Threefold Way for Typical Entanglement

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arXiv:2410.11309

Keywords

Physics: typical entanglement, symmetry fractionalization, SPT phases

Mathematics: random matrices, unitary-antiunitary representations of groups

Background

Quantum Entanglement

Entanglement $\simeq$ cannot be decomposed into tensor products of the local basis

$$\text{2 qubit:} \quad \frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} \quad (\neq |\psi_A\rangle \otimes |\psi_B\rangle)$$

$$\text{3 qubit:} \quad \frac{|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle}{\sqrt{2}}, \ \frac{|1_A 0_B 0_C\rangle + |0_A 1_B 0_C\rangle + |0_A 0_B 1_C\rangle}{\sqrt{3}}$$

$$\vdots$$

Entanglement entropy is a way to evaluate the strength of bipartite entanglement.

Partitioning the entire system into system (**sys**) and environment (**env**), we have

$$\rho_{\mathrm{sys}} = \mathrm{Tr}_{\mathrm{env}} |\Psi\rangle\langle\Psi|, \quad S_{\mathrm{EE}} = -\,\mathrm{Tr}_{\mathrm{sys}}\left[\rho_{\mathrm{sys}} \ln \rho_{\mathrm{sys}}\right].$$

The **larger S_{EE}**, the **more information is lost** in the environment.

The Black Hole Information Paradox

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2

3

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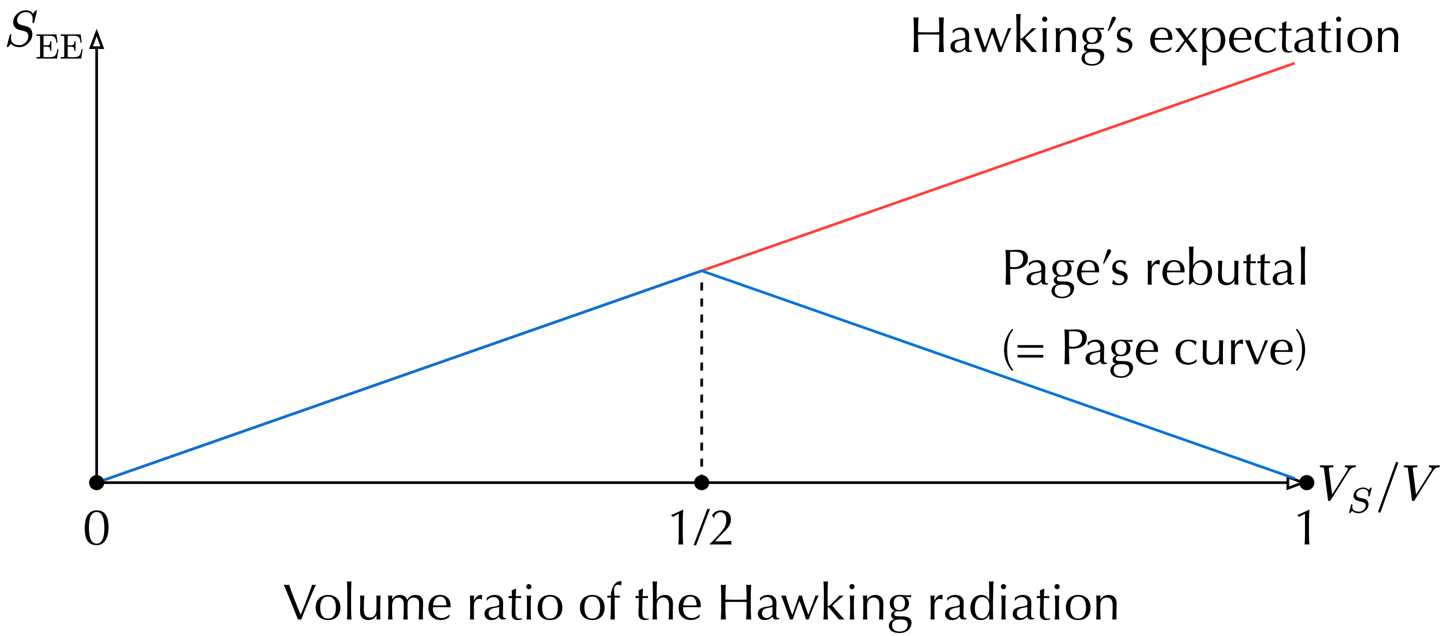
Page curve

