Calculus 1
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Integrals of Exponential and Trigonometric Functions. Integrals Producing Logarithmic Functions.

Integrals of exponential functions. Since the derivative of e^x is e^x, e^x is an antiderivative of e^x. Thus

$$\int e^x\,dx = e^x + c$$

Recall that the exponential function with base a^x can be represented with the base e as $e^{\ln a^x} = e^{x\ln a}$. With substitution $u = x\ln a$ and using the above formula for the integral of e^x, we have that

$$\int a^x dx = \int e^{x\ln a} dx = \int e^u \frac{du}{\ln a} = \frac{1}{\ln a}\int e^u du = \frac{1}{\ln a}e^u + c = \frac{1}{\ln a}e^{x\ln a} + c = \frac{1}{\ln a}a^x + c.$$

Integrals producing logarithmic functions. Recall that the Power Rule formula for integral of x^n is valid just for $n \neq 1$ because of zero in denominator of $\frac{1}{n+1}x^{n+1}$ when $n = -1$. Thus, this rule does not apply to the integral $\int \frac{1}{x}dx$. However, this integral can be evaluated using the fact that derivative of $\ln x$ is $\frac{1}{x}$. Since $\ln x$ is defined just for $x > 0$, we have that $\ln x$ is an antiderivative of $\frac{1}{x}$ for $x > 0$.

If x is negative, the derivative of $\ln(-x)$ is $\frac{1}{-x}(-1) = \frac{1}{x}$ so that we can conclude that $\ln|x|$ is an antiderivation of $\frac{1}{x}$ both for $x > 0$ and $x < 0$. Thus,

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

Be careful about the following.

1. The formula $\int \frac{1}{x}\,dx = \ln|x| + c$ does <u>not</u> imply that $\int \frac{1}{x^2}\,dx = \ln|x^2| + c$. Use the power rule for $\int x^{-2}dx$ to get the answer $\frac{-1}{x} + c$.
2. The fact that $\int \frac{1}{x^2}\,dx = \frac{1}{-2+1}x^{-2+1} + c$ does <u>not</u> imply that $\int \frac{1}{x}\,dx = \frac{1}{-1+1}x^{-1+1} + c$. Use the formula $\int \frac{1}{x}\,dx = \ln|x| + c$ for the integrand $\frac{1}{x}$.

Integrals producing trigonometric functions.
Since the derivative of $\sin x$ is $\cos x$, $\sin x$ is an antiderivative of $\cos x$. Also, since the derivative of $\cos x$ is $-\sin x$, $\cos x$ is an antiderivative of $-\sin x$ so that $-\cos x$ is an antiderivative of $\sin x$.

$$\int \sin x\,dx = -\cos x + c \qquad \text{and} \qquad \int \cos x\,dx = \sin x + c$$

To integrate other trigonometric functions, you can convert them to sine and cosine functions and use the formulas above.

We summarize the formulas for integration of functions in the table below and illustrate their use in examples below.

y	x^n	e^x	a^x	$\frac{1}{x}$	$\sin x$	$\cos x$
$\int y\, dx$	$\frac{1}{n+1}x^{n+1}$	e^x	$\frac{1}{\ln a}\ a^x$	$\ln\|x\|$	$-\cos x$	$\sin x$

Example 1. Find the integral $\int xe^{x^2+1}\, dx$.

Solution. Identify the inner function $u = x^2 + 1$. Find the differential $du = 2xdx$ and solve for $dx = \frac{du}{2x}$.

Substitute $x^2 + 1$ with u and dx with $\frac{du}{2x}$. Obtain the following $\int xe^u\ \frac{du}{2x}$. Cancel x and factor $\frac{1}{2}$ out of the integral. The integrand e^u is now simple and you can integrate it using the formula for integral of e^x. Obtain

$$\frac{1}{2}\int e^u\ du = \frac{1}{2}\ e^u + c = \frac{1}{2}\ e^{x^2+1} + c.$$

Example 2. Find the integral $\int 2^{3x+1}\, dx$.

Solution. Use the substitution $u = 3x + 1 \Rightarrow du = 3dx \Rightarrow \frac{du}{3} = dx$. The integral becomes $\int 2^u \frac{du}{3} = \frac{1}{3}\int 2^u\ du$. The integrand 2^u is now simple and you can integrate it using the formula for integral of a^x with $a = 2$. Obtain

$$\frac{1}{3}\int 2^u\ du = \frac{1}{3}\ \frac{1}{\ln 2}2^u + c = \frac{1}{3\ln 2}2^{3x+1} + c.$$

Example 3. Find the integral $\int \frac{x^2+4}{x}dx$.

