Solutions – Chapter 5

Transform Analysis of Linear Time-Invariant Systems

5.1.

$$y[n] = \begin{cases} 1, & 0 \le n \le 10, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

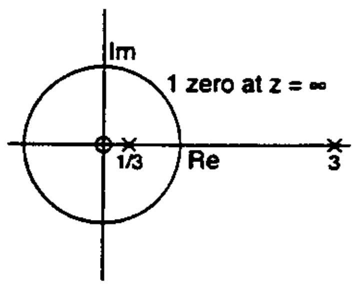
$$Y(e^{j\omega}) = e^{-j5\omega} \frac{\sin \frac{11}{2}\omega}{\sin \frac{\omega}{2}}$$

This $Y(e^{j\omega})$ is full band. Therefore, since $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, the only possible $x[n]$ and ω_c that could produce $y[n]$ is $x[n] = y[n]$ and $\omega_c = \pi$.

5.2. We have $y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$ or $z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$. So,

$$\begin{aligned} H(z) &= \frac{1}{z^{-1} - \frac{10}{3} + z} \\ &= \frac{z}{(z-\frac{1}{3})(z-3)} \\ &= \frac{-\frac{1}{8}}{z-\frac{1}{3}} + \frac{\frac{9}{8}}{z-3} \end{aligned}$$

(a)



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1-3z^{-1}}$$

Stable $\Rightarrow$ ROC is $\frac{1}{3} \le |z| \le 3$. Therefore,

$$h[n] = -\frac{1}{8}\left(\frac{1}{3}\right)^{n-1} u[n-1] - \frac{9}{8}(3)^{n-1}u[-n]$$

5.3.

$$y[n-1] + \frac{1}{3}y[n-2] = x[n]$$

$$z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} + \frac{1}{3}z^{-2}} \\ H(z) &= \frac{z}{1 + \frac{1}{3}z^{-1}} \end{aligned}$$

i) $\frac{1}{3} < |z|$, $h[n] = (-\frac{1}{3})^{n+1}u[n+1] \Rightarrow$ **answer (a)**
ii) $\frac{1}{3} > |z|$,

$$\begin{aligned} h[n] &= -\left(-\frac{1}{3}\right)^{n+1} u[-n-2] \\ &= -\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n} u[-n-2] \\ &= \frac{1}{3}\left(-\frac{1}{3}\right)^{n} u[-n-2] \Rightarrow \textbf{answer (d)} \end{aligned}$$

5.4. (a)

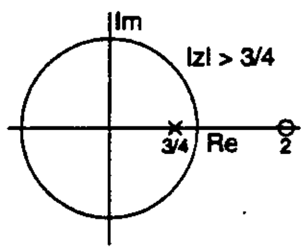
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z}{z-2}, \qquad \frac{1}{2} < |z| < 2$$

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4} < |z|$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{-\frac{3}{2}z^{-1}} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

(d) The ROC is outside $|z| = \frac{3}{4}$, which includes the unit circle. Therefore the system is stable. The $h[n]$ we found in part (b) tells us the system is also causal.

5.5.

$$y[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] + u[n]$$

$$Y(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$x[n] = u[n]$$

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - \frac{19}{6}z^{-1} + \frac{2}{3}z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}, \quad |z| > \frac{1}{3}$$

(a) Cross multiplying and equating z^{-1} with a delay in time:

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = 3x[n] - \frac{19}{6}x[n-1] + \frac{2}{3}x[n-2]$$

(b) Using partial fractions on $H(z)$ we get:

$$H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{z^{-1}}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}} - \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} + 1, \quad |z| > \frac{1}{3}$$

So,

$$h[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^{n-1} u[n-1] + \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n-1] + \delta[n]$$

(c) Since the ROC of $H(z)$ includes $|z| = 1$ the system is stable.

5.6. (a)

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$X(z) = \frac{-\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1 - 2z^{-1}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

(b)

$$Y(z) = \frac{1 - z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$

This has the same poles as the input, therefore the ROC is still $\frac{1}{2} < |z| < 2$.

(c)

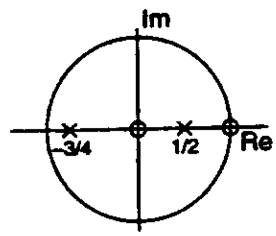
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \Leftrightarrow h[n] = \delta[n] - \delta[n-2]$$

5.7. (a)

$$x[n] = 5u[n] \Leftrightarrow X(z) = \frac{5}{1-z^{-1}}, \quad |z| > 1$$

$$y[n] = \left(2\left(\frac{1}{2}\right)^n + 3\left(-\frac{3}{4}\right)^n\right)u[n] \Leftrightarrow Y(z) = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{3}{1+\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}, \quad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} = \frac{-\frac{2}{5}}{(1-\frac{1}{2}z^{-1})} + \frac{\frac{7}{5}}{(1+\frac{3}{4}z^{-1})}, \quad |z| > \frac{3}{4}$$

$$h[n] = -\frac{2}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{5}\left(-\frac{3}{4}\right)^n u[n]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}$$

$$Y(z) + \frac{1}{4}z^{-1}Y(z) - \frac{3}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

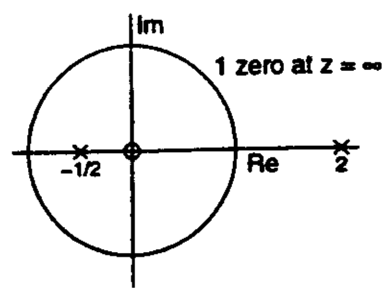
$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - x[n-1]$$

5.8. (a)

$$\begin{aligned} y[n] &= \frac{3}{2}y[n-1] + y[n-2] + x[n-1] \\ Y(z) &= \frac{3}{2}z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \end{aligned}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-\frac{3}{2}z^{-1}-z^{-2}} = \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}, \quad |z| > 2$$



(b)

$$
\begin{aligned}
H(z) &= \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{A}{(1-2z^{-1})} + \frac{B}{(1+\frac{1}{2}z^{-1})}, \quad |z| > 2 \\
A &= \left.\frac{z^{-1}}{(1+\frac{1}{2}z^{-1})}\right|_{z^{-1}=\frac{1}{2}} = \frac{2}{5} \\
B &= \left.\frac{z^{-1}}{(1-2z^{-1})}\right|_{z^{-1}=-2} = -\frac{2}{5}
\end{aligned}
$$

$$h[n] = \frac{2}{5}\left[(2)^n - \left(-\frac{1}{2}\right)^n\right]u[n]$$

(c) Use ROC of $\frac{1}{2} < |z| < 2$ since the ROC must include $|z| = 1$ for a stable system.

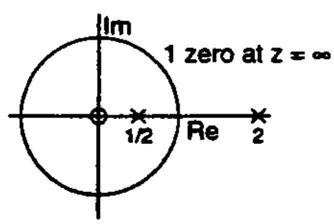
$$h[n] = -\frac{2}{5}(2)^n u[-n-1] - \frac{2}{5}\left(-\frac{1}{2}\right)^n u[n]$$

5.9.

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

$$z^{-1}Y(z) - \frac{5}{2}Y(z) + zY(z) = X(z)$$

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\
&= \frac{z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= \frac{\frac{2}{3}}{1-2z^{-1}} - \frac{\frac{2}{3}}{1-\frac{1}{2}z^{-1}}
\end{aligned}
$$



Three regions of convergence:

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

Includes $|z| = 1$, so this is stable.

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}(2)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

ROC outside of largest pole, so this is causal.

5.10. Figure P5.16 shows two zeros and three poles inside the unit circle. Since the number of poles must equal the number of zeros, there must be an additional zero at $z = \infty$.

$H(z)$ is causal, so the ROC lies outside the largest pole and includes the unit circle. Therefore, the system is also stable.

The inverse system switches poles and zeros. The inverse system could have a ROC that includes $|z| = 1$, making it stable. However, the zero at $z = \infty$ of $H(z)$ is a pole for $H_i(z)$, so the system $H_i(z)$ cannot be causal.

5.11. (a) *It cannot be determined.* The ROC might or might not include the unit circle.

(b) *It cannot be determined.* The ROC might or might not include $z = \infty$.

(c) *False.* Given that the system is causal, we know that the ROC must be outside the outermost pole. Since the outermost pole is outside the unit circle, the ROC will not include the unit circle, and thus the system is not stable.

(d) *True.* If the system is stable, the ROC must include the unit circle. Because there are poles both inside and outside the unit circle, any ROC including the unit circle must be a ring. A ring-shaped ROC means that we have a two-sided system.

5.12. (a) *Yes.* The poles $z = \pm j(0.9)$ are inside the unit circle so the system is stable.

(b) First, factor $H(z)$ into two parts. The first should be minimum phase and therefore have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

$$H(z) = \underbrace{\frac{1 + 0.2z^{-1}}{1 + 0.81z^{-2}}}_{\text{minimum phase}} \cdot \underbrace{\frac{1 - 9z^{-2}}{1}}_{\substack{\text{poles \& zeros} \\ \text{outside unit circle}}}$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1 - \frac{1}{9}z^{-2})$ in both parts of the equation above the first part will remain minimum phase and the second will become allpass.

$$\begin{aligned} H(z) &= \frac{(1 + 0.2z^{-1})(1 - \frac{1}{9}z^{-2})}{1 + 0.81z^{-2}} \cdot \frac{1 - 9z^{-2}}{1 - \frac{1}{9}z^{-2}} \\ &= H_1(z)H_{ap}(z) \end{aligned}$$

5.13. *An aside*: Technically, this problem is not well defined, since a pole/zero plot does not uniquely determine a system. That is, many system functions can have the same pole/zero plot. For example, consider the systems

$$\begin{aligned} H_1(z) &= z^{-1} \\ H_2(z) &= 2z^{-1} \end{aligned}$$

Both of these systems have the same pole/zero plot, namely a pole at zero and a zero at infinity. Clearly, the system $H_1(z)$ is allpass, as it passes all frequencies with unity gain (it is simply a unit delay). However, one could ask whether $H_2(z)$ is allpass. Looking at the standard definition of an allpass system provided in this chapter, the answer would be no, since the system does not pass all frequencies with *unity* gain.

A broader definition of an allpass system would be a system for which the system magnitude response $|H(e^{j\omega})| = a$, where a is a real constant. Such a system would pass all frequencies, and scale the output by a constant factor a. In a practical setting, this definition of an allpass system is satisfactory. Under this definition, both systems $H_1(z)$ and $H_2(z)$ would be considered allpass.

For this problem, it is assumed that none of the poles or zeros shown in the pole/zero plots are scaled, so this issue of using the proper definition of an allpass system does not apply. The standard definition of an allpass system is used.

Solution:

(a) Yes, the system is allpass, since it is of the appropriate form.

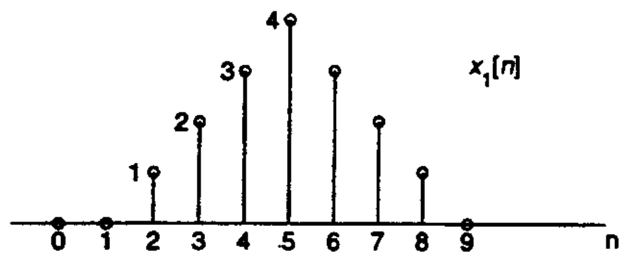
(b) No, the system is not allpass, since the zero does not occur at the conjugate reciprocal location of the pole.

(c) Yes, the system is allpass, since it is of the appropriate form.

(d) Yes, the system is allpass. This system consists of an allpass system in cascade with a pole at zero. The pole at zero is simply a delay, and does not change the magnitude spectrum.

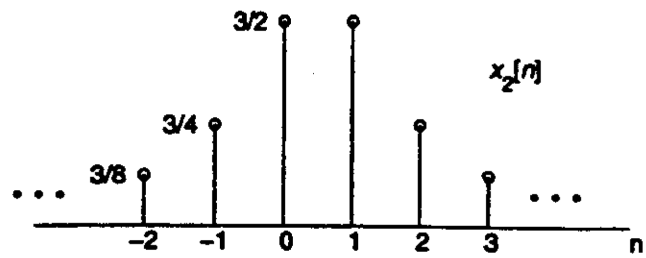
5.14. (a) By the symmetry of $x_1[n]$ we know it has linear phase. The symmetry is around $n = 5$ so the continuous phase of $X_1(e^{j\omega})$ is $\arg[X_1(e^{j\omega})] = -5\omega$. Thus,

$$\text{grd}[X_1(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_1(e^{j\omega})]\} = -\frac{d}{d\omega}\{-5\omega\} = 5$$



(b) By the symmetry of $x_2[n]$ we know it has linear phase. The symmetry is around $n = 1/2$ so we know the phase of $X_2(e^{j\omega})$ is $\arg[X_2(e^{j\omega}] = -\omega/2$. Thus,

$$\text{grd}[X_2(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_2(e^{j\omega})]\} = -\frac{d}{d\omega}\left\{-\frac{\omega}{2}\right\} = \frac{1}{2}$$



5.15. (a) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 2 + e^{-j\omega} + 2e^{-2j\omega} \\ &= e^{-j\omega}(2e^{j\omega} + 1 + 2e^{-j\omega}) \\ &= (1 + 4\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 1 + 4\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase but not Linear Phase since $A(\omega)$ is not always positive.

(b) This sequence has no even or odd symmetry, so it does not possess generalized linear phase.

(c) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 1 + 3e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 3 + e^{-j\omega}) \\ &= (3 + 2\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 3 + 2\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase & Linear Phase.

(d) $h[n]$ has even symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 + e^{-j\omega} \\ &= e^{-j(1/2)\omega}(e^{j(1/2)\omega} + e^{-j(1/2)\omega}) \\ &= 2\cos(\omega/2)e^{-j(1/2)\omega} \end{aligned}$$

$$A(\omega) = 2\cos(\omega/2),\ \alpha = \frac{1}{2},\ \beta = 0$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

(e) $h[n]$ has odd symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} - e^{-j\omega}) \\ &= e^{-j\omega}2j\sin\omega \\ &= (2\sin\omega)e^{-j\omega + j\frac{\pi}{2}} \end{aligned}$$

$$A(\omega) = 2\sin\omega,\ \alpha = 1,\ \beta = \frac{\pi}{2}$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

5.16. The causality of the system cannot be determined from the figure. A causal system $h_1[n]$ that has a linear phase response $\angle H(e^{j\omega}) = -\alpha w$, is:

$$\begin{aligned} h_1[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\ H_1(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-j2\omega} \\ &= e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) \\ &= e^{-j\omega}(2 + 2\cos(\omega)) \\ |H_1(e^{j\omega})| &= 2 + 2\cos(\omega) \\ \angle H_1(e^{j\omega}) &= -\omega \end{aligned}$$

An example of a non-causal system with the same phase response is:

$$\begin{aligned} h_2[n] &= \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3] \\ H_2(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\ &= e^{-j\omega}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\ &= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\ |H_2(e^{j\omega})| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\ \angle H_2(e^{j\omega}) &= -\omega \end{aligned}$$

Thus, both the causal sequence $h_1[n]$ and the non-causal sequence $h_2[n]$ have a linear phase response $\angle H(e^{j\omega}) = -\alpha\omega$, where $\alpha = 1$.

