EEM 401 Digital Signal Processing

The Z-Transform

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The z-Transform

- Counterpart of the Laplace transform for discrete-time signals
- Generalization of the Fourier Transform
  - Fourier Transform does not exist for all signals
- The z-Transform is often time more convenient to use
- Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

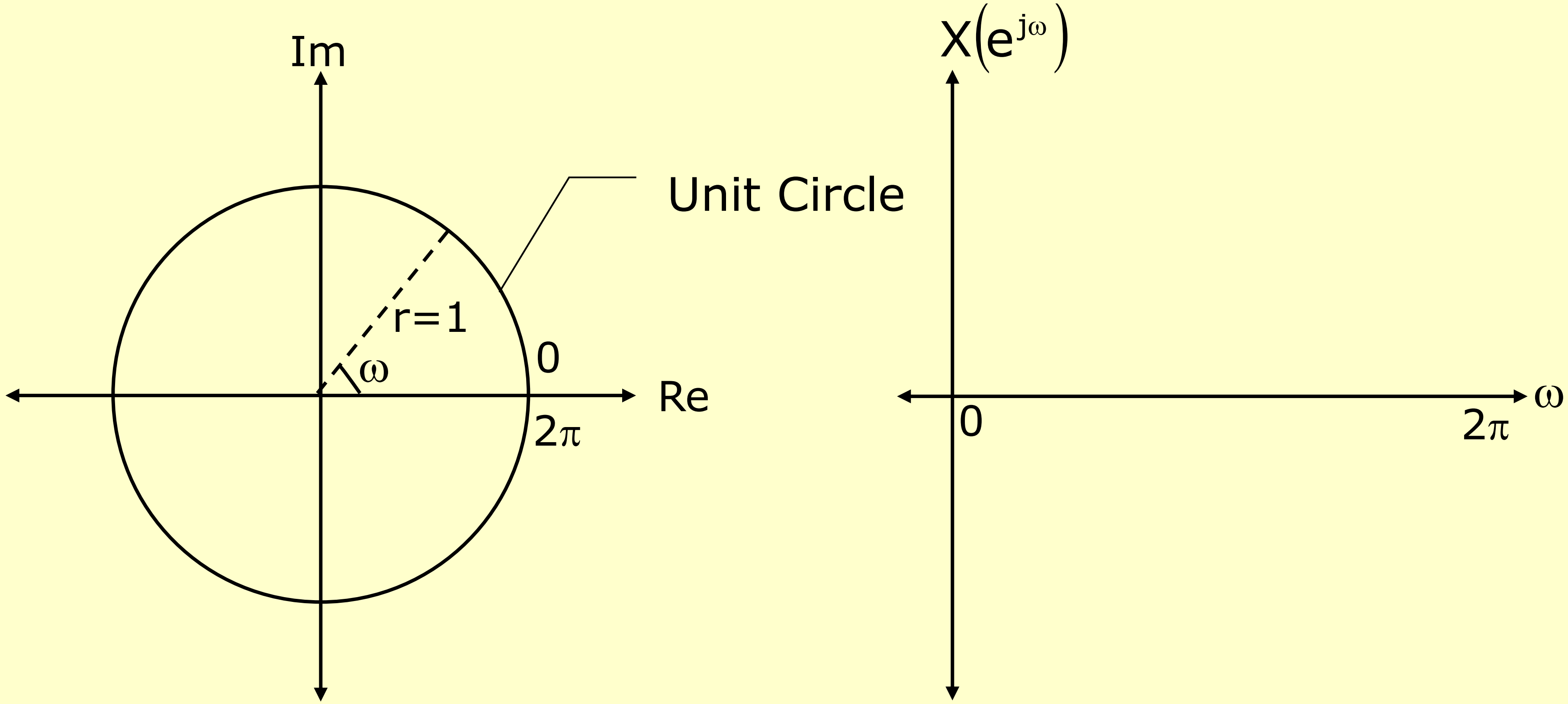
- Compare to DTFT definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- z is a complex variable that can be represented as $z = r\, e^{j\omega}$
- Substituting $z = e^{j\omega}$ will reduce the z-transform to DTFT

The z-transform and the DTFT

- The z-transform is a function of the complex z variable
- Convenient to describe on the complex z-plane
- If we plot $z=e^{j\omega}$ for $\omega=0$ to 2π we get the unit circle



The z-transform

Aşağıdaki dizilerin z dönüşümlerini inceleyiniz.

