Solutions – Chapter 4

Sampling of Continuous-Time Signals

4.1.

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(2\pi(100)n\frac{1}{400}\right) \\ &= \sin\left(\frac{\pi}{2}n\right) \end{aligned}$$

4.2. The discrete-time sequence

$$x[n] = \cos(\frac{\pi n}{4})$$

results by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_o t).$$

Since $\omega = \Omega T$ and $T = 1/1000$ seconds, the signal frequency could be:

$$\Omega_o = \frac{\pi}{4} \cdot 1000 = 250\pi$$

or possibly:

$$\Omega_o = (2\pi + \frac{\pi}{4}) \cdot 1000 = 2250\pi.$$

4.3. (a) Since $x[n] = x_c(nT)$,

$$\begin{aligned} \frac{\pi n}{3} &= 4000\pi nT \\ T &= \frac{1}{12000} \end{aligned}$$

(b) No. For example, since

$$\cos(\frac{\pi}{3}n) = \cos(\frac{7\pi}{3}n),$$

T can be 7/12000.

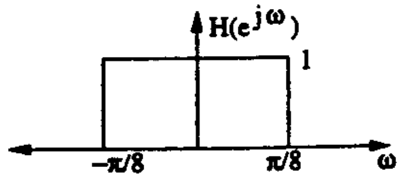
4.4. (a) Letting $T = 1/100$ gives

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n\frac{1}{100}\right) + \cos\left(40\pi n\frac{1}{100}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

(b) No, another choice is $T = 11/100$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n\frac{11}{100}\right) + \cos\left(40\pi n\frac{11}{100}\right) \\ &= \sin\left(\frac{11\pi n}{5}\right) + \cos\left(\frac{22\pi n}{5}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

4.5. A plot of $H(e^{j\omega})$ appears below.



(a)

$$x_c(t) = 0, \quad , |\Omega| \geq 2\pi \cdot 5000$$

The Nyquist rate is 2 times the highest frequency. $\Rightarrow T = \frac{1}{10,000}$ sec. This avoids all aliasing in the C/D converter.

(b)

$$\begin{aligned} \frac{1}{T} &= 10kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{10,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 625\text{rad/sec} \\ f_c &= 625Hz \end{aligned}$$

(c)

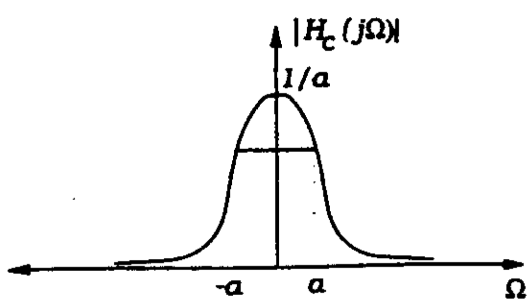
$$\begin{aligned} \frac{1}{T} &= 20kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{20,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 1250\text{rad/sec} \\ f_c &= 1250Hz \end{aligned}$$

4.6. (a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}\,dt \\ &= \int_{0}^{\infty} a^{-at}e^{-j\Omega t}\,dt \\ &= \frac{1}{a+j\Omega} \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = \left(\frac{1}{a^2+\Omega^2}\right)^{\frac{1}{2}}.$$



(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

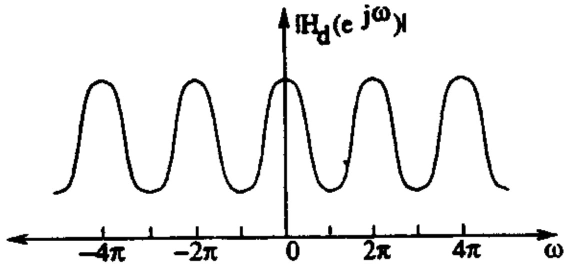
$$h_d[n] = Te^{-anT}u[n]$$

$$H_d(e^{j\omega}) = \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} = \frac{T}{1-e^{-aT}e^{-j\omega}}$$

Taking the magnitude of this response

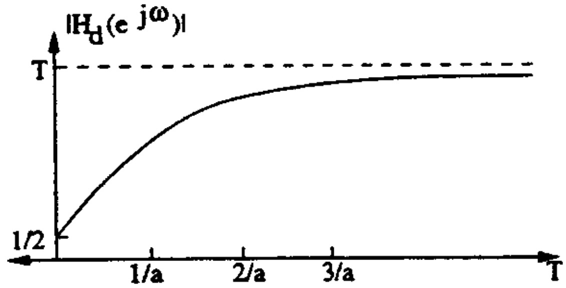
$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.



(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$|H_d(e^{j\pi})| = \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} = \frac{T}{1+e^{-aT}}.$$



4.7. The continuous-time signal contains an attenuated replica of the original signal with a delay of τ_d.

$$x_c(t) = s_c(t) + \alpha s_c(t-\tau_d)$$

(a) Taking the Fourier transform of the analog signal:

$$X_c(j\Omega) = S_c(j\Omega)\cdot(1+\alpha e^{-j\tau_d\Omega})$$

Note that $X_c(j\Omega)$ is zero for $|\Omega| > \pi/T$. Sampling the continuous-time signal yields the discrete-time sequence, $x[n]$. The Fourier transform of the sequence is

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} S_c\left(\frac{j\omega}{T}+j\frac{2\pi r}{T}\right) + \frac{\alpha}{T}\sum_{r=-\infty}^{\infty} S_c\left(\frac{j\omega}{T}+j\frac{2\pi r}{T}\right)e^{-j\tau_d\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)}.$$

(b) The desired response:

$$H(j\Omega) = \begin{cases} 1+\alpha e^{-j\tau_d\Omega}, & \text{for } |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

Using $\omega = \Omega T$, we obtain a discrete-time system which simulates the above response:

$$H(e^{j\omega}) = 1 + \alpha e^{-j\frac{\tau_d \omega}{T}}$$

(c) We need to take the inverse Fourier transform of the discrete-time impulse response of part (b).

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+\alpha e^{-j\frac{\tau_d\omega}{T}})e^{j\omega n}\,d\omega \end{aligned}$$

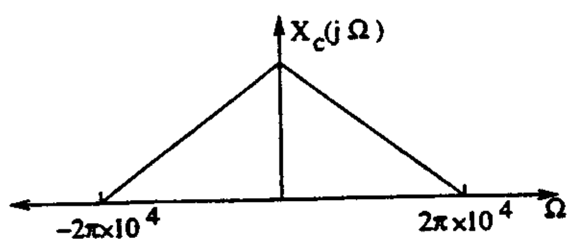
(i) Consider the case when $\tau_d = T$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-1)})\,d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-1)]}{\pi(n-1)} \\ &= \delta[n] + \alpha\delta[n-1] \end{aligned}$$

(ii) For $\tau_d = T/2$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-\frac{1}{2})})\,d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \\ &= \delta[n] + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \end{aligned}$$

4.8. A plot of $X_c(j\Omega)$ appears below.



(a) For $x_c(t)$ to be recoverable from $x[n]$, the transform of the discrete signal must have no aliasing. When sampling, the radian frequency is related to the analog frequency by

$$\omega = \Omega T.$$

No aliasing will occur if the sampling interval satisfies the Nyquist Criterion. Thus, for the band-limited signal, $x_c(t)$, we should select T as:

$$T \le \frac{1}{2\times 10^4}.$$

(b) Assuming that the system is linear and time-invariant, the convolution sum describes the input-output relationship.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We are given

$$\begin{aligned} y[n] &= T \sum_{k=-\infty}^{n} x[k] \\ &= T \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

Hence, we may infer that the impulse response of the system

$$h[n] = T \cdot u[n].$$

(c) We use the expression for $y[n]$ as given and examine the limit

$$\begin{aligned} \lim_{n\to\infty} y[n] &= \lim_{n\to\infty} T \cdot \sum_{k=-\infty}^{n} x[k] \\ &= T \cdot \sum_{k=-\infty}^{\infty} x[k] \end{aligned}$$

Recall the analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Hence,

$$\lim_{n\to\infty} y[n] = T \cdot X(e^{j\omega})|_{\omega=0}$$

(d) We use the result from part (c). Noting that

$$X(e^{j\omega}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c\left(\frac{j\omega}{T} + \frac{j2\pi r}{T}\right).$$

Thus, we have

$$T \cdot X(e^{j\omega})|_{\omega=0} = \sum_{r=-\infty}^{\infty} X_c\left(\frac{j2\pi r}{T}\right)$$

From the given information, we seek a value of T such that:

$$\begin{aligned} \sum_{r=-\infty}^{\infty} X_c\left(\frac{j2\pi r}{T}\right) &= \int_{-\infty}^{\infty} x_c(t)\,dt \\ &= X_c(j\Omega)|_{\Omega=0} \end{aligned}$$

For the final equality to be true, there must be no contribution from the terms for which $r \neq 0$. That is, we require no aliasing at $\Omega = 0$. Since we are only interested in preserving the spectral component at $\Omega = 0$, we may sample at a rate which is lower than the Nyquist rate. The maximum value of T to satisfy these conditions is

$$T \leq \frac{1}{1 \times 10^4}.$$

4.9. (a) Since $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, $X(e^{j\omega})$ is periodic with period π.

(b) Using the inverse DTFT,

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{(2\pi)} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{(2\pi)} X(e^{j(\omega-\pi)})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{(2\pi)} X(e^{j\omega})e^{j(\omega+\pi)n}d\omega \\ &= \frac{1}{2\pi}e^{j\pi n}\int_{(2\pi)} X(e^{j\omega})e^{j\omega n}d\omega \\ &= (-1)^n x[n]. \end{aligned}$$

All odd samples of $x[n] = 0$, because $x[n] = -x[n]$. Hence $x[3] = 0$.

(c) Yes, $y[n]$ contains all even samples of $x[n]$, and all odd samples of $x[n]$ are 0.

$$x[n] = \begin{cases} y[n/2], & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

4.10. Use $x[n] = x_c(nT)$, and simplify:

(a) $x[n] = \cos(2\pi n/3)$.

(b) $x[n] = \sin(4\pi n/3) = -\sin(2\pi n/3)$

(c) $x[n] = \frac{\sin(2\pi n/5)}{\pi n/5000}$

4.11. (a) Pick T such that

$$x[n] = x_c(nT) = \sin(10\pi nT) = \sin(\pi n/4) \qquad \Longrightarrow T = 1/40$$

There are other choices. For example, by realizing that $\sin(\pi n/4) = \sin(9\pi n/4)$, we find $T = 9/40$.

(b) Choose $T = 1/20$ to make $x[n] = x_c(nT)$. This is unique.

4.12. (a) Notice first that $H(e^{j\omega}) = 10j\omega, -\pi \le \omega < \pi$.

(i) After sampling,

$$\begin{aligned} x[n] &= \cos(\frac{3\pi}{5}n), \\ y[n] &= |H(e^{j\frac{3\pi}{5}})|\cos(\frac{3\pi}{5}n + \angle H(e^{j\frac{3\pi}{5}})) \\ &= 6\pi\cos(\frac{3\pi}{5}n + \frac{\pi}{2}) \\ &= -6\pi\sin(\frac{3\pi}{5}n) \\ y_c(t) &= -6\pi\sin(6\pi t). \end{aligned}$$

(ii) After sampling, $x[n] = \cos(\frac{7\pi}{5}n) = \cos(\frac{3\pi}{5}n)$, so again, $y_c(t) = -6\pi\sin(6\pi t)$.

(b) $y_c(t)$ is what you would expect from a differentiator in the first case but not in the second case. This is because aliasing has occurred in the second case.