5.17. A minimum phase system is one which has all its poles and zeros inside the unit circle. It is causal, stable, and has a causal and stable inverse.

(a) $H_1(z)$ has a zero outside the unit circle at $z = 2$ so it is not minimum phase.

(b) $H_2(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(c) $H_3(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(d) $H_4(z)$ has a zero outside the unit circle at $z = \infty$ so it is not minimum phase. Moreover, the inverse system would not be causal due to the pole at infinity.

5.18. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_{min}(z)$ will be inside the unit circle. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

(a) Simply reflect the zero at $z = 2$ to its conjugate reciprocal location at $z = \frac{1}{2}$. Then, determine the scale factor.

$$H_{min}(z) = 2\left(\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{3}z^{-1}}\right)$$

(b) First, simply reflect the zero at $z = -3$ to its conjugate reciprocal location at $z = -\frac{1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z) = 3\frac{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}\left(1 + \frac{1}{3}z^{-1}\right)}$$

The $\left(1 + \frac{1}{3}z^{-1}\right)$ terms cancel, leaving

$$H_{min}(z) = 3\frac{\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}}$$

Note that the term $\frac{1}{z^{-1}}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus, the remaining term has a zero inside the unit circle, and is therefore minimum phase. As a result, we are left with the system

$$H_{min}(z) = 3\left(1 - \frac{1}{2}z^{-1}\right)$$

(c) Simply reflect the zero at 3 to its conjugate reciprocal location at $\frac{1}{3}$ and reflect the pole at $\frac{4}{3}$ to its conjugate reciprocal location at $\frac{3}{4}$. Then, determine the scale factor.

$$H_{min}(z) = \frac{9}{4}\frac{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{3}{4}z^{-1}\right)^2}$$

5.19. Due to the symmetry of the impulse responses, all the systems have generalized linear phase of $\arg[H(e^{j\omega})] = \beta - n_o\omega$ where n_o is the point of symmetry in the impulse response graphs. The group delay is

$$\text{grd}\left[H_i(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H_i(e^{j\omega})\right]\right\} = -\frac{d}{d\omega}\left\{\beta - n_o\omega\right\} = n_o$$

To find each system's group delay we need only find the point of symmetry n_o in each system's impulse response.

$$\begin{aligned} \text{grd}\left[H_1(e^{j\omega})\right] &= 2 & \text{grd}\left[H_4(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_2(e^{j\omega})\right] &= 1.5 & \text{grd}\left[H_5(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_3(e^{j\omega})\right] &= 2 & \text{grd}\left[H_6(e^{j\omega})\right] &= 3.5 \end{aligned}$$

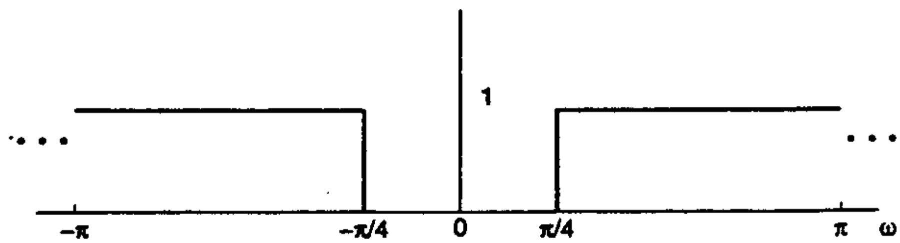
5.20. (a) *Yes.* The system function could be a generalized linear phase system implemented by a linear constant-coefficient differential equation (LCCDE) with real coefficients. The zeros come in a set of four: a zero, its conjugate, and the two conjugate reciprocals. The pole-zero plot could correspond to a Type I FIR linear phase system.

(b) *No.* This system function could not be a generalized linear phase system implemented by a LCCDE with real coefficients. Since the LCCDE has real coefficients, its poles and zeros must come in conjugate pairs. However, the zeros in this pole-zero plot do not have corresponding conjugate zeros.

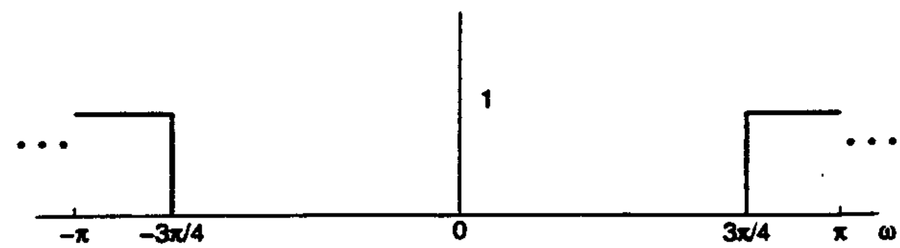
(c) *Yes.* The system function could be a generalized linear phase system implemented by a LCCDE with real coefficients. The pole-zero plot could correspond to a Type II FIR linear phase system.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

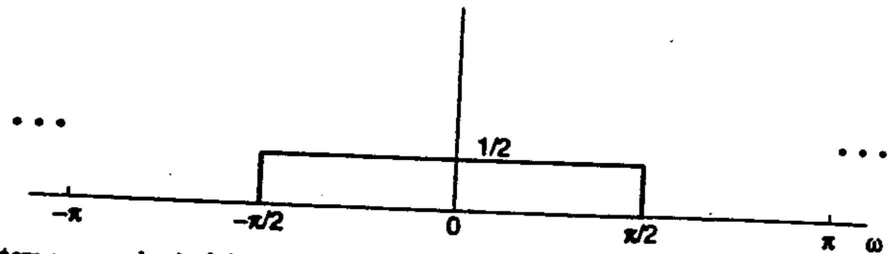
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



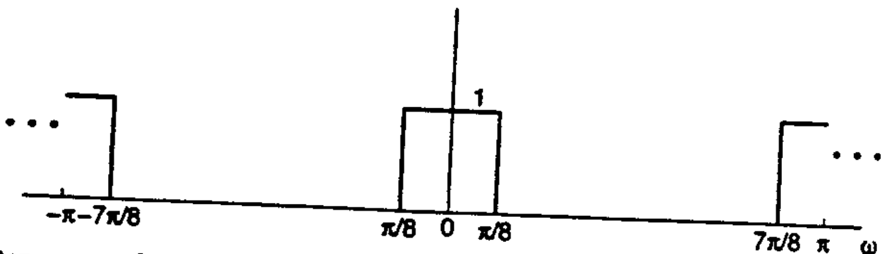
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



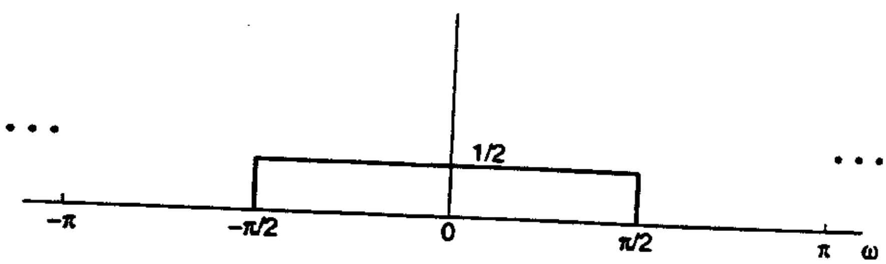
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.



5.22.

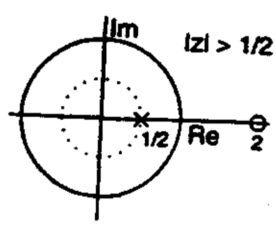
$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = \frac{Y(z)}{X(z)}, \quad \text{causal, so ROC is } |z| > a$$

(a) Cross multiplying and taking the inverse transform

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

(b) Since $H(z)$ is causal, we know that the ROC is $|z| > a$. For stability, the ROC must include the unit circle. So, $H(z)$ is stable for $|a| < 1$.

(c) $a = \frac{1}{2}$



(d)

$$H(z) = \frac{1}{1-az^{-1}} - \frac{a^{-1}z^{-1}}{1-az^{-1}}, \qquad |z| > a$$

$$h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

(e)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}} \cdot \frac{1-a^{-1}e^{j\omega}}{1-ae^{j\omega}}$$

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{1+\frac{1}{a^2}-\frac{2}{a}\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\frac{a^2+1-2a\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a} \end{aligned}$$

5.23. (a) Type I:

$$A(\omega) = \sum_{n=0}^{M/2} a[n]\cos\omega n$$

$\cos 0 = 1$, $\cos\pi = -1$, so there are no restrictions.

Type II:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n]\cos\omega\left(n-\frac{1}{2}\right)$$

$\cos 0 = 1$, $\cos\left(n\pi - \frac{\pi}{2}\right) = 0$. So $H(e^{j\pi}) = 0$.

Type III:

$$A(\omega) = \sum_{n=0}^{M/2} c[n]\sin\omega n$$

$\sin 0 = 0$, $\sin n\pi = 0$, so $H(e^{j0}) = H(e^{j\pi}) = 0$.

Type IV:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n]\sin\omega\left(n-\frac{1}{2}\right)$$

$\sin 0 = 0$, $\sin\left(n\pi - \frac{\pi}{2}\right) \neq 0$, so just $H(e^{j0}) = 0$.

(b)

	Type I	Type II	Type III	Type IV
Lowpass	Y	Y	N	N
Bandpass	Y	Y	Y	Y
Highpass	Y	N	N	Y
Bandstop	Y	N	N	N
Differentiator	Y	N	N	Y

5.24. (a) Taking the z-transform of both sides and rearranging

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4}z^{-2}}$$

Since the poles and zeros {2 poles at $z = \pm 1/2$, 2 zeros at $z = \pm 2$} occur in conjugate reciprocal pairs the system is allpass. This property is easy to recognize since, as in the system above, the coefficients of the numerator and denominator z-polynomials get reversed (and in general conjugated).

(b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof.

$$\begin{aligned}
\sum_{n=0}^{N-1} |y[n]|^2 &= \sum_{n=-\infty}^{\infty} |y[n]|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2\, d\omega && \text{(by Parseval's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})X(e^{j\omega})|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega && (|H(e^{j\omega})|^2 = 1 \text{ since } h[n] \text{ is allpass}) \\
&= \sum_{n=-\infty}^{\infty} |x[n]|^2 && \text{(by Parseval's theorem)} \\
&= \sum_{n=0}^{N-1} |x[n]|^2 \\
&= 5
\end{aligned}$$

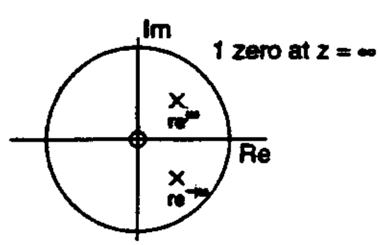
5.25. *The statement is false.* A non-causal system can indeed have a positive constant group delay. For example, consider the non-causal system

$$h[n] = \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3]$$

This system has the frequency response

$$\begin{aligned}
H(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\
&= e^{-j\omega}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\
&= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\
|H(e^{j\omega})| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\
\angle H(e^{j\omega}) &= -\omega \\
\text{grd}[H(e^{j\omega})] &= 1
\end{aligned}$$

5.26. (a) A labeled pole-zero diagram appears below.



The table of common z-transform pairs gives us

$$(r^n \sin\omega_0 n)u[n] \longleftrightarrow \frac{(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>r$$

which enables us to derive $h[n]$.

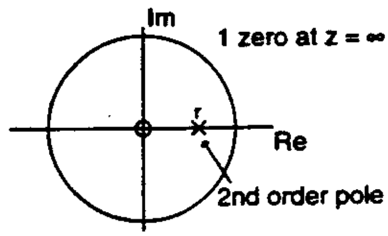
$$h[n] = \left(\frac{1}{\sin\omega_0}\right)(r^n \sin\omega_0 n)u[n]$$

(b) When $\omega_0 = 0$

$$H(z) = \frac{rz^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} = \frac{rz^{-1}}{(1-rz^{-1})^2}, \quad |z|>r$$

Again, using a table lookup gives us

$$h[n] = nr^n u[n]$$



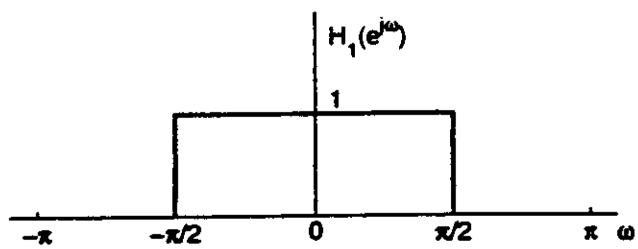
5.27. Making use of some DTFT properties can aide in the solution of this problem. First, note that

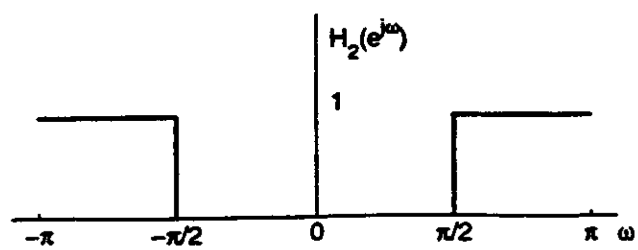
$$\begin{aligned} h_2[n] &= (-1)^n h_1[n] \\ h_2[n] &= e^{-j\pi n} h_1[n] \end{aligned}$$

Using the DTFT property that states that modulation in the time domain corresponds to a shift in the frequency domain,

$$H_2(e^{j\omega}) = H_1(e^{j(\omega+\pi)})$$

Consequently, $H_2(e^{j\omega})$ is simply $H_1(e^{j\omega})$ shifted by π. The ideal low pass filter has now become the ideal high pass filter, as shown below.





