Logarithm

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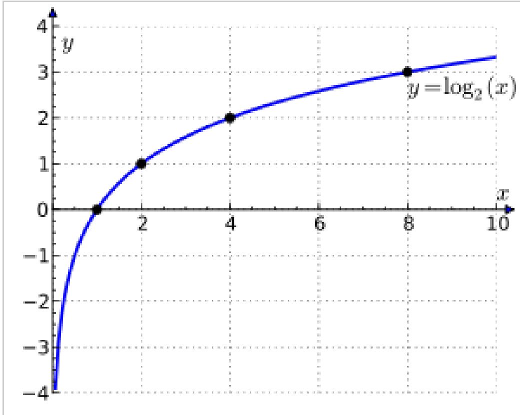


The graph of the logarithm to base 2 crosses the x axis (horizontal axis) at 1 and passes through the points with coordinates (2, 1), (4, 2), and (8, 3). For example, $\log_2(8) = 3$, because $2^3 = 8$. The graph gets arbitrarily close to the y axis, but does not meet or intersect it.

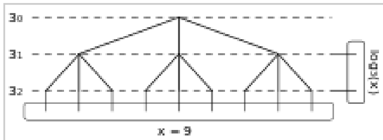
In mathematics, the **logarithm** is the inverse operation to exponentiation. That means the logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. In simple cases the logarithm counts repeated multiplication. For example, the base 10 logarithm of 1000 is 3, as 10 to the power 3 is 1000 ($1000 = 10 \times 10 \times 10 = 10^3$); the multiplication is repeated three times. More generally, exponentiation allows any positive real number to be raised to any real power, always producing a positive result, so the logarithm can be calculated for any two positive real numbers b and x where b is not equal to 1. The logarithm of x to *base* b, denoted $\log_b(x)$, is the unique real number y such that

$$b^y = x.$$

For example, as $64 = 2^6$, we have

$$\log_2(64) = 6$$

The logarithm to base 10 (that is $b = 10$) is called the common logarithm and has many applications in science and engineering. The natural logarithm has the number e (≈ 2.718) as its base; its use is widespread in mathematics and physics, because of its simpler derivative. The binary logarithm uses base 2 (that is $b = 2$) and is commonly used in computer science.



A full 3-ary tree can be used to visualize the exponents of 3 and how the logarithm function relates to them.

Logarithms were introduced by John Napier in the early 17th century as a means to simplify calculations. They were rapidly adopted by navigators, scientists, engineers, and others to perform computations more easily, using slide rules and logarithm tables. Tedious multi-digit multiplication steps can be replaced by table look-ups and simpler addition because of the fact — important in its own right — that the logarithm of a product is the sum of the logarithms of the factors:

$$\log_b(xy) = \log_b(x) + \log_b(y),$$

provided that b, x and y are all positive and $b \neq 1$. The present-day notion of logarithms comes from Leonhard Euler, who connected them to the exponential function in the 18th century.

Logarithmic scales reduce wide-ranging quantities to tiny scopes. For example, the decibel is a unit quantifying signal power log-ratios and amplitude log-ratios (of which sound pressure is a common example). In chemistry, pH is a logarithmic measure for the acidity of an aqueous solution. Logarithms are commonplace in scientific formulae, and in measurements of the complexity of algorithms and of geometric objects called fractals. They describe musical intervals, appear in formulas counting prime numbers, inform some models in psychophysics, and can aid in forensic accounting.

In the same way as the logarithm reverses exponentiation, the complex logarithm is the inverse function of the exponential function applied to complex numbers. The discrete logarithm is another variant; it has uses in public-key cryptography.

Contents

Motivation and definition

The idea of logarithms is to reverse the operation of exponentiation, that is, raising a number to a power. For example, the third power (or cube) of 2 is 8, because 8 is the product of three factors of 2:

$$2^3 = 2 \times 2 \times 2 = 8.$$

It follows that the logarithm of 8 with respect to base 2 is 3, so $\log_2 8 = 3$.

Exponentiation

The third power of some number b is the product of three factors of b. More generally, raising b to the n-th power, where n is a natural number, is done by multiplying n factors of b. The n-th power of b is written b^n, so that

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}.$$

Exponentiation may be extended to b^y, where b is a positive number and the *exponent* y is any real number. For example, b^{-1} is the reciprocal of b, that is, $1/b$. (For further details, including the formula $b^{m+n} = b^m \cdot b^n$, see exponentiation or [1] for an elementary treatise.)

Definition

The *logarithm* of a positive real number x with respect to base b, a positive real number not equal to 1[nb 1], is the exponent by which b must be raised to yield x. In other words, the logarithm of x to base b is the solution y to the equation[2]

$$b^y = x.$$

The logarithm is denoted "$\log_b(x)$" (pronounced as "the logarithm of x to base b" or "the base-b logarithm of x"). In the equation $y = \log_b(x)$, the value y is the answer to the question "To what power must b be raised, in order to yield x?". This question can also be addressed (with a richer answer) for complex numbers, which is done in section "Complex logarithm", and this answer is much more extensively investigated in the page for the complex logarithm.

Examples

For example, $\log_2(16) = 4$, since $2^4 = 2 \times 2 \times 2 \times 2 = 16$. Logarithms can also be negative:

$$\log_2\left(\frac{1}{2}\right) = -1,$$

since

$$2^{-1} = \frac{1}{2^1} = \frac{1}{2}.$$

A third example: $\log_{10}(150)$ is approximately 2.176, which lies between 2 and 3, just as 150 lies between $10^2 = 100$ and $10^3 = 1000$. Finally, for any base b, $\log_b(b) = 1$ and $\log_b(1) = 0$, since $b^1 = b$ and $b^0 = 1$, respectively.

Logarithmic identities

Several important formulas, sometimes called *logarithmic identities* or *log laws*, relate logarithms to one another.[3]

Product, quotient, power and root

The logarithm of a product is the sum of the logarithms of the numbers being multiplied; the logarithm of the ratio of two numbers is the difference of the logarithms. The logarithm of the p-th power of a number is p times the logarithm of the number itself; the logarithm of a p-th root is the logarithm of the number divided by p. The following table lists these identities with examples. Each of the identities can be derived after substitution of the logarithm definitions $x = b^{\log_b(x)}$ or $y = b^{\log_b(y)}$ in the left hand sides.

	Formula	Example
product	$\log_b(xy) = \log_b(x) + \log_b(y)$	$\log_3(243) = \log_3(9 \cdot 27) = \log_3(9) + \log_3(27) = 2 + 3 = 5$
quotient	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$\log_2(16) = \log_2\left(\frac{64}{4}\right) = \log_2(64) - \log_2(4) = 6 - 2 = 4$
power	$\log_b(x^p) = p\log_b(x)$	$\log_2(64) = \log_2(2^6) = 6\log_2(2) = 6$
root	$\log_b \sqrt[p]{x} = \frac{\log_b(x)}{p}$	$\log_{10}\sqrt{1000} = \frac{1}{2}\log_{10}1000 = \frac{3}{2} = 1.5$

Change of base

The logarithm $\log_b(x)$ can be computed from the logarithms of x and b with respect to an arbitrary base k using the following formula:

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}.$$

Typical scientific calculators calculate the logarithms to bases 10 and e.[4] Logarithms with respect to any base b can be determined using either of these two logarithms by the previous formula:

$$\log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)} = \frac{\log_e(x)}{\log_e(b)}.$$

Given a number x and its logarithm $\log_b(x)$ to an unknown base b, the base is given by:

$$b = x^{\frac{1}{\log_b(x)}}.$$

Particular bases

Among all choices for the base, three are particularly common. These are $b = 10$, $b = e$ (the irrational mathematical constant ≈ 2.71828), and $b = 2$. In mathematical analysis, the logarithm to base e is widespread because of its particular analytical properties explained below. On the other hand, base-10 logarithms are easy to use for manual calculations in the decimal number system:[5]

$$\log_{10}(10x) = \log_{10}(10) + \log_{10}(x) = 1 + \log_{10}(x).$$

Thus, $\log_{10}(x)$ is related to the number of decimal digits of a positive integer x: the number of digits is the smallest integer strictly bigger than $\log_{10}(x)$.[6] For example, $\log_{10}(1430)$ is approximately 3.15. The next integer is 4, which is the number of digits of 1430. Both the natural logarithm and the logarithm to base two are used in information theory, corresponding to the use of nats or bits as the fundamental units of information, respectively.[7] Binary logarithms are also used in computer science, where the binary system is ubiquitous, in music theory, where a pitch ratio of two (the octave) is ubiquitous and the cent is the binary logarithm (scaled by 1200) of the ratio between two adjacent equally-tempered pitches, and in photography to measure exposure values.[8]