4.13. (a)

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{20}t) \\ y_c(t) &= \sin(\frac{\pi}{20}(t-5)) \\ &= \sin(\frac{\pi}{20}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(b) We get the same result as before:

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{10}t) \\ y_c(t) &= \sin(\frac{\pi}{10}(t-2.5)) \\ &= \sin(\frac{\pi}{10}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(c) The sampling period T is not limited by the continuous time system $h_c(t)$.

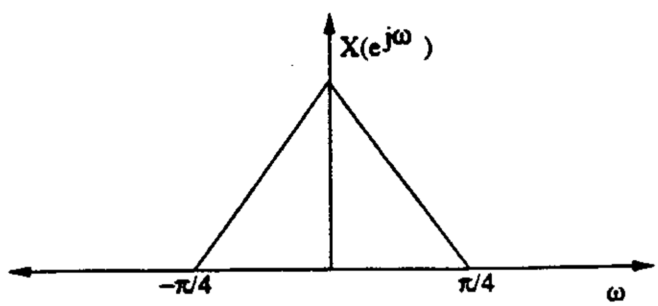
4.14. There is no loss of information if $X(e^{j\omega/2})$ and $X(e^{j(\omega/2-\pi)})$ do not overlap. This is true for (b), (d), (e).

4.15. The output $x_r[n] = x[n]$ if no aliasing occurs as result of downsampling. That is, $X(e^{j\omega}) = 0$ for $\pi/3 \le |\omega| \le \pi$.

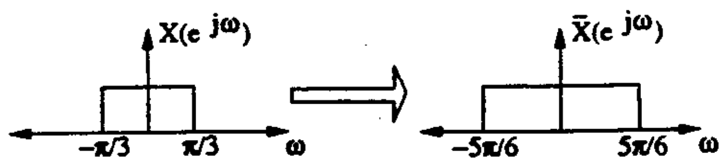
(a) $x[n] = \cos(\pi n/4)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/4$, so there is no aliasing. $x_r[n] = x[n]$.

(b) $x[n] = \cos(\pi n/2)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/2$, so there is aliasing. $x_r[n] \neq x[n]$.

(c) A sketch of $X(e^{j\omega})$ is shown below. Clearly there will be no aliasing and $x_r[n] = x[n]$.



4.16. (a) In the frequency domain, we have



$$\frac{M}{L} = \frac{5\pi/6}{\pi/3} = \frac{5}{2}$$

This is unique.

(b) One choice is

$$\frac{M}{L} = \frac{\pi/2}{3\pi/4} = \frac{2}{3}$$

However, this is not unique. We can also write $\tilde{x}_d[n] = \cos(\frac{5\pi}{2}n)$, so another choice is

$$\frac{M}{L} = \frac{5\pi/2}{3\pi/4} = \frac{10}{3}$$

4.17. (a) In the frequency domain,

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < 2\pi/3 \\ 0, & 2\pi/3 < |\omega| < \pi \end{cases}$$

After the sampling rate change,

$$\tilde{X}_d(e^{j\omega}) = \begin{cases} 4/3, & |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| < \pi \end{cases},$$

which leads to

$$x[n] = \frac{4}{3}\frac{\sin(\pi n/2)}{\pi n}$$

(b) Upsampling by 3 and low-pass filtering $x[n] = \sin(3\pi n/4)$ results in $\sin(\pi n/4)$. Downsampling by 5 gives us $\tilde{x}_d[n] = \sin(5\pi n/4) = -\sin(3\pi n/4)$.

4.18. For the condition to be satisfied, we have to ensure that $\omega_0/L \leq \min(\pi/L, \pi/M)$, so that the lowpass filtering does not cut out part of the spectrum.

(a) $\omega_0/2 \leq \pi/3 \Longrightarrow \omega_{0,max} = 2\pi/3$.

(b) $\omega_0/3 \leq \pi/5 \Longrightarrow \omega_{0,max} = 3\pi/5$.

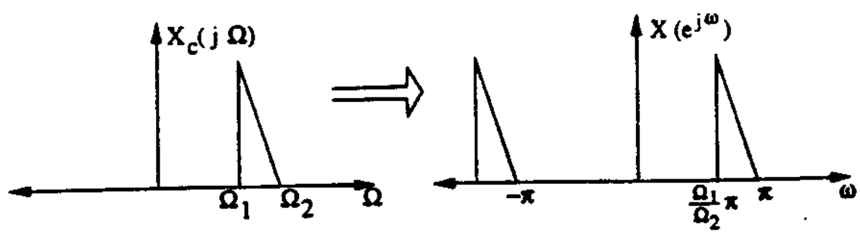
(c) Since $L > M$, there is no chance of aliasing. Hence $\omega_{0,max} = \pi$.

4.19. The nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0$.

4.20. (a) The Nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0 \Longrightarrow F_s \geq 2000$.

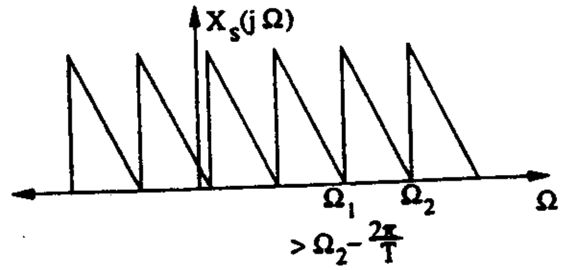
(b) We'd have to sample so that $X(e^{j\omega})$ lies between $|\omega| < \pi/2$. So $F_s \geq 4000$.

4.21. (a) Keeping in mind that after sampling, $\omega = \Omega T$, the Fourier transform of $x[n]$ is

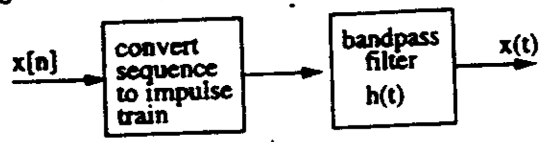


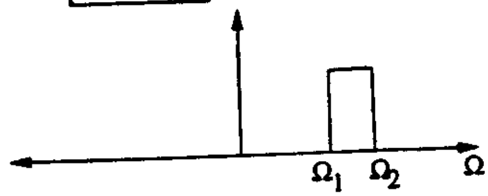
(b) A straight-forward application of the Nyquist criterion would lead to an incorrect conclusion that the sampling rate is at least twice the maximum frequency of $x_c(t)$, or $2\Omega_2$. However, since the spectrum is bandpass, we only need to ensure that the replications in frequency which occur as a result of sampling do not overlap with the original. (See the following figure of $X_s(j\Omega)$.) Therefore, we only need to ensure

$$\Omega_2 - \frac{2\pi}{T} < \Omega_1 \Longrightarrow T < \frac{2\pi}{\Delta\Omega}$$



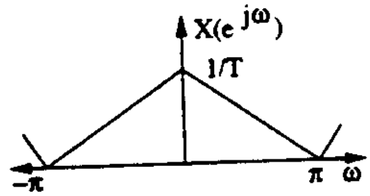
(c) The block diagram along with the frequency response of $h(t)$ is shown here:



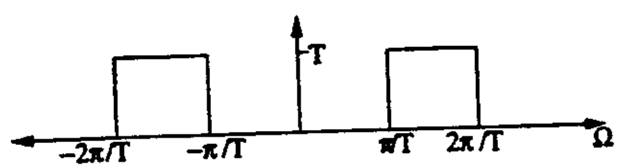


4.22. (a)

$$\omega = \Omega T, \quad T = \frac{2\pi}{\Omega_0}$$



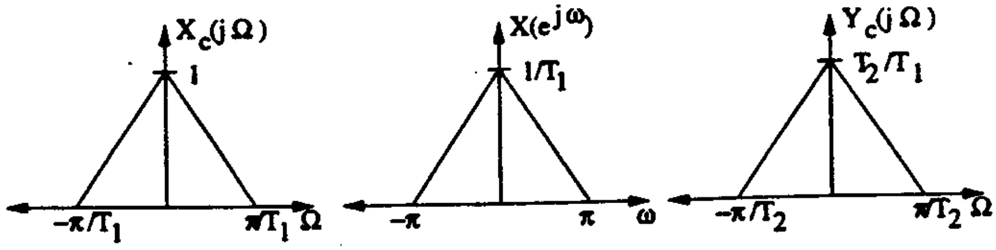
(b) To recover simply filter out the undesired parts of $X(e^{j\omega})$.



(c)

$$T \leq \frac{2\pi}{\Omega_0}$$

4.23. In the frequency domain, we have



$$x_c(t) = 0, \quad |\Omega| \geq \frac{\pi}{T_1}$$

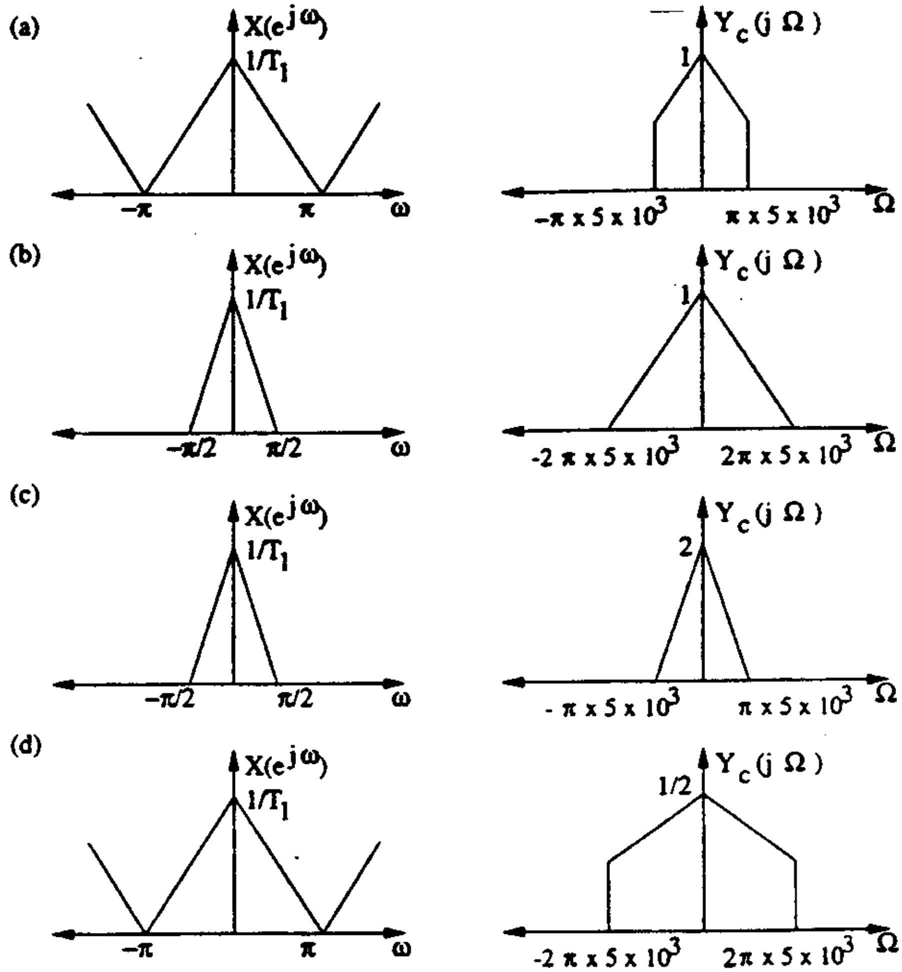
Therefore, since we are sampling this $x_c(t)$ at the Nyquist frequency $x[n]$ will be full band and unaliased.