Solution. Simplify the integral as $\int \frac{x^2+4}{x}dx = \int \frac{x^2}{x} + \frac{4}{x}dx = \int(x + \frac{4}{x})dx$. You can integrate term by term and factor 4 in front of the second integral. Evaluate the second integral using the formula that produces $\ln|x|$.

$$\int(x + \frac{4}{x})dx = \int xdx + 4\int \frac{1}{x}dx = \frac{x^2}{2} + 4\ln|x| + c.$$

Example 4. Find the integral $\int(9 + 2\sin\frac{\pi t}{5})\ dt$.

Solution. Use the substitution $u = \frac{\pi t}{5} \Rightarrow du = \frac{\pi}{5}dt \Rightarrow \frac{5du}{\pi} = dt$. The integral becomes

$$\int(9+2\sin u)\frac{5du}{\pi} = \frac{5}{\pi}\int(9+2\sin u)du = \frac{5}{\pi}(9u-2\cos u)+c = \frac{5}{\pi}(9\frac{\pi t}{5}-2\cos\frac{\pi t}{5})+c = 9t-\frac{10}{\pi}\cos\frac{\pi t}{5}+c.$$

Alternatively, separate the integral into a sum of two as $\int 9dt + 2\int \sin\frac{\pi t}{5})\ dt$ and use the substitution $u = \frac{\pi t}{5}$ just for the second part. Obtain the same answer as above.

Example 5. Find the integral $\int \tan x dx$.

Solution. Recall that $\tan x = \frac{\sin x}{\cos x}$. The denominator $\cos x$ has derivative $-\sin x$ which is (up to a constant multiple) in the numerator. This points out to using the substitution $u = \cos x$. Then $du = -\sin x dx \Rightarrow dx = \frac{du}{-\sin x}$ and the integral reduces to

$$\int \frac{\sin x}{\cos x}dx = \int \frac{\sin x}{u}\frac{du}{-\sin x} = -\int \frac{1}{u}\, du = -\ln|u| + c = \ln|\cos x| + c.$$

Practice Problems.

1. Evaluate the following integrals. In problems (d) and (k) a and b are arbitrary constants.

(a) $\int e^{2x}\, dx$ (b) $\int 5^{4x+7}\, dx$ (c) $\int x\, 3^{2x^2+1} dx$

(d) $\int bx\, e^{ax^2+1}\, dx$ (e) $\int (e^{2x}+e^{-2x})\, dx$ (f) $\int \frac{e^x+1}{e^x}\, dx$

(g) $\int \frac{e^x}{e^x+1}\, dx$ (h) $\int \frac{e^{2x}}{e^x+1}\, dx$ (i) $\int \frac{1}{3x+5}\, dx$

(j) $\int \frac{x-1}{x^2}\, dx$ (k) $\int \frac{ax^2}{bx^3+1}\, dx$ (l) $\int \cos(3x+1)\, dx$

(m) $\int x \sin x^2\, dx$ (n) $\int \sin^3 x \cos x\, dx$ (o) $\int \frac{\cos x}{\sin x+3} dx$

2. An oscillator's velocity depends on time t in seconds as $v(t) = \sin \omega t$ m/s where $\omega = 2$ (in 1/second). If the oscillator is at a distance of 1 m from the equilibrium position when $t = 0$, determine the position as a function of the time t.

3. This problem deals with functions called the hyperbolic sine and the hyperbolic cosine. These functions occur in the solutions of some differential equations that appear in electromagnetic theory, heat transfer, fluid dynamics, and special relativity. Hyperbolic sine and cosine are defined as follows.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

(a) Find derivatives of $\sinh x$ and $\cosh x$ and express your answers in terms of $\sinh x$ and $\cosh x$. Use those formulas to find derivatives of $y = x \sinh x$ and $y = \cosh(x^2)$.

(b) From part (a), obtain the antiderivatives of $\sinh x$ and $\cosh x$ and use them to show that $\int \sinh(ax)dx = \frac{1}{a}\cosh(ax) + c$.

Solutions.

1. (a) Use the substitution $u = 2x \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$. The integral becomes $\int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{2x} + c$.

(b) Use the substitution $u = 4x+7 \Rightarrow du = 4dx \Rightarrow \frac{du}{4} = dx$. The integral becomes $\int 5^u \frac{du}{4} = \frac{1}{4}\int 5^u du = \frac{1}{4}\frac{1}{\ln 5}5^u + c = \frac{1}{4\ln 5}5^{4x+7} + c$.

(c) Use the substitution $u = 2x^2+1 \Rightarrow du = 4xdx \Rightarrow \frac{du}{4x} = dx$. The integral becomes $\int x3^u \frac{du}{4x} = \frac{1}{4}\int 3^u du = \frac{1}{4\ln 3}3^u + c = \frac{1}{4\ln 3}3^{2x^2+1} + c$.

(d) Use the substitution $u = ax^2+1 \Rightarrow du = 2axdx \Rightarrow \frac{du}{2ax} = dx$. The integral becomes $b\int xe^u \frac{du}{2ax} = \frac{b}{2a}\int e^u du = \frac{b}{2a}e^u + c = \frac{b}{2a}e^{ax^2+1} + c$.

