

Threefold Way for Typical Entanglement

Haruki Yagi¹, Ken Mochizuki^{1,2}, Zongping Gong¹

¹ Department of Applied Physics, Graduate School of Engineering, The University of Tokyo

² Nonequilibrium Quantum Statistical Mechanics RIKEN Hakubi Research Team, RIKEN Cluster for Pioneering Research

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Keywords

Physics: typical entanglement, symmetry fractionalization, SPT phases

Mathematics: random matrices, unitary-antiunitary representations of groups

Background

Quantum Entanglement

Entanglement \simeq cannot be decomposed into tensor products of the local basis

$$\text{2 qubit: } \frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} \quad (\neq |\psi_A\rangle \otimes |\psi_B\rangle)$$

$$\text{3 qubit: } \frac{|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle}{\sqrt{2}}, \quad \frac{|1_A 0_B 0_C\rangle + |0_A 1_B 0_C\rangle + |0_A 0_B 1_C\rangle}{\sqrt{3}}$$

\vdots

Entanglement entropy is a way to evaluate the strength of bipartite entanglement.

Partitioning the entire system into system (**sys**) and environment (**env**), we have

$$\rho_{\text{sys}} = \text{Tr}_{\text{env}} |\Psi\rangle\langle\Psi|, \quad S_{\text{EE}} = -\text{Tr}_{\text{sys}} [\rho_{\text{sys}} \ln \rho_{\text{sys}}].$$

The **larger** S_{EE} , the **more information is lost** in the environment.

The Black Hole Information Paradox

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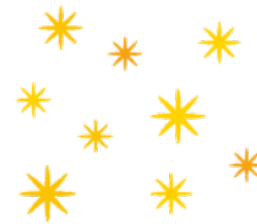
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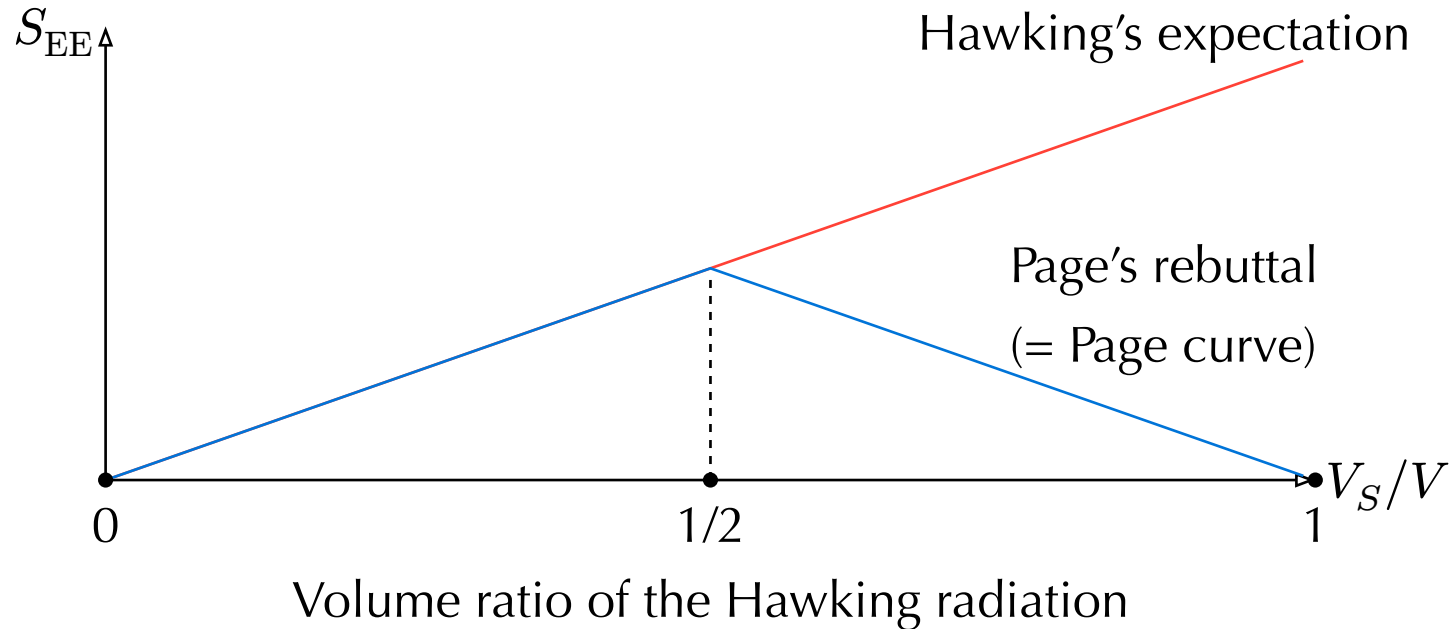
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Page curve

