Discrete-Time Signals: Time-Domain Representation - II

SECTION 2

These lecture slides are based on "Digital Signal Processing: A Computer-Based Approach, 4th ed." textbook by S.K. Mitra and its instructor materials. U.Sezen

Contents

Classification of Sequences

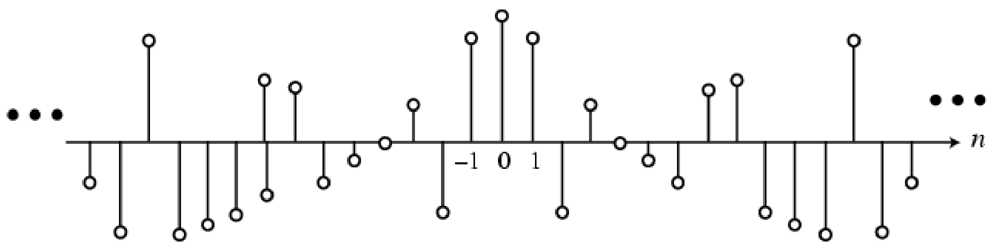
- There are several types of classification
- One classification is in terms of the number of samples defining the sequence
- Another classification is based on its symmetry with respect to time index $n = 0$
- Other classifications in terms of its other properties, such as periodicity, summability, energy and power

Classification of Sequences Based on Symmetry

- **Conjugate-symmetric sequence** is defined as:

$$x[n] = x^*[-n]$$

- If $x[n]$ is real, then it is an **even sequence**

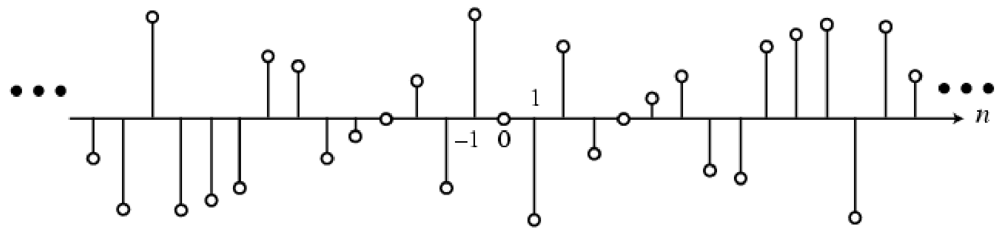


An even sequence

- **Conjugate-antisymmetric sequence** is defined as:

$$x[n] = -x^*[-n]$$

- If $x[n]$ is real, then it is an **odd sequence**



An odd sequence

- It follows from the definition that for a **conjugate-symmetric sequence** $\{x[n]\}$, $x[0]$ must be a real number
- Likewise, it follows from the definition that for a **conjugate anti-symmetric sequence** $\{y[n]\}$, $y[0]$ must be an imaginary number
- From the above, it also follows that for an **odd sequence** $\{w[n]\}$, $w[0] = 0$

- Any complex sequence can be expressed as a sum of its **conjugate-symmetric part** and its **conjugate-antisymmetric part**:

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \frac{1}{2}(x[n] + x^*[-n])$$
$$x_{ca}[n] = \frac{1}{2}(x[n] - x^*[-n])$$

- As indicated above, computation of conjugate-symmetric and conjugate anti-symmetric parts of a sequence involves **conjugation**, **time-reversal**, **addition**, and **multiplication** operations

- The decomposition of a finite-length sequence into a sum of **conjugate-symmetric** and **conjugate anti-symmetric** sequences is possible if the parent sequence is an **odd sequence** defined for a **symmetric interval**, i.e.,

$$-M \leq n \leq M$$

Example: Consider the length-7 sequence defined for $-3 \leq n \leq 3$

$$\{g[n]\} = \{0, 1 + j4, -2 + j3, \underset{\uparrow}{4 - j2}, -5 - j6, -j2, 3\}$$

- Its conjugate sequence is then given by

$$\{g^*[n]\} = \{0, 1 - j4, -2 - j3, \underset{\uparrow}{4 + j2}, -5 + j6, j2, 3\}$$

- The time-reversed version of the above is

$$\{g^*[-n]\} = \{3, j2, -5 + j6, \underset{\uparrow}{4 + j2}, -2 - j3, 1 - j4, 0\}$$

- Therefore $g_{cs}[n] = \frac{1}{2}(g[n] + g^*[-n])$

$$\{g_{cs}[n]\} = \{1.5, 0.5 + j3, -3.5 + j4.5, \underset{\uparrow}{4}, -3.5 - j4.5, 0.5 - j3, 1.5\}$$

- Likewise $g_{ca}[n] = \frac{1}{2}(g[n] - g^*[-n])$

$$\{g_{ca}[n]\} = \{-1.5, 0.5 + j, 1.5 - j1.5, \underset{\uparrow}{-j2}, -1.5 - j1.5, -0.5 - j, 1.5\}$$

- It can be easily verified that
  $g_{cs}[n] = g^*_{cs}[-n]$ and $g_{ca}[n] = -g^*_{ca}[-n]$

- Any real sequence can be expressed as a sum of its **even part** and its **odd part**:

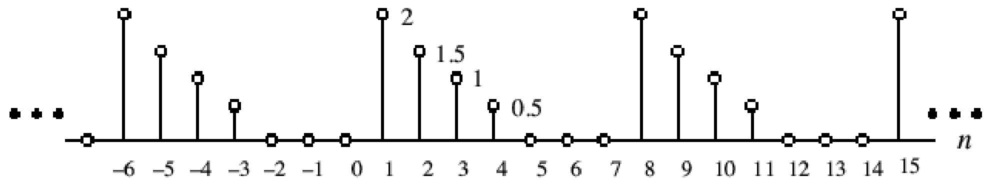
$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$$
$$x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$$

Classification of Sequences Based on Periodicity

- A sequence $\tilde{x}[n]$ satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called a **periodic sequence** with a period N where N is a positive integer and k is any integer
- Smallest value of N satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called the **fundamental period**



A periodic sequence

- A sequence not satisfying the periodicity condition is called an **aperiodic sequence**

- If $\tilde{x}_a[n]$ and $\tilde{x}_b[n]$ are two periodic sequences with fundamental periods N_a and N_b, respectively, then the sequence

$$\tilde{y}[n] = \tilde{x}_a[n] + \tilde{x}_b[n]$$

is a periodic sequence with a fundamental period N given by

$$N = \frac{N_a N_b}{\gcd(N_a, N_b)}$$

where $\gcd(\cdot)$ is the greatest common divisor operator.