The following table lists common notations for logarithms to these bases and the fields where they are used. Many disciplines write $\log(x)$ instead of $\log_b(x)$, when the intended base can be determined from the context. The notation ${}^b\log(x)$ also occurs.[9] The "ISO notation" column lists designations suggested by the International Organization for Standardization (ISO 31-11).[10]

Base b	Name for $\log_b(x)$	ISO notation	Other notations	Used in
2	binary logarithm	$\text{lb}(x)$[11]	$\text{ld}(x)$, $\log(x)$, $\lg(x)$,[12] $\log2(x)$	computer science, information theory, music theory, photography
e	natural logarithm	$\ln(x)$[nb 2]	$\log(x)$ (in mathematics [16] and many programming languages[nb 3])	mathematics, physics, chemistry, statistics, economics, information theory, and some engineering fields
10	common logarithm	$\lg(x)$	$\log(x)$, $\log_{10}(x)$ (in engineering, biology, astronomy)	various engineering fields (see decibel and see below), logarithm tables, handheld calculators, spectroscopy

History

The **history of logarithm** in seventeenth century Europe is the discovery of a new function that extended the realm of analysis beyond the scope of algebraic methods. The method of logarithms was publicly propounded by John Napier in 1614, in a book titled *Mirifici Logarithmorum Canonis Descriptio* (*Description of the Wonderful Rule of Logarithms*).[17][18] Prior to Napier's invention, there had been other techniques of similar scopes, such as the prosthaphaeresis or the use of tables of progressions, extensively developed by Jost Bürgi around 1600.[19][20]

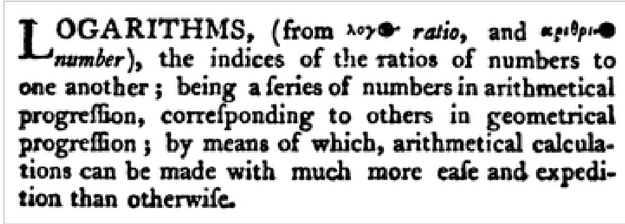
The common logarithm of a number is the index of that power of ten which equals the number.[21] Speaking of a number as requiring so many figures is a rough allusion to common logarithm, and was referred to by Archimedes as the "order of a number".[22] The first real logarithms were heuristic methods to turn multiplication into addition, thus facilitating rapid computation. Some of these methods used tables derived from trigonometric identities.[23] Such methods are called prosthaphaeresis.

Invention of the function now known as natural logarithm began as an attempt to perform a quadrature of a rectangular hyperbola by Gregoire de Saint Vincent, a Belgian Jesuit residing in Prague. Archimedes had written The Quadrature of the Parabola in the third century BC, but a quadrature for the hyperbola eluded all efforts until Saint-Vincent published his results in 1647. The relation that the logarithm provides between a geometric progression in its argument and an arithmetic progression of values, prompted A. A. de Sarasa to make the connection of Saint-Vincent’s quadrature and the tradition of logarithms in prosthaphaeresis, leading to the term "hyperbolic logarithm", a synonym for natural logarithm. Soon the new function was appreciated by Christiaan Huygens, Patavii, and James Gregory. The notation Log y was adopted by Leibniz in 1675,[24] and the next year he connected it to the integral $\int \frac{dy}{y}$.

Logarithm tables, slide rules, and historical applications

By simplifying difficult calculations, logarithms contributed to the advance of science, and especially of astronomy. They were critical to advances in surveying, celestial navigation, and other domains. Pierre-Simon Laplace called logarithms

> "...[a]n admirable artifice which, by reducing to a few days the labour of many months, doubles the life of the astronomer, and spares him the errors and disgust inseparable from long calculations."[25]


LOGARITHMS, (from λογ● ratio, and αριθμο● number), the indices of the ratios of numbers to one another; being a ſeries of numbers in arithmetical progreſſion, correſponding to others in geometrical progreſſion; by means of which, arithmetical calculations can be made with much more eaſe and expedition than otherwiſe.

The 1797 *Encyclopædia Britannica* explanation of logarithms

A key tool that enabled the practical use of logarithms before calculators and computers was the *table of logarithms*.[26] The first such table was compiled by Henry Briggs in 1617, immediately after Napier's invention. Subsequently, tables with increasing scope were written. These tables listed the values of $\log_b(x)$ and b^x for any number x in a certain range, at a certain precision, for a certain base b (usually $b = 10$). For example, Briggs' first table contained the common logarithms of all integers in the range 1–1000, with a precision of 14 digits. As the function $f(x) = b^x$ is the inverse function of $\log_b(x)$, it has been called the **antilogarithm**.[27] The product and quotient of two positive numbers c and d were routinely calculated as the sum and difference of their logarithms. The product cd or quotient c/d came from looking up the antilogarithm of the sum or difference, also via the same table:

$$cd = b^{\log_b(c)}\, b^{\log_b(d)} = b^{\log_b(c)+\log_b(d)}$$

and

$$\frac{c}{d} = cd^{-1} = b^{\log_b(c)-\log_b(d)}.$$

For manual calculations that demand any appreciable precision, performing the lookups of the two logarithms, calculating their sum or difference, and looking up the antilogarithm is much faster than performing the multiplication by earlier methods such as prosthaphaeresis, which relies on trigonometric identities. Calculations of powers and roots are reduced to multiplications or divisions and look-ups by

$$c^d = \left(b^{\log_b(c)}\right)^d = b^{d\log_b(c)}$$

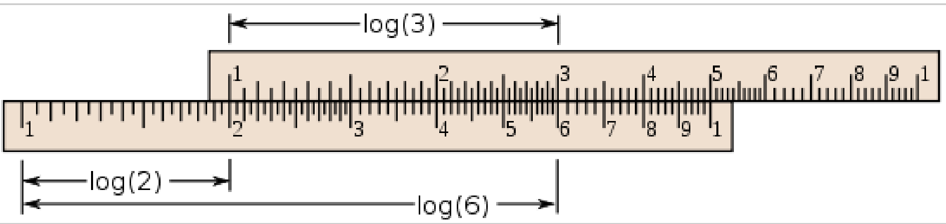
and

$$\sqrt[d]{c} = c^{\frac{1}{d}} = b^{\frac{1}{d}\log_b(c)}.$$

Many logarithm tables give logarithms by separately providing the characteristic and mantissa of x, that is to say, the integer part and the fractional part of $\log_{10}(x)$.[28] The characteristic of $10 \cdot x$ is one plus the characteristic of x, and their significands are the same. This extends the scope of logarithm tables: given a table listing $\log_{10}(x)$ for all integers x ranging from 1 to 1000, the logarithm of 3542 is approximated by

$$\log_{10}(3542) = \log_{10}(10 \cdot 354.2) = 1 + \log_{10}(354.2) \approx 1 + \log_{10}(354).$$ Greater accuracy can be obtained by interpolation.

Another critical application was the slide rule, a pair of logarithmically divided scales used for calculation, as illustrated here:



Schematic depiction of a slide rule. Starting from 2 on the lower scale, add the distance to 3 on the upper scale to reach the product 6. The slide rule works because it is marked such that the distance from 1 to x is proportional to the logarithm of x.

The non-sliding logarithmic scale, Gunter's rule, was invented shortly after Napier's invention. William Oughtred enhanced it to create the slide rule—a pair of logarithmic scales movable with respect to each other. Numbers are placed on sliding scales at distances proportional to the differences between their logarithms. Sliding the upper scale appropriately amounts to mechanically adding logarithms. For example, adding the distance from 1 to 2 on the lower scale to the distance from 1 to 3 on the upper scale yields a product of 6, which is read off at the lower part. The slide rule was an essential calculating tool for engineers and scientists until the 1970s, because it allows, at the expense of precision, much faster computation than techniques based on tables.[29]

Analytic properties

A deeper study of logarithms requires the concept of a *function*. A function is a rule that, given one number, produces another number.[30] An example is the function producing the x-th power of b from any real number x, where the base b is a fixed number. This function is written

$$f(x) = b^x.$$

Logarithmic function

To justify the definition of logarithms, it is necessary to show that the equation

$$b^x = y$$

has a solution x and that this solution is unique, provided that y is positive and that b is positive and unequal to 1. A proof of that fact requires the intermediate value theorem from elementary calculus.[31] This theorem states that a continuous function that produces two values m and n also produces any value that lies between m and n. A function is *continuous* if it does not "jump", that is, if its graph can be drawn without lifting the pen.

This property can be shown to hold for the function $f(x) = b^x$. Because f takes arbitrarily large and arbitrarily small positive values, any number $y > 0$ lies between $f(x_0)$ and $f(x_1)$ for suitable x_0 and x_1. Hence, the intermediate value theorem ensures that the equation $f(x) = y$ has a solution. Moreover, there is only one solution to this equation, because the function f is strictly increasing (for $b > 1$), or strictly decreasing (for $0 < b < 1$).[32]

The unique solution x is the logarithm of y to base b, $\log_b(y)$. The function that assigns to y its logarithm is called *logarithm function* or *logarithmic function* (or just *logarithm*).

