Entanglement Spectrum Resolved by Loop Symmetries

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Haruki Yagi, Zongping Gong

Department of Applied Physics, The University of Tokyo, Tokyo 113-8656, Japan

Take-home messages:

We provide a **rigorous analysis** of the **entanglement structure** of pure states with non-invertible Wilson-loop symmetries described by $\mathrm{Rep}(G)$.

Our framework can be extended to **topological gauge theories** in arbitrary dimensions.

It provides an expression for **topological entanglement entropy** for **arbitrary manifolds and bipartitions**.

In 2D systems, the **Li-Haldane correspondence** holds for the gapped boundary.

Outline

Interrupt me anytime with questions!

[Review] Entanglement and Quantum Phases

Entanglement of Pure States: Linear Algebra

We will discuss entanglement only in pure states.

Definition 1 (Entanglement for pure states)

Divide the system into N subsystems.

The entire state ψ is **not entangled** iff it can be written as

$$|\psi\rangle = |\psi_1\rangle|\psi_2\rangle\cdots|\psi_N\rangle. \tag{1}$$

Otherwise, the state ψ is **entangled**.

♣

How can we detect it for a given state and a given partition?

It is hard to answer this question for arbitrary N.

However, the **entanglement spectrum** is a good diagnostic in the case $N = 2$.

Entanglement of Pure States: Linear Algebra

Definition 2 (Entanglement spectrum)

Let $\{|k_i\rangle\}$ be an orthonormal basis of subsystem $i \in \{1, 2\}$. Expand the state $|\psi\rangle$ as

$$|\psi\rangle = \sum_{k_1,k_2} \psi_{k_1,k_2} |k_1\rangle|k_2\rangle. \tag{2}$$

Define a rectangular matrix W as

$$W = \sum_{k_1,k_2} \psi_{k_1,k_2} |k_1\rangle\langle k_2|. \tag{3}$$

The **entanglement spectrum** $\{\sqrt{\lambda_j}\}$ is the set of singular values of W.

Proposition 3

For $N = 2$, the state ψ is not entangled $\Leftrightarrow$ $\sqrt{\lambda_1} = 1, \sqrt{\lambda_{j\geq 2}} = 0$.

Entanglement of Pure States: Linear Algebra

In the case $N = 2$, the **entanglement entropy** is also of interest.

Definition 4 (Entanglement entropy)

1. Obtain the matrix W by the procedure above.
2. Define a square matrix ρ as $\rho = WW^\dagger$.
3. The **entanglement entropy** S between subsystems 1 and 2 is defined as

$$S = -\operatorname{Tr} \rho \ln \rho. \tag{4}$$

♣

The entanglement entropy S is a diagnostic of bipartite entanglement.

Theorem 5

For $N = 2$, $S = 0 \Leftrightarrow \psi$ is not entangled. Equivalently, $S \neq 0 \Leftrightarrow \psi$ is entangled.

♡

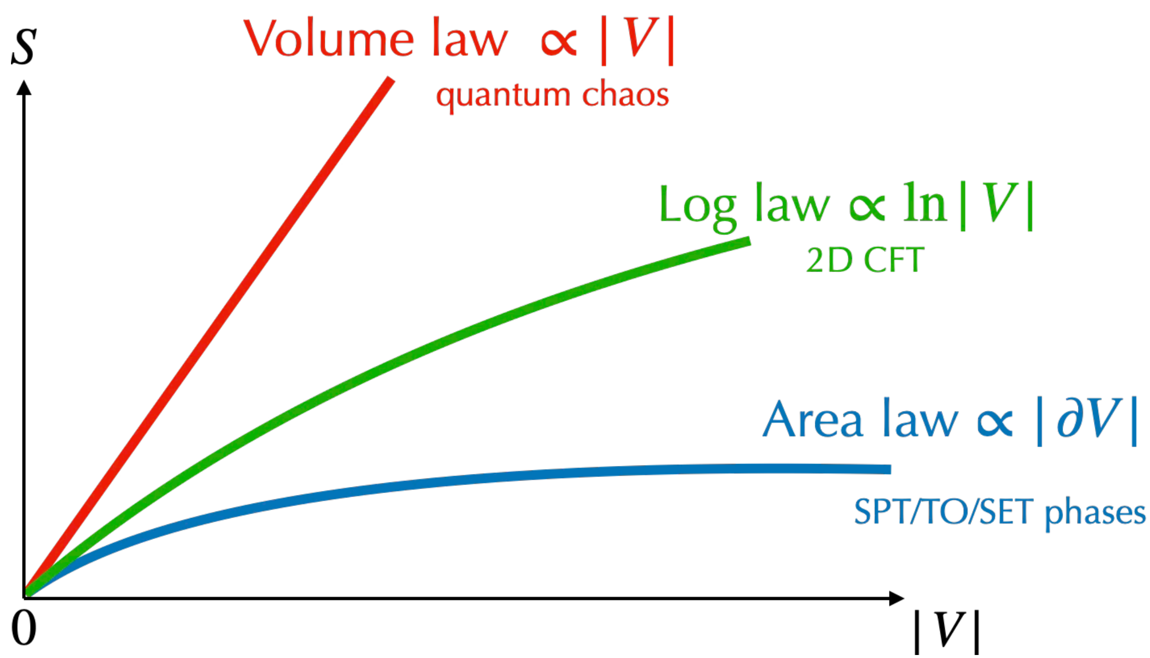
Let's apply this to quantum phases!

Entanglement of Pure States: Quantum Phases

Definition 6 (Page curve [Pag93])

The **Page curve** is a plot with the x-axis showing the subsystem volume (ratio) and the y-axis the entanglement entropy. ♣

Its behavior is generally categorized into **area law, log law, or volume law**.



Entanglement of Pure States: Quantum Phases

Proposition 7 (Calabrese-Cardy formula [CC04, CC09, HLW94])

In $\mathbf{(1+1)}$**D CFTs**, the entanglement entropy takes the form

$$S = \frac{c}{6} \ln|V| + \cdots \tag{5}$$

- c: the **central charge** of the theory
- V: subsystem

Proposition 8 (Topological entanglement entropy [KP06, LW06])

In $\mathbf{(2+1)}$**D TQFTs**, the entanglement entropy (of connected ∂V) takes the form

$$S = \alpha|\partial V| - \ln \mathcal{D} + \cdots \tag{6}$$

- α: a non-universal coefficient
- $\mathcal{D}$: the **total quantum dimension** of the theory

Entanglement in Pure States: Resolved(?) by Symmetries

Entanglement entropy extracts important information from CFT and TQFT.

Then, can we do a similar thing for **SPT phases**? Here **symmetry** comes into play.

Definition 9 (Group symmetry of the density matrix)

Fix a group G and a (possibly anomalous) representation D.

The density matrix ρ has G **symmetry** if $[D(g), \rho] = 0$ for ${}^{\forall} g \in G$.

♣

Proposition 10 (Symmetry-resolved entanglement)

Group symmetry **resolves** the density matrix ρ into irreducible sectors:

$$\rho = \bigoplus_{\alpha \in \mathrm{Rep}(G)} \mathbb{1}_{d_\alpha} \otimes \rho_\alpha. \tag{7}$$

♠

How about entanglement entropy?

Entanglement in Pure States: Resolved(?) by Symmetries

Remark

The entanglement entropy of SPT phases has a general expression known as **symmetry-resolved entanglement entropy**, but it **does not extract particularly important information** (e.g. topological index). See e.g. [AS20, Kus+23].

However, the entanglement spectrum **degeneracy reflects the topological class** [Pol+10, YMG25].

Remark

Moreover, the entanglement **spectral statistics** can capture the presence of chaotic behavior in quantum many-body dynamics [GNR16].

The entanglement **spectrum** is much more **important**!

Section Summary

> **Important**
>
> Entanglement aids **extracting characteristics of quantum phases**.

> **Important**
>
> **Entanglement entropy is sometimes not enough** to capture quantum phases.
> In such cases, **entanglement spectrum** may provide a good criterion.

> **Important**
>
> **Group symmetry resolves the density matrix** into irrep sectors.

However…

We encounter *generalized* symmetries in QFTs.

Do they resolve the entanglement spectrum?

And how?

[Review] Generalized Symmetries

Global Symmetry & Topological Defect

Henceforth, let d denote the spacetime dimension.

Consider a global symmetry with group G.

The following figures depict how symmetry acts in $d = 3$.