- If $\tilde{x}_a[n]$ and $\tilde{x}_b[n]$ are two periodic sequences with fundamental periods N_a and N_b, respectively, then the sequence

$$\tilde{y}[n] = \tilde{x}_a[n] \cdot \tilde{x}_b[n]$$

is a periodic sequence with a fundamental period N given by

$$N = \frac{N_a N_b}{\gcd(N_a, N_b)}$$

Classification of Sequences: Energy and Power Signals

- Total **energy** of a sequence $x[n]$ is defined by

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- An infinite length sequence with finite sample values may or may not have finite energy
- A finite length sequence with finite sample values has finite energy

Classification of Sequences

- **Example:** The infinite-length sequence

$$x[n] = \begin{cases} \dfrac{1}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has an energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$$

which converges to $\dfrac{\pi^2}{6}$, indicating that $x[n]$ has **finite energy**

- **Example:** The infinite-length sequence

$$y[n] = \begin{cases} \dfrac{1}{\sqrt{n}}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has an energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)$$

which does not converge indicating that $y[n]$ has **infinite energy**

Classification of Sequences: Energy and Power Signals

- The **average power** of an **aperiodic sequence** is defined by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

- Let's define the **energy** of a sequence $x[n]$ over a finite interval $-K \leq n \leq K$ as

$$\mathcal{E}_{x,K} = \sum_{n=-K}^{K} |x[n]|^2$$

then

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \mathcal{E}_{x,K}$$

- The **average power** of a **periodic sequence** $\tilde{x}[n]$ with a period N by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

- The average power of an infinite-length sequence may be finite or infinite

- **Example:** Consider the causal sequence defined by

$$x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Note: $x[n]$ has infinite energy

  - Its average power is given by

$$\begin{aligned} P_x &= \lim_{K \to \infty} \frac{1}{2K+1} \left(9 \sum_{n=0}^{K} 1 \right) \\ &= \lim_{K \to \infty} \frac{9(K+1)}{2K+1} \\ &= 4.5 \end{aligned}$$

- An infinite energy signal with finite average power is called a **power signal**

  Example: A periodic sequence which has a finite average power but infinite energy

- A finite energy signal with zero average power is called an **energy signal**

  Example: A finite-length sequence which has finite energy but zero average power

Other Types of Classifications

- A sequence $x[n]$ is said to be **bounded** if

$$|x[n]| \leq B_x < \infty$$

Example: The sequence $x[n] = \cos(0.3\pi n)$ is a bounded sequence as

$$|x[n]| = |\cos(0.3\pi n)| \leq 1$$

- A sequence $x[n]$ is said to be **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Example: The causal sequence

$$y[n] = \begin{cases} 0.3^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

is an absolutely summable sequence as

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |y[n]| &= \sum_{n=0}^{\infty} 0.3^n \\ &= \frac{1}{1-0.3} \\ &= 1.42857 < \infty \end{aligned}$$

- A sequence $x[n]$ is said to be **square-summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Example: The sequence

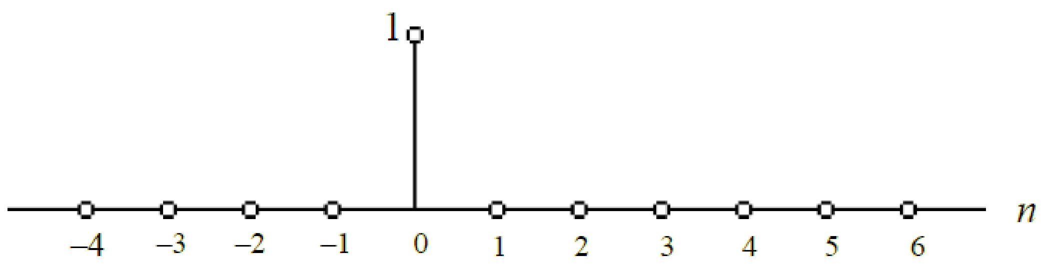
$$h[n] = \frac{\sin(0.4n)}{\pi n}$$

is square-summable but not absolutely summable

Basic Sequences

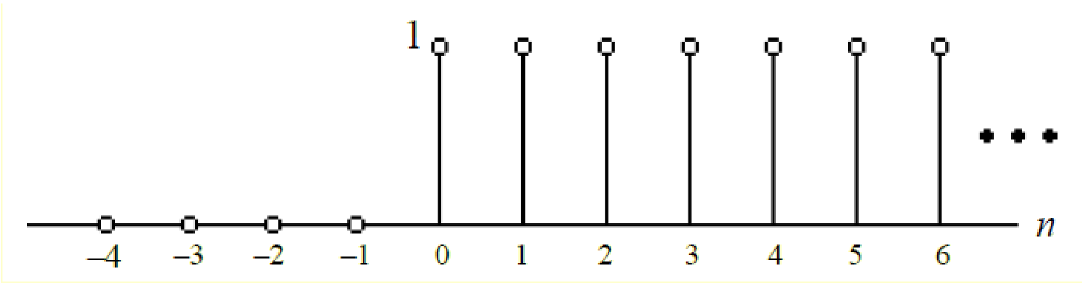
- **Unit sample sequence**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