$$x[n] = x_c(nT_1)$$

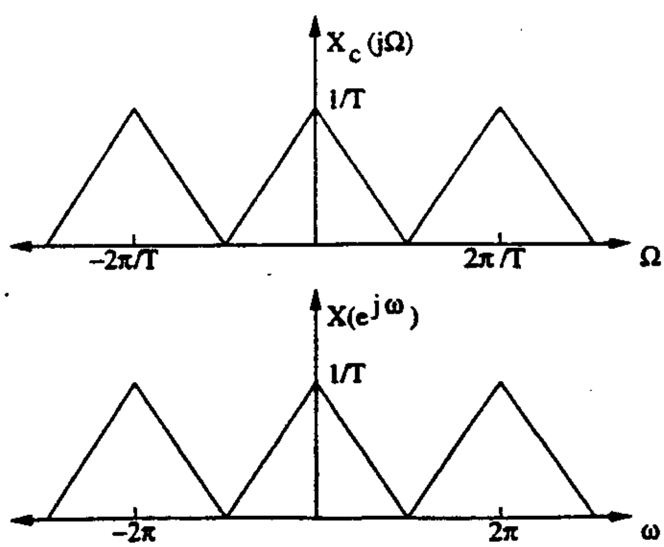
$y_c(t)$ is a band-limited interpolation of $x[n]$ at a different period. Since no aliasing occurs at $x[n]$, the spectrum of $y_c(t)$ will be a frequency axis scaling of the spectrum of $x_c(t)$ for $T_1 > T_2$ or $T_1 < T_2$. As we show in the figure,

$$y_c(t) = \frac{T_2}{T_1} x_c\left(\frac{T_2}{T_1} t\right)$$

4.24. The Fourier transform of $y_c(t)$ is sketched below for each case.



4.25. (a) $x_s(t) = x_c(t)s(t) \Rightarrow X_s(j\Omega) * s(j\Omega)$



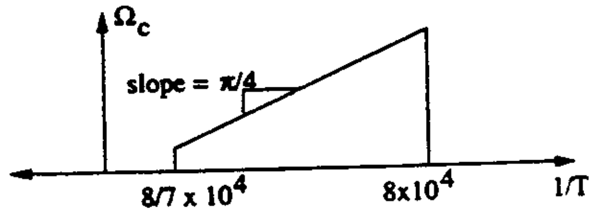
(b) Since $H_d(e^{j\omega})$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$, we don't care about any signal aliasing that occurs in the region $\frac{\pi}{4} \le \omega \le \pi$. We require:

$$\begin{aligned} \frac{2\pi}{T} - 2\pi \cdot 10000 &\ge \frac{\pi}{4T} \\ \frac{1}{T} &\ge \frac{8}{7} \cdot 10000 \\ T &\le \frac{7}{8} \times 10^{-4}\text{sec} \end{aligned}$$

Also, once all of the signal lies in the range $|\omega| \le \frac{\pi}{4}$, the filter will be ineffective, i.e., $\frac{\pi}{4} \le T(2\pi \times 10^4)$. So, $T \ge 12.5\mu$sec.

(c)

$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_c = \frac{\pi}{4T}$$



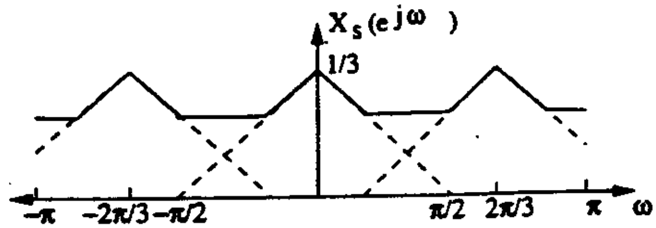
4.26. First we show that $X_s(e^{j\omega})$ is just a sum of shifted versions of $X(e^{j\omega})$:

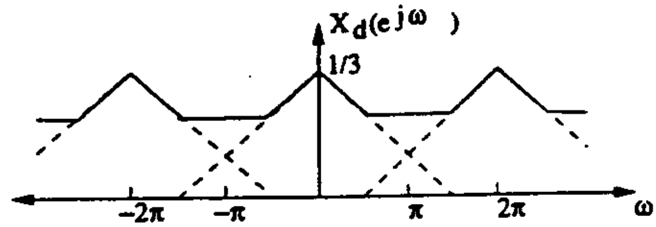
$$\begin{aligned} x_s[n] &= \begin{cases} x[n], & n = Mk, \quad k = 0, \pm 1, \pm 2 \\ 0, & \text{otherwise} \end{cases} \\ &= \left(\frac{1}{M} \sum_{k=0}^{M-1} e^{j(2\pi kn/M)} \right) x[n] \\ X_s(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_s[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{M} \sum_{k=0}^{M-1} x[n] e^{j(2\pi kn/M)} e^{-j\omega n} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n] e^{-j[\omega-(2\pi k/M)]n} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j[\omega-(2\pi k/M)]} \right) \end{aligned}$$

Additionally, $X_d(e^{j\omega})$ is simply $X_s(e^{j\omega})$ with the frequency axis expanded by a factor of M:

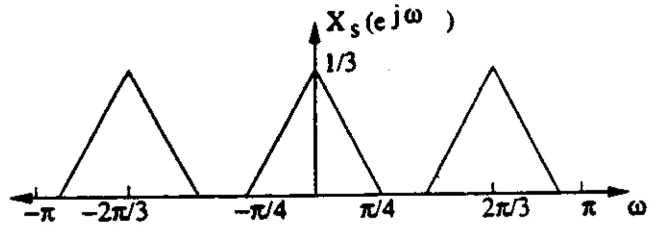
$$\begin{aligned} X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} X_s[Mn] e^{-j\omega n} \\ &= \sum_{l=-\infty}^{\infty} x_s[l] e^{-j(\omega/M)l} \\ &= X_s\left(e^{j(\omega/M)} \right) \end{aligned}$$

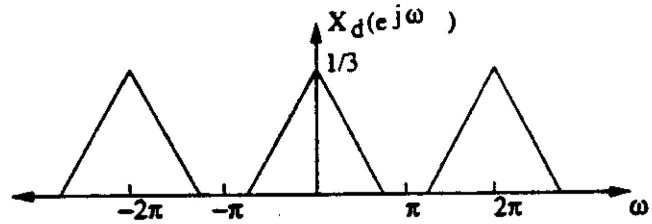
(a) (i) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/2$.





(ii) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/4$.





(b) From the definition of $X_s(e^{j\omega})$, we see that there will be no aliasing if the signal is bandlimited to π/M. In this problem, $M = 3$. Thus the maximum value of ω_H is $\pi/3$.

4.27. Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

When we upsample, the added samples are zeros, so the upsampled signal $x_u[n]$ has the same energy as the original $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x_u[n]|^2,$$

and by Parseval's theorem:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_u(e^{j\omega})|^2 d\omega.$$

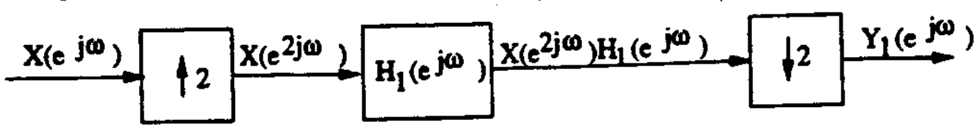
Hence the amplitude of the Fourier transform does not change.

When we downsample, the downsampled signal $x_d[n]$ has less energy than the original $x[n]$ because some samples are discarded. Hence the amplitude of the Fourier transform will change after downsampling.

4.28. (a) Yes, the system is linear because each of the subblocks is linear. The C/D step is defined by $x[n] = x_c(nT)$, which is clearly linear. The DT system is an LTI system. The D/C step consists of converting the sequence to impulses and of CT LTI filtering, both of which are linear.

(b) No, the system is not time-invariant.
For example, suppose that $h[n] = \delta[n]$, $T = 5$ and $x_c(t) = 1$ for $-1 \le t \le 1$. Such a system would result in $x[n] = \delta[n]$ and $y_c(t) = \text{sinc}(\pi/5)$. Now suppose we delay the input to be $x_c(t-2)$. Now $x[n] = 0$ and $y_c(t) = 0$.

4.29. We can analyze the system in the frequency domain:



$Y_1(e^{j\omega})$ is $X(e^{2j\omega})H_1(e^{j\omega})$ downsampled by 2:

$$\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{2j\omega/2})H_1(e^{j\omega/2}) + X(e^{(2j(\omega-2\pi)/2)})H_1(e^{j(\omega-2\pi)/2})\right\} \\
&= \frac{1}{2}\left\{X(e^{j\omega})H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)})H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} \\
&= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} X(e^{j\omega}) \\
&= H_2(e^{j\omega})X(e^{j\omega}) \\
H_2(e^{j\omega}) &= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\}
\end{aligned}$$

4.30.

$$\begin{array}{ll}
X_c(j\Omega) = 0 & |\Omega| \ge 4000\pi \\
Y(j\Omega) = |\Omega| X_c(j\Omega), & 1000\pi \le |\Omega| \le 2000\pi
\end{array}$$

Since only half the frequency band of $X_c(j\Omega)$ is needed, we can alias everything past $\Omega = 2000\pi$. Hence, $T = 1/3000$ s.

Now that T is set, figure out $H(e^{j\omega})$ band edges.

$$\begin{array}{lll}
\omega_1 = \Omega_1 T & \Rightarrow \omega_1 = 2\pi \cdot 500 \cdot \frac{1}{3000} & \Rightarrow \omega_1 = \frac{\pi}{3} \\
\omega_2 = \Omega_2 T & \Rightarrow \omega_2 = 2\pi \cdot 1000 \cdot \frac{1}{3000} & \Rightarrow \omega_2 = \frac{2\pi}{3}
\end{array}$$

$$H(e^{j\omega}) = \begin{cases} |\omega| & \frac{\pi}{3} \le |\omega| \le \frac{2\pi}{3} \\ 0 & 0 \le |\omega| < \frac{\pi}{3}, \frac{2\pi}{3} < |\omega| \le \pi \end{cases}$$

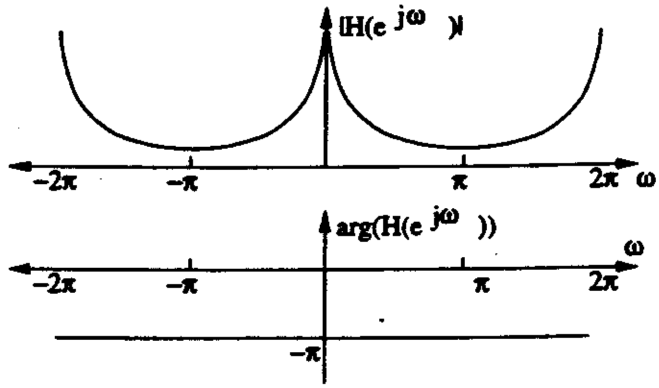
4.31.

$$X_c(j\Omega) = 0, \quad |\Omega| > \frac{\pi}{T}$$

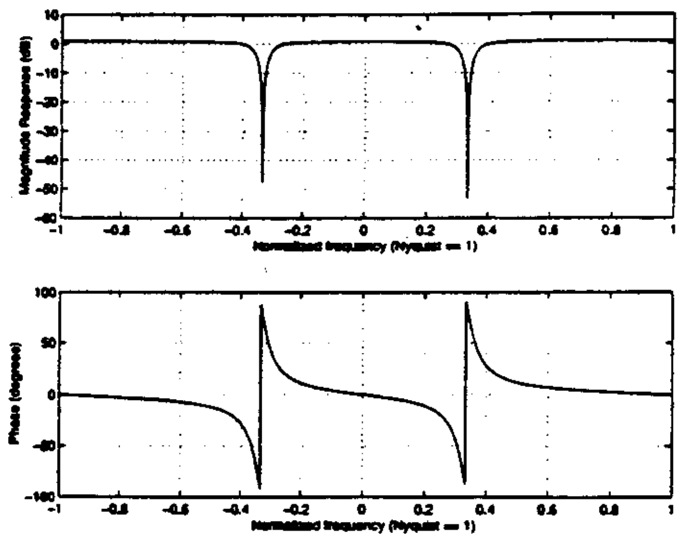
$$y_r(t) = \int_{-\infty}^{t} x_c(\tau)d\tau \Longrightarrow H_c(j\Omega) = \frac{1}{j\Omega}$$

In discrete-time, we want

$$H(e^{j\omega}) = \begin{cases} \frac{1}{j\omega}, & -\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$



4.32. (a) The highest frequency is $\pi/T = \pi \times 10000$.



