Solutions – Chapter 10

Fourier Analysis of Signals

Using the Discrete Fourier Transform

10.1. (a) Using the relation

$$\Omega_k = \frac{2\pi k}{NT},$$

we find that the index $k = 150$ in $X[k]$ corresponds to a continuous time frequency of

$$\begin{aligned} \Omega_{150} &= \frac{2\pi(150)}{(1000)(10^{-4})} \\ &= 2\pi(1500) \text{ rad/s} \end{aligned}$$

(b) For this part, it is important to realize that the $k = 800$ index corresponds to a *negative* continuous-time frequency. Since the DFT is periodic in k with period N,

$$\begin{aligned} \Omega_{800} &= \frac{2\pi(800 - 1000)}{1000(10^{-4})} \\ &= -2\pi(2000) \text{ rad/s} \end{aligned}$$

10.2. Using the relation

$$\Omega_k = \frac{2\pi k}{NT}$$

or

$$f_k = \frac{k}{NT}$$

we find that the equivalent analog spacing between frequencies is

$$\Delta f = \frac{1}{NT}$$

Thus, in addition to the constraint that N is a power of 2, there are two conditions which must be met:

$$\begin{aligned} \tfrac{1}{T} &> 10,000 \text{ Hz} \quad \text{(to avoid aliasing)} \\ \tfrac{1}{NT} &< 5 \text{ Hz} \quad \text{(given)} \end{aligned}$$

These conditions can be expressed in the form

$$10,000 < \frac{1}{T} < 5N$$

The minimal $N = 2^\nu$ that satisfies the relationship is

$$N = 2048$$

for which

$$10,000 \text{ Hz} < \frac{1}{T} < 10,240 \text{ Hz}$$

Thus, $F_{\min} = 10,000$ Hz, and $F_{\max} = 10,240$ Hz.

10.3. (a) The length of a window is

$$\begin{aligned} L &= \left(16,000 \, \frac{\text{samples}}{\text{sec}}\right)(20 \times 10^{-3} \text{ sec}) \\ &= 320 \text{ samples} \end{aligned}$$

(b) The *frame rate* is the number of frames of data processed per second, or equivalently, the number of DFT computations done per second. Since the window is advanced 40 samples between computations of the DFT, the frame rate is

$$\begin{aligned} \text{frame rate} &= \left(16,000 \, \frac{\text{samples}}{\text{sec}}\right)\left(\frac{1 \text{ frame processed}}{40 \text{ samples}}\right) \\ &= 400 \, \frac{\text{frames}}{\text{sec}} \end{aligned}$$

(c) The most straightforward solution to this problem is to say that since the window length L is 320, we need $N \geq L$ in order to do the DFT. Therefore, a value of $N = 512$ meets the criteria of $N \geq L$, $N = 2^\nu$. However, since the windows overlap, we can find a smaller N.

Since the window advances 40 samples between computations, we really only need 40 *valid* samples for each DFT in order to reconstruct the original input signal. If we time alias the windowed data, we can use a smaller DFT length than the window length. With $N = 256$, 64 samples will be time aliased, and remaining 192 samples will be valid. However, with $N = 128$, all the samples will be aliased. Therefore, the minimum size of N is 256.

(d) Using the relation

$$\Delta f = \frac{1}{NT},$$

the frequency spacing for $N = 512$ is

$$\Delta f = \frac{16,000}{512} = 31.25 \text{ Hz}$$

and for $N = 256$ is

$$\Delta f = \frac{16,000}{256} = 62.5 \text{ Hz}$$

10.4. (a) Since $x[n]$ is real, $X[k]$ must be conjugate symmetric.

$$X[k] = X^*[((-k))_N]$$

We can use this conjugate symmetry property to find $X[k]$ for $k = 200$.

$$\begin{aligned} X[((-k))_N] &= X^*[k] \\ X[((-800))_{1000}] &= (1+j)^* \\ X[200] &= 1-j \end{aligned}$$

