From K. Ogata, "Modern Control Engineering", Prentice Hall.

Observed-state feedback control system with minimum-order observer. For the case of the observed-state feedback control system with full-order state observer, we have shown that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole placement design alone, plus the poles due to the observer design alone. Hence the pole placement design and the full-order observer design are independent of each other.

For the observed-state feedback control sytem with minimum-order observer, the same conclusion applies. The system characteristic equation can be derived as

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| \, |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e\mathbf{A}_{ab}| = 0$$

(See Problem A–10–12 for the detail.) The closed-loop poles of the observed-state feedback control system with a minimum-order observer comprise the closed-loop poles due to pole placement [the eigenvalues of matrix $(\mathbf{A} - \mathbf{B}\mathbf{K})$] and the closed-loop poles due to the minimum-order observer [the eigenvalues of matrix $(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})$]. Therefore, the pole placement design and the design of the minimum-order observer are independent of each other.

10–4 DESIGN OF SERVO SYSTEMS

In Section 4–7, we discussed the system types according to the number of the integrators in the feedforward transfer function. The type 1 system has one integrator in the feedforward path and the system will exhibit no steady-state error in the step response. In this section we shall discuss the pole placement approach to the design of type 1 servo systems. Here we shall limit our systems each to have a scalar control signal u and a scalar output y.

In Chapter 7 we discussed I–PD control systems. In the I–PD control system, an integrator is placed in the feedforward path to integrate the error signal, and the proportional and derivative controls were inserted in the minor loop. Figure 10–14 shows a block diagram of an I–PD control of a plant $G_p(s)$, where we assume the plant has no integrator. Figure 10–15 shows a block diagram of a type 1 servo system. It is a more general case of the

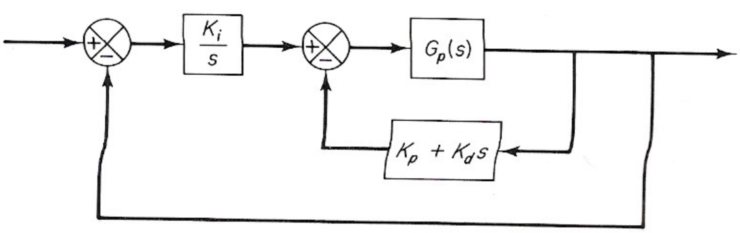

Figure 10–14 I-PD control of plant $G_p(s)$.

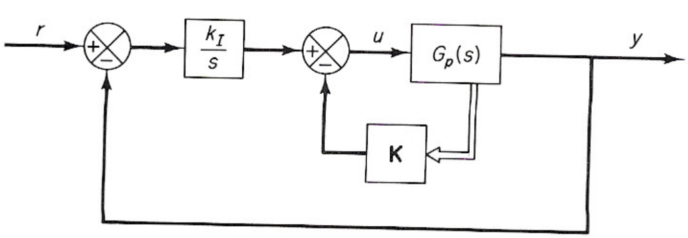

Figure 10–15 Type 1 servo system.

I–PD control of the plant $G_p(s)$. Such a configuration of the type 1 servo system is commonly encountered in practice. (Other configurations are also used in practice.) In the servo system shown in Figure 10–15, the integral control action together with state feedback scheme is used to properly stabilize the system. This system will exhibit no steady-state error in the response to the step input.

In what follows we shall first discuss a problem of designing a type 1 servo system where the plant involves an integrator. Then we shall discuss the design of type 1 servo systems where the plant has no integrator.

Type 1 servo system when plant has an integrator. Assume that the plant is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{10–86}$$

$$y = \mathbf{C}\mathbf{x} \tag{10–87}$$

where $\mathbf{x}$ = state vector for the plant (n-vector)
u = control signal (scalar)
y = output signal (scalar)
$\mathbf{A}$ = $n \times n$ constant matrix
$\mathbf{B}$ = $n \times 1$ constant matrix
$\mathbf{C}$ = $1 \times n$ constant matrix

As stated earlier, we assume that both the control signal u and the output signal y are scalars. By a proper choice of a set of state variables, it is possible to choose the output to be equal to one of the state variables. (See the method presented in Section 2–2 for obtaining a state-space representation of the transfer function system in which the output y becomes equal to x_1.)