Hawking: Information would be lost 😭 / Page: Information would be preserved 🤔

Page curve: **sys = the Hawking radiation, env = BH**

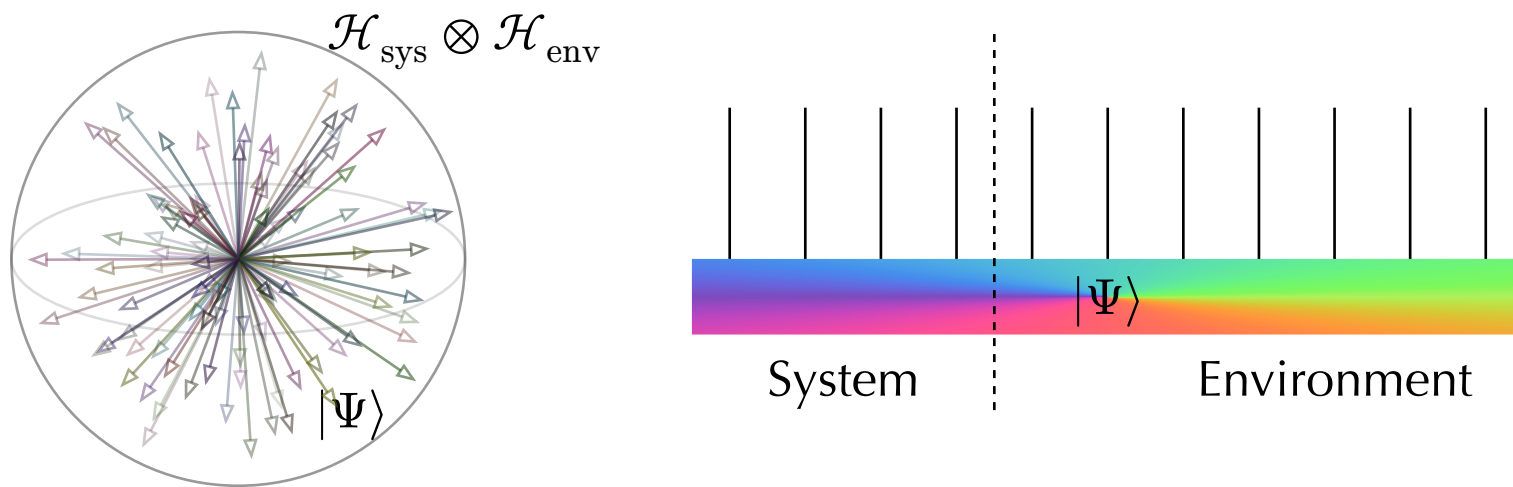


Page's consideration:

The entire dynamics should be **unitary** and **maximally chaotic**.

$\Rightarrow |\Psi\rangle \sim U|0\rangle$, where U is a random unitary matrix.

Typical Entanglement



- **Typical** = **uniformly random** sampling on the entire Hilbert space \sim Haar-random
- **Entanglement spectrum** = eigvals of the reduced density matrix $\rho_{\text{sys}} = \text{Tr}_{\text{env}} |\psi\rangle\langle\psi|$
- **Typical entanglement** = entanglement of uniformly random-sampled state

The random matrix theory (RMT) is useful to evaluate typical entanglement. (Page, 1993; Sánchez-Ruiz, 1995; Sen, 1996).

Let's see how it works.

Laguerre Unitary Ensemble of Random Matrices

Preparing Haar-random pure state from the entire Hilbert space:

$$|\Psi\rangle = \sum_{\text{sys,env}} W_{\text{sys,env}} |\text{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\text{sys,env}} \sim \mathcal{CN}(0, 1).$$

$$|\Psi\rangle \sim \text{Haar} \Leftrightarrow \rho_{\text{sys}} = \begin{array}{|c|} \hline W \\ \hline \end{array} \begin{array}{|c|} \hline W^\dagger \\ \hline \end{array}.$$

(Zyczkowski and Sommers, 2001; Nechita, 2007)

$\Rightarrow \rho_{\text{sys}}$ follows the **Laguerre unitary ensemble (LUE)** of RMT.

