İsim:
Öğrenci No:

**ARASINAV (30%)**

**EEM 401**
**SAYISAL SİNYAL İŞLEME** 15 Kasım 2012

Lütfen, tüm soruları yanıtlayınız ve yaptığınız varsayımları açıklayınız. İyi şanslar!

**RELEVANT INFORMATION**

The Discrete-Time Fourier Transform (DTFT) of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse discrete-time Fourier transform is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Commonly used DTFT pairs

| **Sequence** | | **DTFT** |
|---|---|---|
| $\delta[n]$ | $\longleftrightarrow$ | $1$ |
| $1$ | $\longleftrightarrow$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$ |
| $\mu[n]$ | $\longleftrightarrow$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$ |
| $\alpha^n\mu[n] \quad (\lvert\alpha\rvert<1)$ | $\longleftrightarrow$ | $\frac{1}{1-\alpha e^{-j\omega}}$ |
| $h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \quad (-\infty<n<\infty)$ | $\longleftrightarrow$ | $H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le \omega_c \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$ |

DTFT theorems

| | $g[n]$<br>$h[n]$ | $G(e^{j\omega})$<br>$H(e^{j\omega})$ |
|---|---|---|
| Linearity | $\alpha\, g[n] + \beta\, h[n]$ | $\alpha\, G(e^{j\omega}) + \beta\, H(e^{j\omega})$ |
| Time-shifting | $g[n-n_0]$ | $e^{-j\omega n_0}\, G(e^{j\omega})$ |
| Frequency-shifting | $e^{j\omega_0 n}\, g[n]$ | $G(e^{j(\omega-\omega_0)})$ |
| Differentiation in frequency | $n\, g[n]$ | $j\frac{dG(e^{j\omega})}{d\omega}$ |
| Convolution | $g[n] \circledast h[n]$ | $G(e^{j\omega})H(e^{j\omega})$ |
| Modulation | $g[n]h[n]$ | $\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$ |
| Parseval's relation | $\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$ | |

Q1. (25 pts) Consider the following sequences

$$x[n] = \left\{2, \ \ 0, \ \ -1, \ \ 6, \ \ -3, \ \ 2\right\}, \quad -3 \le n \le 2$$
$$g[n] = \left\{2, \ \ -7, \ \ -3, \ \ 0, \ \ 1\right\}, \quad -4 \le n \le 0$$
$$w[n] = \left\{3, \ \ 6, \ \ -1, \ \ 2, \ \ 6, \ \ 5\right\}, \quad -2 \le n \le 3$$

The sample values of each of the above sequences outside the ranges specified are all zeros.

a) Express the sequences $x[n]$, $g[n]$ and $w[n]$ as a linear combination of delayed unit sample sequences (unit sample sequence: $\delta[n]$).

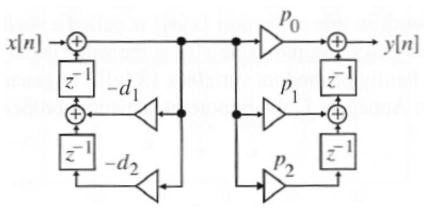
b) Express the sequences $x[n]$ and $g[n]$ as a linear combination of delayed unit step sequences (unit step sequence: $\mu[n]$).

c) Compute the linear convolution $y[n] = x[n-2] \circledast g[n+1]$

d) Compute the autocorrelation, $r_{gg}$, of the sequence $g[n]$. What can you say about the autocorrelation of $g[n-2]$ in terms of $r_{gg}$?

Q2. (10 pts) Let $y[n]$ be the sequence obtained by a linear combination of two causal finite-length sequences $h[n]$ and $x[n]$, i.e. $y[n] = x[n] \circledast h[n]$. If $y[n] = \left\{0, -15, -7, 9, -4, 2\right\}$ and $h[n] = \left\{5, -1, 1\right\}$, then determine $x[n]$.

Q3. (5 pts) Considering the block diagram below, develop the expression for $y[n]$.




Q4. (10 pts) A continuous-time sinusoidal signal $x_a(t) = \cos(\Omega_0 t)$ is sampled at $t = nT$, $-\infty < n < \infty$, generating the discrete-time sequence $x[n] = x_a(nT) = \cos(\Omega_0 nT)$. For what values of $T$ is $x[n]$ a periodic sequence? What is the fundamental period, $N_F$, of $x[n]$ if $\Omega_0 = 30$ radians and sampling period $T = \pi/6$ seconds? Can $x_a(t)$ be recovered from $x[n]$?