Figure 10–16 shows a general configuration of the type 1 servo system where the plant has an integrator. Here we assumed that $y = x_1$. In the present analysis we assume that the reference input r is a step function. In this system we use the following state feedback control scheme:

$$u = -[0 \quad k_2 \quad k_3 \quad \cdots \quad k_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + k_1(r - x_1)$$

$$= -\mathbf{K}\mathbf{x} + k_1 r \tag{10–88}$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n] \tag{10–89}$$

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by Equations (10–86) and (10–88), or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r \tag{10–90}$$

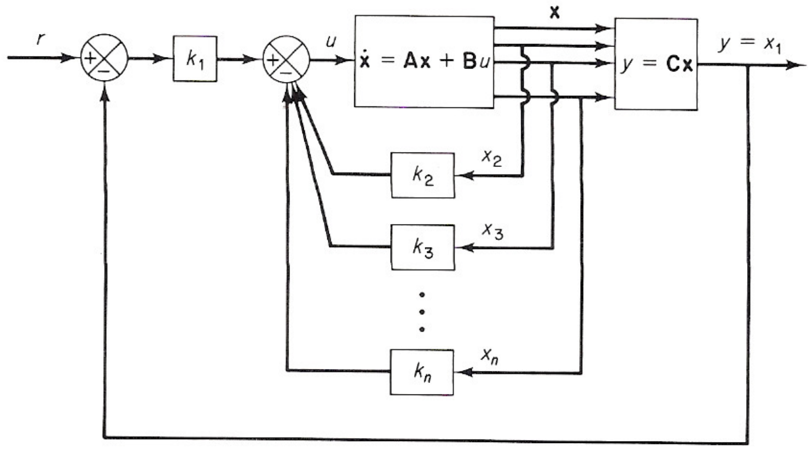

Figure 10–16
Type 1 servo system where the plant has an integrator.

We shall design the type 1 servo system such that the closed-loop poles are located at desired positions. The designed system will be an asymptotically stable system, and $y(\infty)$ will approach constant value r and $u(\infty)$ will approach zero.

Notice that at steady state we have

$$\dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(\infty) + \mathbf{B}k_1 r(\infty) \tag{10–91}$$

Noting that $r(t)$ is a step input, we have $r(\infty) = r(t) = r$ (constant) for $t > 0$. By subtracting Equation (10–91) from Equation (10–90), we obtain

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})[\mathbf{x}(t) - \mathbf{x}(\infty)] \tag{10–92}$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then Equation (10–92) becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e} \tag{10–93}$$

Equation (10–93) describes the error dynamics.

The design of the type 1 servo system here becomes that of a design of an asymptotically stable regulator system such that $\mathbf{e}(t)$ approaches zero, given any initial condition $\mathbf{e}(0)$. If the system defined by Equation (10–86) is completely state controllable, then by specifying the desired eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ for the matrix $\mathbf{A} - \mathbf{BK}$, matrix $\mathbf{K}$ can be determined by the pole placement technique presented in Section 10–2.

The steady-state values of $\mathbf{x}(t)$ and $u(t)$ can be found as follows: At steady state ($t = \infty$), we have, from Equation (10–90),

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = (\mathbf{A} - \mathbf{BK})\mathbf{x}(\infty) + \mathbf{B}k_1 r$$

Since the desired eigenvalues of $\mathbf{A} - \mathbf{BK}$ are all in the left-half s plane, the inverse of matrix $\mathbf{A} - \mathbf{BK}$ exists. Consequently, $\mathbf{x}(\infty)$ can be determined as

$$\mathbf{x}(\infty) = -(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}k_1 r$$

Also, $u(\infty)$ can be obtained as

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r = 0$$

(See Example 10–7 to verify this last equation.)

EXAMPLE 10–7 Consider the design of a type 1 servo system where the plant transfer function has the integrator

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

It is desired to design a type 1 servo system such that the closed-loop poles are at $-2 \pm j3.464$ and -10. Assume that the system configuration is the same as that shown in Figure 10–16 and the reference input r is a step function.

Define state variables x_1, x_2, and x_3 as follows:

$$x_1 = y$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

Then the state space representation of the system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (10\text{–}94)$$

$$y = \mathbf{C}\mathbf{x} \qquad (10\text{–}95)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0]$$

Referring to Figure 10–16 and noting that $n = 3$, the control signal u is given by

$$u = -(k_2 x_2 + k_3 x_3) + k_1(r - x_1) = -\mathbf{K}\mathbf{x} + k_1 r \qquad (10\text{–}96)$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

Our problem here is to determine the state feedback gain matrix $\mathbf{K}$ by the pole placement approach.

Let us examine the controllability matrix for the system. The rank of

$$\mathbf{M} = [\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

is 3. Hence the plant is completely state controllable. The characteristic equation for the system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{vmatrix}$$

$$= s^3 + 3s^2 + 2s$$

$$= s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Hence

$$a_1 = 3, \qquad a_2 = 2, \qquad a_3 = 0$$

For the determination of the matrix $\mathbf{K}$ by the pole placement approach, we shall use Equation (10–13), rewritten as

$$\mathbf{K} = [\alpha_3 - a_3 \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1]\mathbf{T}^{-1} \tag{10–97}$$

Since the state equation for the system, Equation (10–94), is already in the controllable canonical form, we have $\mathbf{T} = \mathbf{I}$.