5.28. (a)

$$H(z) = \frac{A}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \;\; h[n] \text{ causal}$$

$$H(1) = 6 \Rightarrow A = 4$$

(b)

$$\begin{aligned} H(z) &= \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \\ &= \frac{(\frac{12}{5})}{1-\frac{1}{2}z^{-1}} + \frac{(\frac{8}{5})}{1+\frac{1}{3}z^{-1}} \\ h[n] &= \frac{12}{15}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(c) (i)

$$x[n] = u[n] - \frac{1}{2}u[n-1] \Leftrightarrow X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}}, \qquad |z| > 1$$

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}} \; \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > 1 \\ &= \frac{4}{(1-z^{-1})(1+\frac{1}{3}z^{-1})} \\ &= \frac{3}{1-z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \\ y[n] &= 3u[n] + \left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(ii)

$$x(t) = 50 + 10\cos(20\pi t) + 30\cos(40\pi t)$$

$$T = \frac{1}{40} \qquad t = nT$$

$$\begin{aligned} x[n] &= 50 + 10\cos\frac{\pi}{2}n + 30\cos\pi n \\ &= 50 + 5e^{j(n\pi/2)} + 5e^{-j(n\pi/2)} + 15e^{jn\pi} + 15e^{-jn\pi} \end{aligned}$$

Using the eigenfunction property:

$$y[n] = 50H(e^{j0}) + 5e^{j(n\pi/2)}H(e^{j(\pi/2)}) + 5e^{-j(n\pi/2)}H(e^{-j(\pi/2)}) + 15e^{jn\pi}H(e^{j\pi}) + 15e^{-jn\pi}H(e^{-j\pi})$$

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}}$$

$H(e^{j0}) = 6$, $H(e^{j(\pi/2)}) = 7\left(\frac{12}{25}\right) - j\frac{12}{25}$, $H(e^{-j(\pi/2)}) = 7\left(\frac{12}{25}\right) + j\frac{12}{25}$,
$H(e^{j\pi}) = 4$, $H(e^{-j\pi}) = 4$

$$y[n] = 300 + 24\sqrt{2}\cos\left(\frac{\pi}{2}n - \tan^{-1}\left(\frac{1}{7}\right)\right) + 120\cos\pi n$$

5.29.

$$\begin{aligned} H(z) &= \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-4z^{-1})} \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}} + \frac{48}{1-4z^{-1}} \end{aligned}$$

Since we know the sequence is not stable, the ROC must not include $|z| = 1$, and since it is two-sided, the ROC must be a ring. This leaves only one possible choice: the ROC is $2 < |z| < 4$.

(a)

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] - 48(4)^n u[-n-1]$$

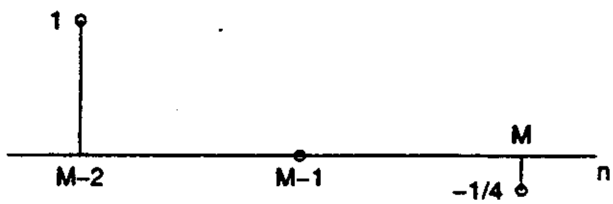
(b)

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}}$$

$$H_2(z) = \frac{48}{1-4z^{-1}}$$

5.30. (a)

$$H(z) = \frac{(z+\frac{1}{2})(z-\frac{1}{2})}{z^M} = z^{-(M-2)}\left(1 - \frac{1}{4}z^{-2}\right)$$



(b)

$$\begin{aligned} w[n] &= x[n-(M-2)] - \frac{1}{4}x[n-M] \\ y[n] &= w[2n] = x[2n-(M-2)] - \frac{1}{4}x[2n-M] \end{aligned}$$

Let $v[n] = x[2n]$,

$$y[n] = v[n-(M-2)/2] - \frac{1}{4}v[n-(M/2)]$$

Therefore,

$$g[n] = \delta[n-(M-2)/2] - \frac{1}{4}\delta[n-(M/2)], \qquad M \text{ even}$$

$$G(z) = z^{-(M-2)/2} - \frac{1}{4}z^{-M/2}$$

5.31. (a)

$$H(z) = \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}, \qquad \text{stable, so the ROC is } \tfrac{1}{2} < |z| < 3$$

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{\frac{4}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \qquad 1 < |z| < 3$$

$$y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n]$$

(b) ROC includes $z = \infty$ so $h[n]$ is causal. Since both $h[n]$ and $x[n]$ are 0 for $n < 0$, we know that $y[n]$ is also 0 for $n < 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1 - \frac{7}{2}z^{-1} + \frac{3}{2}z^{-2}}$$

$$Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$y[n] = x[n-2] + \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2]$$

Since $y[n] = 0$ for $n < 0$, recursion can be done:

$$y[0] = 0, \qquad y[1] = 0, \qquad y[2] = 1$$

(c)

$$H_i(z) = \frac{1}{H(z)} = z^2 - \frac{7}{2}z + \frac{3}{2}, \qquad \text{ROC: entire z-plane}$$

$$h_i[n] = \delta[n+2] - \frac{7}{2}\delta[n+1] + \frac{3}{2}\delta[n]$$

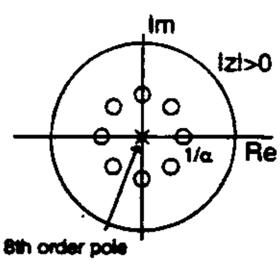
5.32. Since $H(e^{j\omega})$ has a zero on the unit circle, its inverse system will have a pole on the unit circle and thus is not stable.

5.33. (a)

$$X(z) = S(z)(1 - e^{-8\alpha}z^{-8})$$

$$H_1(z) = 1 - e^{-8\alpha}z^{-8}$$

There are 8 zeros at $z = e^{-\alpha}e^{j\frac{\pi}{4}k}$ for $k = 0, \ldots, 7$ and 8 poles at the origin.



(b)

$$Y(z) = H_2(z)X(z) = H_2(z)H_1(z)S(z)$$

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - e^{-8\alpha}z^{-8}}$$

$|z| > e^{-\alpha}$ stable and causal, $\quad |z| < e^{-\alpha}$ not causal or stable

(c) Only the causal $h_2[n]$ is stable, therefore only it can be used to recover $s[n]$.

$$h[n] = \begin{cases} e^{-\alpha n}, & n = 0, 8, 16, \ldots \\ 0, & \text{otherwise} \end{cases}$$

(d)

$$s[n] = \delta[n] \Rightarrow x[n] = \delta[n] - e^{-8\alpha}\delta[n-8]$$

$$\begin{aligned} x[n] * h_2[n] &= \delta[n] - e^{-8\alpha}\delta[n-8] \\ &\quad + e^{-8\alpha}(\delta[n-8] - e^{-8\alpha}\delta[n-16]) \\ &\quad + e^{-16\alpha}(\delta[n-16] - e^{-8\alpha}\delta[n-32]) + \cdots \\ &= \delta[n] \end{aligned}$$

5.34.

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

(a)

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{2-\frac{5}{6}z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}$$

Since $h[n], x[n] = 0$ for $n < 0$ we can assume initial rest conditions.

$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + 2x[n] - \frac{5}{6}x[n-1]$$

(b)

$$h_1[n] = \begin{cases} h[n], & n \le 10^9 \\ 0, & n > 10^9 \end{cases}$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{N-1} h[m]z^{-m}, \quad N = 10^9 + 1$$

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

(d) For IIR, we have 4 multiplies and 3 adds per output point. This gives us a total of $4N$ multiplies and $3N$ adds. So, IIR grows with order N. For FIR, we have N multiplies and $N-1$ adds for the n^{th} output point, so this configuration has order N^2.

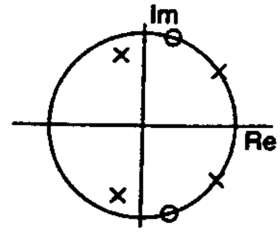
5.35. (a)

$$20\log_{10}|H(e^{j(\pi/5)})| = \infty \Rightarrow \text{pole at } e^{j(\pi/5)}$$

$$20\log_{10}|H(e^{j(2\pi/5)})| = -\infty \Rightarrow \text{zero at } e^{j(2\pi/5)}$$

Resonance at $\omega = \frac{3\pi}{5} \Rightarrow$ pole inside unit circle here.

Since the impulse response is real, the poles and zeros must be in conjugate pairs. The remaining 2 zeros are at zero (the number of poles always equals the number of zeros).



(b) Since $H(z)$ has poles, we know $h[n]$ is IIR.

(c) Since $h[n]$ is causal and IIR, it cannot be symmetric, and thus cannot have linear phase.

(d) Since there is a pole at $|z| = 1$, the ROC does not include the unit circle. This means the system is not stable.

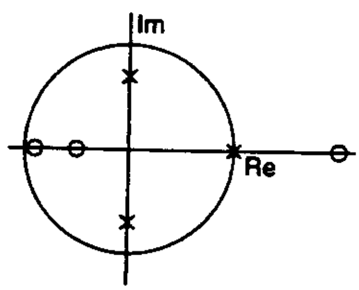
5.36. (a)

$$\begin{aligned} H(z) &= \frac{(1-2z^{-1})(1+\frac{1}{2}z^{-1})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})} \\ &= \frac{1-0.6z^{-1}-2.35z^{-2}-0.9z^{-3}}{1-z^{-1}+0.49z^{-2}-0.49z^{-3}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Cross multiplying and taking the inverse z-transform gives

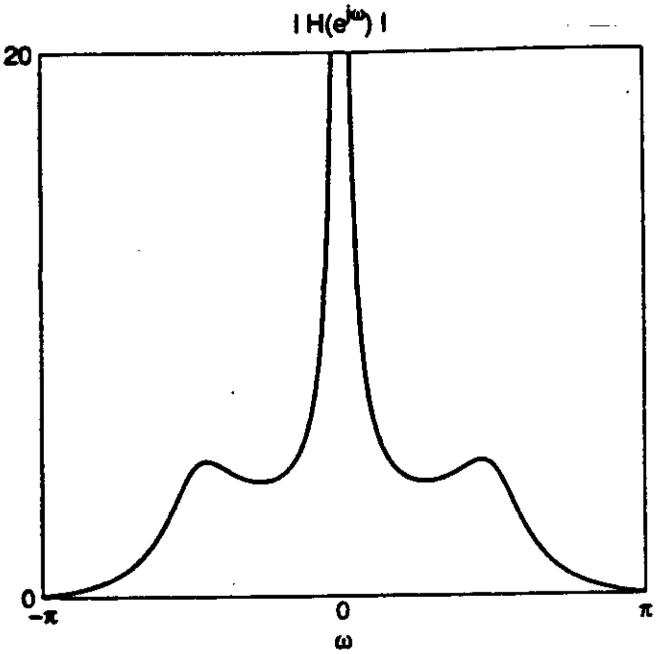
$$y[n]-y[n-1]+0.49y[n-2]-0.49y[n-3] = x[n]-0.6x[n-1]-2.35x[n-2]-0.9x[n-3]$$

(b)



Note that since $h[n]$ is causal, ROC is $|z| > 1$.

(c)

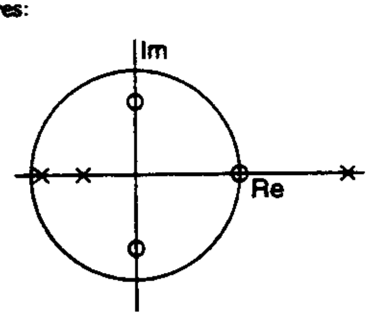


(d) (i) The system is not stable since the ROC does not include $|z| = 1$.

(ii) Because $h[n]$ is not stable, $h[n]$ does not approach a constant as $n \to \infty$.

(iii) We can see peaks at $\omega = \pm\frac{\pi}{2}$ in the graph of $|H(e^{j\omega})|$ shown in part (c), so this is false.

(iv) Swapping poles and zeros gives:



There is a ROC that includes the unit circle ($0.9 < |z| < 2$). However, this stable system would be two sided, so we must conclude the statement is false.

5.37.

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z)}{(1-\frac{1}{6}z)} = \frac{6}{5}\frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-5z^{-1})}{(1-6z^{-1})}$$

$$\alpha^n x[n] \Leftrightarrow X(\alpha^{-1}z) = \frac{6}{5}\frac{(1-\frac{1}{2}\alpha z^{-1})(1-\frac{1}{4}\alpha z^{-1})(1-5\alpha z^{-1})}{(1-6\alpha z^{-1})}$$

A minimum phase sequence has all poles and zeros inside the unit circle.

$$|\alpha/2| < 1 \quad\Rightarrow\quad |\alpha| < 2$$
$$|\alpha/4| < 1 \quad\Rightarrow\quad |\alpha| < 4$$
$$|5\alpha| < 1 \quad\Rightarrow\quad |\alpha| < \frac{1}{5}$$
$$|6\alpha| < 1 \quad\Rightarrow\quad |\alpha| < \frac{1}{6}$$

Therefore, $\alpha^n x[n]$ is real and minimum phase iff α is real and $|\alpha| < \frac{1}{6}$.

5.38. (a) The causal systems have conjugate zero pairs inside or outside the unit circle. Therefore

$$
\begin{aligned}
H(z) &= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1}) \\
H_1(z) &= (0.9)^2(1.25)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1}) \cdot \\
&\quad (1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1}) \\
H_2(z) &= (0.9)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1}) \cdot \\
&\quad (1-1.25e^{-j0.8\pi}z^{-1}) \\
H_3(z) &= (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1}) \cdot \\
&\quad (1-0.8e^{-j0.8\pi}z^{-1})
\end{aligned}
$$

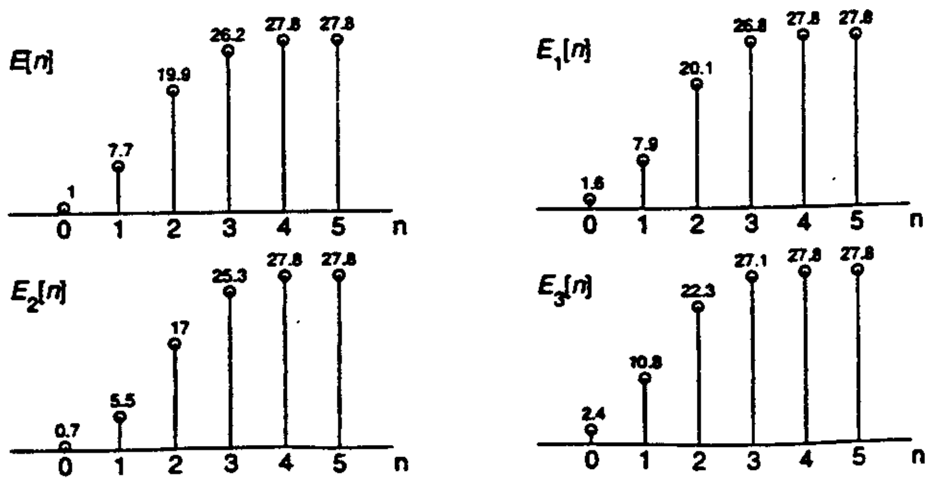
$H_2(z)$ has all its zeros outside the unit circle, and is a maximum phase sequence. $H_3(z)$ has all its zeros inside the unit circle, and thus is a minimum phase sequence.

