5.4 Exponential Functions: Differentiation and Integration

TOOTLIFTST:

Exponential functions are of the form $f(x) = Ab^x$. We will, in this section, look at a specific type of exponential function where the base, *b*, is $e \approx 2.718...$. This function is called the Natural Exponential Function

$$f(x) = e^x$$

Like all exponential functions, it is one-to-one, therefore, it has an inverse. It has a *y*-intercept of 1, a Horizontal Asymptote on the *x*-axis, and is monotonic increasing.

This inverse of $f(x) = e^x$ is the natural log function $g(x) = \ln x$. This means that $y = e^x$ if and only if $x = \ln y$.

Also, we get the following relationships

$\ln\left(e^x\right) = x$ and $e^{\ln x} = x$

Here are a couple of examples that utilize these properties.

Remember from Precalculus, that to solve exponential equations, all you had to do was take the natural log of both sides, regardless of what the exponential base was. For example:

Solve $7 = e^{x+1}$

The first thing you were trained to do was to look if you could use the GTBTS method (Getting the bases the same method.) In this case, it is not going to be easy to write 7 as *e* to the somethingth power, so we take the natural log of both sides.

$$\ln 7 = \ln e^{x+1}$$
$$\ln 7 = (x+1)\ln e$$
$$\ln 7 = (x+1)(1)$$
$$x = \ln 7 - 1 \approx 0.946$$

Here's another example of solving an exponential equation.

Solve $\ln(2x-3) = 5$

Notice here, there is already a natural log in the problem, and we CANNOT distribute it. In order to "get at" the *x* inside the log, we must first get rid of it by exponentiating both sides.

$e^{\ln(2x-3)} = e^5$ (This is like the inverse of the GTBTS method)

$2x - 3 = e^5$

$2x = e^5 + 3$

$x = \dfrac{e^5 + 3}{2} \approx 75.707$

Now for the calculus part of *e*.

One of the most intriguing, bizarre, and useful characteristics of $f(x) = e^x$ is the fact that ***it is its own derivative***!! Please reread the last sentence, then read on. $f(x) = e^x$ is the fact that ***it is its own derivative***!! That's like someone giving birth to himself! Perhaps like asexual reproduction by budding, who knows? Perhaps the *e* really stands for *e*asy.?!

Here's the official rule:

The Derivative of the Natural Exponential Function

Let *u* be a differential function of *x*.

Then 1. $\dfrac{d}{dx}e^x = e^x$ and 2. $\dfrac{d}{dx}e^u = e^u\left(\dfrac{du}{dx}\right)$ (this is the good ol' chain rule again.)

Graph $f(x) = e^x$ and then graph the derivative by analyzing the slopes. You will quickly convince yourself of the above. But why doesn't it work for other bases?

Let's try some quick derivatives:

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a. $\dfrac{d}{dx}e^{2x-1} = \underbrace{\left(e^{2x-1}\right)}_{itself}\underbrace{\left(\dfrac{d}{dx}(2x-1)\right)}_{chain\ rule} = \underbrace{2e^{2x-1}}_{simplified}$

b. $\dfrac{d}{dx}e^{-3/x} = \left(e^{-3/x}\right)\left(\dfrac{3}{x^2}\right) = \dfrac{3e^{-3/x}}{x^2}$

We can now use this rule in conjunction with other rules to find useful information:

c. Find the Relative Extrema of $f(x) = xe^x$

$f'(x) = (1)(e^x) + (x)(e^x)$ by the product rule

$f'(x) = e^x(1+x) = 0$

$e^x = 0$ **or** $1 + x = 0$

because e^x will never equal zero (HA @ $y = 0$), the only solution is $x = -1$, which is a relative min by the first derivative test. The actual point is $(-1, -e^{-1})$.

See how easy it was going forward, that is, differentiating? It's almost as easy going backwards, or integrating. All it requires you to do is to notice the same patterns you have grown accustomed to looking for.

Here's the official rules:

Integration rules for Natural Exponential Functions

Let u be a differentiable function of x.