By substituting Equation (10–96) into Equation (10–94), we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + k_1 r) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r \tag{10–98}$$

where the input r is a step function. Then, as t approaches infinity, $\mathbf{x}(t)$ approaches $\mathbf{x}(\infty)$, a constant vector. At steady state, we have

$$\dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}k_1 r \tag{10–99}$$

Subtracting Equation (10–99) from Equation (10–98), we have

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{B}\mathbf{K})[\mathbf{x}(t) - \mathbf{x}(\infty)]$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}(t) \tag{10–100}$$

Equation (10–100) defines the error dynamics.

Since the desired eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ are

$$\mu_1 = -2 + j3.464, \qquad \mu_2 = -2 - j3.464, \qquad \mu_3 = -10$$

we have the desired characteristic equation as follows:

$$\begin{aligned}(s - \mu_1)(s - \mu_2)(s - \mu_3) &= (s + 2 - j3.464)(s + 2 + j3.464)(s + 10) \\ &= s^3 + 14s^2 + 56s + 160 \\ &= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0\end{aligned}$$

Hence

$$\alpha_1 = 14, \qquad \alpha_2 = 56, \qquad \alpha_3 = 160$$

The state feedback gain matrix $\mathbf{K}$ is given by Equation (10–97), or

$$\begin{aligned}\mathbf{K} &= [\alpha_3 - a_3 \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1]\mathbf{T}^{-1} \\ &= [160 - 0 \;\vdots\; 56 - 2 \;\vdots\; 14 - 3]\mathbf{I} \\ &= [160 \quad 54 \quad 11]\end{aligned}$$

The step response of this system can be obtained easily by a computer simulation. Since

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [160 \quad 54 \quad 11] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

the state equation for the designed system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 160 \end{bmatrix} r$$

and the output equation is

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The unit-step response curve $y(t)$ versus t obtained in the computer simulation is shown in Figure 10–17.

Notice that $\dot{\mathbf{x}}(\infty) = \mathbf{0}$. Hence we have, from Equation (10–99),

$$(\mathbf{A} - \mathbf{BK})\mathbf{x}(\infty) = -\mathbf{B}k_1 r$$

Since

$$(\mathbf{A} - \mathbf{BK})^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{7}{20} & -\frac{7}{80} & -\frac{1}{160} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we have

$$\mathbf{x}(\infty) = -(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}k_1 r = -\begin{bmatrix} -\frac{7}{20} & -\frac{7}{80} & -\frac{1}{160} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (160)r$$

$$= \begin{bmatrix} \frac{1}{160} \\ 0 \\ 0 \end{bmatrix} (160)r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

Clearly, $x_1(\infty) = y(\infty) = r$. There is no steady-state error in the step response.

Note that since

$$u(\infty) = -\mathbf{Kx}(\infty) + k_1 r(\infty) = -\mathbf{Kx}(\infty) + k_1 r$$

we have

$$u(\infty) = -[160 \quad 54 \quad 11] \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} + 160r$$

$$= -[160 \quad 54 \quad 11] \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + 160r$$

$$= -160r + 160r = 0$$

At steady state the control signal u becomes zero.

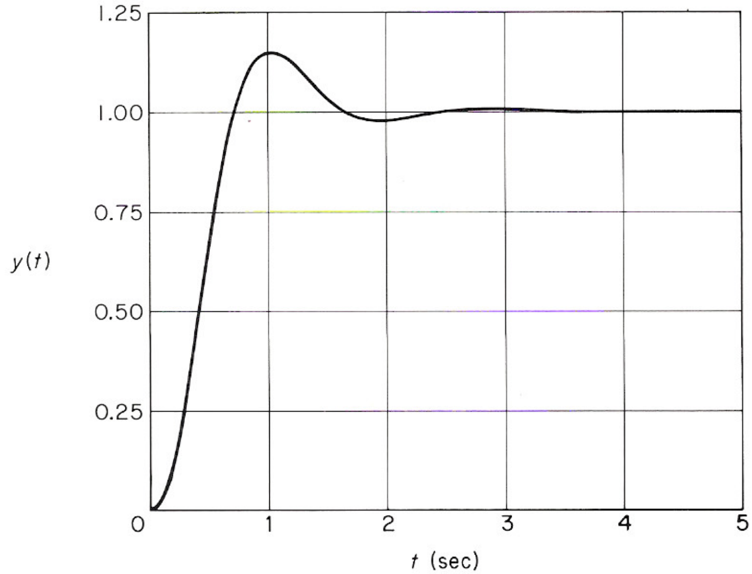

Figure 10–17 Unit-step response curve $y(t)$ versus t for the system designed in Example 10–7.