Hawking: Information would be lost 😭 / Page: Information would be preserved 🧐

Page curve: **sys = the Hawking radiation, env = BH**

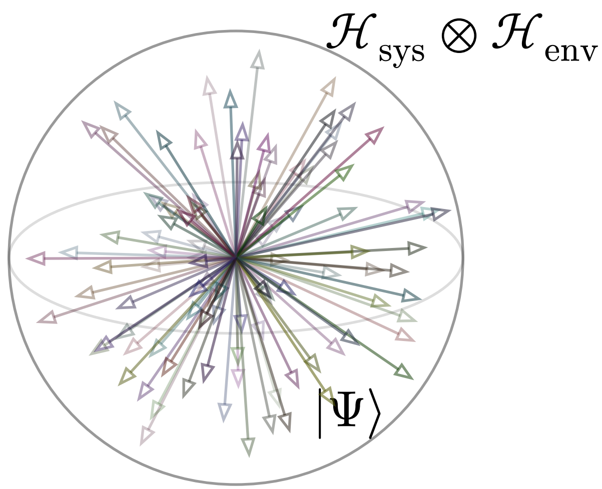


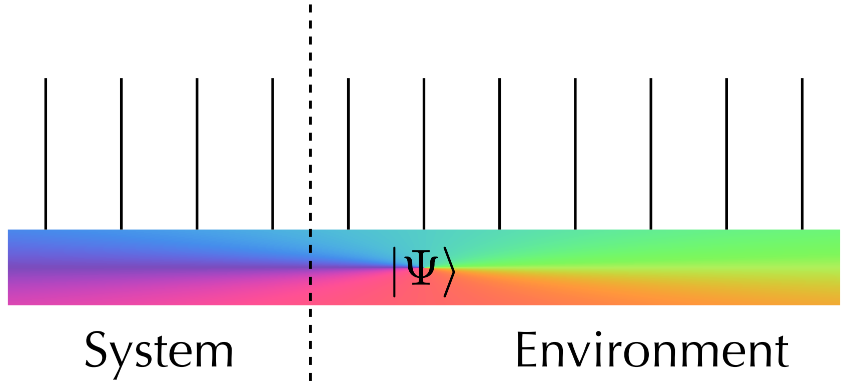
Page's consideration:

The entire dynamics should be **unitary** and **maximally chaotic**.

$\Rightarrow$ $|\Psi\rangle \sim U|0\rangle$, where U is a random unitary matrix.

Typical Entanglement





- **Typical** = **uniformly random** sampling on the entire Hilbert space $\sim$ Haar-random
- **Entanglement spectrum** = eigvals of the reduced density matrix $\rho_{\text{sys}} = \mathrm{Tr}_{\text{env}} |\psi\rangle\langle\psi|$
- **Typical entanglement** = entanglement of uniformly random-sampled state

The random matrix theory (RMT) is useful to evaluate typical entanglement. (Page, 1993; Sánchez-Ruiz, 1995; Sen, 1996).

Let's see how it works.

Laguerre Unitary Ensemble of Random Matrices

Preparing Haar-random pure state from the entire Hilbert space:

$$|\Psi\rangle = \sum_{\text{sys,env}} W_{\text{sys,env}} |\text{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\text{sys,env}} \sim \mathcal{CN}(0,1).$$

$$|\Psi\rangle \sim \text{Haar} \Leftrightarrow \rho_{\text{sys}} = W \; W^\dagger \; .$$

(Zyczkowski and Sommers, 2001; Nechita, 2007)

$\Rightarrow \rho_{\text{sys}}$ follows the ***Laguerre unitary ensemble (LUE)*** of RMT.

The joint probability p distribution of eigenvalues $\{\lambda_i\}$ of ρ_{sys} is

$$p(\lambda_1, ..., \lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{n-m} e^{-\lambda_i} \prod_{1\leq i<j\leq m} |\lambda_i - \lambda_j|^2.$$

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$|\Psi\rangle = \sum_{\mathrm{sys,env}} W_{\mathrm{sys,env}} |\mathrm{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\mathrm{sys,env}} \sim \mathcal{CN}(0,1)$$

What if we change *complex* random variables to *real* random variables?

