Derivation of the arithmetic properties of edges and arcs

Online Resource 1 for "Polyarc-bounded complex interval arithmetic" paper

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S1 Unary operations

Operand equation

For $\boldsymbol{A} \in \mathcal{I}(\mathbb{C})$, $\boldsymbol{\Gamma} \subset \partial \boldsymbol{A}$

$$\begin{aligned}
\boldsymbol{\Gamma} \quad &= \left\{ F(s) \middle| s \in \boldsymbol{s} \right\} \\
&= \left\{ x+iy \middle| f(x,y)=0, s(x,y) \in \boldsymbol{s} \right\} \\
&= \left\{ x+iy \middle| \mathring{f}(\rho,\theta)=0, \mathring{s}(\rho,\theta) \in \boldsymbol{s} \right\}
\end{aligned}$$

S1.1 Negative

Result equation

$$\begin{aligned}
-\boldsymbol{\Gamma} \quad &= \left\{ -F(s) \middle| s \in \boldsymbol{s} \right\} \\
&= \left[-x-iy \middle| f(x,y)=0, s(x,y) \in \boldsymbol{s} \right] \\
&= \left[x+iy \middle| f(-x,-y)=0, s(-x,-y) \in \boldsymbol{s} \right]
\end{aligned}$$

S1.1.1 Edge negative

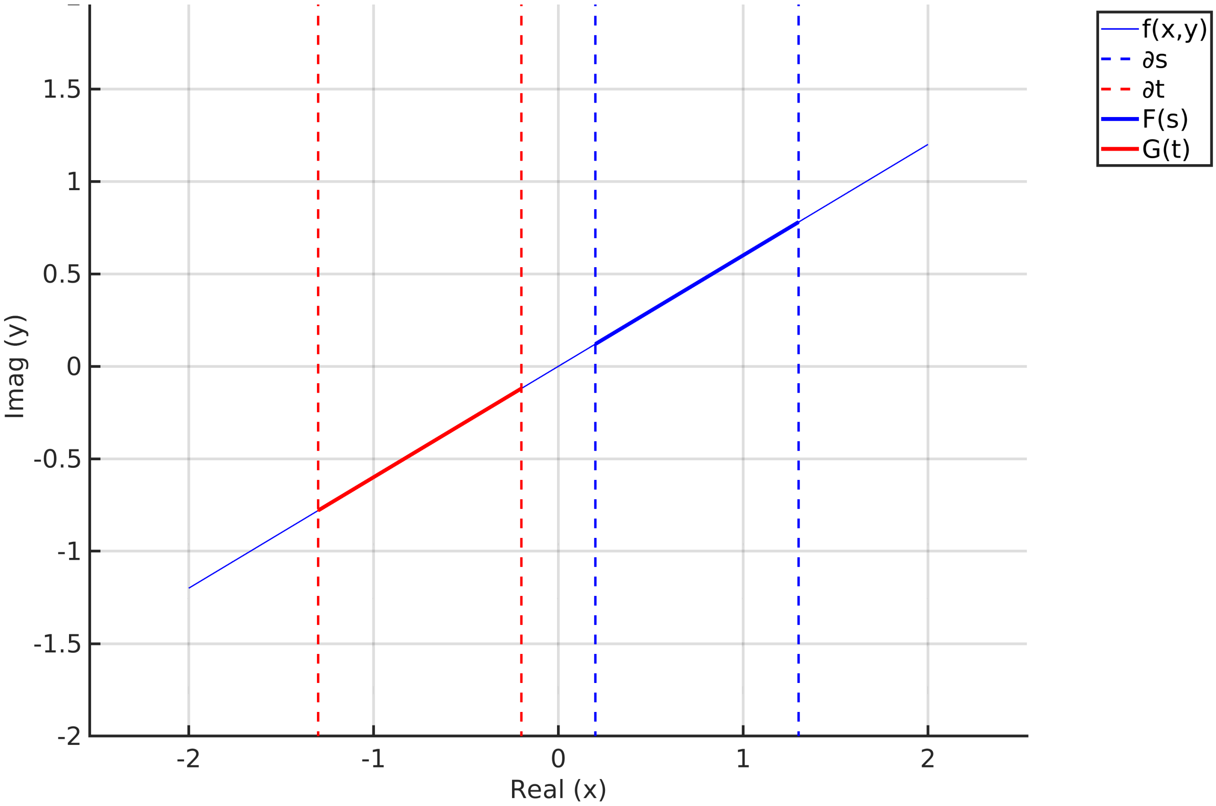

Figure 1: Negative of an edge, where $G(t) = -F(s)$. Parameters: $a = 0.6$, $\boldsymbol{s} = (0.2, 1.3)$.

$$\begin{aligned} f(x,y) &= ax - y \\ s(x,y) &= x \end{aligned}$$

$$\begin{aligned} -\boldsymbol{\Gamma} &= \left[-x - iy \middle| ax - y = 0, x \in \boldsymbol{s}\right] \\ &= \left[x + iy \middle| ax - y = 0, -x \in \boldsymbol{s}\right] \end{aligned}$$

S1.1.2 Arc negative

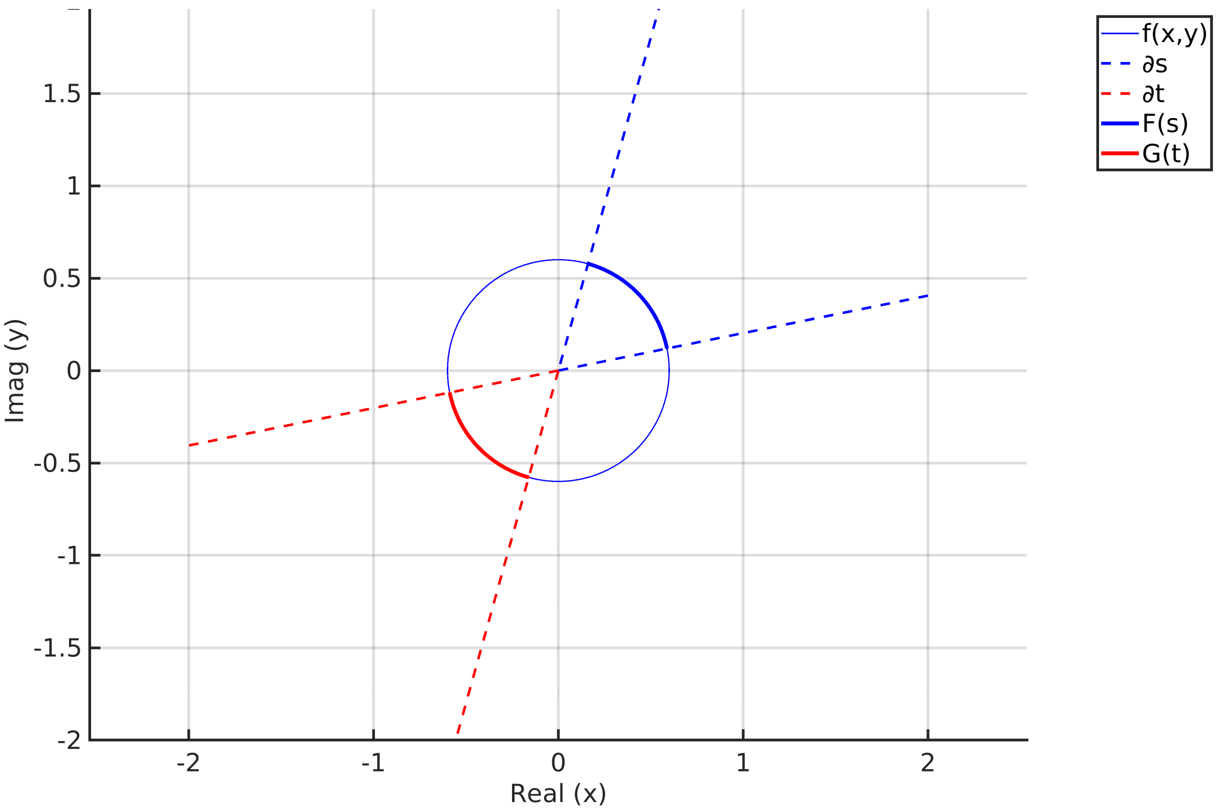

Figure 2: Negative of an arc, where $G(t) = -F(s)$. Parameters: r=0.6, s=(0.2,1.3).

$$
\begin{aligned}
f(x,y) &= x^2 - y^2 - r^2 \\
s(x,y) &= \operatorname{atan2}(y,x)
\end{aligned}
$$

$$
\begin{aligned}
-\mathbf{\Gamma} &= \left\{-x - \mathrm{i}y \middle| x^2 + y^2 - r^2 = 0, \operatorname{atan2}(-y,-x) \in \boldsymbol{s}\right\} \\
&= \left\{x + \mathrm{i}y \middle| x^2 + y^2 - r^2 = 0, \operatorname{atan2}(y,x) - \pi \in \boldsymbol{s}\right\}
\end{aligned}
$$

S1.2 Reciprocal

$$
\begin{aligned}
\mathbf{\Gamma}^{-1} \quad &= \left\{ \tfrac{1}{F(s)} \middle| s \in \boldsymbol{s} \right\} \\
&= \left[\tfrac{1}{\rho} e^{-i\theta} \middle| \mathring{f}(\rho, \theta) = 0, \mathring{s}(\rho, \theta) \in \boldsymbol{s} \right] \\
&= \left[\rho e^{i\theta} \middle| \mathring{f}(\tfrac{1}{\rho}, -\theta) = 0, \mathring{s}(\tfrac{1}{\rho}, -\theta) \in \boldsymbol{s} \right]
\end{aligned}
$$

S1.2.1 Edge reciprocal

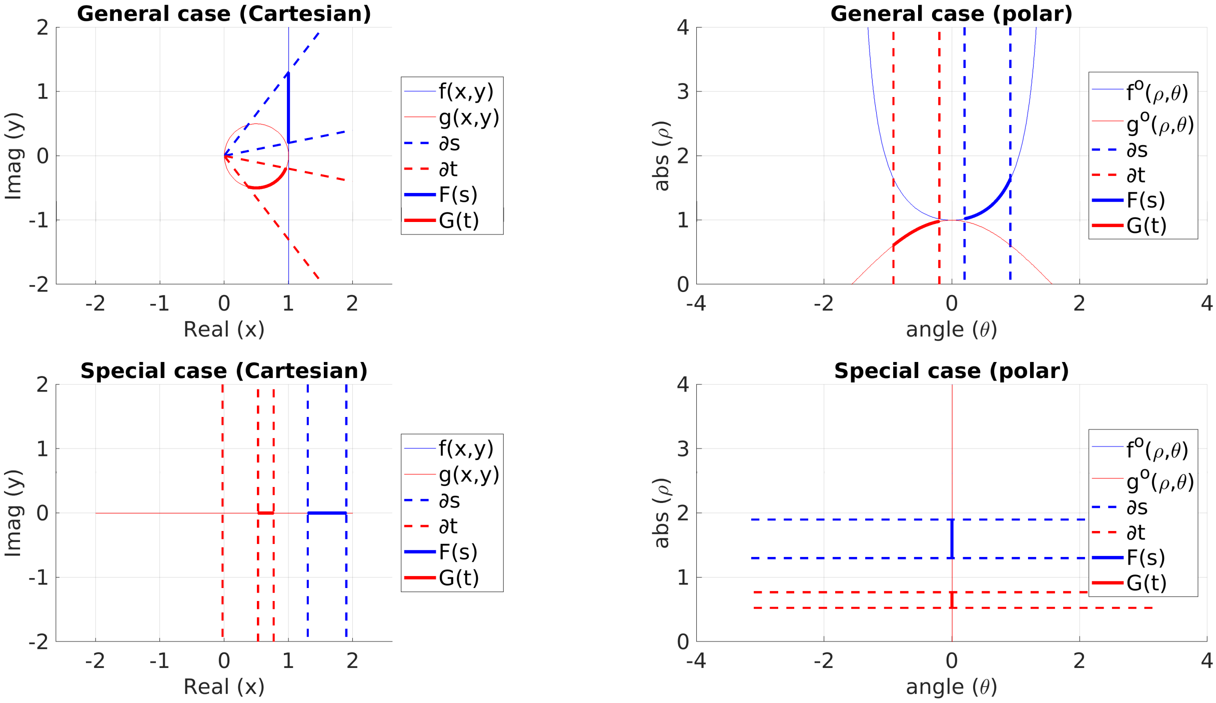

Figure 3: Reciprocal of an edge, where $G(t) = 1/F(s)$, $g(x,y) = 1/f(x,y)$, and $\mathring{g}(\rho,\theta) = 1/\mathring{f}(\rho,\theta)$. Parameters: $\boldsymbol{s} = (0.2, 1.3)$ in the general case, and $\boldsymbol{s} = (1.3, 1.9)$ in the special case.

S1.2.1.1 General case

$$
\begin{aligned}
f(x,y) &= x-1\\
s(x,y) &= y\\
\mathring{f}(\rho,\theta) &= \tfrac{1}{\rho} - \cos(\theta)\\
\mathring{s}(\rho,\theta) &= \rho\sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Gamma}^{-1} &= \left\{\rho e^{i\theta} \middle| \rho - \cos(-\theta) = 0, \tfrac{1}{\rho}\sin(-\theta) \in \boldsymbol{s}\right\}\\
&= \left\{\rho e^{i\theta} \middle| \rho - \cos(\theta) = 0, -\tfrac{1}{\rho}\sin(\theta) \in \boldsymbol{s}\right\}\\
&= \left\{x+iy \middle| \sqrt{x^2+y^2} - \cos(\operatorname{atan2}(y,x)) = 0, -\tfrac{1}{\sqrt{x^2+y^2}}\sin(\operatorname{atan2}(y,x)) \in \boldsymbol{s}\right\}\\
&= \left\{x+iy \middle| \sqrt{x^2+y^2} - \tfrac{x}{\sqrt{x^2+y^2}} = 0, -\tfrac{1}{\sqrt{x^2+y^2}}\tfrac{y}{\sqrt{x^2+y^2}} \in \boldsymbol{s}\right\}\\
&= \left\{x+iy \middle| x^2+y^2-x = 0, -\tfrac{y}{x^2+y^2} \in \boldsymbol{s}\right\}\\
&= \left\{x+iy \middle| (x-\tfrac{1}{2})^2 + y^2 - (\tfrac{1}{2})^2 = 0, -\tfrac{y}{x^2+y^2} \in \boldsymbol{s}\right\}
\end{aligned}
$$

S1.2.1.2 Special case: the edge is on a zero crossing line

$$
\begin{aligned}
f(x, y) &= y \\
s(x, y) &= x \\
\mathring{f}(\rho, \theta) &= \tan(\theta) \\
\mathring{s}(\rho, \theta) &= \rho
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Gamma}^{-1} &= \left\{ \rho e^{i\theta} \middle| \tan(-\theta) = 0, \tfrac{1}{\rho} \in \boldsymbol{s} \right\} \\
&= \left\{ \rho e^{i\theta} \middle| -\tan(\theta) = 0, \tfrac{1}{\rho} \in \boldsymbol{s} \right\} \\
&= \left\{ x + iy \middle| -\tan(\mathrm{atan2}(y, x)) = 0, -\tfrac{1}{\sqrt{x^2+y^2}} \in \boldsymbol{s} \right\} \\
&= \left\{ x + iy \middle| y = 0, \tfrac{1}{x} \in \boldsymbol{s} \right\}
\end{aligned}
$$

S1.2.2 Arc reciprocal

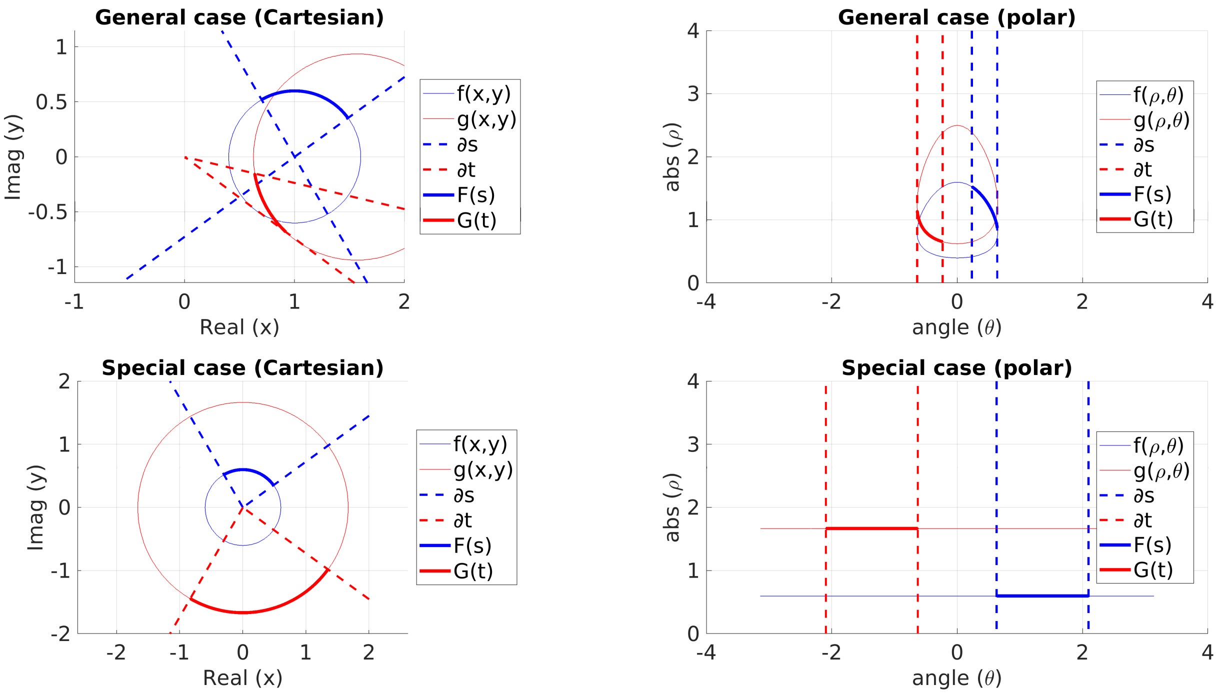

Figure 4: Reciprocal of an arc, where $G(t) = 1/F(s)$, $g(x,y) = 1/f(x,y)$, and $\mathring{g}(\rho,\theta) = 1/\mathring{f}(\rho,\theta)$. Parameters: $r = 0.6$, $\boldsymbol{s} = (\pi/5, 2\pi/3)$ in both cases.