Design of type 1 servo system where the plant has no integrator. Since the plant has no integrator (type 0 plant), the basic principle of the design of a type 1 servo system is to insert an integrator in the feedforward path between the error comparator and the plant as shown in Figure 10–18. (The block diagram of Figure 10–18 is a basic form of the type 1 servo system where the plant has no integrator.) From the diagram we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{10–101}$$

$$y = \mathbf{C}\mathbf{x} \tag{10–102}$$

$$u = -\mathbf{K}\mathbf{x} + k_1\xi \tag{10–103}$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x} \tag{10–104}$$

where $\mathbf{x}$ = state vector of the plant (n-vector)
u = control signal (scalar)
y = output signal (scalar)
ξ = output of the integrator (state variable of the system, scalar)
r = reference input signal (step function, scalar)
$\mathbf{A}$ = $n \times n$ constant matrix
$\mathbf{B}$ = $n \times 1$ constant matrix
$\mathbf{C}$ = $1 \times n$ constant matrix

We assume that the plant given by Equation (10–101) is completely state controllable. The transfer function of the plant can be given by

$$G_p(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

To avoid the possibility of the inserted integrator being canceled by the zero at the origin of the plant, we assume that $G_p(s)$ has no zero at the origin.

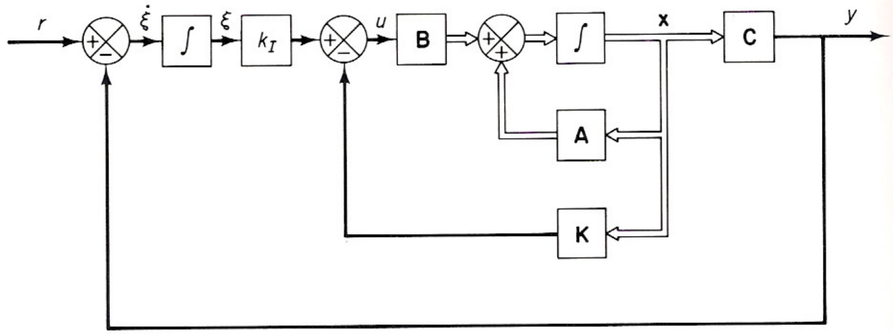

Figure 10–18
Type 1 servo system.

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by an equation that is a combination of Equations (10–101) and (10–104):

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \tag{10–105}$$

We shall design an asymptotically stable system such that $\mathbf{x}(\infty)$, $\xi(\infty)$, and $u(\infty)$ approach constant values, respectively. Then, at steady state $\dot{\xi}(t) = 0$, and we get $y(\infty) = r$.

Notice that at steady state we have

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty) \tag{10–106}$$

Noting that $r(t)$ is a step input, we have $r(\infty) = r(t) = r$ (constant) for $t > 0$. By subtracting Equation (10–106) from Equation (10–105), we obtain

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)] \tag{10–107}$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$

$$\xi(t) - \xi(\infty) = \xi_e(t)$$

$$u(t) - u(\infty) = u_e(t)$$

Then Equation (10–107) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t) \tag{10–108}$$

where

$$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t) \tag{10–109}$$

Define a new $(n + 1)$th-order error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Then Equation (10–108) becomes

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \tag{10–110}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

and Equation (10–109) becomes

$$u_e = -\hat{\mathbf{K}}\mathbf{e} \tag{10–111}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \mid -k_I]$$

The basic idea of designing the type 1 servo system here is to design a stable $(n + 1)$th-order regulator system that will bring the new error vector $\mathbf{e}(t)$ to zero, given any initial condition $\mathbf{e}(0)$.

Equations (10–110) and (10–111) describe the dynamics of the $(n + 1)$th-order regulator system. If the system defined by Equation (10–110) is completely state controllable, then, by specifying the desired characteristic equation for the system, matrix $\hat{\mathbf{K}}$ can be determined by the pole placement technique presented in Section 10–2.

The steady-state values of $\mathbf{x}(t)$, $\xi(t)$, and $u(t)$ can be found as follows: At steady state $(t = \infty)$, from Equations (10–101) and (10–104), we have

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{A}\mathbf{x}(\infty) + \mathbf{B}u(\infty)$$

$$\dot{\xi}(\infty) = 0 = r - \mathbf{C}\mathbf{x}(\infty)$$

which can be combined into one vector-matrix equation:

$$\begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ u(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ r \end{bmatrix}$$

If matrix $\mathbf{P}$, defined by

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} \tag{10–112}$$

is of rank $n + 1$, then its inverse exists and

$$\begin{bmatrix} \mathbf{x}(\infty) \\ u(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ -r \end{bmatrix}$$