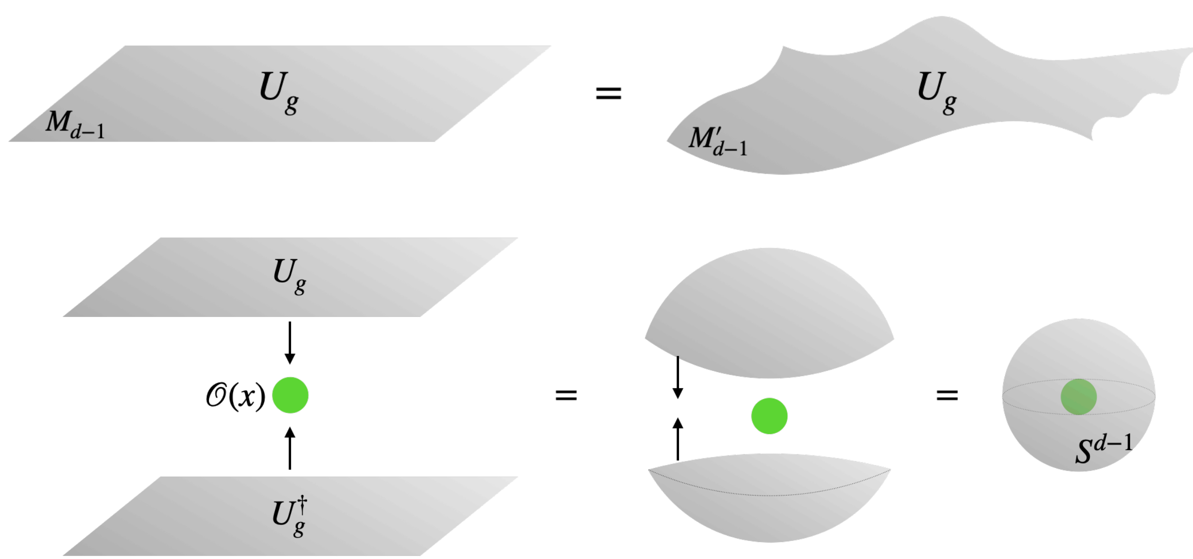
A coarse-grained picture of the consecutive actions of $g, g' \in G$:

Action on the state ϕ:

Global Symmetry & Topological Defect

Action on the operator:

The symmetry operator can be topologically deformed:



Global Symmetry & Topological Defect

Now we understand that

$$\{\text{Global Symmetry}\} \subset \{\text{Topological Defect}\}.$$

The notion of **generalized symmetry** simply identifies the two notions[1]:

$$\boldsymbol{\{\text{Generalized Symmetry}\} = \{\text{Topological Defect}\}.}$$

- Seminal papers: [BT18, FRS02, Gai+15]
- Review papers: [Bha+23, Cho+23, McG23, Sch23, Sha24]

[1]Except subsystem symmetry and modulated symmetry. Including them, the concept is a little bit enlarged as

$$\boldsymbol{\{\text{Generalized Symmetry}\} \supset \{\text{Topological Defect}\}.}$$

Global Symmetry & Topological Defect

Examples of generalized symmetry[1]:

Higher-form symmetry.

The dimension of the manifold on which the symmetry is defined is not $d-1$.

Non-invertible symmetry.

The symmetry operators do not form a group.

[1] There are also subsystem symmetry and modulated symmetry, and research has recently revived in contexts such as fractons.

Higher-Form Symmetry

Definition 11 (p-form symmetry)

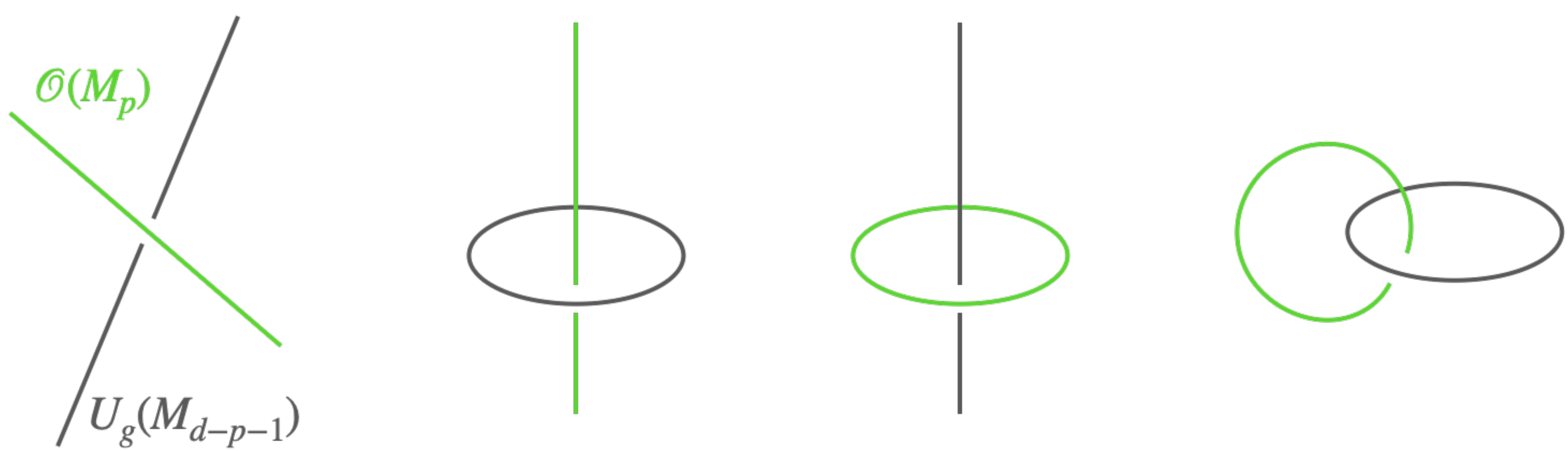
A ***p*-form symmetry**

$= (d - p - 1)$-dimensional topological defect

$=$ **codimension-$(p+1)$ topological defect.**

A p-form symmetry with $p \geq 1$ is called a higher-form symmetry. ♣

A 1-form symmetry in a $d = 3$ QFT:



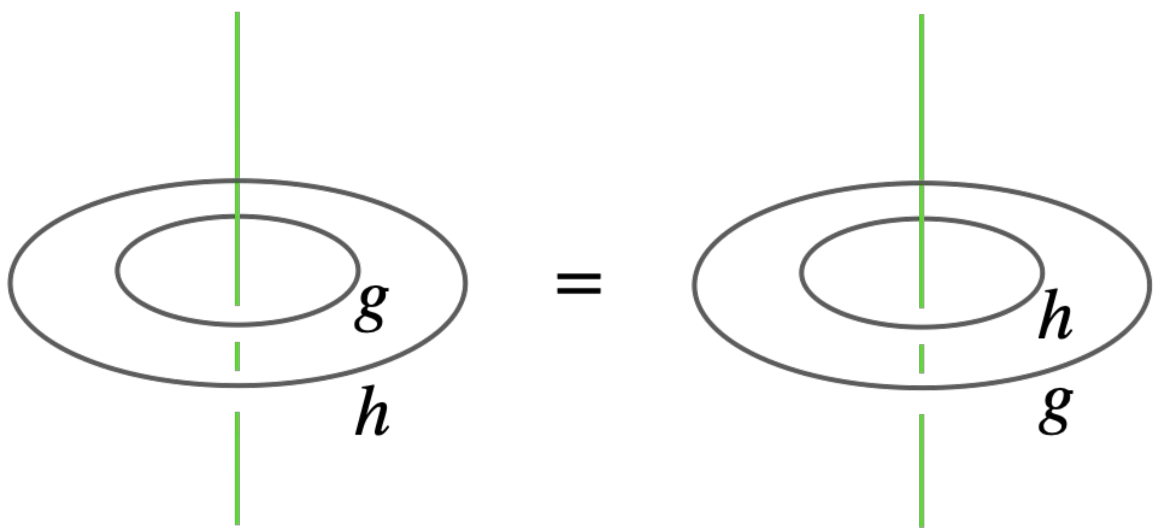
Higher-Form Symmetry

Theorem 12

Higher homotopy groups are all Abelian.

Corollary 12.1

Every higher-form group symmetry is Abelian.

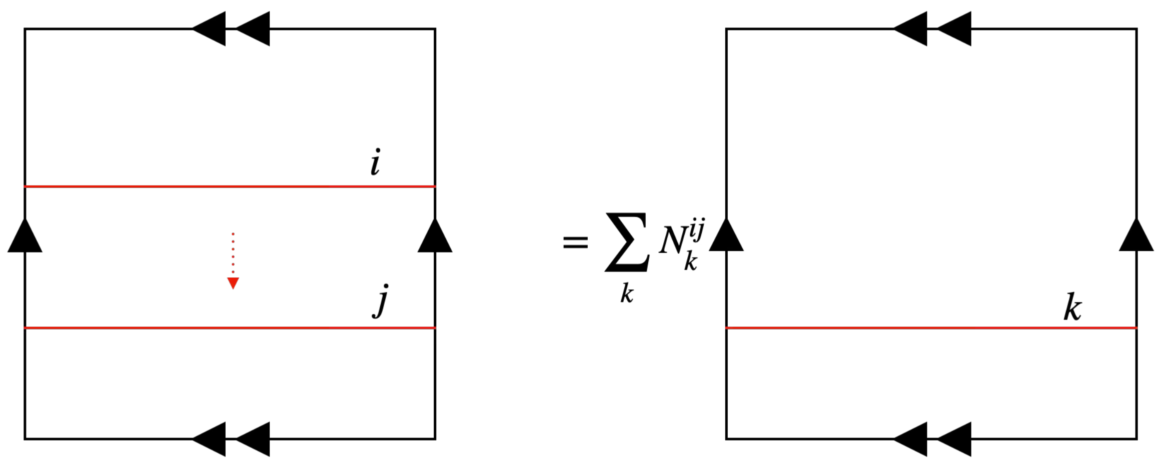


Non-Invertible Symmetry

Definition 13 (Non-invertible symmetry)

A symmetry whose actions **do not form a group**, but a **fusion category**. ♣

Example. In 2D CFTs, the **Verlinde lines** form a non-invertible symmetry.

