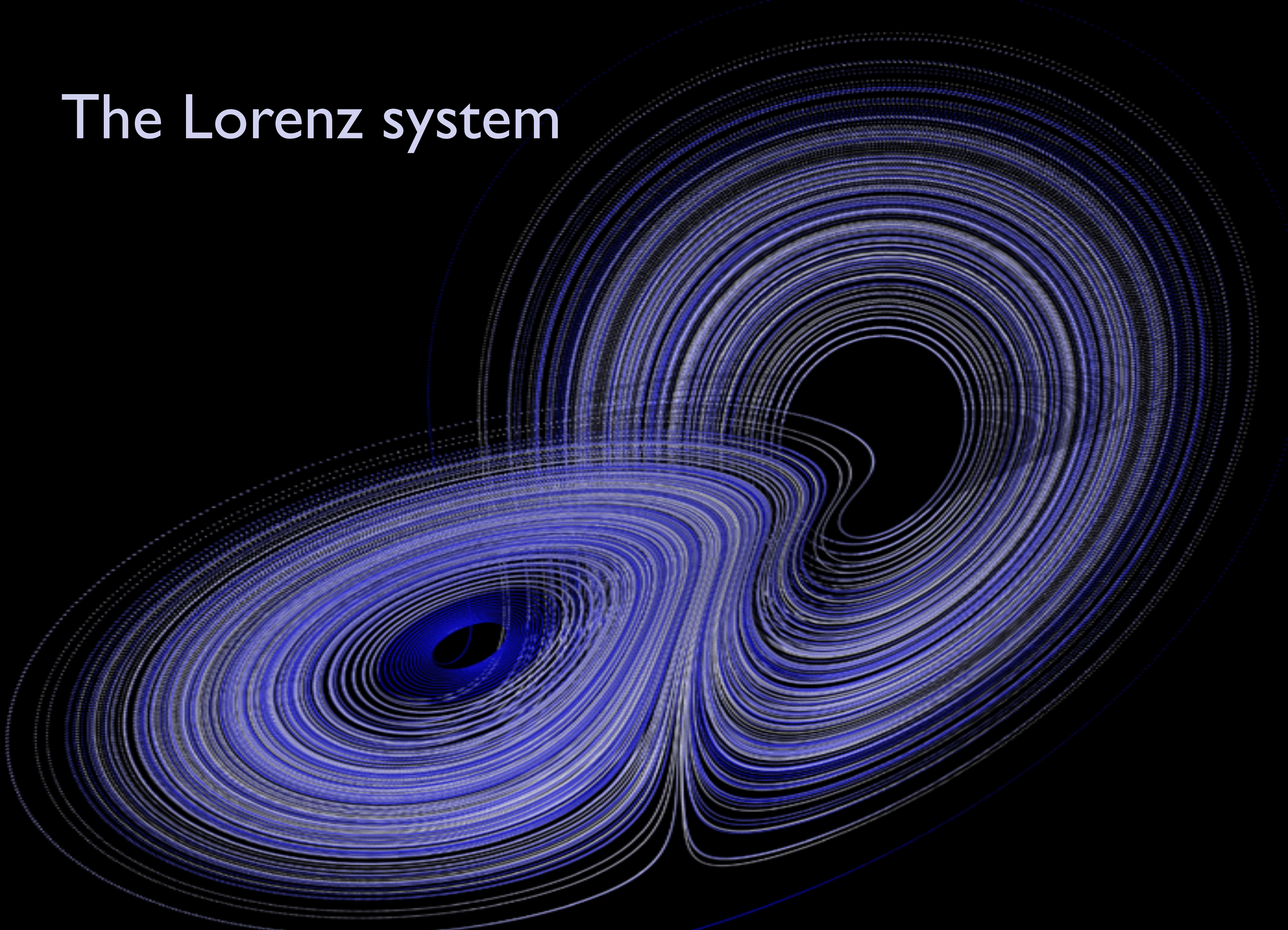
The Lorenz system

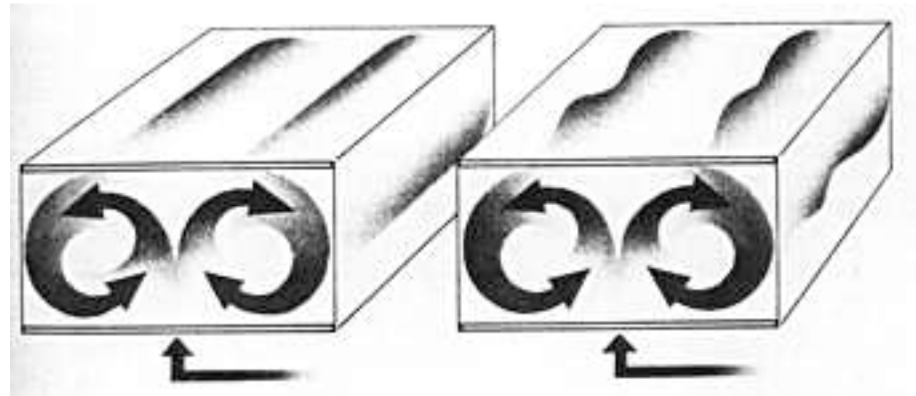


Edward Lorenz

- Professor of Meteorology at the Massachusetts Institute of Technology
- In 1963 derived a three dimensional system in efforts to model long range predictions for the weather
- The weather is complicated! A theoretical simplification was necessary

The Lorenz system

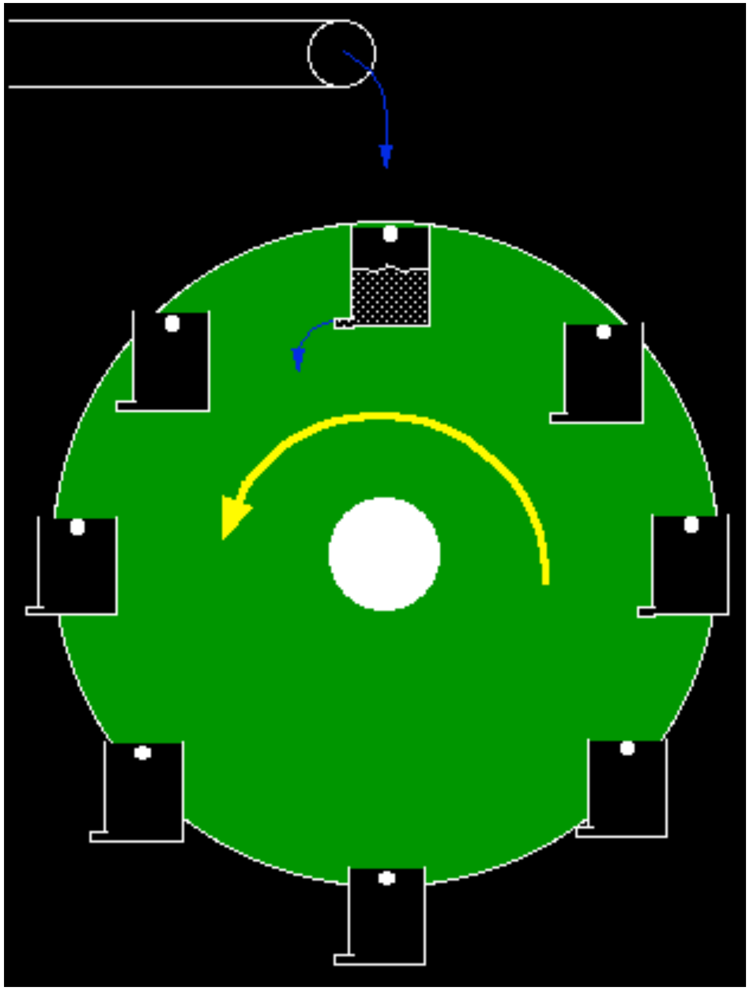
- Le temperature delle due superfici sono fissate
- Assenza di flusso attraverso le 2 superfici



- The Lorenz systems describes the motion of a fluid between two layers at different temperature. Specifically, the fluid is heated uniformly from below and cooled uniformly from above.
- By rising the temperature difeeerence between the two surfaces, we observe initially a linear temperature gradient, and then the formation of Rayleigh-Bernard convection cells. After convection, turbolent regime is also observed.