Also, from Equation (10–103) we have

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_I\xi(\infty)$$

and therefore we have

$$\xi(\infty) = \frac{1}{k_I}[u(\infty) + \mathbf{K}\mathbf{x}(\infty)]$$

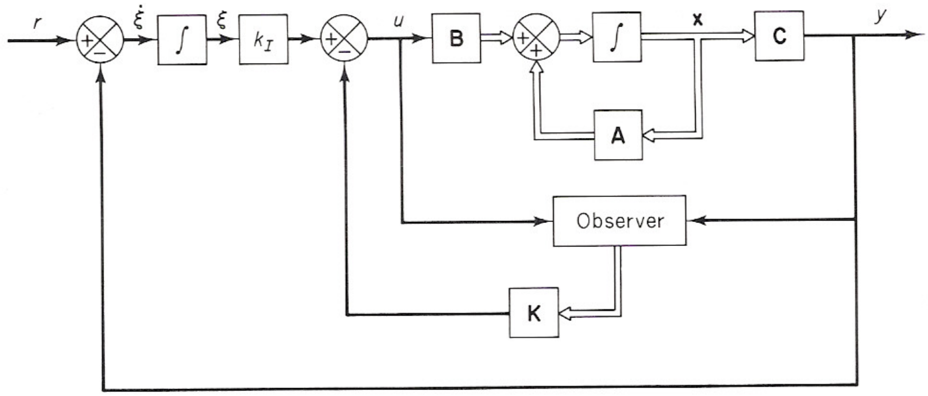

Figure 10–19
Type 1 servo system with state observer.

It is noted that, if matrix **P** given by Equation (10–112) has rank $n + 1$, then the system defined by Equation (10–110) becomes completely state controllable (see Problem A–10–14). Therefore, if the rank of matrix **P** given by Equation (10–112) is $n + 1$, then the solution to this problem can be obtained by the pole placement approach.

The state error equation can be obtained by substituting Equation (10–111) into Equation (10–110).

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e} \tag{10–113}$$

If the desired eigenvalues of matrix $\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}$ (that is, the desired closed-loop poles) are specified as $\mu_1, \mu_2, \ldots, \mu_{n+1}$, then the state feedback gain matrix **K** and the integral gain constant k_I can be determined. In the actual design, it is necessary to consider several different matrices $\hat{\mathbf{K}}$ (which correspond to several different sets of desired eigenvalues) and carry out computer simulations to find the one that yields the best overall system performance. Then choose the best one as the matrix $\hat{\mathbf{K}}$.

As is usually the case, not all state variables can be directly measurable. If this is the case, we need to use a state observer. Figure 10–19 shows a block diagram of a type 1 servo system with a state observer.

EXAMPLE 10–8

Referring to Example 10–2, consider the inverted pendulum system shown in Figure 10–3. In this example, we are concerned only with the motion of the pendulum and motion of the cart on the plane of the page. We assume that the pendulum angle θ and the angular velocity $\dot{\theta}$ are small so that $\sin\theta \doteqdot \theta$, $\cos\theta \doteqdot 1$, and $\theta\dot{\theta}^2 \doteqdot 0$. We also assume the same numerical values for M, m, and l as we used in Example 10–2.

It is desired to keep the inverted pendulum upright as much as possible and yet control the position of the cart, for instance, move the cart in a step fashion. To control the position of the cart, we need to build a type 1 servo system. The inverted-pendulum system mounted on a cart does not have an integrator. Therefore, we feed the position signal y (which indicates the position of the cart) back to the input and insert an integrator in the feedforward path, as shown in Figure 10–20.

As in the case of Example 10–2, we define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$x_3 = x$$

$$x_4 = \dot{x}$$

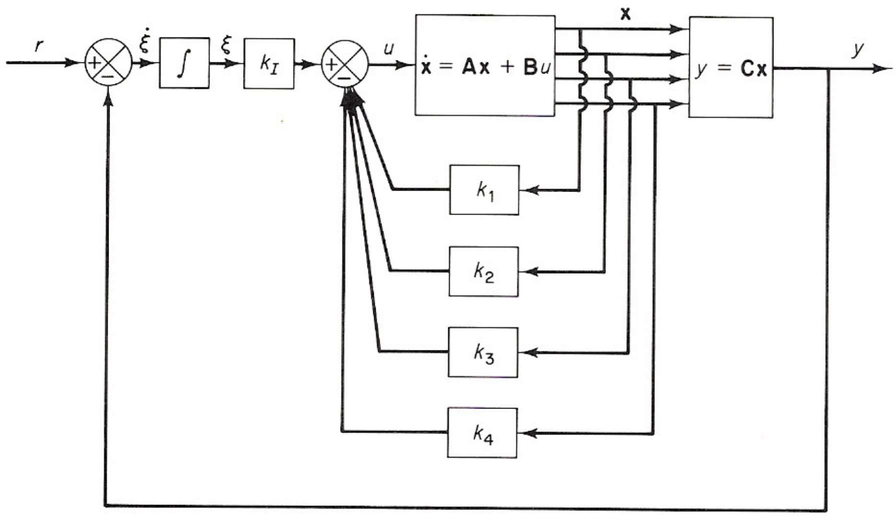

Figure 10–20 Inverted-pendulum control system. (Type 1 servo system where the plant has no integrator.)

