

# Exponential function

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In mathematics, an exponential function is a function of the form

$$f(x) = b^x$$

in which the input variable  $x$  occurs as an exponent. A function of the form  $f(x) = b^{x + c}$  is also considered an exponential function, and a function of the form  $f(x) = a \cdot b^x$  can be re-written as  $f(x) = b^{x + c}$  by the use of logarithms and so is an exponential function.

Exponential functions are uniquely characterized by the fact that the growth rate of such a function is directly proportional to the value of the function. This proportionality can be expressed by saying

$$\frac{d}{dx}b^x = b^x \times \text{constant},$$

where "constant" means a quantity that does not change as the variable  $x$  changes.

For just one base  $b$  this "constant" factor is equal to 1, and that base is the number  $e \approx 2.71828...$ :

$$\frac{d}{dx}e^x = e^x \times 1.$$

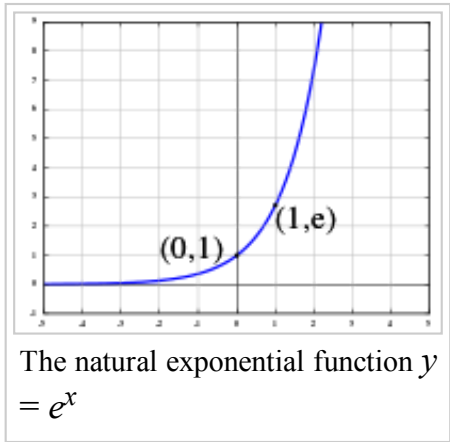
This equality makes it possible to reduce some questions in mathematical analysis of exponential functions to the analysis of this one exponential function, conventionally called the "natural exponential function"<sup>[1][2]</sup> and denoted by

$$x \mapsto \exp(x).$$

The exponential function models a relationship in which a constant change in the independent variable gives the same proportional change (i.e. percentage increase or decrease) in the dependent variable. The function is often written as  $\exp(x)$ , especially when it is impractical to write the independent variable as a superscript. The exponential function is widely used in physics, chemistry, engineering, mathematical biology, economics and mathematics.

The graph of  $y = e^x$  is upward-sloping, and increases faster as  $x$  increases. The graph always lies above the  $x$ -axis but can get arbitrarily close to it for negative  $x$ ; thus, the  $x$ -axis is a horizontal asymptote. The slope of the tangent to the graph at each point is equal to its  $y$ -coordinate at that point. The inverse function is the natural logarithm  $\ln(x)$ ; because of this, some old texts<sup>[3]</sup> refer to the exponential function as the **antilogarithm**.

In general, the variable  $x$  can be any real or complex number or even an entirely different kind of mathematical object; see the formal definition below.



## Exponential function

Representation	$e^x$
Inverse	$\ln x$
Derivative	$e^x$
Indefinite Integral	$e^x + C$



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## Formal definition

The exponential function  $e^x$  can be characterized in a variety of equivalent ways. In particular it may be defined by the following power series:<sup>[4]</sup>

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Using an alternate definition for the exponential function leads to the same result when expanded as a Taylor series.

Less commonly,  $e^x$  is defined as the solution  $y$  to the equation

$$x = \int_1^y \frac{1}{t} dt$$

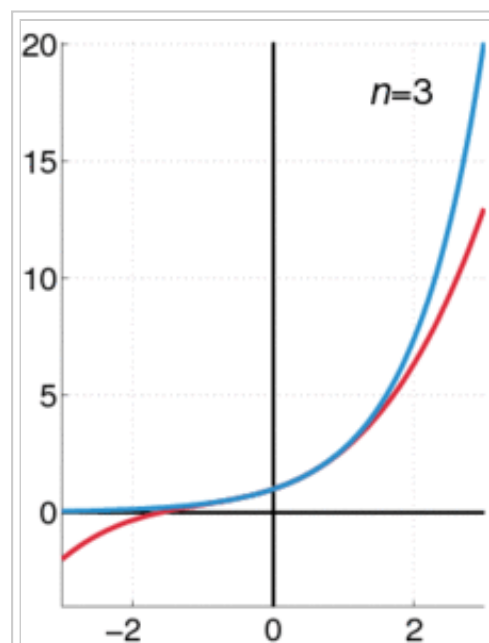
It is also the following limit:<sup>[5]</sup>

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

## Overview

The exponential function arises whenever a quantity grows or decays at a rate proportional to its current value. One such situation is continuously compounded interest, and in fact it was this that led Jacob Bernoulli in 1683<sup>[6]</sup> to the number

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$



The exponential function (in blue), and the sum of the first  $n + 1$  terms of the power series on the left (in red).