(b) Since an N-point DFT is periodic in k with period N, we know that

$$X[800] = 1 + j$$

implies that

$$X[-200] = 1 + j$$

Using the relation

$$\Omega_k = \frac{2\pi k}{NT},$$

we find

$$\begin{aligned} \Omega_{-200} &= \frac{-2\pi(200)}{(1000)(1/20,000)} \\ &= -2\pi(4000) \text{ rad/s} \\ \Omega_{200} &= \frac{2\pi(200)}{(1000)(1/20,000)} \\ &= 2\pi(4000) \text{ rad/s} \end{aligned}$$

Consequently,

$$\begin{aligned} X_c(j\,\Omega)\,|_{\Omega=-2\pi(4000)} &= \frac{1+j}{20,000} \\ X_c(j\,\Omega)\,|_{\Omega=2\pi(4000)} &= \frac{1-j}{20,000} \end{aligned}$$

Note that both expressions for $X_c(j\,\Omega)$ have been multiplied by the sampling period $T = 1/20,000$ because sampling the continuous-time signal $x_c(t)$ involves multiplication by $1/T$.

10.5. (a) After windowing, we have

$$\begin{aligned} x[n] &= \cos(\Omega_0 T n) \\ &= \frac{1}{2}\left[e^{j\Omega_0 T n} + e^{-j\Omega_0 T n}\right] \\ &= \frac{1}{2}\left[e^{j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n} + e^{-j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n}\right] \end{aligned}$$

for $n = 0, \ldots, N-1$ and $x[n] = 0$ outside this range. Using the DFT properties we get

$$X[k] = \frac{N}{2}\delta[((k - \tfrac{N\Omega_0 T}{2\pi}))_N] + \frac{N}{2}\delta[((k + \tfrac{N\Omega_0 T}{2\pi}))_N]$$

If we choose

$$T = \frac{2\pi}{N\Omega_0}k_0$$

then

$$X[k] = \frac{N}{2}\delta[k - k_0] + \frac{N}{2}\delta[k - (N - k_0)],$$

which is nonzero for $X[k_0]$ and $X[N - k_0]$, but zero everywhere else.

(b) *No*, the choice for T is not unique since we can choose the integer k_0.

10.6. Since $x[n]$ is real, $X[k]$ must be conjugate symmetric.

$$\begin{aligned} X[k] &= X^*[((-k))_N] \\ X[((-k))_N] &= X^*[k] \end{aligned}$$

Therefore,

$$\begin{aligned} X[((-900))_{1000}] &= (1)^* \\ X[100] &= 1 \\ X[((-420))_{1000}] &= (5)^* \\ X[580] &= 5 \end{aligned}$$

Note that the $k = 900$ and $k = 580$ correspond to negative frequencies of Ω. Since the DFT is periodic in k with period N, we use $k = 900 - 1000 = -100$ and $k = 580 - 1000 = -420$, respectively, in the equations below.

Starting with

$$\Omega_k = \frac{2\pi k}{NT},$$

we find

$$\begin{aligned} \Omega_{-100} &= \frac{2\pi(-100)}{(1000)(1/10,000)} \\ &= -2\pi(1000) \text{ rad/s} \\ \Omega_{100} &= \frac{2\pi(100)}{(1000)(1/10,000)} \\ &= 2\pi(1000) \text{ rad/s} \\ \Omega_{-420} &= \frac{2\pi(-420)}{(1000)(1/10,000)} \\ &= -2\pi(4200) \text{ rad/s} \\ \Omega_{420} &= \frac{2\pi(420)}{(1000)(1/10,000)} \\ &= 2\pi(4200) \text{ rad/s} \end{aligned}$$

Consequently,

$$\begin{aligned} X_c(j\,\Omega)\,|_{\Omega=-2\pi(1000)} &= \frac{1}{10,000} \\ X_c(j\,\Omega)\,|_{\Omega=2\pi(1000)} &= \frac{1}{10,000} \\ X_c(j\,\Omega)\,|_{\Omega=-2\pi(4200)} &= \frac{1}{2,000} \\ X_c(j\,\Omega)\,|_{\Omega=2\pi(4200)} &= \frac{1}{2,000} \end{aligned}$$

Note that all expressions for $X_c(j\,\Omega)$ have been multiplied by the sampling period $T = 1/10,000$ because sampling the continuous-time signal $x_c(t)$ involves multiplication by $1/T$.