The joint probability p distribution of eigenvalues $\{\lambda_i\}$ of ρ_{sys} is

$$p(\lambda_1, \dots, \lambda_m) \propto \prod_{i=1}^m \lambda_i^{n-m} e^{-\lambda_i} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j|^2.$$

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$|\Psi\rangle = \sum_{\text{sys,env}} W_{\text{sys,env}} |\text{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\text{sys,env}} \sim \mathcal{CN}(0, 1)$$

What if we change *complex* random variables to *real* random variables?

$|\Psi\rangle$ and ρ_{sys} obtain **Time Reversal Symmetry** (TRS) of $\mathcal{T} = K$:

$$|\Psi\rangle \rightarrow \mathcal{T}|\Psi\rangle = K|\Psi\rangle = |\Psi\rangle,$$

$$\rho_{\text{sys}} \rightarrow \mathcal{T} \rho_{\text{sys}} \mathcal{T}^{-1} = K \rho_{\text{sys}} K = \overline{\rho_{\text{sys}}} = \rho_{\text{sys}}.$$

Note that there is **the other & nonequivalent kind of TRS!**

Time Reversal Symmetry

Integer spin: $\mathcal{T}_+^2 = +\mathbb{1}$. ex. $\mathcal{T}_+ = K$

Half-integer spin: $\mathcal{T}_-^2 = -\mathbb{1}$. ex. $\mathcal{T}_- = \sigma_y K$

Let's look at the eigenvalue statistics of the density matrix with TRS.

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$p(\lambda_1, \dots, \lambda_m) \propto \prod_{i=1}^m \lambda_i^{\frac{\beta}{2}(n-m+1)-1} e^{-\frac{\beta}{2}\lambda_i} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j|^\beta$$

No TRS: $\beta = 2$.

Imposing TRS: $\mathcal{T} \rho_{\text{sys}} \mathcal{T}^{-1} = \rho_{\text{sys}}$ allows $\beta = 1$ and/or $\beta = 4$.

Threefold way of Laguerre ensemble

$\mathcal{T}_+^2 = +\mathbb{1}$ ($\mathcal{T}_+ \simeq K$)	No TRS	$\mathcal{T}_-^2 = -\mathbb{1}$ ($\mathcal{T}_- \simeq \sigma_y K$)
$\beta = 1$	$\beta = 2$	$\beta = 4$
Laguerre orthogonal , LOE	unitary , LUE	symplectic ensemble LSE

Kramers' theorem and prohibition of $\mathcal{T}^2 = -\mathbb{1}$ TRS eigenstate

TRS for ρ_{sys} : $\mathcal{T} \rho_{\text{sys}} \mathcal{T}^{-1} = \rho_{\text{sys}}$ always has solutions. However, TRS for $|\Psi\rangle$ is ill-defined:

Theorem: $\mathcal{T}^2 = -\mathbb{1}$ TRS cannot have the eigenstate $\mathcal{T}_-|\Psi\rangle = |\Psi\rangle$.

Proof: Time reversal operator \mathcal{T} is anti-unitary, thus $\langle \mathcal{T}_-a | \mathcal{T}_-b \rangle = \overline{\langle a | b \rangle}$.

$$\langle \psi | \mathcal{T}_- \psi \rangle = \overline{\langle \mathcal{T}_- \psi | \mathcal{T}_-^2 \psi \rangle} = \langle \mathcal{T}_-^2 \psi | \mathcal{T}_- \psi \rangle = -\langle \psi | \mathcal{T}_- \psi \rangle$$

This implies $\langle \psi | \mathcal{T}_- \psi \rangle = 0$ for an arbitrary state, thus $|\psi\rangle$ is orthogonal to $\mathcal{T}|\psi\rangle$. ■

	$\mathcal{T}_+^2 = +\mathbb{1}$ ($\mathcal{T}_+ \simeq K$)	No TRS	$\mathcal{T}_-^2 = -\mathbb{1}$ ($\mathcal{T}_- \simeq \sigma_y K$)
	$\beta = 1$	$\beta = 2$	$\beta = 4$
Laguerre	orthogonal,	unitary,	symplectic ensemble
	LOE	LUE	LSE
$\mathcal{T} \Psi\rangle = \Psi\rangle :$	real vector	complex vector	<i>unknown!</i>

Key Questions and Solutions

Q1. Possible to construct $\rho_{\text{sys}} \sim \text{LSE}$?

Q2. Beyond threefold way if general symmetries?

Q1. Possible to construct $|\Psi\rangle$ whose $\rho_{\text{sys}} \sim \text{LSE}$?

A1. Yes, by *fractionalization* of TRS.

Q2. Beyond threefold way if general symmetries?

A2. Never. Direct sum of threefold way.

