$

where Eq. (EC.53) comes from the fact that if the demographic transfer to Scenario 1, we must have $s^0_{2\ell}+\beta s^0_{1\ell}\ge(1-\beta)s^0_{1w}$. Eq. (EC.54) comes from the fact that $-1+\frac{1-4\beta}{1-\beta}$ and $-2\beta+\frac{1-4\beta}{1-\beta\beta}$ are both negative when $\beta\in(0,1)$, and to make $s^0_{2\ell}+\beta s_{1\ell}\ge(1-\beta)s^0_{1w}$, one of $s^0_{2\ell}$ and $s^0_{1\ell}$ has to be positive.

Finally, consider Scenario 3: $s^0_{2w}+s^0_{2\ell}\ge s^0_{1w}+s^0_{1\ell}$, $s^0_{2\ell}\ge s^0_{1w}$, and $s^0_{2w}<s^0_{1\ell}$. The shadow prices in this scenario is 5.5,3,9,4.5. The PTW system shift $1w$ (1ℓ) player to $2w$ (2ℓ), so with PTW the initial demographic is either in Scenario 1 or Scenario 3. If it remains in Scenario 3, then the total value change will be $(5.5-9)\beta s^0_{1w}+(3-4.5)s^0_{1\ell}<0$ we cause now $s^0_{1\ell}>s^0_{2w}\ge 0$. If the demographic transfer to Scenario 1, to total value change would be

$$
\begin{aligned}
&5(s^0_{2w}+\beta s^0_{1w})+3(s^0_{2\ell}+\beta s^0_{1\ell})+9(1-\beta)s^0_{1w}+5(1-\beta)s^0_{1\ell}-5.5s^0_{2w}-3s^0_{2\ell}-9s^0_{1w}-4.5s^0_{1\ell}\\
=&-0.5s^0_{2w}-4\beta s^0_{1w}+(0.5-2\beta)s^0_{1\ell}\\
\le&-0.5s^0_{2w}-4\beta s^0_{1w}+(0.5-2\beta)\frac{s^0_{2w}+\beta s^0_{1w}}{1-\beta} && \text{(EC.55)}\\
=&\left(-0.5+\frac{0.5-2\beta}{1-\beta}\right)s^0_{2w}+\left(-4\beta+\frac{(0.5-2\beta)\beta}{1-\beta}\right)s^0_{1w}<0, && \text{(EC.56)}
\end{aligned}
$$

where Eq. (EC.55) comes from the fact that if the demographic transfer to Scenario 1, we must have $s^0_{2w}+\beta s^0_{1w}\ge(1-\beta)s^0_{1\ell}$. The inequality in Eq. (EC.54) comes from the fact that $-0.5+\frac{0.5-2\beta}{1-\beta}$ and $-4\beta+\frac{(0.5-2\beta)\beta}{1-\beta}$ are both negative when $\beta\in(0,1)$, and to make $s^0_{2w}+\beta s^0_{1w}\ge(1-\beta)s^0_{1\ell}$, one of s^0_{2w} and s^0_{1w} has to be positive.

We have shown that $V^*(\beta,0,\mathbf{s}^0)<V^*(0,0,\mathbf{s}^0)$ when there are positive amount of low players. It is easy to see that $V^*(\beta,r,\mathbf{s}^0)$ increases monotonically with r, and goes to infinity as r goes to infinity. Hence, there exists a threshold $\bar r>0$ such that $V^*(\beta,r,\mathbf{s}^0)>V^*(0,0,\mathbf{s}^0)$ if and only if $r>\bar r$.

(d) Fix $s^0_{2w}/s^0_{2\ell}$ and $s^0_{1w}/s^0_{1\ell}$ and vary $(s^0_{2w}+s^0_{2\ell})/(s^0_{1w}+s^0_{1\ell})$. We show that if the ratio of high- over low-skilled players is sufficiently small, there is $V^*(\beta,r,\mathbf{s}^0)\ge V^*(0,0,\mathbf{s}^0)$ even if $r=0$.