$|\Psi\rangle$ and ρ_{sys} obtain **Time Reversal Symmetry** (TRS) of $\mathcal{T} = K$:

$$|\Psi\rangle \to \mathcal{T}|\Psi\rangle = K|\Psi\rangle = |\Psi\rangle,$$
$$\rho_{\mathrm{sys}} \to \mathcal{T}\rho_{\mathrm{sys}}\mathcal{T}^{-1} = K\rho_{\mathrm{sys}}K = \overline{\rho_{\mathrm{sys}}} = \rho_{\mathrm{sys}}.$$

Note that there is **the other & nonequivalent kind of TRS!**

Time Reversal Symmetry

Integer spin: $\mathcal{T}_+^2 = +\mathbb{1}$. ex. $\mathcal{T}_+ = K$

Half-integer spin: $\mathcal{T}_-^2 = -\mathbb{1}$. ex. $\mathcal{T}_- = \sigma_y K$

Let's look at the eigenvalue statistics of the density matrix with TRS.

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$p(\lambda_1, ..., \lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{\frac{\beta}{2}(n-m+1)-1} e^{-\frac{\beta}{2}\lambda_i} \prod_{1\leq i<j\leq m} |\lambda_i - \lambda_j|^\beta$$

No TRS: $\beta = 2$.

Imposing TRS: $\mathcal{T}\rho_{\text{sys}}\mathcal{T}^{-1} = \rho_{\text{sys}}$ allows $\beta = 1$ and/or $\beta = 4$.

Threefold way of Laguerre ensemble

	$\underset{(\mathcal{T}_+ \simeq K)}{\mathcal{T}_+^2 = +\mathbb{1}}$	No TRS	$\underset{(\mathcal{T}_- \simeq \sigma_y K)}{\mathcal{T}_-^2 = -\mathbb{1}}$	
	$\beta = 1$	$\beta = 2$	$\beta = 4$	
Laguerre	**orthogonal**,	**unitary**,	**symplectic**	ensemble
	LOE	**LUE**	**LSE**	

Kramers' theorem and prohibition of $\mathcal{T}_-^2 = -\mathbb{1}$ TRS eigenstate

TRS for ρ_{sys}: $\mathcal{T}\rho_{\text{sys}}\mathcal{T}^{-1} = \rho_{\text{sys}}$ always has solutions. However, TRS for $|\Psi\rangle$ is ill-defined:

> **Theorem**: $\mathcal{T}_-^2 = -\mathbb{1}$ **TRS cannot have the eigenstate** $\mathcal{T}_-|\Psi\rangle = |\Psi\rangle$.

Proof: Time reversal operator $\mathcal{T}$ is anti-unitary, thus $\langle \mathcal{T}_-a|\mathcal{T}_-b\rangle = \overline{\langle a|b\rangle}$.

$$\langle\psi|\mathcal{T}_-\psi\rangle = \overline{\langle \mathcal{T}_-\psi|\mathcal{T}_-^2\psi\rangle} = \langle \mathcal{T}_-^2\psi|\mathcal{T}_-\psi\rangle = -\langle\psi|\mathcal{T}_-\psi\rangle$$

This implies $\langle\psi|\mathcal{T}_-\psi\rangle = 0$ for an arbitary state, thus $|\psi\rangle$ is orthogonal to $\mathcal{T}|\psi\rangle$. ■

	$\mathcal{T}_+^2 = +\mathbb{1}$ $(\mathcal{T}_+ \simeq K)$	No TRS	$\mathcal{T}_-^2 = -\mathbb{1}$ $(\mathcal{T}_- \simeq \sigma_y K)$	
	$\beta = 1$	$\beta = 2$	$\beta = 4$	
Laguerre	**orthogonal**,	**unitary**,	**symplectic**	ensemble
	LOE	**LUE**	**LSE**	
$\mathcal{T}\|\Psi\rangle = \|\Psi\rangle$:	real vector	complex vector	*unknown!*	

Key Questions and Solutions

Q1. Possible to construct $\rho_{\mathrm{sys}} \sim$ LSE ?

Q2. Beyond threefold way if general symmetries?

Q1. Possible to construct $|\Psi\rangle$ whose $\rho_{\rm sys} \sim$ LSE ?

A1. Yes, by *fractionalization* of TRS.