10.7. The Hamming window's mainlobe is $\Delta\omega_{ml} = \frac{8\pi}{L-1}$ radians wide. We want

$$\begin{aligned} \Delta\omega_{ml} &\leq \frac{\pi}{100} \\ \frac{8\pi}{L-1} &\leq \frac{\pi}{100} \\ L &\geq 801 \end{aligned}$$

Because the window length is constrained to be a power of 2, we see that

$$L_{\text{min}} = 1024$$

10.8. *All windows expect the Blackman satisfy the criteria.* Using the table, and noting that the window length $N = M + 1$, we find

Rectangular:

$$\begin{aligned} \Delta w_{ml} &= \frac{4\pi}{M+1} \\ &= \frac{4\pi}{256} \\ &= \frac{\pi}{64} \leq \frac{\pi}{25} \text{ rad} \end{aligned}$$

The resolution of the rectangular window satisfies the criteria.

Bartlett, Hanning, Hamming:

$$\begin{aligned} \Delta w_{ml} &= \frac{8\pi}{M} \\ &= \frac{8\pi}{255} \\ &= \frac{\pi}{31.875} \leq \frac{\pi}{25} \text{ rad} \end{aligned}$$

The resolution of the Bartlett, Hanning, and Hamming windows satisfies the criteria.

Blackman:

$$\begin{aligned} \Delta w_{ml} &= \frac{12\pi}{M} \\ &= \frac{12\pi}{255} \\ &= \frac{\pi}{21.25} \not\leq \frac{\pi}{25} \text{ rad} \end{aligned}$$

The Blackman window does not have a frequency resolution of at least $\pi/25$ radians. Therefore, this window does not satisfy the criteria.

10.9. The rectangular window's mainlobe is

$$\Delta\omega_{ml} = \frac{4\pi}{L} = \frac{4\pi}{64}$$

radians wide. The difference in frequency between each cosine must be greater than this amount to be resolved. If they are not separated enough, the mainlobes from each cosine will overlap too much and only a single peak will be seen. The separation of the cosines for each signal is

$$\begin{aligned} \Delta\omega_1 &= \left|\frac{\pi}{4} - \frac{17\pi}{64}\right| = \frac{\pi}{64} \\ \Delta\omega_2 &= \left|\frac{\pi}{4} - \frac{21\pi}{64}\right| = \frac{5\pi}{64} \\ \Delta\omega_3 &= \left|\frac{\pi}{4} - \frac{21\pi}{64}\right| = \frac{5\pi}{64} \end{aligned}$$

Clearly, the cosines in $x_1[n]$ are too closely spaced in frequency to produce distinct peaks.

In $x_3[n]$, we have a small amplitude cosine which will be obscurred by the large sidelobes from the rectangular window. The peak will therefore not be visible.

The only signal from which we would expect to see two distinct peaks is $x_2[n]$.