(b)

(c) To filter the 60Hz out,

$$\omega_0 = T\Omega = \frac{1}{10,000} \cdot 2\pi \cdot 60 = \frac{3\pi}{250}$$

4.33.

$$\begin{aligned} y[n] &= x^2[n] \\ Y(e^{j\omega}) &= X(e^{j\omega}) * X(e^{j\omega}) \end{aligned}$$

therefore, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does if no aliasing occurs.
If $Y(e^{j\omega}) \neq 0, \quad -\pi < \omega < \pi$, then $X(e^{j\omega}) \neq 0, \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2}$ and so $X(j\Omega) = 0, \quad |\Omega| \geq 2\pi(1000)$.
Since $\omega = \Omega T$,

$$\begin{aligned} \frac{\pi}{2} &\geq T \cdot 2\pi(1000) \\ T &\leq \frac{1}{4000} \end{aligned}$$

4.34. (a) Since there is no aliasing involved in this process, we may choose T to be any value. Choose $T=1$ for simplicity. $X_c(j\Omega)=0, |\Omega|\geq \pi/T$. Since $Y_c(j\Omega)=H_c(j\Omega)X_c(j\Omega)$, $Y_c(j\Omega)=0, |\Omega|\geq\pi/T$. Therefore, there will be no aliasing problems in going from $y_c(t)$ to $y[n]$.

Recall the relationship $\omega=\Omega T$. We can simply use this in our system conversion:

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega/2} \\ H(j\Omega) &= e^{-j\Omega T/2} \\ &= e^{-j\Omega/2}, \quad T=1 \end{aligned}$$

Note that the choice of T and therefore $H(j\Omega)$ is not unique.

(b)

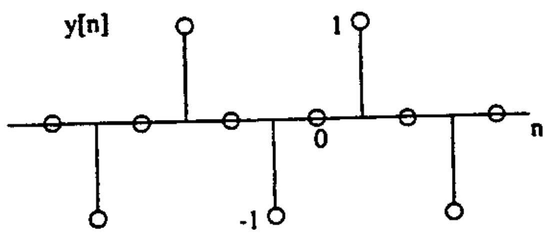
$$\begin{aligned} \cos\left(\frac{5\pi}{2}n-\frac{\pi}{4}\right) &= \frac{1}{2}\left[e^{j(\frac{5\pi}{2}n-\frac{\pi}{4})}+e^{-j(\frac{5\pi}{2}n-\frac{\pi}{4})}\right] \\ &= \frac{1}{2}e^{-j(\pi/4)}e^{j(5\pi/2)n}+\frac{1}{2}e^{j(\pi/4)}e^{-j(5\pi/2)n} \end{aligned}$$

Since $H(e^{j\omega})$ is an LTI system, we can find the response to each of the two eigenfunctions separately.

$$y[n]=\frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n}+\frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n}$$

Since $H(e^{j\omega})$ is defined for $0\leq|\omega|\leq\pi$ we must evaluate the frequency at the baseband, i.e., $5\pi/2 \Rightarrow 5\pi/2-2\pi=\pi/2$. Therefore,

$$\begin{aligned} y[n] &= \frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n}+\frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n} \\ &= \frac{1}{2}\left(e^{j[(5\pi/2)n-(\pi/2)]}+e^{-j[(5\pi/2)n-(\pi/2)]}\right) \\ &= \cos\left(\frac{5\pi}{2}n-\frac{\pi}{2}\right) \end{aligned}$$



4.35. The frequency response $H(e^{j\omega})=H_c(j\Omega/T)$. Finding that

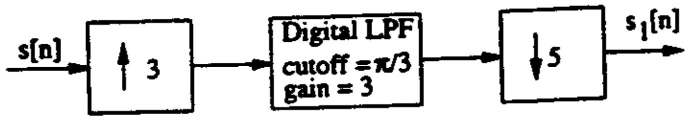
$$H_c(j\Omega)=\frac{1}{(j\Omega)^2+4(j\Omega)+3},$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{(10j\omega)^2+4(10j\omega)+3} \\ &= \frac{1}{-100\omega^2+3+40j\omega} \end{aligned}$$

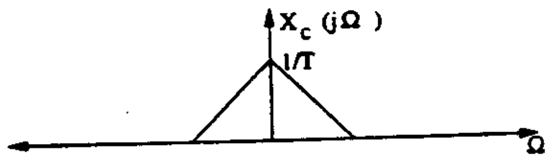
4.36. (a) Since $\Omega T=\omega$, $\quad (2\pi\cdot 100)T=\frac{\pi}{2}\Rightarrow T=\frac{1}{400}$

(b) The downsampler has $M = 2$. Since $x[n]$ is bandlimited to $\frac{\pi}{M}$, there will be no aliasing. The frequency axis simply expands by a factor of 2.
For $y_c(t) = x_c(t) \Leftrightarrow Y_c(j\Omega) = X_c(j\Omega)$.
Therefore $\Omega T' \Rightarrow 2\pi \cdot 100T' \Rightarrow T' = \frac{1}{200}$.

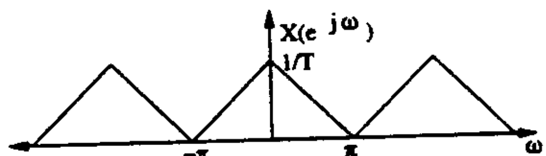
4.37. In both systems, the speech was filtered first so that the subsequent sampling results in no aliasing. Therefore, going $s[n]$ to $s_1[n]$ basically requires changing the sampling rate by a factor of 3kHz/5kHz = 3/5. This is done with the following system:



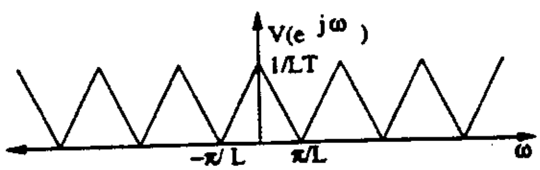
4.38. $X_c(j\Omega)$ is drawn below.



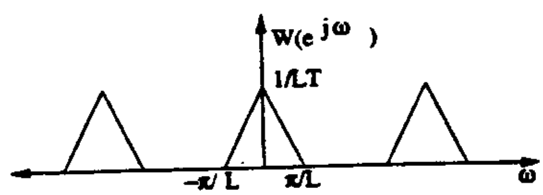
$x_c(t)$ is sampled at sampling period T, so there is no aliasing in $x[n]$.



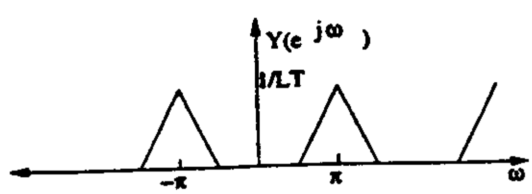
Inserting $L - 1$ zeros between samples compresses the frequency axis.



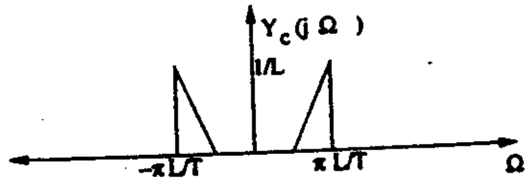
The filter $H(e^{j\omega})$ removes frequency components between π/L and π.



The multiplication by $(-1)^n$ shifts the center of the frequency band from 0 to π.



The D/C conversion maps the range $-\pi$ to π to the range $-\pi/T$ to π/T.



4.39. (a)

$$h[n] = 0, \quad |n| > (RL-1)$$

Therefore, for causal system delay by $RL-1$ samples.

(b) General interpolator condition:

$$\begin{aligned} h[0] &= 1 \\ h[kL] &= 0, \quad k = \pm 1, \pm 2, \ldots \end{aligned}$$

(c)

$$y[n] = \sum_{k=-(RL-1)}^{(RL-1)} h[k]v[n-k] = h[0]v[n] + \sum_{k=1}^{RL-1} h[n](v[n-k] + v[n+k])$$

This requires only RL-1 multiplies, (assuming $h[0] = 1$.)

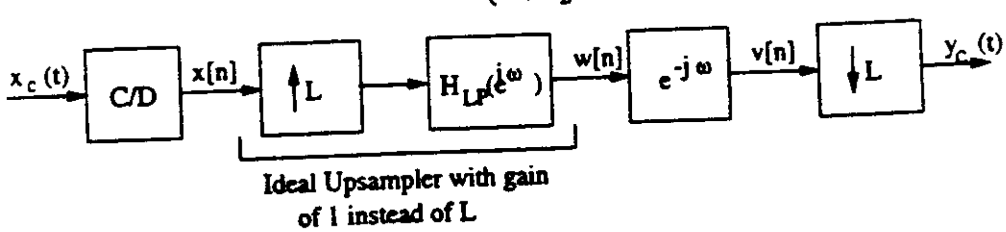
(d)

$$y[n] = \sum_{k=n-(RL-1)}^{n+(RL-1)} v[k]h[n-k]$$

If $n = mL$ (m an integer), then we don't have any multiplications since $h[0] = 1$ and the other non-zero samples of $v[k]$ hit at the zeros $h[n]$. Otherwise the impulse response spans $2RL-1$ samples of $v[n]$, but only $2R$ of these are non-zero. Therefore, there are $2R$ multiplies.

4.40. Split $H(e^{j\omega})$ into a lowpass and a delay.

$$\begin{aligned} H(e^{j\omega}) &= H_{LP}(e^{j\omega})e^{-j\omega} \\ H_{LP}(e^{j\omega}) &= \begin{cases} 1, & |\omega| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \leq \pi \end{cases} \end{aligned}$$



Then we analyze the system as follows:

$$\begin{aligned} x[n] &= x_c(nT) \quad \text{no aliasing assumed} \\ w[n] &= \frac{1}{L}x_c(n\frac{T}{L}) \quad \text{rate change} \\ v[n] &= w[n-1] = \frac{1}{L}x_c\left(n\frac{T}{L} - \frac{T}{L}\right), \quad \text{delay at higher rate} \\ y[n] &= v[nL] = \frac{1}{L}x_c\left(nT - \frac{T}{L}\right) \end{aligned}$$

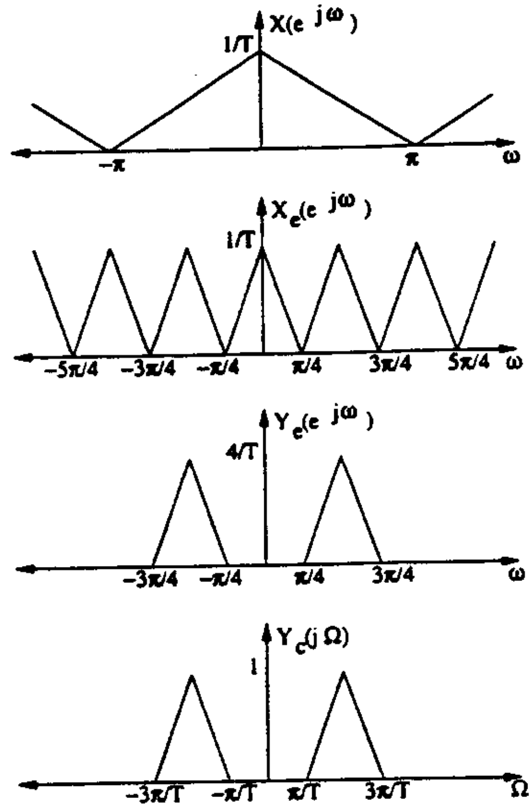
4.41. (a) See figures below.

