

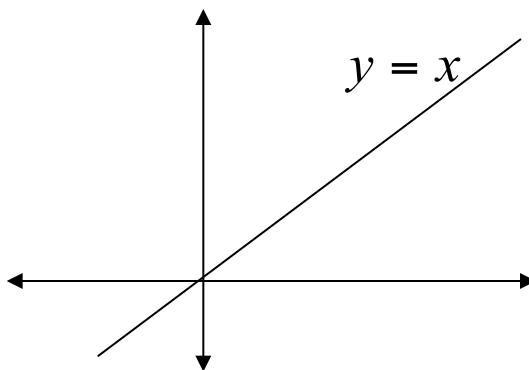
# 8. FUNGSI TRANSENDEN

## 8.1 Invers Fungsi

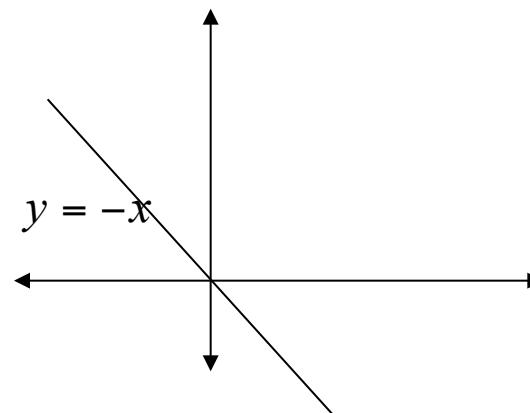
Misalkan  $f : D_f \rightarrow R_f$  dengan

$$x \mapsto y = f(x)$$

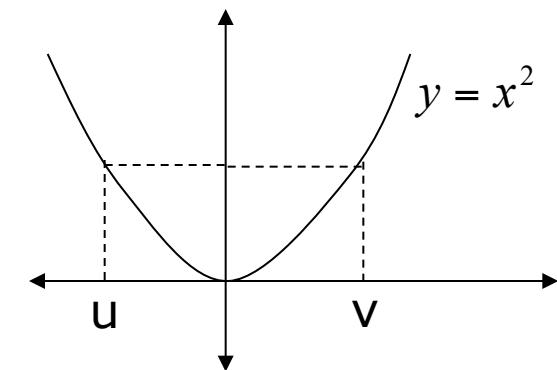
**Definisi 8.1** Fungsi  $y = f(x)$  disebut satu-satu jika  $f(u) = f(v)$  maka  $u = v$   
atau jika  $u \neq v$  maka  $f(u) \neq f(v)$



fungsi  $y=x$  satu-satu



fungsi  $y=-x$  satu-satu

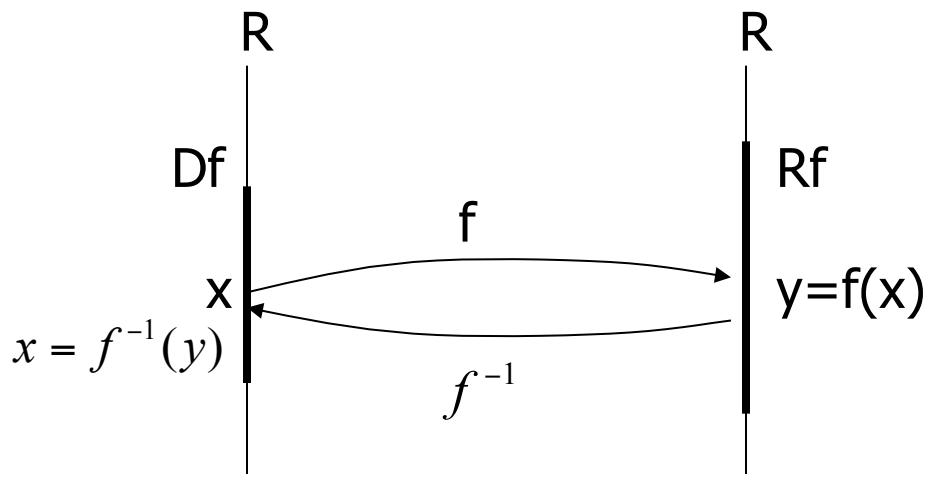


fungsi  $y = x^2$  tidak satu-satu

Secara geometri grafik fungsi satu-satu dan garis yang sejajar dengan sumbu x berpotongan di satu titik.

Teorema : Jika fungsi  $f$  satu-satu maka  $f$  mempunyai invers  
notasi  $f^{-1}$

$$f^{-1} : R_f \rightarrow D_f \\ y \mapsto x = f^{-1}(y)$$



Berlaku hubungan

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

$$D_{f^{-1}} = R_f, \quad R_{f^{-1}} = D_f$$

Teorema : jika  $f$  monoton murni(selalu naik/selalu turun) maka  $f$  mempunyai invers

$$f(x) = x$$

$$f'(x) = 1 > 0, \forall x \in R$$

$f$  selalu naik

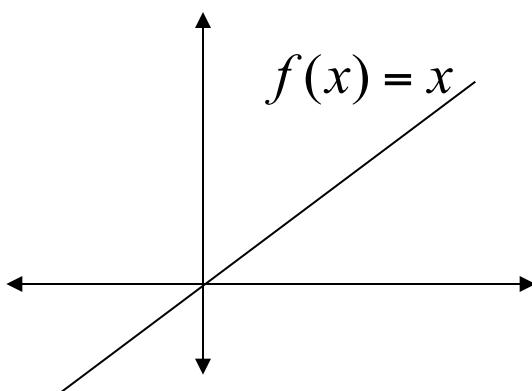
$$f(x) = -x$$

$$f'(x) = -1 < 0, \forall x \in R$$

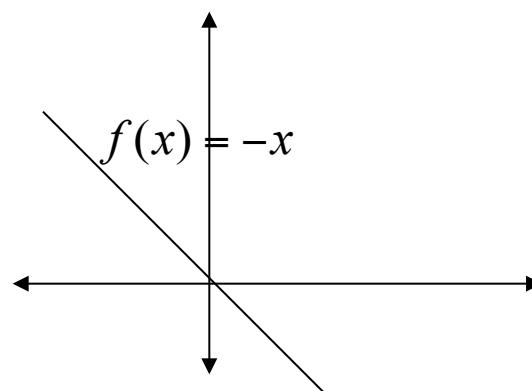
$f$  selalu turun

$$f'(x) = 2x = \begin{cases} > 0, x > 0 \\ < 0, x < 0 \end{cases}$$

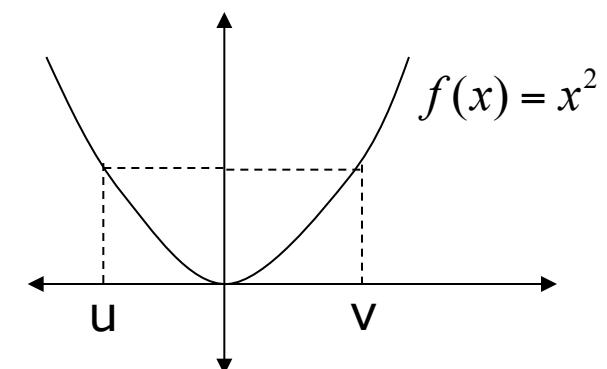
$f$  naik untuk  $x > 0$   
turun untuk  $x < 0$