Q2. Beyond threefold way if general symmetries?

A2. Never. Direct sum of threefold way.

1. Exploring the LSE-Realizing System

Requirement	$\mathcal{T}_+\rho_{\rm sys}\mathcal{T}_+^{-1} = \rho_{\rm sys}$	$\mathcal{T}_-\rho_{\rm sys}\mathcal{T}_-^{-1} = \rho_{\rm sys}$
Existence of pure state	Yes, $\mathcal{T}_+\|\Psi\rangle = \|\Psi\rangle$	**No**, $\mathcal{T}_-\|\Psi\rangle \neq \|\Psi\rangle$
Pure state	real vector	**???**

$$\Downarrow$$

Fractionalization of TRS

$$\mathcal{T}^2 = \left[\mathcal{T}_{\rm sys} \otimes \mathcal{T}_{\rm env}\right]^2 = +\mathbb{1}$$

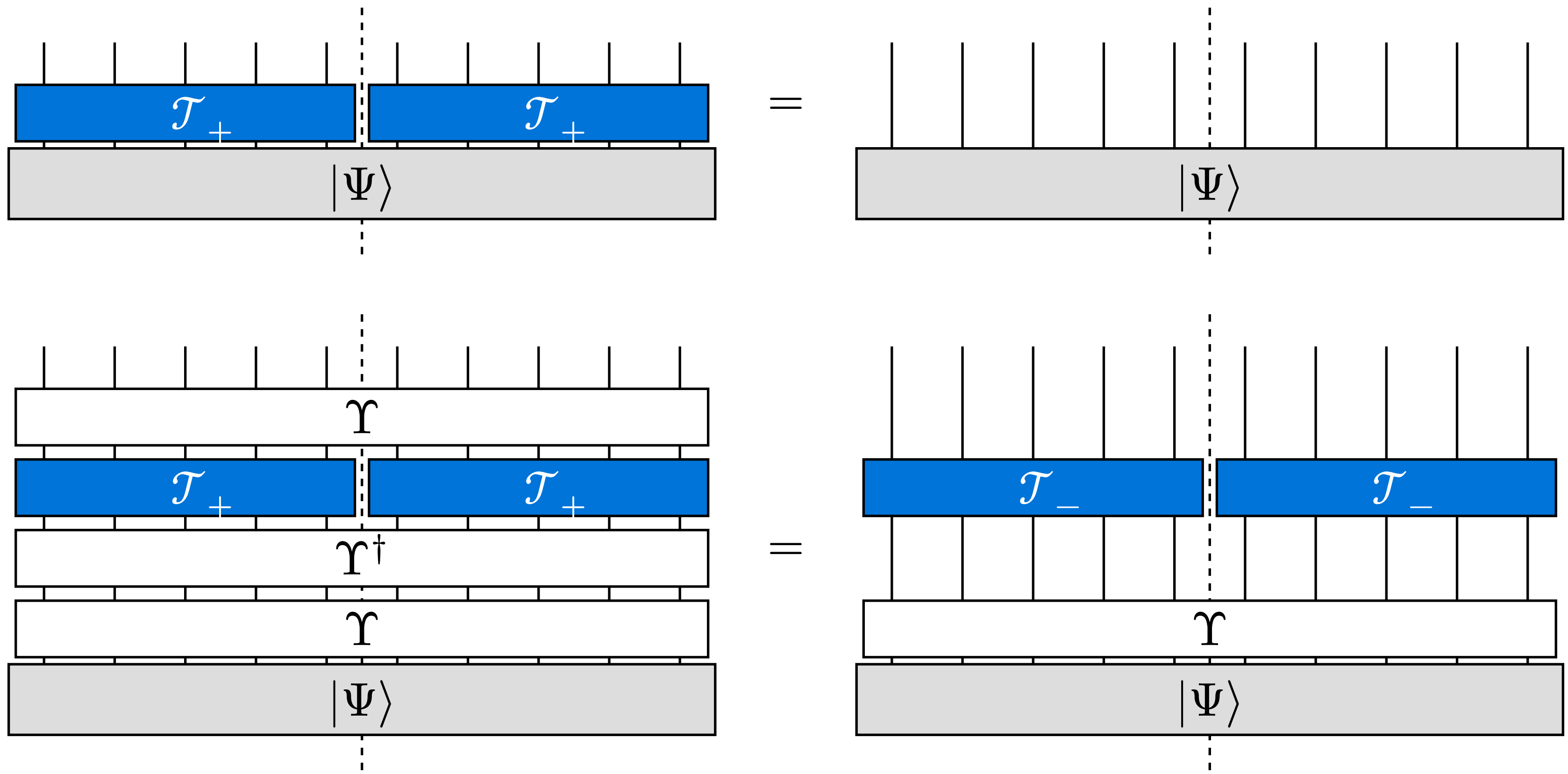
$$\swarrow \quad \mathcal{T} = \Upsilon\left(\mathcal{T}_{\rm sys} \otimes \mathcal{T}_{\rm env}\right)\Upsilon^\dagger \quad \searrow$$