(e) Separate into two integrals $\int e^{2x}dx + \int e^{-2x}dx$ and use the substitution $u = 2x$ for the first and the substitution $v = -2x$ for the second. Obtain $\int e^u \frac{du}{2} + \int e^v \frac{dv}{-2} = \frac{e^u}{2} + \frac{e^v}{-2} = \frac{1}{2}(e^{2x} - e^{-2x}) + c$.

(f) Simplify the function as $\int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx = \int(1 + e^{-x})dx$. You can use substitution $u = -x$ for the second term. Obtain $x - e^{-x} + c$ or $x - \frac{1}{e^x} + c$.

(g) Consider the denominator as the inner function and use the substitution $u = e^x+1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^x}{e^x+1}\, dx = \int \frac{e^x}{u} \frac{du}{e^x} = \int \frac{1}{u} du = \ln|u| + c = \ln|e^x+1| + c$.

(h) Similarly as in the previous problem, consider the denominator as the inner function and use the substitution $u = e^x + 1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^{2x}}{e^x+1}\, dx = \int \frac{e^{2x}}{u} \frac{du}{e^x} = \int \frac{e^x}{u} du$. Use the substitution relation $u = e^x + 1$ to solve for e^x and express it in terms of u as $e^x = u - 1$. Thus the integral becomes $\int \frac{u-1}{u} du = \int(1 - \frac{1}{u})du$. Integrate term by term to get $u - \ln|u| + c = e^x + 1 - \ln|e^x + 1| + c$.

(i) Use the substitution $u = 3x + 5 \Rightarrow du = 3dx \Rightarrow \frac{du}{3} = dx$. The integral becomes $\int \frac{1}{u} \frac{du}{3} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|3x + 5| + c$.

(j) Simplify the integral as follows and integrate term by term. $\int \frac{x-1}{x^2}\, dx = \int(\frac{x}{x^2} - \frac{1}{x^2})dx = \int(\frac{1}{x} - x^{-2})dx = \ln|x| - \frac{1}{-1}x^{-1} + c = \ln|x| + \frac{1}{x} + c$.

(k) Use the substitution $u = bx^3 + 1 \Rightarrow du = 3bx^2 dx \Rightarrow \frac{du}{3bx^2} = dx$. The integral becomes $\int \frac{ax^2}{u} \frac{du}{3bx^2} = \frac{a}{3b} \int \frac{1}{u} du = \frac{a}{3b} \ln|u| + c = \frac{a}{3b} \ln|bx^3 + 1| + c$.

(l) Use the substitution $u = 3x + 1$. Obtain $\frac{1}{3}\sin(3x + 1) + c$.

(m) Use the substitution $u = x^2$. Obtain $\frac{-1}{2}\cos x^2 + c$.

(n) Note that the integrand can be written as $(\sin x)^3 \cos x$ and $\cos x$ is derivative of the inner $\sin x$. This points out to using the substitution $u = \sin x$. Then $du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$ and the integral reduces to $\int u^3 \cos x \,\frac{du}{\cos x} = \int u^3\, du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$.

(o) The denominator $\sin x + 3$ has derivative $\cos x$ which is in the numerator so this points out to using the substitution $u = \sin x + 3$. Then $du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$ and the integral reduces to $\int \frac{\cos x}{u} \frac{du}{\cos x} = \int \frac{1}{u}\, du = \ln|u| + c = \ln|\sin x + 3| + c$.

2. $s(t) = \int v(t)dt = \int \sin 2x\, dx$. Using the substitution $u = 2x$ obtain $\frac{1}{2}\int \sin u du = \frac{-1}{2}\cos u + c = \frac{-1}{2}\cos(2x) + c$. Determine c using that $s(0) = 1 \Rightarrow 1 = \frac{-1}{2}\cos(0) + c \Rightarrow c = 1 + \frac{1}{2} = \frac{3}{2}$. Thus $s(t) = \frac{-1}{2}\cos(2x) + \frac{3}{2}$.

3. (a) The derivative of $\sinh x = \frac{1}{2}(e^x - e^{-x})$ is $\frac{1}{2}(e^x - e^{-x}(-1)) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$. Similarly, obtain that the derivative of $\cosh x$ is $\sinh x$. Using the product rule obtain that the derivative of $y = x \sinh x$ is $y' = \sinh x + x \cosh x$. Using the chain rule obtain that the derivative of $y = \cosh(x^2)$ is $y' = \sinh(x^2)(2x) = 2x \sinh x$.

(b) Part (a) implies that $\int \sinh x\, dx = \cosh x + c$. Integrate $\int \sinh(ax)\, dx$ using substitution $u = ax$. Then $dx = \frac{du}{a}$ and the integral becomes $\frac{1}{a}\int \sinh u\, du = \frac{1}{a}\cosh u + c = \frac{1}{a}\cosh(ax) + c$.