$f^{-1}$  ada



$f^{-1}$  ada



$f^{-1}$  tidak ada

Contoh : Diketahui  $f(x) = \frac{x-1}{x+2}$

- Periksa apakah f mempunyai invers
- Jika ada, tentukan inversnya

Jawab

$$a. \quad f'(x) = \frac{1.(x+2) - 1.(x-1)}{(x+2)^2} = \frac{3}{(x+2)^2} > 0, \forall x \in Df$$

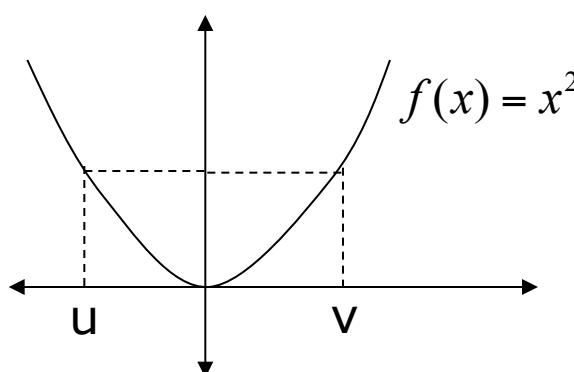
Karena f selalu naik(monoton murni) maka f mempunyai invers

$$b. \quad \text{Misal } y = \frac{x-1}{x+2}$$

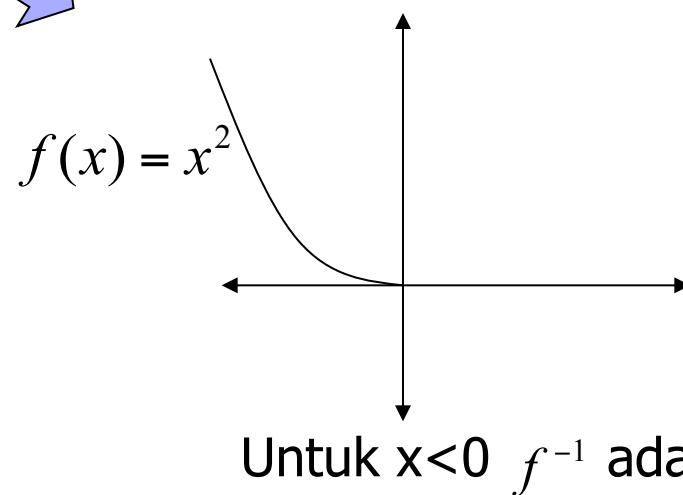
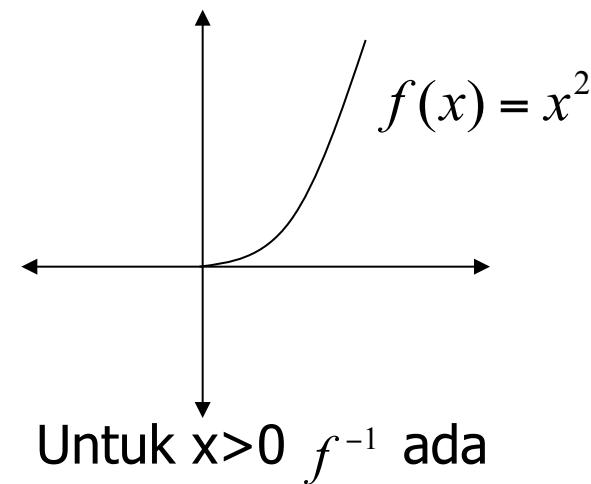
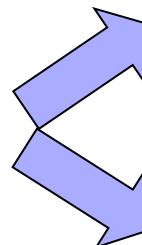
$$xy + 2y = x - 1 \Leftrightarrow xy - x = -2y - 1 \Rightarrow x = \frac{-2y-1}{y-1}$$

$$f^{-1}(y) = \frac{-2y-1}{y-1} \Rightarrow f^{-1}(x) = \frac{-2x-1}{x-1}$$

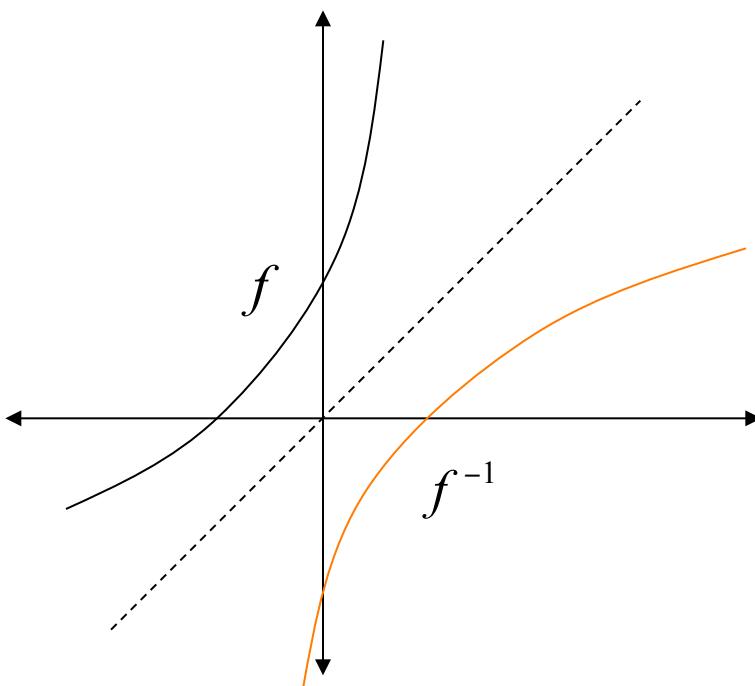
Suatu fungsi yang tidak mempunyai invers pada daerah asalnya dapat dibuat mempunyai invers dengan cara membatasi daerah asalnya.



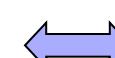
Untuk  $x \in R$   $f^{-1}$  tidak ada



# Grafik fungsi invers



Titik  $(x,y)$  terletak  
pada grafik  $f$



Titik  $(y,x)$  terletak  
pada grafik  $f^{-1}$

Titik  $(x,y)$  dan  $(y,x)$  simetri terhadap garis  $y=x$

Grafik  $f$  dan  $f^{-1}$  simetri terhadap garis  $y=x$

## Turunan fungsi invers

**Teorema** Misalkan fungsi  $f$  monoton murni dan mempunyai turunan pada selang  $I$ . Jika  $f'(x) \neq 0, x \in I$  maka  $f^{-1}$  dapat diturunkan di  $y=f(x)$  dan

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Bentuk diatas dapat juga dituliskan sebagai

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Contoh Diketahui  $f(x) = x^5 + 2x + 1$  tentukan  $(f^{-1})'(4)$