$$\boxed{\mathcal{T}_{\rm sys}^2 = -\mathbb{1}} \qquad \boxed{\mathcal{T}_{\rm env}^2 = -\mathbb{1}}$$

- Υ fractionalizes TRS pure state of LOE $|\Psi\rangle$.
- **We found $\Upsilon = \frac{1-i}{2}\left[\mathbb{1}_4 - i\sigma_y \otimes \sigma_y\right]$ and proved $\Upsilon|\Psi\rangle$ follows LSE.**

1. Exploring the LSE-Realizing System

Fractionalization of TRS: $\Upsilon(\mathcal{T}_+ \otimes \mathcal{T}_+)\Upsilon^\dagger = \mathcal{T}_- \otimes \mathcal{T}_-$



2. Imposing General (Finite) Symmetries

- *Groups* (mathematics) describe symmetries.

Group G
Set with an assosiative multiplication. Identity and inverse exist for every elements.

  - Examples: $\mathbb{Z}_m,\ \mathbb{Z}_m \times \mathbb{Z}_n,\ C_{3v},\ Q_8,\ \ \mathbb{Z}_{\mathbf{2}}^{\mathcal{T}}$ **(TRS)**, $\cdots$

- Constraints to the states / operators are given by *unitary-antiunitary representations*.

Unitary-antiunitary representation D of G

$$D(a)D(b) = D(ab)$$

for $\forall a, b \in G$.

- *Projective representations* describe *anomalous symmetries* and (1+1)D SPT phases.

Projective representation $\mathcal{D}$ of G

$$\mathcal{D}(a)\mathcal{D}(b) = \omega(a,b)\mathcal{D}(ab)$$

for $\forall a, b \in G,\ \omega : G \times G \to U(1)$.

2. Imposing General (Finite) Symmetries

Symmetry Fractionalization

$$\forall g, g' \in G,\ \ D(g)D(g') = D(gg')\ \text{(regular rep)}$$

$$\swarrow \quad \Omega(D \otimes D)\Omega^\dagger = \mathcal{D} \otimes \mathcal{D}' \quad \searrow$$

$$\mathcal{D}(g)\mathcal{D}(g') = \omega(g, g')\mathcal{D}(gg') \qquad \mathcal{D}'(g)\mathcal{D}'(g') = \overline{\omega(g, g')}\mathcal{D}'(gg')$$

The eqiuvalence relation between reps is defined by phase modulation;

$$\mathcal{D}_{\mathrm{new}}(g) = e^{\mathrm{i}\phi(g)}\mathcal{D}(g)$$

Then, reps are classified by **2nd order group cohomology $\boldsymbol{H^2(G, U(1))}$.**

- Regular reps of different classes lead to different irreducible decompositions.

2. Imposing General (Finite) Symmetries

- Groups can be extended by *(semi)direct product*.

Direct product and semidirect product of groups

Direct: $G = G_1 \times G_2$, $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)$

Semidirect: $G = G_1 \rtimes G_2$, $(a_1, a_2)(b_1, b_2) = \big(a_1 f_{a_2}(b_1), a_2 b_2\big)$

- Any unitary-antiunitary reps can be decomposed into a set of *irreducible reps*.

Irreducible representations

Reps for which all the elements in a unitary transformation cannot be further decomposed into direct sums at the same time are called an irreducible reps.