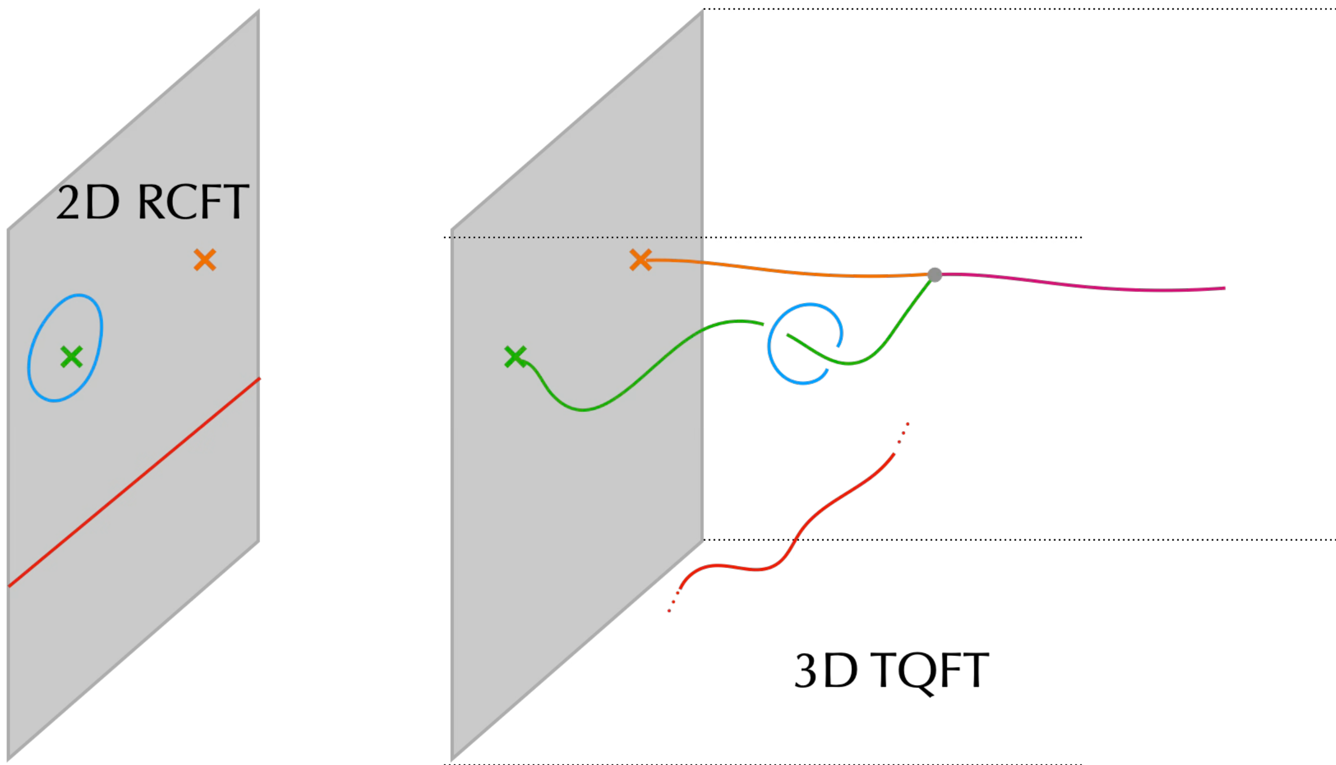


Given a modular S matrix, the fusion coefficient is given by the **Verlinde formula**:

$$N_k^{ij} = \sum_l \frac{S_{il} S_{jl} \overline{S_{lk}}}{S_{0l}}. \tag{8}$$

Non-Invertible Symmetry

Example. **Anyons and Verlinde lines in 2D RCFT** form (generally non-invertible) **1-form symmetry in 3D TQFT**[1] [FMT24, FRS02, Sim23, Wit89].



[1] This correspondence is called *topological holography, SymTFT, categorical symmetry*, and so on...

Dual Theory and Dual Symmetry

We now introduce **gauging** and **dual symmetry**, which are used in our paper.

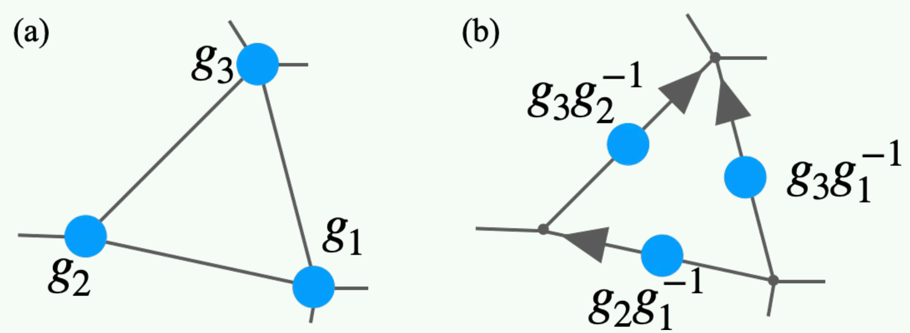
Let us start with the Kramers-Wannier transformation, which is familiar to us.

Definition 14 (Kramers-Wannier transformation)

Fix a directed graph (V, E). Place G-spins on each vertices in V.

A directed edge $e \in E$ connects two vertices (v_e^+, v_e^-).

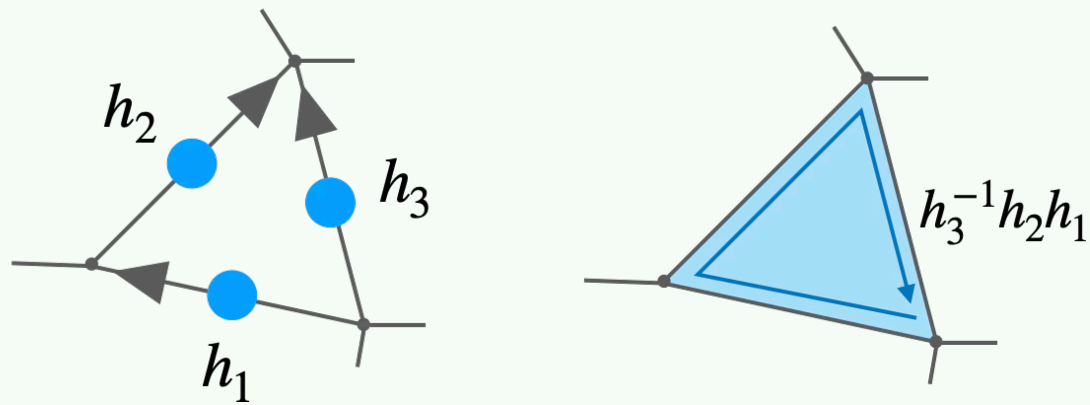
The **Kramers-Wannier (KW) transformation** is a procedure to assign $g_{v_e^+} g_{v_e^-}^{-1}$ to e.



This is nothing but the **"exterior derivative"** $\delta^0 : C^0(M, G) \to C^1(M, G)$ in **non-Abelian cohomology** [Olu58]. ♣

Definition 15

Similarly, we define the exterior derivative $\delta^1 : C^1(M,G) \to C^2(M,G)$:



♣

Remark

$$(\delta^1 \circ \delta^0)(*) = 1. \tag{9}$$

Namely, the KW transformation makes **holonomies along arbitrary loops trivial**.

Dual Theory and Dual Symmetry

We focus on the theory of cochains $c^1 \in C^1(M, G)$ which satisfy $\delta^1 c^1 = 1$:

namely, the Hilbert space dimension is the number of **cocycles**: $\dim = |Z^1(M, G)|$.

Remark

Unlike the KW-transformation, the holonomies along **non-contractible loops** can be **nontrivial**. Contractible loops have trivial holonomies (called *flat* configurations).

Remark

In such a theory, the Wilson loops are topological operators.
We call this a **loop symmetry**[1].
The Wilson loops form fusion rules described by a fusion category $\mathrm{Rep}(G)$.

[1] Formally, this is nothing but $(d-2)$-form $\mathrm{Rep}(G)$ symmetry.

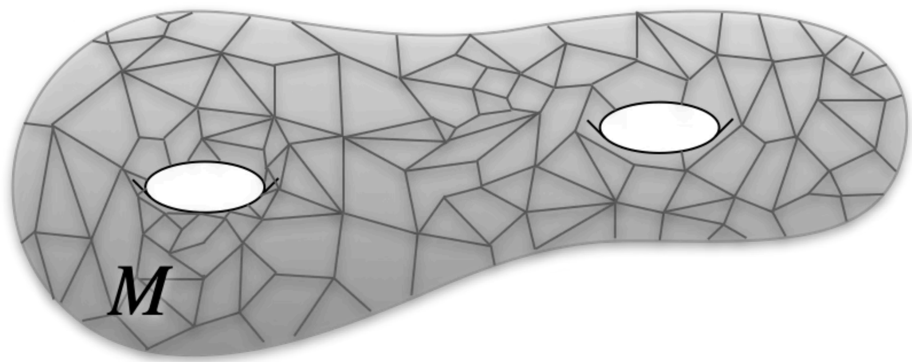
Question to be answered:

How does the loop symmetry resolve W?

[Results] Setup & General Algorithm

Detailed Setup - Defining the Theory

1. Fix a compact manifold M on which the theory is defined.[1]
2. Take a "good discretization" of M consisting of the set of vertices V, directed edges E, and plaquettes P.