Jawab :

$$f'(x) = 5x^4 + 2, y=4 \text{ jika hanya jika } x=1$$

$$\rightarrow (f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{7}$$

## Soal Latihan

Tentukan fungsi invers ( bila ada ) dari

$$1. \quad f(x) = x + \frac{1}{x} , \quad x > 0$$

$$2. \quad f(x) = \sqrt[3]{2x - 1}$$

$$3. \quad f(x) = \sqrt[5]{4x + 2}$$

$$4. \quad f(x) = \frac{5}{x^2 + 1} , \quad x \geq 0$$

$$5. \quad f(x) = \frac{x + 1}{x - 1}$$

$$6. \quad f(x) = \frac{2x - 3}{x + 2}$$

## 8.2 Fungsi Logaritma Asli

- Fungsi Logaritma asli (  $\ln$  ) didefinisikan sebagai :

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

- Dengan Teorema Dasar Kalkulus II, diperoleh :

$$D_x [\ln x] = D_x \left( \int_1^x \frac{1}{t} dt \right) = \frac{1}{x}$$

- Secara umum, jika  $u = u(x)$  maka

$$D_x [\ln u] = D_x \left( \int_1^{u(x)} \frac{1}{t} dt \right) = \frac{1}{u} \frac{du}{dx}$$

Contoh : Diberikan  $f(x) = \ln(\sin(4x + 2))$

maka  $f'(x) = \frac{1}{\sin(4x + 2)} D_x(\sin(4x + 2)) = 4 \cot(4x + 2)$



Jika  $y = \ln |x|, x \neq 0$

$$= \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases} \rightarrow y = \ln x \rightarrow y' = \frac{1}{x}$$

$$\rightarrow y = \ln(-x) \rightarrow y' = \frac{-1}{-x} = \frac{1}{x}$$

Jadi,  $\frac{d}{dx}(\ln |x|) = \frac{1}{x}, x \neq 0.$

Dari sini diperoleh :  $\int \frac{1}{x} dx = \ln |x| + C$

### Sifat-sifat Ln :

1.  $\ln 1 = 0$
2.  $\ln(ab) = \ln a + \ln b$
3.  $\ln(a/b) = \ln(a) - \ln(b)$
4.  $\ln a^r = r \ln a$

Contoh: Hitung  $\int_0^4 \frac{x^2}{x^3 + 2} dx$

jawab

Misal  $u = x^3 + 2 \rightarrow du = 3x^2 dx$

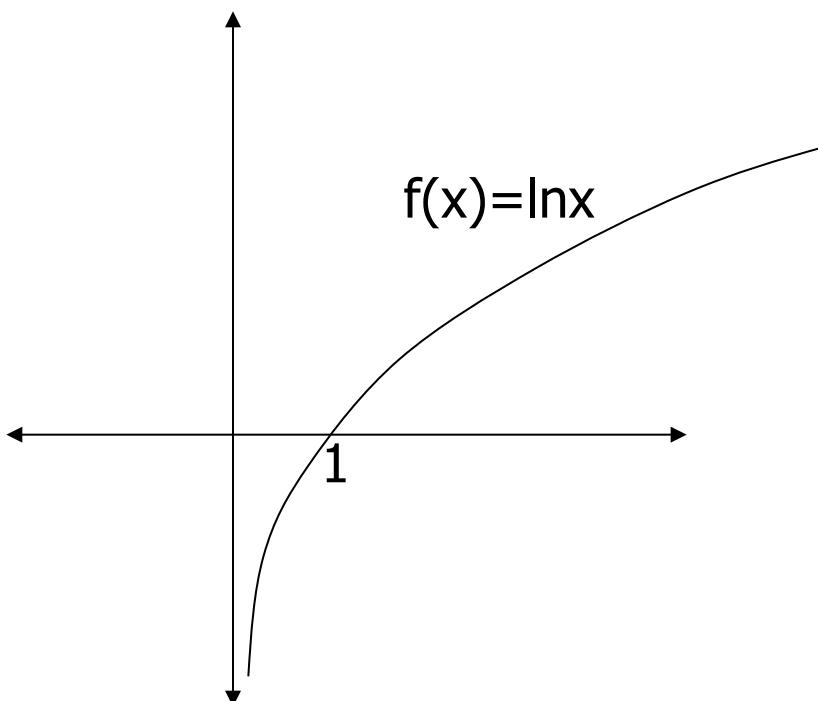
$$\begin{aligned}\int \frac{x^2}{x^3 + 2} dx &= \int \frac{x^2}{u} \frac{du}{3x^2} = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |u| + c \\ &= \frac{1}{3} \ln |x^3 + 2| + c\end{aligned}$$

sehingga

$$\int_0^4 \frac{x^2}{x^3 + 2} dx = \left. \frac{1}{3} \ln |x^3 + 2| \right|_0^4 = \frac{1}{3} (\ln 66 - \ln 2) = \frac{1}{3} \ln 33.$$

# Grafik fungsi logaritma asli

Diketahui



a.  $f(x) = \ln x = \int_1^x \frac{dt}{t}, x > 0$

b.  $f'(x) = \frac{1}{x} > 0 \forall x \in D_f$

$f$  selalu monoton naik pada  $D_f$

c.  $f''(x) = -\frac{1}{x^2} < 0 \forall x \in D_f$

Grafik selalu cekung kebawah

d.  $f(1) = 0$

## 8.3 Fungsi Eksponen Asli

- Karena  $D_x[\ln x] = \frac{1}{x} > 0$  untuk  $x > 0$ , maka fungsi logaritma asli monoton murni, sehingga mempunyai invers. Invers dari fungsi logaritma asli disebut **fungsi eksponen asli, notasi exp**. Jadi berlaku hubungan

$$y = \exp(x) \Leftrightarrow x = \ln y$$

- Dari sini didapat :  $y = \exp(\ln y)$  dan  $x = \ln(\exp(x))$
- **Definisi 8.2** Bilangan  $e$  adalah bilangan Real positif yang bersifat  $\ln e = 1$ .

Dari sifat (iv) fungsi logaritma diperoleh

$$e^r = \exp(\ln e^r) = \exp r \ln e = \exp r \quad \longrightarrow \quad \exp(x) = e^x$$

## Turunan dan integral fungsi eksponen asli

Dengan menggunakan turunan fungsi invers

Dari hubungan

$$y = e^x \quad \Leftrightarrow \quad x = \ln y$$

↓

$$\frac{dy}{dx} = \frac{1}{dx/dy} = y = e^x \quad \leftarrow \quad \frac{dx}{dy} = \frac{1}{y}$$

Jadi,  $D_x(e^x) = e^x$

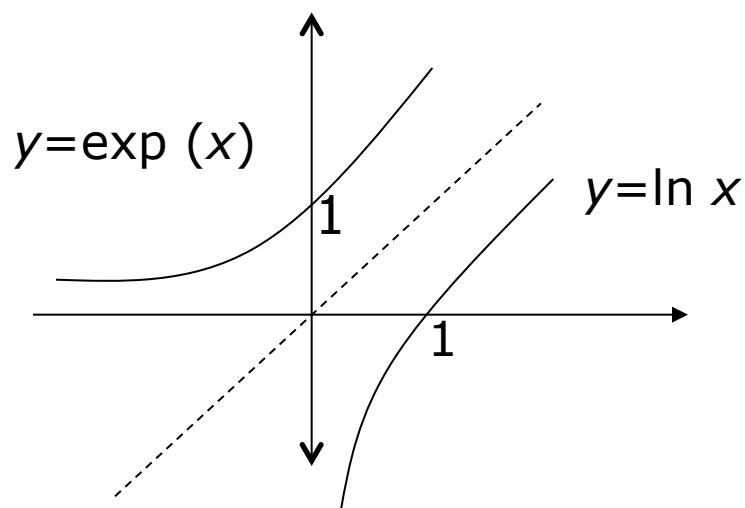
Secara umum  $D_x(e^{u(x)}) = e^u \cdot u'$