(b)

$$
\begin{aligned}
H(z) &= 1 + 2.5788z^{-1} + 3.4975z^{-2} + 2.5074z^{-3} + 1.2656z^{-4} \\
h[n] &= \delta[n] + 2.5788\delta[n-1] + 3.4975\delta[n-2] + 2.5074\delta[n-3] + 1.2656\delta[n-4] \\
H_1(z) &= 1.2656 + 2.5074z^{-1} + 3.4975z^{-2} + 2.5788z^{-3} + z^{-4} \\
h_1[n] &= 1.2656\delta[n] + 2.5074\delta[n-1] + 3.4975\delta[n-2] + 2.5788\delta[n-3] + \delta[n-4] \\
H_2(z) &= 0.81 + 2.1945z^{-1} + 3.3906z^{-2} + 2.8917z^{-3} + 1.5625z^{-4} \\
h_2[n] &= 0.81\delta[n] + 2.1945\delta[n-1] + 3.3906\delta[n-2] + 2.8917\delta[n-3] + 1.5625\delta[n-4] \\
H_3(z) &= 1.5625 + 2.8917z^{-1} + 3.3906z^{-2} + 2.1945z^{-3} + 0.81z^{-4} \\
h_3[n] &= 1.5625\delta[n] + 2.8917\delta[n-1] + 3.3906\delta[n-2] + 2.1945\delta[n-3] + 0.81\delta[n-4]
\end{aligned}
$$

(c)

n	$E(n)$	$E_1(n)$	$E_2(n)$	$E_3(n)$
0	1.0	1.6	0.7	2.4
1	7.7	7.9	5.5	10.8
2	19.9	20.1	17.0	22.3
3	26.2	26.8	25.3	27.1
4	27.8	27.8	27.8	27.8
5	27.8	27.8	27.8	27.8



The plot of $E_3[n]$ corresponds to the minimum phase sequence.

5.39. All zeros inside the unit circle means the sequence is minimum phase. Since

$$\sum_{n=0}^{M} |h_{min}[n]|^2 \geq \sum_{n=0}^{M} |h[n]|^2$$

is true for all M, we can use $M = 0$ and just compute $h^2[0]$. The largest result will be the minimum phase sequence.

A	B	C	D	E	F	G	H
44.5	28.4	1.8	2.8	1.8	177.7	113.8	7.1

The answer is F.

5.40.

(i) A zero phase sequence has all its poles and zeros in conjugate reciprocal pairs. Generalized linear phase systems are zero phase systems with additional poles or zeros at $z = 0, \infty, 1$ or -1.

(ii) A stable system's ROC includes the unit circle.

(a) The poles are not in conjugate reciprocal pairs, so this does not have zero or generalized linear phase. $H_i(z)$ has a pole at $z = 0$ and perhaps $z = \infty$. Therefore, the ROC is $0 < |z| < \infty$, which means the inverse is stable. If the ROC includes $z = \infty$, the inverse will also be causal.

(b) Since the poles are not conjugate reciprocal pairs, this does not have zero or generalized linear phase either. $H_i(z)$ has poles inside the unit circle, so ROC is $|z| > \frac{2}{3}$ to match the ROC of $H(z)$. Therefore, the inverse is both stable and causal.

(c) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. The inverse has poles both inside and outside the unit circle. Therefore, a stable non-causal inverse exists.

(d) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. Since the poles of the inverse system are on the unit circle a stable inverse does not exist.

5.41. Convolving two symmetric sequences yields another symmetric sequence. A symmetric sequence convolved with an antisymmetric sequence gives an antisymmetric sequence. If you convolve two antisymmetric sequences, you will get a symmetric sequence.

$$A: \; h_1[n] * h_2[n] * h_3[n] = (h_1[n] * h_2[n]) * h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, $\quad (-1 \leq n \leq 7)$
$(h_1[n] * h_2[n]) * h_3[n]$ is antisymmetric about $n = 3$, $\quad (-3 \leq n \leq 9)$

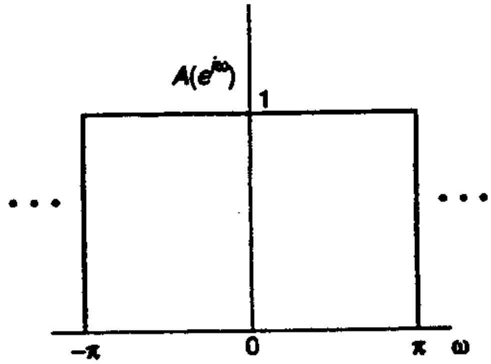
Thus, system A has generalized linear phase

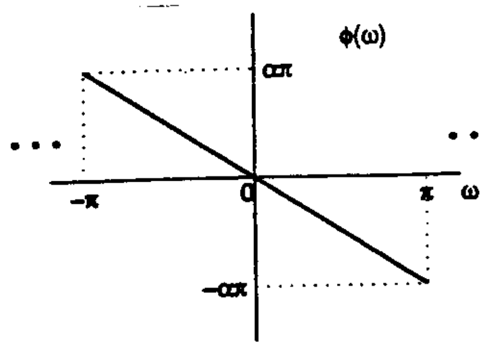
$$B: \; (h_1[n] * h_2[n]) + h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, as we noted above. Adding $h_3[n]$ to this sequence will destroy all symmtery, so this does not have generalized linear phase.

5.42. (a)

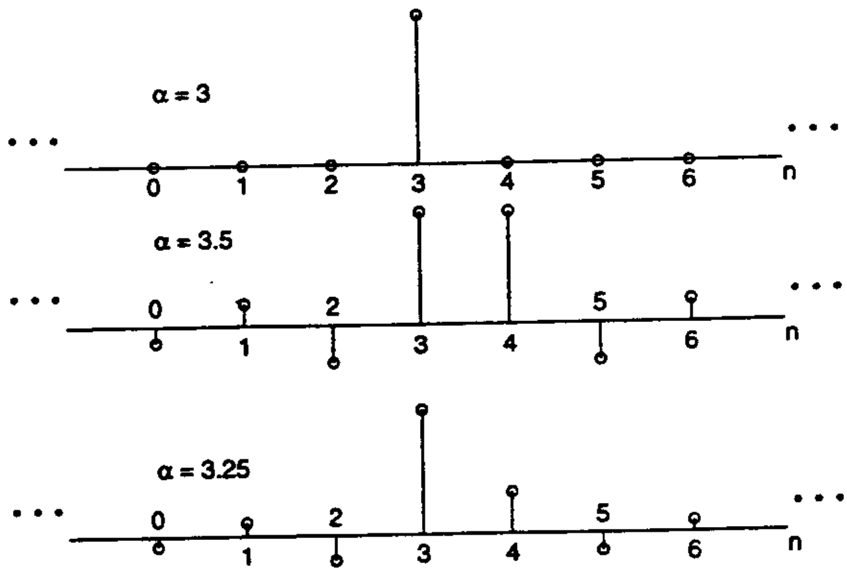
$$A(e^{j\omega}) = 1, \quad |\omega| < \pi$$
$$\phi(\omega) = -\alpha\omega, \quad |\omega| < \pi$$





(b)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\omega}e^{j\omega n}d\omega = \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}$$



(c) If α is an integer, then $h[n]$ is symmetric about about the point $n = \alpha$. If $\alpha = \frac{M}{2}$, where M is odd, then $h[n]$ is symmetric about $\frac{M}{2}$, which is not a point of the sequence. For α in general, $h[n]$ will not be symmetric.

5.43. Type I: Symmetric, M Even, Odd Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=(M+2)/2}^{M} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega(M/2)} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} + h[M/2]e^{-j\omega(M/2)}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} + h[M/2]\right)\\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} 2h[m]\cos\omega((M/2)-m) + h[M/2]\right)\\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\cos\omega n + h[M/2]\right)
\end{aligned}
$$

Let

$$
a[n] = \begin{cases} h[M/2], & n = 0 \\ 2h[(M/2)-n], & n = 1,\ldots,M/2 \end{cases}
$$

Then

$$
H(e^{j\omega}) = e^{-j\omega(M/2)}\sum_{n=0}^{M/2} a[n]\cos\omega n
$$

and we have

$$
A(\omega) = \sum_{n=0}^{M/2} a[n]\cos(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = 0
$$

Type II: Symmetric, M Odd, Even Length

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n}\\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n}\\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)}\\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right)\\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} 2h[m]\cos\omega((M/2)-m)\right)\\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\cos\omega(n-(1/2))\right)
\end{aligned}
$$

Let

$$
b[n] = 2h[(M+1)/2-n], \quad n = 1,\ldots,(M+1)/2
$$

Then

$$
H(e^{j\omega}) = e^{-j\omega(M/2)}\sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2))
$$

and we have

$$
A(\omega) = \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = 0
$$

Type III: Antisymmetric, M Even, Odd Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + 0 + \sum_{(M+2)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)} \left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} \right) \\
&= e^{-j\omega(M/2)} \left(j \sum_{m=0}^{(M-2)/2} 2h[m]\sin\omega((M/2)-m) \right) \\
&= e^{-j\omega(M/2)}e^{j(\pi/2)} \left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\sin\omega n \right)
\end{aligned}$$

Let

$$c[n] = h[(M/2)-n], \quad n = 1, \ldots, M/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)}e^{j(\pi/2)} \sum_{n=1}^{M/2} c[n]\sin\omega n$$

and we have

$$A(\omega) = \sum_{n=1}^{M/2} c[n]\sin(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

Type IV: Antisymmetric, M Odd, Even Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)} \left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)} \right) \\
&= e^{-j\omega(M/2)} \left(j \sum_{m=0}^{(M-1)/2} 2h[m]\sin\omega((M/2)-m) \right) \\
&= e^{-j\omega(M/2)}e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\sin\omega(n-(1/2))
\end{aligned}$$

Let

$$d[n] = 2h[(M+1)/2 - n], \quad n = 1, \ldots, (M+1)/2$$

Then

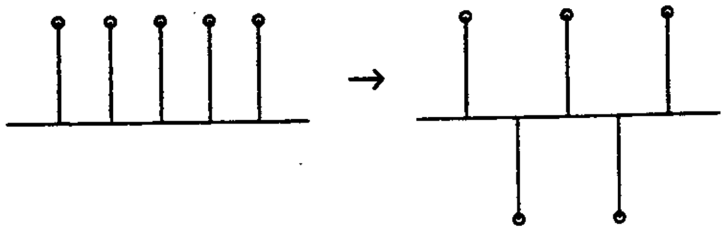
$$H(e^{j\omega}) = e^{-j\omega(M/2)}e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} d[n] \sin \omega(n - (1/2))$$

and we have

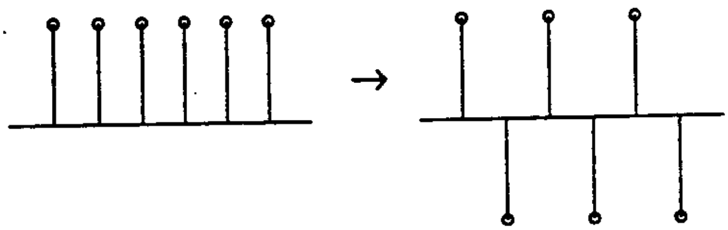
$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n] \sin \omega(n - (1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

5.44. Filter Types II and III cannot be highpass filters since they both must have a zero at $z = 1$.

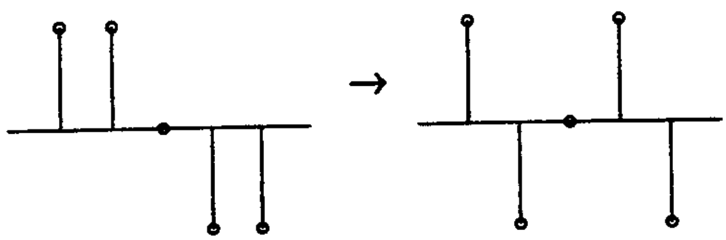
Type I $\rightarrow$ Type I could be highpass:



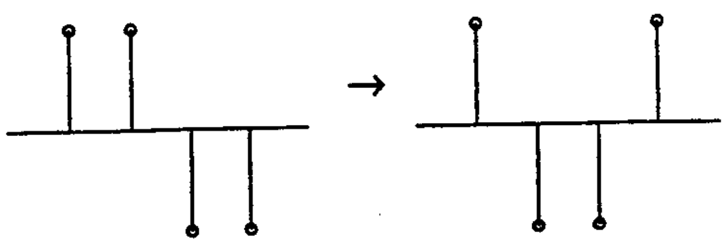
Type II $\rightarrow$ Type IV can be highpass:



Type III $\rightarrow$ Type III cannot be highpass:

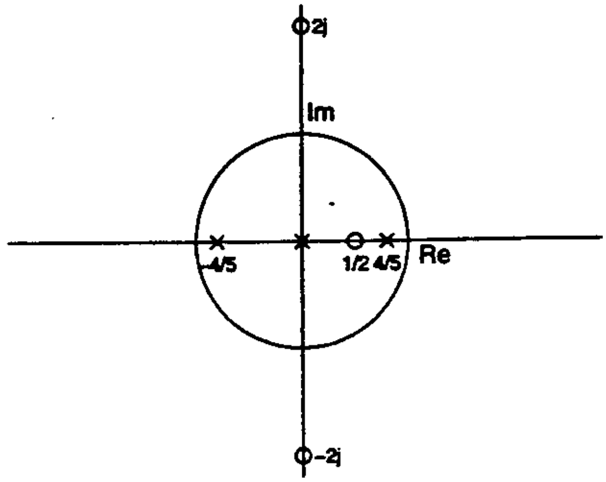


Type IV $\rightarrow$ Type II cannot be highpass:



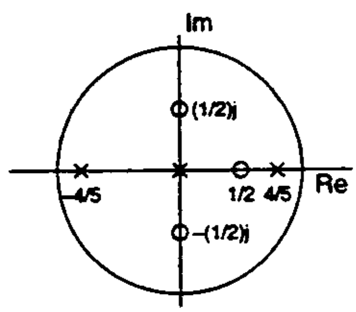
5.45.

$$H(z) = \frac{(1 - 0.5z^{-1})(1 + 2jz^{-1})(1 - 2jz^{-1})}{(1 - 0.8z^{-1})(1 + 0.8z^{-1})}$$

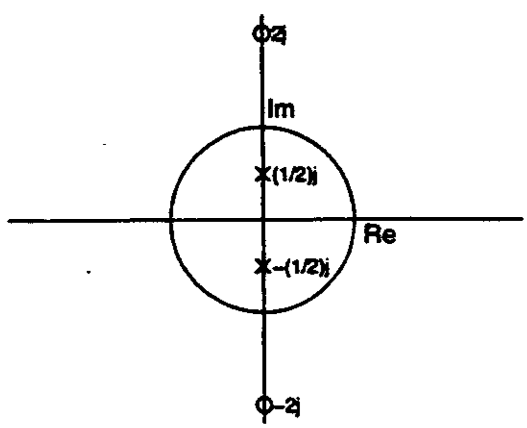


(a) A minimum phase system has all poles and zeros inside $|z| = 1$

$$H_1(z) = \frac{(1 - 0.5z^{-1})(1 + \frac{1}{4}z^{-2})}{(1 - 0.64z^{-2})}$$

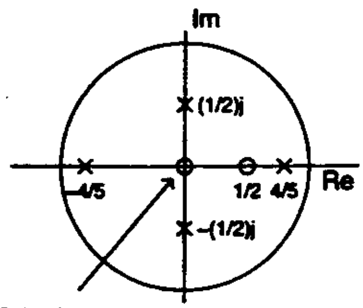


$$H_{ap}(z) = \frac{(1 + 4z^{-2})}{(1 + \frac{1}{4}z^{-2})}$$

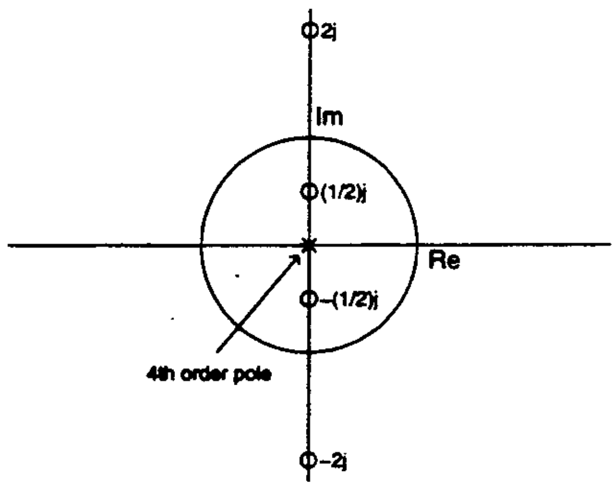


(b) A generalized linear phase system has zeros and poles at $z = 1, -1, 0$ or ∞ or in conjugate reciprocal pairs.

$$H_2(z) = \frac{(1-0.5z^{-1})}{(1-0.64z^{-2})(1+\frac{1}{4}z^{-2})}$$



$$H_{lin}(z) = (1+\tfrac{1}{4}z^{-2})(1+4z^{-2})$$

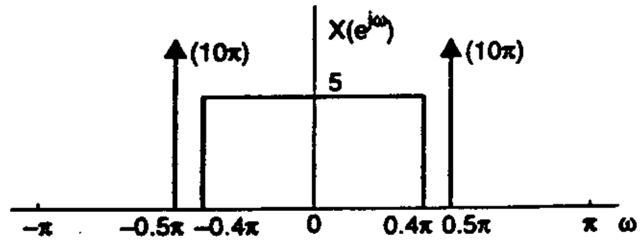


5.46. (a) Minimum phase systems have all poles and zeros inside $|z| = 1$. Allpass systems have pole-zero pairs at conjugate reciprocal locations. Generalized linear phase systems have pole pairs and zero pairs in conjugate reciprocal locations and at $z = 0, 1, -1$ and ∞. This implies that all the poles and zeros of $H_{min}(z)$ are second-order. When the allpass filter flips a pole or zero outside the unit circle, one is left in the conjugate reciprocal location, giving us linear phase.

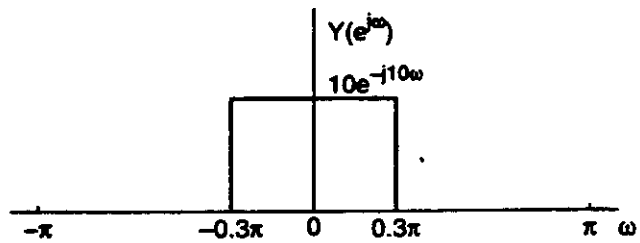
(b) We know that $h[n]$ is length 8 and therefore has 7 zeros. Since it is an even length generalized linear phase filter with real coefficients and odd symmetry it must be a Type IV filter. It therefore has the property that its zeros come in conjugate reciprocal pairs stated mathematically as $H(z) = H(1/z^*)$. The zero at $z = -2$ implies a zero at $z = -\frac{1}{2}$, while the zero at $z = 0.8e^{j(\pi/4)}$ implies zeros at $z = 0.8e^{-j(\pi/4)}$, $z = 1.25e^{j(\pi/4)}$ and $z = 1.25e^{-j(\pi/4)}$. Because it is a IV filter, it also must have a zero at $z = 1$. Putting all this together gives us

$$\begin{aligned} H(z) &= (1+2z^{-1})(1+0.5z^{-1})(1-0.8e^{j(\pi/4)}z^{-1})(1-0.8e^{-j(\pi/4)}z^{-1}) \cdot \\ &\quad (1-1.25e^{j(\pi/4)}z^{-1})(1-1.25e^{-j(\pi/4)}z^{-1})(1-z^{-1}) \end{aligned}$$

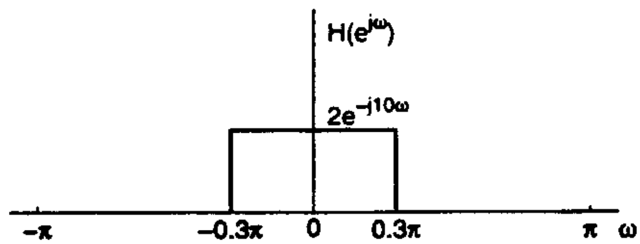
5.47. The input $x[n]$ in the frequency domain looks like



while the corresponding output $y[n]$ looks like



Therefore, the filter must be



In the time domain this is

$$h[n] = \frac{2\sin[0.3\pi(n-10)]}{\pi(n-10)}$$

5.48. (a)

Property	Applies?	Comments
Stable	No	For a stable, causal system, all poles must be inside the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	No	Allpass systems have poles and zeros in conjugate reciprocal pairs.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	No	This system is not in the appropriate form.

(b)

Property	Applies?	Comments
Stable	Yes	The ROC for this system function, $\|z\| > 0$, contains the unit circle. (Note there is 7th order pole at $z = 0$).
IIR	No	The system has poles only at $z = 0$.
FIR	Yes	The system has poles only at $z = 0$.
Minimum Phase	No	By definition, a minimum phase system must have all its poles and zeros located *inside* the unit circle.
Allpass	No	Note that the zeros on the unit circle will cause the magnitude spectrum to drop zero at certain frequencies. Clearly, this system is not allpass.
Generalized Linear Phase	Yes	This is the pole/zero plot of a type II FIR linear phase system.
Positive Group Delay for all w	Yes	This system is causal and linear phase. Consequently, its group delay is a positive constant.

(c)

Property	Applies?	Comments
Stable	Yes	All poles are inside the unit circle. Since the system is causal, the ROC includes the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	Yes	The poles inside the unit circle have corresponding zeros located at conjugate reciprocal locations.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	Yes	Stable allpass systems have positive group delay for all w.

5.49. (a) *Yes*. By the region of convergence we know there are no poles at $z = \infty$ and it therefore must be causal. Another way to see this is to use long division to write $H_1(z)$ as

$$H_1(z) = \frac{1 - z^{-5}}{1 - z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\ , |z| > 0$$

(b) $h_1[n]$ is a causal rectangular pulse of length 5. If we convolve $h_1[n]$ with another causal rectangular pulse of length N we will get a triangular pulse of length $N + 5 - 1 = N + 4$. The triangular pulse is symmetric around its apex and thus has linear phase. To make the triangular pulse $g[n]$ have at least 9 nonzero samples we can choose $N = 5$ or let $h_2[n] = h_1[n]$.

Proof:

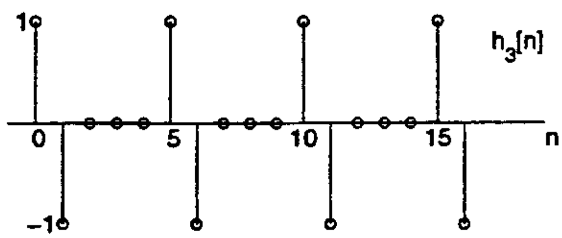
$$G(e^{j\omega}) \;=\; H_1(e^{j\omega})H_2(e^{j\omega}) = H_1^2(e^{j\omega})$$

$$
\begin{aligned}
&= \left[\frac{1-e^{-j5\omega}}{1-e^{-j\omega}}\right]^2 \\
&= \left[\frac{e^{-j\omega 5/2}\left(e^{j\omega 5/2}-e^{-j\omega 5/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}\right]^2 \\
&= \frac{\sin^2(5\omega/2)}{\sin^2(\omega/2)}\, e^{-j4\omega}
\end{aligned}
$$

(c) The required values for $h_3[n]$ can intuitively be worked out using the flip and slide idea of convolution. Here is a second way to get the answer. Pick $h_3[n]$ to be the inverse system for $h_1[n]$ and then simplify using the geometric series as follows.

$$
\begin{aligned}
H_3(z) &= \frac{1-z^{-1}}{1-z^{-5}} \\
&= (1-z^{-1})\left[1+z^{-5}+z^{-10}+z^{-15}+\cdots\right] \\
&= 1-z^{-1}+z^{-5}-z^{-6}+z^{-10}-z^{-11}+z^{-15}-z^{-16}+\cdots
\end{aligned}
$$

This choice for $h_3[n]$ will make $q[n]=\delta[n]$ for all n. However, since we only need equality for $0 \le n \le 19$ truncating the infinite series will give us the desired result. The final answer is shown below.



5.50. (a) This system does not necessarily have generalized linear phase. The phase response,

$$G_1(e^{j\omega}) = \tan^{-1}\left(\frac{Im(H_1(e^{j\omega})+H_2(e^{j\omega}))}{Re(H_1(e^{j\omega})+H_2(e^{j\omega}))}\right)$$

is not necessarily linear. As a counter-example, consider the systems

$$
\begin{aligned}
h_1[n] &= \delta[n]+\delta[n-1] \\
h_2[n] &= 2\delta[n]-2\delta[n-1] \\
g_1[n] &= h_1[n]+h_2[n] = 3\delta[n]-\delta[n-1] \\
G_1(e^{j\omega}) &= 3-e^{-j\omega} = 3-\cos\omega + j\sin\omega \\
\angle G_1(e^{j\omega}) &= \tan^{-1}\left(\frac{\sin\omega}{3-\cos\omega}\right)
\end{aligned}
$$

Clearly, $G_1(e^{j\omega})$ does not have linear phase.

(b) This system must have generalized linear phase.

$$
\begin{aligned}
G_2(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\
|G_2(e^{j\omega})| &= |H_1(e^{j\omega})|\,|H_2(e^{j\omega})| \\
\angle G_2(e^{j\omega}) &= \angle H_1(e^{j\omega}) + \angle H_2(e^{j\omega})
\end{aligned}
$$

The sum of two linear phase responses is also a linear phase response.

(c) This system does not necessarily have linear phase. Using properties of the DTFT, the circular convolution of $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ is related to the product of $h_1[n]$ and $h_2[n]$. Consider the systems

$$\begin{aligned} h_1[n] &= \delta[n] + \delta[n-1] \\ h_2[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\ g_3[n] &= h_1[n]h_2[n] = \delta[n] + 2\delta[n-1] \\ G_3(e^{j\omega}) &= 1 + 2e^{-j\omega} = 1 + 2\cos\omega - j2\sin\omega \\ \angle G_3(e^{j\omega}) &= \tan^{-1}\left(\frac{2\sin\omega}{1+2\cos\omega}\right) \end{aligned}$$

Clearly, $G_3(e^{j\omega})$ does not have linear phase.

5.51. *False.* Let $h[n]$ equal

$$h[n] = \frac{\sin\omega_c(n-4.3)}{\pi(n-4.3)} \longleftrightarrow H(e^{j\omega}) = \begin{cases} e^{-4.3\omega}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Proof: Although the group delay is constant ($\text{grd}\left[H(e^{j\omega})\right] = 4.3$) the resulting M is not an integer.

$$\begin{aligned} h[n] &= \pm h[M-n] \\ H(e^{j\omega}) &= \pm e^{jM\omega} H(e^{-j\omega}) \\ e^{-j4.3\omega} &= \pm e^{j(M+4.3)\omega}, \qquad |\omega| < \omega_c \\ M &= -8.6 \end{aligned}$$

5.52. The type II FIR system $H_{II}(z)$ has generalized linear phase. Therefore, it can be written in the form

$$H_{II}(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where M is an odd integer and $A_e(e^{j\omega})$ is a real, even, periodic function of ω. Note that the system $G(z) = (1 - z^{-1})$ is a type IV generalized linear phase system.

$$\begin{aligned} G(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\ &= e^{-j\omega/2}(2j\sin(\omega/2)) \\ &= 2\sin(\omega/2)e^{-j\omega/2 + j\pi/2} \\ &= A_o(e^{j\omega})e^{-j\omega/2 + j\pi/2} \\ A_o(e^{j\omega}) &= 2\sin(\omega/2) \\ \angle G(e^{j\omega}) &= -\frac{\omega}{2} + \frac{\pi}{2} \end{aligned}$$

The cascade of $H_{II}(z)$ with $G(z)$ results in a generalized linear phase system $H(z)$.

$$\begin{aligned} H(e^{j\omega}) &= A_e(e^{j\omega})A_o(e^{j\omega})e^{-j\omega M/2}e^{-j\omega/2 + j\pi/2} \\ &= A'_o(e^{j\omega})e^{j\omega M'/2 + j\pi/2} \end{aligned}$$

where $A'_o(e^{j\omega})$ is a real, odd, periodic function of ω and M' is an even integer.