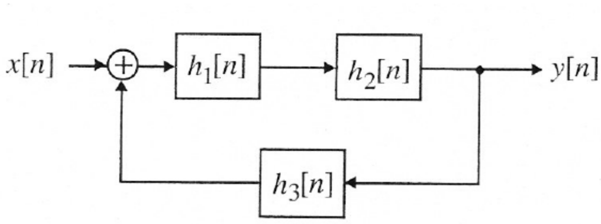
Q5. (5 pts) Determine the DTFT of the causal sequence $x[n] = A\,\alpha^{n-1}\cos(\omega_0 n + \phi)\mu[n-1]$, where $A, \alpha, \omega_0$, and $\phi$ are real, and $|\alpha| < 1$, i.e., $X(e^{j\omega}) = ?$

Q6. (5 pts) Determine the inverse DTFT of $X(e^{j\omega}) = -4 + 3\cos\omega + 4\cos 2\omega$, i.e., $x[n] = ?$

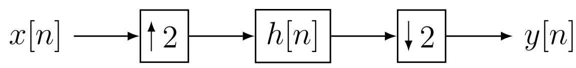
Q7. (15 pts) A continuous-time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies 300 Hz, 500 Hz, 1.2 kHz, 2.15 kHz and 3.5 kHz. The signal $x_a(t)$ is sampled at a 8-kHz rate, and the sampled sequence is then passed through an ideal lowpass filter with a cutoff frequency of 900 Hz, generating a continuous-time signal $y_a(t)$. What are the frequency components present in the reconstructed signal $y_a(t)$? Explain your answer clearly.

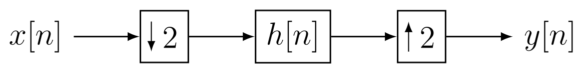
Q8. (15 pts) Determine the overall impulse response, $h[n]$, of the system of figure below, where the impulse responses of the component systems are given as:

$$h_1[n] = 2\,\delta[n-2] + 3\,\delta[n+1]$$
$$h_2[n] = \delta[n-1] - 2\,\delta[n+2]$$
$$h_3[n] = 5\,\delta[n-5] - 7\,\delta[n-3] + 2\,\delta[n-1] + \delta[n] - 3\,\delta[n+1].$$




Q9. (10 pts) Are the overall systems, given in the figures (a) and (b) below, linear and time-invariant? Justify your answer by giving a brief proof and explanation.

a) 


b) 

# SOLUTIONS

Q1. a) (5 pts)

$$x[n] = 2\,\delta[n+3] - \delta[n+1] + 6\,\delta[n] - 3\,\delta[n-1] + 2\,\delta[n-2]$$
$$g[n] = 2\,\delta[n+4] - 7\,\delta[n+3] - 3\,\delta[n+2] + \delta[n]$$
$$w[n] = 3\,\delta[n+2] + 6\,\delta[n+1] - \delta[n] + 2\,\delta[n-1] + 6\,\delta[n-2] + 5\,\delta[n-3]$$

b) (5 pts) Noting that $\delta[n] = \mu[n] - \mu[n-1]$

$$\begin{aligned} x[n] &= 2(\mu[n+3] - \mu[n+2]) - (\mu[n+1] - \mu[n]) + 6(\mu[n] - \mu[n-1]) - 3(\mu[n-1] - \mu[n-2]) + 2(\mu[n-2] - \mu[n-3]) \\ &= 2\mu[n+3] - 2\mu[n+2] - \mu[n+1] + 6\mu[n] - 9\mu[n-1] + 5\mu[n-2] - 2\mu[n-3] \\ g[n] &= 2(\mu[n+4] - \mu[n+3]) - 7(\mu[n+3] - \mu[n+2]) - 3(\mu[n+2] - \mu[n+1]) + (\mu[n] - \mu[n-1]) \\ &= 2\mu[n+4] - 9\mu[n+3] + 4\mu[n+2] + 3\mu[n+1] + \mu[n] - \mu[n-1] \end{aligned}$$

c) (10 pts) $y[n] = \left\{4,\ \ -14,\ \ -8,\ \ 19,\ \ -43,\ \ 7,\ \ -6,\ \ 0,\ \ -3,\ \ 2\right\},\quad -6 \le n \le 3$

d) (5 pts) $r_{gg}[\ell] = \left\{2,\ \ -7,\ \ -9,\ \ 7,\ \ 63,\ \ 7,\ \ -9,\ \ -7,\ \ 2\right\},\quad -4 \le \ell \le 4$

Autocorrelation of $g[n-2]$ is the same as the autocorrelation of $g[n]$.

Q2. (10 pts) $x[n] = \left\{0,\ \ -3,\ \ -2,\ \ 2\right\},\quad 0 \le n \le 3$

Q3. (5 pts) $y[n] = p_0\,x[n] + p_1\,x[n-1] + p_2\,x[n-2] - d_1\,y[n-1] - d_2\,y[n-2]$

Q4. (10 pts) As $x[n] = \cos(\Omega_0 nT)$, if $x[n]$ is periodic with a period $N$, i.e. $x[n] = x[n+N]$, then

$$\cos(\Omega_0 nT) = \cos(\Omega_0 nT + \Omega_0 NT)$$

This implies that $\Omega_0 NT = 2\pi r$ with $r$ to be any nonzero integer. Thus, $x[n]$ is periodic sequence if $T = \dfrac{2\pi r}{\Omega_0 N}$.