Sehingga

$$\int e^x dx = e^x + C$$

# Grafik fungsi eksponen asli

Karena fungsi eksponen asli merupakan invers dari fungsi logaritma asli maka grafik fungsi eksponen asli diperoleh dengan cara mencerminkan grafik fungsi logaritma asli terhadap garis  $y=x$



Contoh

$$D_x(e^{3x \ln x}) = e^{3x \ln x} \cdot D_x(3x \ln x) = e^{3x \ln x} (3 \ln x + 3).$$

## Contoh Hitung

$$\int \frac{e^{3/x}}{x^2} dx$$

Jawab :

Misalkan  $u = \frac{3}{x} \rightarrow du = -\frac{3}{x^2} dx \rightarrow \frac{1}{x^2} dx = -\frac{1}{3} du$

Sehingga

$$\int \frac{e^{3/x}}{x^2} dx = \int -\frac{1}{3} e^u du = -\frac{1}{3} e^u + C = -\frac{1}{3} e^{3/x} + C.$$

# Penggunaan fungsi logaritma dan eksponen asli

Menghitung turunan fungsi berpangkat fungsi

Diketahui  $f(x) = (g(x))^{h(x)}$ ,  $f'(x) = ?$

$$\ln(f(x)) = h(x) \ln(g(x))$$

$$D_x(\ln(f(x))) = D_x(h(x) \ln(g(x)))$$

$$\frac{f'(x)}{f(x)} = h'(x) \ln(g(x)) + \frac{h(x)}{g(x)} g'(x)$$

$$f'(x) = \left( h'(x) \ln(g(x)) + \frac{h(x)}{g(x)} g'(x) \right) f(x)$$

## Contoh

Tentukan turunan fungsi  $f(x) = (\sin x)^{4x}$

Jawab

Ubah bentuk fungsi pangkat fungsi menjadi perkalian fungsi dengan menggunakan fungsi logaritma asli

$$\ln f(x) = \ln(\sin x)^{4x} = 4x \ln(\sin(x))$$

Turunkan kedua ruas

$$D_x(\ln f(x)) = D_x(4x \ln(\sin(x)))$$

$$\frac{f'(x)}{f(x)} = 4 \ln(\sin(x)) + \frac{4x}{\sin x} \cos x = 4 \ln(\sin(x)) + 4x \cot x$$

$$f'(x) = (4 \ln(\sin(x)) + 4x \cot x)(\sin x)^{4x}$$

## B. Selesaikan integral tak tentu berikut

1.  $\int \frac{4}{2x+1} dx$

2.  $\int \frac{\ln^2 3x}{x} dx$

3.  $\int \frac{x^3}{x^2 + 1} dx$

4.  $\int \frac{\tan(\ln x)}{x} dx$

5.  $\int \frac{2}{x(\ln x)^2} dx$

6.  $\int \frac{4x+2}{x^2+x+5} dx$

7.  $\int (x+3)e^{x^2+6x} dx$

8.  $\int e^{-x} \sec^2(2 - e^{-x}) dx$

9.  $\int (\cos x) e^{\sin x} dx$

10.  $\int e^{2 \ln x} dx$

11.  $\int x^2 e^{2x^3} dx$

12.  $\int \frac{e^{2x}}{e^x + 3} dx$

13.  $\int \frac{e^{3x}}{(1 - 2e^{3x})^2} dx$

### C. Selesaikan integral tentu berikut

1. 
$$\int_1^4 \frac{3}{1-2x} dx$$

2. 
$$\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$$

3. 
$$\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x + 4} dx$$

4. 
$$\int_0^{\ln 5} e^x (3 - 4e^x) dx$$

5. 
$$\int_0^1 e^{2x+3} dx$$

6. 
$$\int_0^{\ln 2} e^{-3x} dx$$

7. 
$$\int_1^2 \frac{e^{3/x}}{x^2} dx$$

8. 
$$\int_0^2 xe^{4-x^2} dx$$

9. 
$$\int_e^{e^2} \frac{dx}{x(\ln x)^2}$$

D. Hitung limit berikut :

$$1. \lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$$

$$2. \lim_{x \rightarrow 0} (1 + \sin 2x)^{\frac{1}{x}}$$

$$3. \lim_{x \rightarrow \infty} \left[ \cos \frac{2}{x} \right]^{x^2}$$

$$4. \lim_{x \rightarrow 0^+} (e^{2x} - 1)^{\frac{1}{\ln x}}$$

$$5. \lim_{x \rightarrow \infty} (1 + x^2)^{\frac{1}{\ln x}}$$

$$6. \lim_{x \rightarrow \infty} (\ln x)^{\frac{1}{x}}$$

$$7. \lim_{x \rightarrow \infty} (3^x + 5^x)^{\frac{1}{x}}$$

$$8. \lim_{x \rightarrow \infty} \left( \frac{x+1}{x+2} \right)^x$$

## 8.5 Fungsi Eksponen Umum

Fungsi  $f(x) = a^x$ ,  $a > 0$  disebut fungsi eksponen umum

Untuk  $a > 0$  dan  $x \in \mathbf{R}$ , definisikan  $a^x = e^{x \ln a}$

Turunan dan integral

$$D_x(a^x) = D_x(e^{x \ln a}) = e^{x \ln a} \ln a = a^x \ln a$$

Jika  $u = u(x)$ , maka

$$D_x(a^u) = D_x(e^{u \ln a}) = e^{u \ln a} \ln a \cdot u' = a^u \cdot u' \ln a$$

Dari sini diperoleh :

∴  $\int a^x dx = \frac{1}{\ln a} a^x + C$

## Sifat-sifat fungsi eksponen umum

Untuk  $a > 0$ ,  $b > 0$ ,  $x$ ,  $y$  bilangan riil berlaku

$$1. \quad a^x a^y = a^{x+y}$$

$$2. \quad \frac{a^x}{a^y} = a^{x-y}$$

$$3. \quad (a^x)^y = a^{xy}$$

$$4. \quad (ab)^x = a^x b^x$$

$$5. \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Contoh

1. Hitung turunan pertama dari

$$f(x) = 3^{2x+1} + 2^{\sin 2x}$$

Jawab :

$$f'(x) = 2 \cdot 3^{2x+1} \ln 3 + 2 \cdot 2^{\sin 2x} \cos 2x \ln 2$$

2. Hitung

$$\int 4^{x^2} \cdot x dx$$

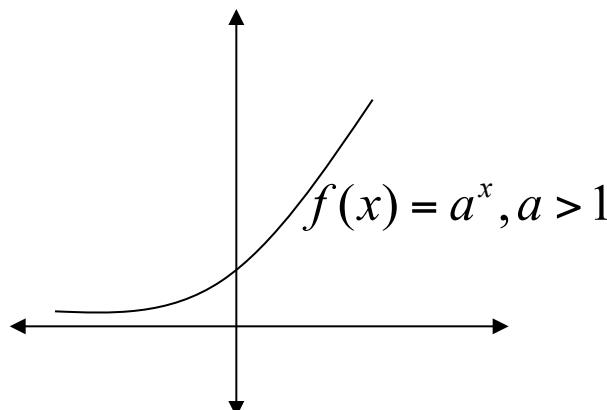
Jawab :

