# Threefold Way for Typical Entanglement (arXiv:2410.11309)


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## 1. The Black Hole Information Paradox

Thermalization is inconsistent (?) with unitary dynamics.

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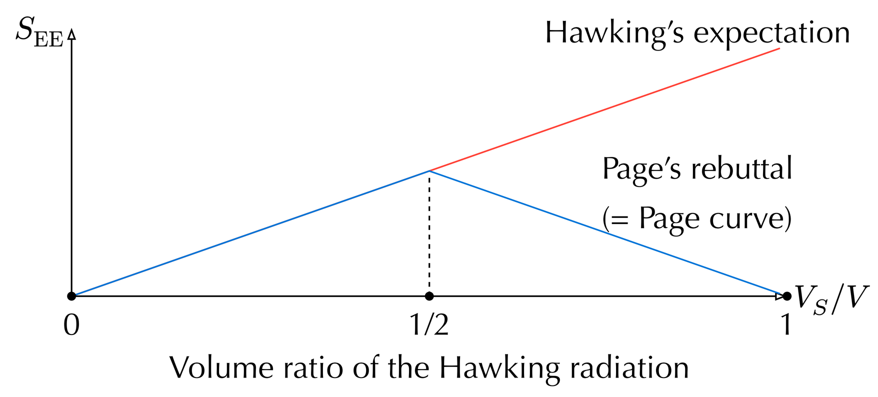
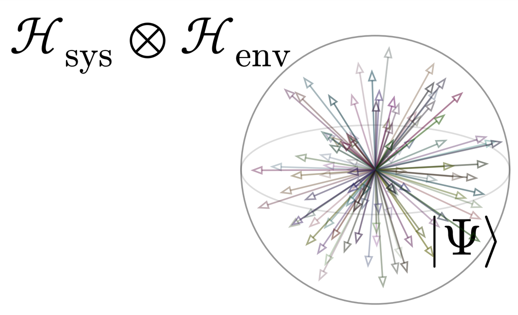


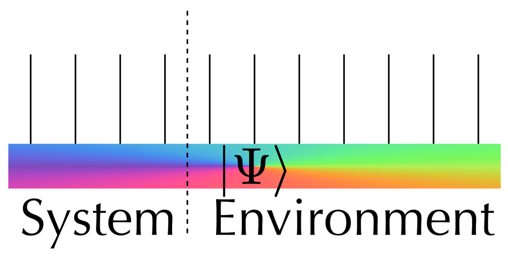

The entire dynamics should be **unitary** and **maximally chaotic**.

$\Rightarrow$ $|\Psi\rangle \sim U|0\rangle$, where $U$ is a random unitary matrix.

This motivates the study of ***typical entanglement***.

## 2. Typical Entanglement




- **Typical** $\simeq$ Haar-random (uniformly random sampling on the entire Hilbert space)
- **Entanglement spectrum** = eigvals of the reduced density matrix
- Random matrix theory is useful to evaluate typical entanglement[1–4].

## 3. Laguerre Unitary Ensemble of Random Matrices

Preparing Haar-random state from the entire Hilbert space:

$$|\Psi\rangle \sim \mathrm{Haar} \Leftrightarrow \rho_{\mathrm{sys}} = W\, W^{\dagger}.$$

where i.i.d. $W_{\mathrm{sys,env}} \sim \mathcal{CN}(0,1)$, $|\Psi\rangle = \sum_{\mathrm{sys,env}} W_{\mathrm{sys,env}}|\mathrm{sys,env}\rangle$ [5,6].

$\Rightarrow$ $\rho_{\mathrm{sys}}$ follows the **Laguerre unitary ensemble (LUE)** of RMT.

The joint probability $p$ distribution of eigenvalues $\{\lambda_i\}$ of $\rho_{\mathrm{sys}}$ is ***Laguerre distribution***:

$$p(\lambda_1,\ldots,\lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{\frac{\beta}{2}(n-m+1)-1} e^{-\frac{\beta}{2}\lambda_i} \prod_{1\le i<j\le m} |\lambda_i-\lambda_j|^{\beta}$$

with $\beta = 2$ for LUE.

## 4. TRS and Threefold Way

What if we change complex random variables to real random variables?

$\Rightarrow$ $|\Psi\rangle$ and $\rho_{\mathrm{sys}}$ obtain **time reversal symmetry** (TRS) of $\mathcal{T} = K$:

$$|\Psi\rangle \to \mathcal{T}|\Psi\rangle = K|\Psi\rangle = |\Psi\rangle,$$

$$\rho_{\mathrm{sys}} \to \mathcal{T}\rho_{\mathrm{sys}}\mathcal{T}^{-1} = K\rho_{\mathrm{sys}}K = \overline{\rho_{\mathrm{sys}}} = \rho_{\mathrm{sys}}.$$

Note that there is the other & nonequivalent kind of TRS!

**Time Reversal Symmetry**

Integer spin: $\mathcal{T}_+^2 = +\mathbb{1}$. ex. $\mathcal{T}_+ = K$

Half-integer spin: $\mathcal{T}_-^2 = -\mathbb{1}$. ex. $\mathcal{T}_- = \sigma_y K$

Imposing TRS: $\mathcal{T}\rho_{\mathrm{sys}}\mathcal{T}^{-1} = \rho_{\mathrm{sys}}$ allows threefold way of Laguerre ensemble:

- $\beta = 1$ (Laguerre orthgonal ensemble)
- $\beta = 4$ (Laguerre symplectic ensemble).