Then 1. $\int e^x dx = e^x + C$ and 2. $\int e^u du = e^u + C$

Ha! So it's not only its own derivative, but its own integral as well. Pretty creepy.

The pattern you are looking for now will involve the function u that is the exponent of the e factor. You will likely call this the "inside" function and checking to see if its derivative is the other factor on the "outside."

Por ejemplo:

$$\int e^{3x-1} dx$$

The derivative of the exponent of e is 3. We do not have a scalar multiple of 3 on the outside, so we will have to make a correction of 1/3 in our guess (which will, or course, be correct.)

$$= \frac{1}{3} e^{3x-1} + C$$ Piece of π!

Now, remember that you can only make corrections of Scalar Multiples, not of variables. If the integral had been $\int e^{x^2}\,dx$, the answer would ***NOT*** be $\frac{1}{2x}e^{x^2}+C$! U-substitution would not work here either. This integral, to you, right now, is not possible.

Here's another example:

$$\int 5xe^{-x^2}\,dx$$

First, rewrite the integral, pulling all existing scalar multiples out front.

$$=5\int xe^{-x^2}\,dx$$

Notice that from the derivative of the exponent of *e*, we are only of by a negative two, so our guess has the correction in it, and ONLY involves the *e* part (remember, the *x* part comes from the chain rule upon differentiation.)

$$=(5)\left(\frac{-1}{2}\right)\left(e^{-x^2}\right)+C=\frac{-5}{2}e^{-x^2}+C$$

Here are some more example of both indefinite and definite integrals. Try to do them quickly on your own before, that's BEFORE, you look at the answer, rewriting first when you need to. Since the answers will be right next to the problems, this will not only be a test of your integral abilities, but also of your Emotional Intelligence!

Don't forget: All the previous rules you have learned can be used too!!

a. $\int \frac{e^{1/x}}{x^2}dx=\int\left(e^{1/x}\right)\left(\frac{1}{x^2}\right)dx=-e^{1/x}+C$

b. $\int \sin x e^{\cos x}dx=-e^{\cos x}+C$

c. $\int_0^1 \frac{e^x}{1+e^x}dx=\ln\left|1+e^x\right|\Big|_0^1=\left(\ln(1+e)-\ln 2\right)\approx 0.620$ (that one was fun)

d. $\int_{-1}^{0} e^x\cos e^x\,dx=\sin e^x\Big|_{-1}^{0}=\sin 1-\sin e^{-1}\approx 0.482$

Did you get them correct? The last two were kinda' tricky, and the last one was very triggy. Notice that in each of them, the "inside" function was not the exponent of *e*, because *e* was not the Main or "outer" function. This is easy to spot if you had trouble: the e^u function appeared MORE THAN ONCE in the integrand, meaning, that it HAD to be the "inner" function itself, so that it can generate itself upon differentiating with the chain rule. Does that make sense? Reread, until it does.

Here's the grand finale, finally.

$\int xe^{x^2} \sec^2 e^{x^2} \tan e^{x^2}\, dx$ OUCH, it's even painful to look at.

This one requires some clever rewriting using the commutative and associative properties of multiplication.

$$\int \left(\sec e^{x^2}\right)\left(\sec e^{x^2} \tan e^{x^2}\right)\left(e^{x^2}\right)(x)dx$$

Notice the derivative of the first factor is the second, and the third factor is the derivative of the angle AND the last factor is the derivative of the angle's exponent (off by only a 2.) This is the chain rule inside of the chain rule which will require the Power rule.

Here's our correct guess (with the correction)

$$= \underbrace{\left(\frac{1}{2}\right)}_{correction} \underbrace{\left(\frac{1}{2}\right)}_{power\ rule} \left(\sec e^{x^2}\right)^2 + C = \frac{\sec^2 e^{x^2}}{4} + C$$

If you don't see it, try *u*-substitution, letting $u = \sec e^{x^2}$, or by taking the derivative of the answer until you see the pattern. Happy integrating.