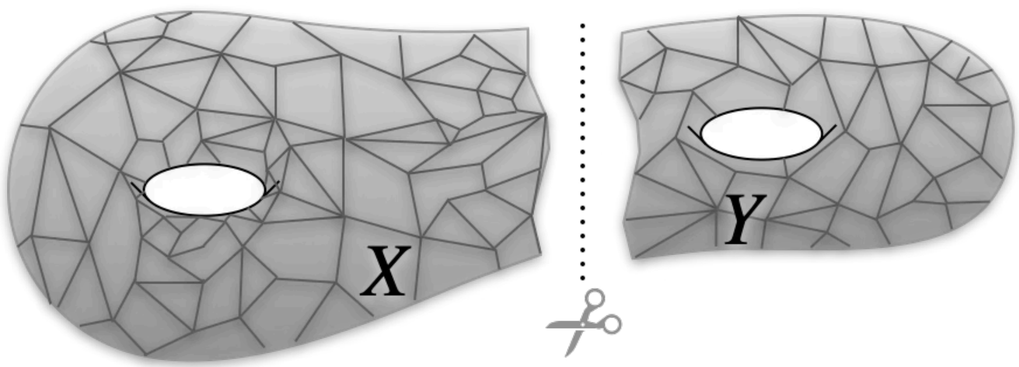


3. Fix a finite (generally non-Abelian) "gauge group" G.
4. Assign elements of G to each oriented edge to satisfy **the locally-flat condition**.
5. Take an (arbitrary) superposition of locally-flat states.

[1]Although the conditions of closedness and connectivity can be relaxed remaining our result unchanged, we implicitly assume them for simplicity for now.

Detailed Setup - Fixing the Bipartition

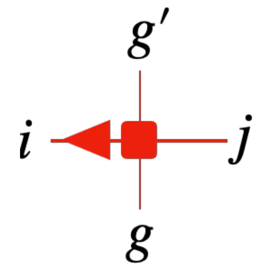
1. Divide M into X and Y.
2. Correspondingly, divide the set of
   - edges E into E_X and E_Y.
   - vertices V into V_X, V_Y and V_∂.



3. Place $|G|$-dimensional qudits (G-spins) on each directed edges.
   - The entire Hilbert space is simply the tensor product $\mathbb{C}^{|G|^{\otimes|E|}}$.
   - The Hilbert space for the subsystem X (Y) is $\mathbb{C}^{|G|^{\otimes|E_X|}}$ $\left(\mathbb{C}^{|G|^{\otimes|E_Y|}}\right)$.

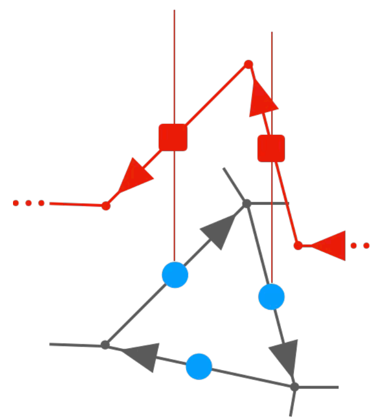
Detailed Setup - Imposing the Symmetry

1. Introduce a matrix product operator (MPO) whose building block is



$$= D^{\alpha}_{ij}(g)\,\delta_{g,g'}$$

The MPO obeys the $\mathrm{Rep}(\mathrm{G})$ fusion rule. The MPO acts as a $(d-2)$-form symmetry operator[1] as



[1]If its orientation aligns with (opposite to) that of the edge, we adopt the building block (with $D^{\alpha}_{ij}(g)$ replaced by $D^{\alpha}_{ij}(g^{-1})$).

Detailed Setup - Imposing the Symmetry

2. Discard all states that do not make the MPO topological[1].

Theorem 16

A loop-symmetric configuration is equivalent to a flat configuration[2].

3. We can expand the $\mathrm{Rep}(G)$ loop-symmetric many-body state as

$$|\Psi\rangle = \sum_{\{g\}_X,\{g\}_Y} \Psi^{\mathrm{flat}}_{\{g\}_X,\{g\}_Y} |\{g\}_X\rangle|\{g\}_Y\rangle. \tag{10}$$

Problem. Find all the blocks and their sizes in

$$W = \sum_{\{g\}_X,\{g\}_Y} \Psi^{\mathrm{flat}}_{\{g\}_X,\{g\}_Y} |\{g\}_X\rangle\langle\{g\}_Y|. \tag{11}$$

---

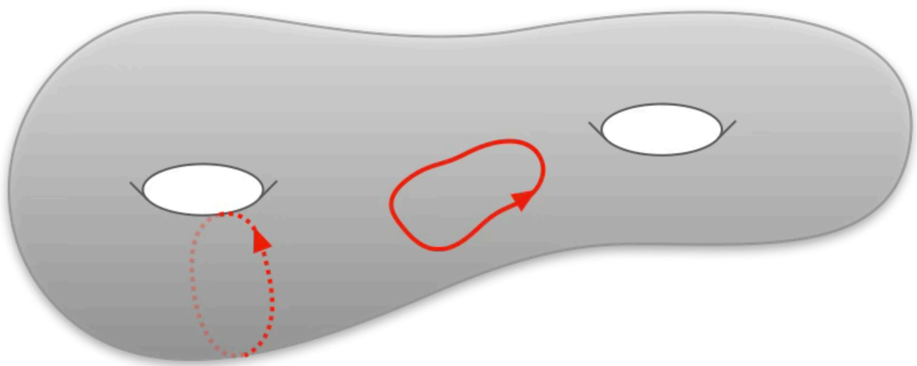
[1]Topological deformation is now interpreted as "pulling through" move on plaquettes.

[2]The Appendix A in our paper [YG25] gives the proof.

Detailed Setup - Imposing the Symmetry

Caution

The flatness condition admits nontrivial holonomy along non-contractible loops (the red dashed loop).



Let us explain our answer to this problem.

Main Result: Algorithm by Category Theory

Theorem 17 (The main result 1 of our paper)

The following commutative diagram is accompanied by the bipartition:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(M,A),G) \arrow[r, "\Pi_X"] \arrow[d, "\Pi_Y"'] & \mathrm{Hom}(\pi_1(X,A),G) \arrow[d, "r_X"'] \\
\mathrm{Hom}(\pi_1(Y,A),G) \arrow[r, "r_Y"] & \mathrm{Hom}(\pi_1(\partial,A),G)
\end{tikzcd}$$

where A is the set of base points (one point per one connected component in ∂).
Define $\mathrm{Im} \subset \mathrm{Hom}(\pi_1(\partial,A),G)$ such that

$$\mathrm{Im} = \{\phi |\ {}^{\exists}\Phi \in \mathrm{Hom}(\pi_1(M,A),G), \phi = r_X \circ \Pi_X(\Phi) = r_Y \circ \Pi_Y(\Phi)\}. \tag{12}$$

Then, we find that W takes the form

$$\boldsymbol{W \in \left(\bigoplus_{\phi\in \mathrm{Im}} \mathbb{C}^{|r_X^{-1}(\phi)| \times |r_Y^{-1}(\phi)|}\right) \otimes \left(\bigoplus_{j=1}^{|G|^{|V_\partial|-|A|}} \mathbb{C}^{|G|^{|V_X|} \times |G|^{|V_Y|}}\right).} \tag{13}$$

Main Result: Algorithm by Category Theory

Absolutely my bad

Sorry for skipping mathematical details!

Category theory is inevitable to explain our result!

Please refer to appendix or our paper.

Main Result: Algorithm by Category Theory

It is worth remarking that:

Corollary 17.1

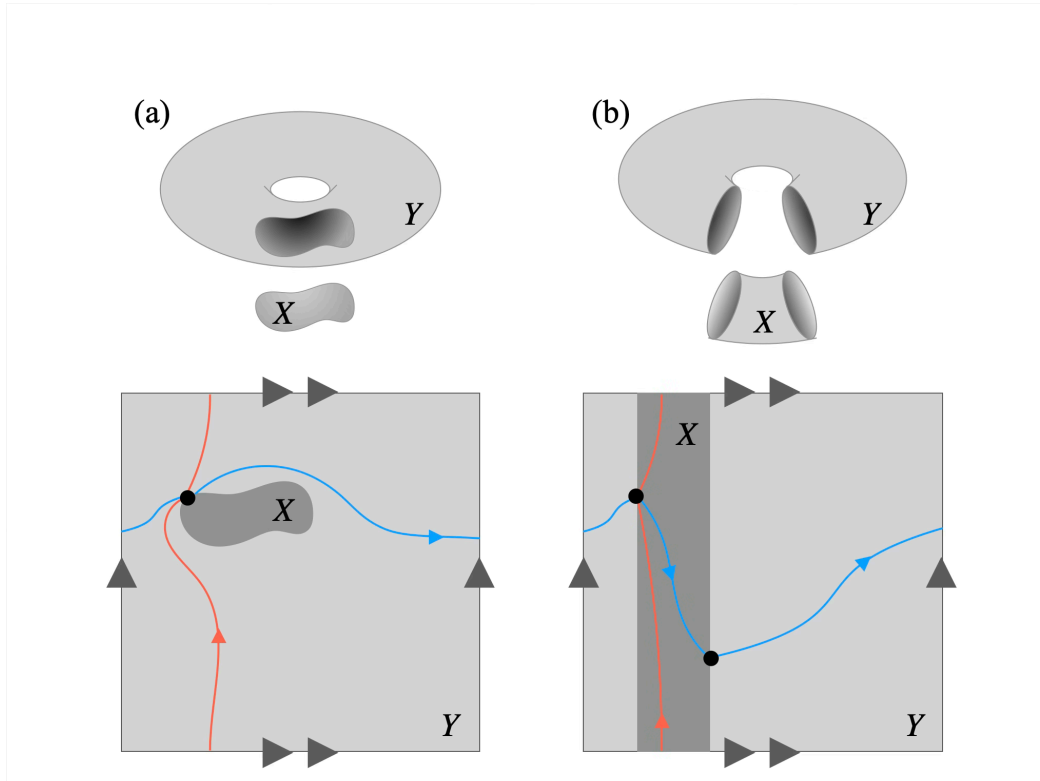
The loop symmetry **does not yield degeneracies in the entanglement spectrum.**

Although we can obtain such an important property, our algorithm seems conceptual. As illustrations, we see how it works in low-dimensional manifolds.

