10. BODE PLOTS

Topics:
- Bode plots

Objectives:
- To be able to describe the response of a system using Bode plots

10.1 INTRODUCTION

When a phasor transform is applied to a transfer function the result can be expressed as a magnitude and angle that are functions of frequency. The magnitude is the gain, and the angle is the phase shift. In the previous chapter these values were calculated for a single frequency and then multiplied by the input values to get an output value. At different frequencies the transfer function value will change. The transfer function gain and phase angle can be plotted as a function of frequency to give an overall picture of system response.

Aside: Consider a 'graphic equalizer' commonly found on home stereo equipment. The spectrum can be adjusted so that high or low tones are emphasized or muted. The position of the sliders adjusts the envelope that the audio signal is filtered through. The sliders trace out a Bode gain plot. In theoretical terms the equalizer can be described with a transfer function. As the slides are moved the transfer function is changed, and the bode plot shifts. In the example below the slides are positioned to pass more of the lower frequencies. The high frequencies would not be passed clearly, and might sound somewhat muffled.

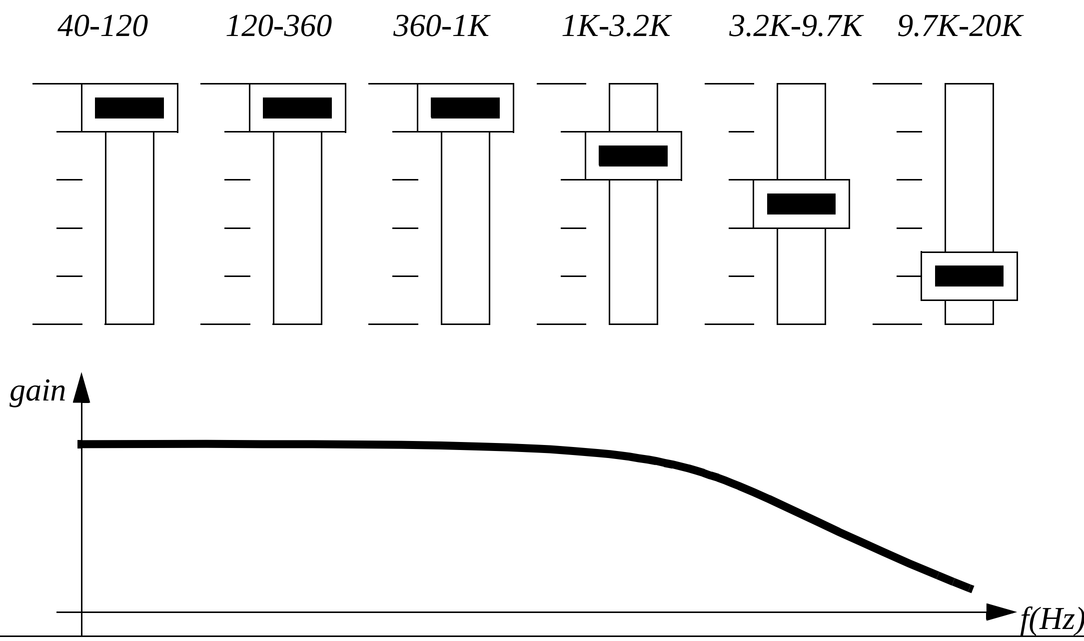


Figure 10.1 Commonly seen Bode plot

The mass-spring-damper transfer function from the previous chapter is expanded in Figure 10.2. In this example the transfer function is multiplied by the complex conjugate to eliminate the complex number in the denominator. The magnitude of the resulting transfer function is the gain, and the phase shift is the angle. Note that to correct for the quadrant of the phase shift angles pi radians is subtracted for certain frequency values.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}$$

$$\frac{x(\omega)}{F(\omega)} = \left[\frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}\right]\left[\frac{(-j)(3000\omega) + (2000 - 1000\omega^2)}{(-j)(3000\omega) + (2000 - 1000\omega^2)}\right]$$

$$\frac{x(\omega)}{F(\omega)} = \frac{(2000 - 1000\omega^2) - j(3000\omega)}{(2000 - 1000\omega^2)^2 + (3000\omega)^2}$$

$$\left|\frac{x(\omega)}{F(\omega)}\right| = \frac{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}{(2000 - 1000\omega^2)^2 + (3000\omega)^2} = \frac{1}{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}$$

$$\theta = \operatorname{atan}\left(\frac{-3000\omega}{-1000\omega^2 + 2000}\right) = \operatorname{atan}\left(\frac{-3\omega}{2-\omega^2}\right) \qquad for\ (\omega \le 2)$$

$$\theta = \operatorname{atan}\left(\left(\frac{-3\omega}{2-\omega^2}\right) - \pi\right) \qquad for\ (\omega > 2)$$

Figure 10.2 A phasor transform example

The results in Figure 10.2 are normally left in variable form so that they may be analyzed for a range of frequencies. An example of this type of analysis is done in Figure 10.3. A set of frequencies is used for calculations. These need to be converted from Hz to rad/s before use. For each one of these the gain and phase angle is calculated. The gain gives a ratio between the input sine wave and output sine wave of the system. The magnitude of the output wave can be calculated by multiplying the input wave magnitude by the gain. (Note: recall this example was used in the previous chapter) The phase angle can be added to the input wave to get the phase of the output wave. Gain is normally converted to 'dB' so that it may cover a larger range of values while still remaining similar numerically. Also note that the frequencies are changed in multiples of tens, or magnitudes.

f(Hz)	*(rad/sec)*	*Gain*	*Gain (dB)*	θ *(rad.)*	θ *(deg.)*
0	*0*				
0.001	*0.006283*				
0.01	*0.06283*				
0.1	*0.6283*				
1	*6.283*				
10	*62.83*				
100	*628.3*				
1000	*6283*				

Note: the gain values will cover many magnitudes of values. To help keep the graphs rational we will "compress" the values by converting them to dB (decibels) using the following formula.

$$gain_{db} = 20\log(gain)$$

Note: negative phase angles mean that the mass motion lags the force.

Note: The frequencies chosen should be chosen to cover the points with the greatest amount of change.

Figure 10.3 A phasor transform example (continued)

In this example gain is defined as x/F. Therefore F is the input to the system, and x is the resulting output. The gain means that for each unit of F input to the system, there will be gain*F=x output. If the input and output are sinusoidal, there is a difference in phase between the input and output wave of θ (the phase angle). This is shown in Figure 10.4, where an input waveform is supplied with three sinusoidal components. For each of the frequencies a gain and phase shift are calculated. These are then used to calculate the resulting output wave, relative to the input wave. The resulting output represents the steady-state response to the sinusoidal output.

Assuming there is excitation from two sinusoidal sources in addition to the static load, as defined by the equation,

$$F(t) = \underbrace{1000} + \underbrace{20\sin(20t)} + \underbrace{10\sin(0.5t)}(N)$$

The result for each component can be evaluated separately

$$\begin{aligned} x(t) = {} & 1000(\_\_\_\_\_\_)\sin\left(0t + \_\_\_\_\_\left(\frac{2\pi}{360}\right)\right) \\ & + 20(\_\_\_\_\_\_)\sin\left(20t + \_\_\_\_\_\left(\frac{2\pi}{360}\right)\right) \\ & + 10(\_\_\_\_\_\_)\sin\left(0.5t - \_\_\_\_\_\left(\frac{2\pi}{360}\right)\right) \end{aligned}$$

gain from calculations or Bode plot

phase angle from calculations or Bode plot

$$x(t) = 0.5 + \_\_\_\_\_\sin(20t + \_\_\_\_\_) + \_\_\_\_\_\sin(0.5t + \_\_\_\_\_)$$

Figure 10.4 A phasor transform example (continued)

10.2 BODE PLOTS

In the previous section we calculated a table of gains and phase angles over a range of frequencies. Graphs of these values are called Bode plots. These plots are normally done on semi-log graph paper, such as that seen in Figure 10.5. Along the longer axis of this paper the scale is logarithmic (base 10). This means that if the paper started at 0.1 on one side, the next major division would be 1, then 10, then 100, and finally 1000 on the other side of the paper. The basic nature of logarithmic scales prevents the frequency from being zero. Along the linear axis (the short one) the gains and phase angles are plotted, normally with two graphs side-by-side on a single sheet of paper.