S1.2.2.1 General case

$$
\begin{aligned}
f(x,y) &= (x-1)^2 + y^2 - r^2 \\
s(x,y) &= \operatorname{atan2}(y, x-1) \\
\mathring{f}(\rho,\theta) &= \rho^2 - 2\rho\cos(\theta) + 1 - r^2 \\
\mathring{s}(\rho,\theta) &= \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Gamma}^{-1} &= \left\{\rho e^{i\theta} \middle| \tfrac{1}{\rho^2} + \tfrac{2}{\rho}\cos(-\theta) + 1 - r^2 = 0, \operatorname{atan2}(\tfrac{1}{\rho} sin(-\theta), \tfrac{1}{\rho}\cos(-\theta) - 1) \in \boldsymbol{s}\right\} \\
&= \left\{\rho e^{i\theta} \middle| 1 + 2\rho\cos(\theta) + \rho^2(1 - r^2) = 0, \operatorname{atan2}(-\tfrac{1}{\rho}\sin(\theta), \tfrac{1}{\rho}\cos(\theta) - 1) \in \boldsymbol{s}\right\} \\
&= \Big\{x + iy \Big| 1 + 2\sqrt{x^2+y^2}\cos(\operatorname{atan2}(y,x)) + (x^2+y^2)(1-r^2) = 0, \\
&\qquad \operatorname{atan2}(-\tfrac{1}{\sqrt{x^2+y^2}}\sin(\operatorname{atan2}(y,x)), \tfrac{1}{\sqrt{x^2+y^2}}\cos(\operatorname{atan2}(y,x)) - 1) \in \boldsymbol{s}\Big\} \\
&= \Big\{x + iy \Big| 1 + 2\sqrt{x^2+y^2}\tfrac{x}{\sqrt{x^2+y^2}} + (x^2+y^2)(1-r^2) = 0, \\
&\qquad \operatorname{atan2}\left(-\tfrac{1}{\sqrt{x^2+y^2}}\tfrac{y}{\sqrt{x^2+y^2}}, \tfrac{1}{\sqrt{x^2+y^2}}\tfrac{x}{\sqrt{x^2+y^2}} - 1\right) \in \boldsymbol{s}\Big\} \\
&= \left\{x + iy \middle| 1 + 2x + (x^2+y^2)(1-r^2) = 0, \operatorname{atan2}\left(-\tfrac{y}{x^2+y^2}, \tfrac{x}{x^2+y^2} - 1\right) \in \boldsymbol{s}\right\} \\
&= \left\{x + iy \middle| x^2 + \tfrac{2x}{1-r^2} + \tfrac{1}{1-r^2} + y^2 = 0, \operatorname{atan2}\left(-\tfrac{y}{x^2+y^2}, \tfrac{x}{x^2+y^2} - 1\right) \in \boldsymbol{s}\right\} \\
&= \left\{x + iy \middle| \left(x - \tfrac{1}{1-r^2}\right)^2 + y^2 - \left(\tfrac{1}{1-r^2}\right)^2 = 0, \operatorname{atan2}\left(-\tfrac{y}{x^2+y^2}, \tfrac{x}{x^2+y^2} - 1\right) \in \boldsymbol{s}\right\}
\end{aligned}
$$

S1.2.2.2 Special case: the arc is on a zero centered circle

$$
\begin{aligned}
f(x,y) &= x^2 + y^2 - r^2 \\
s(x,y) &= \operatorname{atan2}(y,x) \\
\mathring{f}(\rho,\theta) &= \rho - r \\
\mathring{s}(\rho,\theta) &= \theta
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Gamma}^{-1} &= \left\{\rho e^{i\theta} \middle| \tfrac{1}{\rho} - r = 0, -\theta \in \boldsymbol{s}\right\} \\
&= \left\{\rho e^{i\theta} \middle| \rho = \tfrac{1}{r}, -\theta \in \boldsymbol{s}\right\} \\
&= \left\{\rho e^{i\theta} \middle| \sqrt{x^2+y^2} = \tfrac{1}{r}, -\operatorname{atan2}(y,x) \in \boldsymbol{s}\right\} \\
&= \left\{\rho e^{i\theta} \middle| x^2 + y^2 - \tfrac{1}{r^2} = 0, -\operatorname{atan2}(y,x) \in \boldsymbol{s}\right\}
\end{aligned}
$$

S2 Binary operations

We use the following algorithm in SageMath to compute the implicit equation of the envelope, as described in Subsection 3.3.5.

```
P.<x,y,u,v,Q,R,X,Y> = PolynomialRing(QQ, 8,order='degrevlex(4),degrevlex(4)')
    #(x,y), (u,v): coordinates of the two operands
    #Q,R: fixed parameters such as radii and slopes
    #(X,Y): coordinates of the result
def imageideal(f,g,o):
        #f,g: equations of the two operands in (x,y) and (u,v) respectively
        #o: operation, 0 is addition, 1 is multiplication
    if o==0:
        re=x+u
        im=y+v
            #addition
    else:
        re=x*u-y*v
        im=x*v+y*u
            #multiplication
    h=jacobian((f,g,re,im),(x,y,u,v)).det()
        #h: equation for the critical points
    I=ideal(f,g,h,re-X,im-Y)
    B=I.groebner_basis()
    Bred=[p for p in B if p(x=0,y=0,u=0,v=0)==p]
        #we eliminate the "heavy" variables, the Groebner basis of image ideal remains
    return len(Bred),Bred
        #returns the number of basis functions and their list
```

Operand equations

For $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{I}(\mathbb{C})$, $\boldsymbol{\Gamma} \subset \partial \boldsymbol{A}$, $\boldsymbol{\Gamma'} \subset \partial \boldsymbol{B}$

$$\begin{aligned}
\boldsymbol{\Gamma} &= \left\{x+iy \middle| f(x,y)=0, s(x,y) \in \boldsymbol{s}\right\} \\
&= \left\{\rho e^{i\theta} \middle| \mathring{f}\,(\rho,\theta)=0, s^\circ\,(\rho,\theta) \in \boldsymbol{s}\right\} \\
&= \left\{F(s) \middle| s \in \boldsymbol{s}\right\}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Gamma'} &= \left\{x+iy \middle| g\,(x,y)=0, t\,(x,y) \in \boldsymbol{t}\right\} \\
&= \left\{\rho e^{i\theta} \middle| \mathring{g}\,(\rho,\theta)=0, t^\circ\,(\rho,\theta) \in \boldsymbol{t}\right\} \\
&= \left\{G(t) \middle| t \in \boldsymbol{t}\right\}
\end{aligned}$$

S2.1 Addition

Parametric combination

$$H(s,t) = \left(\Re\left(F(s)+G(t)\right), \Im\left(F(s)+G(t)\right)\right) = \big(\Re(F)(s)+\Re(G)(t), \Im(F)(s)+\Im(G)(t)\big)$$

$$J(s,t) = \begin{vmatrix} \frac{\partial H_{\Re}(s,t)}{\partial s} & \frac{\partial H_{\Re}(s,t)}{\partial t} \\ \frac{\partial H_{\Im}(s,t)}{\partial s} & \frac{\partial H_{\Im}(s,t)}{\partial t} \end{vmatrix} = \begin{vmatrix} \Re(F)'(s) & \Re(G)'(t) \\ \Im(F)'(s) & \Im(G)'(t) \end{vmatrix}$$

$$J(s,t) = 0 \Longrightarrow \text{envelope}$$

Implicit combination

$$\begin{aligned} \varphi_{\oplus}(x,y,u,v) &= (x+u, y+v) \\ \widetilde{h}(x,y,u,v) &= \big|\mathrm{Jac}\big(f, g, \Re(\varphi_{\oplus}), \Im(\varphi_{\oplus})\big)\big| \\ &= \begin{vmatrix} \partial f/\partial x & \partial f/\partial y & 0 & 0 \\ 0 & 0 & \partial g/\partial u & \partial g/\partial v \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \\ I &= (f, g, \widetilde{h}, \Re(\varphi_{\oplus}) - X, \Im(\varphi_{\oplus}) - Y) \\ h &= \text{Gröbner basis elements involving only } X, Y \end{aligned}$$

Mixed combination

$$u(x,y,t) = s\left(x - G^{\Re}(t), y - G^{\Im}(t)\right)$$

$$\begin{aligned} \hat{h}(x,y,t) &= f\left(x - G^{\Re}(t), y - G^{\Im}(t)\right) \\ \frac{\partial \hat{h}}{\partial t} &= 0 \Longrightarrow t(x,y) \\ h(x,y) &= \hat{h}(x,y,t(x,y)) = 0 \end{aligned}$$

$$\begin{aligned} x(s,t) &= \Re\left(F(s)+G(t)\right) \\ y(s,t) &= \Im\left(F(s)+G(t)\right) \\ J(s,t) &= h\left(x(s,t), y(s,t)\right) \end{aligned}$$

S2.1.1 Edge plus edge

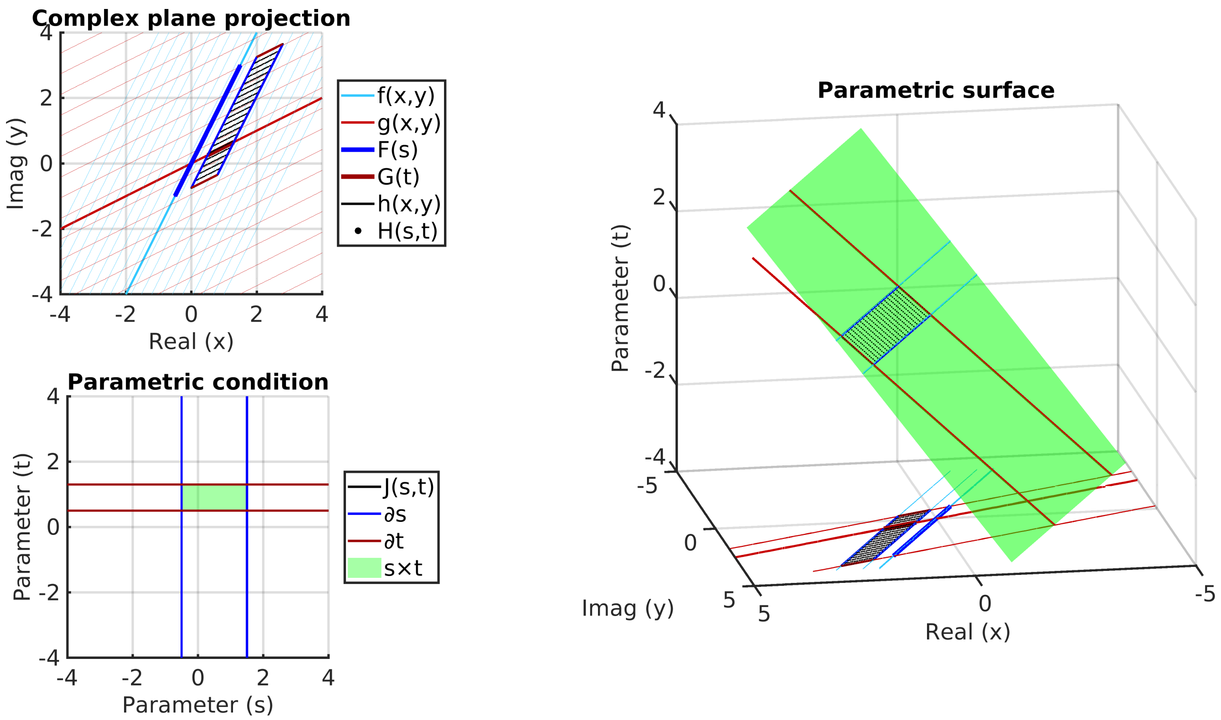

Figure 5: Addition of two edges. Parameters: $a_1 = 2.0$, $a_2 = 0.5$, $s = (-0.5, 1.5)$, $t = (0.5, 1.3)$

Operand equations

$$
\begin{aligned}
f(x,y) &= a_1 x - y \\
s(x,y) &= x \\
F(s) &= s + i a_1 s = (s, a_1 s)
\end{aligned}
$$

$$
\begin{aligned}
g\,(x,y) &= a_2 x - y \\
t\,(x,y) &= x \\
G(t) &= t + i a_2 t = (t, a_2 t)
\end{aligned}
$$

Parametric combination

$$
H(s,t) = (s + t, a_1 s + a_2 t)
$$

$$
J(s,t) = \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1
$$

$$
\text{Envelope: } \boxed{\begin{cases} \text{all } (s,t) & \text{if } a_1 = a_2 \\ \emptyset & \text{if } a_1 \neq a_2 \end{cases}}
$$

Implicit combination

$$\text{Envelope: } \boxed{\begin{cases} a_2x - y & \text{if } a_1 = a_2 \\ 1 & \text{if } a_1 \neq a_2 \end{cases}}$$

Mixed combination

$$\hat{h}(x, y, t) = a_1 (x - t) - (y - a_2 t)$$

$$\frac{\partial \hat{h}}{\partial t} = a_2 - a_1$$

$$\frac{\partial \hat{h}}{\partial t} = 0 \Longrightarrow a_1 = a_2$$

$$\text{Envelope: } \boxed{\begin{cases} a_1x - y = 0 & \text{if } a_1 = a_2 \\ \emptyset & \text{if } a_1 \neq a_2 \end{cases}}$$

S2.1.2 Arc plus edge

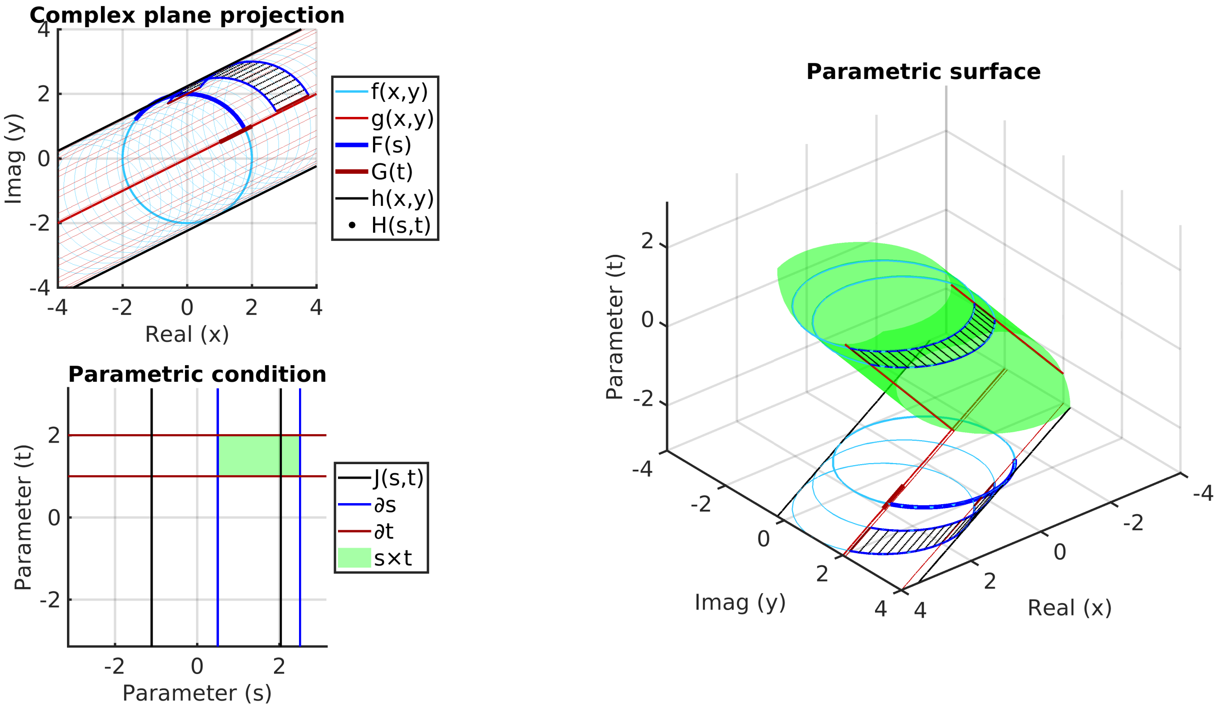

Figure 6: Addition of an arc and an edge. Parameters: $r = 2.0$, $a = 0.5$, $s = (0.5, 2.5)$, $t = (1.0, 2.0)$

Operand equations

$$f(x, y) = x^2 + y^2 - r^2$$
$$s(x, y) = \operatorname{atan2}(y, x)$$
$$F(s) = re^{is} = (r\cos(s), r\sin(s))$$

$$g(x, y) = \mathrm{ax} - y$$
$$t(x, y) = x$$
$$G(t) = t + iat = (t, at)$$

Parametric combination

$$H(s,t) = (r\cos(s) + t, r\sin(s) + at)$$
$$J(s,t) = \begin{vmatrix} -r\sin(s) & 1 \\ r\cos(s) & a \end{vmatrix}$$
$$= -r\,(a\sin(s) + \cos(s))$$