The function $\log_b(x)$ is essentially characterized by the above product formula

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

More precisely, the logarithm to any base $b > 1$ is the only increasing function f from the positive reals to the reals satisfying $f(b) = 1$ and [33]

$$f(xy) = f(x) + f(y).$$

Inverse function

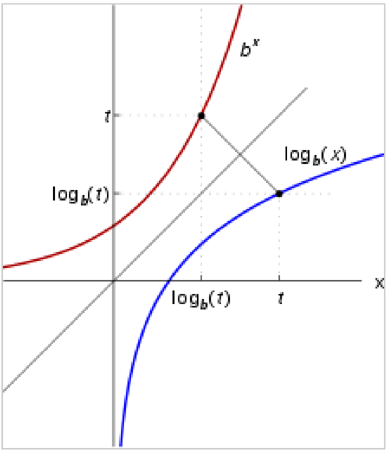
The formula for the logarithm of a power says in particular that for any number x,

$$\log_b\left(b^x\right) = x \log_b(b) = x.$$

In prose, taking the x-th power of b and then the base-b logarithm gives back x. Conversely, given a positive number y, the formula

$$b^{\log_b(y)} = y$$

says that first taking the logarithm and then exponentiating gives back y. Thus, the two possible ways of combining (or composing) logarithms and exponentiation give back the original number. Therefore, the logarithm to base b is the *inverse function* of $f(x) = b^x$.[34]



The graph of the logarithm function $\log_b(x)$ (blue) is obtained by reflecting the graph of the function b^x (red) at the diagonal line ($x = y$).

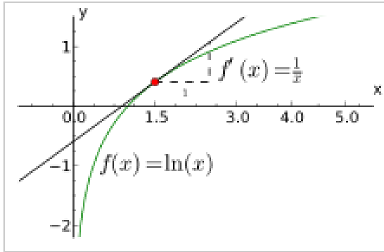
Inverse functions are closely related to the original functions. Their graphs correspond to each other upon exchanging the x- and the y-coordinates (or upon reflection at the diagonal line $x = y$), as shown at the right: a point (t, $u = b^t$) on the graph of f yields a point (u, $t = \log_b u$) on the graph of the logarithm and vice versa. As a consequence, $\log_b(x)$ diverges to infinity (gets bigger than any given number) if x grows to infinity, provided that b is greater than one. In that case, $\log_b(x)$ is an increasing function. For $b < 1$, $\log_b(x)$ tends to minus infinity instead. When x approaches zero, $\log_b(x)$ goes to minus infinity for $b > 1$ (plus infinity for $b < 1$, respectively).

Derivative and antiderivative

Analytic properties of functions pass to their inverses.[31] Thus, as $f(x) = b^x$ is a continuous and differentiable function, so is $\log_b(y)$. Roughly, a continuous function is differentiable if its graph has no sharp "corners". Moreover, as the derivative of $f(x)$ evaluates to $\ln(b)b^x$ by the properties of the exponential function, the chain rule implies that the derivative of $\log_b(x)$ is given by[32][35]

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$

That is, the slope of the tangent touching the graph of the base-b logarithm at the point $(x, \log_b(x))$ equals $1/(x \ln(b))$.



The graph of the natural logarithm (green) and its tangent at $x = 1.5$ (black)

The derivative of $\ln(x)$ is $1/x$; this implies that $\ln(x)$ is the unique antiderivative of $1/x$ that has the value 0 for x =1. This is this very simple formula that motivated to qualify as "natural" the natural logarithm; this is also one of the main reasons of the importance of the constant e.

The derivative with a generalised functional argument $f(x)$ is

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}.$$

The quotient at the right hand side is called the logarithmic derivative of f. Computing $f'(x)$ by means of the derivative of $\ln(f(x))$ is known as logarithmic differentiation.[36] The antiderivative of the natural logarithm $\ln(x)$ is:[37]

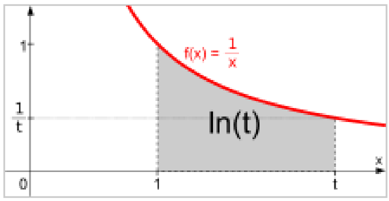
$$\int \ln(x)\,dx = x\ln(x) - x + C.$$

Related formulas, such as antiderivatives of logarithms to other bases can be derived from this equation using the change of bases.[38]

Integral representation of the natural logarithm

The natural logarithm of t equals the integral of $1/x\ dx$ from 1 to t:

$$\ln(t) = \int_1^t \frac{1}{x}\,dx.$$

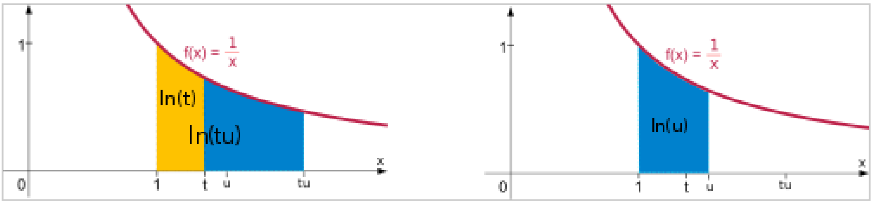


The natural logarithm of t is the shaded area underneath the graph of the function $f(x) = 1/x$ (reciprocal of x).

In other words, $\ln(t)$ equals the area between the x axis and the graph of the function $1/x$, ranging from $x = 1$ to $x = t$ (figure at the right). This is a consequence of the fundamental theorem of calculus and the fact that derivative of $\ln(x)$ is $1/x$. The right hand side of this equation can serve as a definition of the natural logarithm. Product and power logarithm formulas can be derived from this definition.[39] For example, the product formula $\ln(tu) = \ln(t) + \ln(u)$ is deduced as:

$$\ln(tu) = \int_1^{tu}\frac{1}{x}\,dx \overset{(1)}{=} \int_1^t \frac{1}{x}\,dx + \int_t^{tu}\frac{1}{x}\,dx \overset{(2)}{=} \ln(t) + \int_1^u \frac{1}{w}\,dw = \ln(t) + \ln(u).$$

The equality (1) splits the integral into two parts, while the equality (2) is a change of variable ($w = x/t$). In the illustration below, the splitting corresponds to dividing the area into the yellow and blue parts. Rescaling the left hand blue area vertically by the factor t and shrinking it by the same factor horizontally does not change its size. Moving it appropriately, the area fits the graph of the function $f(x) = 1/x$ again. Therefore, the left hand blue area, which is the integral of $f(x)$ from t to tu is the same as the integral from 1 to u. This justifies the equality (2) with a more geometric proof.



A visual proof of the product formula of the natural logarithm

The power formula $\ln(t^r) = r \ln(t)$ may be derived in a similar way:

$$\ln(t^r) = \int_1^{t^r} \frac{1}{x} dx = \int_1^t \frac{1}{w^r}\left(rw^{r-1}\, dw\right) = r \int_1^t \frac{1}{w}\, dw = r \ln(t).$$

The second equality uses a change of variables (integration by substitution), $w = x^{1/r}$.

The sum over the reciprocals of natural numbers,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k},$$

is called the harmonic series. It is closely tied to the natural logarithm: as n tends to infinity, the difference,

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n),$$

converges (i.e., gets arbitrarily close) to a number known as the Euler–Mascheroni constant. This relation aids in analyzing the performance of algorithms such as quicksort.[40]

There is also another integral representation of the logarithm that is useful in some situations.

$$\ln(x) = -\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{dt}{t}\left(e^{-xt} - e^{-t}\right)$$

This can be verified by showing that it has the same value at $x = 1$, and the same derivative.