Thus, the resulting system $H(e^{j\omega})$ has the form of a type III FIR generalized linear phase system. It is antisymmetric, has odd length (M is even), and has generalized linear phase.

5.53. For all of the following we know that the poles and zeros are real or occur in complex conjugate pairs since each impulse response is real. Since they are causal we also know that none have poles at infinity.

(a) • Since $h_1[n]$ is real there are complex conjugate poles at $z = 0.9e^{\pm j\pi/3}$.

• If $x[n] = u[n]$

$$Y(z) = H_1(z)X(z) = \frac{H_1(z)}{1 - z^{-1}}$$

We can perform a partial fraction expansion on $Y(z)$ and find a term $(1)^n u[n]$ due to the pole at $z = 1$. Since $y[n]$ eventually decays to zero this term must be cancelled by a zero. Thus, the filter must have a zero at $z = 1$.

• The length of the impulse response is infinite.

(b) • Linear phase and a real impulse response implies that zeros occur at conjugate reciprocal locations so there are zeros at $z = z_1, 1/z_1, z_1^*, 1/z_1^*$ where $z_1 = 0.8e^{j\pi/4}$.

• Since $h_2[n]$ is both causal and linear phase it must be a Type I, II, III, or IV FIR filter. Therefore the filter's poles only occur at $z = 0$.

• Since the $\arg\{H_2(e^{j\omega})\} = -2.5\omega$ we can narrow down the filter to a Type II or Type IV filter. This also tells us that the length of the impulse response is 6 and that there are 5 zeros. Since the number of poles always equal the number of zeros, we have 5 poles at $z = 0$.

• Since $20\log|H_2(e^{j0})| = -\infty$ we must have a zero at $z = 1$. This narrows down the filter type even more from a Type II or Type IV filter to just a Type IV filter.

With all the information above we can determine $H_2(z)$ completely (up to a scale factor)

$$H_2(z) = A(1 - z^{-1})(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 1.25e^{j\pi/4}z^{-1})(1 - 1.25e^{-j\pi/4}z^{-1})$$

(c) Since $H_3(z)$ is allpass we know the poles and zeros occur in conjugate reciprocal locations. The impulse response is infinite and in general looks like

$$H_3(z) = \frac{(z^{-1} - 0.8e^{j\pi/4})(z^{-1} - 0.8e^{-j\pi/4})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} H_{ap}(z)$$

5.54. (a) To be rational, X(z) must be of the form

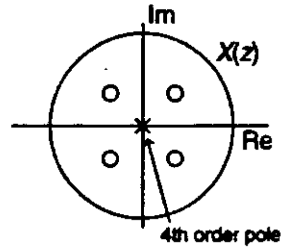
$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Because x[n] is real, its zeros must appear in conjugate pairs. Consequently, there are two more zeros, at $z = \frac{1}{2}e^{-j\pi/4}$, and $z = \frac{1}{2}e^{-j3\pi/4}$. Since x[n] is zero outside $0 \le n \le 4$, there are only four zeros (and poles) in the system function. Therefore, the system function can be written as

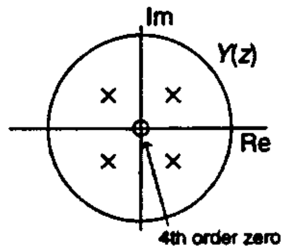
$$X(z) = \left(1 - \frac{1}{2}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{j3\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j3\pi/4}z^{-1}\right)$$

Clearly, X(z) is rational.

(b) A sketch of the pole-zero plot for $X(z)$ is shown below. Note that the ROC for X(z) is $|z| > 0$.



(c) A sketch of the pole-zero plot for $Y(z)$ is shown below. Note that the ROC for Y(z) is $|z| > \frac{1}{2}$.



5.55. • Since $x[n]$ is real the poles & zeros come in complex conjugate pairs.

- From (1) we know there are no poles except at zero or infinity.
- From (3) and the fact that $x[n]$ is finite we know that the signal has generalized linear phase.
- From (3) and (4) we have $\alpha = 2$. This and the fact that there are no poles in the finite plane except the five at zero (deduced from (1) and (2)) tells us the form of $X(z)$ must be

$$X(z) = x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} + x[5]z^{-5}$$

The phase changes by π at $\omega = 0$ and π so there must be a zero on the unit circle at $z = \pm 1$. The zero at $z = 1$ tells us $\sum x[n] = 0$. The zero at $z = -1$ tells us $\sum(-1)^n x[n] = 0$.
We can also conclude $x[n]$ must be a Type III filter since the length of $x[n]$ is odd and there is a zero at both $z = \pm 1$. $x[n]$ must therefore be antisymmetric around $n = 2$ and $x[2] = 0$.

- From (5) and Parseval's theorem we have $\sum |x[n]|^2 = 28$.
- From (6)

$$\begin{aligned} y[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega = 4 \\ &= x[n] * u[n]\,|_{n=0} = x[-1] + x[0] \end{aligned}$$

$$\begin{aligned} y[1] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega}d\omega = 6 \\ &= x[n] * u[n]\,|_{n=1} = x[-1] + x[0] + x[1] \end{aligned}$$

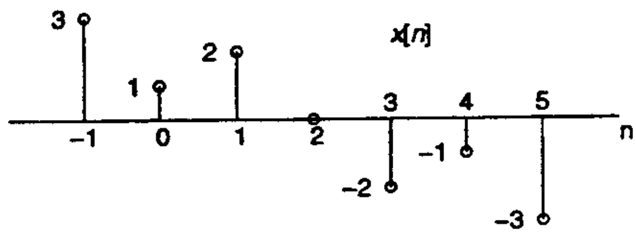
- The conclusion from (7) that $\sum(-1)^n x[n] = 0$ we already derived earlier.
- Since the DTFT $\{x_e[n]\} = \mathcal{R}e\left\{X(e^{j\omega})\right\}$ we have

$$\begin{aligned} \frac{x[5] + x[-5]}{2} &= -\frac{3}{2} \\ x[5] &= -3 + x[-5] \\ x[5] &= -3 \end{aligned}$$

Summarizing the above we have the following (dependent) equations

(1) $x[-1]+x[0]+x[1]+x[2]+x[3]+x[4]+x[5]=0$
(2) $-x[-1]+x[0]-x[1]+x[2]-x[3]+x[4]-x[5]=0$
(3) $x[2]=0$
(4) $x[-1]=-x[5]$
(5) $x[0]=-x[4]$
(6) $x[1]=-x[3]$
(7) $x[-1]^2+x[0]^2+x[1]^2+x[2]^2+x[3]^2+x[4]^2+x[5]^2=28$
(8) $x[-1]+x[0]=4$
(9) $x[-1]+x[0]+x[1]=6$
(10) $x[5]=-3$

$x[n]$ is easily obtained from solving the equations in the following order: (3),(10),(4),(8),(5),(9), and (6).



5.56. (a) The LTI system S_2 is characterized as a lowpass filter.
The z-transform of $h_1[n]$ is found below.

$$\begin{aligned} y[n]-y[n-1]+\frac{1}{4}y[n-2] &= x[n] \\ Y(z)-Y(z)z^{-1}+\frac{1}{4}Y(z)z^{-2} &= X(z) \\ Y(z)\left(1-z^{-1}+\frac{1}{4}z^{-2}\right) &= X(z) \end{aligned}$$

$$H_1(z)=\frac{1}{(1-z^{-1}+\frac{1}{4}z^{-2})}=\frac{1}{(1-\frac{1}{2}z^{-1})^2}$$

This system function has a second order pole at $z=\frac{1}{2}$. (There is also a second order zero at $z=0$). Evaluating this pole-zero plot on the unit circle yields a low pass filter, as the second order pole boosts the low frequencies.

Since

$$\begin{aligned} H_2(e^{j\omega}) &= H_1(-e^{j\omega}) \\ H_2(z) &= H_1(-z) \end{aligned}$$

If we replace all references to z in $H_1(z)$ with $-z$, we will get $H_2(z)$.

$$H_2(z)=\frac{1}{(1+\frac{1}{2}z^{-1})^2}$$

Consequently, $H_2(z)$ has two poles at $z=-\frac{1}{2}$. (There is also a second order zero at $z=0$). Evaluating this pole-zero plot on the unit circle yields a high pass filter, as the second order pole now boosts the high frequencies.

(b) The LTI system S_3 is characterized as a highpass filter. $H_3(e^{j\omega})$ is the inverse system of $H_1(e^{j\omega})$, since $H_3(e^{j\omega})H_1(e^{j\omega}) = 1$. Consequently, $H_3(z)H_1(z) = 1$.

As shown in part (a), $H_1(z)$ has a second order pole at $z = \frac{1}{2}$, and a second order zero at $z = 0$. Thus, $H_3(z)$ has a second order zero at $z = \frac{1}{2}$, and a second order pole at $z = 0$. Evaluating this pole-zero plot on the unit circle yields a high pass filter, as the second order zero attenuates the low frequencies.

S_3 is a minimum phase filter, since its poles and zeros are located inside the unit circle. However, because the zeros of S_3 do not occur in conjugate reciprocal pairs, S_3 cannot be classified as one of the four types of FIR filters with generalized linear phase.

(c) First, we compute the system function $H_4(z)$

$$\begin{aligned} y[n] + \alpha_1 y[n-1] + \alpha_2 y[n-2] &= \beta_0 x[n] \\ Y(z) + \alpha_1 Y(z)z^{-1} + \alpha_2 Y(z)z^{-2} &= \beta_0 X(z) \end{aligned}$$

$$H_4(z) = \frac{\beta_0}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

S_4 is a stable and noncausal LTI system. Therefore, its poles must be located *outside* the unit circle, and its ROC must be an interior region that includes the unit circle. We place a second order pole at $z = 2$, which is the (conjugate) reciprocal location of the second order pole of $H_1(z)$ at $z = \frac{1}{2}$. This gives

$$\begin{aligned} H_4(z) &= \frac{\beta_0}{(1 - 2z^{-1})^2} \\ &= \frac{\beta_0}{(1 - 4z^{-1} + 4z^{-2})} \end{aligned}$$

In order for

$$|H_4(e^{j\omega})| = |H_1(e^{j\omega})|$$

an appropriate value of β_0 must be found. Consider the case when $z = 1$. Then,

$$\begin{aligned} \left|\frac{\beta_0}{(1 - 4z^{-1} + 4z^{-2})}\right| &= \left|\frac{1}{(1 - z^{-1} + \frac{1}{4}z^{-2})}\right| \\ \left|\frac{\beta_0}{(1 - 4 + 4)}\right| &= \left|\frac{1}{(1 - 1 + \frac{1}{4})}\right| \\ |\beta_0| &= 4 \end{aligned}$$

The values $\alpha_1 = -4$, $\alpha_2 = 4$, and $\beta_0 = 4$ satisfy the criteria. Note that $\beta_0 = -4$ also is a valid solution.

(d) If $h_5[n] * h_1[n]$ is FIR, then the poles of $H_1(z)$ must be cancelled by zeros of $H_5(z)$. Thus, we expect a second order zero of $H_5(z)$ at $z = \frac{1}{2}$. Therefore, $H_5(z)$ will have the term $(1 - \frac{1}{2}z^{-1})^2$.

In order for the filter $h_5[n]$ to be zero phase, it must satisfy the symmetry property $h_5[n] = h_5^*[-n]$, which means that $H_5(z) = H_5^*(z)$. For this property to be satisfied, we need two more zeros located at $z = 2$. In addition, we want these zeros to correspond to a noncausal sequence. Therefore, $H_5(z)$ will also have the term $(1 - \frac{1}{2}z)^2$.

Combining these two results,

$$\begin{aligned} H_5(z) &= \left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{2}z\right)^2 \\ &= \left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z + \frac{1}{4}z^2\right) \\ &= \frac{1}{4}z^{-2} - \frac{5}{4}z^{-1} + \frac{33}{16} - \frac{5}{4}z + \frac{1}{4}z^2 \end{aligned}$$

Taking the inverse z-transform yields

$$h_5[n] = \frac{1}{4}\delta[n-2] - \frac{5}{4}\delta[n-1] + \frac{33}{16}\delta[n] - \frac{5}{4}\delta[n+1] + \frac{1}{4}\delta[n+2]$$

5.57. (a)

$$x[n] = s[n]\cos\omega_0 n = \frac{1}{2}s[n]e^{j\omega_0 n} + \frac{1}{2}s[n]e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{1}{2}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}S(e^{j(\omega+\omega_0)})$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)} \\ &= s[n]\cos(\omega_0 n - \phi_0) \end{aligned}$$

(b) This time,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}e^{-j\omega n_d}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}e^{-j\omega n_d}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \delta[n-n_d] * \left(\frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)}\right) \\ &= \delta[n-n_d] * s[n]\cos(\omega_0 n - \phi_0) \\ &= s[n-n_d]\cos(\omega_0 n - \omega_0 n_d - \phi_0) \end{aligned}$$

Therefore, if $\phi_1 = \phi_0 + \omega_0 n_d$ then

$$y[n] = s[n-n_d]\cos(\omega_0 n - \phi_1)$$

for narrowband $s[n]$.

(c)

$$\begin{aligned} \tau_{gr} &= -\frac{d}{d\omega}arg[H(e^{j\omega})] = -\frac{d}{d\omega}[-\phi_0 - \omega n_d] = n_d \\ \tau_{ph} &= -\frac{1}{\omega}arg[H(e^{j\omega})] = -\frac{1}{\omega}[-\phi_0 - \omega n_d] = \frac{\phi_0}{\omega} - n_d \end{aligned}$$

$$y[n] = s[n - \tau_{gr}(\omega_0)]\cos[\omega_0(n - \tau_{ph}(\omega_0))]$$

(d) The effect would be the same as the following:

(i) Bandlimit interpolate the composite signal to a C–T signal with some rate T.

(ii) Delay the envelope by $T \cdot \tau_{gr}$, and delay the carrier by $T \cdot \tau_{ph}$.