1. Exploring the LSE-Realizing System

Requirement	$\mathcal{T}_+ \rho_{\text{sys}} \mathcal{T}_+^{-1} = \rho_{\text{sys}}$	$\mathcal{T}_- \rho_{\text{sys}} \mathcal{T}_-^{-1} = \rho_{\text{sys}}$
Existence of pure state	Yes, $\mathcal{T}_+ \Psi\rangle = \Psi\rangle$	No , $\mathcal{T}_- \Psi\rangle \neq \Psi\rangle$
Pure state	real vector	???

\Downarrow

Fractionalization of TRS

$$\mathcal{T}^2 = [\mathcal{T}_{\text{sys}} \otimes \mathcal{T}_{\text{env}}]^2 = +\mathbb{1}$$

$$\swarrow \quad \mathcal{T} = \Upsilon (\mathcal{T}_{\text{sys}} \otimes \mathcal{T}_{\text{env}}) \Upsilon^\dagger \quad \searrow$$

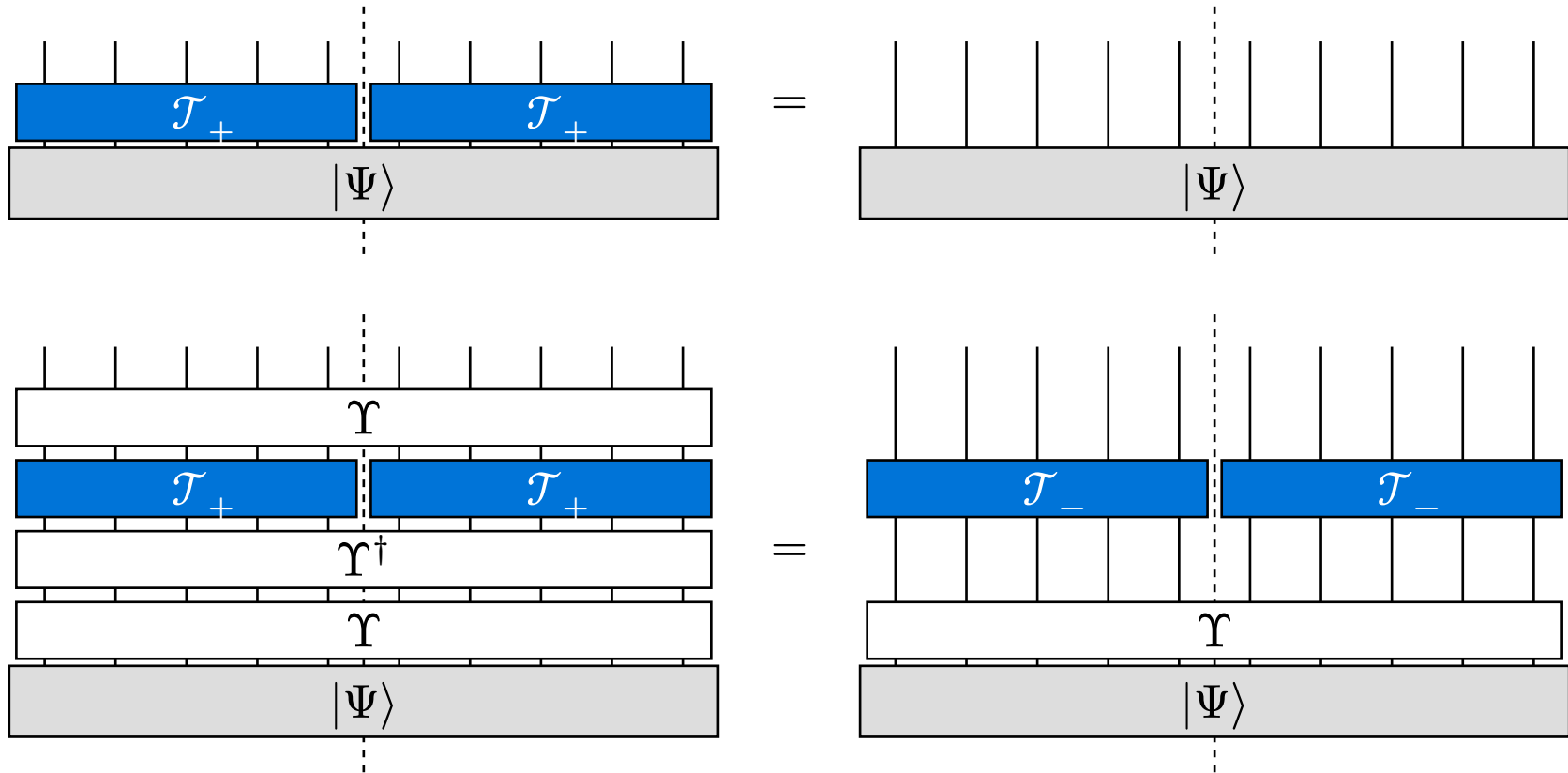
$\mathcal{T}_{\text{sys}}^2 = -\mathbb{1}$