Misal  $u = x^2 \rightarrow du = 2x dx \rightarrow dx = \frac{1}{2x} du$

$\Rightarrow \int 4^{x^2} \cdot x dx = \int 4^u \frac{du}{2} = \frac{1}{2} \frac{4^u}{\ln 4} + C = \frac{4^{x^2}}{2 \ln 4} + C$

# Grafik fungsi eksponen umum

Diketahui



a.  $f(x) = a^x, a > 0$

$$Df = (-\infty, \infty)$$

b.  $f'(x) = a^x \ln a = \begin{cases} a^x \ln a < 0, & 0 < a < 1 \\ a^x \ln a > 0, & a > 1 \end{cases}$

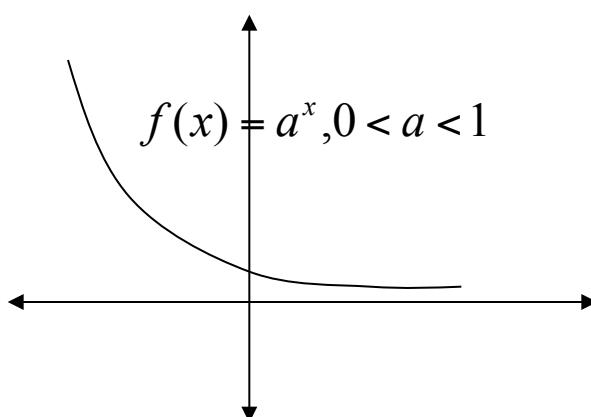
f monoton naik jika  $a > 1$

monoton turun jika  $0 < a < 1$

c.  $f''(x) = a^x (\ln a)^2 > 0 \quad \forall x \in D_f$

Grafik f selalu cekung keatas

d.  $f(0) = 1$



## 8.6 Fungsi Logaritma Umum

Karena fungsi eksponen umum monoton murni maka ada Inversnya. Invers dari fungsi eksponen umum disebut fungsi Logaritma Umum ( logaritma dengan bilangan pokok  $a$  ), notasi  ${}^a \log x$  , sehingga berlaku :

$$y = {}^a \log x \Leftrightarrow x = a^y$$

Dari hubungan ini, didapat

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a} \Rightarrow {}^a \log x = \frac{\ln x}{\ln a}$$

Sehingga  $D_x({}^a \log x) = D_x\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}$

Jika  $u=u(x)$ , maka  $D_x({}^a \log u) = D_x\left(\frac{\ln u}{\ln a}\right) = \frac{u'}{u \ln a}$

Contoh Tentukan turunan pertama dari

$$1. \quad f(x) = {}^3 \log(x^2 + 1)$$

$$2. \quad f(x) = {}^4 \log\left(\frac{x+1}{x-1}\right)$$

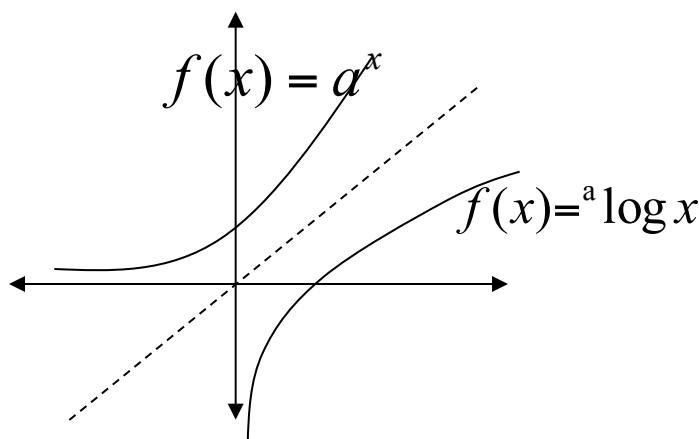
Jawab :

$$\begin{aligned}
 1. \quad f(x) &= {}^3 \log(x^2 + 1) = \frac{\ln(x^2 + 1)}{\ln 3} & \Rightarrow & \quad f'(x) = \frac{2x}{x^2 + 1} \frac{1}{\ln 3} \\
 2. \quad f(x) &= {}^4 \log\left(\frac{x+1}{x-1}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)}{\ln 4} & \Rightarrow & \quad f'(x) = \frac{1}{\ln 4} \frac{1}{\left(\frac{x+1}{x-1}\right)} D_x\left(\frac{x+1}{x-1}\right) \\
 & & & = \frac{1}{\ln 4} \frac{x-1}{x+1} \frac{x-1 - (x+1)}{(x-1)^2} \\
 & & & = \frac{1}{\ln 4} \frac{-2}{(x+1)(x-1)}
 \end{aligned}$$

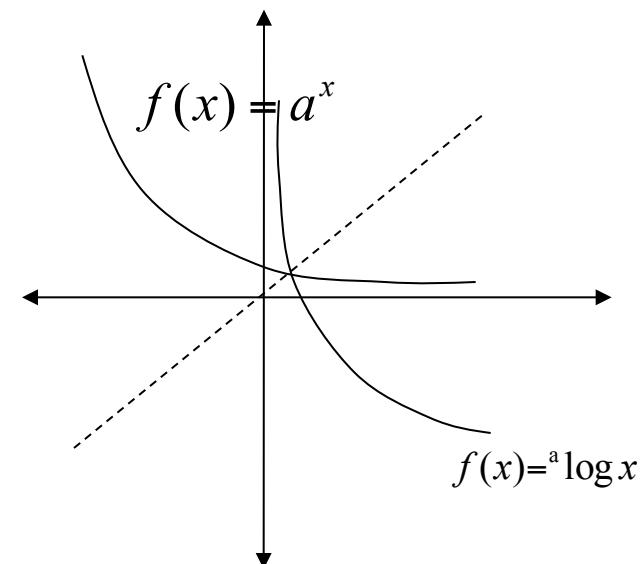
## Grafik fungsi logaritma umum

Grafik fungsi logaritma umum diperoleh dengan mencerminkan grafik fungsi eksponen umum terhadap garis  $y=x$