- **Unit step sequence**

$$\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

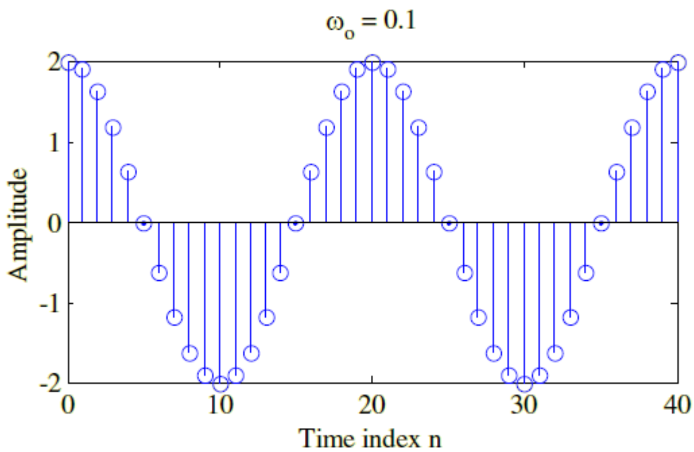


- **Real sinusoidal sequence**

$$x[n] = A\cos(\omega_0 n + \phi)$$

where A is the amplitude, ω_0 is the angular frequency, and ϕ is the phase of $x[n]$

Example:



- **Exponential sequence**

$$x[n] = A\alpha^n, \quad -\infty < n < \infty$$

where A and α are real or complex numbers

- If we write $\alpha = e^{(\sigma_0 + j\omega_0)}$ and $A = |A|e^{j\phi}$ then we can express

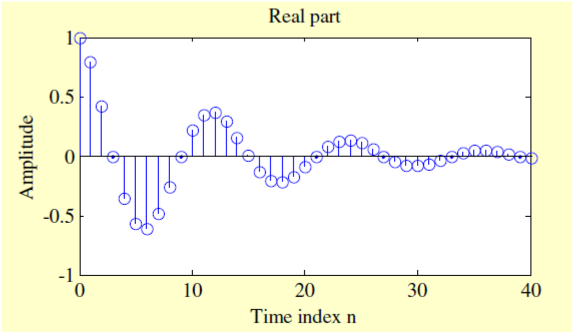
$$x[n] = |A|e^{j\phi}e^{(\sigma_0+j\omega_0)n} = x_{re}[n] + jx_{im}[n]$$

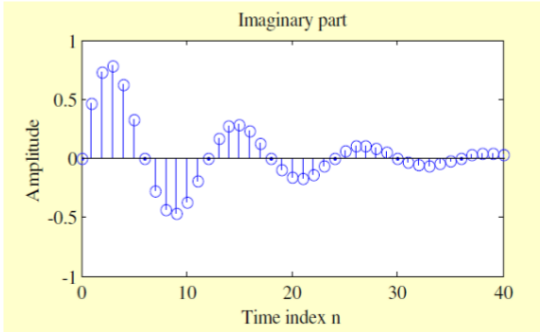
where

$$x_{re}[n] = |A|e^{\sigma_0 n}\cos(\omega_0 n + \phi)$$
$$x_{im}[n] = |A|e^{\sigma_0 n}\sin(\omega_0 n + \phi)$$

- $x_{re}[n]$ and $x_{im}[n]$ of a complex exponential sequence are sinusodial sequences with constant $(\sigma_0 = 0)$, growing $(\sigma_0 > 0)$, and decaying $(\sigma_0 < 0)$ amplitudes for $n > 0$

Example:





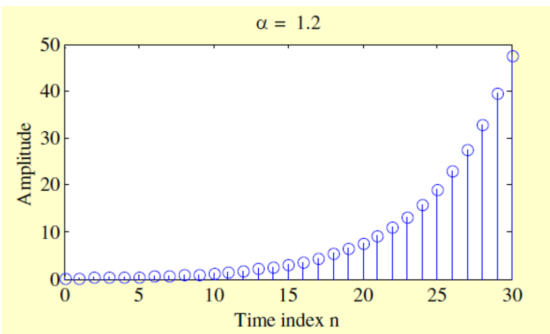
$$x[n] = \exp\left(-\frac{1}{12} + j\frac{\pi}{6}\right)n$$

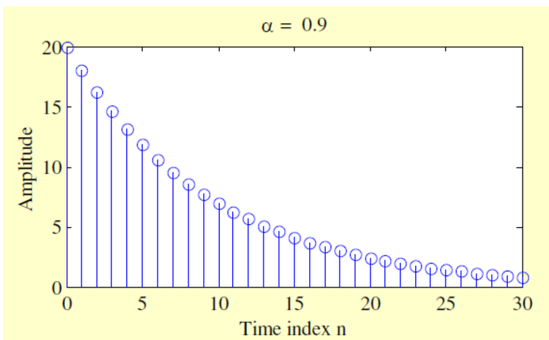
- **Real exponential sequence**

$$x[n] = A\alpha^n, \quad -\infty < n < \infty$$

where A and α are real numbers

Example:





- Sinusoidal sequence $A\cos(\omega_0 n + \phi)$ and complex exponential sequence $B\exp(j\omega_0 n)$ are periodic sequences of period N if $\omega N = 2\pi r$ where N and r are positive integers
- Smallest value of N satisfying $\omega N = 2\pi r$ is the **fundamental period** of the sequence