$\mathcal{T}_{\text{env}}^2 = -\mathbb{1}$

- Υ fractionalizes TRS pure state of LOE $|\Psi\rangle$.
- **We found $\Upsilon = \frac{1-i}{2} [\mathbb{1}_4 - i\sigma_y \otimes \sigma_y]$ and proved $\Upsilon|\Psi\rangle$ follows LSE.**

1. Exploring the LSE-Realizing System

$$\text{Fractionalization of TRS: } \Upsilon(\mathcal{T}_+ \otimes \mathcal{T}_+)\Upsilon^\dagger = \mathcal{T}_- \otimes \mathcal{T}_-$$



2. Imposing General (Finite) Symmetries

- *Groups* (mathematics) describe symmetries.

Group G

Set with an associative multiplication. Identity and inverse exist for every elements.

- Examples: \mathbb{Z}_m , $\mathbb{Z}_m \times \mathbb{Z}_n$, C_{3v} , Q_8 , $\mathbb{Z}_2^{\mathcal{T}}$ (TRS), ...
- Constraints to the states / operators are given by *unitary-antiunitary representations*.

Unitary-antiunitary representation D of G

$$D(a)D(b) = D(ab) \\ \text{for } \forall a, b \in G.$$

- *Projective representations* describe *anomalous symmetries* and (1+1)D SPT phases.

Projective representation \mathcal{D} of G

$$\mathcal{D}(a)\mathcal{D}(b) = \omega(a, b)\mathcal{D}(ab) \\ \text{for } \forall a, b \in G, \omega : G \times G \rightarrow U(1).$$

2. Imposing General (Finite) Symmetries

Symmetry Fractionalization

$$\forall g, g' \in G, D(g)D(g') = D(gg') \text{ (regular rep)}$$

$$\swarrow \quad \Omega(D \otimes D)\Omega^\dagger = \mathcal{D} \otimes \mathcal{D}' \quad \searrow$$

$$\mathcal{D}(g)\mathcal{D}(g') = \omega(g, g')\mathcal{D}(gg')$$

$$\mathcal{D}'(g)\mathcal{D}'(g') = \overline{\omega(g, g')}\mathcal{D}'(gg')$$

The equivalence relation between reps is defined by phase modulation;

$$\mathcal{D}_{\text{new}}(g) = e^{i\phi(g)} \mathcal{D}(g)$$

Then, reps are classified by **2nd order group cohomology** $H^2(G, U(1))$.

- Regular reps of different classes lead to different irreducible decompositions.

2. Imposing General (Finite) Symmetries

- Groups can be extended by *(semi)direct product*.

Direct product and semidirect product of groups

Direct: $G = G_1 \times G_2$, $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)$

Semidirect: $G = G_1 \rtimes G_2$, $(a_1, a_2)(b_1, b_2) = (a_1 f_{a_2}(b_1), a_2 b_2)$

- Any unitary-antiunitary reps can be decomposed into a set of *irreducible reps*.

Irreducible representations

Reps for which all the elements in a unitary transformation cannot be further decomposed into direct sums at the same time are called an irreducible reps.

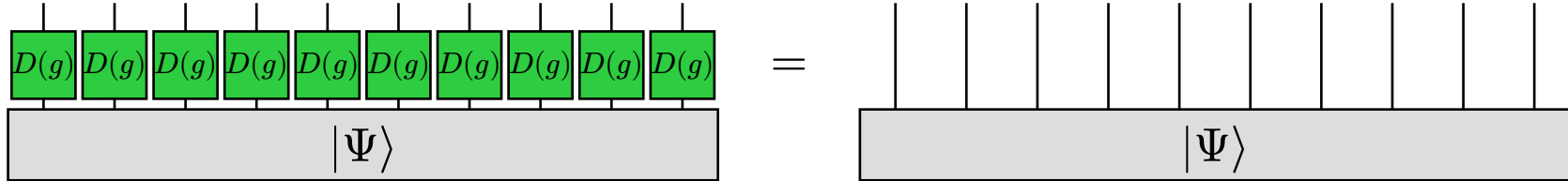
- The Regular representation* can be chosen for the most natural rep.