bode plots - 10.6

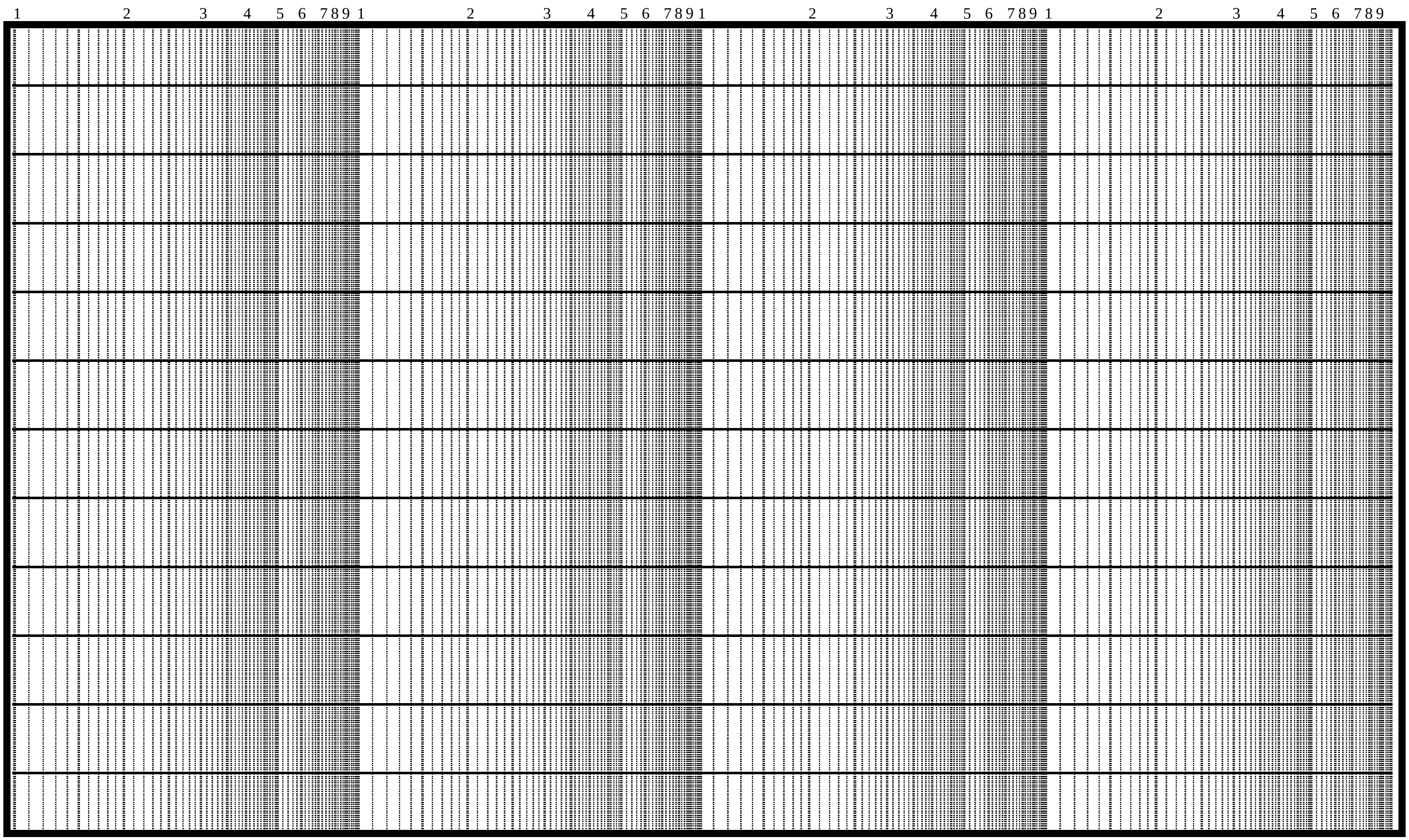


Figure 10.5 4 cycle semi-log graph paper

Plot the points from Figure 10.3 in the semi-log graph paper in Figure 10.5. The general layout is pictured below.

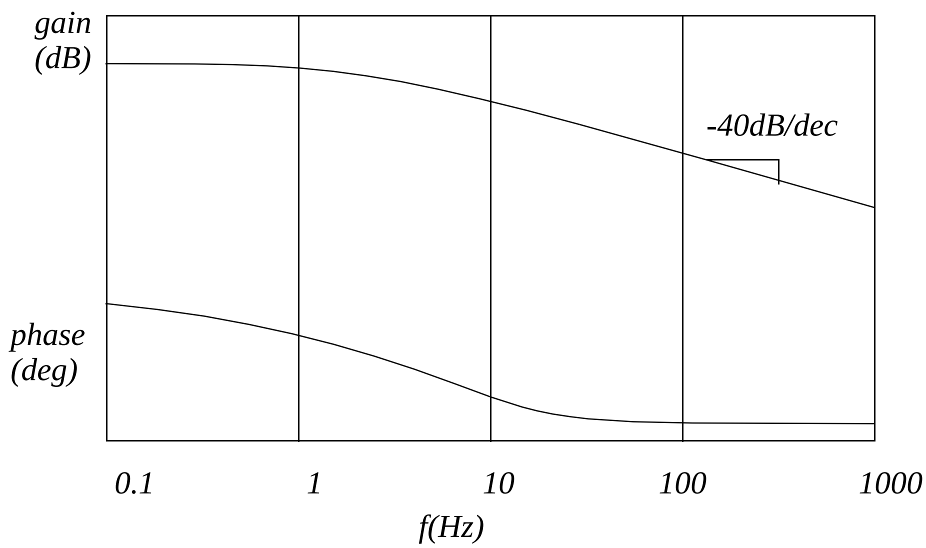


Figure 10.6 Drill problem: Plot the points from Figure 10.3 on graph paper

```
steps_per_dec = 6;
decades = 6;
start_freq = 0.1;

// the transfer function
function foo=G(w)
	D = %i * w;
	foo = (D + 5) / (D^2 + 100*D + 10000);
endfunction

// this section writes the values to a datafile that may be graphed in a spreadsheet
fd = mopen("data.txt", "w");
for step = 0:(steps_per_dec * decades),
	f = start_freq * 10 ^ (step / steps_per_dec); // calculate the next frequency
	w = f * 2 * %pi);				// convert the frequency to radians
	[gain, phase] = polar(G(w));// find the gain and convert it to mag and angle
	gaindb = 20 * log10(gain);// convert magnitude to dB
	phasedeg = 180 * phase / %pi;// convert to degrees
	mfprint(fd, "%f, %f, %f \n", f, gaindb, phasedeg);
end
mclose(fd);

// to graph it directly the following is used
D = poly(0, 'D');
h = syslin('c', (D + 5) / (D^2 + 100*D + 10000) ) ;
bode(h, 0.1, 1000, 'Sample Transfer Function');
```

Figure 10.7 Aside: Bode Plot Example with Scilab

Use computer software, such as Mathcad or a spreadsheet, to calculate the points in Figure 10.2, and then draw Bode plots. Most software will offer options for making one axis use a log10 scale.

Figure 10.8 Drill problem: Plot the points from Figure 10.2 with a computer

Draw the Bode plot for the transfer function by hand or with computer.

$$\frac{D+3}{D^2 + 10000D + 10000}$$

Figure 10.9 Drill problem: Draw the Bode plot for gain and phase

An approximate technique for constructing a gain Bode plot is shown in Figure 10.10. This method involves looking at the transfer function and reducing it to roots in the numerator and denominator. Once in that form, a straight line approximation for each term can be drawn on the graph. An initial gain is also calculated to shift the results up or down. When done, the straight line segments are added to produce a more complex straight line curve. A smooth curve is then drawn over top of this curve.

Bode plots for transfer functions can be approximated with the following steps.

1. Plot the straight line pieces.

a) The gain at 0rad/sec is calculated and used to find an initial offset. For example this transfer function starts at 10(D+1)/(D+1000)=10(0+1)/(0+1000)=0.01=-40dB.

b) Put the transfer function in root form to identify corner frequencies. For example (D+1)/(D+1000) will have corner frequencies at 1 and 1000 rad/sec.

c) Curves that turn up or down are drawn for each corner frequency. At each corner frequency a numerator term causes the graph to turn up, each term in the dominator causes the graph to turn down. The slope up or down is generally +/- 20dB/decade for each term. Also note that squared (second-order) terms would have a slope of +/-40dB/decade.

2. The effect of each term is added up to give the resulting straight line approximation.

3. When the smooth curve is drawn, there should commonly be a 3dB difference at the corner frequencies. In second-order systems the damping coefficient make may the corner flatter or peaked.

Figure 10.10 The method for Bode graph straight line gain approximation

Note: Some of the straight line approximation issues are discussed below.

Why is there 3dB between a first order corner and the smooth plot, and the phase angle is 45 degrees of the way to +/- 90 degrees.

$$G(j\omega) = \frac{1}{\omega_c + j\omega}$$

the initial gain is

$$G(0) = \frac{1}{\omega_c + j0} = \frac{1}{\omega_c}$$

at the corner frequency $\omega_c = \omega$

$$G(j\omega) = \frac{1}{\omega_c + j\omega_c} = \frac{1}{\omega_c\sqrt{2}\angle\frac{\pi}{4}} = \frac{1}{\omega_c\sqrt{2}}\angle-\frac{\pi}{4}$$

Therefore the difference is

$$diff = \frac{G(j\omega)}{G(0)} = \frac{1}{\sqrt{2}}\angle-\frac{\pi}{4} = 20\log\left(\frac{1}{\sqrt{2}}\right)\angle-\frac{\pi}{4} = -3.01dB\angle-\frac{\pi}{4}$$

Why does a first order pole go down at 20dB/dec?

$$|G(j\omega)| = \left|\frac{1}{\omega_c + j\omega}\right|$$

before the corner frequency, $\omega_c > j\omega$

$$|G(j\omega)| = \left|\frac{1}{\omega_c}\right| = 20\log(\omega_c^{-1}) = -20\log(\omega_c)$$

after the corner frequency, $\omega_c < j\omega$

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = 20\log(\omega^{-1}) = -20\log(\omega)$$

each time the frequency increases by a multiple of 10, the log value becomes 1 larger, thus resulting in a gain change of -20 dB.