10.10. The equivalent continuous-time frequency spacing is

$$\Delta f = \frac{1}{NT}$$

Thus, to satisfy the criterion that the frequency spacing between consecutive DFT samples is 1 Hz or less we must have

$$\begin{aligned} \Delta f &\leq 1 \\ \frac{1}{NT} &\leq 1 \\ T &\geq \frac{1}{N} \\ T &\geq \frac{1}{1024} \text{ sec} \end{aligned}$$

However, we must also satisfy the Sampling Theorem to avoid aliasing. We therefore have the addition restriction that,

$$\begin{aligned} \frac{1}{T} &\geq 200 \text{ Hz} \\ T &\leq \frac{1}{200} \text{ sec} \end{aligned}$$

Putting the two constraints together we find

$$\frac{1}{1024} \leq T \leq \frac{1}{200}$$

$$T_{\min} = \frac{1}{1024} \text{ sec}$$

10.11. The equivalent frequency spacing is

$$\Delta\Omega = \frac{2\pi}{NT} = \frac{2\pi}{(8192)(50\mu s)} = 15.34 \text{ rad/s}$$

or

$$\Delta f = \frac{\Delta\Omega}{2\pi} = 2.44 \text{ Hz}$$

10.12. The equivalent frequency spacing is

$$\Delta f = \frac{1}{NT}$$

Thus, the minimum DFT length N such that adjacent samples of $X[k]$ correspond to a frequency spacing of 5 Hz or less in the original continuous-time signal is

$$\begin{aligned} \Delta f &\leq 5 \\ \frac{1}{NT} &\leq 5 \\ N &\geq \frac{1}{5T} \\ &\geq \frac{8000}{5} \\ &\geq 1600 \text{ samples} \end{aligned}$$

10.13. Since $w[n]$ is the rectangular window and we are using $N = 36$ we have

$$\begin{aligned} X_r[k] &= \sum_{m=0}^{35} x[rR+m]e^{-j(2\pi/36)km} \\ &= \text{DFT}\{x[rR+n]\} \end{aligned}$$

Because $x[n]$ is zero outside the range $0 \leq n \leq 71$, $X_r[k]$ will be zero except when $r = 0$ or $r = 1$. When $r = 0$, the 36 points in the sum of the DFT only include the section

$$\cos(\pi n/6) = \frac{e^{j(\frac{2\pi}{36})3n} + e^{-j(\frac{2\pi}{36})3n}}{2}$$

of $x[n]$. Therefore, we can use the properties of the DFT to find

$$\begin{aligned} X_0[k] &= \tfrac{36}{2}\delta[((k-3))_{36}] + \tfrac{36}{2}\delta[((k+3))_{36}] \\ &= 18\delta[k-3] + 18\delta[k-33] \end{aligned}$$

When $r = 1$, the 36 points in the sum of the DFT only include the section

$$\cos(\pi n/2) = \frac{e^{j(\frac{2\pi}{36})9n} + e^{-j(\frac{2\pi}{36})9n}}{2}$$

of $x[n]$. Therefore, we can use the properties of the DFT to find

$$\begin{aligned} X_1[k] &= \tfrac{36}{2}\delta[((k-9))_{36}] + \tfrac{36}{2}\delta[((k+9))_{36}] \\ &= 18\delta[k-9] + 18\delta[k-27] \end{aligned}$$

Putting it all together we get

$$X_r[k] = \begin{cases} 18(\delta[k-3] + \delta[k-33]), & r = 0 \\ 18(\delta[k-9] + \delta[k-27]), & r = 1 \\ 0, & \text{otherwise} \end{cases}$$

10.14. *The signals* $x_2[n]$, $x_3[n]$, *and* $x_6[n]$ *could be* $x[n]$, as described below.

Looking at the figure, it is clear that there are two nonzero DFT coefficients at $k = 8$, and $k = 16$. These correspond to frequencies

$$\begin{aligned}\omega_1 &= \frac{(2\pi)(8)}{128}\\ &= \frac{\pi}{8}\text{ rad}\\ \omega_2 &= \frac{(2\pi)(16)}{128}\\ &= \frac{\pi}{4}\text{ rad}\end{aligned}$$

Also notice that the magnitude of the DFT coefficient at $k = 16$ is about 3 times that of the DFT coefficient at $k = 8$.