Regular representation

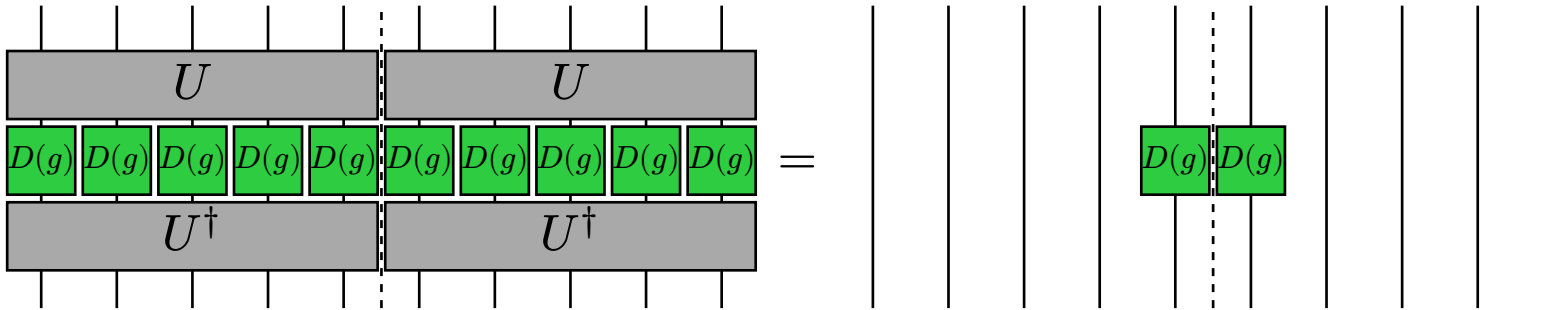
$$D(g) = [\delta(g_i g g_j^{-1})] \text{ where } \delta(e) = 1 \text{ and otherwise } \delta(g) = 0$$

2. Imposing General (Finite) Symmetries

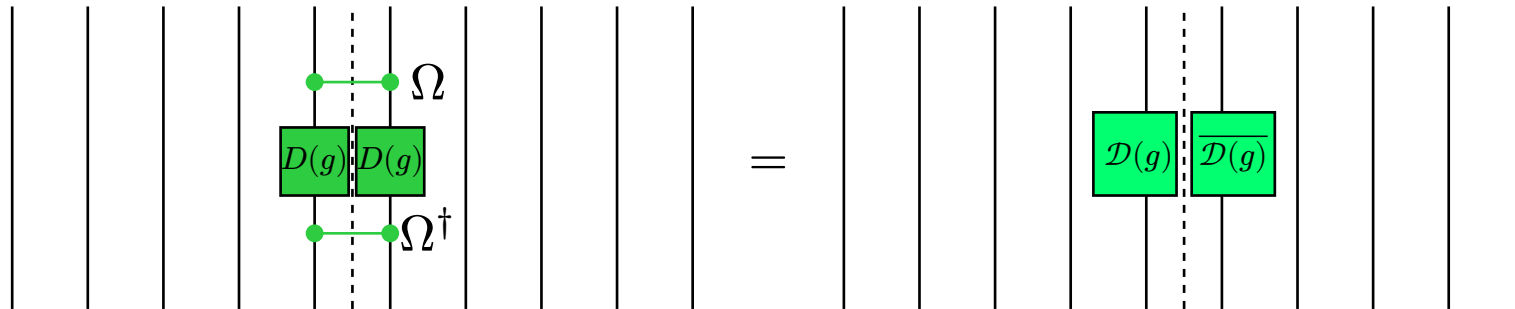
- We consider **random G -symmetric states** $|\Psi\rangle$, where $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$.



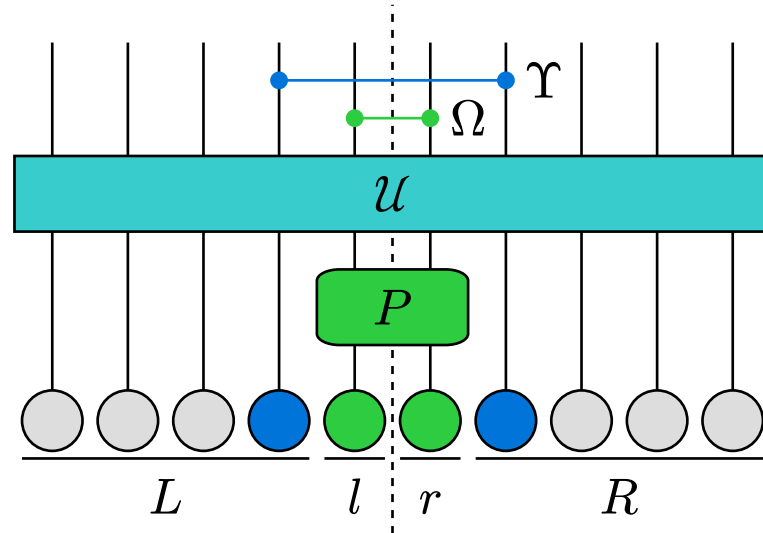
- We concentrate onsite symmetries s.t. the entanglement spectrum is not changed.



