Discrete-Time Signals: Frequency-Domain Representation - II

CHAPTER 3

These lecture slides are based on "Digital Signal Processing: A Computer-Based Approach, 4th ed." textbook by S.K. Mitra and its instructor materials. U.Sezen

Contents

Digital Processing of Continuous-Time Signals
- Effect of Sampling in the Frequency Domain
- Recovery of the Analog Signal
- Implication of the Sampling Process
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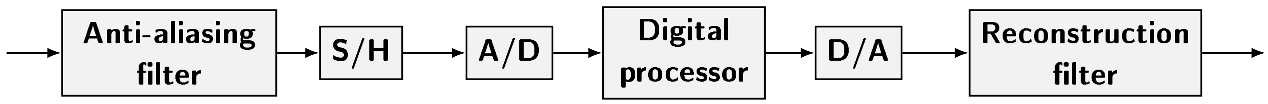
Digital Processing of Continuous-Time Signals

- Digital processing of a continuous-time signal involves the following basic steps:
  1. Conversion of the continuous-time signal into a discrete-time signal,
  2. Processing of the discrete-time signal,
  3. Conversion of the processed discrete-time signal back into a continuous-time signal

- Conversion of a continuous-time signal into digital form is carried out by an **analog-to-digital (A/D) converter**
- The reverse operation of converting a digital signal into a continuous-time signal is performed by a **digital-to-analog (D/A) converter**
- Since the A/D conversion takes a finite amount of time, a **sample-and-hold (S/H) circuit** is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation

- To prevent aliasing, an analog **anti-aliasing filter** is employed before the S/H circuit
- To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog **reconstruction filter** is used
- Complete block-diagram:



Note: Both the anti-aliasing filter and the reconstruction filter are analog lowpass filters

- As indicated earlier, discrete-time signals in many applications are generated by sampling continuous-time signals
- We have seen earlier that identical discrete-time signals may result from the sampling of more than one distinct continuous-time function
- In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal

- However, under certain conditions, it is possible to relate a unique continuous-time signal to a given discrete-time signal
- If these conditions hold, then it is possible to recover the original continuous-time signal from its sampled values
- We next develop this correspondence and the associated conditions

Effect of Sampling in the Frequency Domain

- Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g[n]$ where

$$g[n] = g_a(nT), \quad -\infty < n < \infty$$

with T being the **sampling period**

- The reciprocal of T is called the **sampling frequency** F_T , i.e.,

$$F_T = \frac{1}{T}$$

- Now, the frequency-domain representation of $g_a(t)$ is given by its continuous-time Fourier transform (**CTFT**):

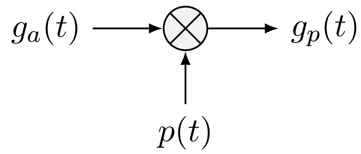
$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

- The frequency-domain representation of $g[n]$ is given by its discrete-time Fourier transform (**DTFT**):

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

- To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a modulation of $g_a(t)$ by a **periodic impulse train** $p(t)$:

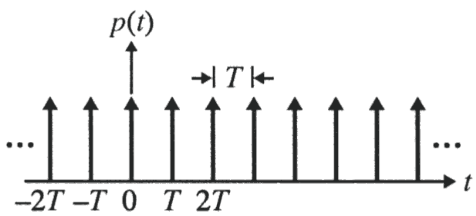
$$g_p(t) = g_a(t)\, p(t)$$



where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

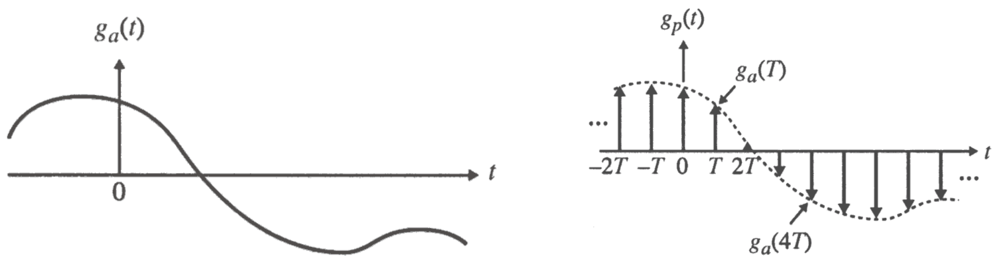
Here, $p(t)$ consists of a train of ideal impulses with a period T as shown below