now known as  $e$ . Later, in 1697, Johann Bernoulli studied the calculus of the exponential function.<sup>[6]</sup>

If a principal amount of 1 earns interest at an annual rate of  $x$  compounded monthly, then the interest earned each month is  $\frac{x}{12}$  times the current value, so each month the total value is multiplied by  $(1 + \frac{x}{12})$ , and the value at the end of the year is  $(1 + \frac{x}{12})^{12}$ . If instead interest is compounded daily, this becomes  $(1 + \frac{x}{365})^{365}$ . Letting the number of time intervals per year grow without bound leads to the limit definition of the exponential function,

$$\exp(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

first given by Euler.<sup>[5]</sup> This is one of a number of characterizations of the exponential function; others involve series or differential equations.

From any of these definitions it can be shown that the exponential function obeys the basic exponentiation identity,

$$\exp(x + y) = \exp(x) \cdot \exp(y)$$

which is why it can be written as  $e^x$ .

The derivative (rate of change) of the exponential function is the exponential function itself. More generally, a function with a rate of change *proportional* to the function itself (rather than equal to it) is expressible in terms of the exponential function. This function property leads to exponential growth and exponential decay.

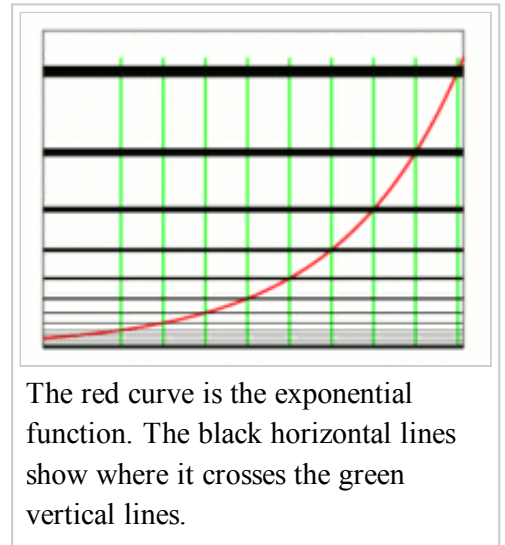
The exponential function extends to an entire function on the complex plane. Euler's formula relates its values at purely imaginary arguments to trigonometric functions. The exponential function also has analogues for which the argument is a matrix, or even an element of a Banach algebra or a Lie algebra.

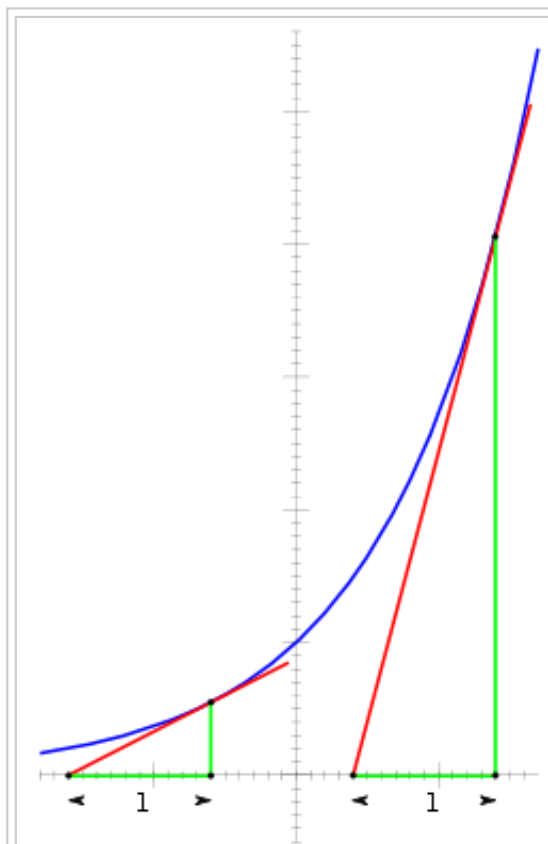
## Derivatives and differential equations

The importance of the exponential function in mathematics and the sciences stems mainly from properties of its derivative. In particular,

$$\frac{d}{dx} e^x = e^x$$

Proof:





The derivative of the exponential function is equal to the value of the function. From any point  $P$  on the curve (blue), let a tangent line (red), and a vertical line (green) with height  $h$  be drawn, forming a right triangle with a base  $b$  on the  $x$ -axis. Since the slope of the red tangent line (the derivative) at  $P$  is equal to the ratio of the triangle's height to the triangle's base (rise over run), and the derivative is equal to the value of the function,  $h$  must be equal to the ratio of  $h$  to  $b$ . Therefore, the base  $b$  must always be 1.