- $x_1[n]$: The second cosine term has a frequency of $.26\pi$ rad, which is neither $\pi/8$ rad or $\pi/4$ rad. Consequently, $x_1[n]$ is not consistent with the information shown in the figure.
- $x_2[n]$: This signal is consistent with the information shown in the figure. The peaks occur at the correct locations, and are scaled properly.
- $x_3[n]$: This signal is consistent with the information shown in the figure. The peaks occur at the correct locations, and are scaled properly.
- $x_4[n]$: This signal has a cosine term with frequency $\pi/16$ rad, which is neither $\pi/8$ rad or $\pi/4$ rad. Consequently, $x_4[n]$ is not consistent with the information shown in the figure.
- $x_5[n]$: This signal has sinusoids with the correct frequencies, but the scale factors on the two terms are not consistent with the information shown in the figure.
- $x_6[n]$: This signal is consistent with the information shown in the figure. Note that phase information is not represented in the DFT magnitude plot.

10.15. The instantaneous frequency of the chirp signal is

$$\omega_i[n] = \omega_0 + \lambda n$$

This describes a line with slope λ and intercept ω_0. Thus,

$$\lambda = \frac{\Delta y}{\Delta x} = \frac{(0.5\pi - 0.25\pi)}{(19000 - 0)} = 41.34 \times 10^{-6}\text{ rad}$$

$$\omega_0 = 0.25\pi\text{ rad}$$

10.16. Using

$$\Delta f = \frac{1}{NT}$$

and assuming no aliasing occured when the continuous-time signal was sampled, we find that the frequency spacing between spectral samples is

$$\begin{aligned}\Delta f &= \frac{1}{(1024)(1/10,000)}\\ &= 9.77\text{ Hz}\end{aligned}$$

or

$$\Delta\Omega = 2\pi\Delta f = 61.4\text{ rad/s}$$

10.17. We should choose *Method 2*.

Method 1: This doubles the number of samples we take of the frequency variable, but does not change the frequency resolution. The size of the main lobe from the window remains the same.

Method 2: *This improves the frequency resolution since the main lobe from the window gets smaller.*

Method 3: This increases the time resolution (the ability to distinguish events in time), but does not affect the frequency resolution.

Method 4: This will decrease the frequency resolution since the main lobe from the window increases. This is a strange thing to do since there are samples of x[n] that do not get used in the transform.

Method 5: This will only improve the resolution if we can ignore any problems due to sidelobe leakage. For example, changing to a rectangular window will improve our ability to resolve two equal amplitude sinusoids. In most cases, however, we need to worry about sidelobe levels. A large sidelobe might mask the presence of a low amplitude signal. Since we do not know ahead of time the nature of the signal we are trying to analyze, changing to a rectangular window may actually make things worse. Thus, in general, changing to a rectangular window will not necessarily increase the frequency resolution.

10.18. *No*, the peaks will not have the same height. The peaks in $V_2(e^{j\omega})$ will be larger than those in $V_1(e^{j\omega})$.

First, note that the Fourier transform of the rectangular window has a higher peak than that of the Hamming window. If this is not obvious, consider Figure 7.21, and recall that the Fourier transform of an L-point window $w[n]$, evaluated at DC ($\omega = 0$), is

$$W(e^{j0}) = \sum_{n=0}^{L-1} w[n]$$

Let the rectangular window be $w_R[n]$, and the Hamming window be $w_H[n]$. It is clear from the figure (where M = L+1) that

$$\sum_{n=0}^{L-1} w_R[n] > \sum_{n=0}^{L-1} w_H[n]$$

Therefore,

$$W_R(e^{j0}) > W_H(e^{j0})$$

Thus, the Fourier transform of the rectangular window has a higher peak than that of the Hamming window.