Figure 10.11 Why the straight line method works

Why does a pole make the phase angle move by -90deg after the corner frequency?

$$G(j\omega) = \frac{1}{\omega_c + j\omega} = \frac{1\angle 0}{\sqrt{\omega_c^2 + \omega^2}\angle \operatorname{atan}\left(\frac{\omega}{\omega_c}\right)} = \frac{1}{\sqrt{\omega_c^2 + \omega^2}}\angle -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right)$$

before the corner frequency, $\omega_c > j\omega$

$$angle(G(j\omega)) = -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right) = -\operatorname{atan}\left(\frac{0}{\omega_c}\right) = 0$$

after the corner frequency, $\omega_c < j\omega$

$$angle(G(j\omega)) = -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right) = -\operatorname{atan}(\infty) = -\frac{\pi}{2}$$

Figure 10.12 Why the straight line method works (cont'd)

Show that the second order transfer function below would result in a slope of +/- 40 dB/decade.

$$G(j\omega) = \frac{1}{(\omega_c + j\omega)^2}$$

Figure 10.13 Drill problem: Slope of second order transfer functions.

An example of the straight line plotting technique is shown in Figure 10.14. In this example the transfer function is first put into a root form. In total there are three roots, 1, 10 and 100 rad/sec. The single root in the numerator will cause the curve to start upward with a slope of 20dB/dec after 1rad/sec. The two roots will cause two curves downwards at -20dB/dec starting at 10 and 100 rad/sec. The initial gain of the transfer function is also calculated, and converted to decibels. The frequency axis is rad/sec by default, but if Hz are used then it is necessary to convert the values.

$$G(D) = \frac{100D + 100}{0.01D^2 + 0.11D + 10} = \frac{10^4(D+1)}{(D+10)(D+100)}$$ *(the equation is put in root form)*

Step 1: Draw lines for each of the terms in the transfer function,

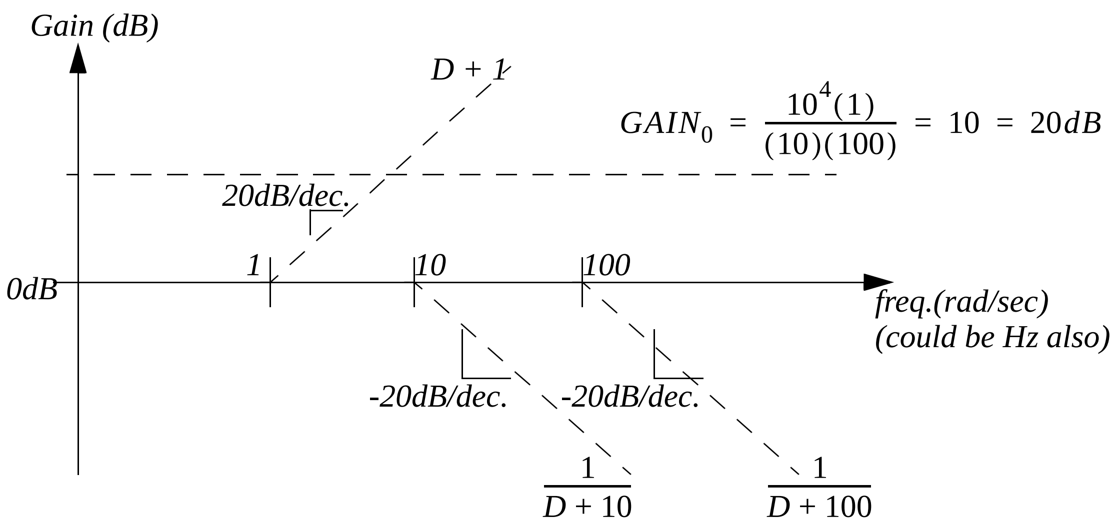


Figure 10.14 An approximate gain plot example

The example is continued in Figure 10.15 where the straight line segments are added to produce a combined straight line curve.

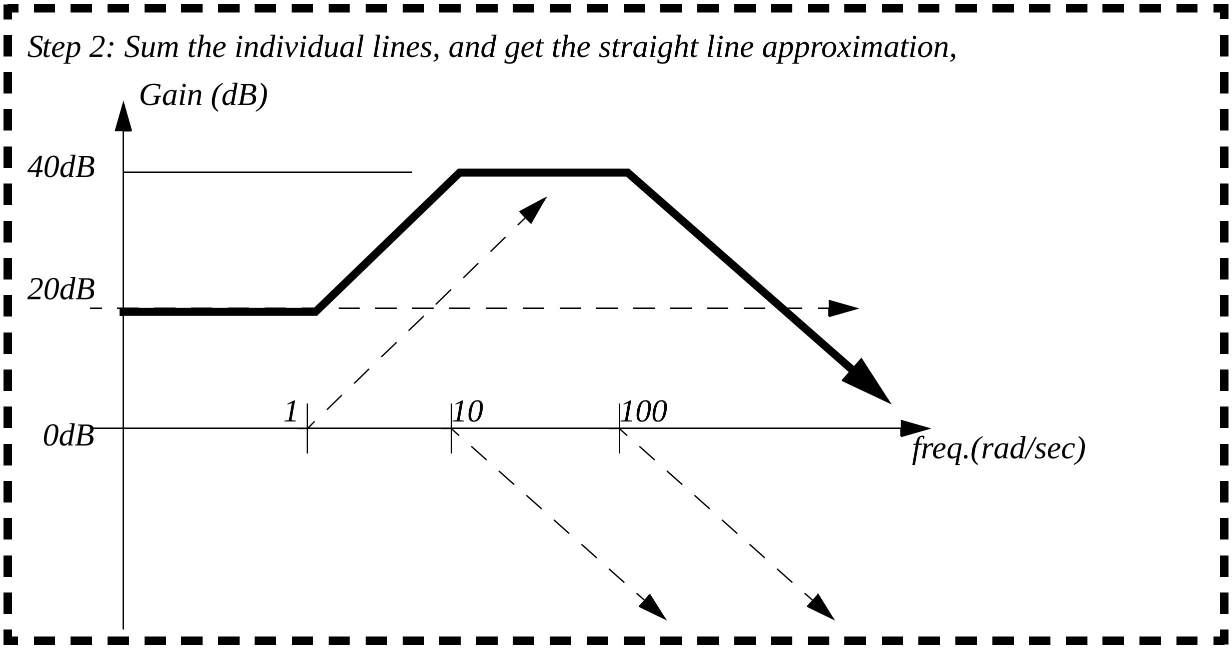


Figure 10.15 An approximate gain plot example (continued)

Finally a smooth curve is fitted to the straight line approximation. When drawing the curve imagine that there are rubber bands at the corners that pull slightly and smooth out. For a simple first-order term there is a 3dB gap between the sharp corner and the function. Higher order functions will be discussed later.

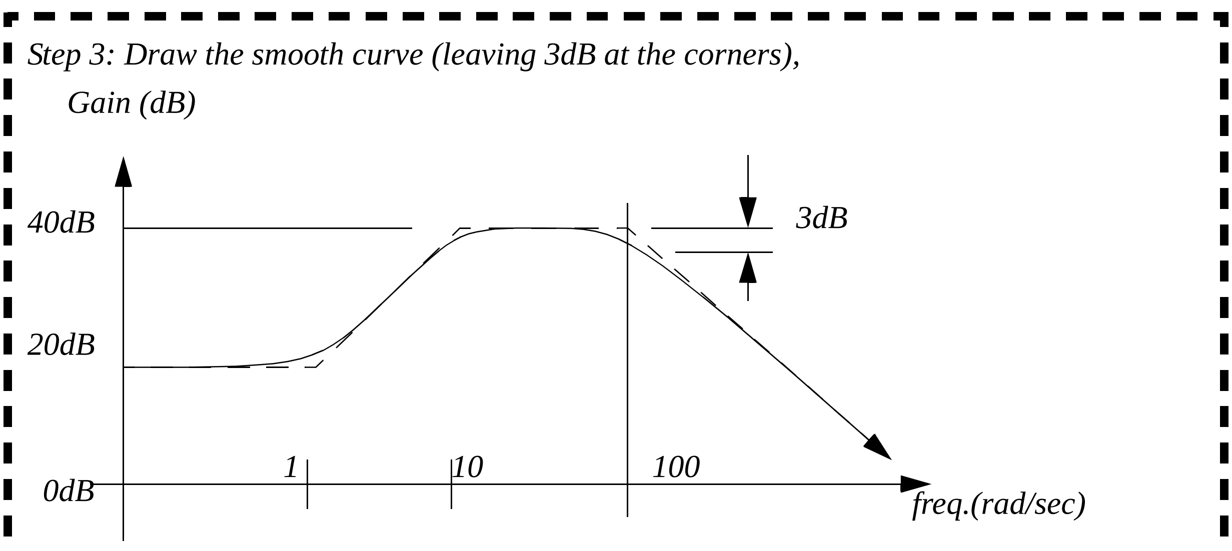


Figure 10.16 An approximate gain plot example (continued)

The process for constructing phase plots is similar to that of gain plots, as seen in Figure 10.18. The transfer function is put into root form, and then straight line phase shifts are drawn for each of the terms. Each term in the numerator will cause a positive shift of 90 degrees, while terms in the denominator cause negative shifts of 90 degrees. The phase shift occurs over two decades, meaning that for a center frequency of 100, the shift would start at 10 and end at 1000. If there are any lone ’D’ terms on the top or bottom, they will each shift the initial value by 90 degrees, otherwise the phase should start at 0degrees.

Gain plots for transfer functions can be approximated with the following steps.