## 5. Prohibition of $\mathcal{T}_-^2 = -\mathbb{1}$ TRS Eigenstate

**Kramers' theorem:** $\mathcal{T}_-^2 = -\mathbb{1}$ **TRS cannot have the eigenstate** $\mathcal{T}_-|\Psi\rangle = |\Psi\rangle$.

*Proof*: $\mathcal{T}$ is anti-unitary, thus $\langle\mathcal{T}\_a|\mathcal{T}\_b\rangle = \overline{\langle a|b\rangle}$.

$$\langle\psi|\mathcal{T}\_\psi\rangle = \overline{\langle\mathcal{T}\_\psi|\mathcal{T}\_^2\psi\rangle} = \langle\mathcal{T}\_^2\psi|\mathcal{T}\_\psi\rangle = -\langle\psi|\mathcal{T}\_\psi\rangle$$

This implies $\langle\psi|\mathcal{T}\_\psi\rangle = 0$, thus $|\psi\rangle$ is orthogonal to $\mathcal{T}|\psi\rangle$. □

## 6. Key Questions

**Threefold way of Laguerre Ensemble**[7]

| | $\mathcal{T}_+^2 = +\mathbb{1}$ $(\mathcal{T}_+ \simeq K)$ | No TRS | $\mathcal{T}_-^2 = -\mathbb{1}$ $(\mathcal{T}_- \simeq \sigma_y K)$ | |
|---|---|---|---|---|
| | $\beta = 1$ | $\beta = 2$ | $\beta = 4$ | |
| Laguerre | **orthogonal**, | **unitary**, | **symplectic** | ensemble |
| | **LOE** | **LUE** | **LSE** | |
| $\mathcal{T}\lvert\Psi\rangle = \lvert\Psi\rangle$ : | real vector | complex vector | *unknown* | |

1. **Possible to construct the pure state that shows $\rho_{\mathrm{sys}} \sim$ LSE ?**
2. **Beyond threefold way if general symmetries?**

## 7. Key Idea: Symmetry Fractionalization

**Fractionalization** of TRS

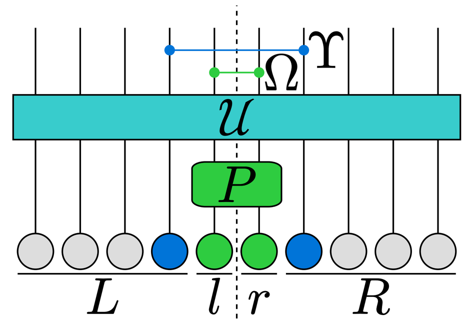
$$\mathcal{T}^2 = \left[\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}}\right]^2 = +\mathbb{1}$$

$$\swarrow\ \mathcal{T} = \Upsilon(\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}})\Upsilon^{\dagger}\ \searrow$$

$\mathcal{T}_{\mathrm{sys}}^2 = -\mathbb{1}$ $\qquad$ $\mathcal{T}_{\mathrm{env}}^2 = -\mathbb{1}$

- **We found $\Upsilon = \frac{1-i}{2}[\mathbb{1}_4 - i(\sigma_y \otimes \sigma_y)]$ and proved $\Upsilon|\Psi\rangle$ follows LSE.**
- Similarly, $\Omega = \sum_{g_l,g_r} \omega(g_r, g_r^{-1}g_l)|g_l,g_r\rangle\langle g_l,g_r|$ **fractionalizes general symmetries**: $\Omega[D(g)\otimes D(g)]\Omega^{\dagger} = \mathcal{D}(g)\otimes\overline{\mathcal{D}(g)}$, $\mathcal{D}(g)\mathcal{D}(g') = \omega(g,g')\mathcal{D}(gg')$.

## 8. General Setup




- $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ ($G_0$: unitary, $\mathbb{Z}_2^{\mathcal{T}}$: anti-unitary).
- $l, r$: $|G_0|$-dimensional qudit (regular representation of $G_0$:$\langle g|g'\rangle = \delta_{g,g'}$).
- $P$: Projection of $|G_0|^2$-dimensional Hilbert space of $l \cup r$ onto $G_0$-symmetric $|G_0|$-dimensional basis $\forall g \in G_0$, $|\psi_g\rangle = \frac{1}{\sqrt{|G_0|}}\sum_{h\in G_0}|hg\rangle|h\rangle$.
- $\mathcal{U} \sim$ Haar measure on the projected $d_L d_R|G_0|$-dimensional space:
$$|\Psi\rangle = \sum_{L,g\in G_0,R} c_{L,g,R}|L\rangle|\psi_g\rangle|R\rangle = \frac{1}{\sqrt{|G_0|}}\sum_{L,g_l,g_r,R} c_{L,g_r^{-1}g_l,R}|L\rangle|g_l\rangle|g_r\rangle|R\rangle$$
- $\Omega$ fractionalizes $G_0$, $\Upsilon$ fractionalizes $\mathbb{Z}_2^{\mathcal{T}}$.

## 9. Results and Conclusions

Entanglement-spectrum statistics of…

$G = G_0$ is

$$\bigoplus_{\alpha}\left[\frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right].$$

$G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ is

$$\left[\bigoplus_{\alpha:R_1}\frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LOE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right] \oplus \left[\bigoplus_{\alpha:R_0}\frac{\mathbb{1}_{2d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right] \oplus \left[\bigoplus_{\alpha:R_{-1}}\frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LSE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right].$$

Until our work

- **The setup which follows LSE have been elusive.**

What this work reveraled are:

- **The LSE setup can be constructed by fractionalizing TRS** of the LOE setup.
- **Extended the setup to general symmetries.**
- Entanglement-spectrum statistics is **direct sum of the threefold way.**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