Envelope: $\boxed{\tan(s) = -1/a}$

Implicit combination

$$h(x, y) = a^2x^2 - a^2r^2 - 2axy + y^2 - r^2$$

$$= (ax - y)^2 - r^2 \left(a^2 + 1\right)$$

$$\text{Envelope: } \boxed{ax - y \pm r\sqrt{(a^2 + 1)} = 0}$$

Mixed combination

$$u(x, y, t) = \text{atan2}\,(y - at, x - t)$$

$$\begin{aligned}
\hat{h}(x, y, t) &= (x - t)^2 + (y - at)^2 - r^2 \\
\frac{\partial \hat{h}}{\partial t} &= 2t + 2x + 2a\,(y - at) \\
\frac{\partial \hat{h}}{\partial t} = 0 &\Longrightarrow t\,(x, y) = \frac{x + \mathrm{ay}}{a^2 + 1} \\
h(x, y) &= \left(x - \frac{x + \mathrm{ay}}{a^2 + 1}\right)^2 + \left(y - a\frac{x + \mathrm{ay}}{a^2 + 1}\right)^2 - r^2 \\
&= \left(\frac{a^2x + x - x - \mathrm{ay}}{a^2 + 1}\right)^2 + \left(\frac{a^2y + y - \mathrm{ax} - a^2y}{a^2 + 1}\right)^2 - r^2 \\
&= \left(a^2x - ay\right)^2 + (y - \mathrm{ax})^2 - r^2\left(a^2 + 1\right)^2 \\
&= a^2(\mathrm{ax} - y)^2 + (ax - y)^2 - r^2\left(a^2 + 1\right)^2 \\
&= \left(a^2 + 1\right)(\mathrm{ax} - y)^2 - r^2\left(a^2 + 1\right)^2 = 0 \\
&\Longrightarrow (ax - y)^2 - r^2\left(a^2 + 1\right) = 0
\end{aligned}$$

$$\text{Envelope: } \boxed{ax - y \pm r\sqrt{a^2 + 1} = 0}$$

$$\begin{aligned}
x\,(s, t) &= r\cos(s) + t \\
y\,(s, t) &= r\sin(s) + at \\
J(s, t) &= (ax\,(s, t) - y\,(s, t))^2 - r^2\left(a^2 + 1\right) \\
&= (a\,(r\cos(s) + t) - (r\sin(s) + at))^2 - r^2\left(a^2 + 1\right) \\
&= (ar\cos(s) - r\sin(s))^2 - r^2\left(a^2 + 1\right) = 0 \\
&\Longrightarrow (ar\cos(s) - r\sin(s))^2 = r^2\left(a^2 + 1\right)
\end{aligned}$$

$$\text{Envelope: } \boxed{(a\cos(s) - \sin(s))^2 = a^2 + 1}$$

S2.1.3 Arc plus arc

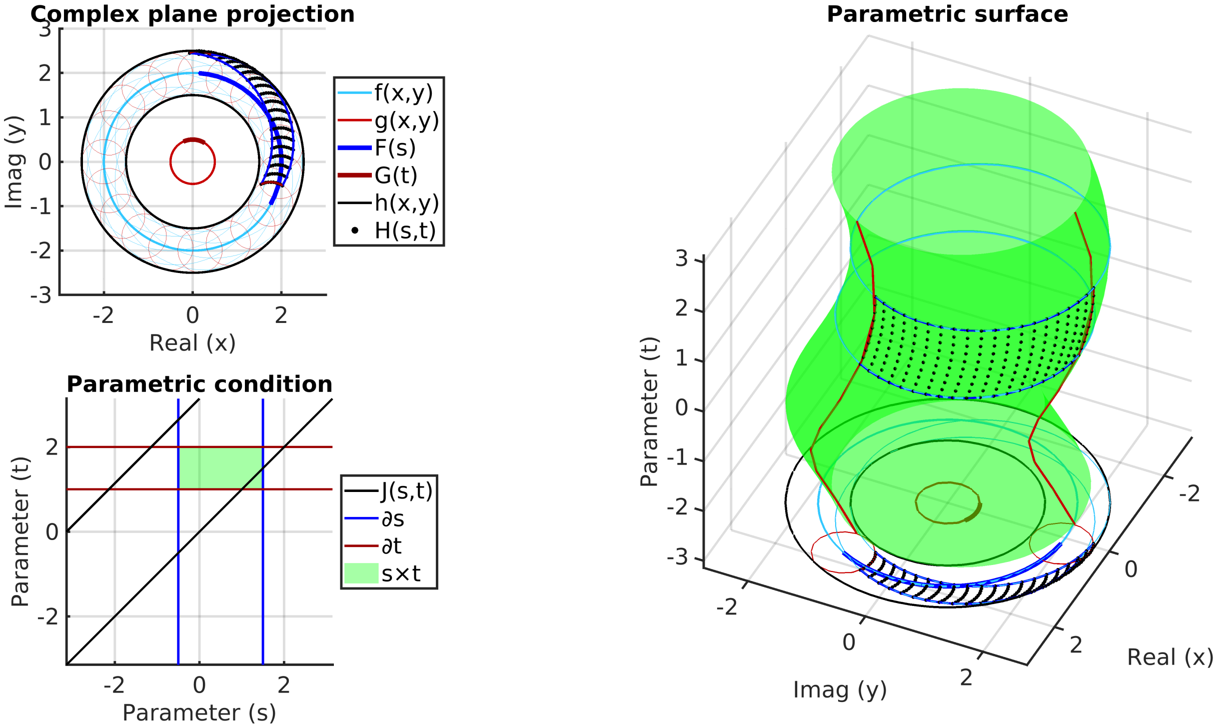

Figure 7: Addition of two arcs. Parameters: $r_1 = 2.0$, $r_2 = 0.5$, $s = (-0.5, 1.5)$, $t = (1.0, 2.0)$

Operand equations

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 - r_1^2 \\
s(x, y) &= \operatorname{atan2}(y, x) \\
F(s) &= r_1 e^{is} = (r_1 \cos(s), r_1 \sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g(x, y) &= x^2 + y^2 - r_2^2 \\
t(x, y) &= \operatorname{atan2}(y, x) \\
G(t) &= r_2 e^{it} = (r_2 \cos(t), r_2 \sin(t))
\end{aligned}
$$

Parametric combination

$$
\begin{aligned}
H(s, t) &= (r_1 \cos(s) + r_2 \cos(t), r_1 \sin(s) + r_2 \sin(t)) \\
J(s, t) &= \begin{vmatrix} -r_1 \sin(s) & -r_2 \sin(t) \\ r_1 \cos(s) & r_2 \cos(t) \end{vmatrix} \\
&= r_1 r_2 (\sin(t) \cos(s) - \sin(s) \cos(t)) \\
&= r_1 r_2 \sin(t - s)
\end{aligned}
$$

Envelope: $\boxed{s = t + k\pi \ (k \in \mathbb{Z})}$

Implicit combination

$$\begin{aligned} h(x,y) &= r_2^4 - 2r_2^2 r_1^2 + r_1^4 - 2r_2^2 x^2 - 2r_1^2 x^2 + x^4 - 2r_2^2 y^2 - 2r_1^2 y^2 + 2x^2y^2 + y^4 \\ &= \left(x^2+y^2-(r_1+r_2)^2\right)\left(x^2+y^2-(r_1-r_2)^2\right) \end{aligned}$$

$$\text{Envelope: } \boxed{x^2+y^2-(r_1 \pm r_2)^2 = 0}$$

Mixed combination

$$u(x,y,t) = \operatorname{atan2}\left(y - r_2\cos(t), x - r_2\sin(t)\right)$$

$$\begin{aligned} \hat{h}(x,y,t) &= (x - r_2\cos(t))^2 + (y - r_2\sin(t))^2 - r_1^2 \\ \frac{\partial \hat{h}}{\partial t} &= 2r_2 x\sin(t) - 2r_2^2\cos(t)\sin(t) - 2r_2 y\cos(t) + 2r_2^2\cos(t)\sin(t) \\ &= 2r_2\left(x\sin(t) - y\cos(t)\right) \\ \frac{\partial \hat{h}}{\partial t} = 0 &\implies x\sin(t) = y\cos(t) \\ h(x,y) &= \left(x - r_2\cos\left(\operatorname{atan}\left(\frac{y}{x}\right)\right)\right)^2 + \left(y - r_2\sin\left(\operatorname{atan}\left(\frac{y}{x}\right)\right)\right)^2 - r_1^2 \\ &= \left(x - \frac{r_2\, x}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{r_2\, y}{\sqrt{x^2+y^2}}\right)^2 - r_1^2 \\ &= x^2 - \frac{2r_2\, x^2}{\sqrt{x^2+y^2}} + \frac{(r_2\, x)^2}{x^2+y^2} + y^2 - \frac{2r_2\, y^2}{\sqrt{x^2+y^2}} + \frac{(r_2\, y)^2}{x^2+y^2} - r_1^2 \\ &= x^2+y^2 - \frac{2r_2\,\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} + \frac{r_2^2\left(x^2+y^2\right)}{x^2+y^2} - r_1^2 \\ &= x^2+y^2 - 2r_2\sqrt{x^2+y^2} + r_2^2 - r_1^2 \\ &= \left(\sqrt{x^2+y^2} - r_2\right)^2 - r_1^2 = 0 \\ &\implies \sqrt{x^2+y^2} - r_2 = \pm r_1 \\ &\implies \sqrt{x^2+y^2} = r_2 \pm r_1 \end{aligned}$$

$$\text{Envelope: } \boxed{x^2+y^2-(r_1 \pm r_2)^2 = 0}$$

$$\begin{aligned} x\left(s,t\right) &= r_1\cos(s) + r_2\cos(t) \\ y\left(s,t\right) &= r_1\sin(s) + r_2\sin(t) \\ J(s,t) &= (r_1\cos(s) + r_2\cos(t))^2 + (r_1\sin(s) + r_2\sin(t))^2 - (r_1 \pm r_2)^2 \\ &= r_1^2\cos^2(s) + 2r_1r_2\cos(s)\cos(t) + r_2^2\cos^2(t) + r_1^2\sin^2(s) \\ &\quad + 2r_1r_2\sin(s)\sin(t) + r_2^2\sin^2(t) - (r_1 \pm r_2)^2 \\ &= r_1^2 + r_2^2 + 2r_1r_2\left(\cos(s)\cos(t) + \sin(s)\sin(t)\right) - r_1^2 \pm 2r_1r_2 - r_2^2 \\ &= 2r_1r_2\cos\left(s-t\right) \pm 2r_1r_2 \\ J(s,t) = 0 &\implies \cos\left(s-t\right) = \pm 1 \end{aligned}$$

$$\text{Envelope: } \boxed{s = t + k\pi\ (k \in \mathbb{Z})}$$

S2.2 Multiplication

Parametric combination

$$\begin{aligned}H(s,t) &= \left(\Re\left(F(s)G(t)\right), \Im\left(F(s)G(t)\right)\right)\\ &= \big(\Re(F)(s)\Re(G)(t) - \Im(F)(s)\Im(G)(t), \Re(F)(s)\Im(G)(t) + \Im(F)(s)\Re(G)(t)\big)\end{aligned}$$

$$\begin{aligned}J(s,t) &= \begin{vmatrix} \frac{\partial H_{\Re}(s,t)}{\partial s} & \frac{\partial H_{\Re}(s,t)}{\partial t} \\ \frac{\partial H_{\Im}(s,t)}{\partial s} & \frac{\partial H_{\Im}(s,t)}{\partial t} \end{vmatrix}\\ &= \begin{vmatrix} \Re(F)'(s)\Re(G)(t) - \Im(F)'(s)\Im(G)(t) & \Re(F)(s)\Re(G)'(t) - \Im(F)(s)\Im(G)'(t) \\ \Re(F)'(s)\Im(G)(t) + \Im(F)'(s)\Re(G)(t) & \Re(F)(s)\Im(G)'(t) + \Im(F)(s)\Re(G)'(t) \end{vmatrix}\\ J(s,t) &= 0 \implies \text{envelope}\end{aligned}$$

Implicit combination

$$\begin{aligned}\varphi_{\otimes}(x,y,u,v) &= (xu - yv, xv + yu)\\ \widetilde{h}(x,y,u,v) &= \big|\mathrm{Jac}\big(f, g, \Re(\varphi_{\otimes}), \Im(\varphi_{\otimes})\big)\big|\\ &= \begin{vmatrix} \partial f/\partial x & \partial f/\partial y & 0 & 0 \\ 0 & 0 & \partial g/\partial u & \partial g/\partial v \\ u & -v & x & -y \\ v & u & y & x \end{vmatrix}\\ I &= (f, g, \widetilde{h}, \Re(\varphi_{\otimes}) - X, \Im(\varphi_{\otimes}) - Y)\\ h &= \text{Gröbner basis elements involving only } X, Y\end{aligned}$$

Mixed combination

$$\begin{aligned}u^{\circ}(\rho,\theta,t) &= s^{\circ}\left(\rho/|G(t)|, \theta - \angle G(t)\right)\\ u(x,y,t) &= u^{\circ}\left(\sqrt{x^2+y^2}, \operatorname{atan2}\left(y,x\right), t\right)\end{aligned}$$

$$\begin{aligned}h(\rho,\theta) &= \mathring{f}\left(\frac{\rho}{|G(t)|}, \theta - \angle G(t)\right)\\ \hat{h}(x,y,t) &= h\left(\rho\left(x,y\right), \theta\left(x,y\right)\right)\\ \frac{\partial \hat{h}}{\partial t} &= 0 \implies t(x,y)\\ h(x,y) &= \hat{h}(x,y,t(x,y))\end{aligned}$$

$$\begin{aligned}x\left(s,t\right) &= \Re\left(F(s)G(t)\right)\\ y\left(s,t\right) &= \Im\left(F(s)G(t)\right)\\ J(s,t) &= h\left(x\left(s,t\right), y\left(s,t\right)\right)\end{aligned}$$

S2.2.1 Edge times edge

S2.2.1.1 General case

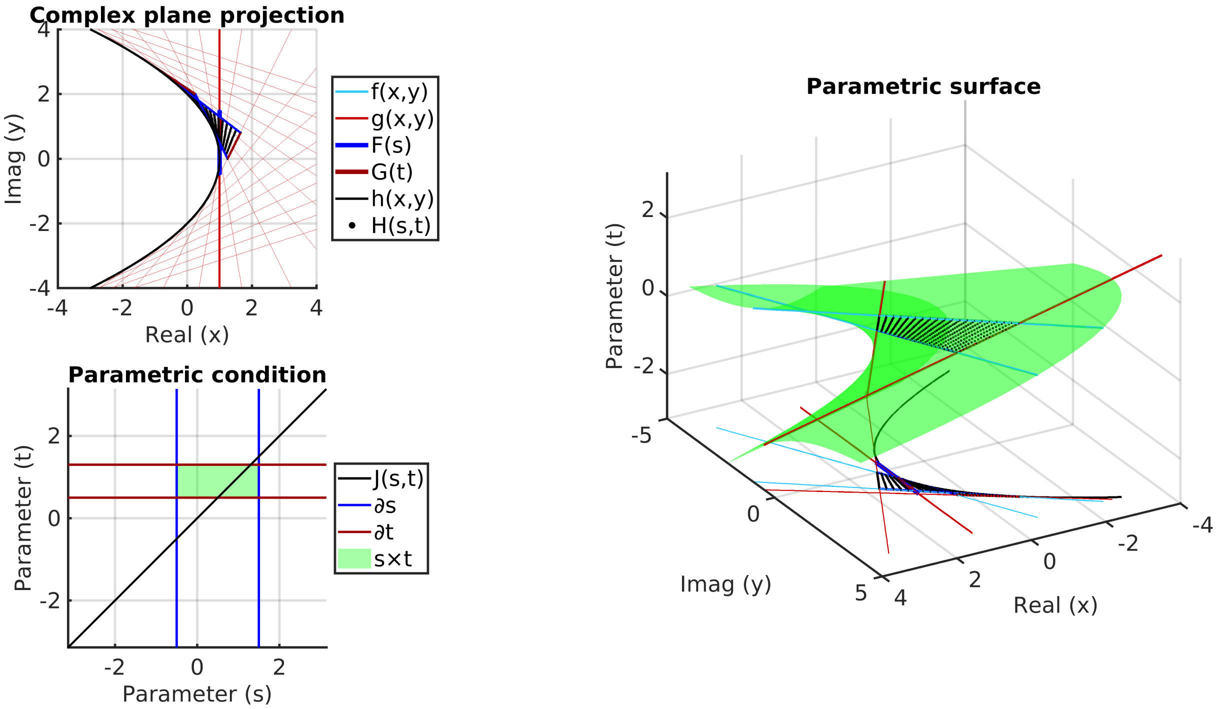

Figure 8: Multiplication of two edges in the general case. Parameters: $s = (-0.5, 1.5)$, $t = (0.5, 1.3)$.