(b) From part(a), we see that

$$Y_c(j\Omega) = X_c(j(\Omega - \frac{2\pi}{T})) + X_c(j(\Omega + \frac{2\pi}{T}))$$

Therefore,

$$y_c(t) = 2x_c(t)\cos(\frac{2\pi}{T}t)$$



4.42. (a) The Nyquist criterion states that $x_c(t)$ can be recovered as long as

$$\frac{2\pi}{T} \geq 2 \times 2\pi(250) \Longrightarrow T \leq \frac{1}{500}.$$

In this case, $T = 1/500$, so the Nyquist criterion is satisfied, and $x_c(t)$ can be recovered.

(b) Yes. A delay in time does not change the bandwidth of the signal. Hence, $y_c(t)$ has the sam bandwidth and same Nyquist sampling rate as $x_c(t)$.

(c) Consider first the following expressions for $X(e^{j\omega})$ and $Y(e^{j\omega})$:

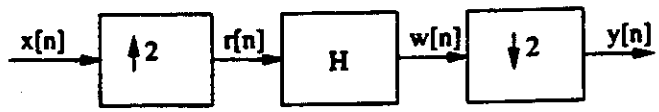
$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{T}X_c(j\Omega)\,|_{\Omega=\frac{\omega}{T}} = \frac{1}{500}X_c(j500\omega) \\
Y(e^{j\omega}) &= \frac{1}{T}Y_c(j\Omega)\,|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}e^{-j\Omega/1000}X_c(j\Omega)\,|_{\Omega=\frac{\omega}{T}} \\
&= \frac{1}{500}e^{-j\omega/2}X_c(j500\omega) \\
&= e^{-j\omega/2}X(e^{j\omega})
\end{aligned}$$

Hence, we let

$$H(e^{j\omega}) = \begin{cases} 2e^{-j\omega}, & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Then, in the following figure,

$$\begin{aligned} R(e^{j\omega}) &= X(e^{j2\omega}) \\ W(e^{j\omega}) &= \begin{cases} 2e^{-j\omega}X(e^{j2\omega}), & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \\ Y(e^{j\omega}) &= e^{-j\omega/2}X(e^{j\omega}) \end{aligned}$$



(d) Yes, from our analysis above,

$$H_2(e^{j\omega}) = e^{-j\omega/2}$$

4.43. (a) Notice first that

$$X_c(j\Omega) = \begin{cases} F_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & |\Omega| \le 400\pi \\ E_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & 400\pi \le |\Omega| \le 800\pi \\ 0, & \text{otherwise} \end{cases}$$

For the given $T = 1/800$, there is no aliasing from the C/D conversion. Hence, the equivalent CT transfer function $H_c(j\Omega)$ can be written as

$$H_c(j\Omega) = \begin{cases} H(e^{j\omega})|_{\omega=\Omega T}, & |\Omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}$$

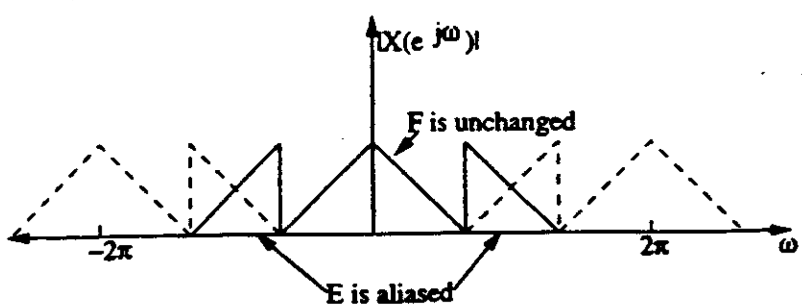
Furthermore, since $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$, the desired tranfer function is

$$H_c(j\Omega) = \begin{cases} e^{j\Omega^3}, & |\Omega| \le 400\pi \\ 0, & \text{otherwise} \end{cases}$$

Combining the two previous equations, we find

$$H(e^{j\omega}) = \begin{cases} e^{j(800\omega)^3}, & |\omega| \le \pi/2 \\ 0, & \pi/2 \le |\omega| \le \pi \end{cases}$$

(b) Some aliasing will occur if $2\pi/T < 1600\pi$. However, this is fine as long as the aliasing affects only $E_c(j\Omega)$ and not $F_c(j\Omega)$, as we show below:



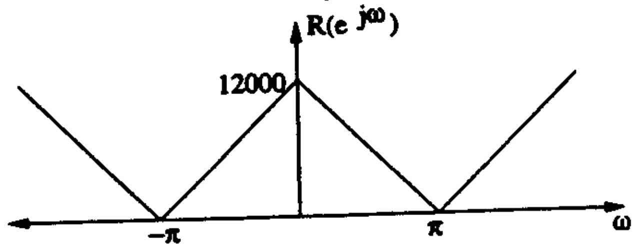
In order for the aliasing to not affect $F_c(j\Omega)$, we require

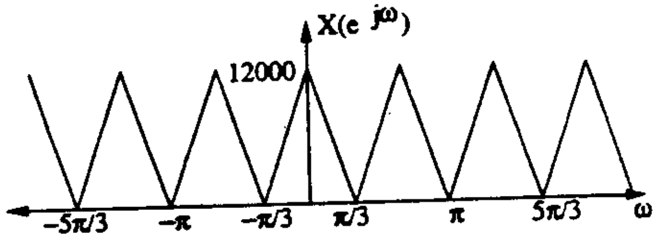
$$\frac{2\pi}{T} - 800\pi \geq 400\pi \Longrightarrow \frac{2\pi}{T} \geq 1200\pi$$

The minimum $\frac{2\pi}{T}$ is 1200π. For this choice, we get

$$H(e^{j\omega}) = \begin{cases} e^{j(600\omega)^2}, & |\omega| \leq 2\pi/3 \\ 0, & 2\pi/3 \leq |\omega| \leq \pi \end{cases}$$

4.44. (a) See the following figure:

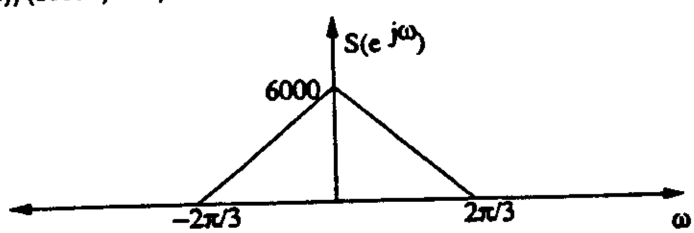




(b) For this to be true, $H(e^{j\omega})$ needs to filter out $X(e^{j\omega})$ for $\pi/3 \leq |\omega| \leq \pi$. Hence let $\omega_0 = \pi/3$. Furthermore, we want

$$\frac{\pi/2}{T_2} = 2\pi(1000) \Longrightarrow T_2 = 1/6000$$

(c) Matching the following figure of $S(e^{j\omega})$ with the figure for $R_c(j\Omega)$, and remembering that $\Omega = \omega/T$, we get $T_3 = (2\pi/3)/(2000\pi) = 1/3000$.



4.45. Notice first that since $x_c(t)$ is time-limited,

$$A = \int_0^{10} x_c(t)dt = \int_{-\infty}^{\infty} x_c(t)dt = X_c(j\Omega)|_{\Omega=0}.$$

To estimate $X_c(j \cdot 0)$ by DT processing, we need to sample only fast enough so that $X_c(j \cdot 0)$ is not aliased. Hence, we pick

$$2\pi/T = 2\pi \times 10^4 \Longrightarrow T = 10^{-4}.$$

The resulting spectrum satisfies

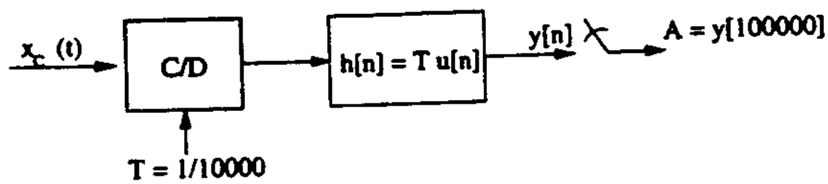
$$X(e^{j\cdot 0}) = \frac{1}{T} X_c(j \cdot 0)$$

Further,

$$X(e^{j\cdot 0}) = \sum_{n=-\infty}^{\infty} x[n].$$

Therefore, we pick $h[n] = Tu[n]$, which makes the system an accumulator. Our estimate $\hat{A}$ is the output $y[n]$ at $n = 10/(10^{-4}) = 10^5$, when all of the non-zero samples of $x[n]$ have been added-up. This is an *exact* estimate given our assumption of both band- and time-limitedness. Since the assumption can never be exactly satisfied, however, this method only gives an approximate estimate for actual signals.

The overall system is as follows:



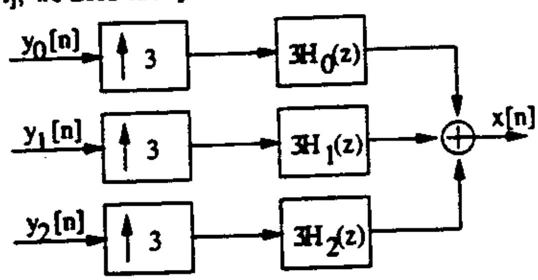
4.46. (a) Notice that

$$\begin{aligned} y_0[n] &= x[3n] \\ y_1[n] &= x[3n+1] \\ y_2[n] &= x[3n+2], \end{aligned}$$

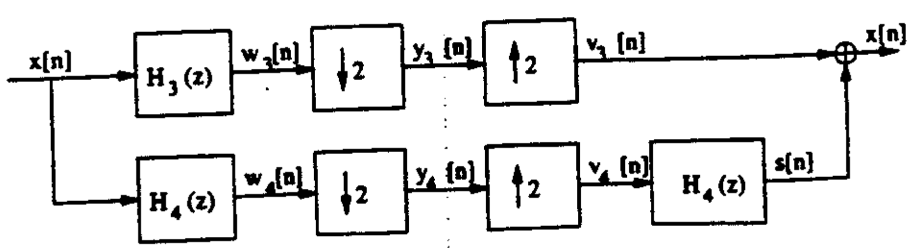
and therefore,

$$x[n] = \begin{cases} y_0[n/3], & n = 3k \\ y_1[(n-1)/3], & n = 3k+1 \\ y_2[(n-2)/3], & n = 3k+2 \end{cases}$$