[Results] Examples

Torus

There are two topologically distinct ways to bipartition the torus.



$$(a)\ W \in \bigoplus_{j=1}^{|G|^{|V_\partial|-1}} \mathbb{C}^{|G|^{|V_X|} \times |G||\mathrm{Rep}(G)||G|^{|V_Y|}}, \quad (b) \bigoplus_{c\in G} \bigoplus_{j=1}^{|[c]||G|^{|V_\partial|-2}} \mathbb{C}^{|C_c||G|^{|V_X|} \times |C_c||G|^{|V_Y|}} \tag{14}$$

General Orientable Surfaces

In general, closed orientable surfaces has a bipartition as follows:

$$\Sigma_{\gamma_X,n} \sqcup_{\sqcup^n S^1} \Sigma_{\gamma_Y,n} \mapsto \Sigma_{\gamma_X+\gamma_Y+n-1}. \tag{15}$$

The matrix W for this is as follows:

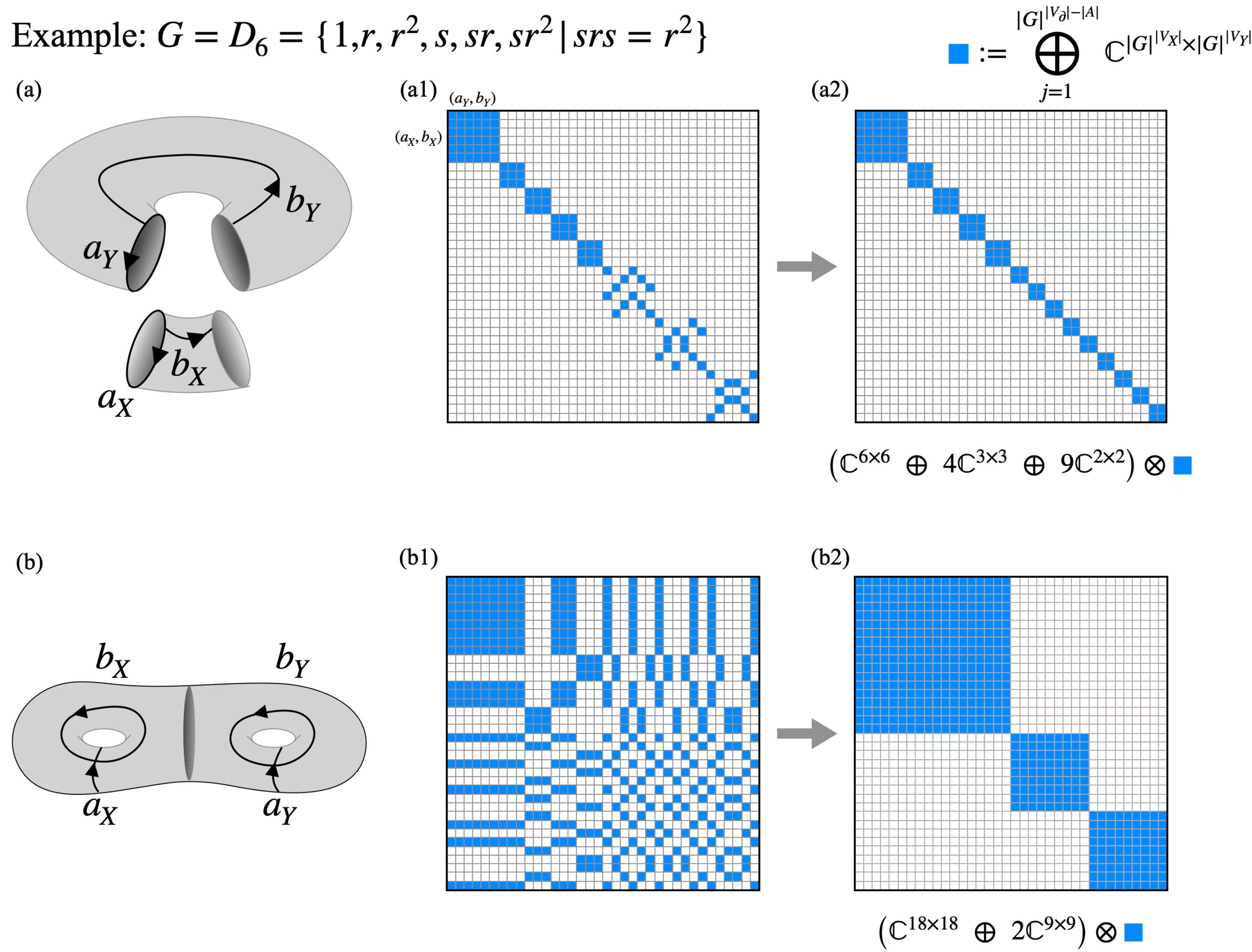
$$W \in \bigoplus_{c\in G^{\times n}} \bigoplus_{j=1}^{|G|^{|V_\partial|-n}} \mathbb{C}^{R_{\gamma_X,n}(c)|G|^{|V_X|} \times R_{\gamma_Y,n}(c)|G|^{|V_Y|}} \tag{16}$$

where $R_{\gamma,n}(c)$ is defined by

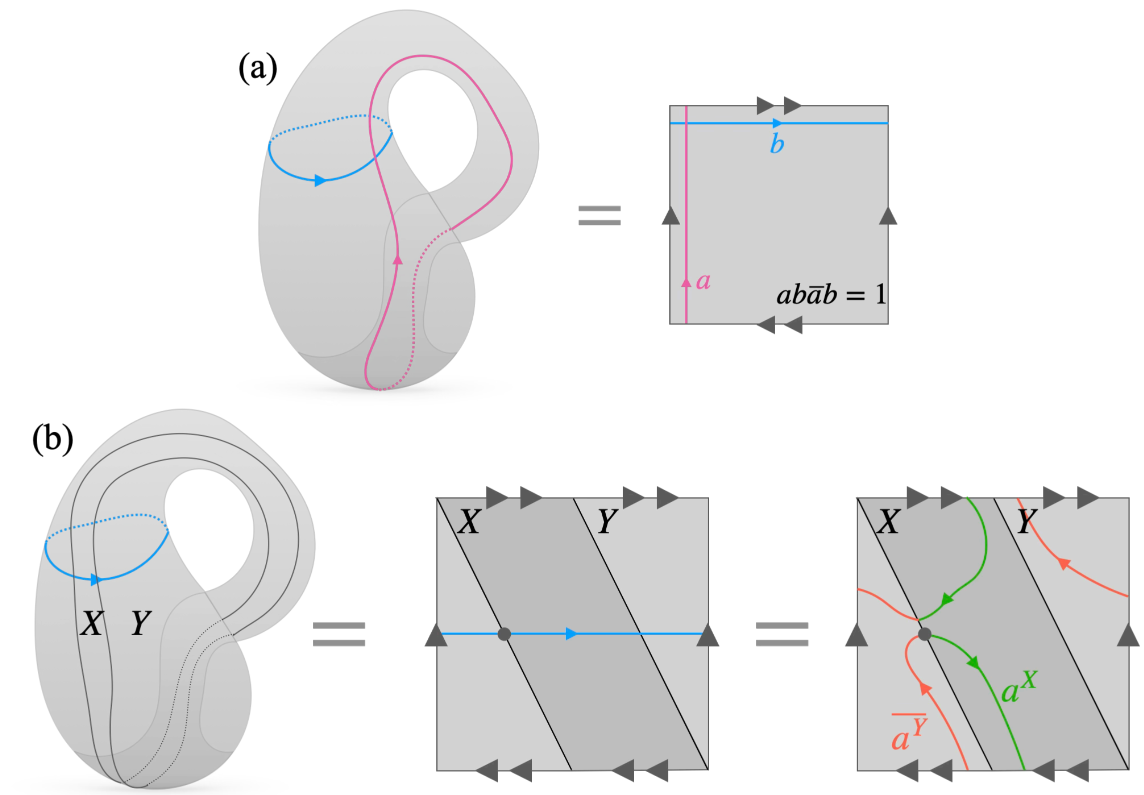
$$R_{\gamma,n}(c) = \sum_{\alpha\in \mathrm{Rep}(G)} \left(\frac{|G|}{d_\alpha}\right)^{2\gamma+n-2} \prod_{i=1}^{n} \chi^\alpha(c_i). \tag{17}$$

👊👊👊 Brute-Force Calculations 👊👊👊

Example: $G = D_6 = \{1, r, r^2, s, sr, sr^2 \mid srs = r^2\}$



The Klein bottle = Möbius band + Möbius band



$$\bigoplus_{c\in G} \bigoplus_{j=1}^{|G|^{|V_\partial|-1}} \mathbb{C}^{K(c)\ |G|^{|V_X|}\ \times K(c)\ |G|^{|V_Y|}}, \quad K(c) = \sum_{\alpha\in\ \mathrm{Rep}(G)} \iota^\alpha \chi^\alpha(c) \tag{18}$$

General Non-Orientable Surfaces

The difference between the orientable and non-orientable cases lies in the presence of the **Frobenius–Schur indicator**.

$$\iota^\alpha = \frac{1}{|G|} \sum_{g \in G} \chi^\alpha(g^2). \tag{19}$$

This indicator is related to time-reversal symmetry in TQFTs[1].