Untuk  $a > 1$



Untuk  $0 < a < 1$



## Soal Latihan

A. Tentukan  $y'$  dari

$$1. \quad y = 3^{2x^4 - 4x}$$

$$2. \quad y = {}^{10}\log(x^2 + 9)$$

$$3. \quad x^3 \log(xy) + y = 2$$

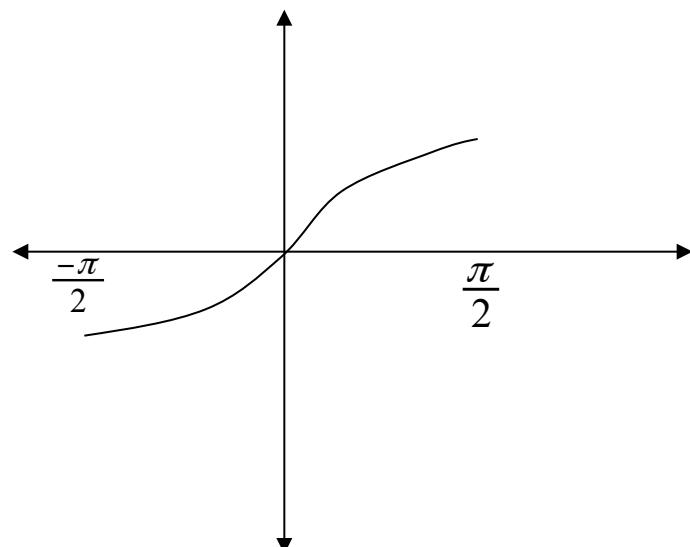
B. Hitung

$$1. \quad \int 10^{5x-1} dx$$

$$2. \quad \int x \cdot 2^{x^2} dx$$

## 8.7 Fungsi Invers Trigonometri

Fungsi trigonometri adalah fungsi yang periodik sehingga tidak satu-satu, jika daerah asalnya dibatasi, fungsi trigonometri bisa dibuat menjadi satu-satu sehingga mempunyai invers.



### a. Invers fungsi sinus

Diketahui  $f(x) = \sin x$ ,  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Karena pada  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$   $f(x)=\sin x$  monoton murni maka inversnya ada. Invers dari fungsi sinus disebut arcus sinus, notasi  $\arcsin(x)$ , atau  $\sin^{-1}(x)$

Sehingga berlaku

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

## Turunan

Dari hubungan  $y = \sin^{-1} x \Leftrightarrow x = \sin y$   $-1 \leq x \leq 1, \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$   
 dan rumus turunan fungsi invers diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}, |x| < 1$$

atau  $D_x(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$

Jika  $u = u(x)$   $D_x(\sin^{-1} u) = \frac{u'}{\sqrt{1 - u^2}}$

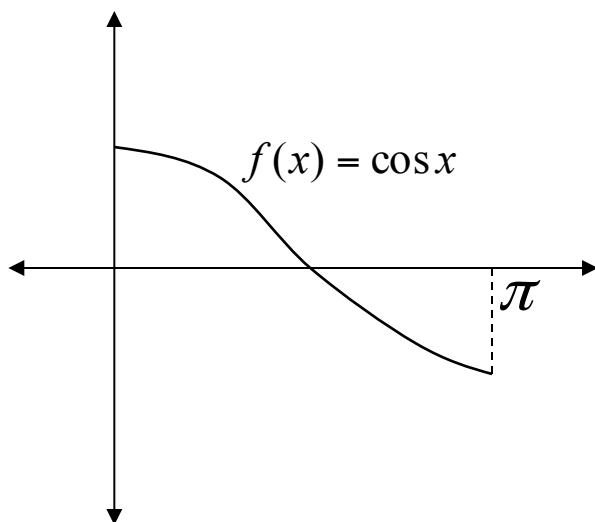
Dari rumus turunan diperoleh

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$$

## b. Invers fungsi cosinus

Fungsi  $f(x) = \cos x$ ,  $0 \leq x \leq \pi$

monoton murni (selalu monoton turun), sehingga mempunyai invers



**Definisi** : Invers fungsi  $\cos x$  disebut  $\text{arc cos} x$ , notasi  $\text{arc cos} x$  atau  $\cos^{-1}(x)$

Berlaku hubungan

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

### Turunan

Dari  $y = \cos^{-1} x \Leftrightarrow x = \cos y$ ,  $-1 \leq x \leq 1, 0 \leq y \leq \pi$  diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$$

atau  $D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

Jika  $u = u(x)$   $D_x(\cos^{-1} u) = \frac{-u'}{\sqrt{1-u^2}}$

Dari rumus turunan diatas diperoleh

$$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$$

Contoh

$$D_x(\sin^{-1}(x^2)) = \frac{1}{\sqrt{1-(x^2)^2}} D_x(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

$$D_x(\cos^{-1}(\tan x)) = \frac{-1}{\sqrt{1-(\tan x)^2}} D_x(\tan x) = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$$

## Contoh Hitung

$$\int \frac{1}{\sqrt{4-x^2}} dx$$

Jawab :

Gunakan rumus

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) + C$$

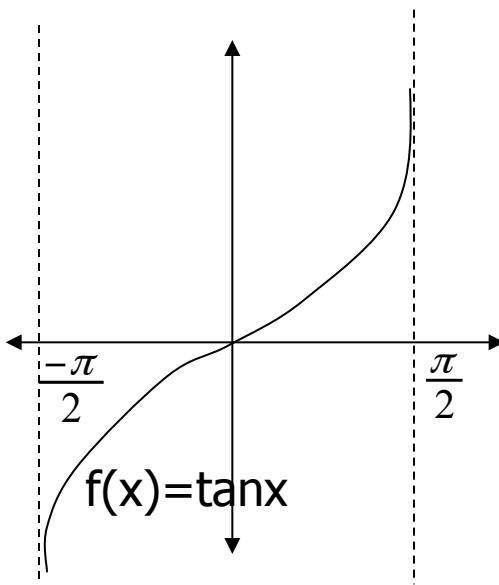
$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{4(1-\frac{x^2}{4})}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(1-(\frac{x}{2})^2}}} dx$$

$$\text{Misal } u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$$

$$\int \frac{1}{\sqrt{4-x^2}} dx = \frac{1}{2} \int \frac{2}{\sqrt{(1-u^2)}} du = \sin^{-1} u + C = \sin^{-1} \left(\frac{x}{2}\right) + C$$

## c. Invers fungsi tangen

Fungsi  $f(x) = \tan x$ ,  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$  Monoton murni (selalu naik)  
sehingga mempunyai invers.



**Definisi** Invers dari  $\tan x$  disebut fungsi arcus  $\tan x$ ,  
notasi  $\text{arc tan } x$  atau  $\tan^{-1}(x)$

Berlaku hubungan

$$y = \tan^{-1} x \Leftrightarrow x = \tan y$$

Turunan

Dari  $y = \tan^{-1} x \Leftrightarrow x = \tan y$ ,  $-\frac{\pi}{2} < y < \frac{\pi}{2}$   
dan turunan fungsi invers diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

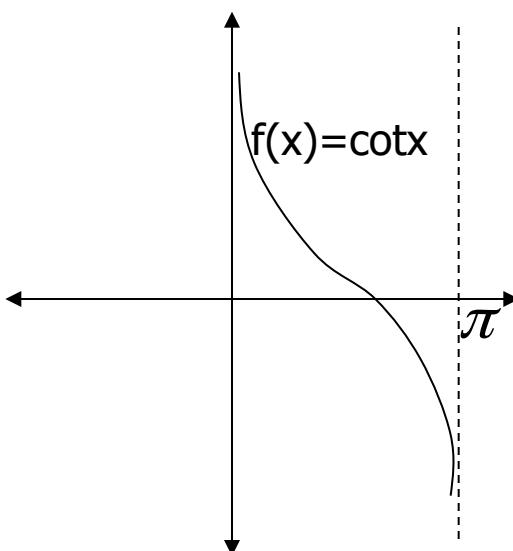
atau  $D_x(\tan^{-1} x) = \frac{1}{1+x^2}$   $\rightarrow \int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Jika  $u=u(x)$   $D_x(\tan^{-1} u) = \frac{u'}{1+u^2}$