(b) Yes. Since the bandwidth of the filters are $2\pi/3$, there is no aliasing introduced by downsampling. Hence to reconstruct $x[n]$, we need the system shown in the following figure:



(c) Yes, $x[n]$ can be reconstructed from $y_3[n]$ and $y_4[n]$ as demonstrated by the following figure:



In the following discussion, let $x_e[n]$ denote the even samples of $x[n]$, and $x_o[n]$ denote the odd samples of $x[n]$:

$$
\begin{aligned}
x_e[n] &= \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
x_o[n] &= \begin{cases} 0, & n \text{ even} \\ x[n], & n \text{ odd} \end{cases}
\end{aligned}
$$

In the figure, $y_3[n] = x[2n]$, and hence,

$$
\begin{aligned}
v_3[n] &= \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= x_e[n]
\end{aligned}
$$

Furthermore, it can be verified using the IDFT that the impulse response $h_4[n]$ corresponding to $H_4(e^{j\omega})$ is

$$
h_4[n] = \begin{cases} -2/(j\pi n), & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}
$$

Notice in particular that every other sample of the impulse response $h_4[n]$ is zero. Also, from the form of $H_4(e^{j\omega})$, it is clear that $H_4(e^{j\omega})H_4(e^{j\omega}) = 1$ and hence $h_4[n] * h_4[n] = \delta[n]$. Therefore,

$$
\begin{aligned}
v_4[n] &= \begin{cases} y_4[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} w_4[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} (x * h_4)[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= x_o[n] * h_4[n]
\end{aligned}
$$

where the last equality follows from the fact that $h_4[n]$ is non-zero only in the odd samples. Now, $s[n] = v_4[n] * h_4[n] = x_o[n] * h_4[n] * h_4[n] = x_o[n]$, and since $x[n] = x_e[n] + x_o[n]$, $s[n] + v_3[n] = x[n]$.

4.47. Sampling random processes

$$
\phi_{x_c x_c}(\tau) = E(x_c(t)x_c^*(t+\tau)) \Leftrightarrow P_{x_c x_c}(\Omega) = \int_{-\infty}^{\infty} \phi_{x_c x_c}(\tau) e^{-j\Omega\tau}\, d\tau
$$

(a)

$$
\begin{aligned}
\phi_{xx}[m] &= E(x[n]x^*[n+m]) = E(x_c(nT)x_c^*(nT+mT)) \\
&= \phi_{x_c x_c}(mT), \quad \text{i.e., sampled autocovariance}
\end{aligned}
$$

(b) Since $\phi_{xx}[m]$ is a sampled $\phi_{x_c x_c}(\tau)$

$$
P_{xx}(\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} P_{x_c x_c}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)
$$

(c) If

$$
P_{x_c x_c} = 0, \text{ for } |\omega| \geq \pi
$$

then

$$
P_{xx}(\omega) = \frac{1}{T} P_{x_c x_c}\left(\frac{\omega}{T}\right), \quad |\omega| \leq \pi
$$

4.48. (a)

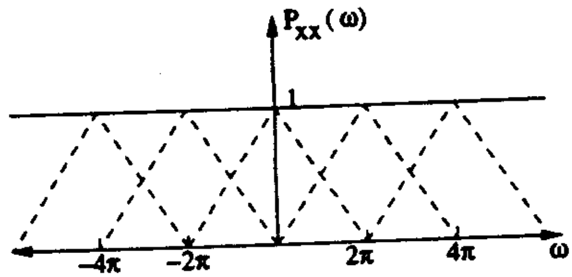
$$\begin{aligned}\phi_{x_c x_c}(\tau) &= E(x_c(t)x_c(t+\tau)\\ \phi_{xx}[m] &= E(x[n]x[n+m]) = E(x_c(nT)x_c(nT+mT))\\ &= \phi_{x_c x_c}(mT)\end{aligned}$$

(b)

$$P_{xx}(\omega) = \frac{1}{T}\sum_{r=-\infty}^{\infty} P_{x_c x_c}\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)$$

Therefore, we require that $\frac{\pi}{T} \geq \Omega_0$.

(c) For the spectrum of Fig P3.8-2 it is clear that if $T = \frac{2\pi}{\Omega_o}$ then the discrete-time power spectrum will be white, as shown in the figure above.

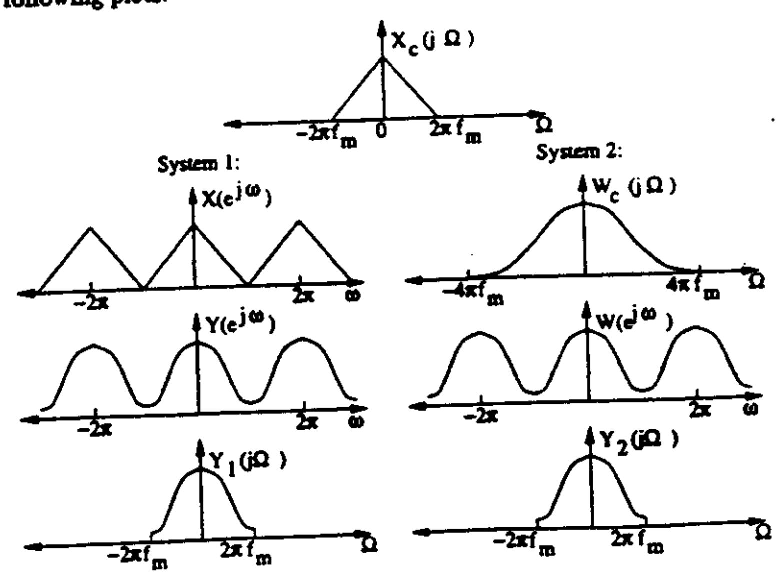


(d) For white discrete-time signal $\Rightarrow \phi_{xx}[m] = 0, \quad m \neq 0$ but $\phi_{xx}[m] = \phi_{x_c x_c}(mT)$. Therefore, any analog signal whose autocorrelation function has zeros equally spaced at intervals of T will yield a white discrete-time sequence is sampled with sampling period T. For example, for Fig P3.8-1:

$$\phi_{x_c x_c}(\tau) = \frac{\sin \Omega_0 T}{T\pi} \Rightarrow \phi_{xx}[m] = \frac{\sin \Omega_0 mT}{\pi mT}$$

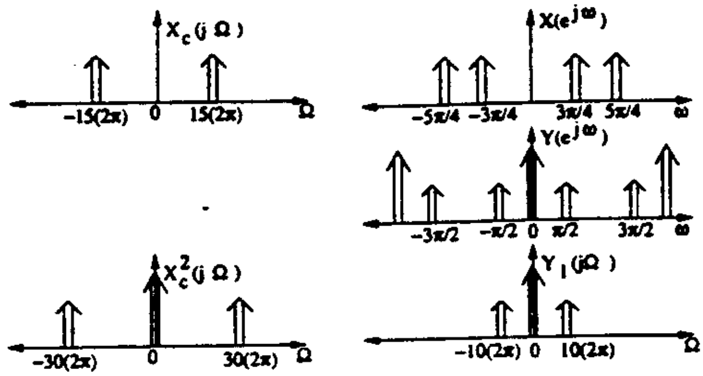
$$\text{if } T = \frac{\pi}{\Omega_0} \quad \phi_{xx}[m] = \frac{\sin \pi m}{\pi^2 m/\Omega_o} = 0, \quad m \neq 0$$

4.49. (a) Consider the following plots.



$y_1(t) = y_2(t)$: Convolution is a linear process. Aliasing is a linear process. Periodic convolution is equivalent to convolution followed by aliasing.
$y_1(t) \neq x^2(t)$: System 2 at Step 1 shows $X_c^2(j\Omega)$. This is clearly not $Y_1(j\Omega)$. $Y_1(j\Omega)$ is an aliased version of $X_c(j\Omega)$

(b) Now,



(c)

$$
\begin{aligned}
x(t) &= A\cos(30\pi t) \\
x^3(t) &= \frac{3}{4}A\cos(30\pi t) + \frac{1}{4}A\cos(3\cdot 30\pi t), \\
v[n] &= \frac{3}{4}A\cos\left(\frac{3}{4}\pi n\right) + \frac{1}{4}A\cos\left(\frac{1}{4}\pi n\right) \\
v[n] &= x^3[n] \\
y[n] &= x[n]
\end{aligned}
$$

We can see here that sometimes aliasing won't be destructive. When aliased sections do not overlap they can be reconstructed.

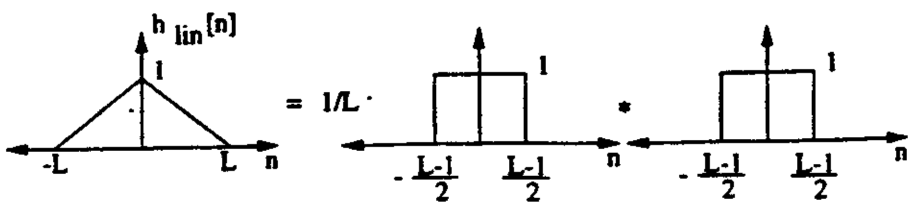
(d) This is the inverse to part (c). Since multiplication in time corresponds to convolution in frequency, a signal $x^2(t)$ has at most two times the bandwidth of $x(t)$. Therefore, $x^{1/2}$ will have at least $\frac{1}{2}$ the bandwidth of $x(t)$. If we run our signal through a box that will raise it to the $1/M$ power, then the sampling rate can be decreased by a factor of M.

4.50. (a)

$$
\begin{aligned}
x_i[n] &= x_u[n] * h_{zoh}[n] \\
h_{zoh}[n] &= \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

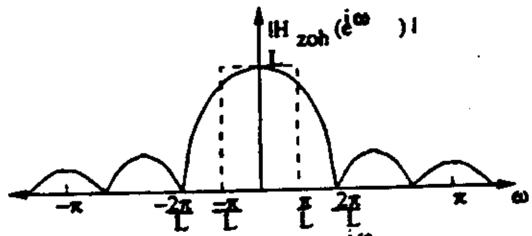
$$H_{zoh}(e^{j\omega}) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j(L-1)\omega/2}$$

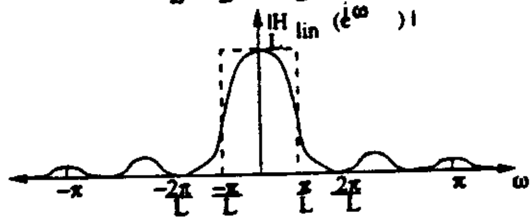
(b) The impulse response $h_{lin}[n]$ corresponds to the convolution of two rectangular sequences, as shown below.



$$H_{lin}(e^{j\omega}) = \frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right)^2$$

(c) The frequency response of zero-order-hold is flatter in the region $[-\pi/L, \pi/L]$, but achieves less out-of-band attenuation.





4.51.

$$\begin{aligned}
\phi_{xx}[n] &= x[n] * x[-n] \\
\Phi_{xx}(e^{j\omega}) &= X(e^{j\omega}) * X^*(e^{j\omega})
\end{aligned}$$

The bandwidth of $\Phi_{xx}(e^{j\omega})$ is no larger than the bandwidth of $X(e^{j\omega})$. Therefore, the outputs of the systems will be the same if $H_2(e^{j\omega})$ is an ideal lowpass filter with a cutoff of π/L.