$$\begin{aligned}
 e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
 \frac{d}{dx}e^x &= \frac{d}{dx} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right) \\
 &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots \\
 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
 &= e^x
 \end{aligned}$$

That is,  $e^x$  is its own derivative and hence is a simple example of a Pfaffian function. Functions of the form  $ce^x$  for constant  $c$  are the only functions with that property (by the Picard–Lindelöf theorem). Other ways of saying the same thing include:

- The slope of the graph at any point is the height of the function at that point.
- The rate of increase of the function at  $x$  is equal to the value of the function at  $x$ .
- The function solves the differential equation  $y' = y$ .
- $\exp$  is a fixed point of derivative as a functional.

If a variable's growth or decay rate is proportional to its size—as is the case in unlimited population growth (see Malthusian catastrophe), continuously compounded interest, or radioactive decay—then the variable can be written as a constant times an exponential function of time. Explicitly for any real constant  $k$ , a function  $f: \mathbf{R} \rightarrow \mathbf{R}$  satisfies  $f' = kf$  if and only if  $f(x) = ce^{kx}$  for some constant  $c$ .

Furthermore, for any differentiable function  $f(x)$ , we find, by the chain rule:

$$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$$

## Continued fractions for $e^x$

A continued fraction for  $e^x$  can be obtained via an identity of Euler:

$$e^x = 1 + \frac{x}{1 - \frac{x}{x + 2 - \frac{2x}{x + 3 - \frac{3x}{x + 4 - \ddots}}}}$$

The following generalized continued fraction for  $e^z$  converges more quickly:<sup>[7]</sup>

$$e^z = 1 + \frac{2z}{2 - z + \frac{z^2}{6 + \frac{z^2}{10 + \frac{z^2}{14 + \ddots}}}}$$

or, by applying the substitution  $z = \frac{x}{y}$ :

$$e^{\frac{x}{y}} = 1 + \frac{2x}{2y - x + \frac{x^2}{6y + \frac{x^2}{10y + \frac{x^2}{14y + \ddots}}}}$$

with a special case for  $z = 2$ :

$$e^2 = 1 + \frac{4}{0 + \frac{2^2}{6 + \frac{2^2}{10 + \frac{2^2}{14 + \ddots}}}} = 7 + \frac{2}{5 + \frac{1}{7 + \frac{1}{9 + \frac{1}{11 + \ddots}}}}$$

This formula also converges, though more slowly, for  $z > 2$ . For example:

$$e^3 = 1 + \frac{6}{-1 + \frac{3^2}{6 + \frac{3^2}{10 + \frac{3^2}{14 + \ddots}}}} = 13 + \frac{54}{7 + \frac{9}{14 + \frac{9}{18 + \frac{9}{22 + \ddots}}}}$$

## Complex plane

As in the real case, the exponential function can be defined on the complex plane in several equivalent forms. One such definition parallels the power series definition for real numbers, where the real variable is replaced by a complex one:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The exponential function is periodic with imaginary period  $2\pi i$  and can be written as

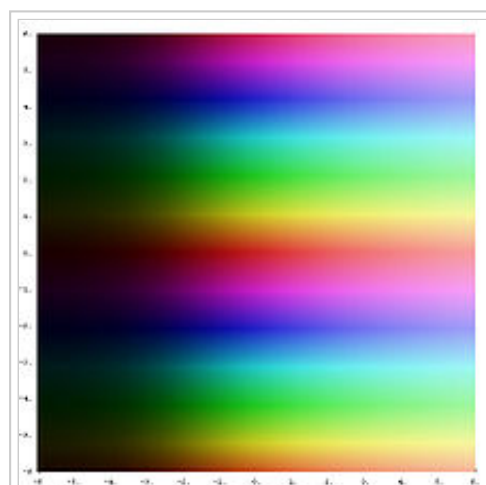
$$e^{a+bi} = e^a (\cos b + i \sin b) = e^a \operatorname{cis}(b)$$

where  $a$  and  $b$  are real values and on the right the real functions must be used if used as a definition<sup>[8]</sup> (see also Euler's formula). This formula connects the exponential function with the trigonometric functions and to the hyperbolic functions.

When considered as a function defined on the complex plane, the exponential function retains the properties

- $e^{z+w} = e^z e^w$
- $e^0 = 1$
- $e^z \neq 0$
- $\frac{d}{dz} e^z = e^z$
- $(e^z)^n = e^{nz}, n \in \mathbb{Z}$

for all  $z$  and  $w$ .