D. Gulick, *Encounters with Chaos*, Mc-Graw Hill, Inc., New York, 1992.

La Lorenz water wheel



http://www.youtube.com/watch?v=zhOBibeW5J0

The equations

- x proportional to the velocity field
- y proportional to the difference of temperature T_1-T_2
- z proportional to the distortion of the vertical profile of temperature
- σ depends on the type of fluid
- b depends on the geometry
- r depends on the Rayleigh, number which influences the change of behavior from conductive to convective

$$\frac{dx}{dt} = \sigma\left(y - x\right)$$

$$\frac{dy}{dt} = -xz + rx - y$$

$$\frac{dz}{dt} = xy - bz$$

σ, r, b are positive parameters

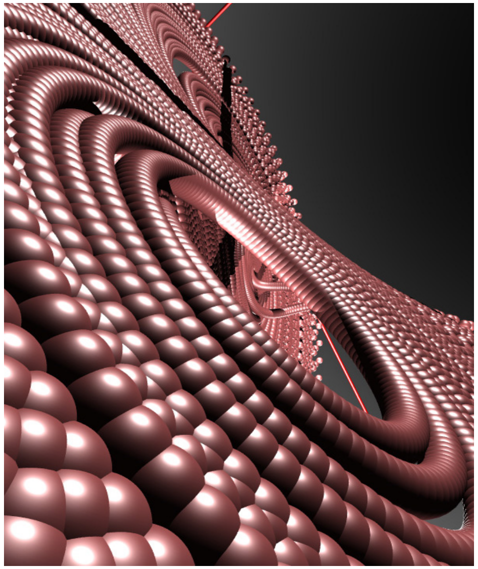
σ=10, b=8/3, r varies in [0,30]

Steady states

$$S1 = (0, 0, 0)$$

$$S2 = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$

$$S3 = \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$



Linear stability of S1

Jacobian

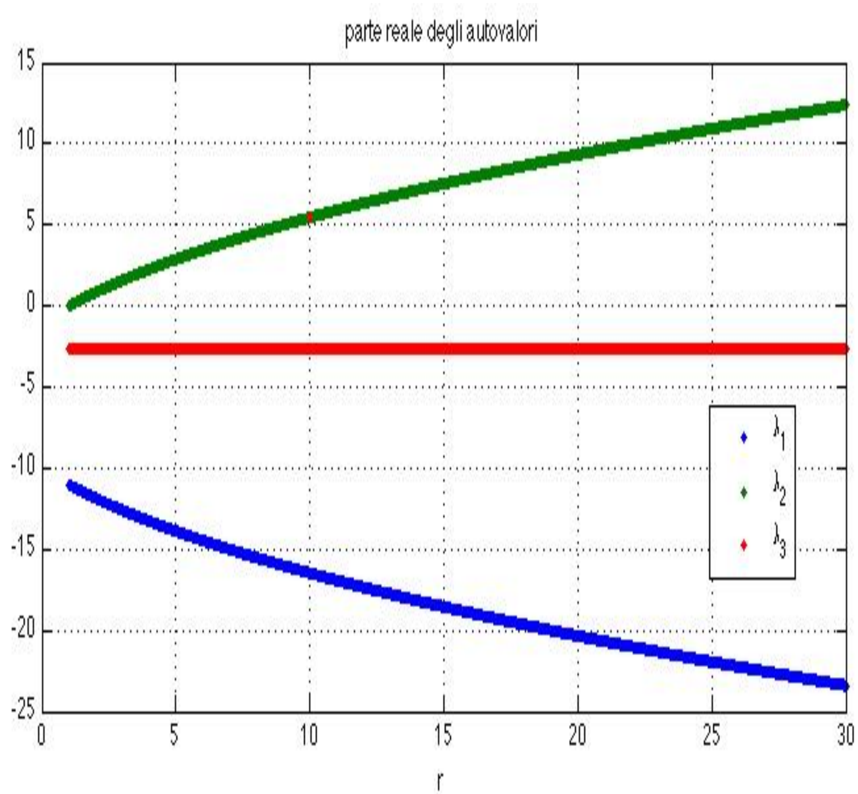
$$J(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