Operand equations

$$
\begin{aligned}
f(x,y) &= x - 1 \\
\mathring{f}\,(\rho,\theta) &= 1/\rho - \cos(\theta) \\
s(x,y) &= y \\
s^{\circ}\,(\rho,\theta) &= \rho \sin(\theta) \\
F(s) &= 1 + is = (1, s)
\end{aligned}
$$

$$
\begin{aligned}
g\,(x,y) &= x - 1 \\
g^{\circ}\,(\rho,\theta) &= 1/\rho - \cos(\theta) \\
t\,(x,y) &= y \\
t^{\circ}\,(\rho,\theta) &= \rho \sin(\theta) \\
G(t) &= 1 + \mathrm{it} = (1, t)
\end{aligned}
$$

Parametric combination

$$H(s,t) = (1 - \mathrm{st}, s + t)$$

$$J(s,t) = \begin{vmatrix} -t & -s \\ 1 & 1 \end{vmatrix}$$
$$= s - t$$
Envelope: $\boxed{s = t}$

Implicit combination

$$h(x,y) = y^2 + 4x - 4$$
Envelope: $\boxed{x = -y^2/4 + 1}$

Mixed combination

$$u^\circ(\rho,\theta,t) = \frac{\rho}{\sqrt{1+t^2}} \sin\left(\theta - \operatorname{atan2}(t,1)\right)$$
$$= \frac{\rho}{\sqrt{1+t^2}} \sin\left(\theta - \operatorname{atan}(t)\right)$$
$$u(x,y,t) = \frac{\sqrt{x^2+y^2}}{\sqrt{1+t^2}} \sin\left(\operatorname{atan2}(y,x) - \operatorname{atan}(t)\right)$$
$$u(x,y,t) = 0 \Longrightarrow \operatorname{atan2}(y,x) - \operatorname{atan}(t) = 0 + k\pi$$

$$h(\rho,\theta) = \frac{\sqrt{t^2+1}}{\rho} - \cos\left(\theta - \operatorname{atan2}(t,1)\right)$$
$$= \frac{\sqrt{t^2+1}}{\rho} - \cos\left(\theta - \operatorname{atan}(t)\right)$$
$$\hat{h}(x,y,t) = \frac{\sqrt{t^2+1}}{\sqrt{x^2+y^2}} - \cos\left(\operatorname{atan2}(y,x) - \operatorname{atan}(t)\right) = 0$$
$$= \frac{\sqrt{t^2+1}}{\sqrt{x^2+y^2}} - \cos\left(\operatorname{atan2}(y,x)\right)\cos\left(\operatorname{atan}(t)\right) - \sin\left(\operatorname{atan2}(y,x)\right)\sin\left(\operatorname{atan}(t)\right) = 0$$
$$= \frac{\sqrt{t^2+1}}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{1}{\sqrt{t^2+1}} - \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{t}{\sqrt{t^2+1}} = 0$$
$$= t^2 + 1 - x - \mathrm{yt}$$
$$\frac{\partial \hat{h}}{\partial t} = 2t - y$$
$$\frac{\partial \hat{h}}{\partial t} = 0 \Longrightarrow t = \frac{y}{2}$$
$$h(x,y) = \frac{y^2}{4} + 1 - x - \frac{y^2}{2}$$
$$h(x,y) = 0 \Longrightarrow x + \frac{y^2}{4} - 1 = 0$$
Envelope: $\boxed{x = -y^2/4 + 1}$
$$x(s,t) = 1 - \mathrm{st}$$
$$y(s,t) = s + t$$
$$J(s,t) = h\left(1 - st, s + t\right)$$

$$
\begin{aligned}
&= 1 - st + \frac{(s+t)^2}{4} - 1 \\
J(s,t) = 0 &\Longrightarrow (s+t)^2 = 4st \\
&\Longrightarrow s^2 + 2st + t^2 = 4st \\
&\Longrightarrow s^2 - 2st + t^2 = 0 \\
&\Longrightarrow (s-t)^2 = 0 \\
&\Longrightarrow s = t
\end{aligned}
$$

Envelope: $\boxed{s = t}$

S2.2.1.2 Special case: one edge is on a zero crossing line

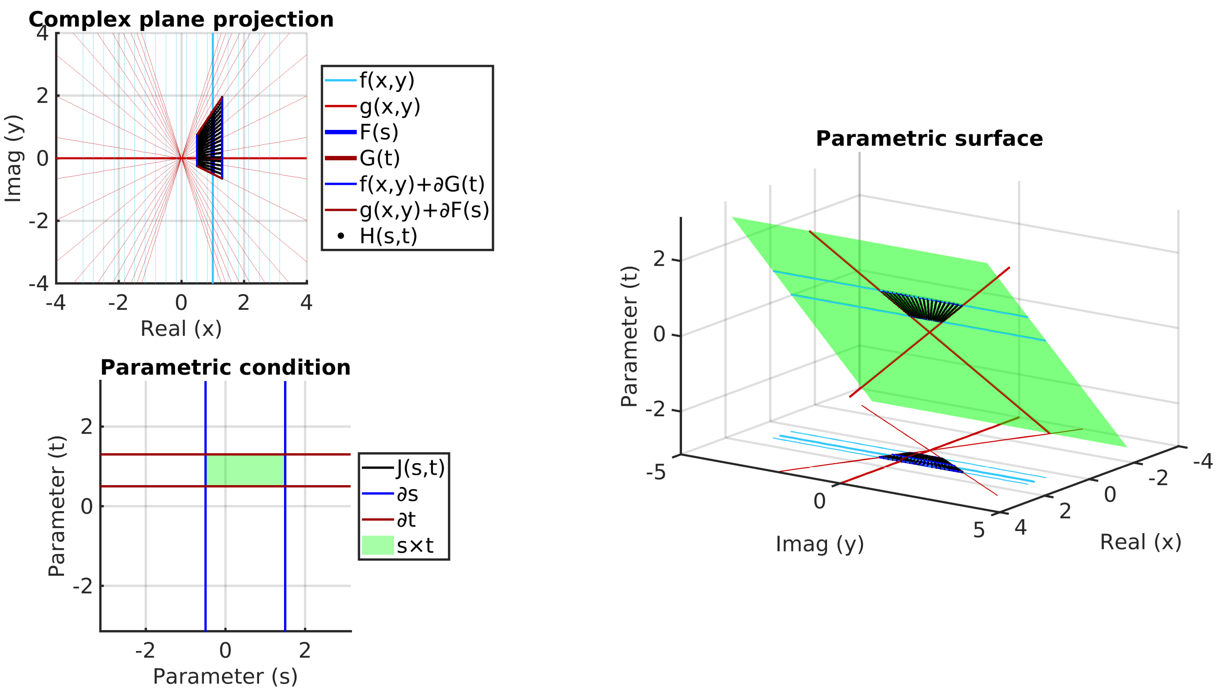

Figure 9: Multiplication of two edges when both are on zero crossing lines. Parameters: $s = (-0.5, 1.5)$, $t = (0.5, 1.3)$.

Operand equations

$$
\begin{aligned}
f(x, y) &= x - 1 \\
\mathring{f}\,(\rho, \theta) &= 1/\rho - \cos(\theta) \\
s(x, y) &= y \\
s^{\circ}\,(\rho, \theta) &= \rho \sin(\theta) \\
F(s) &= 1 + is = (1, s)
\end{aligned}
$$

$$
\begin{aligned}
g\,(x, y) &= y \\
g^{\circ}\,(\rho, \theta) &= \tan(\theta) \\
t\,(x, y) &= x \\
t^{\circ}\,(\rho, \theta) &= \rho \\
G(t) &= t = (t, 0)
\end{aligned}
$$

Parametric combination

$$H(s, t) = (s, st)$$

$$
\begin{aligned}
J(s,t) &= \begin{vmatrix} 1 & 0 \\ t & s \end{vmatrix} \\
&= s
\end{aligned}
$$

Envelope: $\boxed{s = 0}$

Implicit combination

$$
\begin{aligned}
h_1(x,y) &= x \\
h_2(x,y) &= y
\end{aligned}
$$

Envelope: $\boxed{(0,0) \text{ (i.e. the origin)}}$

Mixed combination

$$
u^\circ(\rho,\theta,t) = \frac{\rho}{t}\sin(\theta)
$$

$$
u(x,y,t) = \frac{\sqrt{x^2+y^2}}{t}\sin\left(\operatorname{atan2}(y,x)\right)
$$

$$
h(\rho,\theta) = \frac{t}{\rho} - \cos(\theta)
$$

$$
\begin{aligned}
\hat{h}(x,y,t) &= \frac{t}{\sqrt{x^2+y^2}} - \cos\left(\operatorname{atan2}(y,x)\right) \\
&= \frac{t-x}{\sqrt{x^2+y^2}}
\end{aligned}
$$

$$
\hat{h}(x,y,t) = 0 \implies t = x
$$

$$
\frac{\partial \hat{h}}{\partial t} = 1
$$

$$
h(x,y) = \hat{h}(x,y,t(x,y)) = \emptyset
$$

Envelope: $\boxed{\emptyset}$

S2.2.1.3 Special case: both edges are on zero crossing lines.

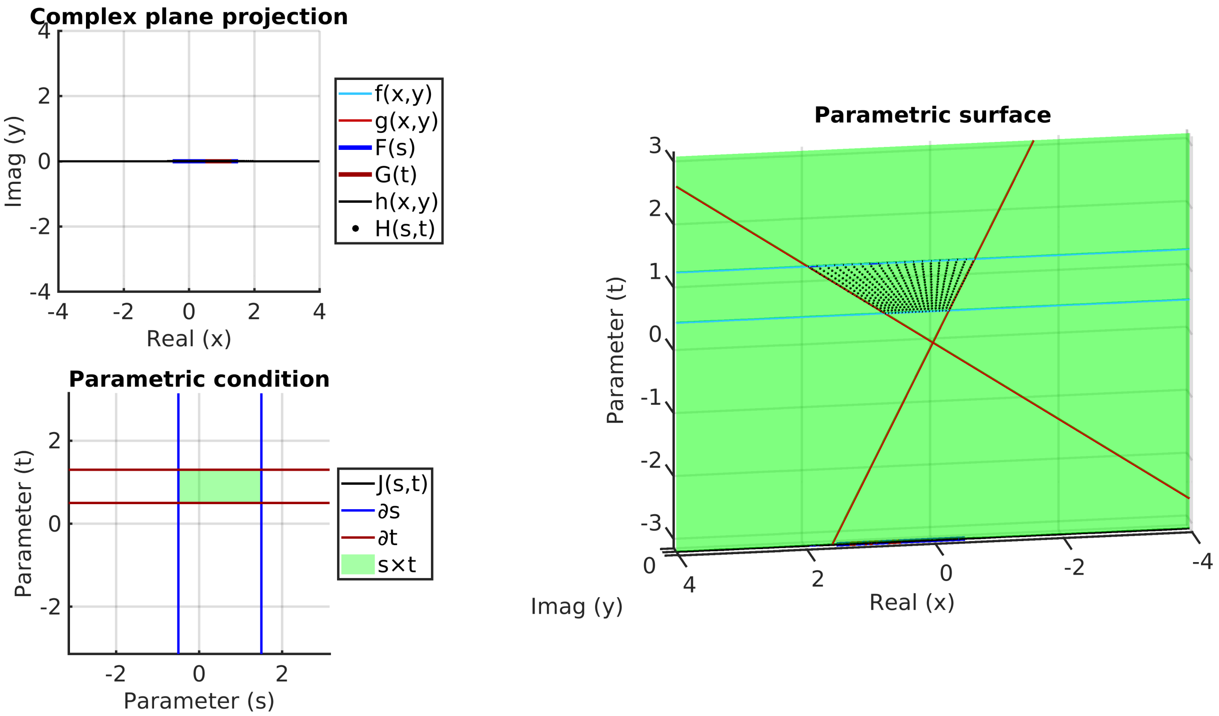

Figure 10: Multiplication of two edges when both are on zero crossing lines. Parameters: $s = (-0.5, 1.5)$, $t = (0.5, 1.3)$.

Operand equations

$$
\begin{aligned}
f(x,y) &= y \\
\mathring{f}(\rho,\theta) &= \tan(\theta) \\
s(x,y) &= x \\
s^{\circ}(\rho,\theta) &= \rho \\
F(s) &= s = (s,0)
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) &= y \\
g^{\circ}(\rho,\theta) &= \tan(\theta) \\
t(x,y) &= x \\
t^{\circ}(\rho,\theta) &= \rho \\
G(t) &= t = (t,0)
\end{aligned}
$$

Parametric combination

$$
H(s,t) = (st, 0)
$$

$$
J(s,t) = \begin{vmatrix} t & s \\ 0 & 0 \end{vmatrix}
$$

$$= 0$$

$$\text{Envelope: } \boxed{\text{all } (s,t)}$$

Implicit combination

$$h(x,y) = y$$

$$\text{Envelope: } \boxed{y = 0}$$

Mixed combination

$$u^\circ(\rho,\theta,t) = \frac{\rho}{t}$$

$$u(x,y,t) = \frac{\sqrt{x^2+y^2}}{t}$$

$$h(\rho,\theta) = \tan\left(\frac{\theta}{t}\right)$$

$$\hat{h}(x,y,t) = \tan\left(\frac{\operatorname{atan2}(y,x)}{t}\right)$$

$$\begin{aligned}\hat{h}(x,y,t) = 0 &\implies \operatorname{atan2}(y,x) = \operatorname{atan}(0)\cdot t\\ &\implies \operatorname{atan2}(y,x) = 0 + k\pi\\ &\implies y = 0\end{aligned}$$

$$\frac{\partial \hat{h}}{\partial t} = 0$$

$$h(x,y) = y$$

$$\text{Envelope: } \boxed{y = 0}$$

S2.2.2 Arc times edge

S2.2.2.1 General case

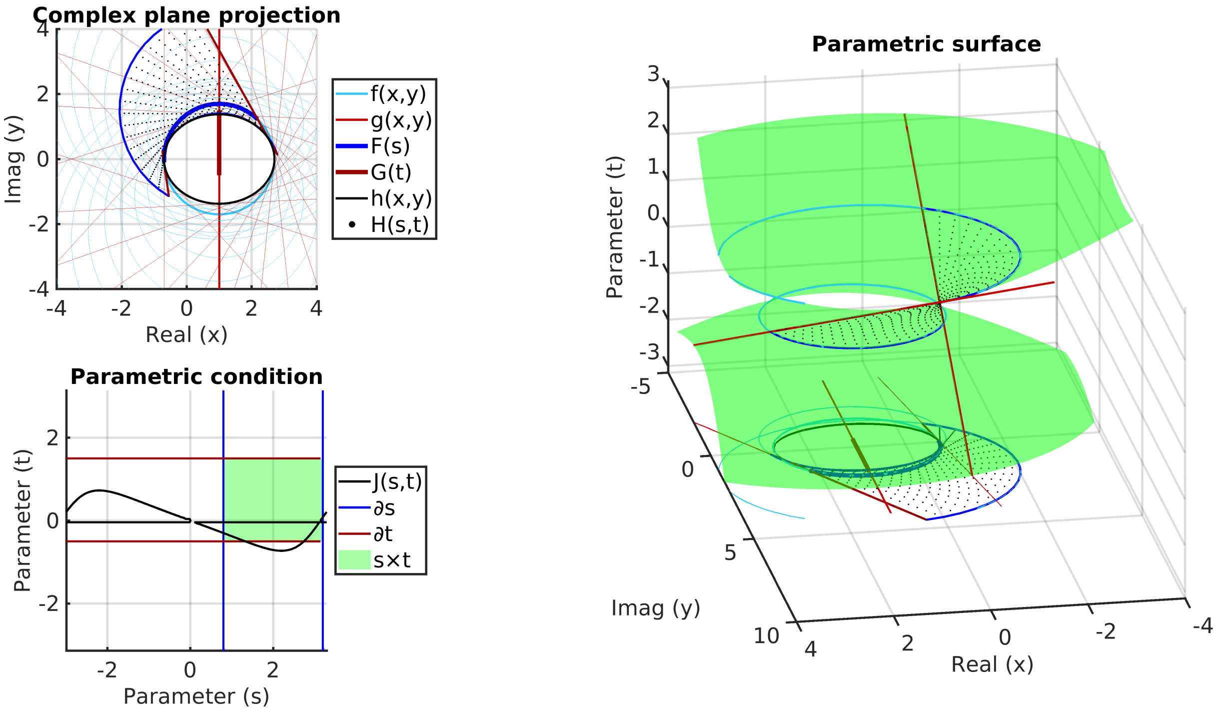

Figure 11: Multiplication of an arc and an edge when neither the edge is on a zero crossing line, nor the arc is on a zero centered circle and the radius is greater than one. Parameters: $r = 1.7$, $s = (0.8, 3.2)$, $t = (-0.5, 1.5)$.

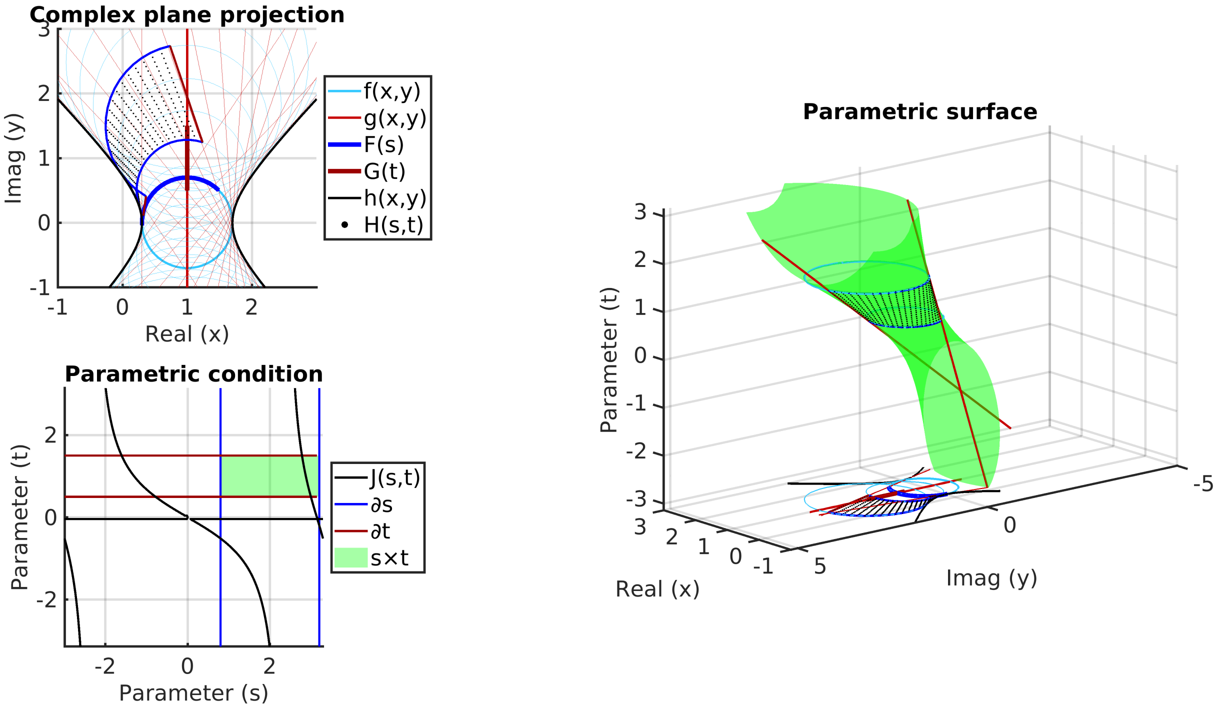

Figure 12: Multiplication of an arc and an edge when neither the edge is on a zero crossing line, nor the arc is on a zero centered circle and the radius is less than one. Parameters: $r = 0.7$, $s = (0.8, 3.2)$, $t = (0.5, 1.5)$.