- The modulation operation yields an impulse train:

$$g_p(t) = g_a(t)\,p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$

- Here, $g_p(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant



- There are two different forms of $G_p(j\Omega)$:
  - One form is given by the weighted sum of the CTFTs of $\delta(t-nT)$

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

  - To derive the second form, we make use of the **Poisson's formula**:

$$\sum_{n=-\infty}^{\infty} \phi(t+nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)e^{jk\Omega_T t}$$

where $\Omega_T = \dfrac{2\pi}{T}$ and $\Phi(j\Omega)$ is the CTFT of $\phi(t)$

For $t = 0$, equation given on the previous slide reduces to

$$\sum_{n=-\infty}^{\infty} \phi(nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)$$

Now, from the frequency shifting property of the CTFT, the CTFT of $g_a(t)e^{-j\Psi t}$ is given by $G_a(j(\Omega + \Psi))$

Substituting $\phi(t) = g_a(t)e^{-j\Psi t}$ in the equation above, we arrive at

$$\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Psi nT} = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(k\Omega_T + \Psi))$$

By replacing Ψ with Ω in the above equation we arrive at the alternative form of the CTFT $G_p(j\Omega)$ of $g_p(t)$

The alternative form of the CTFT of $g_p(t)$ is given by

$$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_T))$$

Therefore, $G_p(j\Omega)$ is a periodic function of Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by $\dfrac{1}{T}$

- The term on the right hand side (RHS) of the previous equation for $k = 0$ is the **baseband** portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$
- The frequency range
$$-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$$
is called the **baseband** or **Nyquist band**

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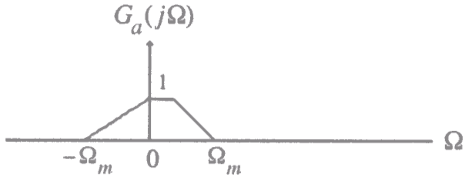
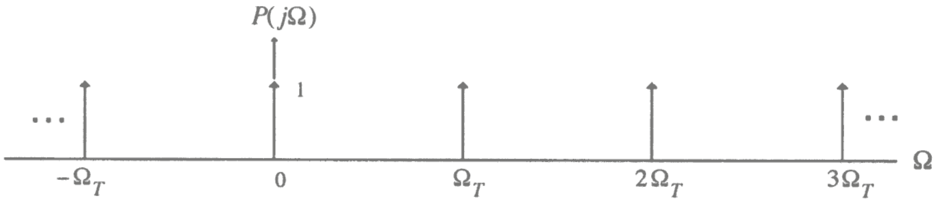
$$-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$$
is called the **baseband** or **Nyquist band**

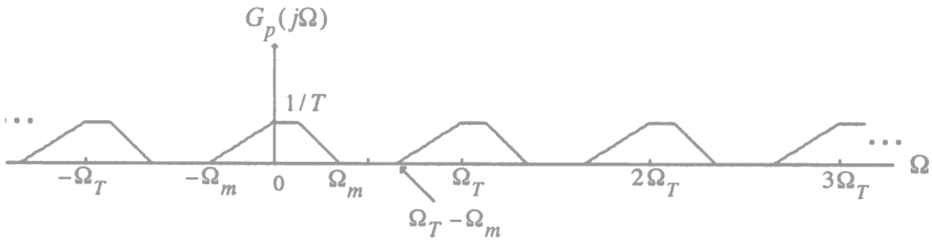
- Assume is a band-limited signal $g_a(t)$ with a CTFT $G_a(j\Omega)$ as shown below



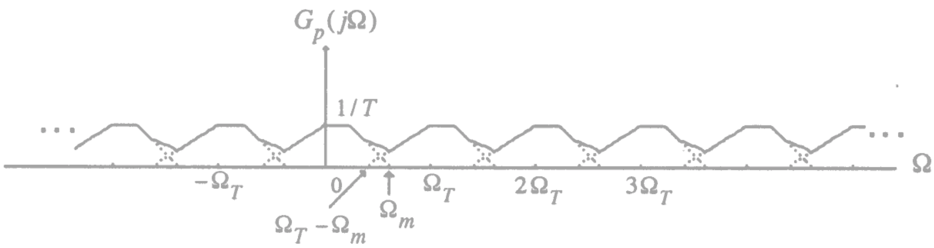
- The spectrum $P(j\Omega)$ of $p(t)$ having a sampling period $T = \dfrac{2\pi}{\Omega_T}$ is indicated below