Transcendence of the logarithm

Real numbers that are not algebraic are called transcendental;[41] for example, π and *e* are such numbers, but $\sqrt{2 - \sqrt{3}}$ is not. Almost all real numbers are transcendental. The logarithm is an example of a transcendental function. The Gelfond–Schneider theorem asserts that logarithms usually take transcendental, i.e., "difficult" values.[42]

Calculation

The logarithm keys (*log* for base-10 and *ln* for base-*e*) on a typical scientific calculator

Logarithms are easy to compute in some cases, such as $\log_{10}(1000) = 3$. In general, logarithms can be calculated using power series or the arithmetic–geometric mean, or be retrieved from a precalculated logarithm table that provides a fixed precision.[43][44] Newton's method, an iterative method to solve equations approximately, can also be used to calculate the logarithm, because its inverse function, the exponential function, can be computed efficiently.[45] Using look-up tables, CORDIC-like methods can be used to compute logarithms if the only available operations are addition and bit shifts.[46][47] Moreover, the binary logarithm algorithm calculates $\mathrm{lb}(x)$ recursively based on repeated squarings of x, taking advantage of the relation

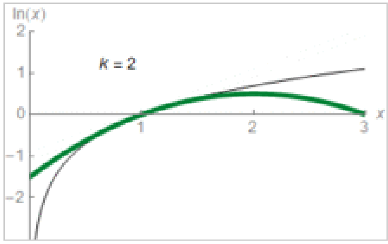
$$\log_2(x^2) = 2 \log_2(x).$$

Power series

Taylor series

For any real number z that satisfies $0 < z < 2$, the following formula holds:[nb 4][48]

$$\ln(z) = \frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$



The Taylor series of ln(z) centered at z = 1. The animation shows the first 10 approximations along with the 99th and 100th. The approximations do not converge beyond a distance of 1 from the center.

This is a shorthand for saying that ln(z) can be approximated to a more and more accurate value by the following expressions:

$$\begin{array}{lllll} (z-1) & & & & \\ (z-1) & - & \frac{(z-1)^2}{2} & & \\ (z-1) & - & \frac{(z-1)^2}{2} & + & \frac{(z-1)^3}{3} \\ \vdots & & & & \end{array}$$

For example, with z = 1.5 the third approximation yields 0.4167, which is about 0.011 greater than ln(1.5) = 0.405465. This series approximates ln(z) with arbitrary precision, provided the number of summands is large enough. In elementary calculus, ln(z) is therefore the *limit* of this series. It is the Taylor series of the natural logarithm at z = 1. The Taylor series of ln z provides a particularly useful approximation to ln(1+z) when z is small, $|z| < 1$, since then

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots \approx z.$$

For example, with z = 0.1 the first-order approximation gives $\ln(1.1) \approx 0.1$, which is less than 5% off the correct value 0.0953.

More efficient series

Another series is based on the area hyperbolic tangent function:

$$\ln(z) = 2 \cdot \operatorname{artanh} \frac{z-1}{z+1} = 2\left(\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right),$$

for any real number $z > 0$.[nb 5][48] Using the Sigma notation, this is also written as

$$\ln(z) = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{z-1}{z+1}\right)^{2n+1}.$$

This series can be derived from the above Taylor series. It converges more quickly than the Taylor series, especially if z is close to 1. For example, for z = 1.5, the first three terms of the second series approximate ln(1.5) with an error of about 3×10^{-6}. The quick convergence for z close to 1 can be taken advantage of in the following way: given a low-accuracy approximation $y \approx \ln(z)$ and putting

$$A = \frac{z}{\exp(y)},$$

the logarithm of z is:

$$\ln(z) = y + \ln(A).$$

The better the initial approximation y is, the closer A is to 1, so its logarithm can be calculated efficiently. A can be calculated using the exponential series, which converges quickly provided y is not too large. Calculating the logarithm of larger z can be reduced to smaller values of z by writing $z = a \cdot 10^b$, so that $\ln(z) = \ln(a) + b \cdot \ln(10)$.

A closely related method can be used to compute the logarithm of integers. From the above series, it follows that:

$$\ln(n+1) = \ln(n) + 2\sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k+1}.$$

If the logarithm of a large integer n is known, then this series yields a fast converging series for log(n+1).

Arithmetic–geometric mean approximation

The arithmetic–geometric mean yields high precision approximations of the natural logarithm. ln(x) is approximated to a precision of 2^{-p} (or p precise bits) by the following formula (due to Carl Friedrich Gauss):[49][50]

$$\ln(x) \approx \frac{\pi}{2M(1, 2^{2-m}/x)} - m\ln(2).$$

Here M(x,y) denotes the arithmetic–geometric mean of x and y. It is obtained by repeatedly calculating the average (x+y)/2 (arithmetic mean) and sqrt(x*y) (geometric mean) of x and y then let those two numbers become the next x and y. The two numbers quickly converge to a common limit which is the value of M(x,y). m is chosen such that

$$x\,2^m > 2^{p/2}.$$

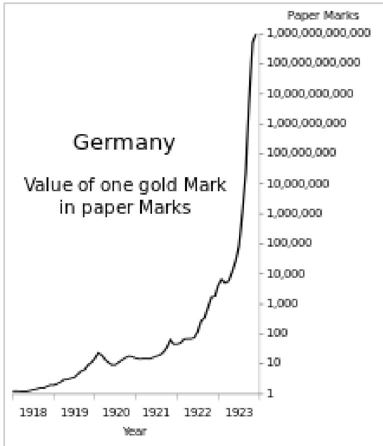
to insure the required precision. A larger m makes the M(x,y) calculation take more steps (the initial x and y are farther apart so it takes more steps to converge) but gives more precision. The constants π and ln(2) can be calculated with quickly converging series.

Applications

A nautilus displaying a logarithmic spiral

Logarithms have many applications inside and outside mathematics. Some of these occurrences are related to the notion of scale invariance. For example, each chamber of the shell of a nautilus is an approximate copy of the next one, scaled by a constant factor. This gives rise to a logarithmic spiral.[51] Benford's law on the distribution of leading digits can also be explained by scale invariance.[52] Logarithms are also linked to self-similarity. For example, logarithms appear in the analysis of algorithms that solve a problem by dividing it into two similar smaller problems and patching their solutions.[53] The dimensions of self-similar geometric shapes, that is, shapes whose parts resemble the overall picture are also based on logarithms. Logarithmic scales are useful for quantifying the relative change of a value as opposed to its absolute difference. Moreover, because the logarithmic function log(x) grows very slowly for large x, logarithmic scales are used to compress large-scale scientific data. Logarithms also occur in numerous scientific formulas, such as the Tsiolkovsky rocket equation, the Fenske equation, or the Nernst equation.

Logarithmic scale



A logarithmic chart depicting the value of one Goldmark in Papiermarks during the German hyperinflation in the 1920s

Scientific quantities are often expressed as logarithms of other quantities, using a *logarithmic scale*. For example, the decibel is a unit of measurement associated with logarithmic-scale quantities. It is based on the common logarithm of ratios—10 times the common logarithm of a power ratio or 20 times the common logarithm of a voltage ratio. It is used to quantify the loss of voltage levels in transmitting electrical signals,[54] to describe power levels of sounds in acoustics,[55] and the absorbance of light in the fields of spectrometry and optics. The signal-to-noise ratio describing the amount of unwanted noise in relation to a (meaningful) signal is also measured in decibels.[56] In a similar vein, the peak signal-to-noise ratio is commonly used to assess the quality of sound and image compression methods using the logarithm.[57]

The strength of an earthquake is measured by taking the common logarithm of the energy emitted at the quake. This is used in the moment magnitude scale or the Richter magnitude scale. For example, a 5.0 earthquake releases 32 times ($10^{1.5}$) and a 6.0 releases 1000 times (10^3) the energy of a 4.0.[58] Another logarithmic scale is apparent magnitude. It measures the brightness of stars logarithmically.[59] Yet another example is pH in chemistry; pH is the negative of the common logarithm of the activity of hydronium ions (the form hydrogen ions H^+ take in water).[60] The activity of hydronium ions in neutral water is 10^{-7} mol·L^{-1}, hence a pH of 7. Vinegar typically has a pH of about 3. The difference of 4 corresponds to a ratio of 10^4 of the activity, that is, vinegar's hydronium ion activity is about 10^{-3} mol·L^{-1}.