1. Plot the straight line segments.

a) Put the transfer function in root form to identify center frequencies. For example (D+1)/(D+1000) will have center frequencies at 1 and 1000 rad/sec. This should have already been done for the gain plot.

b) The phase at 0rad/sec is determined by looking for any individual D terms. Effectively they have a root of 0rad/sec. Each of these in the numerator will shift the starting phase angle up by 90deg. Each in the denominator will shift the start down by 90 deg. For example the transfer function 10(D+1)/(D+1000) would start at 0 deg while 10D(D+1)/(D+1000) would start at +90deg.

c) Curves that turn up or down are drawn around each center frequency. Again terms in the numerator cause the curve to go up 90 deg, terms in the denominator cause the curves to go down 90 deg. Curves begin to shift one decade before the center frequency, and finish one decade after.

2. The effect of each term is added up to give the resulting straight line approximation.

3. The smooth curve is drawn.

Figure 10.17 The method for Bode graph straight line gain approximation

The previous example started in Figure 10.14 is continued in Figure 10.18 to develop a phase plot using the approximate technique. There are three roots for the transfer function. None of these are zero, so the phase plot starts at zero degrees. The root in the numerator causes a shift of positive 90 deg, starting one decade before 1rad/sec and ending one decade later. The two roots in the denominator cause a shift downward.

$$G(D) = \frac{10^4(D+1)}{(D+10)(D+100)}$$

Step 1: Draw lines for each of the terms in the transfer function

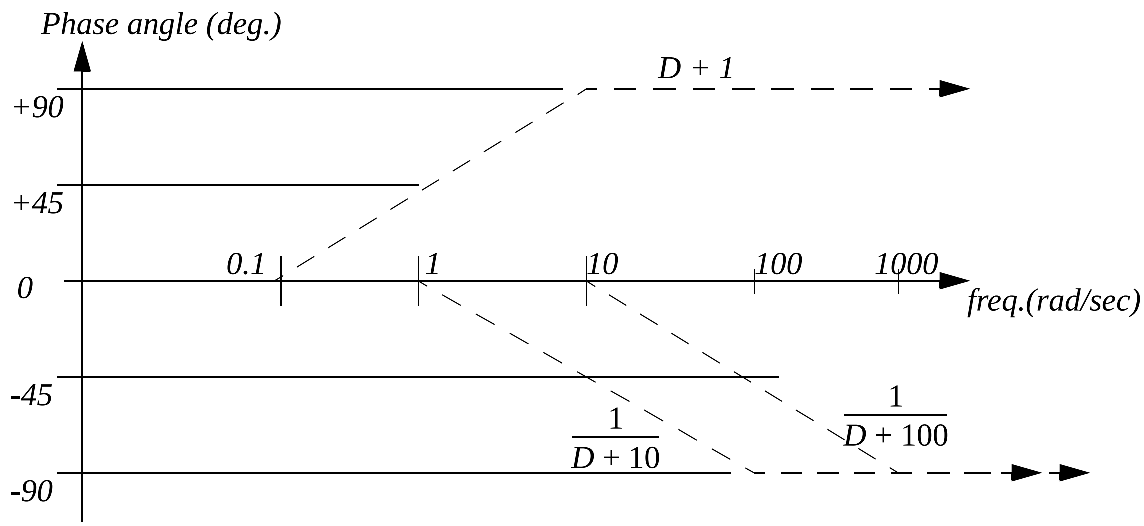


Figure 10.18 An approximate phase plot example

The straight line segments for the phase plot are added in Figure 10.19 to produce a straight line approximation of the final plot. A smooth line approximation is drawn using the straight line as a guide. Again, the concept of an rubber band will smooth the curve.

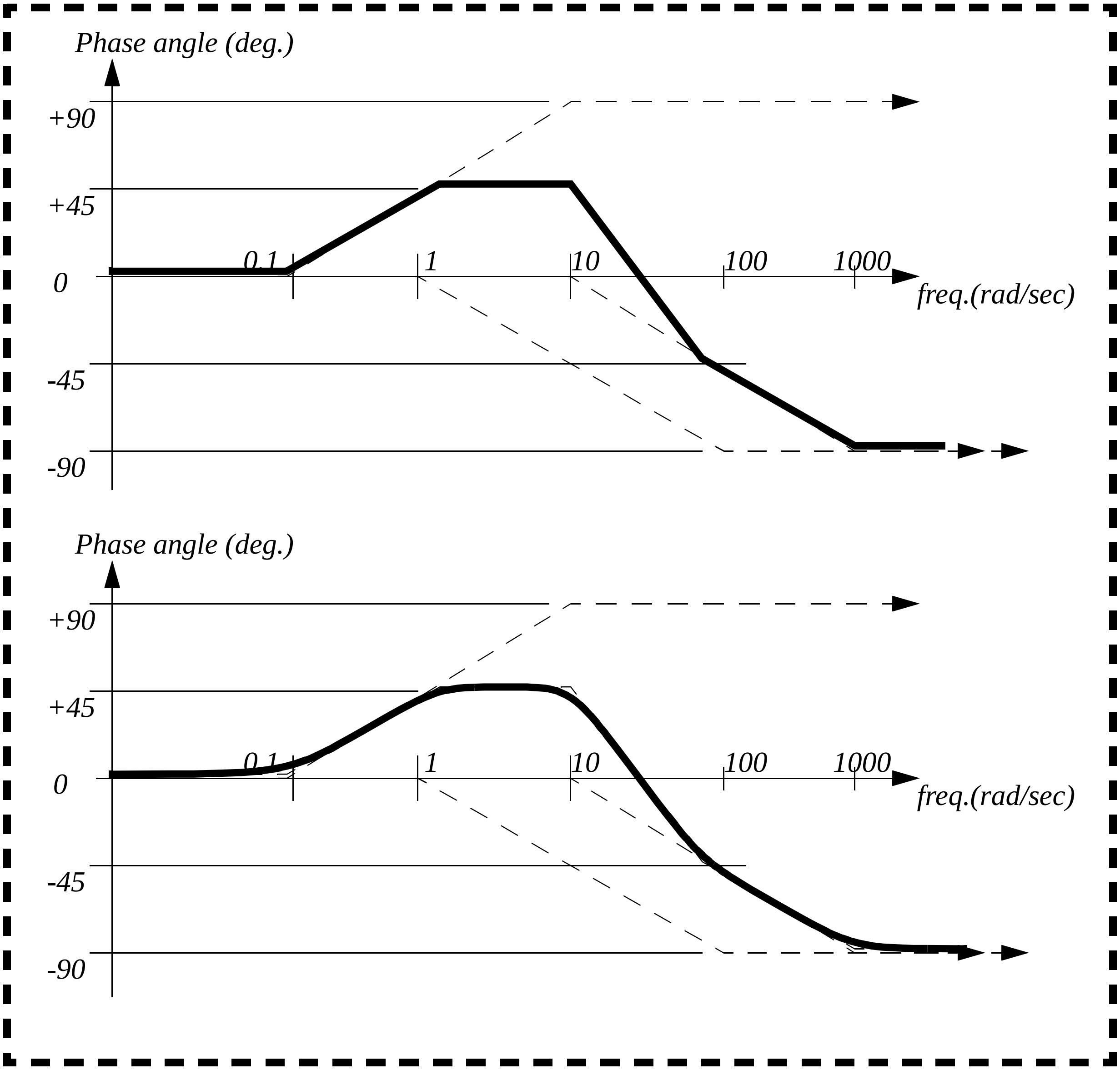
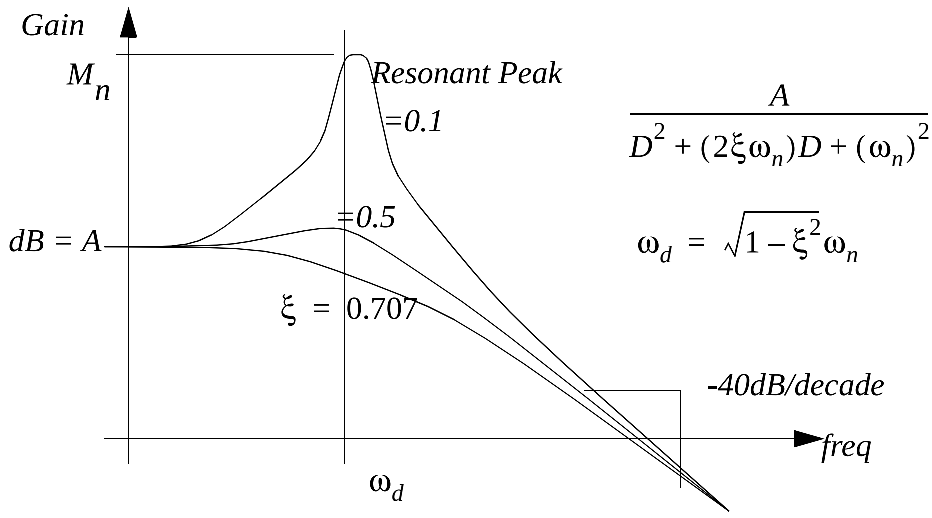


Figure 10.19 An approximate phase plot example (continued)

The previous example used a transfer function with real roots. In a second-order system with double real roots (overdamped) the curve can be drawn with two overlapping straight line approximations. If the roots for the transfer function are complex (under-damped the corner frequencies will become peaked. This can be handled by determining the damping coefficient and natural frequency as shown in Figure 10.20. The peak will occur at the damped frequency. The peaking effect will become more pronounced as the damping coefficient goes from 0.707 to 0 where the peak will be infinite.



Note: If ζ is less than 1 the roots become complex, and the Bode plots get a peak. This can be seen mathematically because the roots of the transfer function become complex.