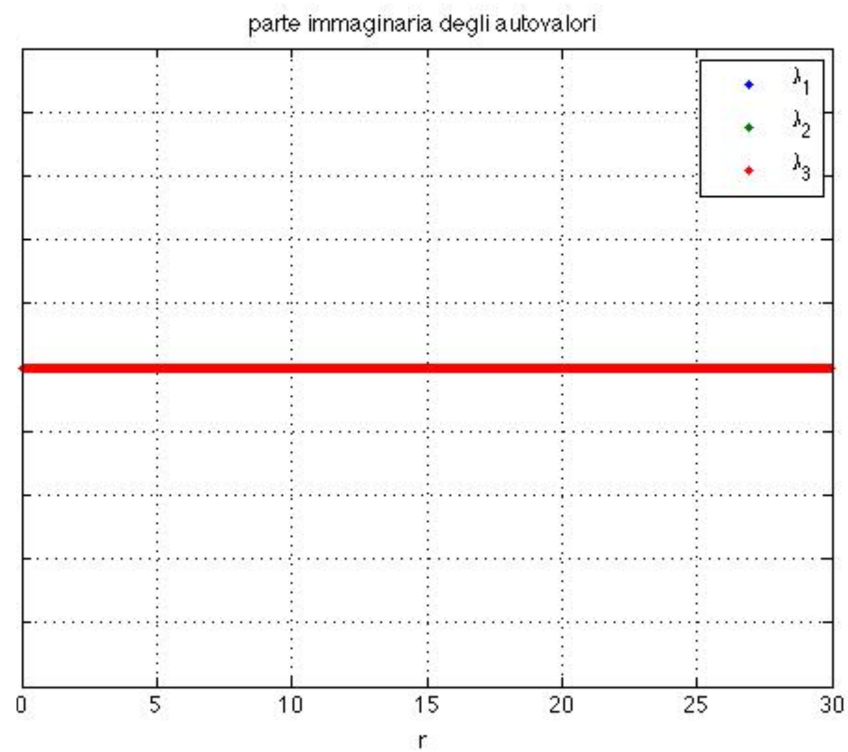
Eigenvalues of *J*

$$\lambda_{1,2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r}}{2}$$

$$\lambda_3 = -b$$

Linear stability of S1



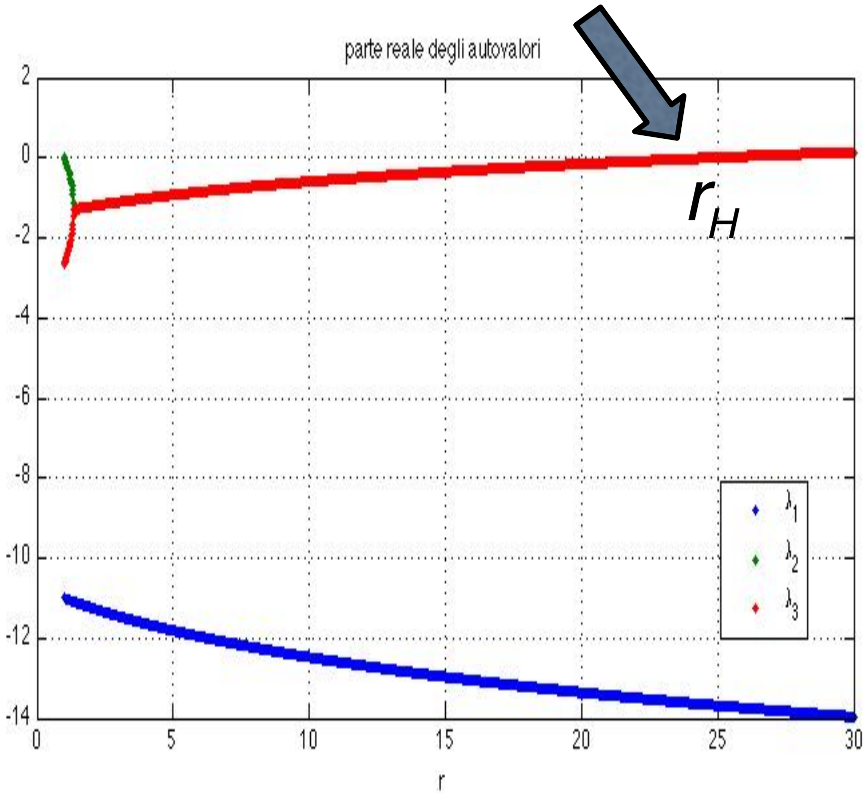


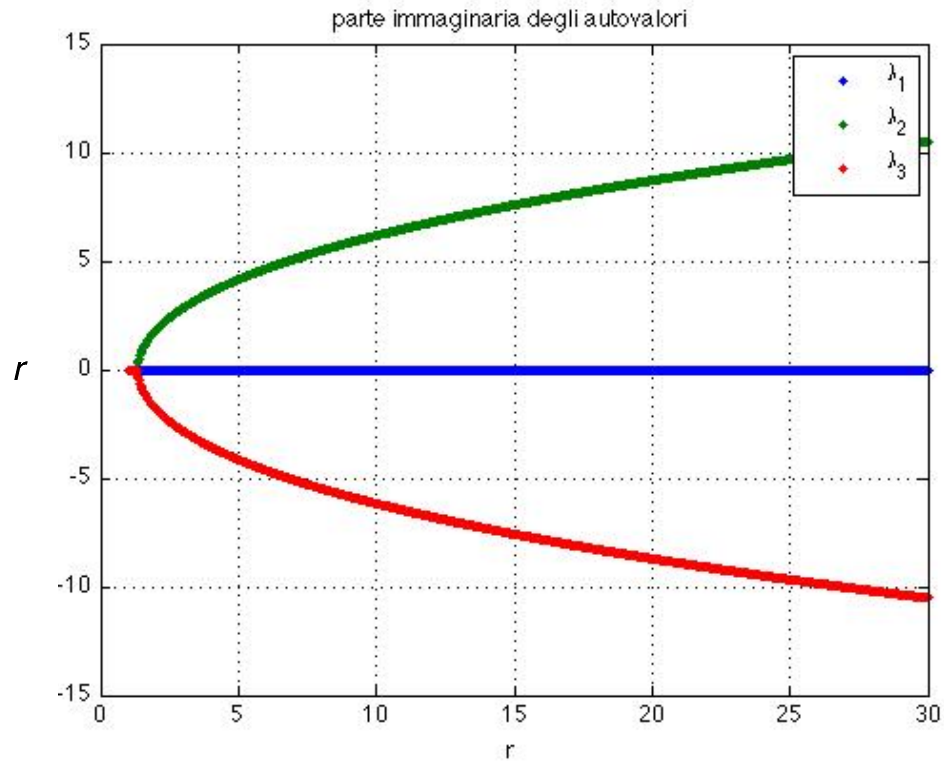
Linear stability of S1

$$\lambda_{1,2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r}}{2}$$

Valori di r	**Comportamento**
$0 < r < 1$	$\sqrt{(\sigma-1)^2 + 4\sigma r} \leq \sigma + 1$ Negative eigenvalues: asymptotic stability
$r = 1$	$\lambda_1 = 0 \quad \lambda_2 = -(\sigma+1) \quad \lambda_3 = -b$ • *Marginal stability*
$r > 1$	$\sqrt{(\sigma-1)^2 + 4\sigma r} > \sigma + 1 \quad \lambda_1 > 0$ *Positive eigenvalues: instability*

Stabilità lineare di S2 e S3





$$r_H = \frac{\sigma\left(\sigma + b + 3\right)}{\sigma - b - 1} = 24.74$$

Linear stability of S2 and S3

The eigenvalues of J(S2) e J(S3) coincide and the associated eigenvalues are linearly independent

Eigenvalues as functions of r

r	λ_1	λ_2	λ_3
5	-11.809	$-0.92872 + i\ 4.1476$	$-0.92872 - i\ 4.1476$
20	-13.357	$-0.15479 + i\ 8.7087$	$-0.15479 - i\ 8.7087$
24.7368	-13.667	$-1.2725\ 10^{-6} + i\ 9.6245$	$-1.2725\ 10^{-6} - i\ 9.6245$
24.74	-13.667	$9.5435\ 10^{-5} + i\ 9.6251$	$9.5435\ 10^{-5} - i\ 9.6251$
28	-13.855	$0.093956 + i\ 10.195$	$0.093956 - i\ 10.195$