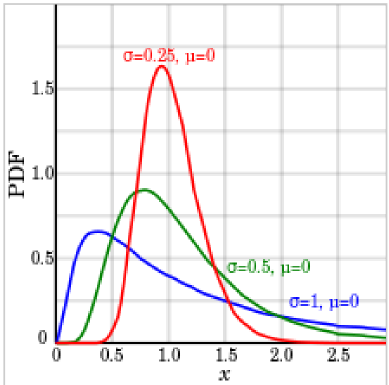
Semilog (log-linear) graphs use the logarithmic scale concept for visualization: one axis, typically the vertical one, is scaled logarithmically. For example, the chart at the right compresses the steep increase from 1 million to 1 trillion to the same space (on the vertical axis) as the increase from 1 to 1 million. In such graphs, exponential functions of the form $f(x) = a \cdot b^x$ appear as straight lines with slope equal to the logarithm of b. Log-log graphs scale both axes logarithmically, which causes functions of the form $f(x) = a \cdot x^k$ to be depicted as straight lines with slope equal to the exponent k. This is applied in visualizing and analyzing power laws.[61]

Psychology

Logarithms occur in several laws describing human perception:[62][63] Hick's law proposes a logarithmic relation between the time individuals take to choose an alternative and the number of choices they have.[64] Fitts's law predicts that the time required to rapidly move to a target area is a logarithmic function of the distance to and the size of the target.[65] In psychophysics, the Weber–Fechner law proposes a logarithmic relationship between stimulus and sensation such as the actual vs. the perceived weight of an item a person is carrying.[66] (This "law", however, is less precise than more recent models, such as the Stevens' power law.[67])

Psychological studies found that individuals with little mathematics education tend to estimate quantities logarithmically, that is, they position a number on an unmarked line according to its logarithm, so that 10 is positioned as close to 100 as 100 is to 1000. Increasing education shifts this to a linear estimate (positioning 1000 10x as far away) in some circumstances, while logarithms are used when the numbers to be plotted are difficult to plot linearly.[68][69]

Probability theory and statistics

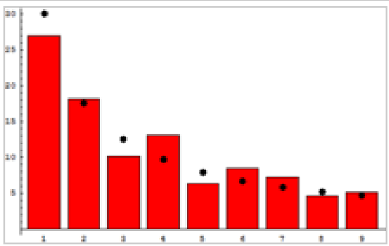


Three probability density functions (PDF) of random variables with log-normal distributions. The location parameter μ, which is zero for all three of the PDFs shown, is the mean of the logarithm of the random variable, not the mean of the variable itself.

Logarithms arise in probability theory: the law of large numbers dictates that, for a fair coin, as the number of coin-tosses increases to infinity, the observed proportion of heads approaches one-half. The fluctuations of this proportion about one-half are described by the law of the iterated logarithm.[70]

Logarithms also occur in log-normal distributions. When the logarithm of a random variable has a normal distribution, the variable is said to have a log-normal distribution.[71] Log-normal distributions are encountered in many fields, wherever a variable is formed as the product of many independent positive random variables, for example in the study of turbulence.[72]

Logarithms are used for maximum-likelihood estimation of parametric statistical models. For such a model, the likelihood function depends on at least one parameter that must be estimated. A maximum of the likelihood function occurs at the same parameter-value as a maximum of the logarithm of the likelihood (the "*log likelihood*"), because the logarithm is an increasing function. The log-likelihood is easier to maximize, especially for the multiplied likelihoods for independent random variables.[73]


Distribution of first digits (in %, red bars) in the population of the 237 countries of the world. Black dots indicate the distribution predicted by Benford's law.

Benford's law describes the occurrence of digits in many data sets, such as heights of buildings. According to Benford's law, the probability that the first decimal-digit of an item in the data sample is d (from 1 to 9) equals $\log_{10}(d + 1) - \log_{10}(d)$, *regardless* of the unit of measurement.[74] Thus, about 30% of the data can be expected to have 1 as first digit, 18% start with 2, etc. Auditors examine deviations from Benford's law to detect fraudulent accounting.[75]

Computational complexity

Analysis of algorithms is a branch of computer science that studies the performance of algorithms (computer programs solving a certain problem).[76] Logarithms are valuable for describing algorithms that divide a problem into smaller ones, and join the solutions of the subproblems.[77]

For example, to find a number in a sorted list, the binary search algorithm checks the middle entry and proceeds with the half before or after the middle entry if the number is still not found. This algorithm requires, on average, $\log_2(N)$ comparisons, where N is the list's length.[78] Similarly, the merge sort algorithm sorts an unsorted list by dividing the list into halves and sorting these first before merging the results. Merge sort algorithms typically require a time approximately proportional to $N \cdot \log(N)$.[79] The base of the logarithm is not specified here, because the result only changes by a constant factor when another base is used. A constant factor is usually disregarded in the analysis of algorithms under the standard uniform cost model.[80]

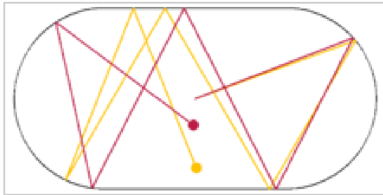
A function $f(x)$ is said to grow logarithmically if $f(x)$ is (exactly or approximately) proportional to the logarithm of x. (Biological descriptions of organism growth, however, use this term for an exponential function.[81]) For example, any natural number N can be represented in binary form in no more than $\log_2(N) + 1$ bits. In other words, the amount of memory needed to store N grows logarithmically with N.

Entropy and chaos

Entropy is broadly a measure of the disorder of some system. In statistical thermodynamics, the entropy S of some physical system is defined as

$$S = -k \sum_i p_i \ln(p_i).$$

The sum is over all possible states i of the system in question, such as the positions of gas particles in a container. Moreover, p_i is the probability that the state i is attained and k is the Boltzmann constant. Similarly, entropy in information theory measures the quantity of information. If a message recipient may expect any one of N possible messages with equal likelihood, then the amount of information conveyed by any one such message is quantified as $\log_2(N)$ bits.[82]

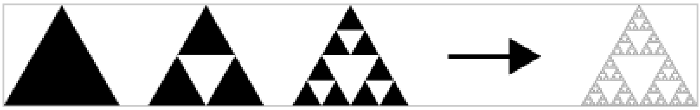


Billiards on an oval billiard table. Two particles, starting at the center with an angle differing by one degree, take paths that diverge chaotically because of reflections at the boundary.

Lyapunov exponents use logarithms to gauge the degree of chaoticity of a dynamical system. For example, for a particle moving on an oval billiard table, even small changes of the initial conditions result in very different paths of the particle. Such systems are chaotic in a deterministic way, because small measurement errors of the initial state predictably lead to largely different final states.[83] At least one Lyapunov exponent of a deterministically chaotic system is positive.

Fractals

Logarithms occur in definitions of the dimension of fractals.[84] Fractals are geometric objects that are self-similar: small parts reproduce, at least roughly, the entire global structure. The Sierpinski triangle (pictured) can be covered by three copies of itself, each having sides half the original length. This makes the Hausdorff dimension of this structure $\ln(3)/\ln(2) \approx 1.58$. Another logarithm-based notion of dimension is obtained by counting the number of boxes needed to cover the fractal in question.

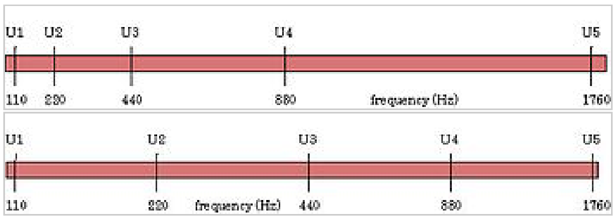


The Sierpinski triangle (at the right) is constructed by repeatedly replacing equilateral triangles by three smaller ones.

Music

Logarithms are related to musical tones and intervals. In equal temperament, the frequency ratio depends only on the interval between two tones, not on the specific frequency, or pitch, of the individual tones. For example, the note *A* has a frequency of 440 Hz and *B-flat* has a frequency of 466 Hz. The interval between *A* and *B-flat* is a semitone, as is the one between *B-flat* and *B* (frequency 493 Hz). Accordingly, the frequency ratios agree:

$$\frac{466}{440} \approx \frac{493}{466} \approx 1.059 \approx \sqrt[12]{2}.$$



Four different octaves shown on a linear scale, then shown on a logarithmic scale (as the ear hears them).

Therefore, logarithms can be used to describe the intervals: an interval is measured in semitones by taking the base-$2^{1/12}$ logarithm of the frequency ratio, while the base-$2^{1/1200}$ logarithm of the frequency ratio expresses the interval in cents, hundredths of a semitone. The latter is used for finer encoding, as it is needed for non-equal temperaments.[85]

Interval (the two tones are played at the same time)	1/12 tone play	Semitone play	Just major third play	Major third play	Tritone play	Octave play
Frequency ratio *r*	$2^{\frac{1}{72}} \approx 1.0097$	$2^{\frac{1}{12}} \approx 1.0595$	$\frac{5}{4} = 1.25$	$2^{\frac{4}{12}} = \sqrt[3]{2} \approx 1.2599$	$2^{\frac{6}{12}} = \sqrt{2} \approx 1.4142$	$2^{\frac{12}{12}} = 2$
Corresponding number of semitones $\log_{\sqrt[12]{2}}(r) = 12\log_2(r)$	$\frac{1}{6}$	1	≈ 3.8631	4	6	12
Corresponding number of cents $\log_{\sqrt[1200]{2}}(r) = 1200\log_2(r)$	$16\frac{2}{3}$	100	≈ 386.31	400	600	1200

Number theory

Natural logarithms are closely linked to counting prime numbers (2, 3, 5, 7, 11, ...), an important topic in number theory. For any integer x, the quantity of prime numbers less than or equal to x is denoted $\pi(x)$. The prime number theorem asserts that $\pi(x)$ is approximately given by

$$\frac{x}{\ln(x)},$$

in the sense that the ratio of $\pi(x)$ and that fraction approaches 1 when x tends to infinity.[86] As a consequence, the probability that a randomly chosen number between 1 and x is prime is inversely proportional to the number of decimal digits of x. A far better estimate of $\pi(x)$ is given by the offset logarithmic integral function Li(x), defined by

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\,dt.$$

The Riemann hypothesis, one of the oldest open mathematical conjectures, can be stated in terms of comparing $\pi(x)$ and Li(x).[87] The Erdős–Kac theorem describing the number of distinct prime factors also involves the natural logarithm.