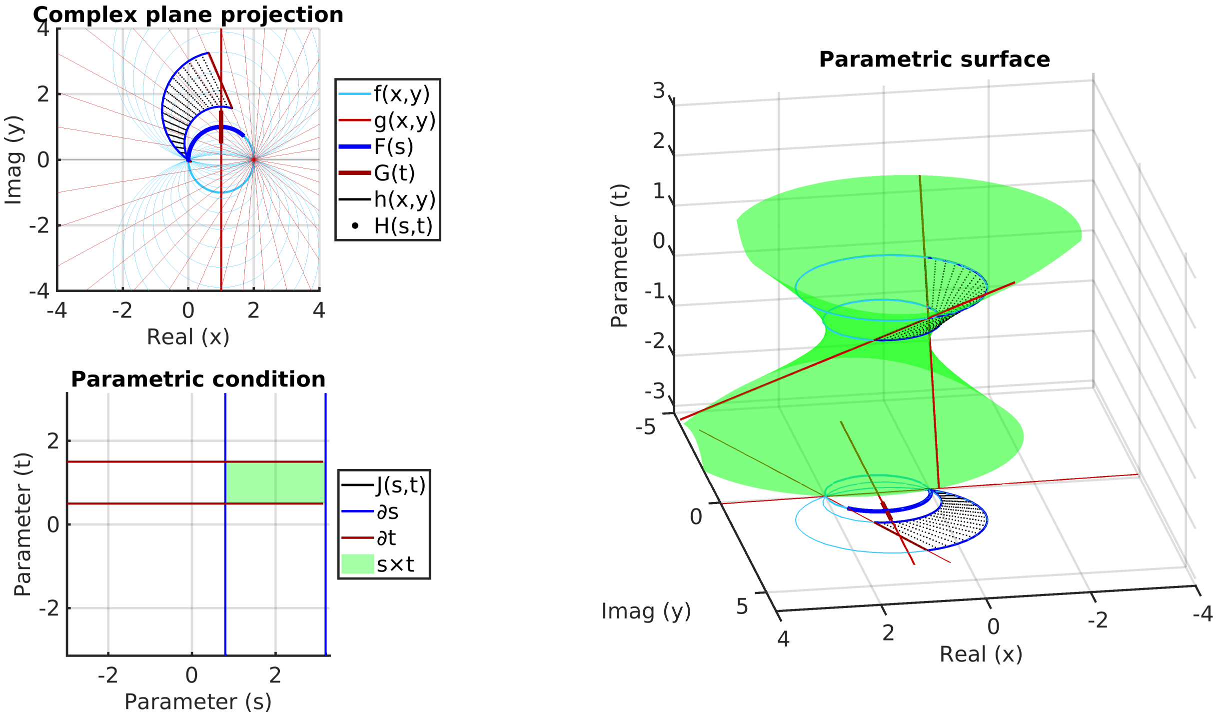

Figure 13: Multiplication of an arc and an edge when neither the edge is on a zero crossing line, nor the arc is on a zero centered circle and the radius is equal to one. Parameters: $r = 1$, $s = (0.8, 3.2)$, $t = (-0.5, 1.5)$.

Operand equations

$$f(x, y) = (x-1)^2 + y^2 - r^2$$
$$\mathring{f}(\rho, \theta) = \rho^2 - 2\rho\cos(\theta) + 1 - r^2$$
$$s(x, y) = \operatorname{atan2}(y, x-1)$$
$$s^\circ(\rho, \theta) = \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1)$$
$$F(s) = re^{is} + 1 = (1 + r\cos(s), r\sin(s))$$

$$g(x, y) = x - 1$$
$$g^\circ(\rho, \theta) = 1/\rho - \cos(\theta)$$
$$t(x, y) = y$$
$$t^\circ(\rho, \theta) = \rho\sin(\theta)$$
$$G(t) = 1 + \mathrm{it} = (1, t)$$

Parametric combination

$$H(s, t) = (1 + r\cos(s) - tr\sin(s), t + rt\cos(s) + r\sin(s))$$

$$J(s, t) = \begin{vmatrix} -r\sin(s) - rt\cos(s) & -r\sin(s) \\ -rt\sin(s) + r\cos(s) & 1 + r\cos(s) \end{vmatrix}$$

$$= -r\sin(s) - rt\cos(s) - r^2 t$$

$$\text{Envelope: } \boxed{\sin(s) + t\cos(s) + tr = 0}$$

Implicit combination

$$\text{Envelope: } \boxed{r^4 - r^2x^2 - r^2y^2 + 2r^2x - 2r^2 + x^2 - 2x + 1 = 0}$$

Mixed combination

$$\begin{aligned}
u^\circ(\rho,\theta,t) &= \operatorname{atan2}\left(\frac{\rho}{\sqrt{1+t^2}}\sin\left(\theta - \operatorname{atan2}(t,1)\right), \frac{\rho}{\sqrt{1+t^2}}\cos\left(\theta - \operatorname{atan2}(t,1)\right) - 1\right) \\
&= \operatorname{atan2}\left(\frac{\rho}{\sqrt{1+t^2}}\sin\left(\theta - \operatorname{atan}(t)\right), \frac{\rho}{\sqrt{1+t^2}}\cos\left(\theta - \operatorname{atan}(t)\right) - 1\right) \\
u(x,y,t) &= \operatorname{atan2}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{1+t^2}}\sin\left(\operatorname{atan2}(y,x) - \operatorname{atan}(t)\right), \frac{\sqrt{x^2+y^2}}{\sqrt{1+t^2}}\cos\left(\operatorname{atan2}(y,x) - \operatorname{atan}(t)\right) - 1\right) \\
u(x,y,t) = 0 &\Longrightarrow \cos\left(\operatorname{atan2}(y,x)\right)\cos\left(\operatorname{atan}(t)\right) + \sin\left(\operatorname{atan2}(y,x)\right)\sin\left(\operatorname{atan}(t)\right) = \frac{\sqrt{1+t^2}}{\sqrt{x^2+y^2}} \\
&\Longrightarrow \frac{x}{\sqrt{x^2+y^2}}\cdot\frac{t}{\sqrt{1+t^2}} + \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{\sqrt{1+t^2}}{\sqrt{x^2+y^2}} \\
&\Longrightarrow xt + y = 1 + t^2
\end{aligned}$$

$$\begin{aligned}
h(\rho,\theta) &= \frac{\rho^2}{1+t^2} - \frac{2\rho}{\sqrt{1+t^2}}\cos\left(\theta - \operatorname{atan}(t)\right) + 1 - r^2 \\
h(\rho,\theta) = 0 &\Longrightarrow \frac{\rho^2}{\sqrt{1+t^2}} - 2\rho\cos\left(\theta - \operatorname{atan}(t)\right) + \sqrt{1+t^2}\left(1 - r^2\right) \\
\hat{h}(x,y,t) &= \frac{x^2+y^2}{\sqrt{1+t^2}} - 2\sqrt{x^2+y^2}\cos\left(\operatorname{atan2}(y,x) - \operatorname{atan}(t)\right) + \sqrt{1+t^2}\left(1-r^2\right) \\
&= \frac{x^2+y^2}{\sqrt{1+t^2}} - 2\sqrt{x^2+y^2}\left(\cos\left(\operatorname{atan2}(y,x)\right)\cos\left(\operatorname{atan}(t)\right) + \sin\left(\operatorname{atan2}(y,x)\right)\sin\left(\operatorname{atan}(t)\right)\right) \\
&\quad + \sqrt{1+t^2}\left(1-r^2\right) \\
\hat{h}(x,y,t) = 0 &\Longrightarrow \frac{x^2+y^2}{\sqrt{1+t^2}} - 2\sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}\cdot\frac{1}{\sqrt{t^2+1}} + \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{t}{\sqrt{t^2+1}}\right) \\
&\quad + \sqrt{1+t^2}\left(1-r^2\right) = 0 \\
&\Longrightarrow \frac{x^2+y^2}{\sqrt{1+t^2}} - 2\frac{x+yt}{\sqrt{t^2+1}} + \sqrt{1+t^2}\left(1-r^2\right) = 0 \\
&\Longrightarrow x^2 + y^2 - 2\left(xt + y\right) + \left(1+t^2\right)\left(1-r^2\right) = 0 \\
&\Longrightarrow (x-1)^2 + (y-t)^2 - r^2\left(1+t^2\right) = 0 \\
\frac{\partial\hat{h}}{\partial t} &= 2t + 2y - 2r^2t \\
\frac{\partial\hat{h}}{\partial t} = 0 &\Longrightarrow t\left(1-r^2\right) = y \\
&\Longrightarrow t = \frac{y}{1-r^2}
\end{aligned}$$

$$
\begin{aligned}
h(x,y) &= (x-1)^2 + \left(y - \frac{y}{1-r^2}\right)^2 - r^2\left(1 + \left(\frac{y}{1-r^2}\right)^2\right)\\
&= (x-1)^2 + \frac{\left(y\left(1-r^2\right)-y\right)^2}{\left(1-r^2\right)^2} - r^2\frac{\left(1-r^2\right)^2+y^2}{\left(1-r^2\right)^2}\\
h(x,y)=0 &\Longrightarrow (x-1)^2\left(1-r^2\right)^2 + y^2\left(1-r^2\right)^2 - 2y^2\left(1-r^2\right) + y^2 - r^2\left(1-r^2\right)^2 - r^2y^2 = 0\\
&\Longrightarrow (x-1)^2\left(1-r^2\right)^2 + y^2\left(1-r^2\right)^2 - 2y^2\left(1-r^2\right) - r^2\left(1-r^2\right)^2 + y^2\left(1-r^2\right) = 0\\
&\Longrightarrow \left[(x-1)^2 + y^2 - r^2\right]\left(1-r^2\right)^2 - y^2\left(1-r^2\right) = 0\\
&\Longrightarrow (x-1)^2 + y^2 - r^2 - \frac{y^2}{1-r^2} = 0\\
&\Longrightarrow (x-1)^2 - \frac{r^2y^2}{1-r^2} - r^2 = 0\\
\text{if } r &= 1\\
h(x,y) &= \left(1-r^2\right)(x-1)^2 - r^2y^2 - \left(1-r^2\right)r^2\\
h(x,y)=0 &\Longrightarrow y = 0\\
h(x,0) &= (x-1)^2 - r^2\\
h(x,y)=0 &\Longrightarrow x = 1 \pm r = \{0,2\}\\
\text{Envelope: } & \boxed{\begin{cases} (x-1)^2 - \frac{r^2y^2}{1-r^2} - r^2 = 0 & \text{if } r \neq 0\\ x = \{0,2\} & \text{if } r = 1\end{cases}}
\end{aligned}
$$

$$
\begin{aligned}
x(s,t) &= 1 + r\cos(s) - tr\sin(s),\\
y(s,t) &= t + tr\cos(s) + r\sin(s)\\
J(s,t) &= h\left(1 + r\cos(s) - tr\sin(s), t + tr\cos(s) + r\sin(s)\right)\\
&= \left(1 + r\cos(s) - tr\sin(s) - 1\right)^2 - \frac{r^2(t + tr\cos(s) + r\sin(s))^2}{1-r^2} - r^2\\
J(s,t)=0 &\Longrightarrow \left(1-r^2\right)r^2(\cos(s) - t\sin(s))^2 - r^2(t + r\left(t\cos(s) + \sin(s)\right))^2 - \left(1-r^2\right)r^2 = 0\\
&\Longrightarrow \left(1-r^2\right)\left(\cos^2(s) - 2t\sin(s)\cos(s) + t^2\sin^2(s) - 1\right) - \left(t + r\left(t\cos(s) + \sin(s)\right)\right)^2 = 0\\
&\Longrightarrow \cos^2(s) - 2t\sin(s)\cos(s) + t^2\sin^2(s) - 1 - r^2\cos^2(s) - 2r^2t\sin(s)\cos(s)\\
&+ r^2t^2\sin^2(s) - r^2 - t^2 - 2rt^2\cos(s) + 2rt\sin(s) - r^2(t\cos(s) + \sin(s))^2 = 0\\
&\Longrightarrow \cos^2(s) - 2t\sin(s)\cos(s) + t^2\sin^2(s) - 1 - r^2\cos^2(s) - 2r^2t\sin(s)\cos(s) + r^2t^2\sin^2(s)\\
&- r^2 - t^2 - 2t^2r\cos(s) + 2tr\sin(s) - r^2t^2\cos^2(s) + 2r^2\text{tsin}(s)\cos(s) + r^2\sin^2(s) = 0\\
&\Longrightarrow r^2t^2 + 2rt^2\cos(s) + 2rt\sin(s) + t^2\cos(s)^2 + 2t\sin(s)\cos(s) + \sin^2(s) = 0\\
&\Longrightarrow r^2t^2 + 2rt\left(\sin(s) + t\cos(s)\right) + (\sin(s) + t\cos(s))^2 = 0\\
&\Longrightarrow [rt + (\sin(s) + t\cos(s))]^2 = 0\\
&\Longrightarrow \frac{\sin(s)}{t} + \cos(s) + r = 0\\
\text{Envelope: } & \boxed{\sin(s) + t\cos(s) + tr = 0}
\end{aligned}
$$

S2.2.2.2 Special case: the edge is on a zero crossing line

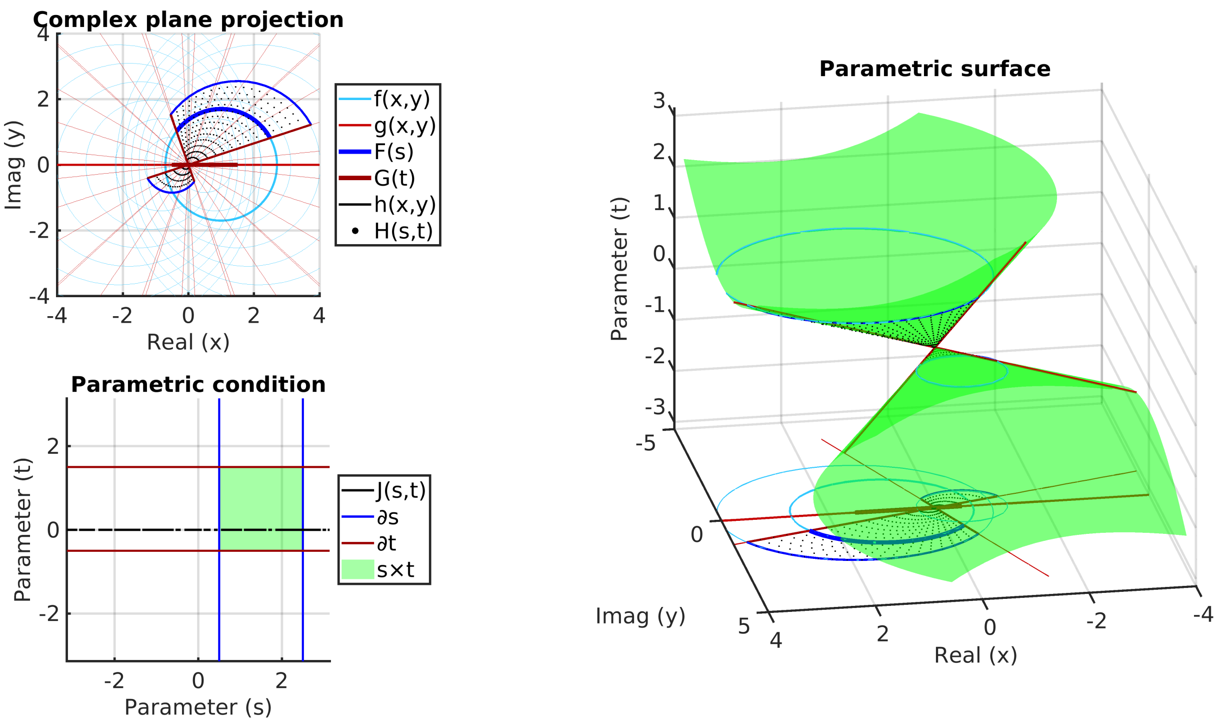

Figure 14: Multiplication of an arc and an edge when the edge is on a zero crossing line, but the arc is not on a zero centered circle and the radius is greater than one. Parameters: $r = 1.7$, $s = (0.5, 2.5)$, $t = (-0.5, 1.5)$.

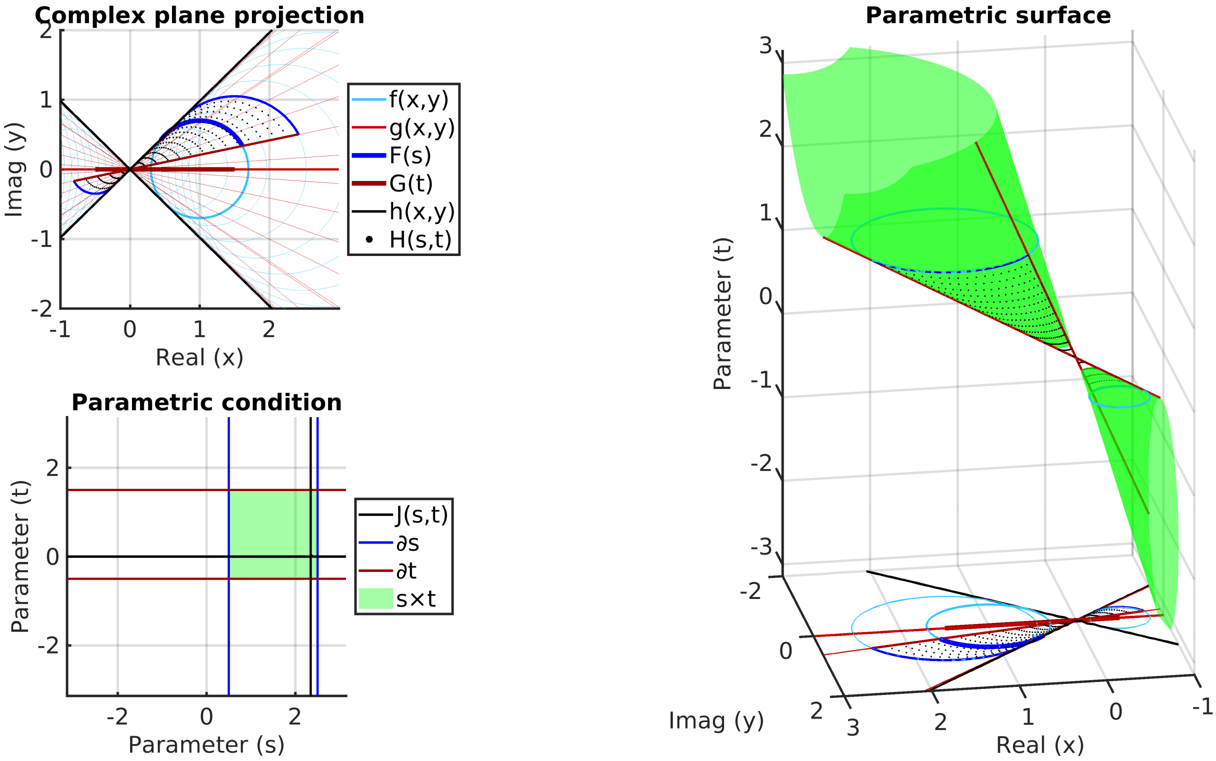

Figure 15: Multiplication of an arc and an edge when the edge is on a zero crossing line, but the arc is not on a zero centered circle and the radius is less than one. Parameters: $r = 0.7$, $s = (0.5, 2.5)$, $t = (-0.5, 1.5)$.