Figure 10.20 Resonant peaks

The approximate techniques do decrease the accuracy of the final solution, but they can be calculated quickly. In addition these curves provide an understanding of the system that makes design easier. For example, a designer will often describe a system with a Bode plot, and then convert this to a desired transfer function.

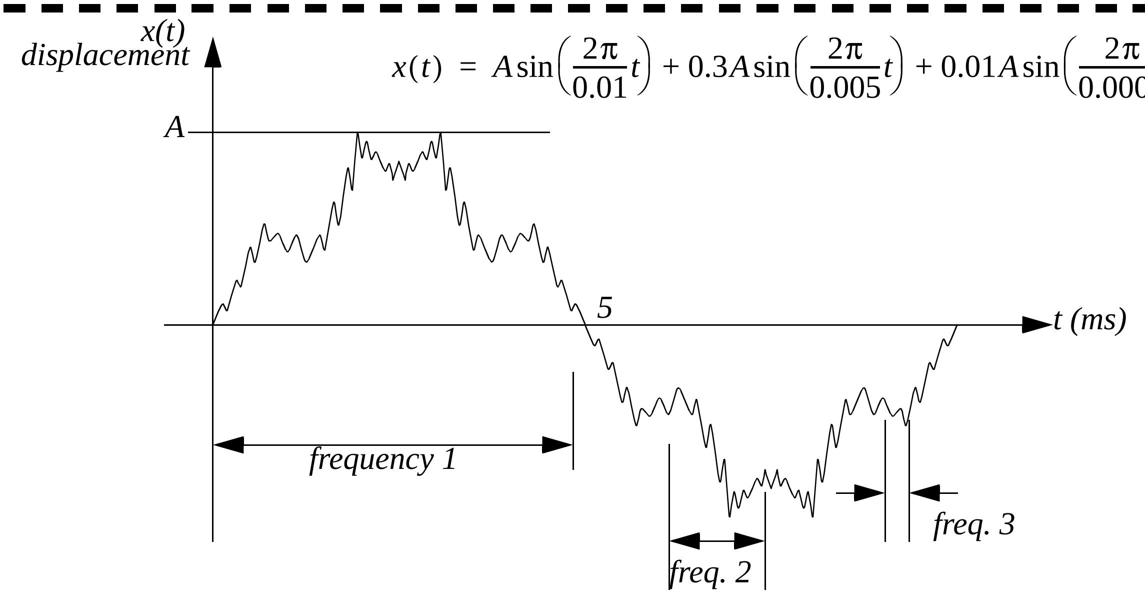
Draw the straight line approximation for the transfer function.

$$\frac{D + 3}{D^2 + 10000D + 10000}$$

Figure 10.21 Drill problem: Draw the straight line approximation

10.3 SIGNAL SPECTRUMS

If a vibration signal is measured and displayed it might look like Figure 10.22. The overall sinusoidal shape is visible, along with a significant amount of 'noise'. When this is considered in greater detail it can be described with the given function. To determine the function other tools are needed to determine the frequencies, and magnitudes of the frequency components.



$$x(t) = A\sin\left(\frac{2\pi}{0.01}t\right) + 0.3A\sin\left(\frac{2\pi}{0.005}t\right) + 0.01A\sin\left(\frac{2\pi}{0.0005}t\right)$$

Figure 10.22 A vibration signal as a function of time

A signal spectrum displays signal magnitude as a function of frequency, instead of time. The time based signal in Figure 10.22 is shown in the spectrum in Figure 10.23. The three frequency components are clearly identifiable spikes. The height of the peaks indicates the relative signal magnitude.

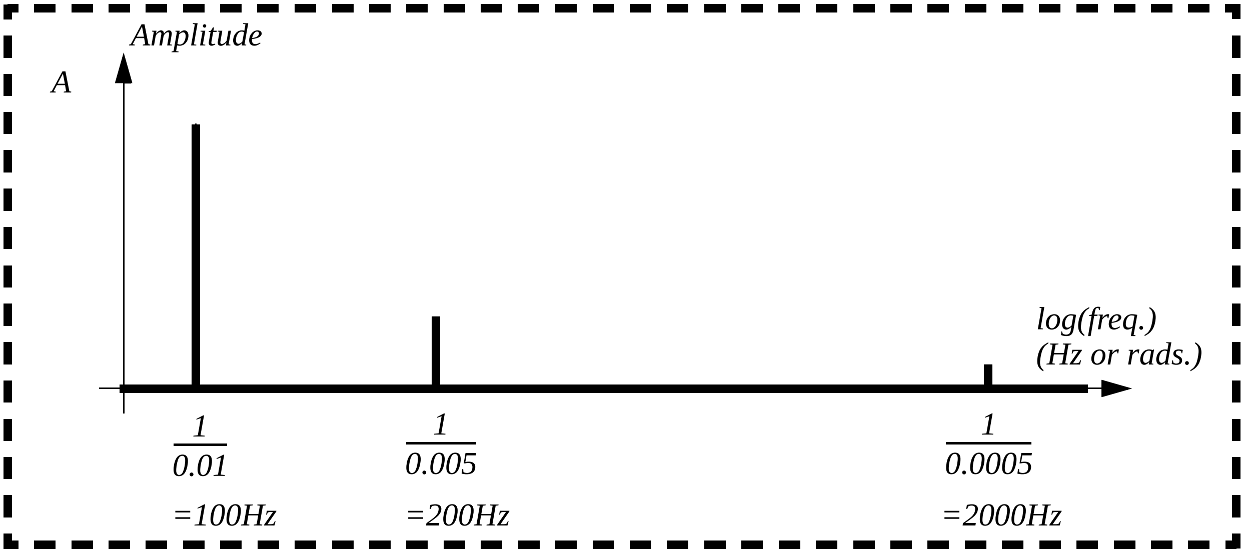


Figure 10.23 The spectrum for the signal in Figure 10.22

10.4 SUMMARY

- Bode plots show gain and phase angle as a function of frequency.
- Bode plots can be constructed by calculating point or with straight line approximations.
- A signal spectrum shows the relative strengths of components at different frequencies.

10.5 PRACTICE PROBLEMS

1. Draw a Bode Plot for both of the transfer functions below using the phasor transform.

$$\frac{(D+1)(D+1000)}{(D+100)^2} \qquad \text{AND} \qquad \frac{5}{D^2}$$

2. Given the transfer function below,

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2}$$

a) draw the straight line approximation of the Bode (gain and phase shift) plots.
b) determine the steady-state output if the input is x(t) = 20 sin(9t+0.3) using the

Bode plot.

3. Use the straightline approximation techniques to draw the Bode plot for the transfer function below.

$$G = \frac{F}{x} = \frac{D + 1000}{D^2 + 5D + 100}$$

4. Given the transfer function below,

$$\frac{V_o}{V_i} = \frac{1000}{D + 1000}$$

a) Find the steady state response of the circuit using phasors (i.e., phasor transforms) if the input is Vi=5sin(100,000t).
b) Draw an approximate Bode plot for the circuit.

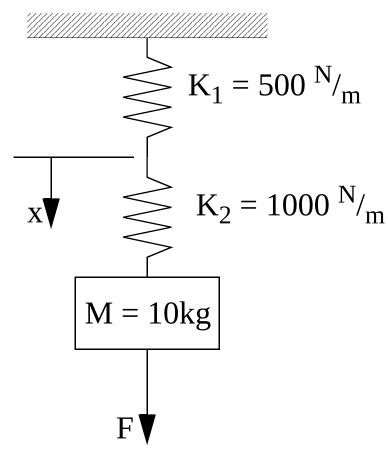
5. For the transfer function,

$$\frac{D(D + 2\pi)}{D^2 + 300\pi D + 62500\pi^2}$$

a) Use the straight line method and the attached log paper to draw an approximate Bode plot.
b) Verify the Bode plot by calculating values at a few points.
c) Use the Bode plot to find the response to an input of 5sin(624t) + 1sin(6.2t).

6. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’. Neglect the effects of gravity.

a) find the transfer function.



b) What is the steady-state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if ‘x’ is the input.
d) Draw the bode plots for the transfer function found in a).
e) Find x(t), given F(t) = 10N for t >= 0 seconds.

f) Find x(t), given F(t) = 10N for t >= 0 seconds considering the effects of gravity.

7. The following differential equation is supplied, with initial conditions.

$$\ddot{y} + \dot{y} + 7y = F \qquad y(0) = 1 \qquad \dot{y}(0) = 0$$

$$F(t) = 10 \qquad t > 0$$

a) Write the equation in state variable form.
b) Solve the differential equation numerically.
c) Solve the differential equation using calculus techniques.
d) Find the frequency response (gain and phase) for the transfer function using the phasor transform. Sketch the bode plots.