Then, referring to Example 10–2 and Figure 10–20, the equations for the inverted pendulum system are

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \tag{10–114}$$

$$y = \mathbf{Cx} \tag{10–115}$$

$$u = -\mathbf{Kx} + k_I\xi \tag{10–116}$$

$$\dot{\xi} = r - y = r - \mathbf{Cx} \tag{10–117}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 0 \quad 1 \quad 0]$$

For the type 1 servo system, we have the state error equation as given by Equation (10–110):

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \tag{10–118}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and the control signal is given by Equation (10–111):

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \mid -k_I] = [k_1 \quad k_2 \quad k_3 \quad k_4 \mid -k_I]$$

We shall determine the necessary state feedback gain matrix $\hat{\mathbf{K}}$ by use of the pole placement technique. We shall use Equation (10–13) for the determination of matrix $\hat{\mathbf{K}}$.

Before we proceed further, we must examine the rank of matrix $\mathbf{P}$, where

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

Matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{10–119}$$

The rank of this matrix is 5. (See Problem A–10–16 for the rank test.) Therefore, the system defined by Equation (10–118) is completely state controllable and arbitrary pole placement is possible. (Refer to Problem A–10–14.) We shall next obtain the characteristic equation for the system given by Equation (10–118).

$$\begin{aligned} |s\mathbf{I} - \hat{\mathbf{A}}| &= \begin{vmatrix} s & -1 & 0 & 0 & 0 \\ -20.601 & s & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0.4905 & 0 & 0 & s & 0 \\ 0 & 0 & 1 & 0 & s \end{vmatrix} \\ &= s^3(s^2 - 20.601) \\ &= s^5 - 20.601s^3 \\ &= s^5 + a_1s^4 + a_2s^3 + a_3s^2 + a_4s + a_5 = 0 \end{aligned}$$

Hence

$$a_1 = 0, \quad a_2 = -20.601, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0$$

To obtain a reasonable speed and damping in the response of the designed system (for example, the settling time of approximately 4 ~ 5 sec and the maximum overshoot of 15% ~ 16% in the step response of the cart), let us choose the desired closed-loop poles at $s = \mu_i$ ($i = 1, 2, 3, 4, 5$), where

$$\mu_1 = -1 + j1.732, \quad \mu_2 = -1 - j1.732, \quad \mu_3 = -5, \quad \mu_4 = -5, \quad \mu_5 = -5$$

(This is a possible set of desired closed-loop poles. Other sets can be chosen.) Then the desired characteristic equation becomes

$$\begin{aligned} &(s - \mu_1)(s - \mu_2)(s - \mu_3)(s - \mu_4)(s - \mu_5) \\ &= (s + 1 - j1.732)(s + 1 + j1.732)(s + 5)(s + 5)(s + 5) \\ &= s^5 + 17s^4 + 109s^3 + 335s^2 + 550s + 500 \\ &= s^5 + \alpha_1s^4 + \alpha_2s^3 + \alpha_3s^2 + \alpha_4s + \alpha_5 = 0 \end{aligned}$$

Hence

$$\alpha_1 = 17, \quad \alpha_2 = 109, \quad \alpha_3 = 335, \quad \alpha_4 = 550, \quad \alpha_5 = 500$$

The next step is to obtain the transformation matrix $\mathbf{T}$ given by Equation (10–4):

$$\mathbf{T} = \mathbf{MW}$$

where **M** and **W** are given by Equations (10–5) and (10–6), respectively:

$$\mathbf{M} = [\hat{\mathbf{B}} \mid \hat{\mathbf{A}}\hat{\mathbf{B}} \mid \hat{\mathbf{A}}^2\hat{\mathbf{B}} \mid \hat{\mathbf{A}}^3\hat{\mathbf{B}} \mid \hat{\mathbf{A}}^4\hat{\mathbf{B}}]$$

$$= \begin{bmatrix} 0 & -1 & 0 & -20.601 & 0 \\ -1 & 0 & -20.601 & 0 & -(20.601)^2 \\ 0 & 0.5 & 0 & 0.4905 & 0 \\ 0.5 & 0 & 0.4905 & 0 & 10.1048 \\ 0 & 0 & -0.5 & 0 & -\ 0.4905 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & 1 \\ a_3 & a_2 & a_1 & 1 & 0 \\ a_2 & a_1 & 1 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -20.601 & 0 & 1 \\ 0 & -20.601 & 0 & 1 & 0 \\ -20.601 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\mathbf{T} = \mathbf{MW} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -9.81 & 0 & 0.5 & 0 \\ 0 & 0 & -9.81 & 0 & 0.5 \\ 9.81 & 0 & -0.5 & 0 & 0 \end{bmatrix}$$