Now recall that the multiplication of two signals in the time domain corresponds to a periodic convolution in the frequency domain. So in the frequency domain, $V_1(e^{j\omega})$ is the convolution of two scaled impulses from the sinusoid, with the Fourier transform of the L-point Hamming window, $W_H(e^{j\omega})$. This results in two scaled copies of $W_H(e^{j\omega})$, centered at the frequencies of the sinusoid. Similarly, $V_2(e^{j\omega})$ consists of two scaled copies of $W_R(e^{j\omega})$, also centered at the frequencies of the sinusoid. The scale factor is the same in both cases, resulting from the Fourier transform of the sinusoid.

Since the peaks of the Fourier transform of the rectangular window are higher than those of the Hamming window, the peaks in $V_2(e^{j\omega})$ will be larger than those in $V_1(e^{j\omega})$.

10.19. Using the approximation given in the chapter

$$L \simeq \frac{24\pi(A_{sl} + 12)}{155\Delta_{ml}} + 1$$

we find for $A_{sl} = 30$ dB and $\Delta_{ml} = \frac{\pi}{40}$ rad,

$$\begin{aligned} L &\simeq \frac{24\pi(30 + 12)}{155(\pi/40)} + 1 \\ &\simeq 261.1 \rightarrow 262 \end{aligned}$$

10.20. (a) The best sidelobe attenuation expected under these constraints is

$$\begin{aligned} L &\simeq \frac{24\pi(A_{sl}+12)}{155\Delta_{ml}}+1 \\ 512 &\simeq \frac{24\pi(A_{sl}+12)}{155(\pi/100)}+1 \\ A_{sl} &\simeq 21 \text{ dB} \end{aligned}$$

(b) The two sinusoidal components are separated by at least $\pi/50$ radians. Since the largest allowable mainlobe width is $\pi/100$ radians, we know that the peak of the DFT magnitude of the weaker sinusoidal component will not be located in the mainlobe of the DFT magnitude of the stronger sinusoidal component. Thus, we only need to consider the sidelobe height of the stronger component.

Converting 21 dB attenuation back from dB gives

$$\begin{aligned} -21 \text{ dB} &= 20\log_{10} m \\ m &= 0.0891 \end{aligned}$$

Since the amplitude of the stronger sinusoidal component is 1, the amplitude of the weaker sinusoidal component must be greater than 0.0891 in order for the weaker sinusoidal component to be seen over the sidelobe of the stronger sinusoidal component.

10.21. We have

$$\begin{aligned} v[n] &= \cos(2\pi n/5)w[n] \\ &= \left[\frac{e^{j2\pi n/5}+e^{-j2\pi n/5}}{2}\right]w[n] \\ V(e^{j\omega}) &= \frac{1}{2}W(e^{j(\omega-2\pi/5)})+\frac{1}{2}W(e^{j(\omega+2\pi/5)}) \end{aligned}$$

The rectangular window's transform is

$$W(e^{j\omega}) = \frac{\sin(16\omega)}{\sin(\omega/2)}e^{-j\omega 31/2}$$

In order to label $V(e^{j\omega})$ correctly, we must find the mainlobe height, strongest sidelobe height, and the first nulls of $W(e^{j\omega})$.

Mainlobe Height of $W(e^{j\omega})$: The peak height is at $\omega = 0$ for which we can use l'hôpital's rule to find

$$W(e^{j0}) = 32\left.\frac{\cos(16\omega)}{\cos(\omega/2)}\right|_{\omega=0} = 32$$