Exponential function on the complex plane. The transition from dark to light colors shows that the magnitude of the exponential function is increasing to the right. The periodic horizontal bands indicate that the exponential function is periodic in the imaginary part of its argument.

The exponential function is an entire function as it is holomorphic over the whole complex plane. It takes on every complex number excepting 0 as value; that is, 0 is a lacunary value of the exponential function. This is an example of Picard's little theorem that any non-constant entire function takes on every complex number as value with at most one value excepted.

Extending the natural logarithm to complex arguments yields the complex logarithm  $\log z$ , which is a multivalued function.

We can then define a more general exponentiation:

$$z^w = e^{w \log z}$$

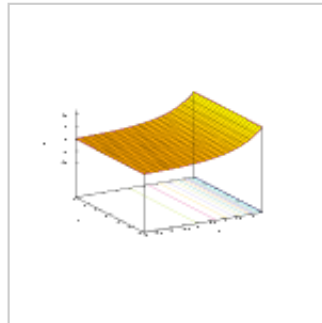
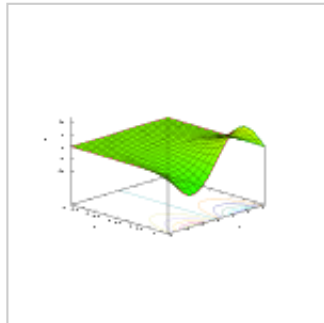
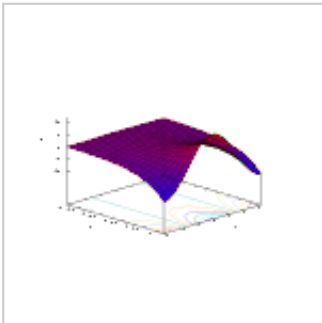
for all complex numbers  $z$  and  $w$ . This is also a multivalued function, even when  $z$  is real. This distinction is problematic, as the multivalued functions  $\log z$  and  $z^w$  are easily confused with their single-valued equivalents when substituting a real number for  $z$ . The rule about multiplying exponents for the case of positive real numbers must be modified in a multivalued context:

$$(e^z)^w \neq e^{zw}, \text{ but rather } (e^z)^w = e^{(z + 2\pi in)w} \text{ multivalued over integers } n$$

See failure of power and logarithm identities for more about problems with combining powers.

The exponential function maps any line in the complex plane to a logarithmic spiral in the complex plane with the center at the origin. Two special cases might be noted: when the original line is parallel to the real axis, the resulting spiral never closes in on itself; when the original line is parallel to the imaginary axis, the resulting spiral is a circle of some radius.

### Plots of the exponential function on the complex plane



$$z = \operatorname{Re}(e^{x+iy})$$

$$z = \operatorname{Im}(e^{x+iy})$$

$$z = |e^{x+iy}|$$

### Computation of $a^b$ where both $a$ and $b$ are complex

Complex exponentiation  $a^b$  can be defined by converting  $a$  to polar coordinates and using the identity  $(e^{\ln(a)})^b = a^b$ :

$$a^b = (re^{\theta i})^b = (e^{\ln(r)+\theta i})^b = e^{(\ln(r)+\theta i)b}$$

However, when  $b$  is not an integer, this function is multivalued, because  $\theta$  is not unique (see failure of power and logarithm identities).

# Matrices and Banach algebras

The power series definition of the exponential function makes sense for square matrices (for which the function is called the matrix exponential) and more generally in any Banach algebra  $B$ . In this setting,  $e^0 = 1$ , and  $e^x$  is invertible with inverse  $e^{-x}$  for any  $x$  in  $B$ . If  $xy = yx$ , then  $e^{x+y} = e^x e^y$ , but this identity can fail for noncommuting  $x$  and  $y$ .

Some alternative definitions lead to the same function. For instance,  $e^x$  can be defined as

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Or  $e^x$  can be defined as  $f(1)$ , where  $f: \mathbf{R} \rightarrow B$  is the solution to the differential equation  $f'(t) = xf(t)$  with initial condition  $f(0) = 1$ .

## Lie algebras

Given a Lie group  $G$  and its associated Lie algebra  $\mathfrak{g}$ , the exponential map is a map  $\mathfrak{g} \mapsto G$  satisfying similar properties. In fact, since  $\mathbf{R}$  is the Lie algebra of the Lie group of all positive real numbers under multiplication, the ordinary exponential function for real arguments is a special case of the Lie algebra situation. Similarly, since the Lie group  $\mathrm{GL}(n, \mathbf{R})$  of invertible  $n \times n$  matrices has as Lie algebra  $\mathbf{M}(n, \mathbf{R})$ , the space of all  $n \times n$  matrices, the exponential function for square matrices is a special case of the Lie algebra exponential map.