- Two possible spectra of $G_p(j\Omega)$ are shown below



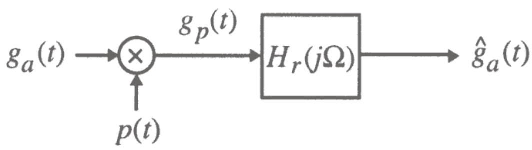
Sampling without aliasing



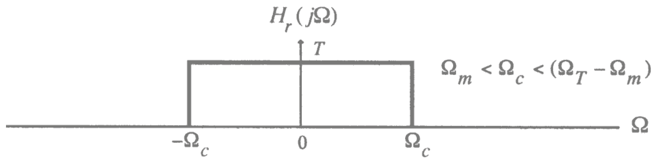
Sampling with aliasing

- It is evident from the top figure on the previous slide that if $\Omega_T > \Omega_m$, there is no overlap between the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$
- On the other hand, as indicated by the figure on the bottom, if $\Omega_T < \Omega_m$, there is an overlap of the spectra of the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$

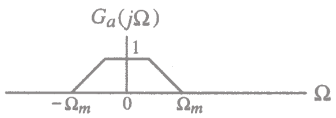
- If $\Omega_T > \Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an **ideal lowpass filter** $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as shown below
- On the other hand, as indicated by the figure below, if $\Omega_T < \Omega_m$, there is an overlap of the spectra of the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$



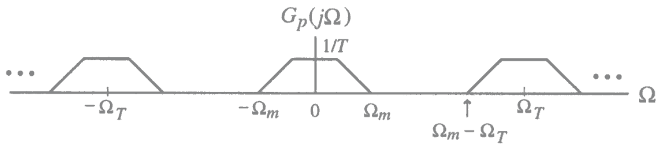
- The spectra of the filter and pertinent signals are shown below



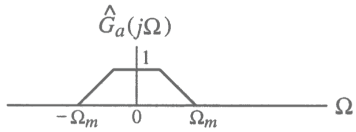
$H_r(j\Omega)$



$G_a(j\Omega)$



$G_p(j\Omega)$



$\hat{G}_a(j\Omega)$

- On the other hand, if $\Omega_T < 2\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_a(j\Omega)$ cannot be separated by filtering to recover $G_a(j\Omega)$ because of the distortion caused by a part of the replicas immediately outside the baseband folded back or **aliased** into the baseband

- **Sampling theorem:** Let $g_a(t)$ be a bandlimited signal with CTFT $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$
- Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty \leq n \leq \infty$ if
$$\Omega_T \geq 2\Omega_m$$
where $\Omega_T = \frac{2\pi}{T}$
- The condition $\Omega_T \geq 2\Omega_m$ is often referred to as the **Nyquist condition**
- The frequency $\frac{\Omega_T}{2}$ is usually referred to as the **folding frequency**

- Given $\{g_a(nT)\}$, we can recover exactly $g_a(t)$ by generating an impulse train
$$g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$
and then passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c satisfying
$$\Omega_m < \Omega_c < (\Omega_T - \Omega_m)$$
- The highest frequency Ω_m contained in $g_a(t)$ is usually called the **Nyquist frequency** since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$ that must be used to fully recover $g_a(t)$ from its sampled version
- The condition $\Omega_T \geq 2\Omega_m$ is often referred to as the **Nyquist condition**
- The frequency $2\Omega_m$ is called the **Nyquist rate**

- **Oversampling:** The sampling frequency is higher than the Nyquist rate
- **Undersampling:** The sampling frequency is lower than the Nyquist rate
- **Critical sampling:** The sampling frequency is equal to the Nyquist rate

  Note: A pure sinusoid may not be recoverable from its critically sampled version

- In digital telephony, a **3.4 kHz** signal bandwidth is acceptable for telephone conversation

  Here, a sampling rate of **8 kHz**, which is greater than twice the signal bandwidth, is used
- In high-quality analog music signal processing, a bandwidth of **20 kHz** has been determined to preserve the fidelity