  To verify the above fact, consider

$$x_1[n] = \cos(\omega_0 n + \phi)$$
$$x_2[n] = \cos(\omega_0 (n + N) + \phi)$$

  Here

$$\cos(\omega_0(n+N)+\phi) = \cos(\omega_0 n + \phi)\cos(\omega_0 N) - \sin(\omega_0 n + \phi)\sin(\omega_0 N)$$

  which will be equal to $\cos(\omega_0 n + \phi) = x_1[n]$ only if

$$\sin(\omega_0 N) = 0 \quad \text{and} \quad \cos(\omega_0 N) = 1$$

  These two conditions are met **if and only if**

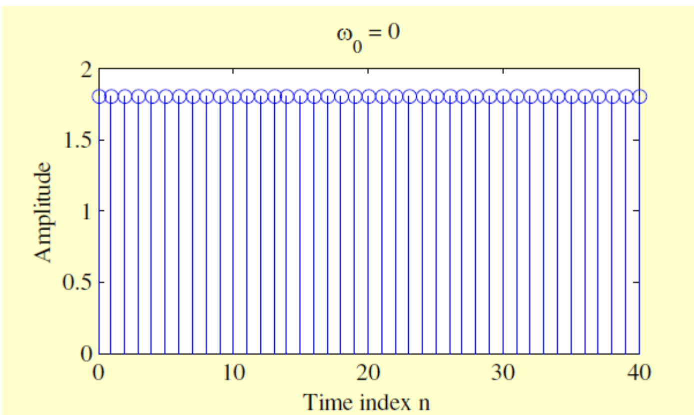
$$\omega_0 N = 2\pi r \quad \text{and} \quad \frac{2\pi}{\omega_0} = \frac{N}{r}$$

If $2\pi/\omega_0$ is a noninteger rational number, then the period will be a multiple of $2\pi/\omega_0$

Otherwise, the sequence is **aperiodic**

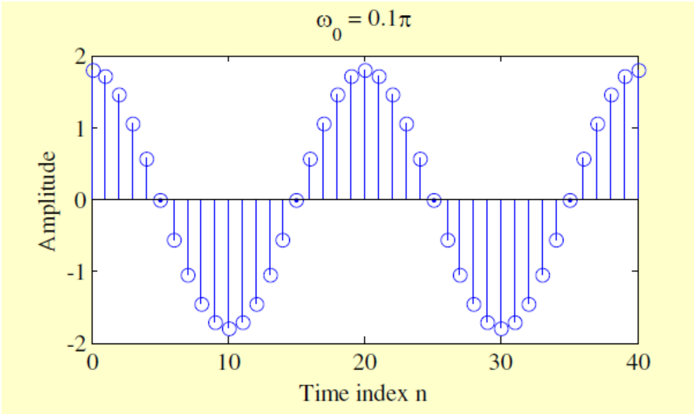
Example: $x[n] = \sin\left(\sqrt{3}n + \phi\right)$ is an aperiodic sequence

- **Example:** See the sequence below



Here $\omega_0 = 0$, hence period $N = \dfrac{2\pi r}{0} = 0$ for $r = 0$

- **Example:** See the sequence below



Here $\omega_0 = 0.1\pi$, hence period $N = \dfrac{2\pi r}{0.1\pi} = 20$ for $r = 1$

- **Property 1:** Consider $x[n] = \exp(j\omega_1 n)$ and $y[n] = \exp(j\omega_2 n)$ with $0 \leq \omega_1 < \pi$ and $2\pi k \leq \omega_2 < 2\pi(k+1)$ where k is any positive integer

  If $w_2 = w_1 + 2\pi k$ then $x[n] = y[n]$

  Thus, $x[n]$ and $y[n]$ are **indistinguishable**

- **Property 2:** The frequency of oscillation of $A\cos(\omega_0 n)$ increases as ω_0 increases from 0 to π, and then decreases ω_0 as increases from π to 2π

  Thus, frequencies in the neighborhood of $\omega = 0$ are called **low frequencies**, whereas, frequencies in the neighborhood of $\omega = \pi$ are called **high frequencies**

- Because of **Property 1**, a frequency ω_0 in the neighborhood of $\omega = 2\pi k$ is indistinguishable from a frequency $\omega_0 - 2\pi k$ in the neighborhood of $\omega = 0$

  and a frequency ω_0 in the neighborhood of $\omega = \pi(2k+1)$ is indistinguishable from a frequency $\omega_0 - \pi(2k+1)$ in the neighborhood of $\omega = \pi$

  Frequencies in the neighborhood of $\omega = 2\pi k$ are usually called **low frequencies**

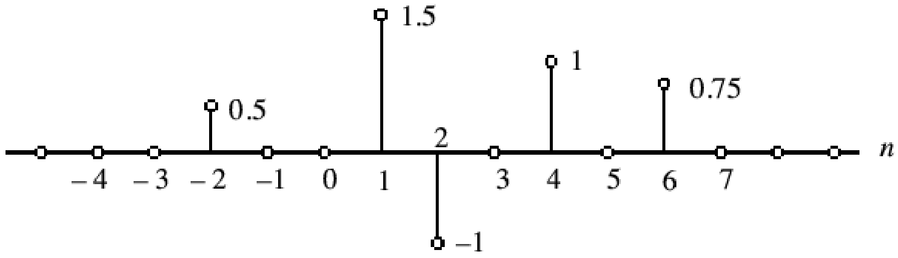
  Frequencies in the neighborhood of $\omega = \pi(2k+1)$ are usually called **high frequencies**

  Example: $v_1[n] = \cos(0.1\pi n) = \cos(1.9\pi n)$ is a low-frequency signal

  Example: $v_2[n] = \cos(0.8\pi n) = \cos(1.2\pi n)$ is a high-frequency signal

- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its **delayed** (or **advanced**) versions

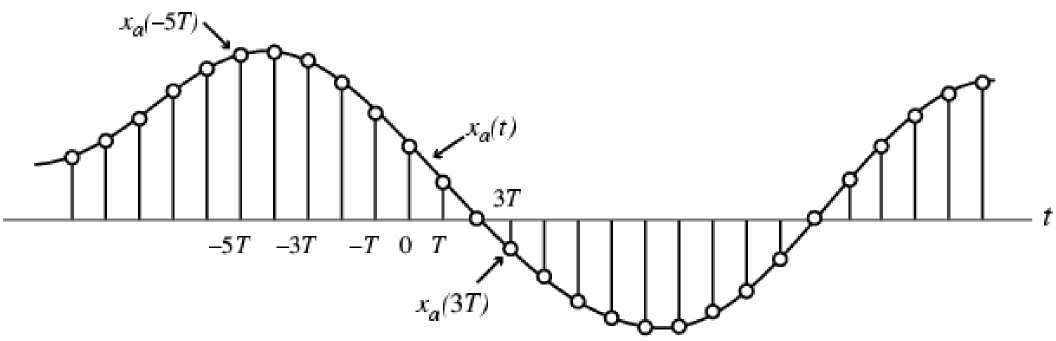
  Example:



$$x[n] = 0.5\,\delta[n+2] + 1.5\,\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\,\delta[n-6]$$