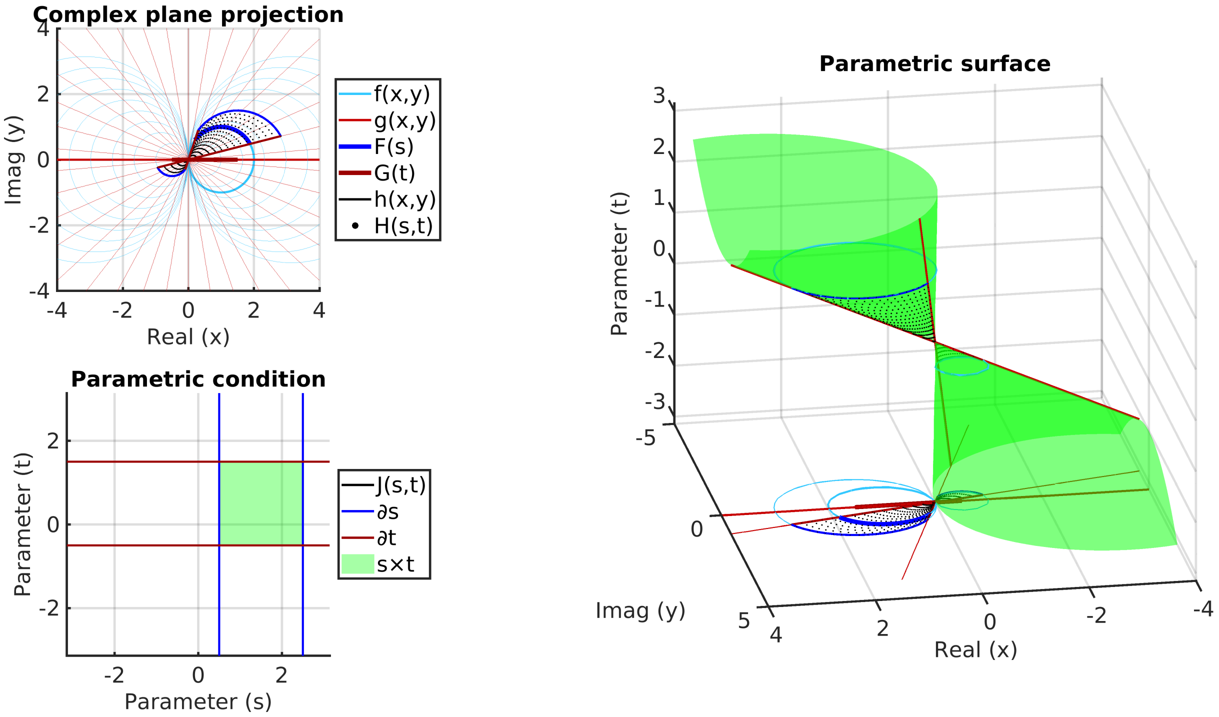

Figure 16: Multiplication of an arc and an edge when the edge is on a zero crossing line, but the arc is not on a zero centered circle and the radius is equal to one. Parameters: $r = 1$, $s = (0.5, 2.5)$, $t = (-0.5, 1.5)$.

Operand equations

$$
\begin{aligned}
f(x, y) &= (x-1)^2 + y^2 - r^2 \\
\mathring{f}(\rho, \theta) &= \rho^2 - 2\rho\cos(\theta) + 1 - r^2 \\
s(x, y) &= \operatorname{atan2}(y, x-1) \\
s^\circ(\rho, \theta) &= \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1) \\
F(s) &= re^{is} + 1 = (1 + r\cos(s), r\sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g(x, y) &= y \\
g^\circ(\rho, \theta) &= \tan(\theta) \\
t(x, y) &= x \\
t^\circ(\rho, \theta) &= \rho \\
G(t) &= t = (t, 0)
\end{aligned}
$$

Parametric combination

$$H(s, t) = (t + rt\cos(s), rt\sin(s))$$

$$J(s, t) = \begin{vmatrix} -rt\sin(s) & 1 + r\cos(s) \\ rt\cos(s) & r\sin(s) \end{vmatrix}$$

$$
\begin{aligned}
&= -r^2 t\sin^2(s) - r^2 t\cos^2(s) - rt\cos(s)\\
&= -rt(r+\cos(s))
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\begin{cases} t=0 & \text{if } r>1 \\ \{\cos(s)=-r\}\cup\{t=0\} & \text{if } r\le 1\end{cases}}
$$

Implicit combination

$$
\begin{aligned}
h(x,y) &= r^2x^2 + r^2y^2 - y^2\\
&= r^2x^2 + (r^2-1)y^2
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\begin{cases} x=y=0 & \text{if } r>1 \\ x=\pm\frac{\sqrt{1-r^2}}{r}y & \text{if } r\le 1\end{cases}}
$$

Mixed combination

$$
\begin{aligned}
u^\circ(\rho,\theta,t) &= \operatorname{atan2}\left(\frac{\rho}{t}\sin(\theta), \frac{\rho}{t}\cos(\theta) - 1\right)\\
u(x,y,t) &= \operatorname{atan2}\left(\frac{\sqrt{x^2+y^2}}{t}\sin(\operatorname{atan2}(y,x)), \frac{\sqrt{x^2+y^2}}{t}\cos(\operatorname{atan2}(y,x)) - 1\right)\\
u(x,y,t)=0 &\implies \frac{\sqrt{x^2+y^2}}{t}\cos(\operatorname{atan2}(y,x)) - 1 = 0, \frac{y}{t}>0\\
&\implies \sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} = t\\
&\implies x=t, \frac{y}{t}>0\\
h(\rho,\theta) &= \frac{\rho^2}{t^2} - 2\frac{\rho}{t}\cos(\theta) + 1 - r^2\\
h(\rho,\theta)=0 &\implies \rho^2 - 2\rho t\cos(\theta) + t^2\left(1-r^2\right) = 0\\
\hat{h}(x,y,t) &= x^2+y^2 - 2\sqrt{x^2+y^2}\,t\cos(\operatorname{atan2}(y,x)) + t^2\left(1-r^2\right)\\
&= x^2+y^2-2tx+t^2-r^2t^2\\
\hat{h}(x,y,t)=0 &\implies (x-t)^2 + y^2 - r^2t^2 = 0\\
\frac{\partial \hat{h}}{\partial t} &= 2(t-x) - 2r^2t\\
\frac{\partial \hat{h}}{\partial t} = 0 &\implies t - x - r^2 t = 0\\
&\implies t\left(1-r^2\right) = x\\
&\implies t = \frac{x}{1-r^2}\\
h(x,y) &= \left(x - \frac{x}{1-r^2}\right)^2 + y^2 - r^2\left(\frac{x}{1-r^2}\right)^2\\
&= \frac{\left(-r^2x\right)^2}{(1-r^2)^2} + y^2 - \frac{r^2x^2}{(1-r^2)^2}\\
h(x,y)=0 &\implies r^4x^2 + y^2\left(1-r^2\right)^2 - r^2x^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow y^2\left(1-r^2\right)^2 = r^2x^2\left(1-r^2\right)\\
&\Longrightarrow y^2\left(1-r^2\right) = r^2x^2\\
&\Longrightarrow \frac{r^2}{1-r^2}x^2 - y^2 = 0\\
&\Longrightarrow x = \pm\frac{\sqrt{1-r^2}}{r}y
\end{aligned}
$$

if $r = 1$

$$
\begin{aligned}
h(x,y) &= r^2x^2 - y^2\left(1-r^2\right)\\
h(x,y) = 0 &\Longrightarrow x = 0\\
h(x,0) &= x^2 - \frac{\left(1-r^2\right)}{r^2}y^2\\
h(x,0) = 0 &\Longrightarrow y = 0
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\begin{cases} x = y = 0 & \text{if } r \geq 1 \\ x = \pm\frac{\sqrt{1-r^2}}{r}y & \text{if } r < 1 \end{cases}}
$$

$$
\begin{aligned}
x(s,t) &= t + rt\cos(s)\\
y(s,t) &= rt\sin(s)\\
J(s,t) &= h\left(t + rt\cos(s), rt\sin(s)\right)\\
&= \frac{r^2}{1-r^2}\left(t + rt\cos(s)\right)^2 - \left(rt\sin(s)\right)^2\\
J(s,t) = 0 &\Longrightarrow r^2\left(t^2 + 2rt^2\cos(s) + r^2t^2\cos^2(s)\right) = \left(1-r^2\right)r^2t^2\sin^2(s)\\
&\Longrightarrow t^2 + 2rt^2\cos(s) + r^2t^2\cos^2(s) = t^2\sin^2(s) - r^2t^2\sin^2(s)\\
&\Longrightarrow t^2 + 2rt^2\cos(s) + r^2t^2 = t^2\sin^2(s)\\
&\Longrightarrow 1 + 2r\cos(s) + r^2 = 1 - \cos^2(s), t = 0\\
&\Longrightarrow \cos^2(s) + 2r\cos(s) + r^2 = 0\\
&\Longrightarrow \left(\cos(s) + r\right)^2 = 0\\
&\Longrightarrow s = \operatorname{acos}(-r), t = 0
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\begin{cases} t = 0 & \text{if } r > 1 \\ \{\cos(s) = -r\} \cup \{t = 0\} & \text{if } r \leq 1 \end{cases}}
$$

S2.2.2.3 Special case: the arc is on a zero centered circle

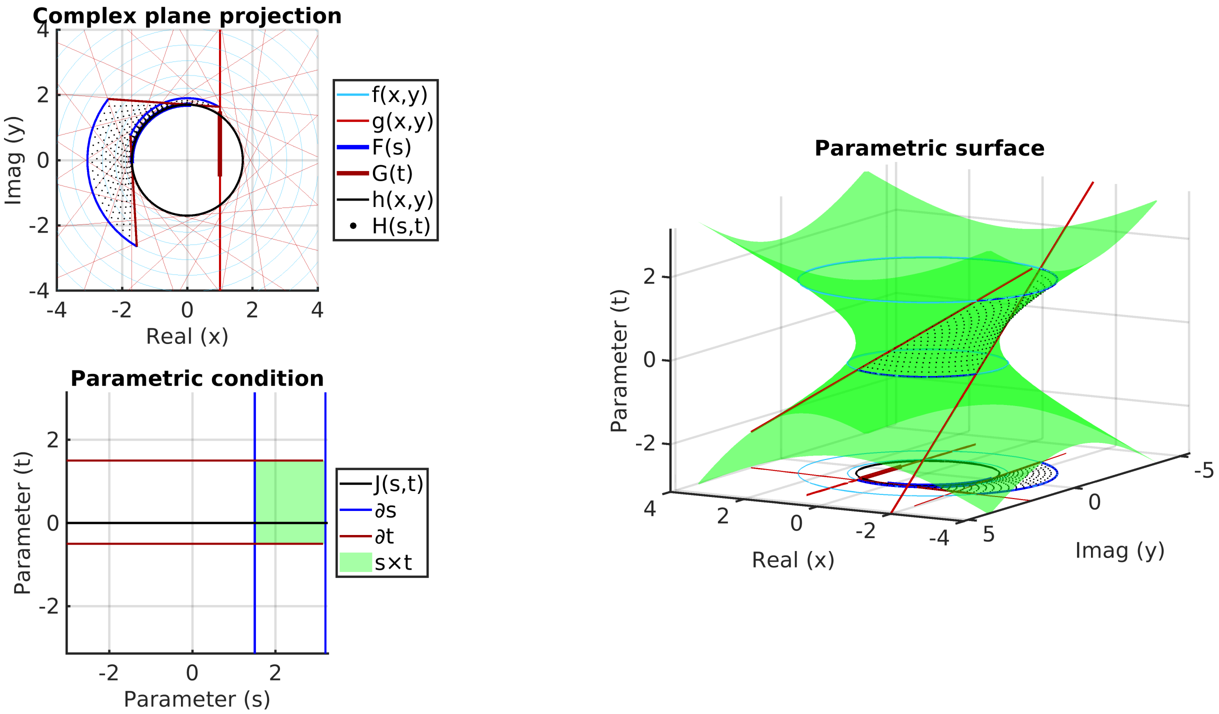

Figure 17: Multiplication of an arc and an edge when the edge is not on a zero crossing line, but the arc is on a zero centered circle. Parameters: $r = 1.7$, $s = (1.5, 3.2)$, $t = (-0.5, 1.5)$.

Operand equations

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 - r^2 \\
\mathring{f}\,(\rho, \theta) &= \rho - r \\
s(x, y) &= \operatorname{atan2}(y, x) \\
s^\circ\,(\rho, \theta) &= \theta \\
F(s) &= re^{is} = (r\cos(s), r\sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g\,(x, y) &= x - 1 \\
g^\circ\,(\rho, \theta) &= 1/\rho - \cos(\theta) \\
t\,(x, y) &= y \\
t^\circ\,(\rho, \theta) &= \rho\sin(\theta) \\
G(t) &= 1 + \mathrm{i}t = (1, t)
\end{aligned}
$$

Parametric combination

$$
H(s, t) = (r\cos(s) - rt\sin(s), rt\cos(s) + r\sin(s))
$$

$$
J(s, t) = \begin{vmatrix} -r\sin(s) - rt\cos(s) & -r\sin(s) \\ -rt\sin(s) + r\cos(s) & r\cos(s) \end{vmatrix}
$$

$$
\begin{aligned}
&= -r^2 t(\sin^2(s) + \cos^2(s)) \\
&= -r^2 t
\end{aligned}
$$

Envelope: $\boxed{t = 0}$

Implicit combination

$$h(x, y) = r^2 - x^2 - y^2$$

Envelope: $\boxed{x^2 + y^2 = r^2}$

Mixed combination

$$u^\circ(\rho, \theta, t) = \theta - \operatorname{atan2}(t, 1)$$

$$
\begin{aligned}
u(x, y, t) &= \operatorname{atan2}(y, x) - \operatorname{atan2}(t, 1) = 0 \\
&\Longrightarrow \operatorname{atan2}(y, x) = \operatorname{atan}(t)
\end{aligned}
$$

$$h(\rho, \theta) = \frac{\rho}{\sqrt{t^2 + 1}} - r$$

$$
\begin{aligned}
\hat{h}(x, y, t) &= \frac{\sqrt{x^2 + y^2}}{\sqrt{t^2 + 1}} - r = 0 \\
&= \frac{x^2 + y^2}{t^2 + 1} = r^2 \\
&\Longrightarrow x^2 + y^2 - r^2 t^2 - r^2 = 0 \\
\frac{\partial \hat{h}}{\partial t} &= -2r^2 t = 0 \\
&\Longrightarrow t = 0 \\
h(x, y) &= x^2 + y^2 - r^2 = 0
\end{aligned}
$$

Envelope: $\boxed{x^2 + y^2 = r^2}$

$$
\begin{aligned}
x(s, t) &= r\cos(s) - rt\sin(s) \\
y(s, t) &= rt\cos(s) + r\sin(s) \\
J(s, t) &= h\left(r\cos(s) - rt\sin(s), rt\cos(s) + r\sin(s)\right) \\
&= \left(r\cos(s) - rt\sin(s)\right)^2 + \left(rt\cos(s) + r\sin(s)\right)^2 - r^2 \\
&= r^2\cos^2(s) - 2r^2 t\sin(s)\cos(s) + r^2 t^2 \sin^2(s) \\
&\quad + r^2 t^2 \cos^2(s) + 2r^2 t\sin(s)\cos(s) + r^2\sin^2(s) - r^2 \\
J(s, t) = 0 &\Longrightarrow r^2 t^2 = 0
\end{aligned}
$$

Envelope: $\boxed{t = 0}$

S2.2.2.4 Special case: the edge is on a zero crossing line and the arc is on a zero centered circle

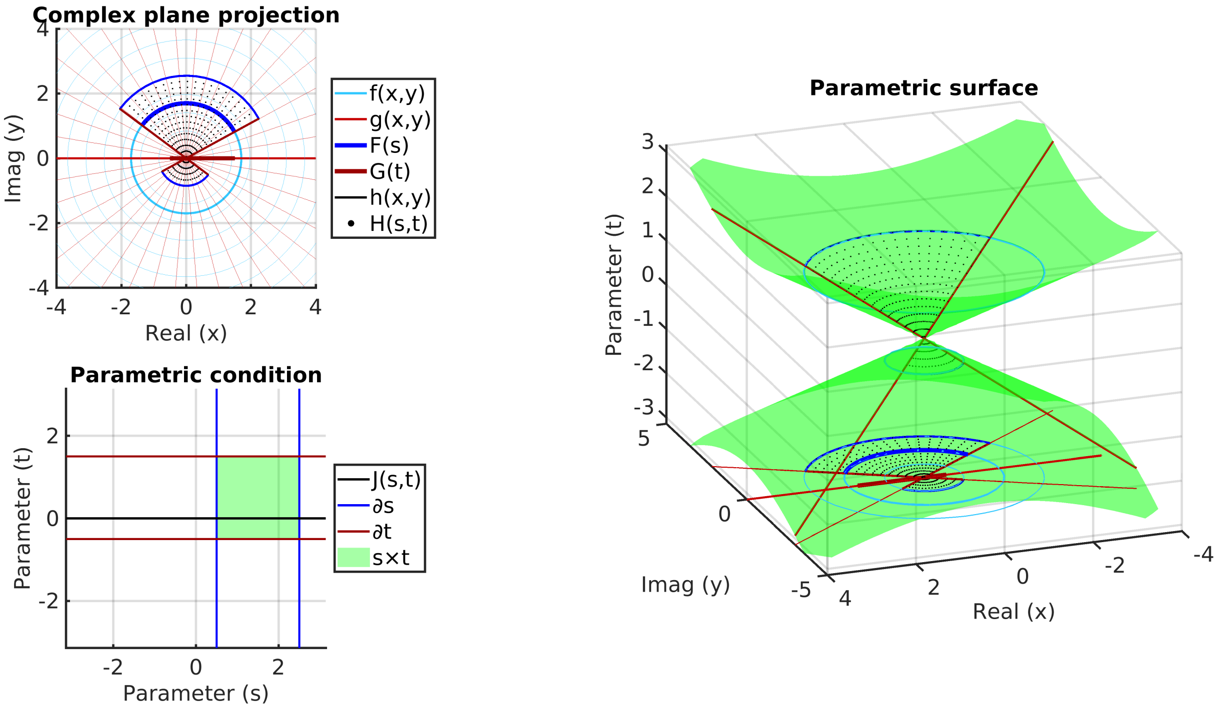

Figure 18: Multiplication of an arc and an edge when the edge is on a zero crossing line, and the arc is on a zero centered circle. Parameters: $r = 1.7$, $s = (0.5, 2.5)$, $t = (-0.5, 1.5)$.