(a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(c) $x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n-k), k > 0$

(g) $x_7(n) = \delta(n+k), k > 0$

Çözüm. (3.1.1) tanımlamasından hareketle

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$ ROC: $z = 0$ hariç tüm z-bölgesi,

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$ ROC: $z = 0$ ve $z = \infty$ hariç tüm z-bölgesi,

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$ ROC: $z = 0$ hariç tüm z-bölgesi,

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$ ROC: $z = 0$ ve $z = \infty$ hariç tüm z-bölgesi,

(e) $X_5(z) = 1$ [i.e., $\delta(n) \stackrel{z}{\longleftrightarrow} 1$] ROC: tüm z-bölgesi,

(f) $X_6(z) = z^{-k}$ [i.e., $\delta(n-k) \stackrel{z}{\longleftrightarrow} z^{-k}$], $k > 0$ ROC: $z = 0$ hariç tüm z-bölgesi,

(g) $X_7(z) = z^{k}$ [i.e., $\delta(n+k) \stackrel{z}{\longleftrightarrow} z^{k}$], $k > 0$ ROC: $z = \infty$ hariç tüm z-bölgesi,

Convergence of the z-Transform

- DTFT does not always converge

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

  - Infinite sum not always finite if x[n] no absolute summable
  - Example: $x[n] = a^n u[n]$ for $|a|>1$ does not have a DTFT

- Complex variable z can be written as r $e^{j\omega}$ so the z-transform

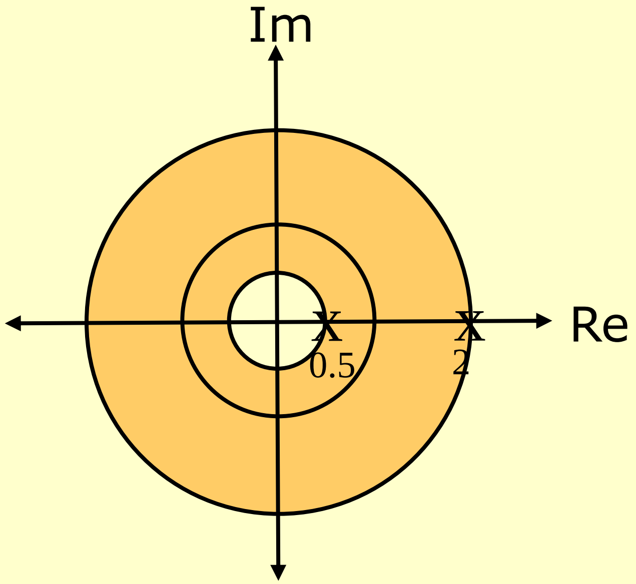
$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]\, r^{-n}\right) e^{-j\omega n}$$

- DTFT of x[n] multiplied with exponential sequence r^{-n}
  - For certain choices of r the sum maybe made finite

$$\sum_{n=-\infty}^{\infty} \left|x[n]\, r^{-n}\right| < \infty$$

Region of Convergence

- The set of values of z for which the z-transform converges
- Each value of r represents a circle of radius r
- The region of convergence is made of circles



- Example: z-transform converges for values of $0.5<r<2$
  - ROC is shown on the left
  - In this example the ROC includes the unit circle, so DTFT exists
- Not all sequence have a z-transform