The logarithm of n factorial, $n! = 1 \cdot 2 \cdot \ldots \cdot n$, is given by

$$\ln(n!) = \ln(1) + \ln(2) + \cdots + \ln(n).$$

This can be used to obtain Stirling's formula, an approximation of $n!$ for large n.[88]

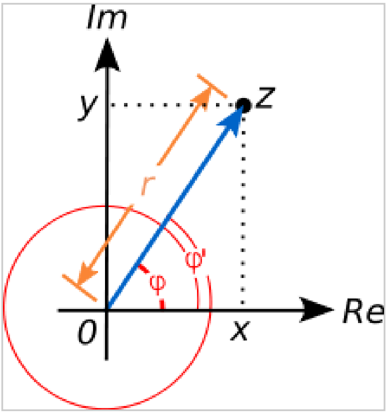
Generalizations

Complex logarithm

The complex numbers a solving the equation

$$e^a = z.$$

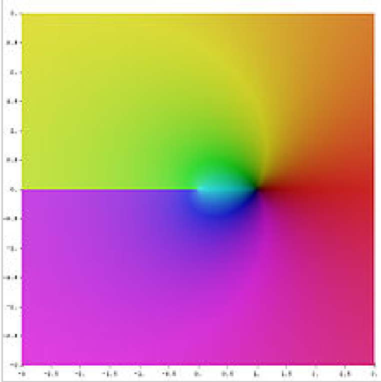
are called *complex logarithms*. Here, z is a complex number. A complex number is commonly represented as $z = x + iy$, where x and y are real numbers and i is the imaginary unit. Such a number can be visualized by a point in the complex plane, as shown at the right. The polar form encodes a non-zero complex number z by its absolute value, that is, the distance r to the origin, and an angle between the x axis and the line passing through the origin and z. This angle is called the argument of z. The absolute value r of z is



Polar form of $z = x + iy$. Both φ and φ' are arguments of z.

$$r = \sqrt{x^2 + y^2}.$$

The argument is not uniquely specified by z: both φ and φ' = φ + 2π are arguments of z because adding 2π radians or 360 degrees[nb 6] to φ corresponds to "winding" around the origin counter-clock-wise by a turn. The resulting complex number is again z, as illustrated at the right. However, exactly one argument φ satisfies −π < φ and φ ≤ π. It is called the *principal argument*, denoted Arg(z), with a capital A.[89] (An alternative normalization is $0 \le \operatorname{Arg}(z) < 2\pi$.[90])



The principal branch of the complex logarithm, Log(z). The black point at $z = 1$ corresponds to absolute value zero and brighter (more saturated) colors refer to bigger absolute values. The hue of the color encodes the argument of Log(z).

Using trigonometric functions sine and cosine, or the complex exponential, respectively, r and φ are such that the following identities hold:[91]

$$\begin{aligned} z &= r(\cos\varphi + i\sin\varphi) \\ &= re^{i\varphi}. \end{aligned}$$

This implies that the a-th power of e equals z, where

$$a = \ln(r) + i(\varphi + 2n\pi),$$

φ is the principal argument Arg(z) and n is an arbitrary integer. Any such a is called a complex logarithm of z. There are infinitely many of them, in contrast to the uniquely defined real logarithm. If $n = 0$, a is called the *principal value* of the logarithm, denoted Log(z). The principal argument of any positive real number x is 0; hence Log(x) is a real number and equals the real (natural) logarithm. However, the above formulas for logarithms of products and powers do *not* generalize to the principal value of the complex logarithm.[92]

The illustration at the right depicts Log(z). The discontinuity, that is, the jump in the hue at the negative part of the x- or real axis, is caused by the jump of the principal argument there. This locus is called a branch cut. This behavior can only be circumvented by dropping the range restriction on φ. Then the argument of z and, consequently, its logarithm become multi-valued functions.

Inverses of other exponential functions

Exponentiation occurs in many areas of mathematics and its inverse function is often referred to as the logarithm. For example, the logarithm of a matrix is the (multi-valued) inverse function of the matrix exponential.[93] Another example is the p-adic logarithm, the inverse function of the p-adic exponential. Both are defined via Taylor series analogous to the real case.[94] In the context of differential geometry, the exponential map maps the tangent space at a point of a manifold to a neighborhood of that point. Its inverse is also called the logarithmic (or log) map.[95]

In the context of finite groups exponentiation is given by repeatedly multiplying one group element b with itself. The discrete logarithm is the integer n solving the equation

$$b^n = x,$$

where x is an element of the group. Carrying out the exponentiation can be done efficiently, but the discrete logarithm is believed to be very hard to calculate in some groups. This asymmetry has important applications in public key cryptography, such as for example in the Diffie–Hellman key exchange, a routine that allows secure exchanges of cryptographic keys over unsecured information channels.[96] Zech's logarithm is related to the discrete logarithm in the multiplicative group of non-zero elements of a finite field.[97]

Further logarithm-like inverse functions include the *double logarithm* ln(ln(x)), the *super- or hyper-4-logarithm* (a slight variation of which is called iterated logarithm in computer science), the Lambert W function, and the logit. They are the inverse functions of the double exponential function, tetration, of $f(w) = we^w$,[98] and of the logistic function, respectively.[99]

Related concepts

From the perspective of group theory, the identity log(cd) = log(c) + log(d) expresses a group isomorphism between positive reals under multiplication and reals under addition. Logarithmic functions are the only continuous isomorphisms between these groups.[100] By means of that isomorphism, the Haar measure (Lebesgue measure) dx on the reals corresponds to the Haar measure dx/x on the positive reals.[101] In complex analysis and algebraic geometry, differential forms of the form df/f are known as forms with logarithmic poles.[102]

The polylogarithm is the function defined by

$$\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

It is related to the natural logarithm by $\mathrm{Li}_1(z) = -\ln(1 - z)$. Moreover, $\mathrm{Li}_s(1)$ equals the Riemann zeta function $\zeta(s)$.[103]

See also

- Cologarithm
- Exponential function
- Decimal exponent (dex)
- Index of logarithm articles

Notes

1. The restrictions on x and b are explained in the section "Analytic properties".
2. Some mathematicians disapprove of this notation. In his 1985 autobiography, Paul Halmos criticized what he considered the "childish ln notation," which he said no mathematician had ever used.[13] The notation was invented by Irving Stringham, a mathematician.[14][15]
3. For example C, Java, Haskell, and BASIC.
4. The same series holds for the principal value of the complex logarithm for complex numbers z satisfying $|z - 1| < 1$.
5. The same series holds for the principal value of the complex logarithm for complex numbers z with positive real part.
6. See radian for the conversion between 2π and 360 degrees.