4.52. The idea here is to exploit the fact that every other sample supplied to $h[n]$ in Fig 3.27-1 is zero. That is,

$$\begin{aligned}
y_1[n] &= h[n] * w[n] = \sum_{n=-\infty}^{\infty} w[n-k]h[k] \\
&= aw[n] + bw[n-1] + cw[n-2] + dw[n-3] + ew[n-4] \\
&= \begin{cases} ax[n/2] + cx[(n/2)-1] + e[(n/2)-2], & n \text{ even} \\ bx[(n/2)-(1/2)] + dx[(n/2)-(3/2)], & n \text{ odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
w_1[n] &= \begin{cases} h_1[n/2] * x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} h_1[0]x[n/2] + h_1[1]x[(n/2)-1] + h_1[2]x[(n/2)-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
w_2[n] &= \begin{cases} h_2[n/2] * x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} h_2[0]x[n/2] + h_2[1]x[(n/2)-1] + h_2[2]x[(n/2)-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\end{aligned}$$

Comparing $w_1[n], w_2[n]$ with $y_1[n]$ above:

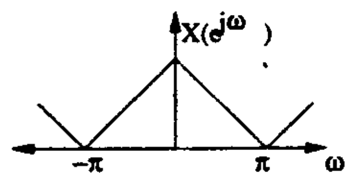
$w[n]$ can give even samples if $h_1[0] = a, h_1[1] = c, h_2[2] = e$. Similarly, $w_2[n]$ can give the odd samples if $h_3[n]$ delays $w_2[n]$ by one sample, i.e., $h_3[0] = 0, h_3[1] = 0, h_3[2] = 0$. Thus

$$w_3[n] = \begin{cases} h_2[0]x[(n-1)/2] + h_2[1]x[(n-1)/2-1] + h_2[2]x[(n-1)/2-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

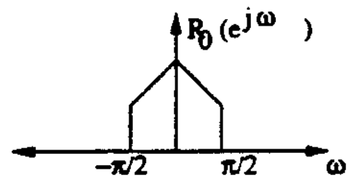
$$h_2[0] = b, \quad h_2[1] = d, \quad h_2[2] = 0$$

4.53. Sketches appear below.

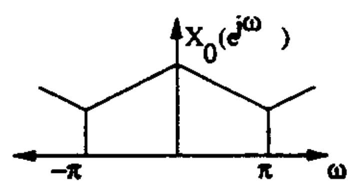
(a) First, $X(e^{j\omega})$ is plotted.



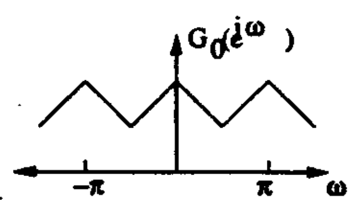
The lowpass filter cuts off at $\frac{\pi}{2}$.



The downsampler expands the frequency axis. Since $R_0(e^{j\omega})$ is bandlimited to $\frac{\pi}{M}$, no aliasing occurs.



The upsampler compresses the frequency axis by a factor of 2.



The lowpass filter cuts off at $\frac{\pi}{2} \Rightarrow Y_0(e^{j\omega}) = R_0(e^{j\omega})$ as sketched above.

(b) $G_0(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega})H_0(e^{j\omega}) + X(e^{j(\omega+\pi)})H_0(e^{j(\omega+\pi)})\right)$

(c)

$$\begin{aligned}
Y_0(e^{j\omega}) &= \frac{1}{2}H_0(e^{j\omega})\left(X(e^{j\omega})H_0(e^{j\omega}) + X(e^{j(\omega+\pi)})H_0(e^{j(\omega+\pi)})\right)\\
Y_1(e^{j\omega}) &= \frac{1}{2}H_1(e^{j\omega})\left(X(e^{j\omega})H_1(e^{j\omega}) + X(e^{j(\omega+\pi)})H_1(e^{j(\omega+\pi)})\right)\\
Y(e^{j\omega}) &= Y_0(e^{j\omega}) - Y_1(e^{j\omega})\\
&= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) - H_1^2(e^{j\omega})\right]\\
&\quad + \frac{1}{2}X(e^{j(\omega+\pi)})\underbrace{\left[H_0(e^{j\omega})H_0(e^{j(\omega+\pi)}) - H_1(e^{j\omega})H_1(e^{j(\omega+\pi)})\right]}_{=0}
\end{aligned}$$

The aliasing terms always cancel. $Y(e^{j\omega})$ is proportional to $X(e^{j\omega})$ if $[H_0^2(e^{j\omega}) - H_1^2(e^{j\omega})]$ is a constant.

$X(e^{j\omega}) = 0, \pi/3 \le |\omega| \le \pi$. $x[n]$ can be thought of as an oversampled signal. The approach is to determine whether n_0 is odd or even, then sample so that n_0 is avoided, upsampled and lowpass filter. This recovers $x[n_0]$.

4.54. (a) In the case where n_0 is not known, we determine whether it is even or odd as follows:

$$\begin{aligned}
\hat{x}[n] &= x[n] - A\delta[n - n_0]\\
\hat{X}(e^{j\omega}) &= X(e^{j\omega}) - Ae^{-j\omega n_0}\\
\hat{X}(e^{j\omega})|_{\omega=\frac{\pi}{2}} &= \sum_n x[n](-j)^n\\
\hat{X}(e^{j(\pi/2)}) &= -A(-j)^{n_0}
\end{aligned}$$

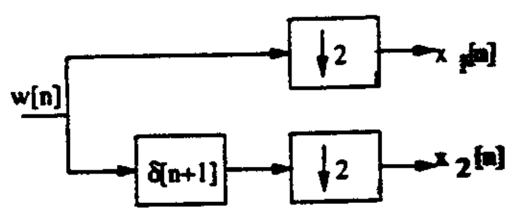
If the result is real, n_0 is even. If the result is imaginary, n_0 is odd

(b) If n_0 is even, sample $\hat{x}[n]$ so that the even-numbered sequence values are set to zero. If n_0 is odd, sample so the odd-numbered samples are set to zero

(c) Filter the sampled sequence with a lowpass filter with cutoff frequency $\pi/3$, and gain 2. This is an exact procedure if ideal filters are used.

4.55. (a)

$$\begin{aligned}
w[n] &= \begin{cases} x_1[n/2], & n \text{ even}\\ x_2[(n-1)/2], & n \text{ odd}\end{cases}\\
x_1[n] &= w[2n]\\
x_2[n] &= w[2n+1]
\end{aligned}$$



The system is linear, time-varying (due to downsampling), non-causal (due to $\delta[n+1]$), and stable.

(b)

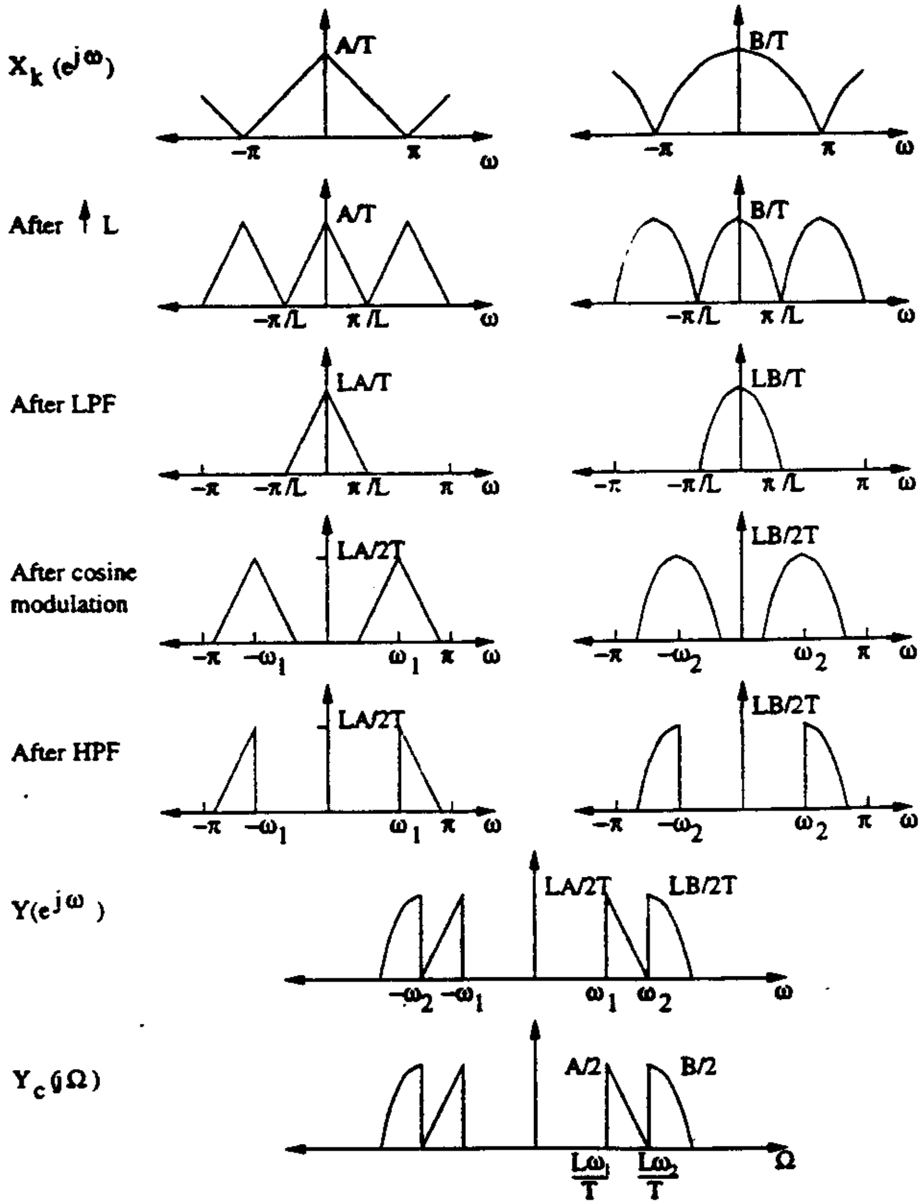
$$T = \frac{\pi}{\Omega_N} = \frac{\pi}{2\pi \times 5000} = 10^{-4}\text{sec}, \quad \frac{L\omega_1}{T} = 2\pi \times 10^5$$

To avoid aliasing in $y_c(t)$:

$$\begin{aligned} \frac{L\omega_1}{T} + \frac{2\pi}{T} &\le \frac{L\pi}{T} \\ \omega_1 &= \frac{20\pi}{L} \\ 20\pi + 2\pi &= L\pi \end{aligned}$$

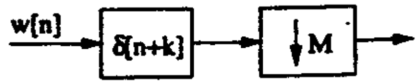
$$L = 22, \quad \omega_1 = 2\pi(\frac{10}{22})$$

(c) The Fourier transforms are sketched below.

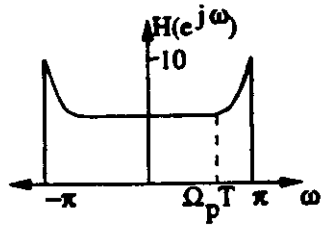


(d) To generalize for M channels, we would use the same modulators, but we would choose a larger value of L to make room for additional spectra above the lower frequency bound. If the lower

bound remained $2\pi \cdot 10^5$, L would become $L = 20 + M$ for M channels.
A branch of the TDM demultiplexing system would be:



4.56. Since we want $W(e^{j\omega})$ to equal $X(e^{j\omega})$, then $H(e^{j\omega})$ must compensate for the drop offs in $H_{aa}(j\Omega)$.