#### d. Invers fungsi cotangen

Fungsi  $f(x) = \cot x$ ,  $0 < x < \pi$  selalu monoton turun(monoton murni)

sehingga mempunyai invers



**Definisi** Invers dari fungsi  $\cot x$  disebut Arcus  $\cot x$ , notasi  $\text{arc cot } x$  atau  $\cot^{-1} x$

Berlaku hubungan

$$y = \cot^{-1} x \Leftrightarrow x = \cot y$$

Turunan

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{\csc^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

atau  $D_x(\cot^{-1} x) = \frac{-1}{1+x^2}$    $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

Jika  $u=u(x)$   $D_x(\cot^{-1} u) = \frac{-u'}{1+u^2}$

Contoh

$$D_x(\tan^{-1}(x^2 + 1)) = \frac{1}{1+(x^2+1)^2} D_x(x^2 + 1) = \frac{2x}{1+(x^2+1)^2}$$

$$D_x(\cot^{-1}(\sin x)) = \frac{-1}{1+(\sin x)^2} D_x(\sin x) = \frac{-\cos x}{1+\sin^2 x}$$

Contoh Hitung

a.  $\int \frac{dx}{4+x^2}$

b.  $\int \frac{dx}{x^2 + 2x + 4}$

## Jawab

$$\text{a. } \int \frac{1}{4+x^2} dx = \int \frac{1}{4(1+\frac{x^2}{4})} dx = \frac{1}{4} \int \frac{1}{1+(\frac{x}{2})^2} dx$$

$$u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$$

$$\int \frac{1}{4+x^2} dx = \frac{1}{4} \int \frac{2}{1+u^2} du = \frac{1}{2} \tan^{-1} u + C$$

$$= \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + C$$

Gunakan rumus

$$\int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$$

$$\begin{aligned}
 b. \quad \int \frac{dx}{x^2 + 2x + 4} &= \int \frac{1}{(x+1)^2 + 3} dx = \int \frac{1}{3\left(1 + \frac{(x+1)^2}{3}\right)} dx \\
 &= \frac{1}{3} \int \frac{1}{1 + \left(\frac{(x+1)}{\sqrt{3}}\right)^2} dx
 \end{aligned}$$

Misal  $u = \frac{x+1}{\sqrt{3}} \rightarrow du = \frac{1}{\sqrt{3}} dx \rightarrow dx = \sqrt{3}du$

Gunakan rumus

$$\int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$$

$$\begin{aligned}
 \int \frac{dx}{x^2 + 2x + 4} &= \frac{1}{3} \int \frac{\sqrt{3}}{1+u^2} du = \frac{1}{\sqrt{3}} \tan^{-1} u + C \\
 &= \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x+1}{\sqrt{3}} \right) + C
 \end{aligned}$$

## e. Invers fungsi secan

Diberikan  $f(x) = \sec x$ ,  $0 \leq x \leq \pi$ ,  $x \neq \frac{\pi}{2}$

$$f'(x) = \sec x \tan x > 0, 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$$



$f(x) = \sec x$  monoton murni



Ada inversnya

**Definisi** Invers dari fungsi  $\sec x$  disebut arcus  $\sec x$ , notasi  $\text{arc sec } x$  atau  $\sec^{-1} x$

Sehingga

$$y = \sec^{-1} x \Leftrightarrow x = \sec y$$

## Turunan

Dari  $y = \sec^{-1} x \Leftrightarrow x = \sec y$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\downarrow \quad \uparrow$$

$$\cos y = \frac{1}{x} \quad \longrightarrow \quad y = \cos^{-1}\left(\frac{1}{x}\right)$$

Sehingga

$$D_x(\sec^{-1} x) = D_x\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \cdot \frac{-1}{x^2} = \frac{|x|}{x^2 \sqrt{x^2 - 1}} = \frac{1}{|x| \sqrt{x^2 - 1}}$$

Jika  $u = u(x)$      $D_x(\sec^{-1} u) = \frac{u'}{|u| \sqrt{u^2 - 1}}$

$$\rightarrow \int \frac{1}{x \sqrt{x^2 - 1}} dx = \sec^{-1} |x| + C$$

## e. Invers fungsi cosecan

Diberikan  $f(x) = \csc x$ ,  $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ ,  $x \neq 0$

$$f'(x) = -\csc x \cot x < 0, \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$$



$f(x) = \sec x$  monoton murni



Ada inversnya

**Definisi** Invers dari fungsi  $\csc x$  disebut arcus  $\csc x$ , notasi  $\text{arc csc } x$  atau  $\csc^{-1} x$

Sehingga

$$y = \csc^{-1} x \Leftrightarrow x = \csc y$$

## Turunan

Dari  $y = \csc^{-1} x \Leftrightarrow x = \csc y$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\begin{array}{ccc} \downarrow & & \uparrow \\ \sin y = \frac{1}{x} & \longrightarrow & y = \sin^{-1}\left(\frac{1}{x}\right) \end{array}$$

Sehingga

$$D_x(\csc^{-1} x) = D_x\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \cdot \frac{-1}{x^2} = \frac{-|x|}{x^2 \sqrt{x^2 - 1}} = \frac{-1}{|x| \sqrt{x^2 - 1}}$$

Jika  $u = u(x)$        $D_x(\sec^{-1} u) = \frac{-u'}{|u| \sqrt{u^2 - 1}}$

$$\rightarrow \int \frac{1}{x \sqrt{x^2 - 1}} dx = -\csc^{-1}|x| + C$$

## Contoh

A. Hitung turunan pertama dari

a.  $f(x) = \sec^{-1}(x^2)$

b.  $f(x) = \sec^{-1}(\tan x)$

Jawab

a.  $f'(x) = \frac{1}{|x^2| \sqrt{(x^2)^2 - 1}} Dx(x^2) = \frac{2x}{x^2 \sqrt{x^4 - 1}} = \frac{2}{x \sqrt{x^4 - 1}}$

b.  $f'(x) = \frac{1}{|\tan x| \sqrt{(\tan x)^2 - 1}} Dx(\tan x) = \frac{\sec^2 x}{|\tan x| \sqrt{\tan^2 x - 1}}$

## B. Hitung

$$\int \frac{1}{x\sqrt{x^2 - 4}} dx$$

Jawab

$$\int \frac{1}{x\sqrt{x^2 - 4}} dx = \int \frac{1}{x\sqrt{4(\frac{x^2}{4} - 1)}} dx = \frac{1}{2} \int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2 - 1}} dx$$

Misal  $u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2 - 4}} dx &= \frac{1}{2} \int \frac{1}{2u\sqrt{u^2 - 1}} 2du = \frac{1}{2} \int \frac{1}{u\sqrt{u^2 - 1}} du \\ &= \frac{1}{2} \sec^{-1} |u| + C = \frac{1}{2} \sec^{-1} \left| \frac{x}{2} \right| + C \end{aligned}$$