Right-Sided Exponential Sequence Example

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

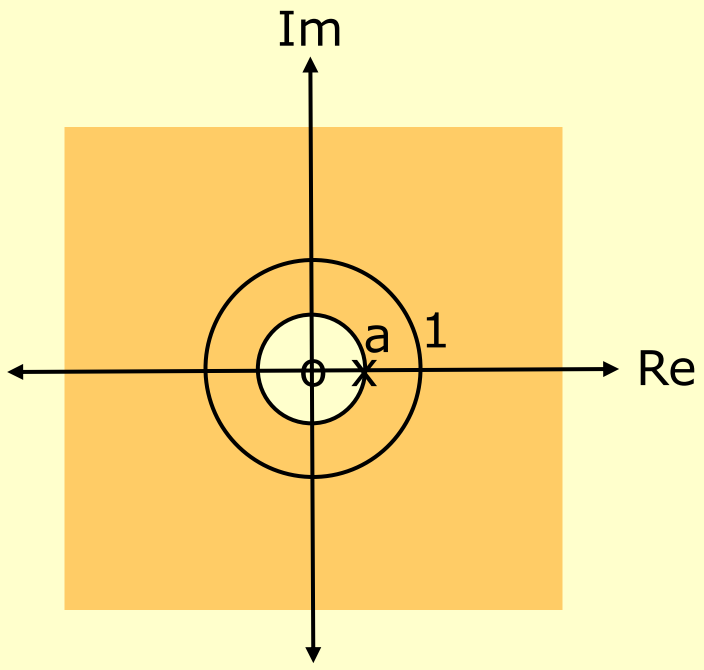
$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1 - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Same Example Alternative Way

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

$$\sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{\left(az^{-1}\right)^0 - \left(az^{-1}\right)^\infty}{1 - az^{-1}}$$

$$|z|>2$$

- For the term with infinite exponential to vanish we need

$$\left|az^{-1}\right| < 1 \quad \Rightarrow \quad |a| < |z|$$

  - Determines the ROC (same as the previous approach)
- In the ROC the sum converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}}$$

Two-Sided Exponential Sequence Example

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

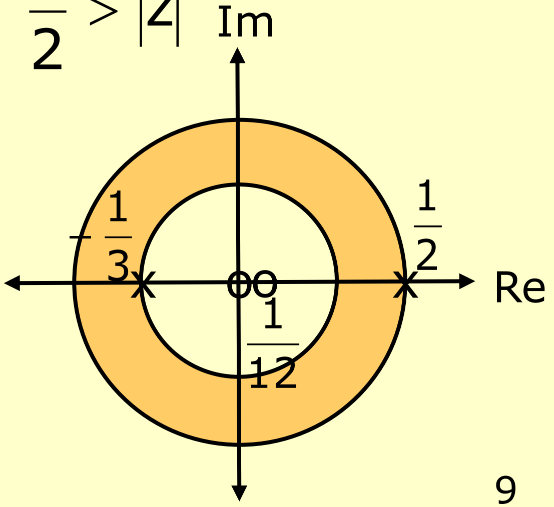
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right| < 1$$

$$\frac{1}{3} < |z|$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right| > 1$$