Operand equations

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 - r^2 \\
\mathring{f}(\rho, \theta) &= \rho - r \\
s(x, y) &= \operatorname{atan2}(y, x) \\
s^\circ(\rho, \theta) &= \theta \\
F(s) &= re^{is} = (r\cos(s), r\sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g(x, y) &= y, g^\circ(\rho, \theta) = \tan(\theta) \\
t(x, y) &= x \\
t^\circ(\rho, \theta) &= \rho \\
G(t) &= t = (t, 0)
\end{aligned}
$$

Parametric combination

$$
\begin{aligned}
H(s, t) &= (rt\cos(s), rt\sin(s)) \\
J(s, t) &= \begin{vmatrix} -rt\sin(s) & r\cos(s) \\ rt\cos(s) & r\sin(s) \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= -r^2 t(\sin^2(s) + \cos^2(s)) \\
&= -r^2 t
\end{aligned}
$$

Envelope: $\boxed{t = 0}$

Implicit combination

$$
\begin{aligned}
h_1(x, y) &= x \\
h_2(x, y) &= y
\end{aligned}
$$

Envelope: $\boxed{(0, 0) \text{ (i.e. the origin)}}$

Mixed combination

$$u^\circ(\rho, \theta, t) = \theta$$

$$
\begin{aligned}
u(x, y, t) &= \operatorname{atan2}(y, x) \\
u(x, y, t) = 0 &\implies y = 0, x > 0
\end{aligned}
$$

$$
\begin{aligned}
h(\rho, \theta) &= \frac{\rho}{t} - r \\
\hat{h}(x, y, t) &= \frac{\sqrt{x^2 + y^2}}{t} - r \\
\hat{h}(x, y, t) = 0 &\implies x^2 + y^2 - r^2 t^2 = 0 \\
\frac{\partial \hat{h}}{\partial t} &= 2r^2 t \\
\frac{\partial \hat{h}}{\partial t} = 0 &\implies t = 0 \\
h(x, y) &= x^2 + y^2 = 0
\end{aligned}
$$

Envelope: $\boxed{(0, 0) \text{ (i.e. the origin)}}$

$$x(s, t) = rt\cos(s)$$

$$y(s, t) = rt\sin(s)$$

$$
\begin{aligned}
J(s, t) &= h\left(rt\cos(s), rt\sin(s)\right) \\
&= \left(rt\cos(s)\right)^2 + \left(rt\sin(s)\right)^2 \\
&= r^2 t^2 \cos^2(s) + r^2 t^2 \sin^2(s) \\
&= r^2 t^2 \\
J(s, t) = 0 &\implies t = 0
\end{aligned}
$$

Envelope: $\boxed{t = 0}$

S2.2.3 Arc times arc

S2.2.3.1 General case

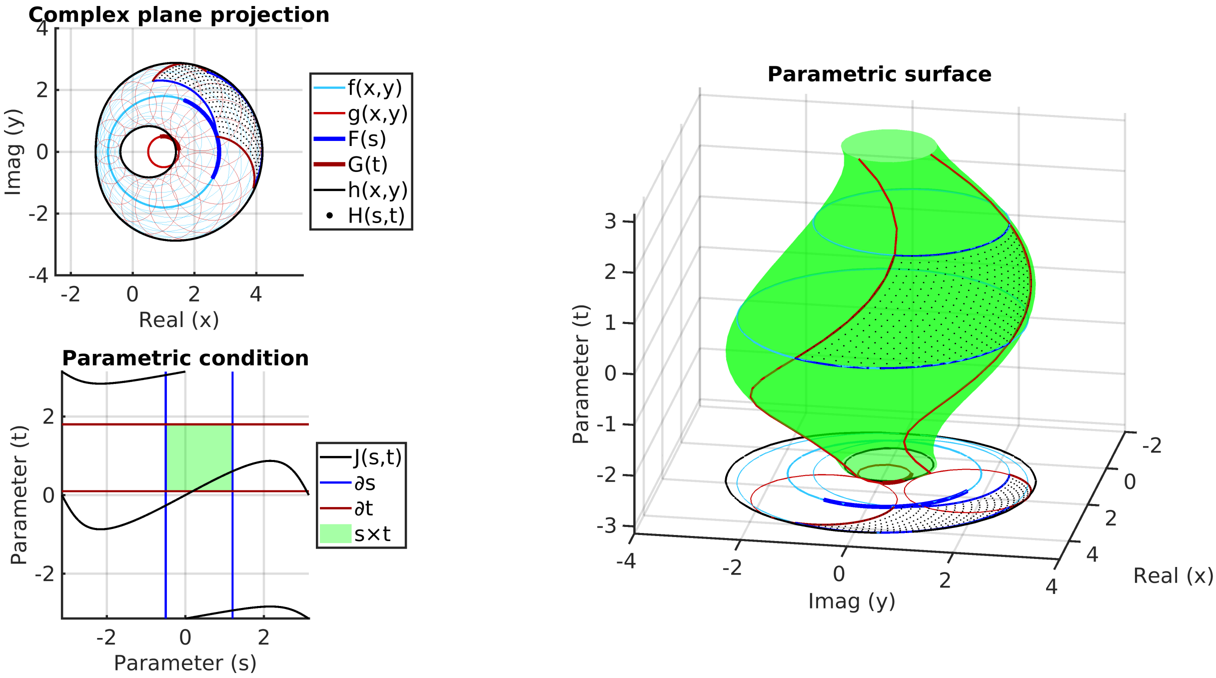

Figure 19: Multiplication of two arcs when none of them is on a zero centered circle. Parameters: $r_1 = 1.8$, $r_2 = 0.5$, $s = (-0.5, 1.2)$, $t = (0.1, 1.8)$.

Operand equations

$$
\begin{aligned}
f(x,y) &= (x-1)^2 + y^2 - r_1^2 \\
\mathring{f}(\rho,\theta) &= \rho^2 - 2\rho\cos(\theta) + 1 - r_1^2 \\
s(x,y) &= \operatorname{atan2}(y, x-1) \\
s^\circ(\rho,\theta) &= \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1) \\
F(s) &= r_1 e^{is} + 1 = (1 + r_1\cos(s), r_1\sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) &= (x-1)^2 + y^2 - r_2^2 \\
g^\circ(\rho,\theta) &= \rho^2 - 2\rho\cos(\theta) + 1 - r_2^2 \\
t(x,y) &= \operatorname{atan2}(y, x-1) \\
t^\circ(\rho,\theta) &= \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1) \\
G(t) &= r_2 e^{\mathrm{it}} + 1 = (1 + r_2\cos(t), r_2\sin(t))
\end{aligned}
$$

Parametric combination

$$H(s,t) = \big(r_1\cos(s) + r_2\cos(t) - r_1 r_2\sin(s)\sin(t) + r_1 r_2\cos(s)\cos(t) + 1,$$

$$
\begin{aligned}
& r_1\sin(s)+r_2\sin(t)+r_1r_2\cos(s)\sin(t)+r_1r_2\sin(s)\cos(t)\big) \\
&= \big(1+r_1\cos(s)+r_2\cos(t)+r_1r_2\cos(s+t), r_1\sin(s)+r_2\sin(t)+r_1r_2\sin(s+t)\big)
\end{aligned}
$$

$$
\begin{aligned}
J(s,t) &= \begin{vmatrix} -r_1\sin(s)-r_1r_2\sin(s+t) & -r_2\sin t - r_1r_2\sin(s+t) \\ r_1\cos(s)+r_1r_2\cos(s+t) & r_2\cos(t)+r_1r_2\cos(s+t) \end{vmatrix} \\
&= r_1r_2\big(\sin(t-s)+r_1\sin((s+t)-s)+r_2\sin(t-(s+t))+r_1r_2\sin((s+t)(s+t))\big) \\
&= r_1r_2\big(\sin(t-s)-r_2\sin(s)+r_1\sin(t)\big)
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\sin(s-t)=r_1\sin(t)-r_2\sin(s)}
$$

Implicit combination

$$
\text{Envelope: } \boxed{\begin{array}{l} y^4+2x^2y^2-4xy^2-2r_2^2r_1^2y^2-2r_1^2y^2-2r_2^2y^2+2y^2 \\ +x^4-4x^3-2r_2^2r_1^2x^2-2r_1^2x^2-2r_2^2x^2+6x^2-4r_2^2r_1^2x+4r_1^2x+4r_2^2x-4x \\ +r_2^4r_1^4-2r_2^2r_1^4+r_1^4-2r_2^4r_1^2+4r_2^2r_1^2-2r_1^2+r_2^4-2r_2^2+1 \end{array}}
$$

Mixed combination

$$
\begin{aligned}
u^\circ(\rho,\theta,t) &= \operatorname{atan2}\Bigg(\frac{\rho}{\sqrt{(1+r_2\cos(t))^2+r_2^2\sin^2(t)}}\sin(\theta-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) \\
&\qquad ,\frac{\rho}{\sqrt{(1+r_2\cos(t))^2+r_2^2\sin^2(t)}}\cos(\theta-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t)))-1\Bigg) \\
&= \operatorname{atan2}\Bigg(\frac{\rho}{\sqrt{r_2^2+2r_2\cos(t)+1}}\sin(\theta-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) \\
&\qquad ,\frac{\rho}{\sqrt{r_2^2+2r_2\cos(t)+1}}\cos(\theta-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t)))-1\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
u(x,y,t) &= \operatorname{atan2}\Bigg(\frac{\sqrt{x^2+y^2}}{\sqrt{r_2^2+2r_2\cos(t)+1}}\sin(\operatorname{atan2}(y,x)-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) \\
&\qquad ,\frac{\sqrt{x^2+y^2}}{\sqrt{r_2^2+2r_2\cos(t)+1}}\cos(\operatorname{atan2}(y,x)-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t)))-1\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
u(x,y,t)=0 &\implies \frac{\sqrt{x^2+y^2}}{\sqrt{r_2^2+2r_2\cos(t)+1}}\cos(\operatorname{atan2}(y,x)-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t)))-1=0 \\
&\implies \cos(\operatorname{atan2}(y,x)-\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) = \frac{\sqrt{r_2^2+2r_2\cos(t)+1}}{\sqrt{x^2+y^2}} \\
&\implies \sin(\operatorname{atan2}(y,x))\sin(\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) \\
&+\cos(\operatorname{atan2}(y,x))\cos(\operatorname{atan2}(r_2\sin(t),1+r_2\cos(t))) = \frac{\sqrt{r_2^2+2r_2\cos(t)+1}}{\sqrt{x^2+y^2}} \\
&\implies \frac{y}{\sqrt{x^2+y^2}}\frac{1+r_2\cos(t)}{\sqrt{r_2^2+2r_2\cos(t)+1}}+\frac{x}{\sqrt{x^2+y^2}}\frac{r_2\sin(t)}{\sqrt{r_2^2+2r_2\cos(t)+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{r_2^2 + 2r_2\cos(t) + 1}}{\sqrt{x^2+y^2}} \\
&\implies y\left(1 + r_2\cos(t)\right) + xr_2\sin(t) = r_2^2 + 2r_2\cos(t) + 1 \\
&\implies y\left(1 + r_2\cos(t)\right) + xr_2\sin(t) - r_2^2 - 2r_2\cos(t) - 1 = 0
\end{aligned}
$$

$$
\begin{aligned}
h(\rho,\theta) &= \frac{\rho^2}{r_2^2 + 2r_2\cos(t) + 1} - \frac{2\rho}{\sqrt{r_2^2 + 2r_2\cos(t) + 1}}\cos\left(\theta - \operatorname{atan2}\left(r_2\sin(t), 1 + r_2\cos(t)\right)\right) \\
&\quad + 1 - r_1^2 = 0 \\
\hat{h}(x,y,t) &= \frac{x^2+y^2}{r_2^2 + 2r_2\cos(t) + 1} - \frac{2\sqrt{x^2+y^2}}{\sqrt{r_2^2 + 2r_2\cos(t) + 1}}\cos\left(\operatorname{atan2}(y,x) - \operatorname{atan2}\left(r_2\sin(t), 1 + r_2\cos(t)\right)\right) \\
&+ 1 - r_1^2 \\
&= \frac{x^2+y^2}{r_2^2 + 2r_2\cos(t) + 1} - \frac{2\sqrt{x^2+y^2}}{\sqrt{r_2^2 + 2r_2\cos(t) + 1}}\left[\frac{y(1 + r_2\cos(t)) + xr_2\sin(t)}{\sqrt{x^2+y^2}\sqrt{r_2^2 + 2r_2\cos(t) + 1}}\cdot\right] + 1 - r_1^2 \\
&= \frac{x^2+y^2}{r_2^2 + 2r_2\cos(t) + 1} - 2\frac{y\left(1 + r_2\cos(t)\right) + xr_2\sin(t)}{r_2^2 + 2r_2\cos(t) + 1} + 1 - r_1^2 \\
\hat{h}(x,y,t) = 0 &\implies x^2 + y^2 - 2y\left(1 + r_2\cos(t)\right) + 2xr_2\sin(t) + \left(1 - r_1^2\right)\left(r_2^2 + 2r_2\cos(t) + 1\right) = 0 \\
&\implies \Big(x - 1 - r_2\cos(t)\Big)^2 + \Big(y - r_2\sin(t)\Big)^2 - r_1^2\left(r_2^2 + 2r_2\cos(t) + 1\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\hat{h}}{\partial t} &= 2r_1^2r_2\sin(t) - 2r_2\cos(t)\left(y - r_2\sin(t)\right) - 2r_2\sin(t)\left(r_2\cos(t) - x + 1\right) \\
\frac{\partial\hat{h}}{\partial t} = 0 &\implies \sin(t)\left(1 - r_1^2 - x\right) + \mathrm{y}\cos(t) = 0 \\
&\implies t = \operatorname{atan}\left(\frac{r_1^2 + x - 1}{y}\right) \\
h(x,y) &= \left(x - 1 - r_2\cos\left(\operatorname{atan}\left(\frac{r_1^2 + x - 1}{y}\right)\right)\right)^2 + \left(y - r_2\sin\left(\operatorname{atan}\left(\frac{r_1^2 + x - 1}{y}\right)\right)\right)^2 \\
&- r_1^2\left(r_2^2 + 2r_2\cos\left(\operatorname{atan}\left(\frac{r_1^2 + x - 1}{y}\right)\right) + 1\right) \\
&= \left(x - 1 - r_2\frac{y}{\sqrt{y^2 + \left(r_1^2 + x + 1\right)^2}}\right)^2 + \left(y - r_2\frac{r_1^2 + x - 1}{\sqrt{y^2 + \left(r_1^2 + x + 1\right)^2}}\right)^2 \\
&- r_1^2\left(r_2^2 + 2r_2\frac{y}{\sqrt{y^2 + \left(r_1^2 + x + 1\right)^2}} + 1\right) \\
h(x,y) = 0 &\implies \left(y - \frac{r_2\,y}{\sqrt{\left({r_1}^2 + x - 1\right)^2 + y^2}}\right)^2 - {r_1}^2\,r_2^2 + \left(x - \frac{r_2\left({r_1}^2 + x - 1\right)}{\sqrt{\left({r_1}^2 + x - 1\right)^2 + y^2}}\right)^2 = 0 \\
&\implies \text{... simplified with Matlab Symbolic Math toolbox}
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\left(\left(x - 1\right)^2 + y^2 - {r_1}^2\left(r_2^2 + 1\right) + r_2^2\right)^2 - 4\,r_2^2\left(\left(x - 1 + {r_1}^2\right)^2 + y^2\right) = 0}
$$

$$
\begin{aligned}
x(s,t) &= (1+r_1\cos(s))\,(1+r_2\cos(t)) - r_1 r_2 \sin(s)\sin(t)\\
y(s,t) &= (1+r_1\cos(s))\,r_2\sin(t) + (1+r_2\cos(t))\,r_1\sin(s)\\
J(s,t) &= h\,((1+r_1\cos(s))\,(1+r_2\cos(t)) - r_1 r_2 \sin(s)\sin(t)\\
&\qquad ,(1+r_1\cos(s))\,r_2\sin(t) + (1+r_2\cos(t))\,r_1\sin(s))\\
&= \Big((r_1\ \sin(s)\ (r_2\ \cos(t)+1) + r_2\ \sin(t)\ (r_1\ \cos(s)+1))^2\\
&+(r_1\,r_2\ \sin(s)\ \sin(t) - (r_1\ \cos(s)+1)\ (r_2\ \cos(t)+1)+1)^2 - {r_1}^2\ \left(r_2^2+1\right) + r_2^2\Big)^2\\
&- 4\,r_2^2\ \Big(((r_1\ \cos(s)+1)\ (r_2\ \cos(t)+1) + {r_1}^2 - r_1\,r_2\ \sin(s)\ \sin(t) - 1)^2\\
&\qquad +(r_1\ \sin(s)\ (r_2\ \cos(t)+1) + r_2\ \sin(t)\ (r_1\ \cos(s)+1))^2\Big)
\end{aligned}
$$

$$
\text{Envelope: } \boxed{\begin{aligned}
4r_2^2 \Big[&\left(r_1 r_2\cos(s) + r_1^2\cos(t) + r_1\cos(s-t) + r_2\right)^2\\
&-(r_1\sin(s) + r_2\sin(t) + r_1 r_2\sin(s+t))^2\\
&-\left(r_1\cos(s) + r_2\cos(t) + r_1 r_2\cos(s+t) + r_1^2\right)^2\Big]
\end{aligned}}
$$