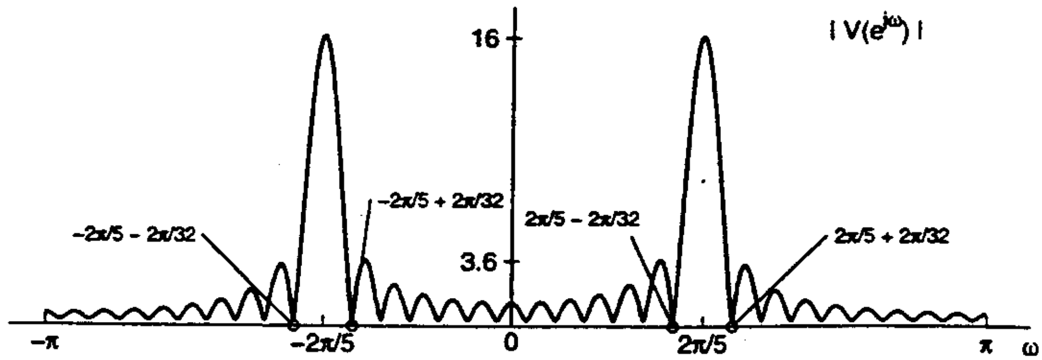
Strongest Sidelobe height of $W(e^{j\omega})$: The strongest sidelobe height for the rectangular window is 13 dB below the main peak height. Therefore, since 13 dB = 0.2239 we have

$$\text{Strongest Sidelobe height} = 0.2239(32) \approx 7.2$$

First Nulls of $W(e^{j\omega})$: The first nulls can be found be noting that $W(e^{j\omega}) = 0$ when $\sin(16\omega) = 0$ Thus, the first nulls occur at

$$\omega = \pm\frac{2\pi}{32}$$

Therefore, $|V(e^{j\omega})|$ looks like

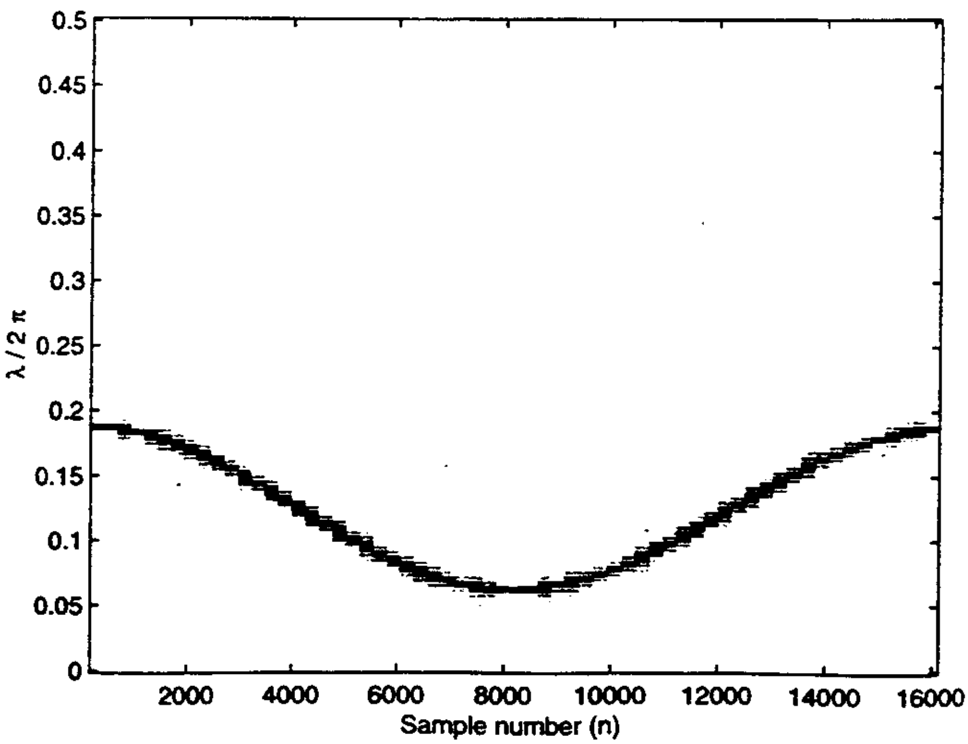


Note that the numbers used above for the heights are not exact because we are adding two copies of $W(e^{j\omega})$ to get $V(e^{j\omega})$ and the exact values for the heights will depend on relative phase and location of the two copies. However, they are a very good approximation and the error is small.