8. You are given the following differential equation for a spring damper pair.

$$5x + \left(\frac{d}{dt}\right)x = F \qquad F(t) = 10\sin(100t)$$

a) Write the transfer function for the differential equation if the input is F.
b) Apply the phasor transform to the transfer function to find magnitude and phase as functions of frequency.
c) Draw a Bode plot for the system using either approximate or exact techniques.
d) Use the Bode plot to find the response to;
$F(t) = 10\sin(100t)$

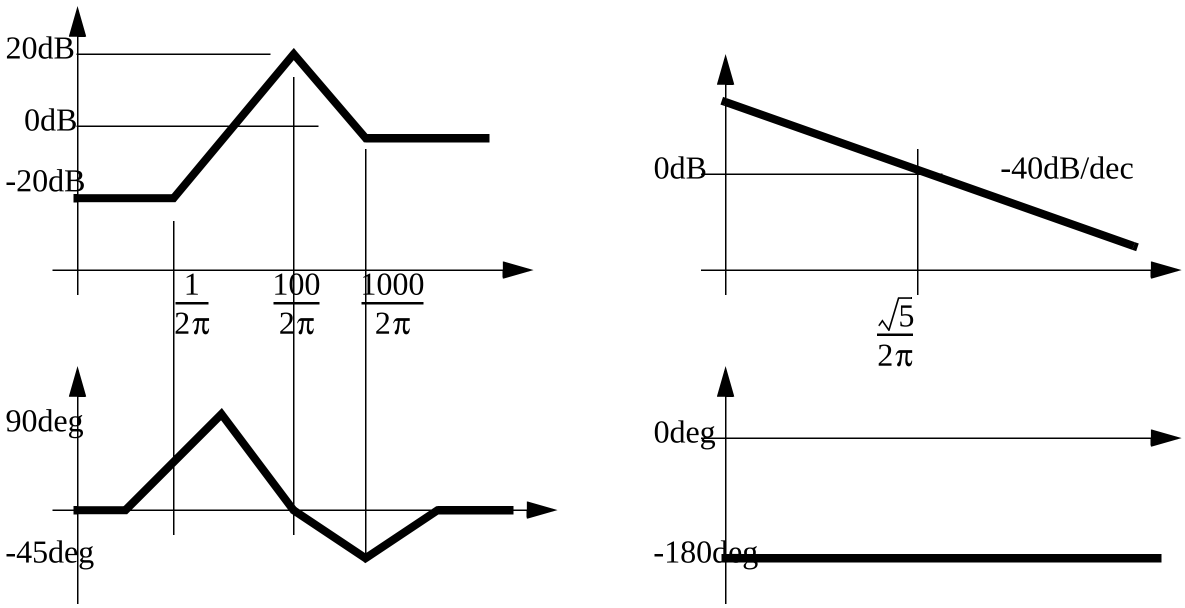
e) Put the differential equation in state variable form and use a calculator to find values in time for the given input.
$F = 10\sin(100t)$

t	x
0.0	
0.002	
0.004	
0.006	
0.008	
0.010	

f) Give the expected ‘x’ response of this first-order system to a step function input for force F = 1N for t > 0 if the system starts at rest. Hint: Use the canonical form.

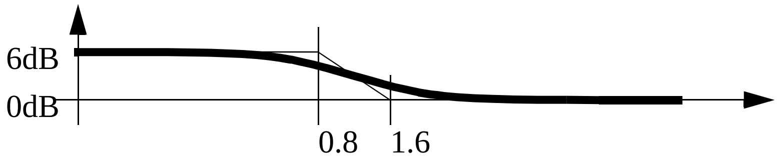
10.6 PRACTICE PROBLEM SOLUTIONS

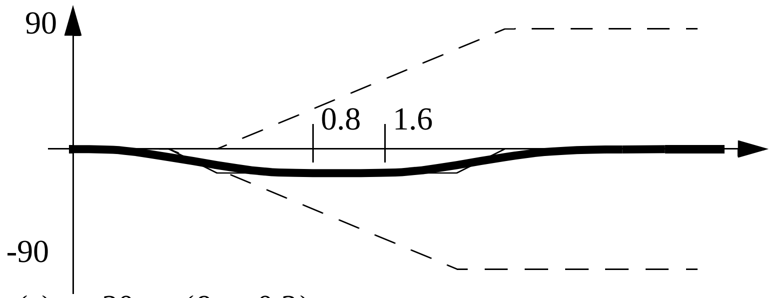
1.



2.

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2} = \frac{(D+10)}{(D+5)}$$





$$x(t) = 20\cos(9t + 0.3)$$

Aside: the numbers should be obtained from the graphs, but I have calculated them

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236 - 0.425j$$

$$\frac{y}{x} = \sqrt{1.236^2 + 0.425^2}\angle\operatorname{atan}\left(\frac{-0.425}{1.236}\right) = 1.307\angle -0.3312rad$$

$$\frac{y}{x} = 2.33dB\angle -9.49°$$

$$y(t) = 20(1.307)\sin(9t + 0.3 + (-0.3312))$$

$$y(t) = 26.1\sin(9t - 0.031)$$

Aside: This can also be done entirely with phasors in cartesian notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236 - 0.425j$$

$$x = 20(\cos(0.3rad) + j\sin(0.3rad)) = 19.1 + 5.91j$$

$$\frac{y}{19.1 + 5.91j} = (1.236 - 0.425j)(19.1 + 5.91j) = 26.1 - 0.813j = 26.1\angle -0.031$$

$$y(t) = 26.11\sin(9t - 0.031)$$

(cont'd

Aside: This can also be done entirely with phasors in polar notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right) = \frac{13.45\angle 0.733}{10.30\angle 1.064} = \frac{13.45}{10.30}\angle 0.733 - 1.064 = 1.31\angle -0.331$$

$$x = 20\angle 0.3$$

$$\frac{y}{20\angle 0.3} = 1.31\angle -0.331$$

$$y = 1.31(20)\angle(-0.331 + 0.3) = 26.2\angle -0.031$$

$$y(t) = 26.2\sin(9t - 0.031)$$

3.

for the numerator, (zero) $1000\frac{rad}{s} = 159Hz$

for the denominator, (poles)

$$D^2 + 2\omega_n\zeta D + \omega_n^2 = D^2 + 5D + 100$$

$$\omega_n^2 = 100$$

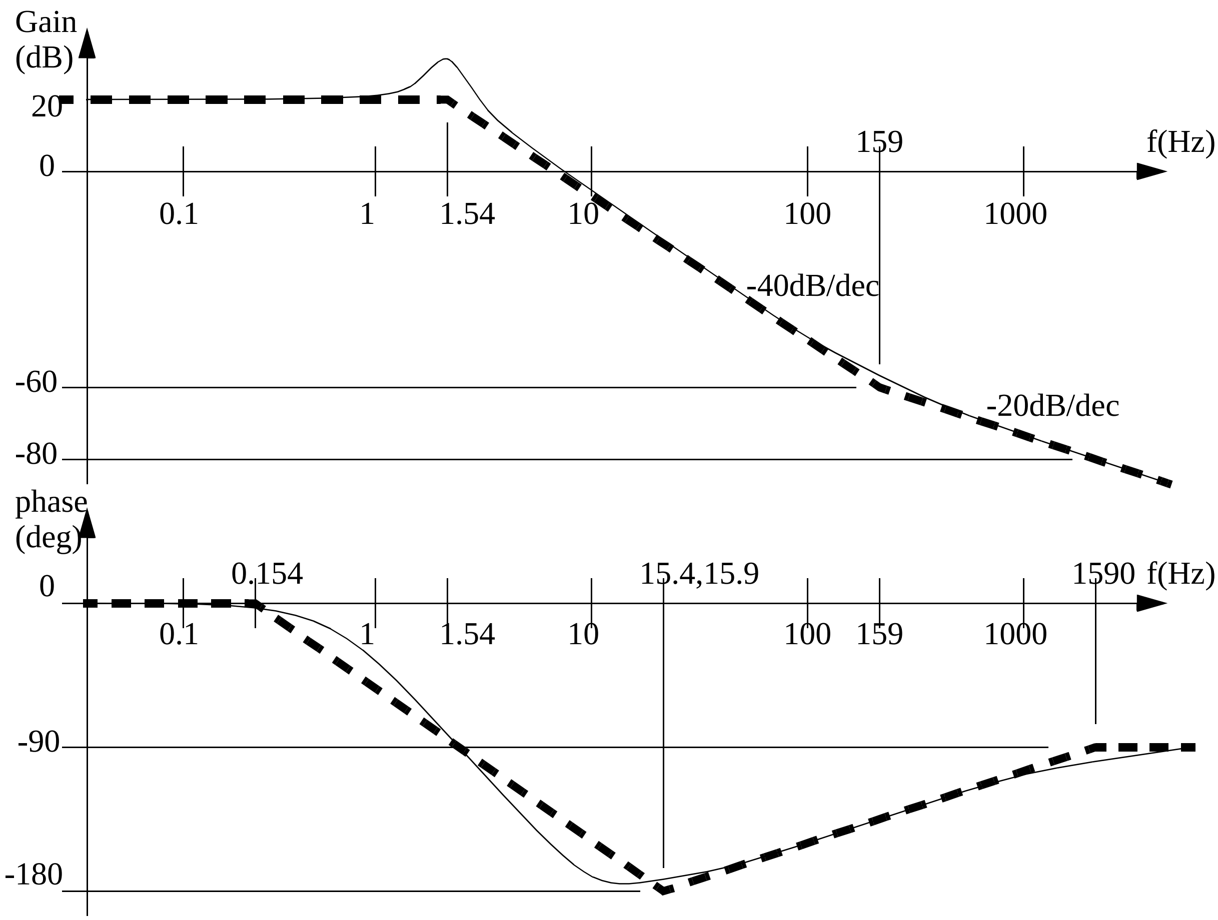
$$\therefore\omega_n = 10\frac{rad}{s} \qquad f_n = 1.59Hz$$

$$2\omega_n\zeta = 5$$

$$\therefore\zeta = \frac{5}{2(10)} = 0.25 \quad \text{(underdamped)}$$

$$f_d = f_n\sqrt{1-\zeta^2} = 1.54Hz$$

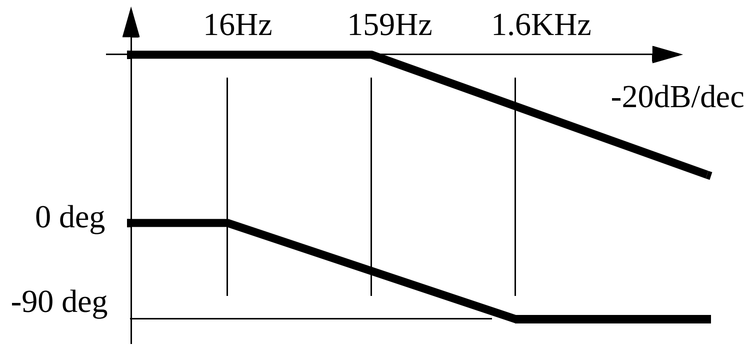
for the initial gain $G(0) = \frac{0 + 1000}{0^2 + 5(0) + 100} = 10 = 20dB$