  Hence, in compact disc (CD) music systems, a sampling rate of **44.1 kHz**, which is slightly higher than twice the signal bandwidth, is used

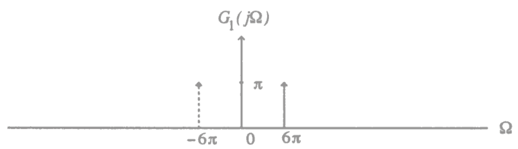
- **Example:** Consider the three continuous-time sinusoidal signals:

$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

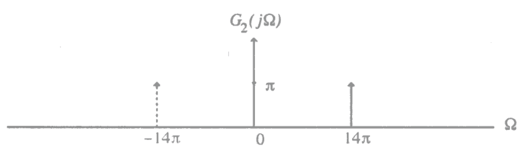
Their corresponding CTFTs are:

$$G_1(j\Omega) = \pi[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)]$$
$$G_2(j\Omega) = \pi[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)]$$
$$G_3(j\Omega) = \pi[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)]$$

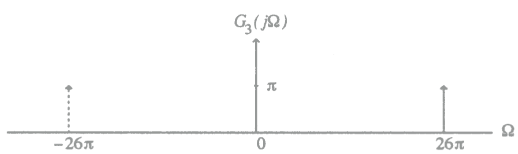
- These three transforms are plotted below



$G_1(j\Omega)$



$G_2(j\Omega)$

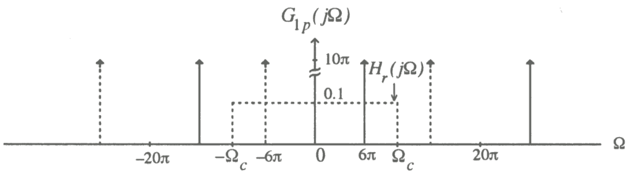


$G_3(j\Omega)$

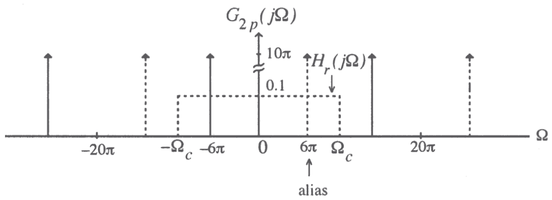
- These continuous-time signals sampled at a rate of $T = 0.1$ sec, i.e., with a sampling frequency $\Omega_T = 20\pi$ rad/sec
- The sampling process generates the continuous-time impulse trains, $q_{1p}(t)$, $q_{2p}(t)$, and $q_{3p}(t)$
- Their corresponding CTFTs are given by

$$G_{\ell p}(j\Omega) = 10 \sum_{k=-\infty}^{\infty} G_{\ell}(j(\Omega - k\Omega_T)), \quad 1 \leq \ell \leq 3$$

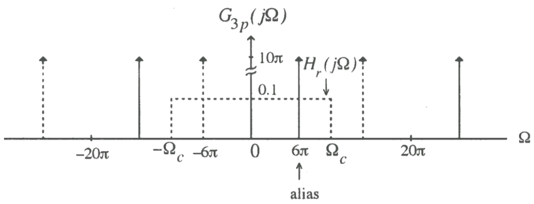
- Plots of the 3 CTFTs are shown below



$G_{1p}(j\Omega)$



$G_{2p}(j\Omega)$



$G_{3p}(j\Omega)$

- These figures also indicate by dotted lines the frequency response of an ideal lowpass filter with a cutoff at $\Omega_c = \dfrac{\Omega_T}{2} = 10\pi$ and a gain $T = 0.1$
- The CTFTs of the lowpass filter output are also shown in these three figures
- In the case of $g_1(t)$, the sampling rate satisfies the Nyquist condition, hence no aliasing
- Moreover, the reconstructed output is precisely the original continuous-time signal
- In the other two cases, the sampling rate does not satisfy the Nyquist condition, resulting in aliasing and the filter outputs are all equal to $\cos(6\pi t)$

- **Note:** In the figure at the bottom, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_2(j\Omega)$ at $\Omega = -14\pi$

  Likewise, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_3(j\Omega)$ at $\Omega = 26\pi$