Please refer to our paper [YG25] for explicit formulas for the block sizes.

[1] See e.g. [Bar+20, FRS04, LT18, Ori25a, Ori25b, TY17].

Theorem 18 (Heegaard splitting)

Every compact, oriented 3-manifold can be obtained by gluing two genus-γ handlebodies via Dehn twists.

Computing the gluing map becomes computationally hard for $\gamma \geq 2$.
We focus on the cases $\gamma = 0, 1$.

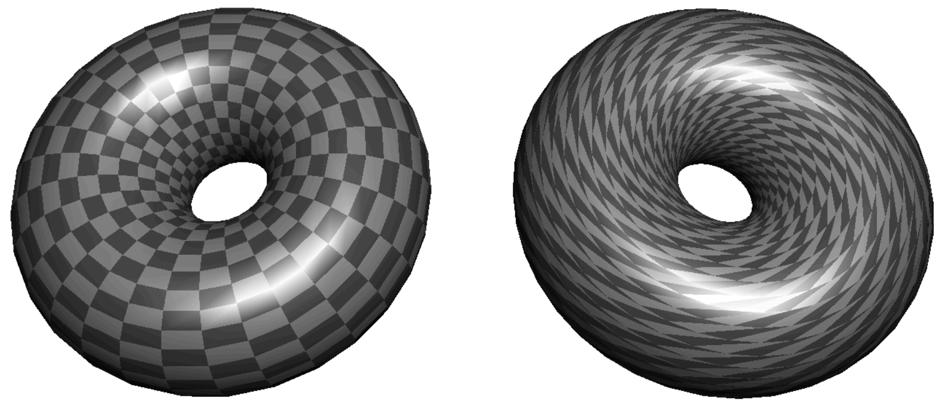
The case $\gamma = 0$ is topologically trivial:

$$|\mathrm{Im}| = |r_X^{-1}(\phi)| = |r_Y^{-1}(\phi)| = 1. \tag{20}$$

How about the case $\boldsymbol{\gamma = 1}$?

Genus–1 Heegaard Splitting

Genus–1 Heegaard splitting
= precisely aligning and identifying the meshes drawn on the two tori:



namely, the **modular group** $\mathrm{SL}(2,\mathbb{Z})$ describes the mapping:

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix}, \quad ps - qr = 1, \quad p, q, r, s \in \mathbb{Z}. \tag{21}$$

Genus–1 Heegaard Splitting

Define the **higher Frobenius-Schur indicator** ν_n^α as

$$\nu_n^\alpha = \frac{1}{|G|}\sum_{g\in G}\chi^\alpha(g^n). \tag{22}$$

Then, the W matrix takes the form

$$W \in \bigoplus_{\phi=1}^{\sum_{\alpha\in \mathrm{Rep(G)}} d_\alpha \nu_q^\alpha} \bigoplus_{j=1}^{|G|^{|V_\partial|-|A|}} \mathbb{C}^{|G|^{|V_X|}\times |G|^{|V_Y|}}. \tag{23}$$

Note that **only q** enters the result and the **preimages have size 1**.

The n-Dimensional Torus

We computed the n-dimensional torus as an example of a higher-dimensional manifold.

The setting:

$$M = \mathbb{T}^n = \left(S^1\right)^n, \quad X = [0,\pi]^{\times k} \times \left(S^1\right)^{\times(n-k)}. \tag{24}$$

The result:[1]

$$\begin{aligned} W &\in \bigoplus_{\phi \in \mathrm{Comm}_{n-1}(G)} \bigoplus_{j=1}^{|[\phi]||G|^{|V_\partial|-2}} \mathbb{C}^{|C_\phi||G|^{|V_X|} \times |C_\phi||G|^{|V_Y|}} \quad (k=1), \\ W &\in \bigoplus_{\phi \in \mathrm{Comm}_{n-k}(G)} \bigoplus_{j=1}^{|G|^{|V_\partial|-1}} \mathbb{C}^{|G|^{|V_X|} \times |\mathrm{Comm}_k(C_\phi)||G|^{|V_Y|}} \quad (k \geq 2). \end{aligned} \tag{25}$$

It is interesting that $k=1$ is the exceptional case.

[1] We defined as $\mathrm{Comm}_{m(G)} = \left\{(g_1, ..., g_m) \in G^{\times m} | g_i g_j = g_j g_i \text{ for } {}^\forall i, j = 1, ..., m\right\}$.

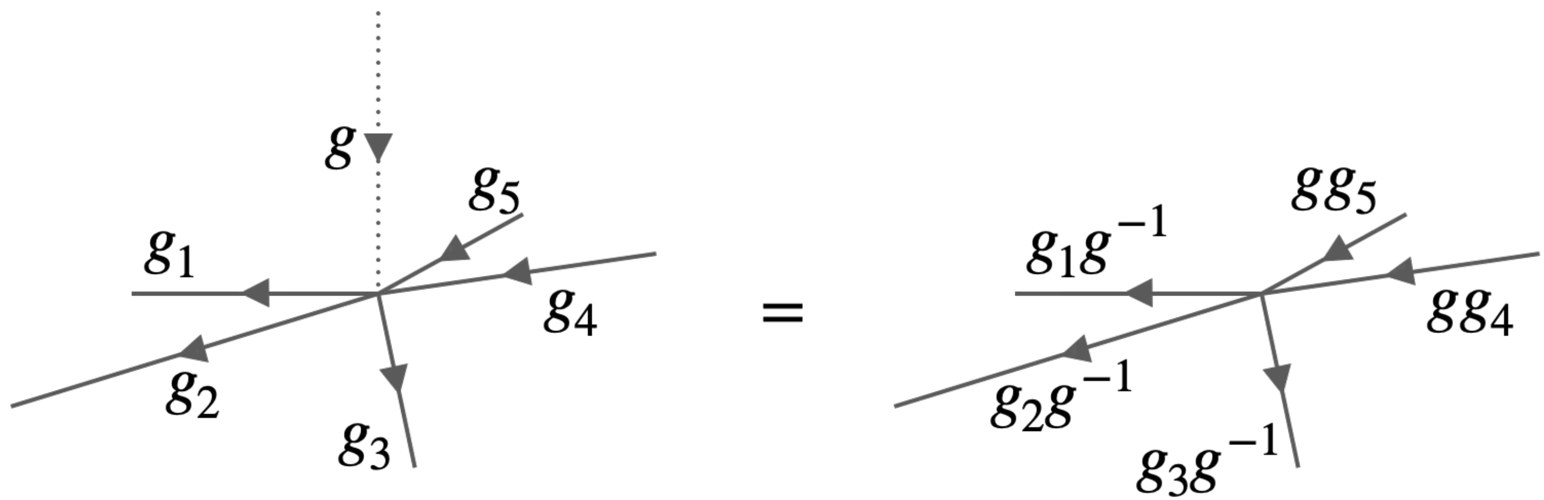
The part 1 is now complete.

Do you have any questions?

[Results] Topological Entanglement

Gauge Invariance and Topological Gauge Theory

Graph gauge transformation:



Topological Gauge Theory

Making the state gauge-invariant leads us to **topological gauge theory**.

$$|\Psi\rangle \mapsto \sum_{\{\text{gauge transf.}\}} \text{gauge transf.} \blacktriangleright |\Psi\rangle \tag{26}$$

Gauge Invariance and Topological Gauge Theory

The Hilbert space dimension of a topological gauge theory is a topological invariant:

$$\dim = |\mathrm{Hom}(\pi_1(M), G)/G|. \tag{27}$$

Here, $*/G$ means identification up to conjugacy: $a \sim gag^{-1}$.
This reflects the Wilson loop is defined up to conjugacy classes.