- Fractionalization of G_0 by Ω can be done independently of $\mathbb{Z}_2^{\mathcal{T}}$.



General Setup



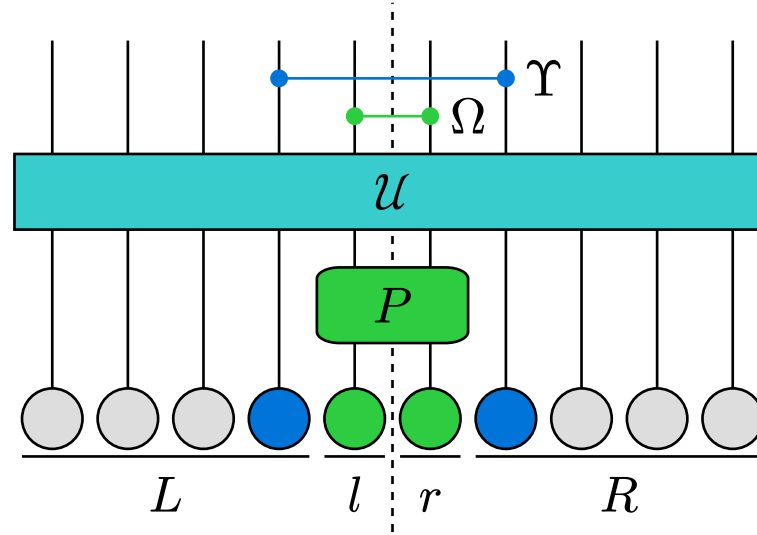
The considered symmetries are $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ (G_0 : unitary, $\mathbb{Z}_2^{\mathcal{T}}$: anti-unitary).

- G_0 : Green circles.
 - l, r : $|G_0|$ -dimensional qudit (Finite. The regular rep of G_0 : $\langle g|g' \rangle = \delta_{g,g'}$)
 - P : Projection of $|G_0|^2$ -dim space $l \cup r$ onto G_0 -symmetric $|G_0|$ -dim basis below:

$$\forall g \in G_0, |\psi_g\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{h \in G_0} |hg\rangle |h\rangle. \quad (D(g) \otimes D(g) |\psi_g\rangle = |\psi_g\rangle).$$

- $\mathbb{Z}_2^{\mathcal{T}}$: Blue circles = 2-dimensional cuts of each subsystem (only when Υ is necessary).

General Setup



- **Randomness:** $\mathcal{U} \sim$ Haar measure on the projected $d_L d_R |G_0|$ -dimensional space:

$$|\Psi\rangle = \sum_{L, g \in G_0, R} c_{L, g, R} |L\rangle |\psi_g\rangle |R\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{L, g_l, g_r, R} c_{L, g_r^{-1} g_l, R} |L\rangle |g_l\rangle |g_r\rangle |R\rangle$$

- **Fractionalization:** Ω fractionalizes G_0 , Υ fractionalizes $\mathbb{Z}_2^{\mathcal{J}}$:

$$\Omega = \sum_{g_l, g_r} \omega(g_r, g_r^{-1} g_l) |g_l, g_r\rangle \langle g_l, g_r|, \quad \Upsilon = \frac{1-i}{2} (\mathbb{1}_4 - i\sigma_y \otimes \sigma_y).$$

Results and Conclusion

Direct Sum into the Threefold Way

Entanglement-spectrum statistics of $G = G_0$ is

$$\bigoplus_{\alpha} \left[\frac{\mathbb{1}_{d_{\alpha}}}{d_{\alpha}} \otimes \mathbf{LUE}_{\alpha}^{d_L d_{\alpha} \times d_R d_{\alpha}} \right],$$