(iii) Sample to a D–T signal at rate T

5.58. (a)

$$m_x = 0 \Rightarrow \phi_{yy}[m] = \Gamma_{yy}[m] \Leftrightarrow \Gamma_{yy}(z) = \Phi_{yy}(z)$$

$$\phi_{yy}[m] = y[n] * y[-n] = x[n] * x[-n] * h[n] * h[-n]$$

$$\Phi_{yy}(z) = X(z)X(z^{-1})H(z)H(z^{-1}) = \Phi_{xx}(z)H(z)H(z^{-1})$$

$$\phi_{xx}[m] = \sigma_x^2\delta[m] \Leftrightarrow \Phi_{xx}(z) = \sigma_x^2$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k], \quad b_0 = 1$$

$$Y(z) = \sum_{k=1}^{N} a_k Y(z) z^{-k} + X(z) + \sum_{k=1}^{M} b_k X(z) z^{-k}$$

$$H(z) = \frac{1 + \sum_{k=1}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = A \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

So,

$$\Gamma_{yy}(z) = \Phi_{xx}(z)H(z)H(z^{-1}) = \sigma_x^2 \frac{\left(1 + \sum_{k=1}^{M} b_k z^{-k}\right)\left(1 + \sum_{k=1}^{M} b_k z^{k}\right)}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)\left(1 - \sum_{k=1}^{N} a_k z^{k}\right)}$$

Or equivalently,

$$\Gamma_{yy}(z) = A^2\sigma_x^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k z)}$$

(b) To "whiten" the signal $y[n]$ we need a system:

$$H_w(z)H_w(z^{-1}) = \frac{1}{H(z)H(z^{-1})}$$

Therefore,

$$H_w(z)H_w(z^{-1}) = \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k z)}{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k z)}$$

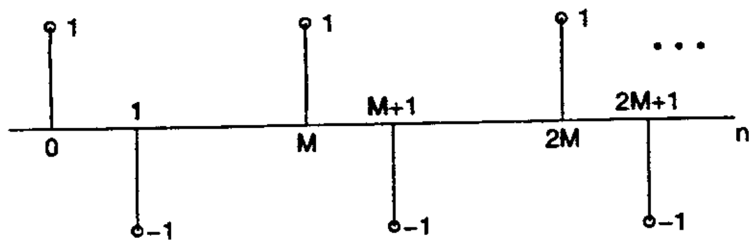
The poles of $H_w(z)$ are the zeros of $H(z)$ and the zeros of $H_w(z)$ are the poles of $H(z)$. We must now decide which N of the $2N$ zeros of $H_w(z)H_w(z^{-1})$ to associate with $H_w(z)$. The remaining N zeros and M poles will be reciprocals and will be associated with $H_w(z^{-1})$. In order for $H_w(z)$ to be stable, we must chose all its poles inside the unit circle. Thus for a pair c_k, c_k^{-1} we chose the one which is inside the unit circle.

(c) There is no real constraint on the zeros of $H_w(z)$, so we can select either d_k or d_k^{-1}. Thus, it is not unique.

5.59. (a)

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}}$$

$$H_i(e^{j\omega}) = \frac{1-e^{-j\omega}}{1-e^{-j\omega M}} \Leftrightarrow h_i[n] = \sum_{k=0}^{\infty} \delta[n-kM] - \delta[n-kM-1]$$



$h_i[n]$ has infinite length, so we can never get a result without infinite sums. Therefore, it is not a real time filter. We can use the transform approach but we must have all the input data available to do this.

(b) The proposed system is a windowed version of $h_i[n]$:

$$h_1[n] * h_2[n] = h_i[n]p[n]$$

Where

$$p[n] = \begin{cases} 1, & 0 \le n \le qM \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] * h[n] * h_i[n]p[n] = w[n]$$

Therefore, if $x[n]$ is shorter than qM points, we can recover it by looking at $w[n]$ in the range $0 \le n \le qM-1$.

(c)

$$H_i(z) = \frac{1}{H(z)} = H_1(z)H_2(z)$$

$$h_1[n] = \sum_{k=0}^{q} \delta[n-kM] \Leftrightarrow H_1(z) = \frac{1-z^{-qM}}{1-z^{-M}}$$

Thus,

$$H_2(z) = \frac{1}{H(z)} \frac{1-z^{-M}}{1-z^{-qM}}$$

Note that

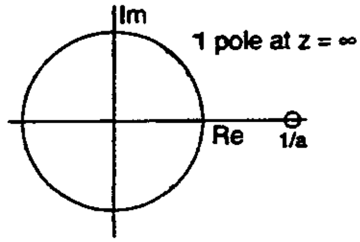
$$\frac{1-z^{-M}}{1-z^{-qM}}$$

has M zeros and qM poles. Since $H_2(z)$ is causal, there are no poles at $z=\infty$. If $H(z)$ has P poles and Z zeros:

$$Z + M \le P + qM$$

5.60. (a)

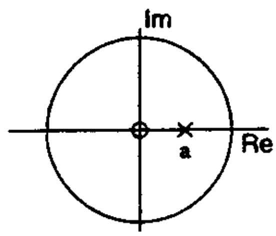
$$H(z) = z - \frac{1}{a} = \frac{az-1}{a} = \frac{a-z^{-1}}{az^{-1}}$$



$$H(e^{j\omega}) = e^{j\omega} - \frac{1}{a} = \cos\omega + j\sin\omega - \frac{1}{a}$$

$$\arg[H(e^{j\omega})] = \tan^{-1}\left(\frac{\sin\omega}{\cos\omega - \frac{1}{a}}\right)$$

(b)



$$G(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$G(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = \frac{1}{1-a\cos\omega + ja\sin\omega}$$

$$\begin{aligned}\arg[G(e^{j\omega})] &= -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)\\ &= \tan^{-1}\left(\frac{a\sin\omega}{a\cos\omega - 1}\right)\\ &= \arg[H(e^{j\omega})]\end{aligned}$$

5.61. (a) Because $h_1[n], h_2[n]$ are minimum phase sequences, all poles and zeros of their transforms must be inside the unit circle.

$$h_1[n] * h_2[n] \leftrightarrow H_1(z)H_2(z)$$

Since $H_1(z)$ and $H_2(z)$ have all their poles and zeros inside the unit circle, their product will also.

(b)

$$h_1[n] + h_2[n] \leftrightarrow H_1(z) + H_2(z)$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} = X_1(z)$$

$$x_2[n] = 2\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{2}{1-\frac{1}{2}z^{-1}} = X_2(z)$$

Both of these are minimum phase, with a zero at $z = 0$ and a pole at $z = \frac{1}{2}$.

$$X_1(z) + X_2(z) = \frac{3}{1-\frac{1}{2}z^{-1}}$$

This is minimum phase, with the same pole and zero as $X_1(z)$ and $X_2(z)$.

$$x_1[n] = 6\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{6}{1-\frac{1}{2}z^{-1}} = X_1(z)$$

$$x_2[n] = -6\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{-6}{1-\frac{1}{3}z^{-1}} = X_2(z)$$

$X_1(z)$ has a pole at $z = \frac{1}{2}$ and a zero at $z = 0$. $X_2(z)$ has a pole at $z = \frac{1}{3}$ and a zero at $z = 0$.

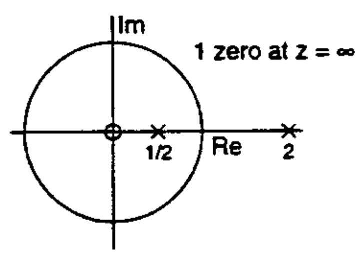
$$X_1(z) + X_2(z) = \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

This has zeros at $z = 0, \infty$ and poles at $z = \frac{1}{2}, \frac{1}{3}$. Therefore, it is not minimum phase.

5.62. (a)

$$r[n] = \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}(2)^n u[-n-1]$$

$$\begin{aligned} R(z) &= \frac{\frac{4}{3}}{1-\frac{1}{2}z^{-1}} - \frac{\frac{4}{3}}{1-2z^{-1}} \\ &= \frac{-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \\ &= \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}, \quad \text{ROC: } \tfrac{1}{2} < |z| < 2 \end{aligned}$$



(b)

$$r[n] = h[n] * h[-n] \Leftrightarrow R(z) = H(z)H(z^{-1})$$

$$R(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}$$

We have two choices from $H(z)$. Since $h[n]$ is minimum phase we need the one which has the pole at $z = \frac{1}{2}$, which is inside the unit circle.

$$H(z) = \frac{\pm 1}{(1-\frac{1}{2}z^{-1})}, \quad \text{ROC: } |z| > \tfrac{1}{2}$$

$$h[n] = \pm\left(\frac{1}{2}\right)^n u[n]$$

5.63. (a) Maximum phase systems are of the form

$$H(z) = \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{M}(z - d_k)}, \qquad |c_k|, |d_k| > 1$$

Since the poles are outside the unit circle, the only stable system will have a ROC of $|z| < \min|d_k|$. This implies the poles will all contribute to the $h[n]$ with terms of the form $-(d_k)^n u[-n-1]$, which are anticausal. The zeros are all positive powers of z, which means they are shifting to left, and $h[n]$ is still anticausal.

(b)

$$H_{min}(z) = h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1})$$

$$H_{max}(z) = h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1}) \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right)$$

$$H_{ap}(z) = \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right)$$

(c)

$$\begin{aligned} H_{max}(z) &= h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1}) \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right) \\ &= h_{min}[0] \prod_{k=1}^{M}(z^{-1} - c_k^*) \\ &= z^{-M} h_{min}[0] \prod_{k=1}^{M}(1 - c_k^* z) \\ &= z^{-M} H_{min}(z^{-1}) \end{aligned}$$

(d)

$$H_{max}(z) = z^{-M} H_{min}(z^{-1})$$

$$h_{max}[n] = \delta[n - M] * h_{min}[-n] = h_{min}[-n + M]$$

5.64. (a) We desire $|H(z)H_c(z)| = 1$, where $H_c(z)$ is stable and causal and $H(z)$ is not minimum phase. So,

$$|H_{ap}(z)H_{min}(z)H_c(z)| = 1$$

Since $|H_{ap}(z)| = 1$, we want

$$|H_{min}(z)H_c(z)| = 1$$

This means we have

$$H_c(z) = \frac{1}{H_{min}(z)}$$

which will be stable and causal since all the zeros of $H_{min}(z)$, which become the poles of $H_c(z)$, are inside the unit circle.

(b) Since

$$H_c(z) = \frac{1}{H_{min}(z)}$$

We have

$$G(z) = H_{ap}(z)$$

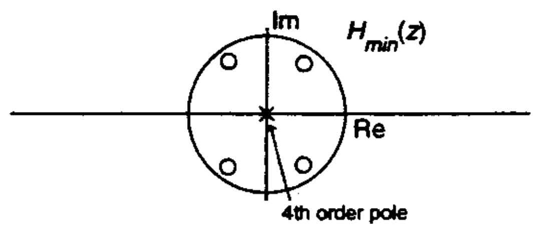
(c)

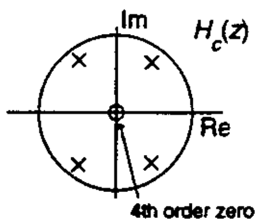
$$H(z) = (1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - 1.2e^{j0.7\pi}z^{-1})(1 - 1.2e^{-j0.7\pi}z^{-1})$$

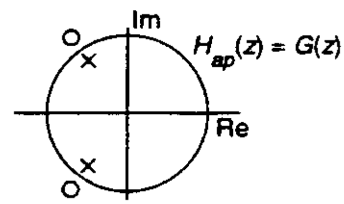
$$H_{min}(z) = (1.44)(1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})$$

$$H_c(z) = \frac{1}{(1.44)(1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$

$$G(z) = H_{ap}(z) = \frac{(z^{-1} - (5/6)e^{-j0.7\pi})(z^{-1} - (5/6)e^{j0.7\pi})}{(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$







5.65.

$$H(z) = H_{min}(z)\frac{z^{-1} - a}{1 - az^{-1}}, \quad |a| < 1$$

Thus,

$$\lim_{z\to\infty} H_{min}(z) = \lim_{z\to\infty} \frac{1 - az^{-1}}{z^{-1} - a}H(z)$$

$$h_{min}[0] = -\frac{1}{a}h[0]$$

Therefore, $|h_{min}[0]| > |h[0]|$ since $|a| < 1$. This process can be repeated if more than one allpass system is cascaded. In each case, the factor for each will be larger than unity in the limit.

5.66. (a) We use the allpass principle and place a pole at $z = z_k$ and a zero at $z = \frac{1}{z_k^*}$.

$$\begin{aligned} H(z) &= H_{min}(z)\frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} \\ &= Q(z)(z^{-1} - z_k^*) \end{aligned}$$

(b)

$$H(z) = Q(z)z^{-1} - z_k^* Q(z)$$
$$h[n] = q[n-1] - z_k^* q[n]$$

$$H_{min}(z) = Q(z) - z_k Q(z) z^{-1}$$
$$h_{min}[n] = q[n] - z_k q[n-1]$$

(c)

$$\begin{aligned}
\varepsilon &= \sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \\
&= \sum_{m=0}^{n} (|q[m]|^2 - z_k q[m-1]q^*[m] - z_k^* q^*[m-1]q[m] + |z_k|^2 |q[m-1]|^2) \\
&\quad - \sum_{m=0}^{n} (|q[m-1]|^2 - z_k^* q^*[m-1]q[m] - z_k q[m-1]q^*[m] + |z_k|^2 |q[m]|^2) \\
&= (1 - |z_k|^2) \sum_{m=0}^{n} (|q[m]|^2 - |q[m-1]|^2) \\
&= (1 - |z_k|^2)|q[n]|^2
\end{aligned}$$

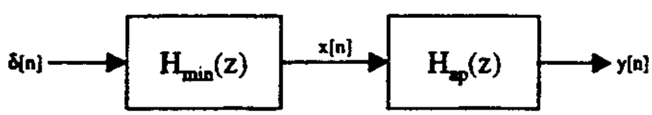
(d)

$$\varepsilon = (1 - |z_k|^2)|q[n]|^2 \geq 0 \quad \forall n \text{ since } |z_k| < 1$$

Then

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \geq 0$$
$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{min}[m]|^2 \quad \forall n$$

5.67. (a) $x[n]$ is real, minimum phase and $x[n] = 0$ for $n < 0$. Consider the system:



$x[n]$ is the impulse response of a minimum phase system. $y[n]$ is the impulse response of a system which has the same frequency response magnitude as that of $x[n]$ but it is not minimum phase. Therefore, the equation applies.

$$\sum_{k=0}^{n} |x[k]|^2 \geq \sum_{k=0}^{n} |y[k]|^2$$

Since $h_{ap}[n]$ is causal and $x[n]$ is causal, $y[n]$ is also causal, and these sums are meaningful.

