Section 4.5 Exponential and Logarithmic Equations

Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent.

EXAMPLE: Solve the equation $2^x = 7$.

Solution 1: We have

$$\begin{aligned} 2^x &= 7 \\ \left[\log_2 2^x \right. &= \left. \log_2 7\right] \\ \left[x \log_2 2 \right. &= \left. \log_2 7\right] \\ x &= \log_2 7 \approx 2.807 \end{aligned}$$

Solution 2: We have

$$\begin{aligned} 2^x &= 7 \\ \ln 2^x &= \ln 7 \\ x \ln 2 &= \ln 7 \\ x &= \frac{\ln 7}{\ln 2} \approx 2.807 \end{aligned}$$

EXAMPLE: Solve the equation $4^{x+1} = 3$.

Solution 1: We have

$$\begin{aligned} 4^{x+1} &= 3 \\ \left[\log_4 4^{x+1} \right. &= \left. \log_4 3\right] \\ \left[(x+1) \log_4 4 \right. &= \left. \log_4 3\right] \\ x+1 &= \log_4 3 \\ x &= \log_4 3 - 1 \approx -0.208 \end{aligned}$$

Solution 2: We have

$$\begin{aligned} 4^{x+1} &= 3 \\ \ln 4^{x+1} &= \ln 3 \\ (x+1) \ln 4 &= \ln 3 \\ x+1 &= \frac{\ln 3}{\ln 4} \\ x &= \frac{\ln 3}{\ln 4} - 1 \approx -0.208 \end{aligned}$$

EXAMPLE: Solve the equation $3^{x-3} = 5$.

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Solution 1: We have

$$\begin{aligned} 3^{x-3} &= 5 \\ x - 3 &= \log_3 5 \\ x &= \log_3 5 + 3 \approx 4.465 \end{aligned}$$

Solution 2: We have

$$\begin{aligned} 3^{x-3} &= 5 \\ \ln 3^{x-3} &= \ln 5 \\ (x-3)\ln 3 &= \ln 5 \\ x - 3 &= \frac{\ln 5}{\ln 3} \\ x &= \frac{\ln 5}{\ln 3} + 3 \approx 4.465 \end{aligned}$$

EXAMPLE: Solve the equation $8e^{2x} = 20$.

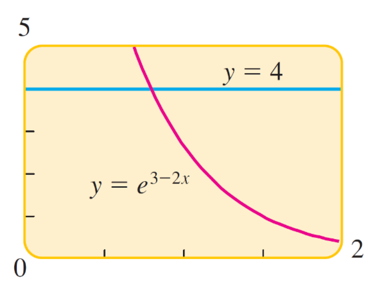
Solution: We have

$$\begin{aligned} 8e^{2x} &= 20 \\ e^{2x} &= \frac{20}{8} = \frac{5}{2} \\ 2x &= \ln\frac{5}{2} \\ x &= \frac{\ln\frac{5}{2}}{2} = \frac{1}{2}\ln\frac{5}{2} \approx 0.458 \end{aligned}$$

EXAMPLE: Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution: We have

$$\begin{aligned} e^{3-2x} &= 4 \\ 3 - 2x &= \ln 4 \\ -2x &= \ln 4 - 3 \\ x &= \frac{\ln 4 - 3}{-2} = \frac{3 - \ln 4}{2} \approx 0.807 \end{aligned}$$



EXAMPLE: Solve the equation $e^{2x} - e^x - 6 = 0$.

Solution 1: We have

$$e^{2x} - e^x - 6 = 0$$

$$(e^x)^2 - e^x - 6 = 0$$

$$(e^x - 3)(e^x + 2) = 0$$

$$e^x - 3 = 0 \qquad \text{or} \qquad e^x + 2 = 0$$

$$e^x = 3 \qquad\qquad e^x = -2$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus, $x = \ln 3 \approx 1.0986$ is the only solution.