## Soal Latihan

A. Tentukan turunan pertama fungsi berikut, sederhanakan jika mungkin

$$1. \quad y = (\sin^{-1} x)^2$$

$$2. \quad y = \tan^{-1}(e^x)$$

$$3. \quad y = \tan^{-1} x \ln x$$

$$4. \quad f(t) = e^{\sec^{-1} t}$$

$$5. \quad y = x^2 \cot^{-1}(3x)$$

$$6. \quad y = \tan^{-1}(x - \sqrt{1 + x^2})$$

## B. Hitung

$$1. \int \frac{dx}{9x^2 + 16}$$

$$2. \int \frac{dx}{4x\sqrt{x^2 - 16}}$$

$$3. \int \frac{dx}{\sqrt{2 - 5x^2}}$$

$$4. \int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$$

$$5. \int \frac{e^x}{e^{2x} + 1} dx$$

$$6. \int \frac{e^{2x}}{\sqrt{1 - e^{4x}}} dx$$

$$7. \int \frac{dx}{x[4 + (\ln x)^2]}$$

## 8.8 Fungsi Hiperbolik

Definisi

a. Fungsi kosinus hiperbolik :

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

b. Fungsi sinus hiperbolik :

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

c. Fungsi tangen hiperbolik :

$$f(x) = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d. Fungsi cotangen hiperbolik :

$$f(x) = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

e. Fungsi secan hiperbolik :

$$f(x) = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x - e^{-x}}$$

f. Fungsi cosecan hiperbolik :

$$f(x) = \operatorname{csc} h x = \frac{1}{\sinh x} = \frac{2}{e^x + e^{-x}}$$

# Persamaan identitas pada fungsi hiperbolik

- $$\left. \begin{array}{l} 1. \cosh x + \sinh x = e^x \\ 2. \cosh x - \sinh x = e^{-x} \end{array} \right\} \quad \begin{array}{l} 3. \cosh^2 x - \sinh^2 x = 1 \\ 4. 1 - \tanh^2 x = \sec h^2 x \\ 5. \coth^2 x - 1 = \csc h^2 x \end{array}$$

## Turunan

$$D_x(\cosh x) = D_x\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \implies \int \sinh x dx = \cosh x + C$$

$$D_x(\sinh x) = D_x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \implies \int \cosh x dx = \sinh x + C$$

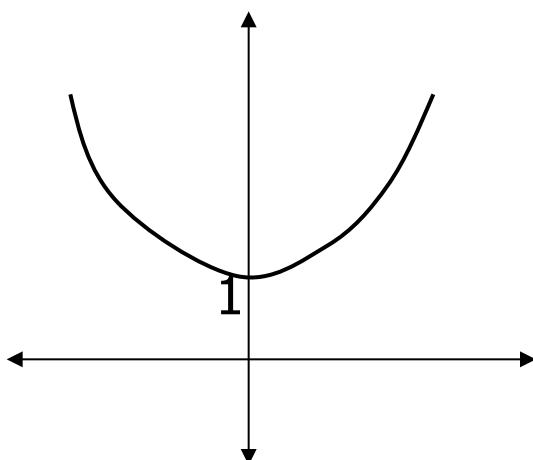
$$D_x(\tanh x) = D_x\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\begin{aligned} D_x(\coth x) &= D_x\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x \end{aligned}$$

$$D_x(\operatorname{sech} x) = D_x\left(\frac{1}{\cosh x}\right) = \frac{-\sinh x}{\cosh^2 x} = -\operatorname{sech} x \operatorname{tanh} x$$

$$D_x(\operatorname{cosech} x) = D_x\left(\frac{1}{\sinh x}\right) = \frac{-\cosh x}{\sinh^2 x} = -\operatorname{cosech} x \operatorname{coth} x$$

# Grafik $f(x) = \cosh x$



Diketahui

$$(i) \quad f(x) = \cosh x = \frac{e^x + e^{-x}}{2}, x \in R$$

$$(ii) \quad f'(x) = \frac{e^x - e^{-x}}{2} = \begin{cases} f'(x) < 0, & x < 0 \\ f'(x) > 0, & x > 0 \end{cases}$$

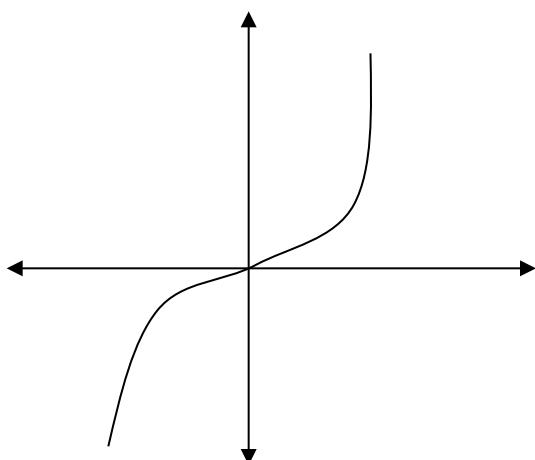
$f$  monoton naik pada  $x > 0$   
 monoton turun pada  $x < 0$

$$(iii) \quad f''(x) = \frac{e^x + e^{-x}}{2} > 0, \forall x \in R$$

Grafik  $f$  selalu cekung keatas

$$(iv) \quad f(0) = 1$$

# Grafik $f(x) = \sinh x$



Diketahui

$$(i) \quad f(x) = \sinh x = \frac{e^x - e^{-x}}{2}, x \in R$$

$$(ii) \quad f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$f$  selalu monoton naik

$$(iii) \quad f''(x) = \frac{e^x - e^{-x}}{2} = \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases}$$

Grafik  $f$  cekung keatas pada  $x > 0$   
cekung kebawah pada  $x < 0$

$$(iv) \quad f(0) = 0$$

## Contoh

Tentukan  $y'$  dari

$$1. \quad y = \tanh(x^2 + 1)$$

$$2. \quad x^2 \sinh x + y^2 = 8$$

Jawab

$$1. \quad y' = \operatorname{sech}^2(x^2 + 1) D_x(x^2 + 1) = 2x \operatorname{sech}^2(x^2 + 1)$$

$$2. \quad D_x(x^2 \sinh x + y^2) = D_x(8)$$

$$2x \sinh x + x^2 \cosh x + 2y y' = 0$$

$$y' = -\frac{2x \sinh x + x^2 \cosh x}{2y}$$

## Soal Latihan

A. Tentukan turunan pertama dari

$$1. \quad f(x) = \tanh 4x$$

$$2. \quad g(x) = \sinh^2 x$$

$$3. \quad g(x) = \frac{1 - \cosh x}{1 + \cosh x}$$

$$4. \quad h(t) = \coth \sqrt{1 + t^2}$$

$$5. \quad g(t) = \ln(\sinh t))$$

$$6. \quad f(x) = x \cosh x^2$$

## B. Hitung integral berikut

$$1. \int \sinh(1 + 4x) dx$$

$$2. \int \sinh x \cosh^2 x dx$$

$$3. \int \tanh x dx$$

$$4. \int \frac{\operatorname{sech}^2 x}{2 + \tanh x} dx$$