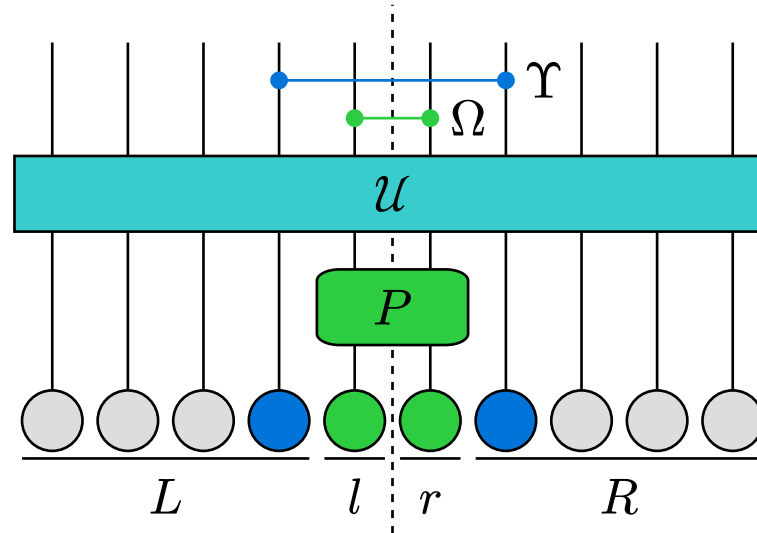
On the other hand, that of $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ is

$$\left[\bigoplus_{\alpha: R_+} \frac{\mathbb{1}_{d_{\alpha}}}{d_{\alpha}} \otimes \mathbf{LOE}_{\alpha}^{d_L d_{\alpha} \times d_R d_{\alpha}} \right] \oplus \left[\bigoplus_{\alpha: R_0} \frac{\mathbb{1}_{2d_{\alpha}}}{d_{\alpha}} \otimes \mathbf{LUE}_{\alpha}^{d_L d_{\alpha} \times d_R d_{\alpha}} \right] \oplus \left[\bigoplus_{\alpha: R_-} \frac{\mathbb{1}_{d_{\alpha}}}{d_{\alpha}} \otimes \mathbf{LSE}_{\alpha}^{d_L d_{\alpha} \times d_R d_{\alpha}} \right].$$

Entanglement-spectrum statistics of random symmetric states is
always able to be decomposed into the direct sum of the threefold way¹.

¹This result is also the Laguerre version of Dyson's Gaussian threefold way (1962).

Conclusion



Until our work

- **The setup which follows LSE have been elusive.**

What this work revealed are:

- **The LSE setup can be constructed by fractionalizing TRS of the LOE setup.**
- **Extended the setup to general symmetries.**
- Entanglement-spectrum statistics is **direct sum of the threefold way.**

References

- Nechita, I. (2007) "Asymptotics of random density matrices," in *Annales Henri Poincaré*, pp. 1521–1538
- Page, D. N. (1993) "Average entropy of a subsystem," *Phys. Rev. Lett.*, 71(9), pp. 1291–1294. Available at: <https://doi.org/10.1103/PhysRevLett.71.1291>
- Sen, S. (1996) "Average Entropy of a Quantum Subsystem," *Phys. Rev. Lett.*, 77(1), pp. 1–3. Available at: <https://doi.org/10.1103/PhysRevLett.77.1>
- Sánchez-Ruiz, J. (1995) "Simple proof of Page's conjecture on the average entropy of a subsystem," *Phys. Rev. E*, 52(5), pp. 5653–5655. Available at: <https://doi.org/10.1103/PhysRevE.52.5653>
- Zyczkowski, K. and Sommers, H.-J. (2001) "Induced measures in the space of mixed quantum states," *Journal of Physics A: Mathematical and General*, 34(35), p. 7111

Supplemental Materials

Some additional information on group reps

- An irrep is either one of *real*, *complex*, or *pseudoreal*.
- **We consider the cases $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$.**
- For $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$, one can define *the indicator* to know irreps real, complex, or pseudoreal.

Indicator of irrep

$$\iota_\alpha = \frac{1}{|G_0|} \sum_{g \in G_0} \omega(\tilde{g}, g) \text{Tr}[\mathcal{D}_\alpha(\tilde{g}) \mathcal{D}_\alpha(g)],$$
$$\iota = \begin{cases} 1 : \text{real} \\ 0 : \text{complex} \\ -1 : \text{pseudoreal} \end{cases}$$

The cocycle (=cohomology class) can be decompled $\omega = \omega_{G_0} \omega_{\mathbb{Z}_2^{\mathcal{T}}}$.

- R_\pm = set irreps that satisfies $\iota = \pm \omega(t, t)$. $\omega(t, t) = 1(-1)$ in the absent (present) of Υ .