Solution 1′: Put $e^x = w$. Then

$$e^{2x} - e^x - 6 = 0$$

$$(e^x)^2 - e^x - 6 = 0$$

$$w^2 - w - 6 = 0$$

$$(w - 3)(w + 2) = 0$$

$$w - 3 = 0 \qquad \text{or} \qquad w + 2 = 0$$

$$w = 3 \qquad\qquad w = -2$$

$$e^x = 3 \qquad\qquad e^x = -2$$

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EXAMPLE: Solve the equation $e^{2x} - 3e^x + 2 = 0$.

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Solution 1: We have

$$e^{2x} - 3e^x + 2 = 0$$

$$(e^x)^2 - 3e^x + 2 = 0$$

$$(e^x - 1)(e^x - 2) = 0$$

$$\begin{array}{rcr} e^x - 1 = 0 & \text{or} & e^x - 2 = 0 \\ e^x = 1 & & e^x = 2 \\ x = 0 & & x = \ln 2 \end{array}$$

Solution $1'$: Put $e^x = w$. Then

$$w^2 - 3w + 2 = 0$$

$$(w - 1)(w - 2) = 0$$

$$\begin{array}{rcr} w - 1 = 0 & \text{or} & w - 2 = 0 \\ w = 1 & & w = 2 \\ e^x = 1 & & e^x = 2 \\ x = 0 & & x = \ln 2 \end{array}$$

EXAMPLE: Solve the equation $7^{2x} - 3 \cdot 7^x + 1 = 0$.

Solution: Put $7^x = w$. Then

$$w^2 - 3w + 1 = 0 \quad \Longrightarrow \quad w = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{3 \pm \sqrt{5}}{2}$$

so $7^x = \dfrac{3 \pm \sqrt{5}}{2}$, therefore $x = \log_7 \left(\dfrac{3 \pm \sqrt{5}}{2} \right)$.

EXAMPLE: Solve the equation $7^{2x} - 7^x - 1 = 0$.

Solution: Put $7^x = w$. Then

$$w^2 - w - 1 = 0 \quad \Longrightarrow \quad w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

Since $\dfrac{1 - \sqrt{5}}{2} < 0$, it follows that

$$7^x = \frac{1 + \sqrt{5}}{2} \quad \Longrightarrow \quad x = \log_7 \left(\frac{1 + \sqrt{5}}{2} \right)$$

EXAMPLE: Solve the equation $3xe^x + x^2 e^x = 0$.

Solution: We have

$$3xe^x + x^2 e^x = 0$$

$$xe^x (3 + x) = 0$$

$$x(3 + x) = 0$$

$$\begin{array}{rcr} x = 0 & \text{or} & 3 + x = 0 \\ x = 0 & & x = -3 \end{array}$$

Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs.

EXAMPLE: Solve the equation $\ln x = 8$.

Solution: We have

$$\begin{aligned} \ln x &= 8 \\ \left[e^{\ln x} \right. &= \left. e^8\right] \\ x &= e^8 \end{aligned}$$

EXAMPLE: Solve the equation $\log_2(x+2) = 5$.

Solution: We have

$$\begin{aligned} \log_2(x+2) &= 5 \\ \left[2^{\log_2(x+2)} \right. &= \left. 2^5\right] \\ x+2 &= 2^5 \\ x &= 2^5 - 2 = 32 - 2 = 30 \end{aligned}$$

EXAMPLE: Solve the equation $\log_7(25-x) = 3$.

Solution: We have

$$\begin{aligned} \log_7(25-x) &= 3 \\ \left[7^{\log_7(25-x)} \right. &= \left. 7^3\right] \\ 25-x &= 7^3 \\ x &= 25 - 7^3 = 25 - 343 = -318 \end{aligned}$$

EXAMPLE: Solve the equation $4 + 3\log(2x) = 16$.

EXAMPLE: Solve the equation $4+3\log(2x)=16$.