(b) As discussed in the book, the group delay for a rational allpass system is always positive. That is,

$$\mathrm{grd}[H_{ap}(e^{j\omega})] \geq 0$$

Therefore, filtering a signal $x[n]$ by such a system will delay the energy in the output $y[n]$. If we require that $x[n]$ is causal, then $y[n]$ will be causal as well, and the equation

$$\sum_{k=0}^{n} |x[k]|^2 \geq \sum_{k=0}^{n} |y[k]|^2$$

applies to the system.

5.68. (a)

$$\begin{aligned} g[n] &= x[n] * h[n] \\ r[n] &= g[-n] * h[n] \end{aligned}$$

$$s[n] = r[-n] = g[n] * h[-n] = x[n] * (h[n] * h[-n])$$

$$\begin{aligned} h_1[n] &= h[n] * h[-n] \\ H_1(e^{j\omega}) &= H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2 \end{aligned}$$

Since $H_1(e^{j\omega})$ is real, it is zero phase.

(b)

$$\begin{aligned} g[n] &= x[n] * h[n] \\ r[n] &= x[-n] * h[n] \end{aligned}$$

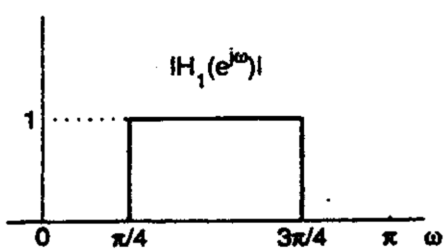
$$y[n] = g[n] + r[-n] = x[n] * h[n] + x[n] * h[-n] = x[n] * (h[n] + h[-n])$$

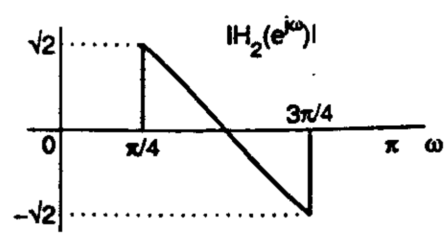
$$\begin{aligned} h_2[n] &= h[n] + h[-n] \\ H_2(e^{j\omega}) &= H(e^{j\omega}) + H^*(e^{j\omega}) \\ &= 2\mathcal{R}e\{H(e^{j\omega})\} \end{aligned}$$

$H_2(e^{j\omega})$ is real, so it is also zero phase.

$$|H_2(e^{j\omega})| = 2|H(e^{j\omega})| \cos\left(\angle H(e^{j\omega})\right)$$

(c)





In general, method A is preferable since method B causes a magnitude distortion which is a function of the (possibly non-linear) phase of $h[n]$.

5.69. *False.* Consider

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{1}{1-\frac{5}{2}z^{-1}+z^{-2}}$$

This system function has poles at $z = 1/2$ and $z = 2$. However, as the following shows it is a generalized linear phase filter.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\frac{5}{2}e^{-j\omega}+e^{-j2\omega}} \\ &= \frac{e^{j\omega}}{e^{j\omega}-\frac{5}{2}+e^{-j\omega}} \\ &= \left(\frac{1}{2\cos w - \frac{5}{2}}\right)e^{j\omega} \end{aligned}$$

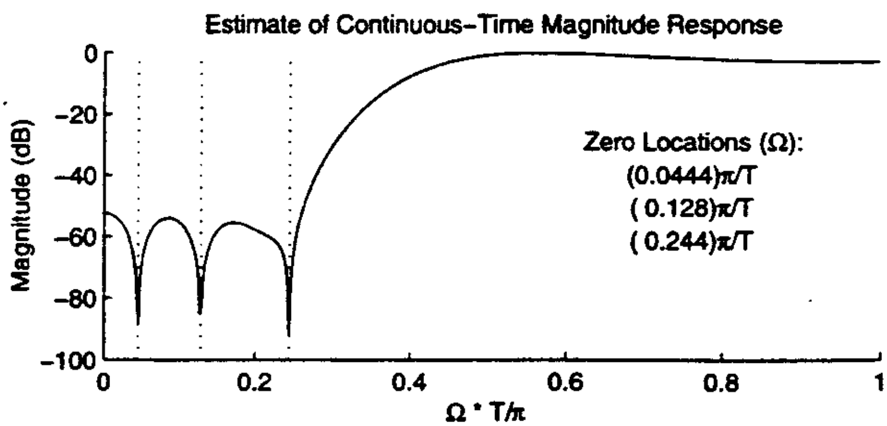
5.70. (a) Since $h[n]$ is a real causal linear phase filter the zeros must occur in sets of 4. Thus, if z_1 is a zero of $H(z)$ then $z_1^*, 1/z_1$ and $1/z_1^*$ must also be zeros. We can use this to find 4 zeros of $H(z)$ from the given information.

$$\begin{array}{rll} z_1, & \text{magnitude} = 0.5, & \text{angle} = 153 \text{ degrees} \\ z_1^*, & \text{magnitude} = 0.5, & \text{angle} = 207 \text{ degrees} \\ 1/z_1, & \text{magnitude} = 2, & \text{angle} = 207 \text{ degrees} \\ 1/z_1^*, & \text{magnitude} = 2, & \text{angle} = 153 \text{ degrees} \end{array}$$

(b) There are 24 zeros so the length of $h[n]$ is 25. Since it is a linear phase filter it has a delay of $(L-1)/2 = (25-1)/2 = 12$ samples. That corresponds to a time delay of

$$\left(0.5\,\frac{\text{ms}}{\text{sample}}\right)(12 \text{ samples}) = 6 \text{ ms}$$

(c) The zero locations used to create the following plot were estimated from the figure using a ruler and a protractor.



5.71. (a) There are many possible solutions to this problem. The idea behind any solution is to have $h[n]$ be an upsampled (by a factor of 2) version of $g[n]$. That is,

$$h[n] = \begin{cases} g[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \textit{otherwise} \end{cases}$$

Thus, $h[n]$ will process only the even-indexed samples. One such system would be described by

$$\begin{aligned} h[n] &= 1+\delta[n-2] \\ g[n] &= 1+\delta[n-1] \\ H(z) &= 1+z^{-2} \\ G(z) &= 1+z^{-1} \end{aligned}$$

(b) As in part a, there are many possible solutions to this problem. The idea behind any solution is to choose an $h[n]$ that cannot be an upsampled (by a factor of 2) version of $g[n]$. Clearly, choosing $h[n]$ to filter odd-indexed samples satisfies this criterion. One such $h[n]$ would be

$$\begin{aligned} h[n] &= 1+\delta[n-1]+\delta[n-2] \\ H(z) &= 1+z^{-1}+z^{-2} \end{aligned}$$

(c) In general, the odd-indexed samples of $h[n]$ must be zero, in order for a $g[n]$ to be found for which $r[n]=y[n]$. Thus, there must not be any odd powers of z^{-1} in $H(z)$.

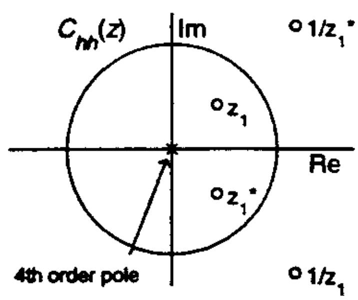
(d) For the conditions determined in part c, $g[n]$ is a downsampled (by a factor of 2) version of $h[n]$. That is,

$$g[n]=h[2n]$$

5.72. (a) *No.* You cannot uniquely recover $h[n]$ from $c_{hh}[l]$.

$$\begin{aligned} c_{hh}[l] &= h[l]*h[-l] \\ C_{hh}(e^{j\omega}) &= H(e^{j\omega})H(e^{-j\omega}) = |H(e^{j\omega})|^2 \\ C_{hh}(z) &= H(z)H^*(1/z^*) \end{aligned}$$

Causality and stability put restrictions on the poles of $H(z)$ (they must be inside the unit circle) but not its zeros. We know the zeros of $C_{hh}(z)$ in general occur in sets of 4. Here is why. A complex conjugate pair of zeros occur in $H(z)$ due to the fact that $h[n]$ is real. These 2 zeros and their conjugate reciprocals occur in $C_{hh}(z)$ due to the formula above for a total of 4. Thus, $H(z)$ is not uniquely determined since we do not know which 2 out of these 4 zeros to factor into $H(z)$. This is illustrated with a simple example below.



Let the above be the pole-zero diagram for $C_{hh}(z)$ and

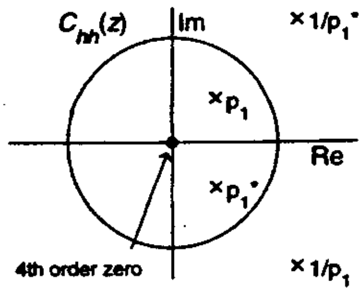
$$\begin{aligned} H_1(z) &= \left(1-z_1z^{-1}\right)\left(1-z_1^*z^{-1}\right) \\ H_2(z) &= \left(1-\frac{1}{z_1}z^{-1}\right)\left(1-\frac{1}{z_1^*}z^{-1}\right) \end{aligned}$$

Since

$$C_{hh}(z)=H_1(z)H_1^*(1/z^*)=H_2(z)H_2^*(1/z^*)$$

we cannot determine whether $h_1[n]$ or $h_2[n]$ generated $c_{hh}[l]$.

(b) *Yes.* The poles of $C_{hh}(z)$ must occur in sets of 4 for the same reasons outlined above for the zeros. However, since the poles of $h[n]$ must be inside the unit circle to be causal and stable we do not have any ambiguity in determining which poles to group into $h[n]$. We always choose the complex conjugate poles inside the unit circle. Here is an example



Let the above be the zero/pole diagram for $C_{hh}(z)$. Then, if $h[n]$ is to be real, causal, and stable $H(z)$ must equal

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

5.73. As shown in the book, the most general form of the system function of an allpass system with a real-valued impulse response is

$$H(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}, \quad |z| \in R_z$$

where R_z is the ROC which includes the unit circle. Correspondingly, the associated inverse system is

$$\begin{aligned}
H_i(z) &= \frac{1}{H(z)} \\
&= \prod_{k=1}^{M_r} \frac{1 - d_k z^{-1}}{z^{-1} - d_k} \prod_{k=1}^{M_c} \frac{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}{(z^{-1} - e_k^*)(z^{-1} - e_k)} \\
&= \prod_{k=1}^{M_r} \frac{z^{-1}(z - d_k)}{z^{-1} - d_k} \prod_{k=1}^{M_c} \frac{z^{-2}(z - e_k)(z - e_k^*)}{(z^{-1} - e_k^*)(z^{-1} - e_k)} \\
&= \prod_{k=1}^{M_r} \frac{z - d_k}{1 - d_k z} \prod_{k=1}^{M_c} \frac{(z - e_k)(z - e_k^*)}{(1 - e_k^* z)(1 - e_k z)} \\
&= H(1/z), \quad |z| \in \frac{1}{R_z}
\end{aligned}$$

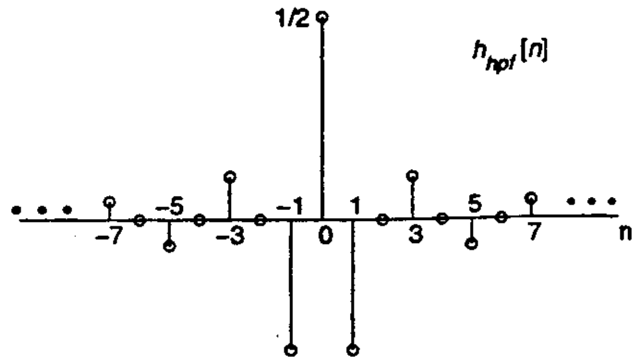
which in the time domain is

$$h_i[n] = h[-n]$$

5.74. We can model $g[n]$ as

$$g[n] = x[n] + \alpha\delta[n - n_0]$$

Now send the corrupted signal $g[n]$ through a highpass filter $h_{hpf}[n]$ with a cutoff of $w_c = \pi/2$.



$$h_{hpf}[n] = (-1)^n \frac{\sin \frac{\pi}{2} n}{\pi n}$$

The highpass filter completely filters out the lowpass signal $x[n]$. The output $y[n]$ is

$$\begin{aligned} y[n] &= (x[n] + \alpha\delta[n-n_0]) * h_{hpf}[n] \\ &= \alpha h_{hpf}[n-n_0] \\ &= \alpha(-1)^{(n-n_0)} \frac{\sin \frac{\pi}{2}(n-n_0)}{\pi(n-n_0)} \end{aligned}$$

$y[n]$ looks similar to the picture of $h_{hpf}[n]$ above except that it is scaled by α and shifted to n_0. Thus,

$$\alpha = 2y[n_0]$$

$$x[n] = g[n] - 2y[n_0]\delta[n-n_0]$$

(a) When n_0 is odd, $y[n] = 0$ at all odd values of n except $n = n_0$. This leads to a procedure to find $x[n]$ from $g[n]$:
   - Filter $g[n]$ with the highpass filter described above.
   - Find the only nonzero value at an odd index in the output $y[n]$. This value is $y[n_0]$.
   - $x[n] = g[n] - 2y[n_0]\delta[n-n_0]$

(b) The only time three consecutive nonzero samples occur in $y[n]$ is at $n = n_0$. The procedure to find $x[n]$ is
   - Filter $g[n]$ with the highpass filter described above.
   - Look for three consecutive nonzero output samples. The middle value is $y[n_0]$.
   - $x[n] = g[n] - 2y[n_0]\delta[n-n_0]$

5.75. Looking at the z-transform of the FIR filter,

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h[n]z^{-n} \\ &= \sum_{n=0}^{N-1} h[N-1-n]z^{-n} \end{aligned}$$

Substituting $m = N-1-n$ into the summation gives

$$H(z) = \sum_{m=N-1}^{0} h[m]z^{m-N+1}$$

$$
\begin{aligned}
&= \sum_{m=0}^{N-1} h[m] z^m z^{-N+1} \\
&= z^{-N+1} \sum_{m=0}^{N-1} h[m] z^m \\
&= z^{-N+1} H(z^{-1})
\end{aligned}
$$

Thus, for such a filter,

$$H(1/z) = z^{N-1} H(z)$$

If z_0 is a zero of $H(z)$, then $H(z_0) = 0$, and

$$H(1/z_0) = z_0^{N-1} H(z_0) = 0$$

Consequently, even-symmetric linear phase FIR filters have zeros that are reciprocal images.