Problem. Find all the blocks and their sizes in W of topological gauge theory

$$W = \sum_{\{g\}_X, \{g\}_Y} \Psi^{\text{flat+gauge}}_{\{g\}_X, \{g\}_Y} |\{g\}_X\rangle\langle\{g\}_Y|. \tag{28}$$

Block Structure and Degeneracies

Theorem 19 (The main result 2 of our paper)

The gauge-invariant W takes the form

$$\bigoplus_{[\phi]\in \mathrm{Im}\ /G^{\times|A|}} \mathbb{1}_{|[\phi]|} \otimes \left(\bigoplus_{\boldsymbol{\alpha}_{[\phi]}} \mathbb{C}^{x_{\boldsymbol{\alpha}_{[\phi]}} \times y_{\boldsymbol{\alpha}_{[\phi]}}} \otimes \mathbb{1}_{d_{\boldsymbol{\alpha}_{[\phi]}}} \right) \otimes \bigoplus_{j=1}^{|G|^{|V_\partial|-|A|}} 1_{|G|^{|V_X|} \times |G|^{|V_Y|}}, \quad (29)$$

where

- $\boldsymbol{\alpha}_{[\phi]}$ is an irrep of the stabilizer group $G_{[\phi]}$ on $[\phi]$,
- $1_{|G|^{|V_X|} \times |G|^{|V_Y|}}$ is the all-ones matrix of size $|G|^{|V_X|} \times |G|^{|V_Y|}$,
- $x_{\boldsymbol{\alpha}_{[\phi]}} = \frac{1}{|G_{[\phi]}|} \sum_{\boldsymbol{g}_{[\phi]}\in G_{[\phi]}} \chi^{\boldsymbol{\alpha}_{[\phi]}}\left(\boldsymbol{g}_{[\phi]}\right) \operatorname{Tr} D^X_{[\phi]}\left(\boldsymbol{g}_{[\phi]}\right)$ and $y_{\boldsymbol{\alpha}_{[\phi]}}$ is similarly defined.

Topological Entanglement Entropy for General Manifolds and Bipartitions

Corollary 19.1

The entanglement spectrum is $|[\phi]|\, d_{\boldsymbol{\alpha}_{[\phi]}}$-fold degenerate.

Corollary 19.2 (General expression of topological entanglement entropy)

The entanglement entropy for an arbitrary manifold/bipartition[1] is given by

$$S = |V_\partial| \ln|G| - |A| \ln|G| - \sum_{[\phi]\in \mathrm{Im}\ /G^{\times|A|}} |[\phi]| \sum_{\boldsymbol{\alpha}_{[\phi]}\in \mathrm{Rep}\left(G_{[\phi]}\right)} d_{\boldsymbol{\alpha}_{[\phi]}} \sum_{i=1}^{x_{\boldsymbol{\alpha}_{[\phi]}}} \frac{\left|\psi_{\boldsymbol{\alpha}_{[\phi]},i}\right|^2}{\mathcal{N}} \ln \frac{\left|\psi_{\boldsymbol{\alpha}_{[\phi]},i}\right|^2}{\mathcal{N}}, \tag{30}$$

where $\mathcal{N}$ is the normalization constant.

[1] To the best of my knowledge, no specific representation of entanglement entropy for 1-form non-Abelian topological gauge theory on a general manifold with a general partition was known.

Verifying the Li-Haldane Conjecture in the Kitaev Quantum Double Model

Proposition 20 (Li-Haldane conjecture [LH08])

The **entanglement** spectrum in **3D TQFT** $\sim$ the **energy** spectrum of the **2D RCFT**. ♠

Definition 21 (Kitaev quantum double model)

The **Kitaev quantum double model** [Kit97] is a finite-G topological gauge theory on a discretized 2D space. ♣

Verifying the Li-Haldane Conjecture in the Kitaev Quantum Double Model

Corollary 21.1 (the Li-Haldane correspondence)

In the Kitaev quantum double model in 2D orientable surface $\Sigma_{\gamma_X,n}$, $x_{\boldsymbol{\alpha}_{[\phi]}}$ given in our paper turns out to be

$$x_{\boldsymbol{\alpha}_{[\phi]}} = \sum_{[g]\in G_{[\phi]}} \sum_{\beta\in \operatorname{Rep}\left(C_{[g]}\right)} S^{-2\gamma_X - n + 2}_{([g],\beta),(1,1)} \prod_{j=1}^{n} S_{([g],\beta),(c_j,\alpha_j)}, \tag{31}$$

where S is the modular S matrix for finite group [CGR00].

This expression equals **the dimension of the conformal block in 2D RCFT** holographically dual to the Kitaev quantum double model [MS89, MS90].

This correspondence holds even when we define the Kitaev quantum double model in **non-orientable surfaces** [Bar+20].

The same holds for Y.

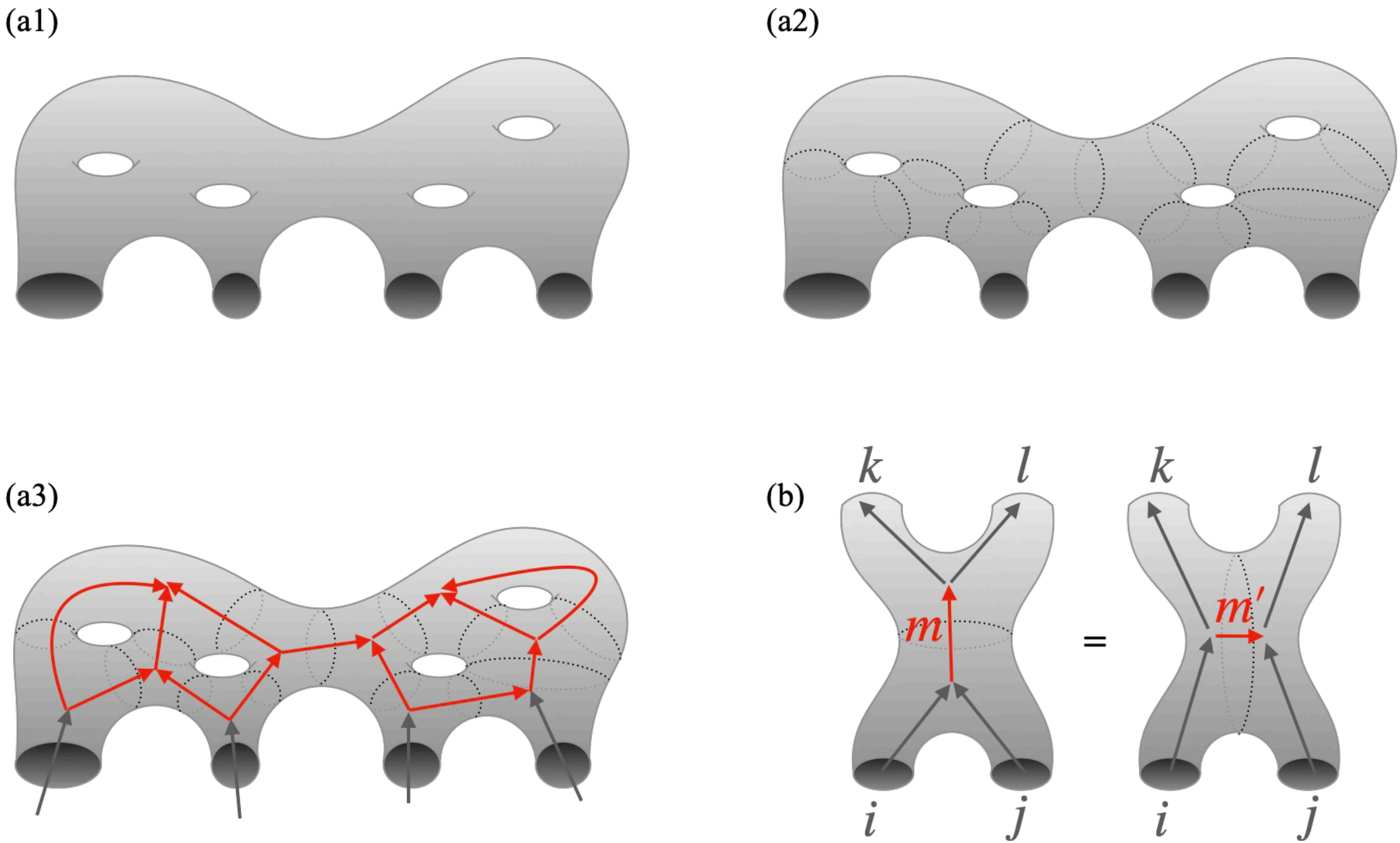
Verifying the Li-Haldane Conjecture in the Kitaev Quantum Double Model

- For $X = \Sigma_{0,1}$, we find $x_{\boldsymbol{\alpha}_{[\phi]}} = \delta_{[c],1}\delta_{\alpha_{[c]},1}$.
  - **Only vacuum** can live in X.
- For $X = \Sigma_{0,2}$, we find $x_{\boldsymbol{\alpha}_{[\phi]}} = \delta_{[c_1],[c_2^{-1}]}\delta_{\alpha_{[c_1]},\overline{\alpha_{[c_2]}}}$.
  - The subsystem X is the **identity** of anyons.
- For $X = \Sigma_{0,3}$, we find

$$
\begin{aligned}
x_{\boldsymbol{\alpha}_{[\phi]}} &= \sum_{[g],\beta_{[g]}} \frac{S_{\left([g],\beta_{[g]}\right),\left([c_1],\alpha_{[c_1]}\right)} S_{\left([g],\beta_{[g]}\right),\left([c_2],\alpha_{[c_2]}\right)} S_{\left([g],\beta_{[g]}\right),\left([c_3],\alpha_{[c_3]}\right)}}{S_{\left([g],\beta_{[g]}\right),(1,1)}} \\
&= N_{\left([c_3^{-1}],\overline{\alpha_{[c_3]}}\right)}^{\left([c_1],\alpha_{[c_1]}\right),\left([c_2],\alpha_{[c_2]}\right)}.
\end{aligned}
\tag{32}
$$

  - Only the fusion following the **Verlinde formula** is allowed.
- For $X = \Sigma_{0,4}$, we find $x_{\boldsymbol{\alpha}_{[\phi]}} = \sum_m N_m^{ij} N_m^{\overline{kl}} = \sum_m N_{m'}^{i\bar{k}} N_{m'}^{j\bar{l}}$.
  - The ***F*–move is trivial**.
- For general orientable surfaces,

Verifying the Li-Haldane Conjecture in the Kitaev Quantum Double Model



Summary and Outlook

1. We explored the entanglement structure in **non-invertible loop-symmetric quantum many-body states**.
   - We developed a **categorical framework** to determine the entanglement.
   - We applied our result to **many examples**.
2. We explored the entanglement structure in **topological gauge theories** in **arbitrary dimensions**.
   - We found a **general formula of topological entanglement entropy**.
   - We verified the **Li-Haldane correspondence of gapped boundary version**.