Solution: We have

$$4+3\log(2x)=16$$
$$3\log(2x)=12$$
$$\log(2x)=4$$
$$2x=10^4$$
$$x=\frac{10^4}{2}=\frac{10,000}{2}=5,000$$

EXAMPLE: Solve the equation $\log(x+2)+\log(x-1)=1$ algebraically and graphically.

Solution: We have

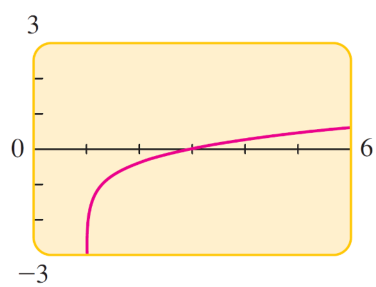
$$\log(x+2)+\log(x-1)=1$$
$$\log[(x+2)(x-1)]=1$$
$$(x+2)(x-1)=10$$
$$x^2+x-2=10$$
$$x^2+x-12=0$$
$$(x+4)(x-3)=0$$
$$x+4=0 \quad \text{or} \quad x-3=0$$
$$x=-4 \qquad\qquad x=3$$

We check these potential solutions in the original equation and find that $x=-4$ is not a solution (because logarithms of negative numbers are undefined), but $x=3$ is a solution.

To solve the equation graphically we rewrite it as

$$\log(x+2)+\log(x-1)-1=0$$

and then graph $y=\log(x+2)+\log(x-1)-1$. The solutions are the x-intercepts of the graph.



EXAMPLE: Solve the following equations

(a) $\log(x+8)+\log(x-1)=1$ (b) $\log(x^2-1)-\log(x+1)=3$

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Solution:

(a) We have

$$\log(x+8)+\log(x-1)=1$$

$$\log[(x+8)(x-1)]=1$$

$$(x+8)(x-1)=10$$

$$x^2+7x-8=10$$

$$x^2+7x-18=0$$

$$(x+9)(x-2)=0$$

$$x+9=0 \quad \text{or} \quad x-2=0$$

$$x=-9 \qquad\qquad x=2$$

We check these potential solutions in the original equation and find that $x=-9$ is not a solution (because logarithms of negative numbers are undefined), but $x=2$ is a solution.

(b) We have

$$\log(x^2-1)-\log(x+1)=3$$

$$\log\frac{x^2-1}{x+1}=3$$

$$\frac{x^2-1}{x+1}=10^3$$

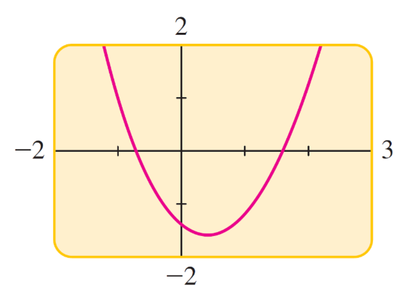
$$\frac{(x-1)(x+1)}{x+1}=1000$$

$$x-1=1000$$

$$x=1001$$

EXAMPLE: Solve the equation $x^2=2\ln(x+2)$ graphically.

Solution: We first move all terms to one side of the equation $x^2-2\ln(x+2)=0$. Then we graph $y=x^2-2\ln(x+2)$. The solutions are the x-intercepts of the graph.



EXAMPLE: Find the solution of the equation, correct to two decimal places.

(a) $10^{x+3}=6^{2x}$ (b) $5\ln(3-x)=4$ (c) $\log_2(x+2)+\log_2(x-1)=2$

EXAMPLE: Find the solution of the equation, correct to two decimal places.