S2.2.3.2 Special case: one arc is on a zero centered circle

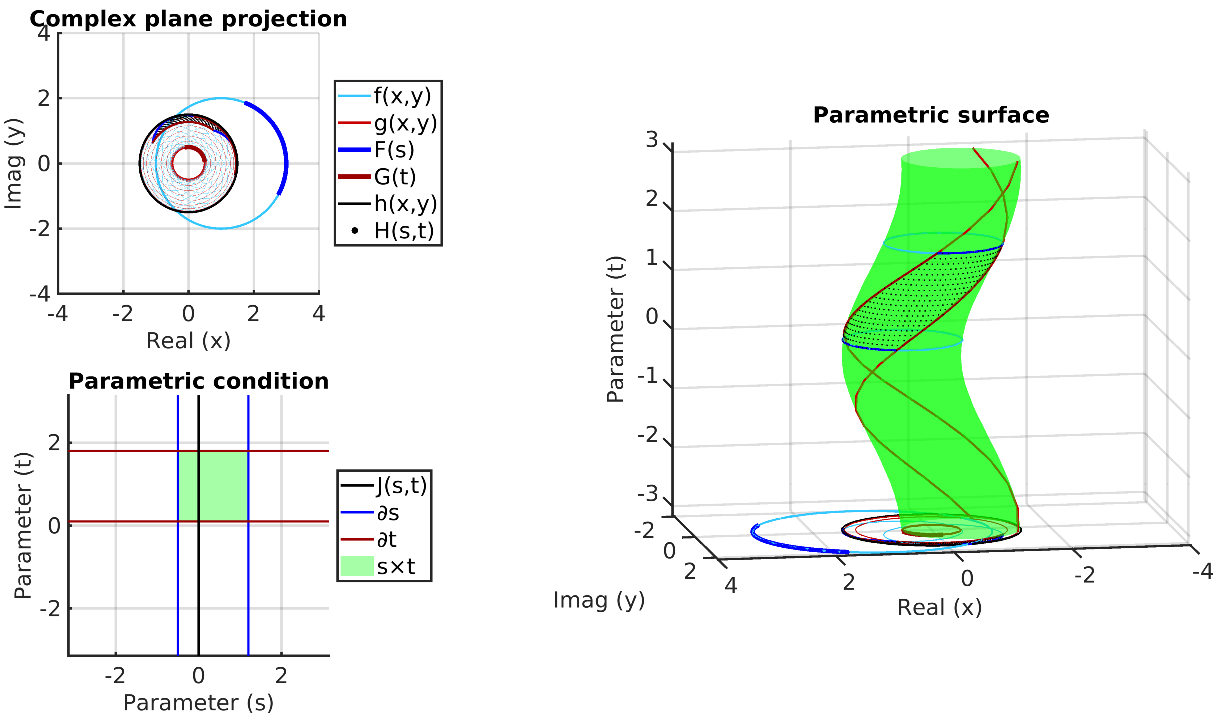

Figure 20: Multiplication of two arcs when one of them is on a zero centered circle. Parameters: $r_1 = 2.0$, $r_2 = 0.5$, $s = (-0.5, 1.2)$, $t = (0.1, 1.8)$.

Operand equations

$$\begin{aligned}
f(x,y) &= (x-1)^2 + y^2 - r_1^2 \\
\mathring{f}\,(\rho,\theta) &= \rho^2 - 2\rho\cos(\theta) + 1 - r_1^2 \\
s(x,y) &= \operatorname{atan2}(y, x-1) \\
s^\circ(\rho,\theta) &= \operatorname{atan2}(\rho\sin(\theta), \rho\cos(\theta) - 1) \\
F(s) &= r_1 e^{is} + 1 = (1 + r_1\cos(s), r_1\sin(s))
\end{aligned}$$

$$\begin{aligned}
g(x,y) &= x^2 + y^2 - r_2^2 \\
g^\circ(\rho,\theta) &= \rho - r_2 \\
t(x,y) &= \operatorname{atan2}(y, x) \\
t^\circ(\rho,\theta) &= \theta \\
G(t) &= r_2 e^{\mathrm{it}} = (r_2\cos(t), r_2\sin(t))
\end{aligned}$$

Parametric combination

$$\begin{aligned}
H(s,t) &= ((r_1\cos(s)+1)\,r_2\cos(t) - r_1 r_2\sin(s)\sin(t), (r_1\cos(s)+1)\,r_2\sin(t) + r_1 r_2\sin(s)\cos(t)) \\
&= r_2\big(\cos(t) + r_1\cos(s+t), \sin(t) + r_1\sin(s+t)\big)
\end{aligned}$$

$$
\begin{aligned}
J(s,t) &= r_2^2 \begin{vmatrix} -\sin(t) - r_1\sin(s+t) & -r_1\sin(s+t) \\ \cos(t) + r_1\cos(s+t) & r_1\cos(s+t) \end{vmatrix} \\
&= r_1 r_2^2(\cos(t)\sin(s+t) - \sin(t)\cos(s+t)) \\
&= r_1 r_2^2 \sin((s+t)-t) \\
&= r_1 r_2^2 \sin(s) \\
\text{Envelope: } & \boxed{s = k\pi \ (k \in \mathbb{Z})}
\end{aligned}
$$

Implicit combination

$$
\begin{aligned}
h(x,y) &= x^4 + y^4 + 2x^2y^2 - 2r_2^2x^2 - 2r_2^2y^2 - 2r_2^2r_1^2x^2 - 2r_2^2r_1^2y^2 - 2r_2^4r_1^2 + r_2^4r_1^4 + r_2^4 \\
&= (x^2+y^2-r_2^2+2r_1r_2^2-r_1^2r_2^2)(x^2+y^2-r_2^2-2r_1r_2^2-r_1^2r_2^2) \\
\text{Envelope: } & \boxed{\left\{x^2+y^2 = (r_2(1-r_1)^2)\right\} \cup \left\{x^2+y^2 = (r_2(1+r_1)^2)\right\}}
\end{aligned}
$$

Mixed combination

$$
\begin{aligned}
u^\circ(\rho,\theta,t) &= \operatorname{atan2}\left(\frac{\rho}{r_2}\sin(\theta-t), \frac{\rho}{r_2}\cos(\theta-t) - 1\right) \\
u^\circ(\rho,\theta,t) = 0 &\implies \frac{\rho}{r_2}\cos(\theta-t) - 1 = 0 \\
&\implies \frac{\rho}{r_2}\left[\cos(\theta)\cos(t) + \sin(\theta)\sin(t)\right] - 1 \\
u(x,y,t) &= \frac{\sqrt{x^2+y^2}}{r_2}\left[\cos(\operatorname{atan2}(y,x))\cos(t) + \sin(\operatorname{atan2}(y,x))\sin(t)\right] - 1 \\
u(x,y,t) = 0 &\implies \sqrt{x^2+y^2}\left(\frac{x\cos(t)}{\sqrt{x^2+y^2}} + \frac{y\sin(t)}{\sqrt{x^2+y^2}}\right) = r_2 \\
&\implies x\cos(t) + y\sin(t) - r_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
h(\rho,\theta) &= \frac{\rho^2}{r_2^2} - 2\frac{\rho}{r_2}\cos(\theta-t) + 1 - r_1^2 \\
h(\rho,\theta) = 0 &\implies \rho^2 - 2r_2\rho\cos(\theta-t) + \left(1-r_1^2\right)r_2 = 0 \\
&\implies \rho^2 - 2r_2\rho\left[\cos(\theta)\cos(t) + \sin(\theta)\sin(t)\right] + \left(1-r_1^2\right)r_2^2 = 0 \\
\hat{h}(x,y,t) &= x^2+y^2 - 2r_2\sqrt{x^2+y^2}\left[\cos(\operatorname{atan2}(y,x))\cos(t) + \sin(\operatorname{atan2}(y,x))\sin(t)\right] + \left(1-r_1^2\right)r_2^2 \\
&= x^2+y^2 - 2r_2\sqrt{x^2+y^2}\left(\frac{x\cos(t)}{\sqrt{x^2+y^2}} + \frac{y\sin(t)}{\sqrt{x^2+y^2}}\right) + \left(1-r_1^2\right)r_2^2 \\
&= x^2+y^2 - 2r_2x\cos(t) - 2r_2y\sin(t) + \left(1-r_1^2\right)r_2^2 \\
\hat{h}(x,y,t) = 0 &\implies (x - r_2\cos(t))^2 + (y - r_2\sin(t))^2 - r_1^2r_2^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\hat{h}}{\partial t} &= 2\,r_2\,\sin(t)\,(x - r_2\,\cos(t)) - 2\,r_2\,\cos(t)\,(y - r_2\,\sin(t)) \\
\frac{\partial\hat{h}}{\partial t} = 0 &\implies 2r_2x\sin(t) = 2r_2y\cos(t)
\end{aligned}
$$

$$
\begin{aligned}
&\implies t = \operatorname{atan}\left(\frac{y}{x}\right)\\
h(x,y) &= \left(x - r_2\cos\left(\operatorname{atan}\left(\frac{y}{x}\right)\right)\right)^2 + \left(y - r_2\sin\left(\operatorname{atan}\left(\frac{y}{x}\right)\right)\right)^2 - r_1^2 r_2^2 = 0\\
&= \left(x - \frac{r_2\, x}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{r_2\, y}{\sqrt{x^2+y^2}}\right)^2 - {r_1}^2\, r_2^2\\
h(x,y) = 0 &\implies x - \frac{r_2 x}{\sqrt{x^2+y^2}} + y - \frac{r_2 y}{\sqrt{x^2+y^2}} \pm r_1 r_2 = 0\\
&\implies x - \frac{r_2 x}{\sqrt{x^2+y^2}} + y - \frac{r_2 y}{\sqrt{x^2+y^2}} \pm r_1 r_2 = 0\\
&\implies \text{... simplified with Matlab Symbolic Math toolbox}\\
\text{Envelope: } &\boxed{x^2 + y^2 - (1 \pm r_1)^2 r_2^2}
\end{aligned}
$$

$$
\begin{aligned}
x(s,t) &= (r_1\cos(s) + 1)\, r_2\cos(t) - r_1 r_2 \sin(s)\sin(t)\\
y(s,t) &= (r_1\cos(s) + 1)\, r_2\sin(t) + r_1 r_2 \sin(s)\cos(t)\\
J(s,t) &= h\left((r_1\cos(s)+1)\, r_2\cos(t) - r_1 r_2\sin(s)\sin(t), (r_1\cos(s)+1)\, r_2\sin(t) + r_1 r_2\sin(s)\cos(t)\right)\\
&= \left(r_2\, \sin(t)\, (r_1\, \cos(s) + 1) + r_1\, r_2\, \cos(t)\, \sin(s)\right)^2 - \left(r_2 + r_1\, r_2\right)^2\\
&+ \left(r_2\, \cos(t)\, (r_1\, \cos(s) + 1) - r_1\, r_2\, \sin(s)\, \sin(t)\right)^2\\
J(s,t) = 0 &\implies 2\, r_1\, r_2^2\, (\cos(s) - 1) = 0\\
\text{Envelope: } &\boxed{s = k\pi\ (k \in \mathbb{Z})}
\end{aligned}
$$

S2.2.3.3 Special case: both arcs are on zero centered circles

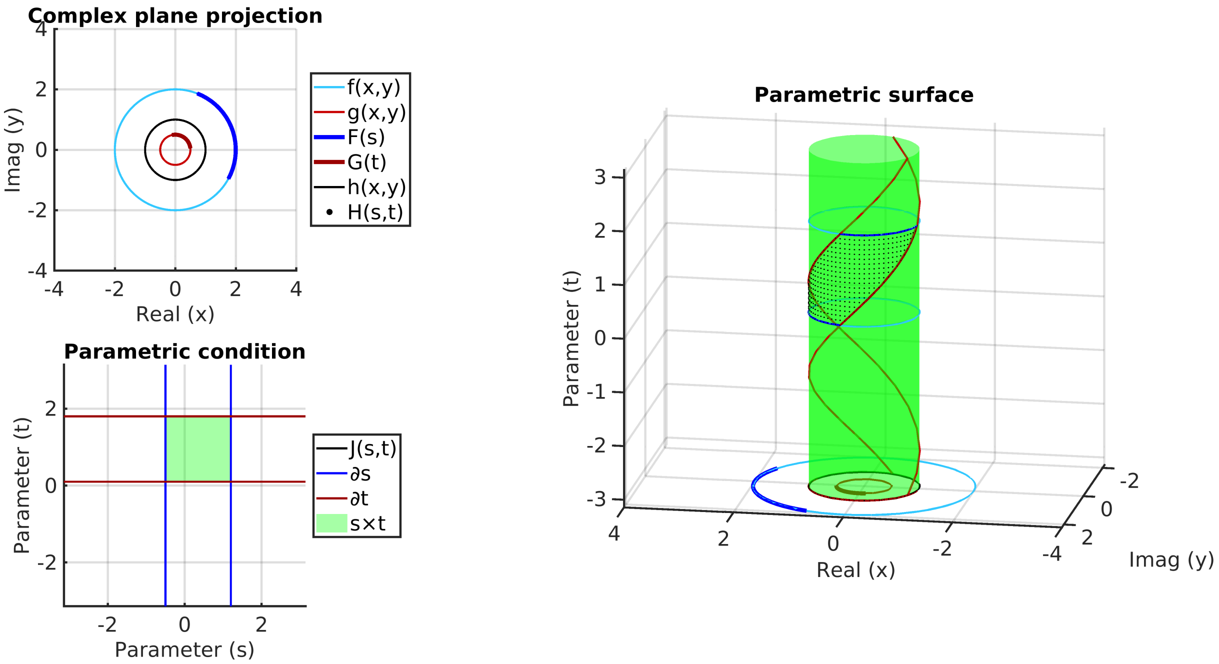

Figure 21: Multiplication of two arcs when both are on zero centered circles. Parameters: $r_1 = 2.0$, $r_2 = 0.5$, $s = (-0.5, 1.2)$, $t = (0.1, 1.8)$.

Operand equations

$$
\begin{aligned}
f(x,y) &= x^2 + y^2 - r_1^2 \\
\mathring{f}(\rho,\theta) &= \rho - r_1 \\
s(x,y) &= \operatorname{atan2}(y,x) \\
s^\circ(\rho,\theta) &= \theta \\
F(s) &= r_1 e^{is} = (r_1\cos(s), r_1\sin(s))
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) &= x^2 + y^2 - r_2^2 \\
g^\circ(\rho,\theta) &= \rho - r_2 \\
t(x,y) &= \operatorname{atan2}(y,x) \\
t^\circ(\rho,\theta) &= \theta \\
G(t) &= r_2 e^{\mathrm{it}} = (r_2\cos(t), r_2\sin(t))
\end{aligned}
$$

Parametric combination

$$
\begin{aligned}
H(s,t) &= (r_1 r_2\cos(s)\cos(t) - r_1 r_2\sin(s)\sin(t), r_1 r_2\cos(s)\sin(t) + r_1 r_2\sin(s)\cos(t)) \\
&= r_1 r_2(cos(s+t), sin(s+t)) \\
J(s,t) &= r_1^2 r_2^2 \begin{vmatrix} -\sin(s+t) & -\sin(s+t) \\ \cos(s+t) & \cos(s+t) \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= 0 \\
\text{Envelope: } &\boxed{\text{all } (s,t)}
\end{aligned}
$$

Implicit combination

$$
\begin{aligned}
h(x,y) &= x^2 + y^2 - r_1^2 r_2^2 \\
\text{Envelope: } &\boxed{x^2 + y^2 = (r_1 r_2)^2}
\end{aligned}
$$

Mixed combination

$$
\begin{aligned}
u^\circ(\rho,\theta,t) &= \theta - t \\
u(x,y,t) &= \operatorname{atan2}(y,x) - t
\end{aligned}
$$

$$
\begin{aligned}
h(\rho,\theta) &= \frac{\rho}{r_2} - r_1 \\
h(\rho,\theta) = 0 &\implies \rho - r_1 r_2 = 0 \\
\hat{h}(x,y,t) &= \sqrt{x^2 + y^2} - r_1 r_2 \\
\hat{h}(x,y,t) = 0 &\implies x^2 + y^2 - r_1^2 r_2^2 = 0 \\
\frac{\partial \hat{h}}{\partial t} &= 0 \\
h(x,y) &= x^2 + y^2 - r_1^2 r_2^2 \\
\text{Envelope: } &\boxed{x^2 + y^2 = (r_1 r_2)^2}
\end{aligned}
$$

$$
\begin{aligned}
x(s,t) &= r_1 r_2 \cos(s)\cos(t) - r_1 r_2 \sin(s)\sin(t) \\
y(s,t) &= r_1 r_2 \cos(s)\sin(t) + r_1 r_2 \sin(s)\cos(t) \\
J(s,t) &= h\left(r_1 r_2 \cos(s)\cos(t) - r_1 r_2 \sin(s)\sin(t), r_1 r_2 \cos(s)\sin(t) + r_1 r_2 \sin(s)\cos(t)\right) \\
&= \left(r_1\, r_2\, \cos(s)\, \sin(t) + r_1\, r_2\, \cos(t)\, \sin(s)\right)^2 + \left(r_1\, r_2\, \sin(s)\, \sin(t) - r_1\, r_2\, \cos(s)\, \cos(t)\right)^2 - {r_1}^2\, r_2^2 \\
&= 0 \\
\text{Envelope: } &\boxed{\text{all } (s,t)}
\end{aligned}
$$