The Sampling Process

- Often, a discrete-time sequence $x[n]$ is developed by uniformly sampling a continuous-time signal $x_a(t)$ as indicated below



  The relation between the two signals is

$$x[n] = x_a(t)|_{t=nT} = x_a(nT), \quad \text{where } n = \ldots, -2, -1, 0, 1, 2, \ldots$$

- Time variable t of $x_a(t)$ is related to the time variable n of $x[n]$ only at discrete-time instants t_n given by

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

with $F_T = \dfrac{1}{T}$ denoting the **sampling frequency** and $\Omega_T = 2\pi F_T$ denoting the **sampling angular frequency**

- Consider the continuous-time signal

$$x_a(t) = A\cos(2\pi f_0 t + \phi) = A\cos(\Omega_0 t + \phi)$$

The corresponding discrete-time signal is

$$\begin{aligned} x[n] &= A\cos(\Omega_0 nT + \phi) = A\cos\left(\frac{2\pi\Omega_0}{\Omega_T} n + \phi\right) \\ &= A\cos(\omega_0 n + \phi) \end{aligned}$$

where

$$\omega_0 = \frac{2\pi\Omega_0}{\Omega_T} = \Omega_0 T$$

is the **normalized digital angular frequency** of $x[n]$

- If the unit of sampling period T is in **seconds**
- The unit of analog frequency f_0 is **hertz** ($\mathbf{Hz}$)
- The unit of normalized analog angular frequency Ω_0 is **radians/second**
- The unit of normalized digital angular frequency ω_0 is **radians/sample**

Aliasing

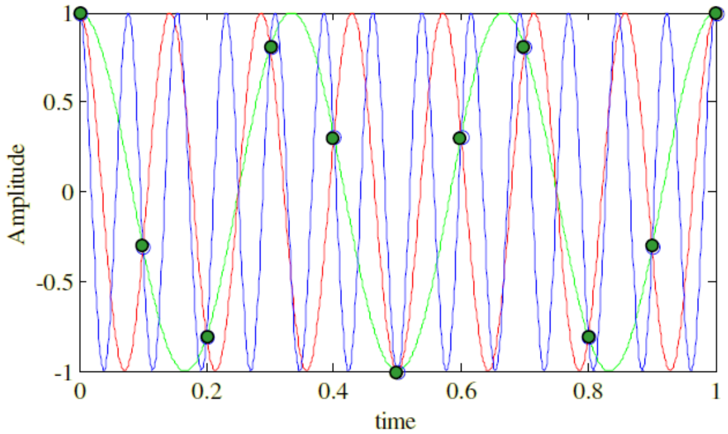
- Consider, the three continuous-time signals

$$
\begin{aligned}
g_1(t) &= \cos(6\pi t) \\
g_2(t) &= \cos(14\pi t) \\
g_3(t) &= \cos(26\pi t)
\end{aligned}
$$

of frequencies $3\ \mathrm{Hz}$, $7\ \mathrm{Hz}$, and $13\ \mathrm{Hz}$, are sampled at a sampling rate of $10\ \mathrm{Hz}$, i.e. with $T = 0.1\ \mathrm{sec}$ generating the three sequences

$$
\begin{aligned}
g_1[n] &= \cos(0.6\pi n) \\
g_2[n] &= \cos(1.4\pi n) \\
g_3[n] &= \cos(2.6\pi n)
\end{aligned}
$$

- Plots of these sequences (shown with circles) and their parent time functions are shown below:



Note that each sequence has exactly the same sample value for any given n

This fact can also be verified by observing that

$$g_2[n] = \cos(1.4\pi n) = \cos((2\pi - 0.6\pi)n) = \cos(0.6\pi n)$$
$$g_3[n] = \cos(2.6\pi n) = \cos((2\pi + 0.6\pi)n) = \cos(0.6\pi n)$$

As a result, all three sequences are identical and it is difficult to associate a unique continuous-time function with each of these sequences

- The above phenomenon of a continuous-time signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called **aliasing**

- Since there are an infinite number of continuous-time signals that can lead to the same sequence when sampled periodically, additional conditions need to be imposed, so that the sequence $\{x[n]\} = \{x_a(nT)\}$ can uniquely represent the parent continuous-time signal $x_a(t)$

  In this case, $x_a(t)$ can be fully recovered from $\{x[n]\}$

- **Example:** Determine the discrete-time signal $v[n]$ obtained by uniformly sampling at a sampling rate of 200 Hz the continuous-time signal

$$\begin{aligned} v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) \\ + 4\cos(500\pi t) + 10\sin(660\pi t) \end{aligned}$$

  Note: $v_a(t)$ is composed of 5 sinusoidal signals of frequencies 30 Hz, 150 Hz, 170 Hz, 250 Hz and 330 Hz

The sampling period is $T = \dfrac{1}{200} = 0.005 \text{ sec}$

The generated discrete-time signal $v[n]$ is thus given by

$$\begin{aligned} v[n] &= 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n) \\ &\quad + 4\cos(2.5\pi n) + 10\sin(3.3\pi n) \\ &= 6\cos(0.3\pi n) + 3\sin((2\pi - 0.5\pi)n) + 2\cos((2\pi - 0.3\pi)n) \\ &\quad + 4\cos((2\pi + 0.5\pi)n) + 10\sin((4\pi - 0.7\pi)n) \\ &= 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) \\ &\quad + 4\cos(0.5\pi n) - 10\sin(0.7\pi n) \\ &= 8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n) \end{aligned}$$