(a) $10^{x+3} = 6^{2x}$ (b) $5\ \ln(3-x) = 4$ (c) $\log_2(x+2) + \log_2(x-1) = 2$

Solution:

(a) We have

$$
\begin{aligned}
10^{x+3} &= 6^{2x} \\
\ln 10^{x+3} &= \ln 6^{2x} \\
(x+3)\ln 10 &= 2x \ln 6 \\
x \ln 10 + 3\ln 10 &= 2x \ln 6 \\
x \ln 10 - 2x \ln 6 &= -3\ln 10 \\
x(\ln 10 - 2\ln 6) &= -3\ln 10 \\
x &= \frac{-3\ln 10}{\ln 10 - 2\ln 6} \approx 5.39
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
5\ln(3-x) &= 4 \\
\ln(3-x) &= \frac{4}{5} \\
3 - x &= e^{4/5} \\
x &= 3 - e^{4/5} \approx 0.77
\end{aligned}
$$

(c) We have

$$
\begin{aligned}
\log_2(x+2) + \log_2(x-1) &= 2 \\
\log_2(x+2)(x-1) &= 2 \\
(x+2)(x-1) &= 4 \\
x^2 + x - 2 &= 4 \\
x^2 + x - 6 &= 0 \\
(x-2)(x+3) &= 0
\end{aligned}
$$

$$
\begin{aligned}
x - 2 &= 0 \qquad \text{or} \qquad & x + 3 &= 0 \\
x &= 2 & x &= -3
\end{aligned}
$$

Since $x = -3$ is not from the domain of $\log_2(x+2) + \log_2(x-1)$, the only answer is $x = 2$.

Applications

EXAMPLE: If I_0 and I denote the intensity of light before and after going through a material and x is the distance (in feet) the light travels in the material, then according to the **Beer-Lambert Law**

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x$$

where k is a constant depending on the type of material.

(a) Solve the equation for I.

(b) For a certain lake $k = 0.025$ and the light intensity is $I_0 = 14$ lumens (lm). Find the light intensity at a depth of 20 ft.

Solution:

(a) We first isolate the logarithmic term.

$$\begin{aligned}
-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) &= x \\
\ln\left(\frac{I}{I_0}\right) &= -kx \\
\frac{I}{I_0} &= e^{-kx} \\
I &= I_0 e^{-kx}
\end{aligned}$$

(b) We find I using the formula from part (a).

$$I = I_0 e^{-kx} = 14e^{(-0.025)(20)} \approx 8.49$$

The light intensity at a depth of 20 ft is about 8.5 lm.

EXAMPLE: A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following method.

(a) Semiannual (b) Continuous

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(a) Semiannual (b) Continuous

Solution:
(a) We use the formula for compound interest

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

with $P = \$5000,\ A(t) = \$10,000,\ r = 0.05,\ n = 2$ and solve the resulting exponential equation for t.

$$5000\left(1 + \frac{0.05}{2}\right)^{2t} = 10,000$$

$$(1.025)^{2t} = 2$$

$$\log 1.025^{2t} = \log 2$$

$$2t \log 1.025 = \log 2$$

$$t = \frac{\log 2}{2 \log 1.025} \approx 14.04$$

The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest

$$A(t) = Pe^{rt}$$

with $P = \$5000,\ A(t) = \$10,000,\ r = 0.05$ and solve the resulting exponential equation for t.

$$5000e^{0.05t} = 10,000$$

$$e^{0.05t} = 2$$

$$0.05t = \ln 2$$

$$t = \frac{\ln 2}{0.05} \approx 13.86$$

The money will double in 13.86 years.

EXAMPLE: A sum of \$1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to \$4000 if interest is compounded continuously.

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Solution: We use the formula for continuously compounded interest

$$A(t) = Pe^{rt}$$

with $P = \$1000,\ A(t) = \$4000,\ r = 0.04$ and solve the resulting exponential equation for t.

$$\begin{aligned} 1000e^{0.04t} &= 4000 \\ e^{0.04t} &= 4 \\ 0.04t &= \ln 4 \\ t &= \frac{\ln 4}{0.04} \approx 34.66 \end{aligned}$$

The amount will be \$4000 in about 34 years and 8 months.