4.

a) $V_o(t) = 0.050\sin(10^5 t - 1.561)$

b)



5.

a)

b)

c) $8.89\sin(624t + 1.571) + 0.141\times10^{-3}\sin(6.2t + 2.356)$

6.

a) $\frac{x}{F} = \frac{0.0667}{D^2 + 33.3}\frac{m}{N}$

b) $\omega = 1 \qquad \therefore D = 1j$

$$\frac{x}{F} = \frac{0.0667}{(1j)^2 + 33.3} = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}$$

$$F(t) = 10\cos(t + 1)N$$

$$\therefore F(\omega) = (10\angle 1 rad)N$$

$$\frac{x}{(10\angle 1 rad)N} = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}$$

$$\therefore x(\omega) = 2.07 \times 10^{-3} \angle 0 rad \frac{m}{N}(10\angle 1 rad)N$$

$$\therefore x(\omega) = (10)2.07 \times 10^{-3} \angle (0 rad + 1 rad)m$$

$$\therefore x(\omega) = 0.0207\angle 1 radm$$

$$\therefore x(t) = 0.0207\cos(t + 1)m$$

c) $\frac{F}{x} = \frac{D^2 + 33.3}{0.0667}\frac{N}{m}$

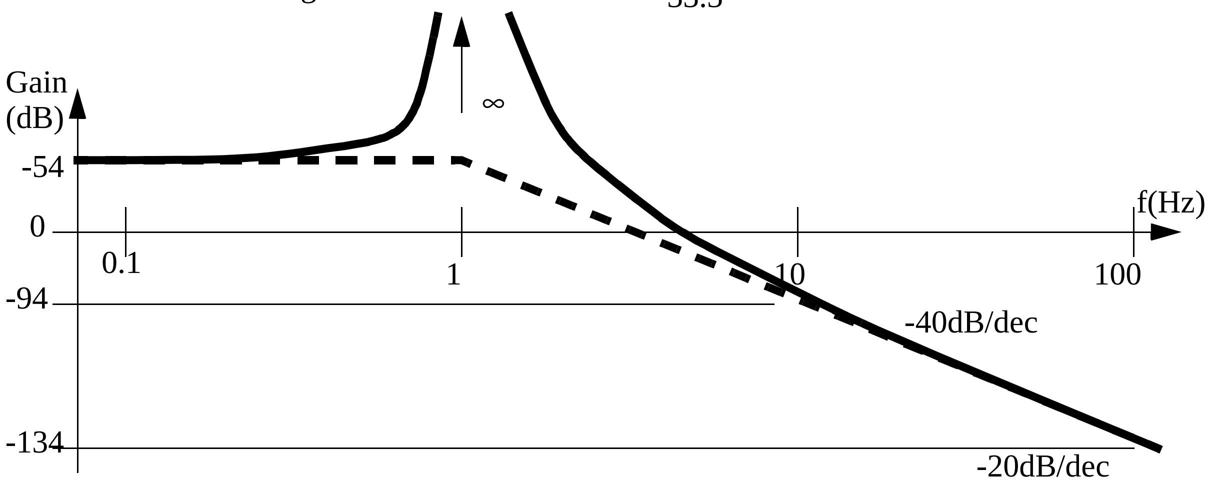
d) for the denominator, (poles) $D^2 + 2\omega_n \zeta D + \omega_n^2 = D^2 + 33.3$

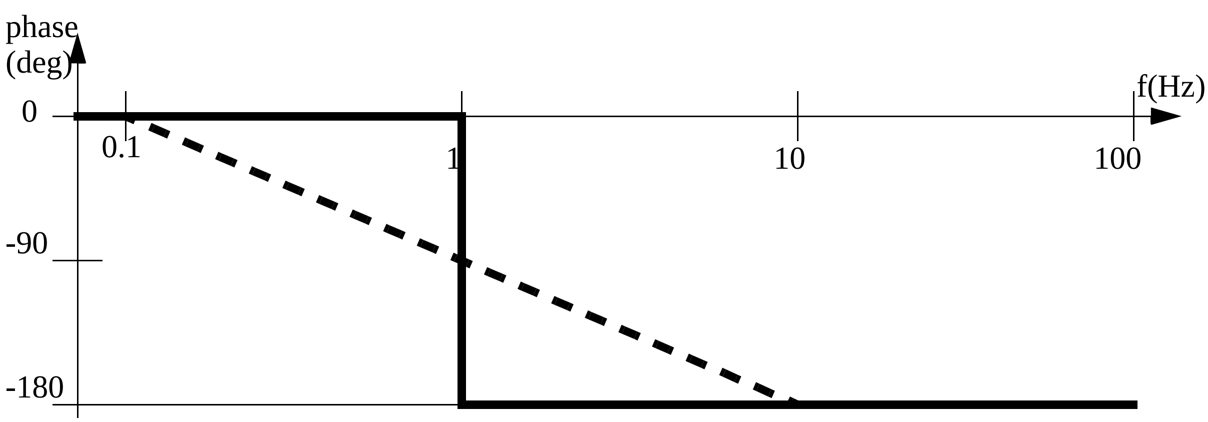
$$\omega_n = 5.77\frac{rad}{s} \qquad f_n = 0.918 Hz$$

$$\zeta = 0 \qquad \text{(undamped)}$$

$$f_d = f_n\sqrt{1-\zeta^2} = 0.918 Hz$$

for the initial gain $G(0) = \frac{0.0667}{33.3} = 2 = -54dB$





e) $x(t) = (-0.020\cos(5.77t) + 0.020)m$

f) $x(t) = (-0.2162\cos(5.77t) + 0.2162)m$

7.

a)
$$\dot{y} = v$$
$$\dot{v} = -v - 7y + F$$

b)
$$\begin{bmatrix} y_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} y_i \\ v_i \end{bmatrix} + h\left[\begin{bmatrix} 0 & 1 \\ -7 & -1 \end{bmatrix}\begin{bmatrix} y_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix}\right]$$

given
$$\begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad F = 10$$

using h=0.001s
$$\begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.001\left[\begin{bmatrix} 0 & 1 \\ -7 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}\right]$$

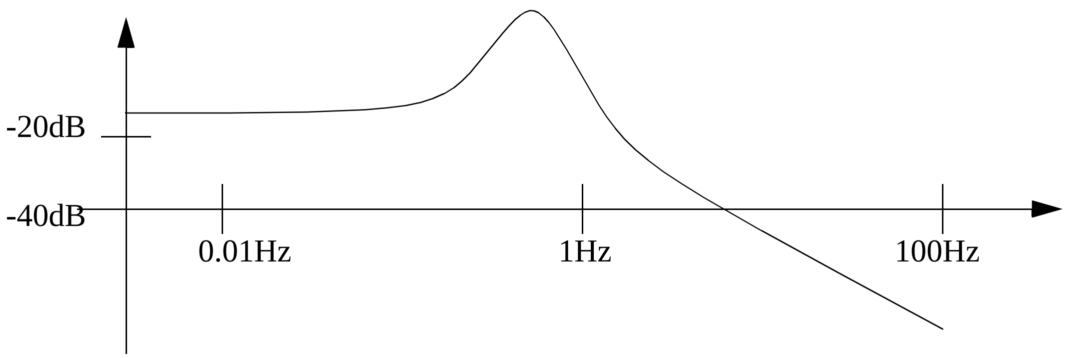
etc.. until
$$\begin{bmatrix} y_{100} \\ v_{100} \end{bmatrix} = \begin{bmatrix} 1.428m \\ 0.000\frac{m}{s} \end{bmatrix}$$

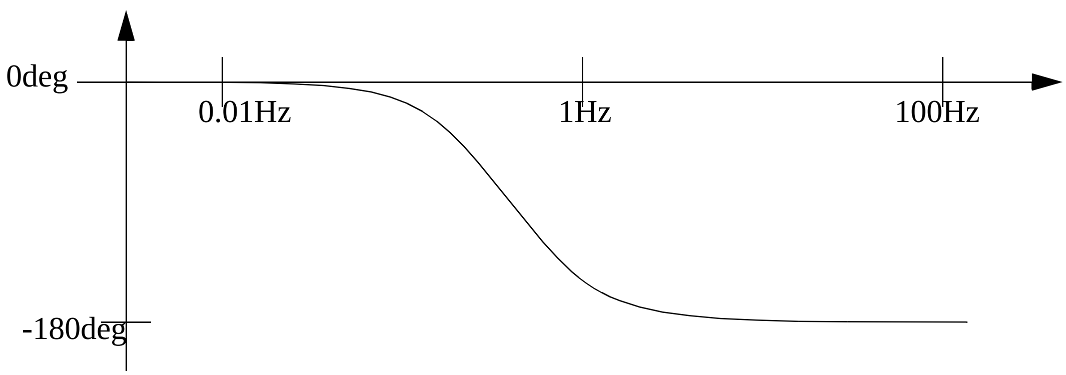
c)
$$y(t) = (-0.437e^{-0.5t}\cos(2.598t - 0.19) + 1.429)m$$

d) $$\frac{y}{F} = \frac{1}{D^2 + D + 7} = \frac{1}{(j\omega)^2 + j\omega + 7} = \frac{1}{(7-\omega^2) + j(\omega)}$$

$$\left|\frac{y}{F}\right| = \frac{1}{\sqrt{(7-\omega^2)^2 + \omega^2}}$$

$$\angle\theta = \frac{\angle 0}{\angle angle(\omega, 7-\omega^2)} = \angle(0 - angle(\omega, 7-\omega^2)) = \angle -angle(\omega, 7-\omega^2)$$