Note: $v[n]$ is composed of 3 discrete-time sinusoidal signals of normalized angular frequencies: 0.3π, 0.5π and 0.7π

Note: An identical discrete-time signal is also generated by uniformly sampling the following continuous-time signals at a 200-Hz sampling rate:

$$\begin{aligned} w_a(t) &= 8\cos(60\pi t) + 5\sin(100\pi t + 0.6435) - 10\sin(140\pi t) \\ g_a(t) &= 2\cos(60\pi t) + 4\cos(100\pi t) + 10\sin(260\pi t) \\ &\quad + 6\cos(460\pi t) + 3\sin(700\pi t) \end{aligned}$$

Recall $\omega_0 = \dfrac{2\pi\Omega_0}{\Omega_T}$, thus

if $\boxed{\Omega_T > 2\Omega_0,}$ then the corresponding normalized digital angular frequency ω_0 of the discrete-time signal obtained by sampling the parent continuous-time sinusoidal signal will be in the range $-\pi < \omega < \pi$,

i.e., **No Aliasing**

- On the other hand, if $\Omega_T < 2\Omega_0$, the normalized digital angular frequency will foldover into a lower digital frequency $w_0 = \left\langle \frac{2\pi\Omega_0}{\Omega_T} \right\rangle_{2\pi}$ in the range $-\pi < \omega < \pi$ because of aliasing
- Hence, to prevent aliasing, the sampling frequency Ω_T should be greater than 2 times the frequency Ω_0 of the sinusoidal signal being sampled

Sampling Theorem

- **Generalization:** Consider an arbitrary continuous-time signal $x_a(t)$ composed of a weighted sum of a number of sinusoidal signals
- $x_a(t)$ can be represented uniquely by its sampled version $\{x[n]\}$ if the sampling frequency Ω_T is chosen to be greater than 2 times the highest frequency contained in $x_a(t)$

  This condition to be satisfied by the sampling frequency to prevent aliasing is called the **sampling theorem**

  A formal proof of this theorem will be presented later

Correlation of Signals

- There are applications where it is necessary to compare one reference signal with one or more signals to determine the similarity between the pair and to determine additional information based on the similarity
- For example, in digital communications, a set of data symbols are represented by a set of unique discrete-time sequences

  If one of these sequences has been transmitted, the receiver has to determine which particular sequence has been received by comparing the received signal with every member of possible sequences from the set

- Similarly, in radar and sonar applications, the received signal reflected from the target is a delayed version of the transmitted signal and by measuring the delay, one can determine the location of the target

  The detection problem gets more complicated in practice, as often the received signal is corrupted by additive random noise

Definitions:

- A measure of similarity between a pair of energy signals, $x[n]$ and $y[n]$, is given by the **cross-correlation sequence** $r_{xy}[\ell]$ defined by

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell], \qquad \text{where } \ell = 0, \pm 1, \pm 2, \ldots$$

  The parameter ℓ called lag, indicates the time-shift between the pair of signals
- $y[n]$ is said to be shifted by ℓ samples to the **right** with respect to the reference sequence $x[n]$ for **positive values** of ℓ, and shifted by ℓ samples to the **left** for **negative values** of ℓ
- The ordering of the subscripts xy in the definition of $r_{xy}[\ell]$ specifies that $x[n]$ is the **reference sequence** which remains fixed in time while $y[n]$ is being shifted with respect to $x[n]$

- If $y[n]$ is made the reference signal and shift $x[n]$ with respect to y[n], then the corresponding cross-correlation sequence is given by

$$\begin{aligned} r_{yx}[\ell] &= \sum_{n=-\infty}^{\infty} y[n]x[n-\ell] \\ &= \sum_{m=-\infty}^{\infty} y[m+\ell]x[m] \\ &= r_{xy}[-\ell] \end{aligned}$$

  Thus, $r_{yx}[\ell]$ is obtained by time-reversing $r_{xy}[\ell]$

Autocorrelation

- The **autocorrelation sequence** of $x[n]$ is given by

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x[n-\ell]$$

obtained by setting $y[n] = x[n]$ in the definition of the cross-correlation sequence $r_{xy}[\ell]$

Note: $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \mathcal{E}_x$, the energy of the real signal $x[n]$

- From the relation $r_{yx}[\ell] = r_{xy}[-\ell]$ it follows that $r_{xx}[\ell] = r_{xx}[-\ell]$ implying that $r_{xx}[\ell]$ is an even function for real $x[n]$

- An examination of

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$$

reveals that the expression for the cross-correlation looks quite similar to that of the linear convolution

This similarity is much clearer if we rewrite the expression for the cross-correlation

$$\begin{aligned} r_{xy}[\ell] &= \sum_{n=-\infty}^{\infty} x[n]y[n-\ell] \\ &= \sum_{n=-\infty}^{\infty} x[n]y[-(\ell-n)] \\ &= x[\ell] \circledast y[-\ell] \end{aligned}$$

Properties of Autocorrelation and Cross-correlation Sequences

- Consider two **finite-energy** sequences $x[n]$ and $y[n]$

  The energy of the combined sequence

  $$a\,x[n] + y[n-\ell]$$

  is also finite and nonnegative for any finite value of a, i.e.,

  $$\sum_{n=-\infty}^{\infty} (a\,x[n] + y[n-\ell])^2 = a^2 \sum_{n=-\infty}^{\infty} x^2[n] + 2a \sum_{n=-\infty}^{\infty} x[x]y[n-\ell] + \sum_{n=-\infty}^{\infty} y^2[n-\ell] \geq 0$$