The identity  $\exp(x+y) = \exp(x)\exp(y)$  can fail for Lie algebra elements  $x$  and  $y$  that do not commute; the Baker–Campbell–Hausdorff formula supplies the necessary correction terms.

## Similar properties of $e$ and the function $e^z$

The function  $e^z$  is not in  $\mathbf{C}(z)$  (i.e., is not the quotient of two polynomials with complex coefficients).

For  $n$  distinct complex numbers  $\{a_1, \dots, a_n\}$ , the set  $\{e^{a_1 z}, \dots, e^{a_n z}\}$  is linearly independent over  $\mathbf{C}(z)$ .

The function  $e^z$  is transcendental over  $\mathbf{C}(z)$ .

## exp and expm1

Some calculators provide a dedicated **exp(x)** function designed to provide a higher precision than achievable by using  $e^x$  directly.<sup>[9][10]</sup>

Based on a proposal by William Kahan and first implemented in the Hewlett-Packard HP-41C calculator in 1979, some scientific calculators, computer algebra systems and programming languages (for example C99<sup>[11]</sup>) support a special **exponential minus 1** function alternatively named  $E^x-1$ ,  $\mathrm{expm1}(x)$ ,<sup>[11]</sup>  $\mathrm{expm}(x)$ ,<sup>[9][10]</sup> or  $\mathrm{exp1m}(x)$  to provide more accurate results for values of  $x$  near zero compared to using



$\exp(x)$ -1 directly.<sup>[9][10][11]</sup> This function is implemented using a different internal algorithm to avoid an intermediate result near 1, thereby allowing both the argument and the result to be near zero.<sup>[9][10]</sup> Similar inverse functions named  $\ln p_1(x)$ ,<sup>[9][10]</sup>  $\ln_1 p(x)$  or  $\log_1 p(x)$ <sup>[11]</sup> exist as well.<sup>[nb 1]</sup>

## See also

- Approximating natural exponents (log base  $e$ )
- Carlitz exponential, a characteristic  $p$  analogue
- Characterizations of the exponential function
- Double exponential function
- $e$  (mathematical constant)
- Exponential decay
- Exponential field
- Exponential growth
- Exponentiation
- Half-exponential function – a compositional square root of an exponential function
- List of exponential topics
- List of integrals of exponential functions
- $p$ -adic exponential function
- Padé table for exponential function – Padé approximation if exponential function by a fraction of polynomial functions
- Tetration

## Notes

1. For a similar approach to reduce round-off errors of calculations for certain input values see trigonometric functions like versine, vercosine, coversine, covercosine, haversine, havercosine, hacoversine, hacovercosine, exsecant and excosecant.

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9. *HP 48G Series – Advanced User's Reference Manual (AUR)* (4 ed.). Hewlett-Packard. December 1994 [1993]. HP 00048-90136, 0-88698-01574-2. Retrieved 2015-09-06.
10. *HP 50g / 49g+ / 48gII graphing calculator advanced user's reference manual (AUR)* (2 ed.). Hewlett-Packard. 2009-07-14 [2005]. HP F2228-90010. Retrieved 2015-10-10. Searchable PDF ([http://holyojo.net/hp/HP\\_50g\\_AUR\\_v2\\_English\\_searchable.pdf](http://holyojo.net/hp/HP_50g_AUR_v2_English_searchable.pdf))

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## External links

- Hazewinkel, Michiel, ed. (2001), "Exponential function", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Complex exponential function (<http://planetmath.org/?op=getobj&from=objects&id=6341>) at PlanetMath.org.
- Derivative of exponential function (<http://planetmath.org/?op=getobj&from=objects&id=9313>) at PlanetMath.org.
- Derivative of exponential function interactive graph (<http://symp1.org/book/examples/interactive-plots/derivative-exponential-function>)
- Weisstein, Eric W., "Exponential Function" (<http://mathworld.wolfram.com/ExponentialFunction.html>), *MathWorld*.
- Taylor Series Expansions of Exponential Functions ([http://www.efunda.com/math/taylor\\_series/exponential.cfm](http://www.efunda.com/math/taylor_series/exponential.cfm)) at efunda.com (<http://www.efunda.com/>)
- Complex exponential interactive graphic (<http://www-math.mit.edu/daimp/ComplexExponential.html>)
- Derivative of  $\exp(x^n)$  by limit definition (<https://www.scribd.com/doc/288367437/A-Compilation-of-Mathematical-Demonstrations>),
- General exponential limit (<https://www.scribd.com/doc/288367437/A-Compilation-of-Mathematical-Demonstrations>),

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