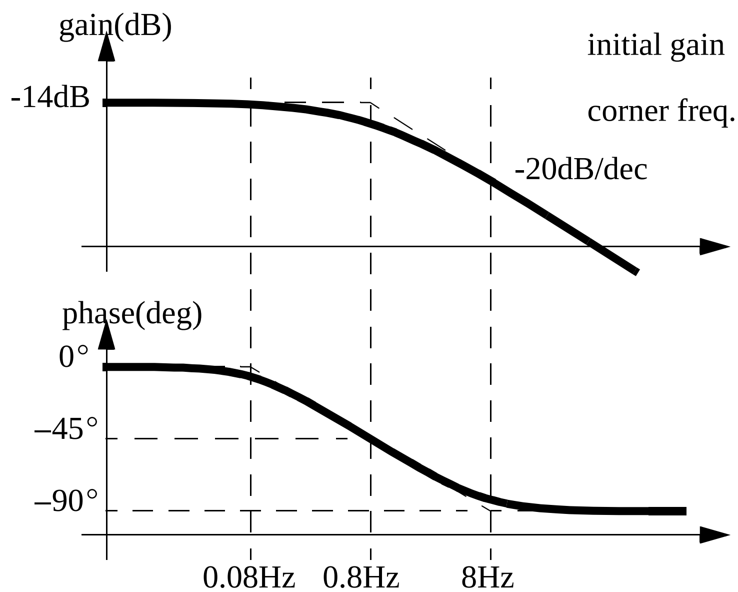


8.

a) $\frac{x}{F} = \frac{1}{5+D}$

b) $\frac{x}{F} = \frac{1}{5+D} = \frac{1}{5+j\omega} = \frac{1\angle 0}{\sqrt{5^2+\omega^2}\angle \mathrm{atan}\left(\frac{\omega}{5}\right)} = \frac{1}{\sqrt{5^2+\omega^2}}\angle -\mathrm{atan}\left(\frac{\omega}{5}\right)$

c)



$$\text{initial gain} = 20\log\left(\frac{1}{5+0}\right) = -14dB$$

$$\text{corner freq.} = \frac{5}{2\pi} = 0.8Hz$$

d) $f = \frac{100}{2\pi} = 16Hz$

From the Bode plot, $gain = -40dB = 0.01$

$$phase = -90° = -\frac{\pi}{2}rad$$

$$x(t) = 10(0.01)\sin\left(100t - \frac{\pi}{2}\right)$$

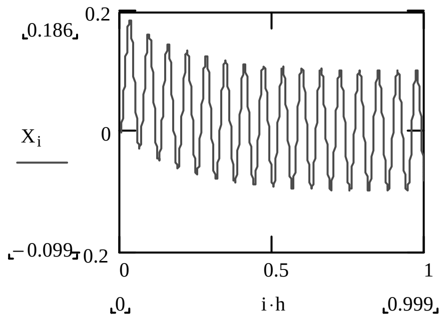
Aside: verified by calculations,

$$\frac{x}{10\angle 0} = \frac{1}{\sqrt{5^2+100^2}}\angle -\mathrm{atan}\left(\frac{100}{5}\right) = 0.00999\angle -1.521$$

$$x = (10\angle 0)(0.00999\angle -1.521) = 0.0999\angle -1.521$$

$$x(t) = 0.0999\sin(100t - 1.521)$$

e)



t	x
0.0	0
0.002	9.98e-4
0.004	5.92e-3
0.006	0.015
0.008	0.026
0.010	0.041

f) $$x = \frac{1}{5} - \frac{e^{-5t}}{5}$$

10.7 ASSIGNMENT PROBLEMS

1. For the transfer functions below, draw the bode plots using computer software.

$$\frac{(D+2\pi)D}{(D+200\pi)(D+0.02\pi)} \qquad \frac{(D+2\pi)}{D^2+50\pi D+10000\pi^2}$$

2. Draw Bode plots for the following functions using straight line approximations.

$$\frac{1}{D+1} \qquad \frac{1}{D^2+1} \qquad \frac{1}{(D+1)^2} \qquad \frac{1}{D^2+2D+2}$$

3. Given the transfer function below,

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2}$$

a) draw the straight line approximation of the bode and phase shift plots.
b) determine the steady state output if the input is x(s) = 20 cos(9t+.3) using the striaght line plots.
c) use an exact method to verify part b).

4. a) Convert the following differential equation to a transfer function.

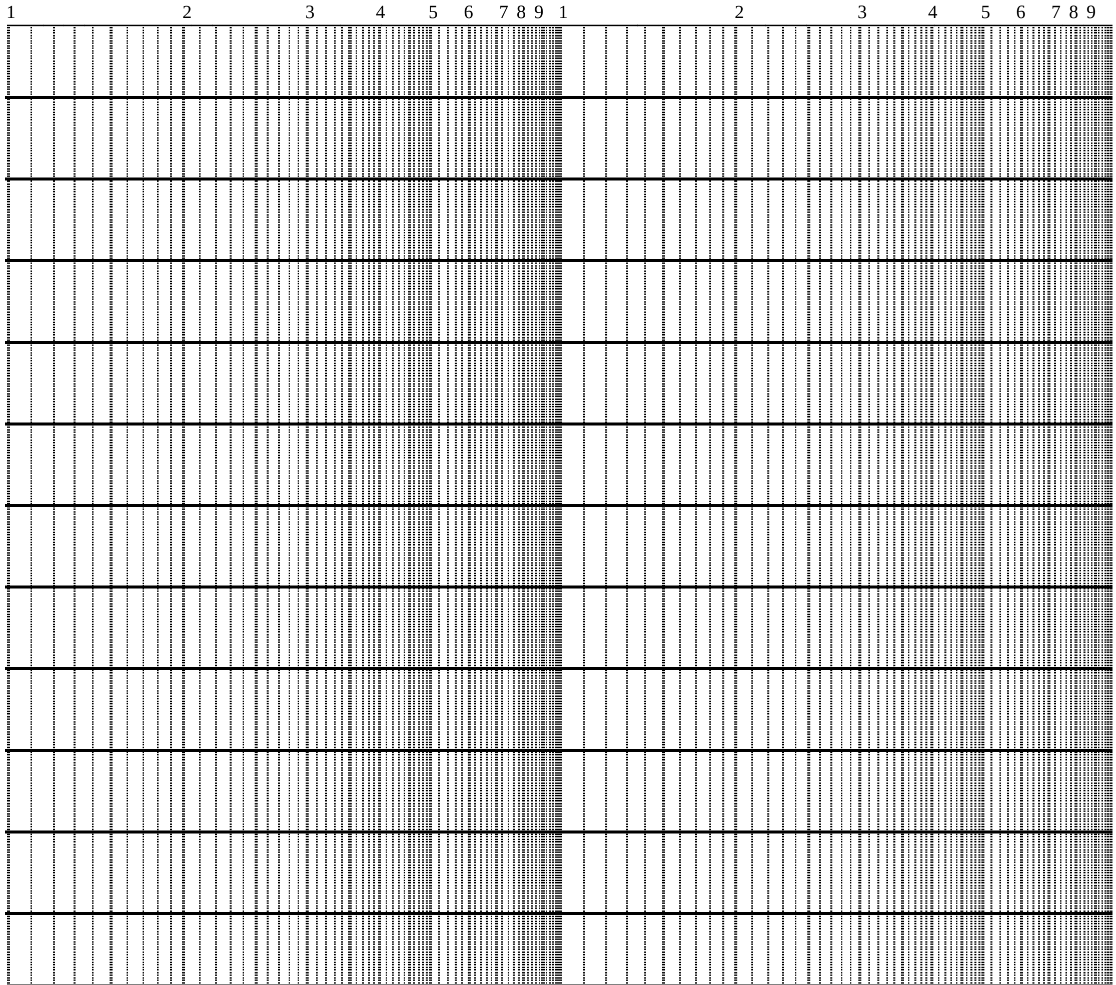
$$5\ddot{x} + 2x = 3F$$

b) Apply a phasor (Fourier) transform to the differential equation and develop equations for the system gain and phase shift as a function of input frequency.
c) Draw a Bode plot using the equations found in part b) on the attached log paper.
d) Draw a straight line approximation of the system transfer function on the attached log paper.

10.8 LOG SCALE GRAPH PAPER

Please notice that there are a few sheets of 2 and 4 cycle log paper attached, make additional copies if required, and if more cycles are required, sheets can be cut and pasted together. Also note that better semi-log paper can be purchased at technical bookstores, as well at most large office supply stores.

bode plots - 10.37



bode plots - 10.38