- We now derive the relation between the DTFT of $g[n]$ and the CTFT of $g_p(t)$

  To this end we compare

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

  with

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

  and make use of

$$g[n] = g_a(nT), \quad -\infty < n < \infty$$

- **Observation:** We have

$$G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\omega/T}$$

  or, equivalently,

$$G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}$$

  From the above observation and

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_T))$$

we arrive at the desired result given by

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_T))\Bigg|_{\Omega=\omega/T}$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\Big(j\frac{\omega}{T} + jk\Omega_T\Big)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} + j\frac{2\pi k}{T}\right)$$

- This relation can be alternately expressed as

$$G(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_T))$$

- From
$$G(e^{j\omega}) = G_p(j\Omega)|_{\Omega=\omega/T}$$
or from
$$G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}$$
it follows that $G(e^{j\omega})$ is obtained from $G_p(j\Omega)$ by applying the mapping $\Omega = \dfrac{\omega}{T}$
- Now, the CTFT $G_p(j\Omega)$ is a periodic function of Ω with a period $\Omega_T = \dfrac{2\pi}{T}$
- Because of the mapping, the DTFT $G(e^{j\omega})$ is a periodic function of ω with a period of 2π

- We now derive the expression for the output $\hat{g}_a(t)$ of the ideal lowpass reconstruction filter $H_r(j\Omega)$ as a function of the samples $g[n]$
- The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse CTFT of $H_r(j\Omega)$:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

- Thus, the impulse response is given by

$$\begin{aligned} h_r(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H_r j\Omega e^{j\Omega t} d\Omega \\ &= \frac{T}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega t} d\Omega \\ &= \frac{\sin(\Omega_c t)}{\Omega_T t/2}, \quad -\infty \leq t \leq \infty \end{aligned}$$

- The input to the lowpass filter is the impulse train $g_p(t)$:

$$g_p(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT)$$

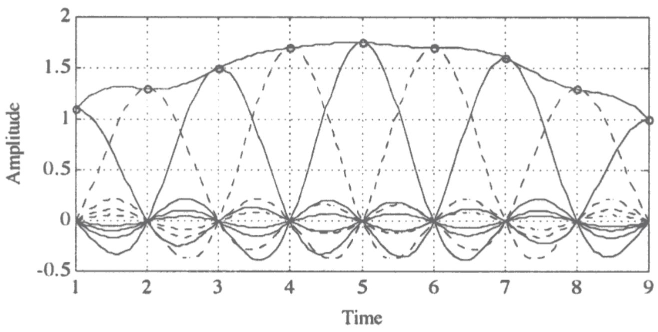
- Therefore, the output $\hat{g}_a(t)$ of the ideal lowpass filter is given by:

$$\hat{g}_a(t) = h_r(t) \circledast g_p(t) = \sum_{n=-\infty}^{\infty} g[n]h_r(t - nT)$$

- Substituting $h_r(t) = \dfrac{\sin(\Omega_c t)}{\Omega_T t/2}$ in the above and assuming for simplicity $\Omega_c = \dfrac{\Omega_T}{2} = \dfrac{\pi}{T}$, we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}$$

- The ideal bandlimited interpolation process is illustrated below



- It can be shown that when $\Omega_c = \dfrac{\Omega_T}{2}$ in

$$h_r(t) = \frac{\sin(\Omega_c t)}{\Omega_T t/2}$$

$h_r(0)$ and for $h_r(nT) = 0$ for $n \neq 0$

- As a result, from

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

we observe

$$\hat{g}_a(rT) = g[r] = g_a(rT)$$

for all integer values of r in the range $-\infty < r < \infty$

- The relation

$$\hat{g}_a(rT) = g[r] = g_a(rT)$$

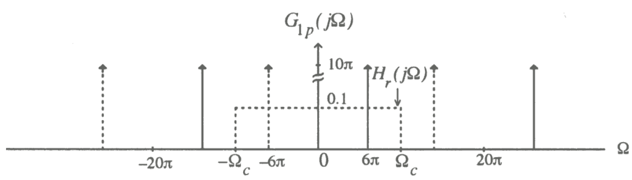
holds for all values of n whether or not the condition of the sampling theorem is satisfied

- However, $\hat{g}_a(t) = g_a(t)$ for all values of t only if the sampling frequency Ω_T satisfies the condition of the sampling theorem

Implication of the Sampling Process

- Consider again the three continuous-time signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$