Consider $\mathbf{s}^0$ such that $(s^0_{2w}+s^0_{2\ell})/(s^0_{1w}+s^0_{1\ell})=0$, i.e., there are only low players. Then the optimal matching is simply SBMM. In presence of PTW system, some of the low player now becomes high player, which enables cross-level matchmaking, and we must have $V^*(\beta,0,\mathbf{s}^0)>V^*(0,0,\mathbf{s}^0)$. That said, even there is no revenue, the engagement is still higher thanks to the change in demographic distribution. For $r\ge 0$, we must have $V^*(\beta,r,\mathbf{s}^0)\ge V^*(\beta,0,\mathbf{s}^0)>V^*(0,0,\mathbf{s}^0)$. Finally, note that when $r=0$, we have $V^*(\beta,r,\mathbf{s}^0)=ENG(\beta,r,\mathbf{s}^0)$, i.e., the value of matchmaking is solely made by player engagement.

Appendix B: Possible Extensions

We point out that our framework eq:LP is flexible enough to allow for the following various practical extensions while still resulting in a nice LP formulation.

1. A draw/tie outcome can be easily added, since our model only depends on the aggregate transition matrix M_{kk}. Let $P^k_{tie}\in[0,1]^{|\mathcal{G}|\times|\mathcal{G}|}$ be the transition matrix of a level k player's engagement state, given that they experienced a tie. Let $p^{win}_{kj},p^{lose}_{kj}$ and p^{tie}_{kj} be the probability of win, lose and tie for a level k player who faces a level j player. Then the aggregate transition matrix is given by $M_{kj}=p^{win}_{kj}P^k_{win}+p^{lose}_{kj}P^k_{lose}+p^{tie}_{kj}P^k_{tie}$. Then the computation follows Eq. (2).

2. If in each period, only *part* fraction of the idle players want to play, then we can simply multiply *part* on the right-hand-side of (FB) and add $(1-part)\mathbf{s}_k^t$ on the right-hand-side of (ED). For Eq. (FB), it now becomes

$$\begin{aligned}\sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f_{kg,jg'}^t &= part\cdot s_{kg}^t,\ k=1,\dots,K,\forall g\in\bar{\mathcal{G}},\\ \sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f_{jg',kg}^t &= part\cdot s_{kg}^t,\ k=1,\dots,K,\forall g\in\bar{\mathcal{G}},\\ f_{kg,jg'}^t &= f_{jg',kg}^t,\ j=1,\dots,K,k=1,\dots,K,\forall g\in\bar{\mathcal{G}},g'\in\bar{\mathcal{G}}\\ f_{kg,jg'}^t &\geq 0,\ j=1,\dots,K,k=1,\dots,K,\forall g\in\bar{\mathcal{G}},g'\in\bar{\mathcal{G}}\end{aligned} \tag{EC.57}$$

For Eq. (ED), it now becomes

$$\mathbf{s}_k^{t+1} = (1-part)\mathbf{s}_k^t + \sum_{j=1,\dots,K}\left(\mathbf{f}_{kj}^t\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.58}$$

3. If the match duration is not one period, we can modify (ED) so that the match flow returns to the demographics after a positive and random delay. Suppose a match may last at most D periods. Let w_d be the probability that a match lasts d periods, and $\sum_{d=1}^{D} w_d = 1$. Then Eq. (ED) can be modified as

$$\mathbf{s}_k^{t+1} = \sum_{d=1}^{D} w_d \sum_{j=1,\dots,K}\left(\mathbf{f}_{kj}^{t+1-d}\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.59}$$

4. New players whose amount are linear functions of past history can be introduced easily by modifying (ED). In the main body, we already consider a special case where the new customer arrival at $t+1$ is proportional to the active players at t. More generally, let $New_k^t(\{\mathbf{f}_{kj}^\tau\}_{\tau=1}^{t-1})$ be a linear function of $\mathbf{f}_{kj}^\tau$, $\tau=1,\dots,t-1$. Then we can modify Eq. (ED) as

$$\mathbf{s}_k^{t+1} = New_k^t(\{\mathbf{f}_{kj}^\tau\}_{\tau=1}^{t-1}) + \sum_{j=1,\dots,K}\left(\mathbf{f}_{kj}^t\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.60}$$