The next step would be…

- multipartite entanglement,
- other generalized symmetries, or
- anomalous theories.

[References]

[Pag93] D. N. Page, Average Entropy of a Subsystem, Physical Review Letters **71**, 1291 (1993).

[HLW94] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and Renormalized Entropy in Conformal Field Theory, Nuclear Physics B **424**, 443 (1994).

[CC09] P. Calabrese and J. Cardy, Entanglement Entropy and Conformal Field Theory, Journal of Physics A: Mathematical and Theoretical **42**, 504005 (2009).

[CC04] P. Calabrese and J. Cardy, Entanglement Entropy and Quantum Field Theory, Journal of Statistical Mechanics: Theory and Experiment **2004**, P6002 (2004).

[KP06] A. Kitaev and J. Preskill, Topological Entanglement Entropy, Physical Review Letters **96**, 110404 (2006).

[LW06] M. Levin and X.-G. Wen, Detecting Topological Order in a Ground State Wave Function, Physical Review Letters **96**, 110405 (2006).

[AS20] D. Azses and E. Sela, Symmetry-resolved entanglement in symmetry-protected topological phases, Phys. Rev. B **102**, 235157 (2020).

[Kus+23] Y. Kusuki, S. Murciano, H. Ooguri, and S. Pal, Symmetry-resolved entanglement entropy, spectra & boundary conformal field theory, Journal of High Energy Physics **2023**, (2023).

[Pol+10] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Entanglement Spectrum of a Topological Phase in One Dimension, Physical Review B **81**, 64439 (2010).

[YMG25] H. Yagi, K. Mochizuki, and Z. Gong, Threefold Way for Typical Entanglement, Physical Review Letters **134**, 150401 (2025).

[GNR16] S. D. Geraedts, R. Nandkishore, and N. Regnault, Many-Body Localization and Thermalization: Insights from the Entanglement Spectrum, Physical Review B **93**, 174202 (2016).

[Gai+15] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized Global Symmetries, Journal of High Energy Physics **2015**, 172 (2015).

[BT18] L. Bhardwaj and Y. Tachikawa, On Finite Symmetries and Their Gauging in Two Dimensions, Journal of High Energy Physics **2018**, 189 (2018).

[FRS02] J. Fuchs, I. Runkel, and C. Schweigert, TFT Construction of RCFT Correlators I: Partition Functions, Nuclear Physics B **646**, 353 (2002).

[Sch23] S. Schafer-Nameki, ICTP Lectures on (Non-)Invertible Generalized Symmetries, (2023).

[Bha+23] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre, and H. Tillim, Lectures on Generalized Symmetries, (2023).

[Sha24] S.-H. Shao, What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries, (2024).

[Cho+23] Y. Choi, B. C. Rayhaun, Y. Sanghavi, and S.-H. Shao, Remarks on Boundaries, Anomalies, and Noninvertible Symmetries, Physical Review D **108**, 125005 (2023).

[McG23] J. McGreevy, Generalized Symmetries in Condensed Matter, Annual Review of Condensed Matter Physics **14**, 57 (2023).

[Wit89] E. Witten, Quantum Field Theory and the Jones Polynomial, Communications in Mathematical Physics **121**, 351 (1989).

[Sim23] S. H. Simon, *Topological Quantum* (Oxford University Press, 2023).

[FMT24] D. S. Freed, G. W. Moore, and C. Teleman, Topological symmetry in quantum field theory, Quantum Topology **15**, 779 (2024).

[Olu58] P. Olum, Non-Abelian Cohomology and Van Kampen's Theorem, Annals of Mathematics **68**, 658 (1958).

[YG25] H. Yagi and Z. Gong, Entanglement Spectrum Resolved by Loop Symmetries, (2025).

[Ori25] I. Orii, On Dimensions of (2+1)D Abelian Bosonic Topological Systems on Unoriented Manifolds, Progress of Theoretical and Experimental Physics **2025**, 53 (2025a).

[Ori25] I. Orii, Generalization of Anomaly Formula for Time Reversal Symmetry in (2+1)D Abelian Bosonic TQFTs, (2025b).

[LT18] Y. Lee and Y. Tachikawa, A Study of Time Reversal Symmetry of Abelian Anyons, Journal of High Energy Physics **2018**, 90 (2018).

[TY17] Y. Tachikawa and K. Yonekura, More on Time-Reversal Anomaly of 2+1d Topological Phases, Physical Review Letters **119**, 111603 (2017).

[Bar+20] M. Barkeshli, P. Bonderson, M. Cheng, C.-M. Jian, and K. Walker, Reflection and Time Reversal Symmetry Enriched Topological Phases of Matter: Path Integrals, Non-orientable Manifolds, and Anomalies, Communications in Mathematical Physics **374**, 1021 (2020).

[FRS04] J. Fuchs, I. Runkel, and C. Schweigert, TFT Construction of RCFT Correlators II: Unoriented World Sheets, Nuclear Physics B **678**, 511 (2004).

[LH08] H. Li and F. D. M. Haldane, Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States, Phys. Rev. Lett. **101**, 10504 (2008).

[Kit97] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, (1997).

[CGR00] A. Coste, T. Gannon, and P. Ruelle, Finite Group Modular Data, Nuclear Physics B **581**, 679 (2000).

[MS89] G. Moore and N. Seiberg, Classical and Quantum Conformal Field Theory, Communications in Mathematical Physics **123**, 177 (1989).

[MS90] G. Moore and N. Seiberg, Lectures on RCFT, Physics, Geometry and Topology 263 (1990).

Thank you for your attention!

[Appendices]

Category Theory

Definition 22 (Category of manifolds)

The category Mfd is a category of manifolds.

The objects are manifolds. The morphisms are the continuous map. ♣

Definition 23 (Groupoid)

A **groupoid**[1] $\mathcal{G}$ is a non-empty set with a product, satisfying

- associativiity of the product,
- existence of identity, and
- existence of inverse.

However, **the product is NOT defined on all pairs** ${}^{\forall}(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$. ♣

[1] The Wikipedia of groupoid is really misleading: do not confuse groupoid with the **magma**.

Definition 24 (Fundamental groupoid)

Fix a manifold and a set of base points A. The **fundamental groupoid** of M about the base points A is a groupoid consists of the homotopically equivalent classes of paths connecting base points and product defined by the path concatenation. ♣

Definition 25

A category Grpd is a category of groupoids.
The objects are groupoids. The morphisms are the homomorphisms. ♣

Definition 26

A category Set is a category of sets.
The objects are sets. The morphisms are the maps. ♣

Mathematics of Our Framework

The splitting can be captured by the commutative diagram in Mfd.

$$\begin{tikzcd} M & X \arrow[l, hook'] \\ Y \arrow[u, hook] & \partial \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Theorem 27 (Seifert-van Kampen theorem)

If one base point per one connected component in ∂, the commutative diagram above can be sent to the category of fundamental groupoids as follows.

$$\begin{tikzcd} \pi_1(M, A) & \pi_1(X, A) \arrow[l, "p_X"'] \\ \pi_1(Y, A) \arrow[u, "p_Y"] & \pi_1(\partial, A) \arrow[l, "i_Y"] \arrow[u, "i_X"'] \end{tikzcd}$$

Mathematics of Our Framework

Proposition 28

Hereafter, $\mathrm{Hom}(*, G) : \mathrm{Grpd} \to \mathrm{Set}$ denotes the set of homomorphisms from $*$ to G. This is a contravariant functor, which sends limits (colimits) to colimits (limits). ♠

We can consider this as "consistent coloring by G". Then, the commutaive diagram used in our result is reproduced.

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(M,A),G) \arrow[r, "\Pi_X"] \arrow[d, "\Pi_Y"'] & \mathrm{Hom}(\pi_1(X,A),G) \arrow[d, "r_X"'] \\
\mathrm{Hom}(\pi_1(Y,A),G) \arrow[r, "r_Y"] & \mathrm{Hom}(\pi_1(\partial,A),G)
\end{tikzcd}$$

To make sure the holonomies are consistent after gluing X and Y to M, we focus on

$$\mathrm{Im} = \{\phi|\ ^\exists\Phi \in \mathrm{Hom}(\pi_1(M,A),G), \phi = r_X \circ \Pi_X(\Phi) = r_Y \circ \Pi_Y(\Phi)\}. \tag{33}$$