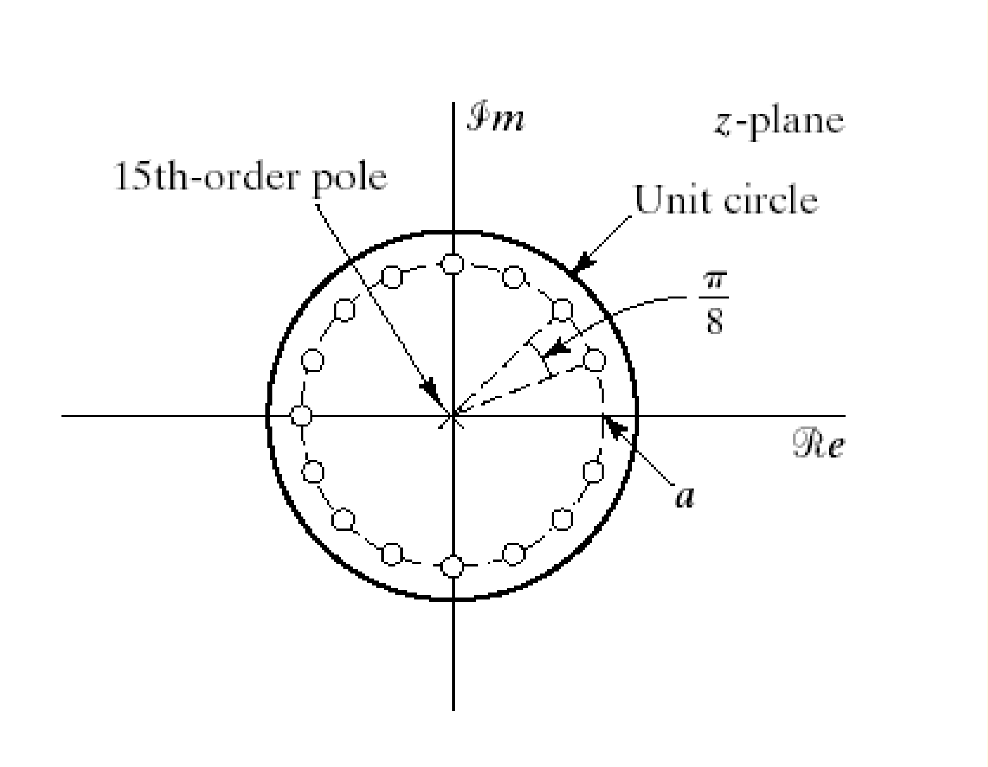
$$\frac{1}{2} > |z|$$



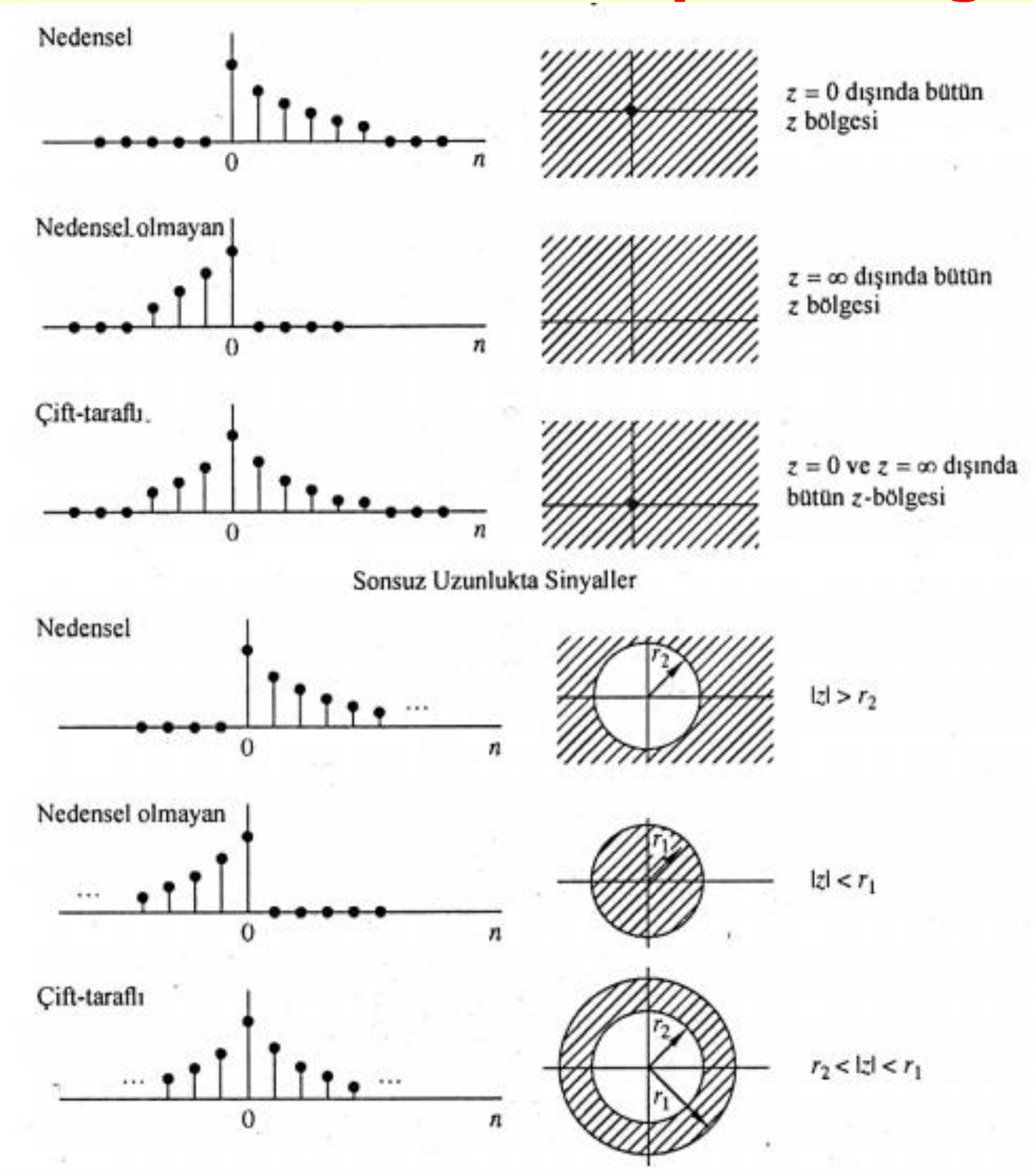
Finite Length Sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