Thus

$$a^2\, r_{xx}[0] + 2a\, r_{xy}[\ell] + r_{yy}[0] \geq 0$$

where $r_{xx}[0] = \mathcal{E}_x > 0$ and $r_{yy}[0] = \mathcal{E}_y > 0$

- We can rewrite the equation on the previous slide as

  $$\begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} r_{xx}[0] & r_{xy}[\ell] \\ r_{xy}[\ell] & r_{yy}[0] \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

  for any finite value of a

  Or, in other words, the matrix

  $$\begin{bmatrix} r_{xx}[0] & r_{xy}[\ell] \\ r_{xy}[\ell] & r_{yy}[0] \end{bmatrix}$$

  is positive semidefinite. i.e.,

  $$r_{xx}[0]\, r_{yy}[0] - r_{xy}^2[\ell] \geq 0$$

or, equivalently

$$|r_{xy}[\ell]| \leq \sqrt{r_{xx}[0]\, r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$$

where $r_{xx}[0] = \mathcal{E}_x > 0$ and $r_{yy}[0] = \mathcal{E}_y > 0$

- The last inequality above provides an upper bound for the cross-correlation samples
- If we set $y[n] = x[n]$, then the inequality reduces to

$$|r_{xx}[\ell]| \leq r_{xx}[0] = \mathcal{E}_x$$

Thus, at **zero lag** ($\ell = 0$), the sample value of the autocorrelation sequence has its maximum value

- Now consider the case

$$y[n] = \pm b\, x[nN]$$

where N is an integer and $b > 0$ is an arbitrary number

In this case $\mathcal{E}_y = b^2\, \mathcal{E}_x$

Therefore $\sqrt{\mathcal{E}_x \mathcal{E}_y} = \sqrt{b^2\, \mathcal{E}_x^2} = b\, \mathcal{E}_x$

- Using the above result in $|r_{xy}[\ell]| \leq \sqrt{r_{xx}[0]\, r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$ we get

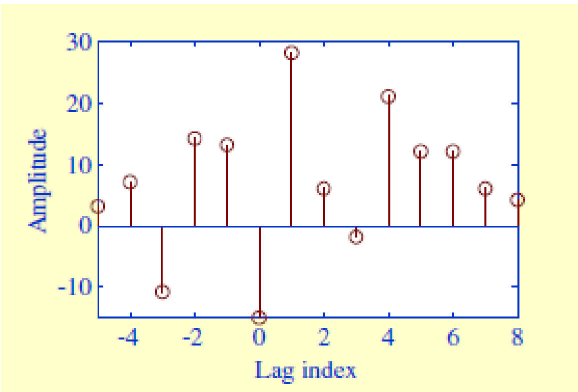
$$-b\, r_{xx}[0] \leq r_{xy}[\ell] \leq b\, r_{xx}[0]$$

Correlation Computation Using MATLAB

- The cross-correlation and autocorrelation sequences can easily be computed using MATLAB
- **Example:** Consider the two finite-length sequences

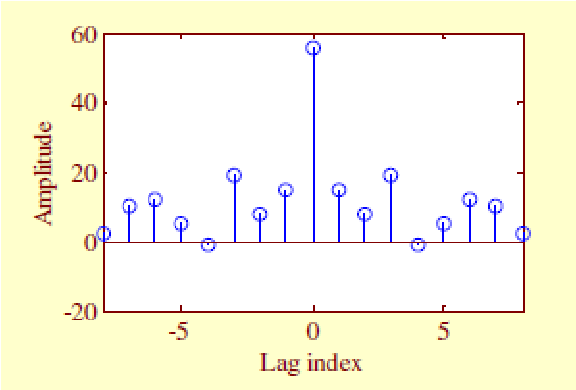
$$x[n] = \begin{bmatrix}1 & 3 & -2 & 1 & 2 & -1 & 4 & 4 & 2\end{bmatrix}$$
$$y[n] = \begin{bmatrix}2 & -1 & 4 & 1 & -2 & 3\end{bmatrix}$$

- The cross-correlation sequence $r_{xy}[\ell]$ computed using `Program_2_7` of the textbook is plotted below



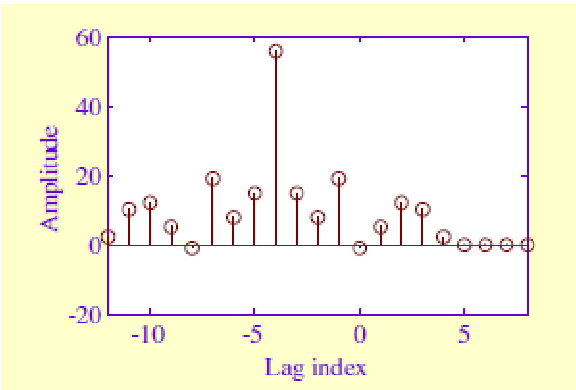
- The autocorrelation sequence $r_{xx}[\ell]$ computed using `Program_2_7` is shown below

  Note: At zero lag, $r_{xx}[0]$ is the maximum



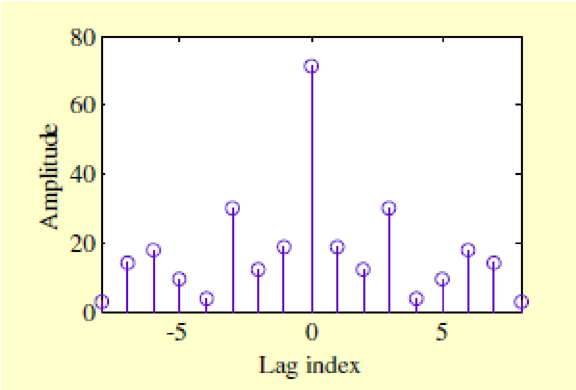
- The plot below shows the cross-correlation of $x[n]$ and $y[n] = x[n - N]$ for $N = 4$

  Note: The peak of the cross-correlation is precisely the value of the delay N



- The plot below shows the autocorrelation of $x[n]$ corrupted with an additive random noise generated using the function `randn`

  Note: The autocorrelation still exhibits a peak at zero lag



- The autocorrelation and the cross-correlation can also be computed using the function `xcorr`

  However, the correlation sequences generated using this function are the time-reversed version of those generated using `Program_2_7` and `Program_2_8`