Close to r=24.74 the real part of the eigenvalues are positive From this value of r all the equilibria are unstable

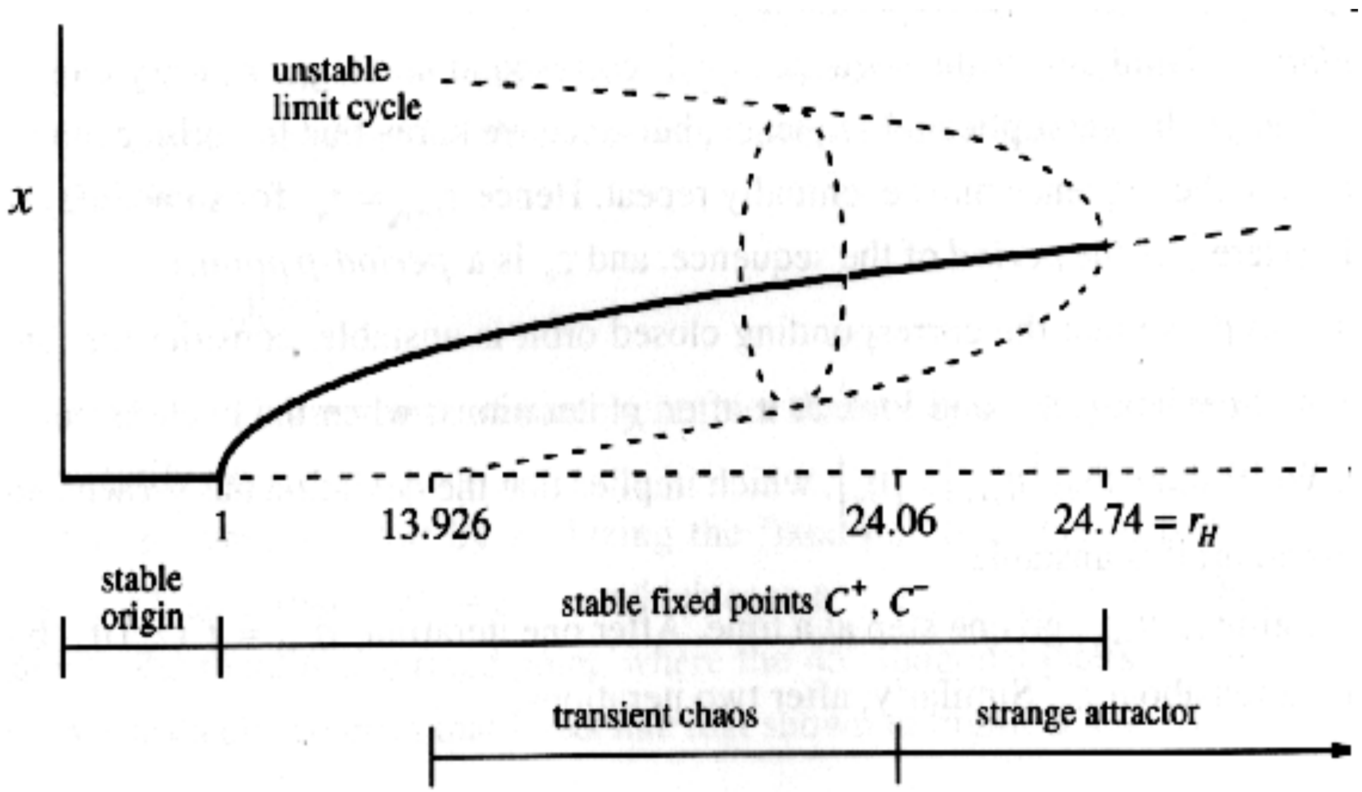
Summary

r	S	λ	stability
	S_1	$\lambda_1,\ \lambda_2,\ \lambda_3 < 0$	attractive
0<r<1	S_2	–	non existence
	S_3	–	non existence
	S_1	$\lambda_1,\ \lambda_2,\ \lambda_3 < 0$	attractive
r=1	S_2	$\lambda_1,\ \lambda_2,\ \lambda_3 < 0$	attractive
	S_3	$\lambda_1,\ \lambda_2,\ \lambda_3 < 0$	attractive
	S_1	$\lambda_1,\ \lambda_2 <0,\ \lambda_3 >0$	repelling
1<r<24,74	S_2	λ_1, $P\varepsilon(\lambda_2)$, $P\varepsilon(\lambda_3) < 0$	attractive
	S_3	λ_1, $P\varepsilon(\lambda_2)$), $P\varepsilon(\lambda_3) < 0$	attractive