The ROC of Some Special Signals

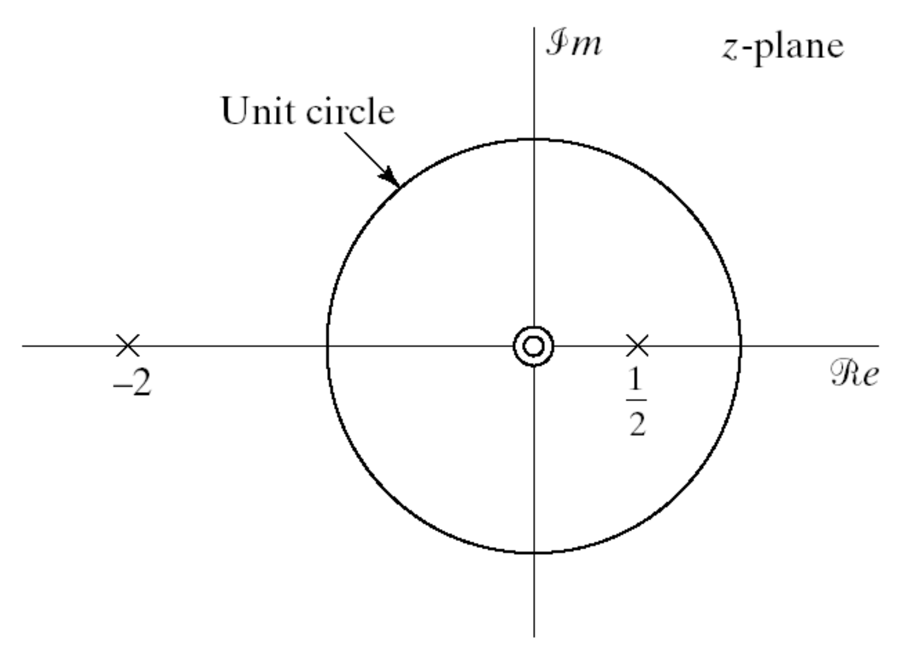


Properties of The ROC of Z-Transform

- The ROC is a ring or disk centered at the origin
- DTFT exists if and only if the ROC includes the unit circle
- The ROC cannot contain any poles
- The ROC for finite-length sequence is the entire z-plane
  - except possibly $z=0$ and $z=\infty$
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including $z=\infty$
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including $z=0$
- The ROC of a two-sided sequence is a ring bounded by poles
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response h[n]
- The z-transform H(z) and the pole-zero plot shown below
- Without any other information h[n] is not uniquely determined
  - $|z|>2$ or $|z|<½$ or $½<|z|<2$
- If system **stable** ROC must include unit-circle: $½<|z|<2$
- If system is **causal** must be right sided: $|z|>2$



Some Common z-Transform Pairs

	Signal, x[n]	Z transform (X(z))	Region of Convergence (ROC)
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$