4.57. (a)

$$E(e) = \int ep(e)de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} ede = \frac{e^2}{2\Delta}\Big|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E(e^2 - 0) = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2 de = \frac{e^3}{3\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

$$r[m,n] = E(e[m]e[n]) = \begin{cases} E(e[m])E(e[n]), & m \neq n \\ E(e^2[n]), & m = n \end{cases}$$

$$r[n,m] = r[n-m] = \frac{\Delta^2}{12}\delta[n-m]$$

(b)

$$\text{SNR} = \frac{\sigma_x^2}{\sigma_e^2} = \frac{12\sigma_x^2}{\Delta^2}$$

(c) Let $e_y[n]$ be the output noise.

$$\begin{aligned}
e_y[n] &= \sum_k h[k]e[n-k] \\
E(e_y^2[n]) &= E\left(\sum_k h[k]e[n-k]\sum_l h[l]e[n-l]\right) = \sum_k\sum_l h[k]h[l]\underbrace{E(e[n-k]e[n-l])}_{\sigma_e^2[k-l]} \\
\sigma_{e_y}^2 &= \sigma_e^2\sum_k h^2[k] \\
&= \sigma_e^2\sum_{k=0}^{\infty}\frac{1}{4}\left(a^k + (-a)^k\right)^2 = \frac{\sigma_e^2}{4}\sum_{k=0}^{\infty}\left(a^{2k} + 2a^k(-a)^k + (-a)^{2k}\right) \\
&= \frac{\sigma_e^2}{2}\left(\sum_{k=0}^{\infty}a^{2k} + \sum_{k=0}^{\infty}(-a^2)^k\right) = \frac{\sigma_e^2}{2}\left(\frac{1}{1-a^2} + \frac{1}{1+a^2}\right) \\
&= \sigma_e^2\left(\frac{1}{1-a^4}\right) = \frac{\Delta^2}{12(1-a^4)}
\end{aligned}$$

The variance of $x[n]$ is weighted similarly so the SNR does not change. $\text{SNR}_{\text{out}} = 12\frac{\sigma_x^2}{\Delta^2}$.

(d) $f[n] = x[n]e[n]$

$$E(f[n]) = E(x[n]e[n]) = E(x[n])E(e[n]) = 0$$
$$\sigma_f^2 = E(f^2[n]) = E(x^2[n]e^2[n]) = E(x^2[n])E(e^2[n]) = \sigma_x^2\sigma_e^2$$
$$r_f[n,m] = E(x[n]x[m]e[n]e[m]) = \underbrace{E(x[n]x[m])}_{\sigma_x^2\delta[n-m]} \cdot \underbrace{E(e[n]e[m])}_{\sigma_e^2\delta[n-m]}$$

(e)

$$\text{SNR} = \frac{\sigma_x^2}{\sigma_f^2} = \frac{\sigma_x^2}{\sigma_x^2\sigma_e^2} = \frac{1}{\sigma_e^2} = \frac{12}{\Delta^2}$$

(f) Using the results of part (c).

$$\sigma_{e_y}^2 = \sigma_f^2\left(\frac{1}{1-a^4}\right) = \frac{\sigma_x^2\sigma_e^2}{1-a^4}$$

Again, the variance of $x[n]$ is weighted by the same factor, so the SNR does not change.

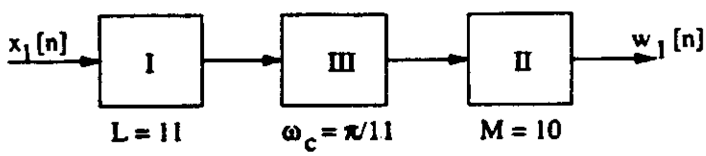
$$\text{SNR}_{\text{out}} = \frac{12}{\Delta^2}.$$

4.58. First, notice that since $y_c(t) = x_1(t)x_2(t)$, $Y_c(j\Omega) = \frac{1}{2\pi}(X_1(j\Omega) * X_2(j\Omega))$, and so $Y_c(j\Omega) = 0$ for $|\Omega| \geq 11\pi/2 \times 10^4$. Hence the Nyquist rate $T = 1/55000$s.

Choose System A and B such that $w_1[n] = ax_1(nT)$ and $w_2[n] = bx_2(nT)$.

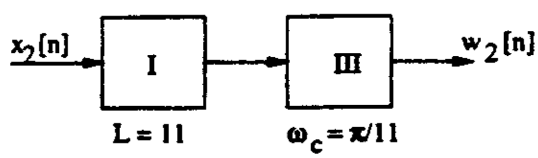
For System A, we need to resample such that

$$\frac{M}{L} = \frac{T}{T_1} = \frac{2\times10^{-5}}{1/55000} = \frac{10}{11}$$



For System B, we need to resample such that

$$\frac{M}{L} = \frac{T}{T_1} = \frac{2\times10^{-4}}{1/55000} = \frac{1}{11}$$



System C is simply the identity system.

4.59. The speech is first sampled at 44.1 kHz, and we wish to resample it so that the sampling rate is at 8 kHz. There are no aliasing effects anywhere in the system. Hence

$$\frac{L}{M} = \frac{44.1}{8} = \frac{441}{80}$$

We simply make $L = 441$, $M = 80$, and $\omega_c = \pi/441$.

4.60. Ω_p, and Ω_s has to be chosen such that

(a) The region $|\Omega| \le \Omega_p$ maps to $|\omega| \le \pi/4$:

$$\Omega_p T = \frac{\pi}{4} \Longrightarrow \Omega_p = 44\pi$$

(b) No aliasing occurs in the region $|\Omega| \le \Omega_p$ during sampling:

$$\frac{2\pi}{T} - \Omega_s = \Omega_p \Longrightarrow \Omega_s = 2\pi(4 \cdot 44) - 44\pi = 308\pi$$

4.61. (a)

$$\begin{aligned} V(z) &= H_1(z)(X(z) - Y(z)) \\ U(z) &= H_2(z)(V(z) - Y(z)) \\ Y(z) &= U(z) + E(z) \\ &= \frac{H_1(z)H_2(z)}{1 + H_2(z)(1 + H_1(z))} X(z) + \frac{1}{1 + H_2(z)(1 + H_1(z))} E(z) \end{aligned}$$

Substituting $H_1(z) = 1/(1 - z^{-1})$ and $H_2(z) = z^{-1}/(1 - z^{-1})$, we find

$$\begin{aligned} H_{xy}(z) &= z^{-1} \\ H_{ey}(z) &= (1 - z^{-1})^2 \end{aligned}$$

Hence the difference equation is $y[n] = x[n-1] + f[n]$, where

$$f[n] = e[n] - 2e[n-1] + e[n-2].$$

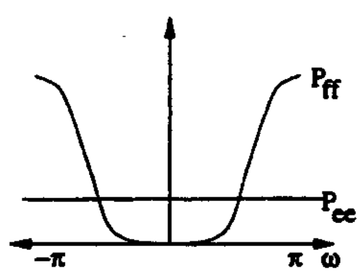
(b)

$$\begin{aligned} P_{ff}(e^{j\omega}) &= \sigma_e^2 |H_{ey}(e^{j\omega})|^2 \\ &= \sigma_e^2 |(1 - e^{-j\omega})^2|^2 \\ &= \sigma_e^2 (1 - e^{-j\omega})^2 (1 - e^{j\omega})^2 \\ &= \sigma_e^2 (2 - 2\cos(\omega))^2 \\ &= \sigma_e^2 (4\sin^2(\omega/2))^2 \\ &= 16\sigma_e^2 \sin^4(\omega/2) \end{aligned}$$

The total noise power σ_f^2 is the autocorrelation of $f[n]$ evaluated at 0:

$$\begin{aligned} \sigma_f^2 &= E[(e[n] - 2e[n-1] + e[n-2])^2] \\ &= E[e^2[n]] + E[-2e^2[n-1]] + E[e[n-2]^2] \\ &= 6\sigma_e^2, \end{aligned}$$

where we have used linearity of expectations, and the fact that since $e[n]$ is white, $E[e[n]e[n-k]] = 0$ for $k \ne 0$.



(c) Since $X(e^{j\omega})$ is bandlimited, $x[n] * h_3[n] = x[n]$. Hence,

$$w[n] = y[n] * h_3[n] = (x[n-1] + f[n]) * h_3[n] = x[n-1] + g[n],$$

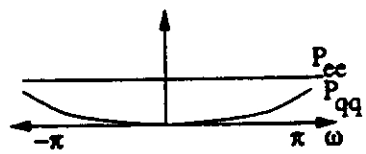
where $g[n]$ is the quantization noise in the region $|\omega| < \pi/M$.

(d) For a small angle x, $\sin x \approx x$. Therefore,

$$\begin{aligned}
\sigma_g^2 &= \frac{1}{2\pi}\int_{\pi/M}^{\pi/M} \sigma_e^2 (2\sin\omega/2)^4 d\omega \\
&\approx \frac{1}{2\pi}\int_{\pi/M}^{\pi/M} \sigma_e^2 (2\omega/2)^4 d\omega \\
&= \frac{\sigma_e^2}{2\pi}\frac{\omega^5}{5}\Big|_{-\pi/M}^{\pi/M} \\
&= \frac{\sigma_e^2 \pi^4}{5M^5}
\end{aligned}$$

(e) $X_c(j\Omega)$ must be sufficiently bandlimited that $X(e^{j\omega}) = X_c(j\Omega T)$ is zero for $|\omega| > \pi/M$. Hence $X_c(j\Omega) = 0$ for $|\Omega| > \pi/MT$.
Assuming that is satisfied, $v_x[n] = x[Mn-1] = x_c(MTn - T)$.
Downsampling does not change the variance of the noise, and hence $\sigma_q^2 = \sigma_g^2$.

$$\begin{aligned}
P_{qq}(e^{j\omega}) &= P_{gg}(e^{j\omega/M}) \\
&= 16\sigma_e^2 \sin^4(\omega/2M)
\end{aligned}$$



4.62. (a) (i) The transfer function from $x[n]$ to $y_x[n]$ is

$$H_{xy}(z) = \frac{\frac{z^{-1}}{1-z^{-1}}}{1 + \frac{z^{-1}}{1-z^{-1}}} = z^{-1}$$

Hence $y_x[n] = x[n-1]$.

(ii) The transfer function from $e[n]$ to $y_e[n]$ is

$$H_{ey}(z) = \frac{1}{1 + \frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1}$$

So

$$\begin{aligned}
P_{y_e}(\omega) &= P_e(\omega) H_{ey}(e^{j\omega}) H_{ey} e^{-j\omega} \\
&= \sigma_e^2 (1 - e^{-j\omega})(1 - e^{j\omega}) \\
&= \sigma_e^2 (2 - 2\cos(\omega))
\end{aligned}$$

(b) (i) $x[n]$ contributes only to $y_1[n]$, but not $y_2[n]$. Therefore

$$\begin{aligned}
y_{1x}[n] &= x[n-1] \\
r_x[n] &= x[n-2]
\end{aligned}$$

(ii) In part(a), the difference equation describing the sigma-delta noise-shaper is

$$y[n] = x[n-1] + e[n] - e[n-1].$$

So here we apply the difference equation to both sigma-delta modulators:

$$\begin{aligned}
y_{1e}[n] &= e_1[n] - e_1[n-1] \\
y_{2e}[n] &= e_1[n-1] + e_2[n] - e_2[n-1] \\
r_e[n] &= y_{1e}[n-1] - (y_{2e}[n] - y_{2e}[n-1]) \\
&= -e_2[n] + 2e_2[n-1] - e_w[n-2] \\
H_{e_2r}(z) &= -(1-z^{-1})^2 \\
P_{r_e}(\omega) &= \sigma_e^2(2 - 2\cos\omega)^2
\end{aligned}$$