References

1. Shirali, Shailesh (2002), *A Primer on Logarithms*, Hyderabad: Universities Press, ISBN 978-81-7371-414-6, esp. section 2
2. Kate, S.K.; Bhapkar, H.R. (2009), *Basics Of Mathematics*, Pune: Technical Publications, ISBN 978-81-8431-755-8, chapter 1
3. All statements in this section can be found in Shailesh Shirali 2002, section 4, (Douglas Downing 2003, p. 275), or Kate & Bhapkar 2009, p. 1-1, for example.
4. Bernstein, Stephen; Bernstein, Ruth (1999), *Schaum's outline of theory and problems of elements of statistics. I, Descriptive statistics and probability*, Schaum's outline series, New York: McGraw-Hill, ISBN 978-0-07-005023-5, p. 21
5. Downing, Douglas (2003), *Algebra the Easy Way*, Barron's Educational Series, Hauppauge, N.Y.: Barron's, ISBN 978-0-7641-1972-9, chapter 17, p. 275
6. Wegener, Ingo (2005), *Complexity theory: exploring the limits of efficient algorithms*, Berlin, New York: Springer-Verlag, ISBN 978-3-540-21045-0, p. 20
7. Van der Lubbe, Jan C. A. (1997), *Information Theory*, Cambridge University Press, p. 3, ISBN 9780521467605
8. Allen, Elizabeth; Triantaphillidou, Sophie (2011), *The Manual of Photography*, Taylor & Francis, p. 228, ISBN 9780240520377
9. Franz Embacher; Petra Oberhuemer, *Mathematisches Lexikon* (in German), mathe online: für Schule, Fachhochschule, Universität unde Selbststudium, retrieved 2011-03-22
10. Taylor, B. N. (1995), *Guide for the Use of the International System of Units (SI)*, US Department of Commerce
11. Gullberg, Jan (1997), *Mathematics: from the birth of numbers.*, New York: W. W. Norton & Co, ISBN 978-0-393-04002-9
12. See footnote 1 in Perl, Yehoshua; Reingold, Edward M. (December 1977). "Understanding the complexity of interpolation search". *Information Processing Letters* **6** (6): 219–222. doi:10.1016/0020-0190(77)90072-2.
13. Paul Halmos (1985), *I Want to Be a Mathematician: An Automathography*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-96078-4
14. Irving Stringham (1893), *Uniplanar algebra: being part I of a propædeutic to the higher mathematical analysis*, The Berkeley Press, p. xiii
15. Roy S. Freedman (2006), *Introduction to Financial Technology*, Amsterdam: Academic Press, p. 59, ISBN 978-0-12-370478-8
16. See Theorem 3.29 in Rudin, Walter (1984). *Principles of mathematical analysis* (3rd ed., International student ed.). Auckland: McGraw-Hill International. ISBN 978-0070856134.
17. Napier, John (1614), *Mirifici Logarithmorum Canonis Descriptio* [*The Description of the Wonderful Rule of Logarithms*] (in Latin), Edinburgh, Scotland: Andrew Hart
18. Hobson, Ernest William (1914), *John Napier and the invention of logarithms, 1614*, Cambridge: The University Press
19. Folkerts, Menso; Launert, Dieter; Thom, Andreas (October 2015), *Jost Bürgi's Method for Calculating Sines*
20. MacTutor Article on Jost Bürgi: http://www-history.mcs.st-and.ac.uk/Biographies/Burgi.html
21. William Gardner (1742) *Tables of Logarithms*
22. R.C. Pierce (1977) "A brief history of logarithm", Two-Year College Mathematics Journal 8(1):22–6.
23. Enrique Gonzales-Velasco (2011) *Journey through Mathematics – Creative Episodes in its History*, §2.4 Hyperbolic logarithms, page 117, Springer ISBN 978-0-387-92153-2
24. Florian Cajori (1913) "History of the exponential and logarithm concepts", American Mathematical Monthly 20: 5, 35, 75, 107, 148, 173, 205.
25. Bryant, Walter W., *A History of Astronomy*, London: Methuen & Co, p. 44
26. Campbell-Kelly, Martin (2003), *The history of mathematical tables: from Sumer to spreadsheets*, Oxford scholarship online, Oxford University Press, ISBN 978-0-19-850841-0, section 2
27. Abramowitz, Milton; Stegun, Irene A., eds. (1972), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (10th ed.), New York: Dover Publications, ISBN 978-0-486-61272-0, section 4.7., p. 89
28. Spiegel, Murray R.; Moyer, R.E. (2006), *Schaum's outline of college algebra*, Schaum's outline series, New York: McGraw-Hill, ISBN 978-0-07-145227-4, p. 264
29. Maor 2009, sections 1, 13
30. Devlin, Keith (2004). *Sets, functions, and logic: an introduction to abstract mathematics*. Chapman & Hall/CRC mathematics (3rd ed.). Boca Raton, Fla: Chapman & Hall/CRC. ISBN 1-58488-449-5., or see the references in function

31. Lang, Serge (1997), *Undergraduate analysis*, Undergraduate Texts in Mathematics (2nd ed.), Berlin, New York: Springer-Verlag, ISBN 978-0-387-94841-6, MR 1476913, section III.3
32. Lang 1997, section IV.2
33. Dieudonné, Jean (1969). *Foundations of Modern Analysis* **1**. Academic Press. p. 84. item (4.3.1)
34. Stewart, James (2007), *Single Variable Calculus: Early Transcendentals*, Belmont: Thomson Brooks/Cole, ISBN 978-0-495-01169-9, section 1.6
35. "Calculation of *d/dx(Log(b,x))*". *Wolfram Alpha*. Wolfram Research. Retrieved 15 March 2011.
36. Kline, Morris (1998), *Calculus: an intuitive and physical approach*, Dover books on mathematics, New York: Dover Publications, ISBN 978-0-486-40453-0, p. 386
37. "Calculation of *Integrate(ln(x))*". *Wolfram Alpha*. Wolfram Research. Retrieved 15 March 2011.
38. Abramowitz & Stegun, eds. 1972, p. 69
39. Courant, Richard (1988), *Differential and integral calculus. Vol. I*, Wiley Classics Library, New York: John Wiley & Sons, ISBN 978-0-471-60842-4, MR 1009558, section III.6
40. Havil, Julian (2003), *Gamma: Exploring Euler's Constant*, Princeton University Press, ISBN 978-0-691-09983-5, sections 11.5 and 13.8
41. Nomizu, Katsumi (1996), *Selected papers on number theory and algebraic geometry* **172**, Providence, RI: AMS Bookstore, p. 21, ISBN 978-0-8218-0445-2
42. Baker, Alan (1975), *Transcendental number theory*, Cambridge University Press, ISBN 978-0-521-20461-3, p. 10
43. Muller, Jean-Michel (2006), *Elementary functions* (2nd ed.), Boston, MA: Birkhäuser Boston, ISBN 978-0-8176-4372-0, sections 4.2.2 (p. 72) and 5.5.2 (p. 95)
44. Hart; Cheney; Lawson; et al. (1968), *Computer Approximations*, SIAM Series in Applied Mathematics, New York: John Wiley, section 6.3, p. 105–111
45. Zhang, M.; Delgado-Frias, J.G.; Vassiliadis, S. (1994), "Table driven Newton scheme for high precision logarithm generation", *IEE Proceedings Computers & Digital Techniques* **141** (5): 281–292, doi:10.1049/ip-cdt:19941268, ISSN 1350-2387, Archived from the original on 29 May 2015, section 1 for an overview
46. Meggitt, J. E. (April 1962), "Pseudo Division and Pseudo Multiplication Processes", *IBM Journal*, doi:10.1147/rd.62.0210
47. Kahan, W. (May 20, 2001), *Pseudo-Division Algorithms for Floating-Point Logarithms and Exponentials*
48. Abramowitz & Stegun, eds. 1972, p. 68
49. Sasaki, T.; Kanada, Y. (1982), "Practically fast multiple-precision evaluation of log(x)", *Journal of Information Processing* **5** (4): 247–250, retrieved 30 March 2011
50. Ahrendt, Timm (1999), *Fast computations of the exponential function*, Lecture notes in computer science **1564**, Berlin, New York: Springer, pp. 302–312, doi:10.1007/3-540-49116-3_28
51. Maor 2009, p. 135
52. Frey, Bruce (2006), *Statistics hacks*, Hacks Series, Sebastopol, CA: O'Reilly, ISBN 978-0-596-10164-0, chapter 6, section 64
53. Ricciardi, Luigi M. (1990), *Lectures in applied mathematics and informatics*, Manchester: Manchester University Press, ISBN 978-0-7190-2671-3, p. 21, section 1.3.2
54. Bakshi, U. A. (2009), *Telecommunication Engineering*, Pune: Technical Publications, ISBN 978-81-8431-725-1, section 5.2
55. Maling, George C. (2007), "Noise", in Rossing, Thomas D., *Springer handbook of acoustics*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-30446-5, section 23.0.2
56. Tashev, Ivan Jelev (2009), *Sound Capture and Processing: Practical Approaches*, New York: John Wiley & Sons, ISBN 978-0-470-31983-3, p. 48
57. Chui, C.K. (1997), *Wavelets: a mathematical tool for signal processing*, SIAM monographs on mathematical modeling and computation, Philadelphia: Society for Industrial and Applied Mathematics, ISBN 978-0-89871-384-8, p. 180
58. Crauder, Bruce; Evans, Benny; Noell, Alan (2008), *Functions and Change: A Modeling Approach to College Algebra* (4th ed.), Boston: Cengage Learning, ISBN 978-0-547-15669-9, section 4.4.
59. Bradt, Hale (2004), *Astronomy methods: a physical approach to astronomical observations*, Cambridge Planetary Science, Cambridge University Press, ISBN 978-0-521-53551-9, section 8.3, p. 231
60. IUPAC (1997), A. D. McNaught, A. Wilkinson, ed., *Compendium of Chemical Terminology ("Gold Book")* (2nd ed.), Oxford: Blackwell Scientific Publications, doi:10.1351/goldbook, ISBN 978-0-9678550-9-7
61. Bird, J. O. (2001), *Newnes engineering mathematics pocket book* (3rd ed.), Oxford: Newnes, ISBN 978-0-7506-4992-6, section 34
62. Goldstein, E. Bruce (2009), *Encyclopedia of Perception*, Encyclopedia of Perception, Thousand Oaks, CA: Sage, ISBN 978-1-4129-4081-8, p. 355–356
63. Matthews, Gerald (2000), *Human performance: cognition, stress, and individual differences*, Human Performance: Cognition, Stress, and Individual Differences, Hove: Psychology Press, ISBN 978-0-415-04406-6, p. 48
64. Welford, A. T. (1968), *Fundamentals of skill*, London: Methuen, ISBN 978-0-416-03000-6, OCLC 219156, p. 61
65. Paul M. Fitts (June 1954), "The information capacity of the human motor system in controlling the amplitude of movement", *Journal of Experimental Psychology* **47** (6): 381–391, doi:10.1037/h0055392, PMID 13174710, reprinted in Paul M. Fitts (1992), "The information capacity of the human motor system in controlling the amplitude of movement" (PDF), *Journal of Experimental Psychology: General* **121** (3): 262–269, doi:10.1037/0096-3445.121.3.262, PMID 1402698, retrieved 30 March 2011
66. Banerjee, J. C. (1994), *Encyclopaedic dictionary of psychological terms*, New Delhi: M.D. Publications, ISBN 978-81-85880-28-0, OCLC 33860167, p. 304
67. Nadel, Lynn (2005), *Encyclopedia of cognitive science*, New York: John Wiley & Sons, ISBN 978-0-470-01619-0, lemmas *Psychophysics* and *Perception: Overview*
68. Siegler, Robert S.; Opfer, John E. (2003), "The Development of Numerical Estimation. Evidence for Multiple Representations of Numerical Quantity" (PDF), *Psychological Science* **14** (3): 237–43, doi:10.1111/1467-9280.02438, PMID 12741747
69. Dehaene, Stanislas; Izard, Véronique; Spelke, Elizabeth; Pica, Pierre (2008), "Log or Linear? Distinct Intuitions of the Number Scale in Western and Amazonian Indigene Cultures", *Science* **320** (5880): 1217–1220, doi:10.1126/science.1156540, PMC 2610411, PMID 18511690
70. Breiman, Leo (1992), *Probability*, Classics in applied mathematics, Philadelphia: Society for Industrial and Applied Mathematics, ISBN 978-0-89871-296-4, section 12.9
71. Aitchison, J.; Brown, J. A. C. (1969), *The lognormal distribution*, Cambridge University Press, ISBN 978-0-521-04011-2, OCLC 301100935