24,74<r<30	λ_1, $P\varepsilon(\lambda_2)$, $P\varepsilon(\lambda_3) >0$ deterministic *chaos*		
r>30	*Chaotic regime with periodic windows*		

Bifurcations

r = 1	S1 become unstable; S2, S3 are stable	Supercritical pitchfork bifurcation
r = 24.74	S2 e S3 become unstable, but no limit cycle is observed	S2 and S3 undergo a subcritical Hopf bifurcation

Bifurcation diagram



The chaotic regime

The trajectories are neither diverging neither converging

There are no attracting steady states or limit cycles of any period

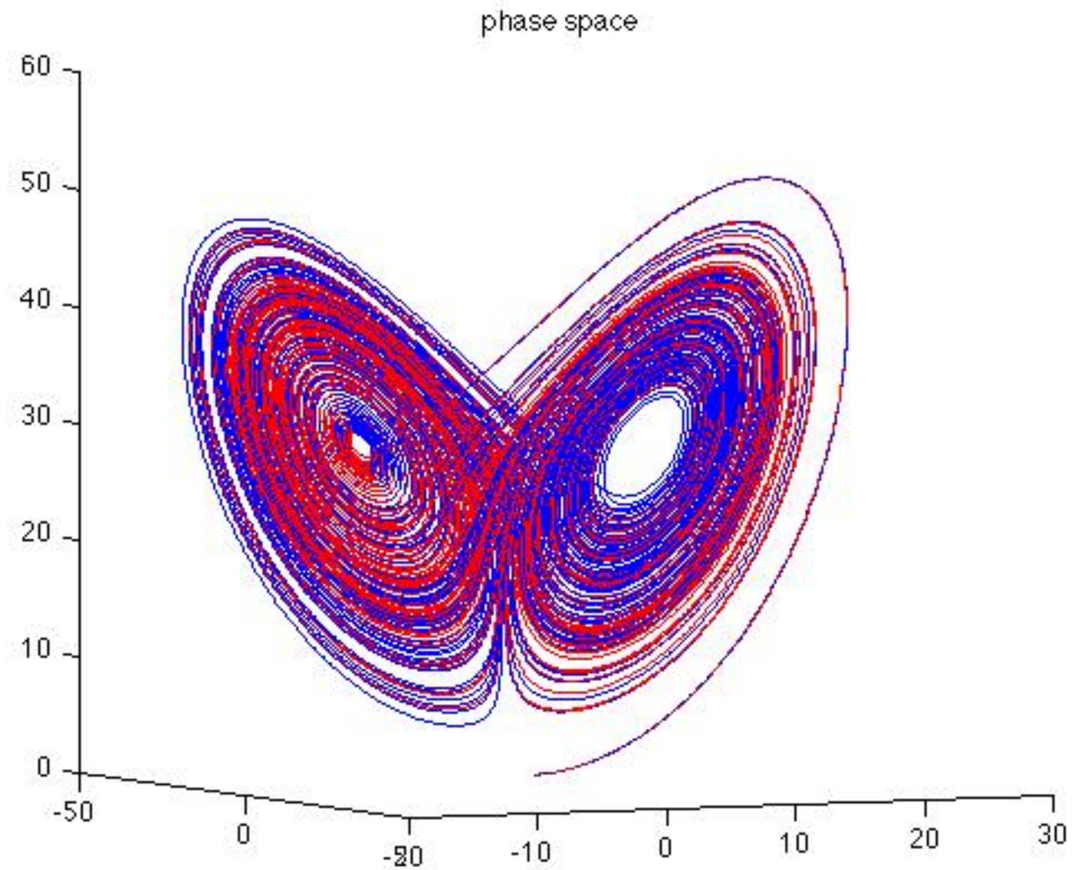
The tractors are repelled from one unstable steady state or limit cycle to another