  The plot of the CTFT $G_{1p}(j\Omega)$ of the sampled version $g_{1p}(t)$ of $g_1(t)$ is shown below



- From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos[(20k \pm 6)\pi t]$ outside the baseband by passing $g_{1p}(t)$ through an ideal analog bandpass filter with a passband centered at $\Omega = (20k \pm 6)\pi$

- For example, to recover the signal $\cos(34\pi t)$, it will be necessary to employ a bandpass filter with a frequency response

$$H_r(j\Omega) = \begin{cases} 0.1, & (34-\Delta)\pi \leq |\Omega| \leq (34+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

where Δ is a small number

- Likewise, we can recover the aliased baseband component $\cos(6\pi t)$, from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response:

$$H_r(j\Omega) = \begin{cases} 0.1, & (6-\Delta)\pi \leq |\Omega| \leq (6+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

- There is no aliasing distortion unless the original continuous-time signal also contains the component $\cos(6\pi t)$
- Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $g_2(t)$ or $g_3(t)$ as the case may be, by employing suitable filters

Sampling of Bandpass Signals

- The conditions developed earlier for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the continuous-time signal is bandlimited in the frequency range from DC to some frequency Ω_m
- Such a continuous-time signal is commonly referred to as a **lowpass signal**
- There are applications where the continuous-time signal is bandlimited to a higher frequency range $\Omega_L \leq |\Omega| \leq \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the **bandpass signal**

- To prevent aliasing a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring

$$\Omega_T \geq 2\Omega_H$$

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if Ω_H is very large, the sampling rate also has to be very large which may not be practical in some situations

- A more practical approach is to use **undersampling**
- Let $\Delta\Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M\,\Delta\Omega$$

- We choose the sampling frequency Ω_T to satisfy the condition

$$\Omega_T = 2\,\Delta\Omega = \frac{2\Omega_H}{M}$$

which is smaller than $2\Omega_H$, the Nyquist rate

- Substitute the above expression for Ω_T in

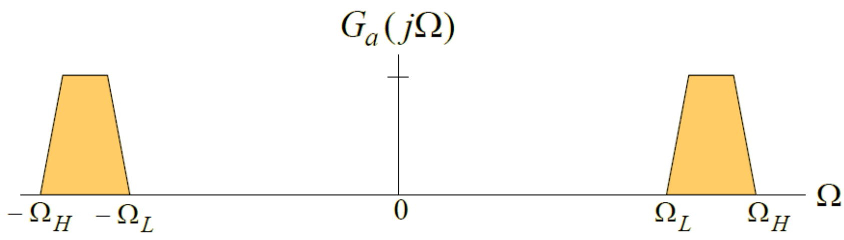
$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + k\Omega_T))$$

- This leads to

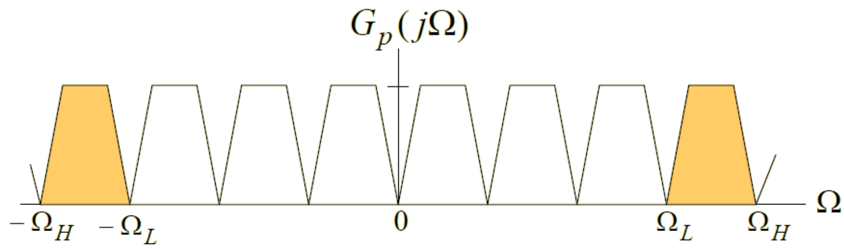
$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega + 2k\Delta\Omega))$$

- As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$ and scaled by $1/T$
- The amount of shift for each value of k ensures that there will be no overlap between all shifted replicas, i.e., **no aliasing**

- Figures below illustrate the idea behind



$G_a(j\Omega)$



$G_p(j\Omega)$

- As can be seen, $g_a(t)$ can be recovered from $g_p(t)$ by passing it through an ideal bandpass filter with a passband given by $\Omega_L \leq |\Omega| \leq \Omega_H$ and a gain of T

- **Note:** Any of the replicas in the lower frequency bands can be retained by passing $g_p(t)$ through bandpass filters with passbands

$$\Omega_L - k\,\Delta\Omega \leq |\Omega| \leq \Omega_H - k\,\Delta\Omega, \quad 1 \leq k \leq M-1$$

providing a translation to lower frequency ranges