The inverse of matrix **T** is

$$\mathbf{T}^{-1} = \begin{bmatrix} 0 & -\dfrac{0.25}{(9.81)^2} & 0 & -\dfrac{0.5}{(9.81)^2} & \dfrac{1}{9.81} \\ -\dfrac{0.5}{9.81} & 0 & -\dfrac{1}{9.81} & 0 & 0 \\ 0 & -\dfrac{0.5}{9.81} & 0 & -\dfrac{1}{9.81} & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Referring to Equation (10–13), matrix $\hat{\mathbf{K}}$ is given by

$$\begin{aligned} \hat{\mathbf{K}} &= [\alpha_5 - a_5 \mid \alpha_4 - a_4 \mid \alpha_3 - a_3 \mid \alpha_2 - a_2 \mid \alpha_1 - a_1]\mathbf{T}^{-1} \\ &= [500 - 0 \mid 550 - 0 \mid 335 - 0 \mid 109 + 20.601 \mid 17 - 0]\mathbf{T}^{-1} \\ &= [500 \mid 550 \mid 335 \mid 129.601 \mid 17]\mathbf{T}^{-1} \\ &= [-157.6336 \quad -35.3733 \quad -56.0652 \quad -36.7466 \quad 50.9684] \\ &= [k_1 \quad k_2 \quad k_3 \quad k_4 \quad -k_I] \end{aligned}$$

Thus we get

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3 \quad k_4] = [-157.6336 \quad -35.3733 \quad -56.0652 \quad -36.7466]$$

and

$$k_I = -50.9684$$

Once we determine the feedback gain matrix **K** and the integral gain constant k_I, the step response in the cart position can be obtained by solving the following equation:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathrm{u} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{10–120}$$

Since

$$u = -\mathbf{K}\mathbf{x} + k_I \xi$$

Equation (10–120) can be written as follows:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}k_I \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -137.0326 & -35.3733 & -56.0652 & -36.7466 & 50.9684 \\ 0 & 0 & 0 & 1 & 0 \\ 78.3263 & 17.6867 & 28.0326 & 18.3733 & -25.4842 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r \tag{10–121}$$

Figure 10–21 shows the response curves $x_1(t)$ versus t, $x_2(t)$ versus t, $x_3(t)$ versus t, $x_4(t)$ versus t, $\xi(t)$ versus t, and $u(t)$ versus t, where the input $r(t)$ to the cart is a step function of magnitude 0.5 [that is, $r(t) = 0.5$ m]. Note that $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$, and $x_4 = \dot{x}$. All initial conditions are set equal to zero.

The step response in $x_3(t)$ [$= x(t)$] shows the settling time of approximately 4.5 sec and the maximum overshoot of approximately 14.8%, as desired. An interesting point in the position curve [$x_3(t)$ versus t curve] is that the cart moves backward for the first 0.6 sec or so to make the pendulum fall forward. Then the cart accelerates to move in the positive direction.

The response curve $x_3(t)$ versus t clearly shows that $x_3(\infty)$ approaches r. Also, $x_1(\infty) = 0$, $x_2(\infty) = 0$, $x_4(\infty) = 0$, and $\xi(\infty) = 0.55$. This result can be verified by the analytical approach as given below. At steady state, from Equations (10–114) and (10–117) we have

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{A}\mathbf{x}(\infty) + \mathbf{B}u(\infty)$$

$$\dot{\xi}(\infty) = 0 = r - \mathbf{C}\mathbf{x}(\infty)$$

which can be combined into

$$\begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ u(\infty) \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Since we earlier found that the rank of matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

is 5, it has the inverse. Hence

$$\begin{bmatrix} \mathbf{x}(\infty) \\ u(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -r \end{bmatrix}$$