The trajectories are confined to a bounded region of the phase space

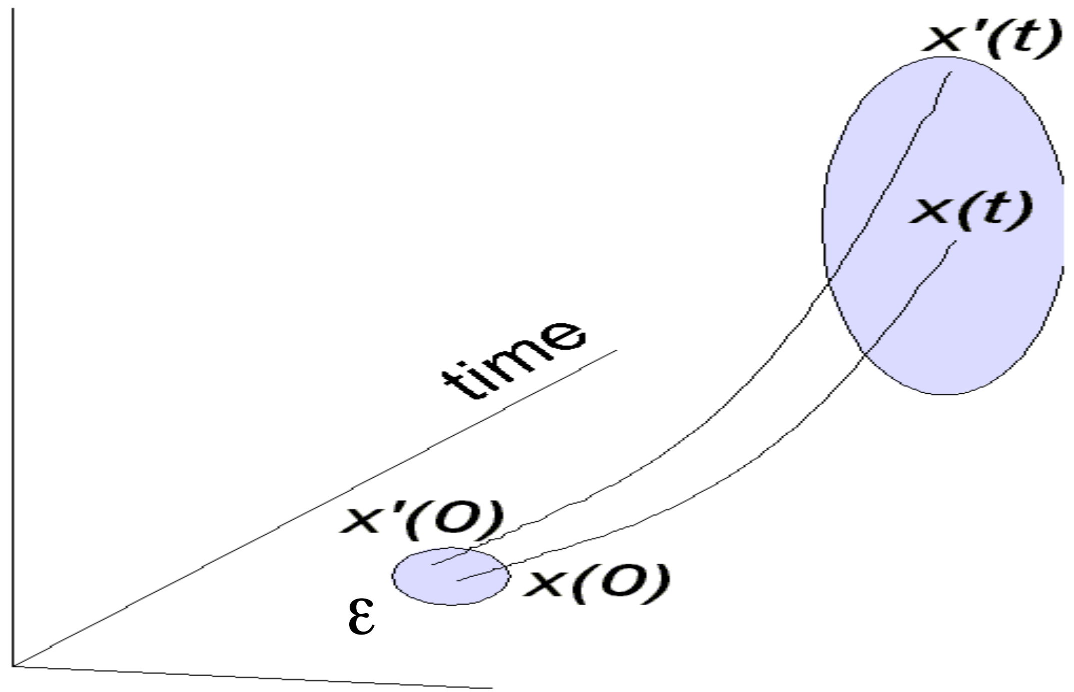
The trajectories can not intersect themselves

The Lorenz attractor

- *“One meteorologist remarked that if the theory were correct, one flap of a seagull's wings would be enough to alter the course of the weather forever”.*



The Lyapunov exponents



$$|\delta x(t)| = |x(t) - x'(t)| \approx \varepsilon e^{\lambda t}$$

$$\varepsilon = \delta x(0) = \delta_0$$

Meaning of Lyapunov exponents

- $\lambda < 0$
  The orbit attracts to a stable fixed point or stable periodic orbit. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems (the damped harmonic oscillator for instance). Such systems exhibit asymptotic stability; the more negative the exponent, the greater the stability.
- $\lambda = 0$
  The orbit is a neutral fixed point. A Lyapunov exponent of zero indicates that the system is in some sort of steady state mode. A physical system with this exponent is conservative. Such systems exhibit Lyapunov stability. Take the case of two identical simple harmonic oscillators with different amplitudes. Because the frequency is independent of the amplitude, a phase portrait of the two oscillators would be a pair of concentric circles. The orbits in this situation would maintain a constant separation, like two flecks of dust fixed in place on a rotating record.
- $\lambda > 0$
  The orbit is unstable and chaotic. Nearby points, no matter how close, will diverge to any arbitrary separation. All neighborhoods in the phase space will eventually be visited. These points are said to be unstable. Although the system is deterministic, there is no order to the orbit that ensues.

Definition of deterministic chaos

Deterministic chaos is an asymptotic behavior produced by a deterministic system showing sensibility to initial conditions

Sensitivity to initial conditions