72. Jean Mathieu and Julian Scott (2000), *An introduction to turbulent flow*, Cambridge University Press, p. 50, ISBN 978-0-521-77538-0
73. Rose, Colin; Smith, Murray D. (2002), *Mathematical statistics with Mathematica*, Springer texts in statistics, Berlin, New York: Springer-Verlag, ISBN 978-0-387-95234-5, section 11.3
74. Tabachnikov, Serge (2005), *Geometry and Billiards*, Providence, R.I.: American Mathematical Society, pp. 36–40, ISBN 978-0-8218-3919-5, section 2.1
75. Durtschi, Cindy; Hillison, William; Pacini, Carl (2004), "The Effective Use of Benford's Law in Detecting Fraud in Accounting Data" (PDF), *Journal of Forensic Accounting* **V**: 17–34
76. Wegener, Ingo (2005), *Complexity theory: exploring the limits of efficient algorithms*, Berlin, New York: Springer-Verlag, ISBN 978-3-540-21045-0, pages 1-2
77. Harel, David; Feldman, Yishai A. (2004), *Algorithmics: the spirit of computing*, New York: Addison-Wesley, ISBN 978-0-321-11784-7, p. 143
78. Knuth, Donald (1998), *The Art of Computer Programming*, Reading, Mass.: Addison-Wesley, ISBN 978-0-201-89685-5, section 6.2.1, pp. 409–426
79. Donald Knuth 1998, section 5.2.4, pp. 158–168
80. Wegener, Ingo (2005), *Complexity theory: exploring the limits of efficient algorithms*, Berlin, New York: Springer-Verlag, p. 20, ISBN 978-3-540-21045-0
81. Mohr, Hans; Schopfer, Peter (1995), *Plant physiology*, Berlin, New York: Springer-Verlag, ISBN 978-3-540-58016-4, chapter 19, p. 298
82. Eco, Umberto (1989), *The open work*, Harvard University Press, ISBN 978-0-674-63976-8, section III.I
83. Sprott, Julien Clinton (2010), *Elegant Chaos: Algebraically Simple Chaotic Flows*, New Jersey: World Scientific, ISBN 978-981-283-881-0, section 1.9
84. Helmberg, Gilbert (2007), *Getting acquainted with fractals*, De Gruyter Textbook, Berlin, New York: Walter de Gruyter, ISBN 978-3-11-019092-2
85. Wright, David (2009), *Mathematics and music*, Providence, RI: AMS Bookstore, ISBN 978-0-8218-4873-9, chapter 5
86. Bateman, P. T.; Diamond, Harold G. (2004), *Analytic number theory: an introductory course*, New Jersey: World Scientific, ISBN 978-981-256-080-3, OCLC 492669517, theorem 4.1
87. P. T. Bateman & Diamond 2004, Theorem 8.15
88. Slomson, Alan B. (1991), *An introduction to combinatorics*, London: CRC Press, ISBN 978-0-412-35370-3, chapter 4
89. Ganguly, S. (2005), *Elements of Complex Analysis*, Kolkata: Academic Publishers, ISBN 978-81-87504-86-3, Definition 1.6.3
90. Nevanlinna, Rolf Herman; Paatero, Veikko (2007), "Introduction to complex analysis", *London: Hilger* (Providence, RI: AMS Bookstore), Bibcode:1974aitc.book.....W, ISBN 978-0-8218-4399-4, section 5.9
91. Moore, Theral Orvis; Hadlock, Edwin H. (1991), *Complex analysis*, Singapore: World Scientific, ISBN 978-981-02-0246-0, section 1.2
92. Wilde, Ivan Francis (2006), *Lecture notes on complex analysis*, London: Imperial College Press, ISBN 978-1-86094-642-4, theorem 6.1.
93. Higham, Nicholas (2008), *Functions of Matrices. Theory and Computation*, Philadelphia, PA: SIAM, ISBN 978-0-89871-646-7, chapter 11.
94. Neukirch, Jürgen (1999), *Algebraic Number Theory*, Grundlehren der mathematischen Wissenschaften **322**, Berlin: Springer-Verlag, ISBN 978-3-540-65399-8, Zbl 0956.11021, MR 1697859, section II.5.
95. Hancock, Edwin R.; Martin, Ralph R.; Sabin, Malcolm A. (2009), *Mathematics of Surfaces XIII: 13th IMA International Conference York, UK, September 7–9, 2009 Proceedings*, Springer, p. 379, ISBN 978-3-642-03595-1
96. Stinson, Douglas Robert (2006), *Cryptography: Theory and Practice* (3rd ed.), London: CRC Press, ISBN 978-1-58488-508-5
97. Lidl, Rudolf; Niederreiter, Harald (1997), *Finite fields*, Cambridge University Press, ISBN 978-0-521-39231-0
98. Corless, R.; Gonnet, G.; Hare, D.; Jeffrey, D.; Knuth, Donald (1996), "On the Lambert *W* function" (PDF), *Advances in Computational Mathematics* (Berlin, New York: Springer-Verlag) **5**: 329–359, doi:10.1007/BF02124750, ISSN 1019-7168
99. Cherkassky, Vladimir; Cherkassky, Vladimir S.; Mulier, Filip (2007), *Learning from data: concepts, theory, and methods*, Wiley series on adaptive and learning systems for signal processing, communications, and control, New York: John Wiley & Sons, ISBN 978-0-471-68182-3, p. 357
100. Bourbaki, Nicolas (1998), *General topology. Chapters 5—10*, Elements of Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-3-540-64563-4, MR 1726872, section V.4.1
101. Ambartzumian, R. V. (1990), *Factorization calculus and geometric probability*, Cambridge University Press, ISBN 978-0-521-34535-4, section 1.4
102. Esnault, Hélène; Viehweg, Eckart (1992), *Lectures on vanishing theorems*, DMV Seminar **20**, Basel, Boston: Birkhäuser Verlag, ISBN 978-3-7643-2822-1, MR 1193913, section 2
103. Apostol, T.M. (2010), "Logarithm", in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, ISBN 978-0521192255, MR 2723248

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- Hazewinkel, Michiel, ed. (2001), "Logarithmic function", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Colin Byfleet, *Educational video on logarithms*, retrieved 2010-10-12
- Edward Wright, *Translation of Napier's work on logarithms*, Archived from the original on 3 December 2002, retrieved 2010-10-12

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