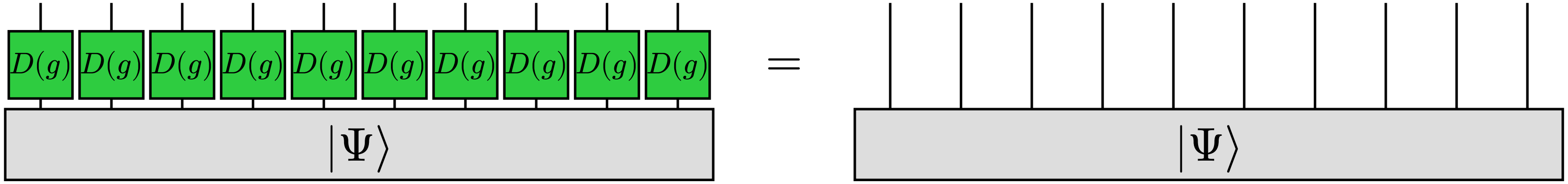
- *The Regular representation* can be chosen for the most natural rep.

Regular representation

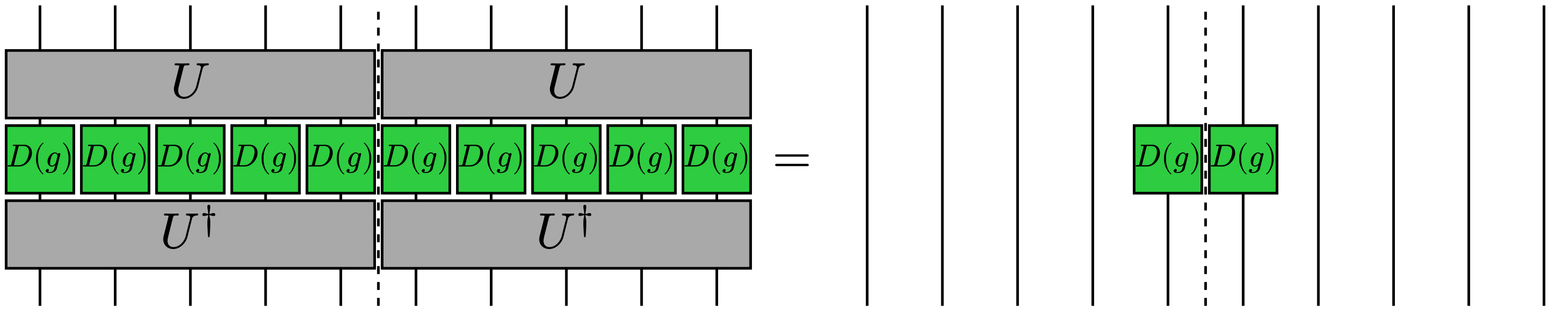
$D(g) = \big[\delta\big(g_i\ g\ g_j^{-1}\big)\big]$ where $\delta(e) = 1$ and otherwise $\delta(g) = 0$

2. Imposing General (Finite) Symmetries

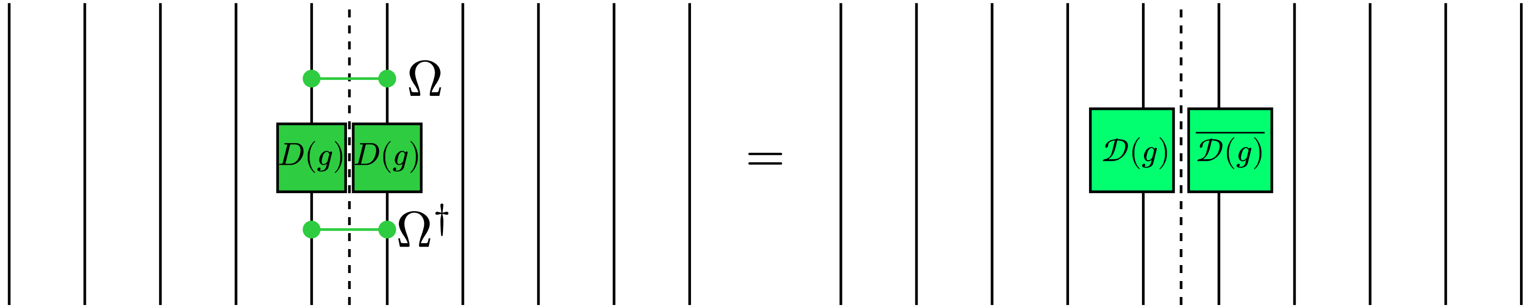
- We consider **random G-symmetric states $|\Psi\rangle$**, where $\boldsymbol{G = G_0}$ **or** $\boldsymbol{G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}}$**.**