Normalized Forms of Correlation

- Normalized forms of autocorrelation and cross-correlation are given by

$$\rho_{xx}[\ell] = \frac{r_{xx}[\ell]}{r_{xx}[0]}$$

$$\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0]r_{yy}[0]}}$$

- They are often used for convenience in comparing and displaying
- **Note:** $|\rho_{xx}[\ell]| \leq 1$ and $|\rho_{xy}[\ell]| \leq 1$ independent of the range of values of $x[n]$ and $y[n]$

Correlation Computation for Power Signals

- The cross-correlation sequence for a pair of power signals, $x[n]$ and $y[n]$, is defined as

$$r_{xy}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]y[n-\ell]$$

- The autocorrelation sequence of a power signal $x[n]$ is given by

$$r_{xx}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]x[n-\ell]$$

Correlation Computation for Periodic Signals

- The cross-correlation sequence for a pair of periodic signals of period N, $\tilde{x}[n]$ and $\tilde{y}[n]$, is defined as

$$r_{\tilde{x}\tilde{y}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{y}[n-\ell]$$

- The autocorrelation sequence of a periodic signal $\tilde{x}[n]$ of period N is given by

$$r_{\tilde{x}\tilde{x}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{x}[n-\ell]$$

- **Note:** Both $r_{\tilde{x}\tilde{y}}[\ell]$ and $r_{\tilde{x}\tilde{x}}[\ell]$ are also periodic signals with a period N

- The periodicity property of the autocorrelation sequence can be exploited to determine the period of a periodic signal that may have been corrupted by an additive random disturbance
- Let $\tilde{x}[n]$ be a periodic signal corrupted by the random noise $d[n]$ resulting in the signal

$$w[n] = \tilde{x}[n] + d[n]$$

which is observed for $0 \le n \le M-1$ and $M \gg N$

The autocorrelation of $w[n]$ is given by

$$\begin{aligned} r_{ww}[\ell] &= \frac{1}{M}\sum_{n=0}^{M-1} w[n]w[n-\ell] \\ &= \frac{1}{M}\sum_{n=0}^{M-1} (\tilde{x}[n]+d[n])(\tilde{x}[n-\ell]+d[n-\ell]) \\ &= \frac{1}{M}\sum_{n=0}^{M-1} \tilde{x}[n]\tilde{x}[n-\ell] + \frac{1}{M}\sum_{n=0}^{M-1} d[n]d[n-\ell] \\ &\quad + \frac{1}{M}\sum_{n=0}^{M-1} \tilde{x}[n]d[n-\ell] + \frac{1}{M}\sum_{n=0}^{M-1} d[n]\tilde{x}[n-\ell] \\ &= r_{\tilde{x}\tilde{x}}[\ell] + r_{dd}[\ell] + r_{\tilde{x}d}[\ell] + r_{d\tilde{x}}[\ell] \end{aligned}$$

In the last equation on the previous slide, $r_{\tilde{x}\tilde{x}}[\ell]$ is a periodic sequence with a period N and hence will have peaks at $\ell = 0, N, 2N, \ldots$ with the same amplitudes, as ℓ approaches M

As $\tilde{x}[n]$ and $d[n]$ are not correlated, samples of cross-correlation sequences $r_{\tilde{x}d}[\ell]$ and $r_{d\tilde{x}d}[\ell]$ are likely to be very small relative to the amplitudes of $r_{\tilde{x}\tilde{x}}[\ell]$

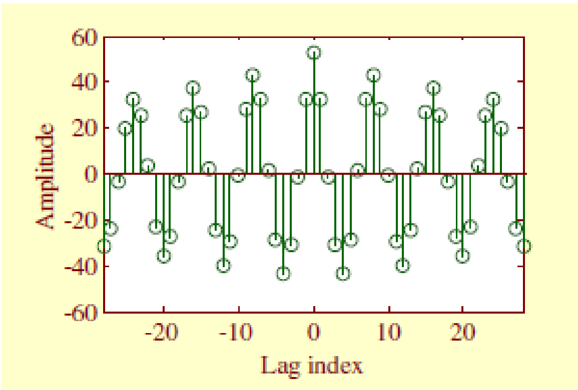
The autocorrelation of $d[n]$ will show a peak at $\ell = 0$ with other samples having rapidly decreasing amplitudes with increasing values of $|\ell|$

- Hence, peaks of $r_{ww}[\ell]$ for $\ell > 0$ are essentially due to the peaks of $r_{\tilde{x}\tilde{x}}[\ell]$ and can be used to determine whether $\tilde{x}[n]$ is a periodic sequence and also its period N if the peaks occur at periodic intervals

Correlation Computation of a Periodic Signal Using MATLAB

- **Example:** Let us determine the period of the sinusoidal sequence $x[n] = \cos(0.125n), 0 \leq n \leq 95$ corrupted by an additive uniformly distributed random noise of amplitude in the range $[-0.5, 0.5]$

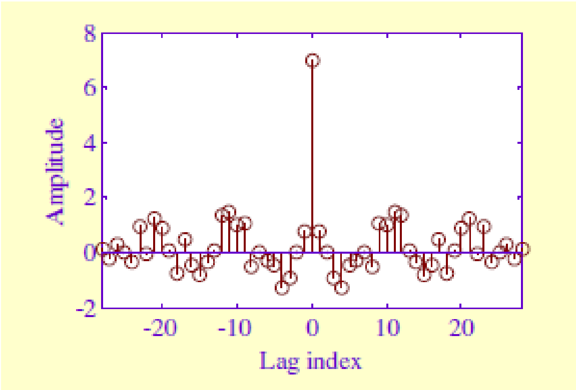
  Using `Program_2_8` of textbook we arrive at the plot of $r_{ww}[\ell]$ shown on the next slide



As can be seen from the plot given above, there is a strong peak at zero lag

However, there are distinct peaks at lags that are multiples of 8 indicating the period of the sinusoidal sequence to be 8 as expected

Figure below shows the plot of $r_{dd}[\ell]$



As can be seen $r_{dd}[\ell]$ shows a very strong peak at only zero lag