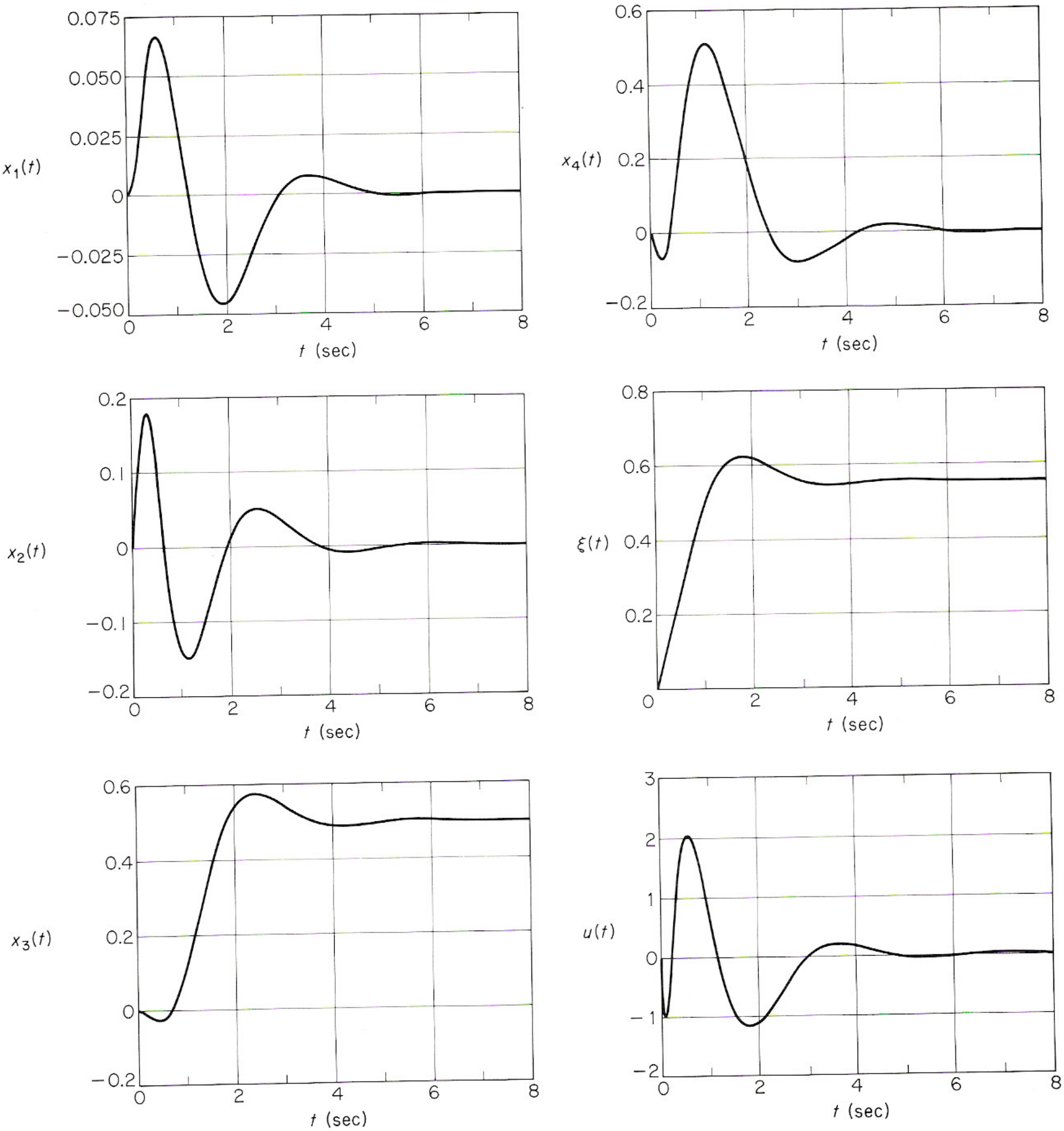

Figure 10–21
Response curves $x_1(t)$ versus t, $x_2(t)$ versus t, $x_3(t)$ versus t, $x_4(t)$ versus t, $\xi(t)$ versus t, and $u(t)$ versus t for the system defined by Eqs. (10–121) and (10–116). (The input to the cart is a step input of magnitude 0.5.)

Referring to Equation (10–119), we have

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \dfrac{0.5}{9.81} & 0 & \dfrac{1}{9.81} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0.05 & 0 & 2.1 & 0 \end{bmatrix}$$

Hence

$$\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \\ x_4(\infty) \\ u(\infty) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{0.5}{9.81} & 0 & \dfrac{1}{9.81} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0.05 & 0 & 2.1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \\ 0 \end{bmatrix}$$

Consequently,

$$y(\infty) = \mathbf{C}\mathbf{x}(\infty) = [0 \quad 0 \quad 1 \quad 0] \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \\ x_4(\infty) \end{bmatrix} = x_3(\infty) = r$$

Since

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{A}\mathbf{x}(\infty) + \mathbf{B}u(\infty)$$

or

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} u(\infty)$$

we get

$$u(\infty) = 0$$

Since $u(\infty) = 0$, we have, from Equation (10–116),

$$u(\infty) = 0 = -\mathbf{K}\mathbf{x}(\infty) + k_I \xi(\infty)$$

and so

$$\xi(\infty) = \frac{1}{k_I}[\mathbf{K}\mathbf{x}(\infty)] = \frac{1}{k_I} k_3 x_3(\infty) = \frac{-56.0652}{-50.9684} r = 1.1r$$

Hence, for $r = 0.5$, we have

$$\xi(\infty) = 0.55$$

as shown in Figure 10–21.

It is noted that as in any design problem, if the speed and damping are not quite satisfactory, then we must modify the desired characteristic equation and determine a new matrix $\hat{\mathbf{K}}$. Computer simulations must be repeated until a satisfactory result is obtained.