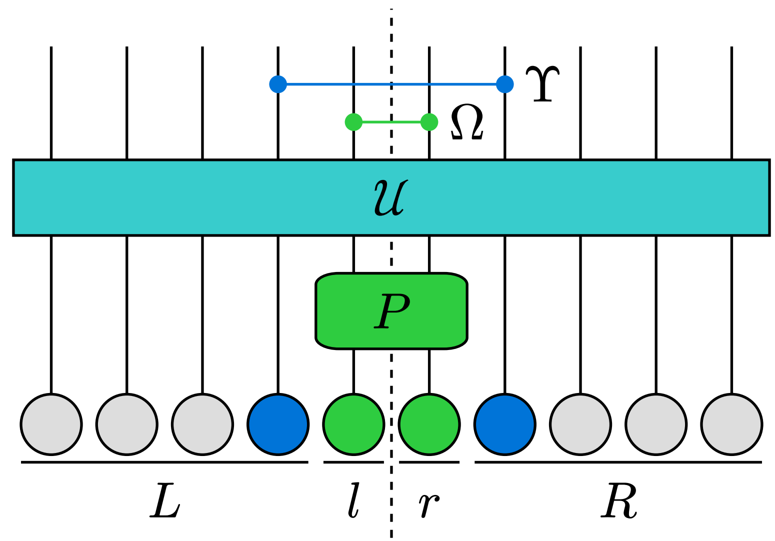
- We concentrate onsite symmetries s.t. the entanglement spectrum is not changed.



- Fractionalization of G_0 by Ω can be done independently of $\mathbb{Z}_2^{\mathcal{T}}$.



General Setup



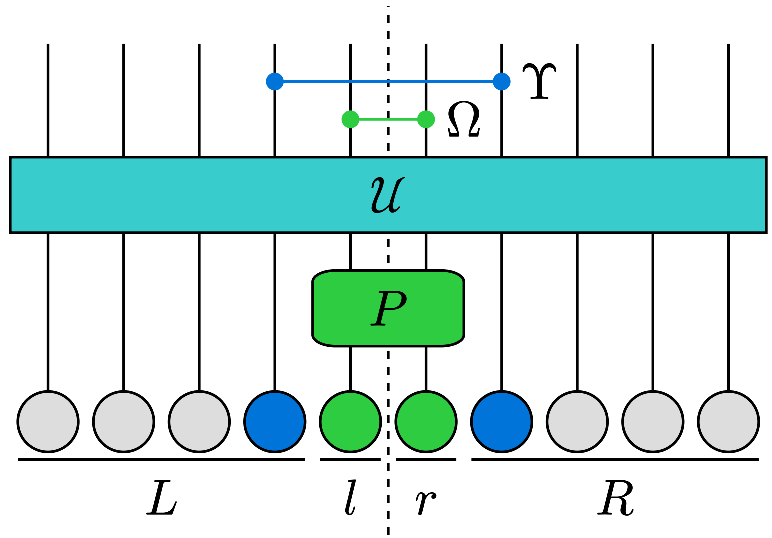
The considered symmetries are $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ (G_0: unitary, $\mathbb{Z}_2^{\mathcal{T}}$: anti-unitary).

- $\boldsymbol{G_0}$: Green circles.
  - l, r: $|G_0|$-dimensional qudit (Finite. The regular rep of G_0: $\langle g|g'\rangle = \delta_{g,g'}$)
  - P: Projection of $|G_0|^2$-dim space $l \cup r$ onto G_0-symmetric $|G_0|$-dim basis below:

$$\forall g \in G_0,\ \left|\psi_g\right\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{h \in G_0} |hg\rangle|h\rangle. \quad \left(D(g) \otimes D(g)\left|\psi_g\right\rangle = \left|\psi_g\right\rangle\right).$$

- $\boldsymbol{\mathbb{Z}_2^{\mathcal{T}}}$: Blue circles = 2-dimensional cuts of each subsystem (only when Υ is necessary).