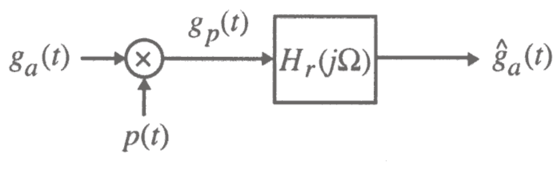
For $\Omega_0 = 30$ and $T = \dfrac{\pi}{6}$ then $\Omega_0 NT = 2\pi r$ reduces to $5N = 2r$, the smallest values $N$ and $r$ which satisfies this equation are $N = 2$ and $r = 5$. Thus $N_F = 2$.

In order $x_a(t)$ to be recovered from $x[n]$, sampling rate $F_T = \dfrac{1}{T}$ must satisfy the Nyquist rate, i.e. $F_T > \dfrac{2\Omega_0}{2\pi}$ or $T < \dfrac{2\pi}{2\Omega_0}$. However, in our case $T = \dfrac{\pi}{6} \not< \dfrac{\pi}{30}$, so aliasing occurs and $x_a(t)$ cannot be recovered from $x[n]$.

Q5. (5 pts) Using the tables in the first page $X(e^{j\omega}) = \dfrac{1}{2}A\left(\dfrac{e^{-j((\omega-w_0)-\theta)}}{1-\alpha\, e^{-j(w-w_0)}} + \dfrac{e^{-j((\omega+w_0)+\theta)}}{1-\alpha\, e^{-j(w+w_0)}}\right)$

Q6. (5 pts) Using the tables in the first page $x[n] = -4\delta[n] + \dfrac{3}{2}(\delta[n+1] + \delta[n-1]) + 2(\delta[n+2] + \delta[n-2])$

Q7. (15 pts) Suppose a signal $g_a(t)$ with a CTFT $G_a(j\Omega)$ is sampled at frequency of $W_T$ (i.e. modulated by a pulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$) as shown in the figure below (including the lowpass filter we are going to apply).



The sampled signal $g_p(t) = g_a(t)p(t)$ will have a CTFT $G_p(j|W)$ as weighted sum of the shifted replicas of $G_a(j|W)$ by the amount of sampling frequency, i.e.

$$G_p(j\Omega) = \frac{1}{T}\sum_{\ell=-\infty}^{\infty} G_a(j(\Omega - \ell\Omega_T))$$

.

Note that, CTFT of a sinusoidal signal ($\cos(\Omega_0 t)$ or $\sin(\Omega_0 t)$) will have to have delta-dirac functions located at two frequencies $|W_0$ and $-\Omega_0$. Thus, CTFT $X_a(j\Omega)$ of $x_a(t)$ (consisting of five sinusoidal signals) will have 10 frequency components at $\mp$300 Hz, $\mp$500 Hz, $\mp$1200 Hz, $\mp$2150 Hz and $\mp$3500 Hz.

So sampling $x_a(t)$ at 8-kHz wil result in the sampled signal $x_p(t)$ with CTFT of $X_p(j\Omega)$ having frequency components at $\mp 300 + 8000\ell$, $\mp 500 + 8000\ell$, $\mp 1200 + 8000\ell$, $\mp 2150 + 8000\ell$ and $\mp 3500 + 8000\ell$ Hz where $\ell$ is an integer. As 8000 Hz is greater than the twice of of the highest frequency present in the signal, i.e. 8000 Hz $>$ 7000 Hz, we see that no aliasing occurs, i.e. none of the five frequencies will have as a lower frequency replica.

So when we apply an ideal lowpass filter with a cutoff frequency of 900 Hz to $x_p(t)$, the frequencies $|f| > 900$ Hz will be filtered out. So, the reconstructed signal $y_a(t)$ will be composed of a linear combination of two sinusoidal signals of frequencies 300 Hz and 500 Hz.

Q8. (15 pts) From the figure, the overall impulse response $h[n]$ will be given by as

$$\begin{aligned} y[n] \circledast (\delta[n] - h_1[n] \circledast h_2[n] \circledast h_3[n]) &= (h_1[n] \circledast h_2[n]) \circledast x[n] \\ y[n] \circledast g_2[n] &= x[n] \circledast g_1[n] \end{aligned}$$

where

$$\begin{aligned} g_1[n] &= h_1[n] \circledast h_2[n] \\ &= -6\delta[n+3] - \delta[n] + 2\delta[n-3] \\ g_3[n] &= \delta[n] - h_1[n] \circledast h_2[n] \circledast h_3[n] \\ &= -18\delta[n+4] + 6\delta[n+3] + 12\delta[n+2] - 3\delta[n+1] - 40\delta[n] + 2\delta[n-1] + 36\delta[n-2] - 9\delta[n-3] \\ &\quad - 4\delta[n-4] + 5\delta[n-5] + 14\delta[n-6] - 10\delta[n-8] \end{aligned}$$

Thus,

$$H(e^{j\omega}) = \frac{G_1(e^{j\omega})}{G_2(e^{j\omega})}$$

and $h[n]$ is given as the IDFT of $H(e^{j\omega})$

$$\begin{aligned} h[n] &= IDTFT\left\{H(e^{j\omega})\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

Q9. a) (5 pts) This system is linear and time-invariant

b) (5 pts) This system is linear, but not time-invariant