General Setup



- **Randomness**: $\mathcal{U} \sim$ Haar measure on the projected $d_L d_R |G_0|$-dimensional space:

$$|\Psi\rangle = \sum_{L,g\in G_0,R} c_{L,g,R}|L\rangle\big|\psi_g\big\rangle|R\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{L,g_l,g_r,R} c_{L,g_r^{-1}g_l,R}|L\rangle|g_l\rangle|g_r\rangle|R\rangle$$

- **Fractionalization**: Ω fractionalizes G_0, Υ fractionalizes $\mathbb{Z}_2^{\mathcal{T}}$:

$$\Omega = \sum_{g_l,g_r} \omega(g_r, g_r^{-1}g_l)|g_l, g_r\rangle\langle g_l, g_r|, \;\; \Upsilon = \frac{1-\mathrm{i}}{2}\big(\mathbb{1}_4 - \mathrm{i}\sigma_y \otimes \sigma_y\big).$$

Results and Conclusion

Direct Sum into the Threefold Way

Entanglement-spectrum statistics of $G = G_0$ is

$$\bigoplus_{\alpha} \left[\frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right],$$

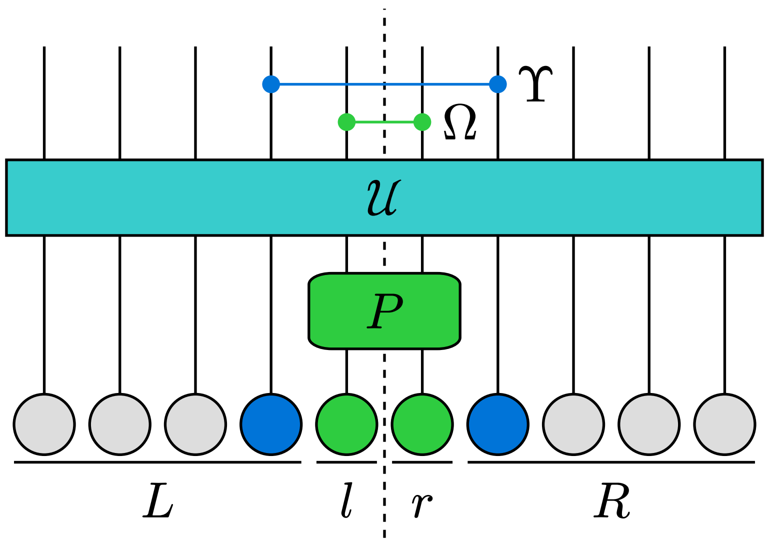
On the other hand, that of $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ is

$$\left[\bigoplus_{\alpha : R_+} \frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LOE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right] \oplus \left[\bigoplus_{\alpha : R_0} \frac{\mathbb{1}_{2 d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right] \oplus \left[\bigoplus_{\alpha : R_-} \frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LSE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right].$$

Entanglement-spectrum statistics of random symmetric states is
always able to be decomposed into the direct sum of the threefold way[1].

[1] This result is also the Laguerre version of Dyson's Gaussian threefold way (1962).

Conclusion



Until our work

- **The setup which follows LSE have been elusive.**

What this work reveraled are:

- **The LSE setup can be constructed by fractionalizing TRS** of the LOE setup.
- **Extended the setup to general symmetries.**
- Entanglement-spectrum statistics is **direct sum of the threefold way.**

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Supplemental Materials

Some additional information on group reps

- An irrep is either one of *real, complex, or pseudoreal*.
- **We consider the cases $\boldsymbol{G = G_0}$ or $\boldsymbol{G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}}$.**
- For $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$, one can define *the indicator* to know irreps real, complex, or pseudoreal.

Indicator of irrep

$$\iota_\alpha = \frac{1}{|G_0|} \sum_{g \in G_0} \omega(\tilde{g}, g) \, \mathrm{Tr}[\mathcal{D}_\alpha(\tilde{g}) \mathcal{D}_\alpha(g)],$$

$$\iota = \begin{cases} 1 : \text{real} \\ 0 : \text{complex} \\ -1 : \text{pseudoreal} \end{cases}$$

The cocycle (=cohomology class) can be decoupled $\omega = \omega_{G_0} \omega_{\mathbb{Z}_2^{\mathcal{T}}}$.

- $R_\pm$ = set irreps that satisfies $\iota = \pm\omega(t, t)$. $\omega(t, t) = 1(-1)$ in the absent (present) of Υ.