10.22. The 'instantaneous frequency' of $x[n]$, denoted as $\lambda[n]$, can be determined by taking the derivative with respect to n of the argument of the cosine term. This gives

$$\begin{aligned}
\lambda[n] &= \frac{d}{dn}\left[\frac{\pi n}{4} + 1000\sin\left(\frac{\pi n}{8000}\right)\right] \\
&= \frac{\pi}{4} + \frac{\pi}{8}\cos\left(\frac{\pi n}{8000}\right) \\
\frac{\lambda[n]}{2\pi} &= \frac{1}{8} + \frac{1}{16}\cos\left(\frac{\pi n}{8000}\right)
\end{aligned}$$

Once $\lambda[n]/2\pi$ is known, it is simple to sketch the spectrogram, shown below.



Here, we see a cosine plot shifted up the frequency $(\lambda/2\pi)$ axis by a constant. As is customary in a spectrogram, only the frequencies $0 \le \lambda/2\pi \le 0.5$ are plotted.

10.23. In this problem, we relate the DFT $X[k]$ of a discrete-time signal $x[n]$ to the continuous-time Fourier transform $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$. Since $x[n]$ is obtained by sampling $x_c(t)$,

$$\begin{aligned} x[n] &= x_c(nT) \\ X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right) \end{aligned}$$

Over one period, assuming no aliasing, this is

$$X(e^{j\omega}) = \frac{1}{T}X_c\left(j\frac{\omega}{T}\right) \qquad \text{for } -\pi \le \omega \le \pi$$

which is equivalent to

$$X(e^{j\omega}) = \begin{cases} \frac{1}{T}X_c\left(j\frac{\omega}{T}\right), & \text{for } 0 \le \omega < \pi \\ \frac{1}{T}X_c\left(j\frac{\omega-2\pi}{T}\right), & \text{for } \pi \le \omega < 2\pi \end{cases}$$

Since the DFT is a sampled version of $X(e^{j\omega})$,

$$X[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N} \qquad \text{for } 0 \le k \le N-1$$

we find

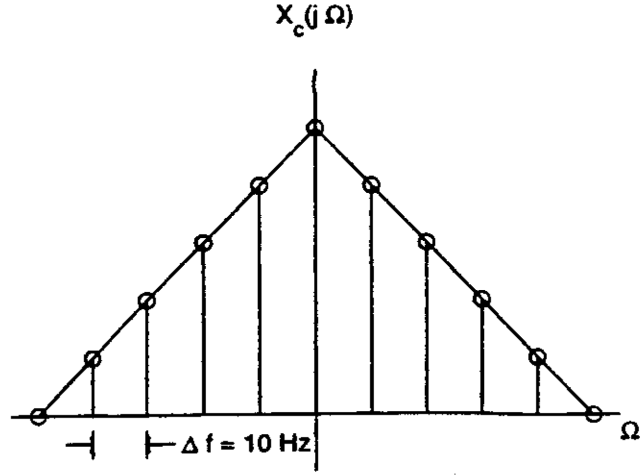
$$X[k] = \begin{cases} \frac{1}{T}X_c\left(j\frac{2\pi k}{NT}\right), & \text{for } 0 \le k < \frac{N}{2} \\ \frac{1}{T}X_c\left(j\frac{2\pi(k-N)}{NT}\right), & \text{for } \frac{N}{2} \le k \le N-1 \end{cases}$$

Breaking up the DFT into two terms like this is necessary to relate the negative frequencies of $X_c(j\Omega)$ to the proper indicies $\frac{N}{2} \le k \le N-1$ in $X[k]$.

Method 1: Using the above equation for $X[k]$, and plugging in values of $N = 4000$, and $T = 25\mu s$, we find

$$X_1[k] = \begin{cases} 40,000X_c(j2\pi \cdot 10 \cdot k), & \text{for } 0 \le k \le 1999 \\ 40,000X_c(j2\pi \cdot 10 \cdot (k-4000)), & \text{for } 2000 \le k \le 3999 \end{cases}$$

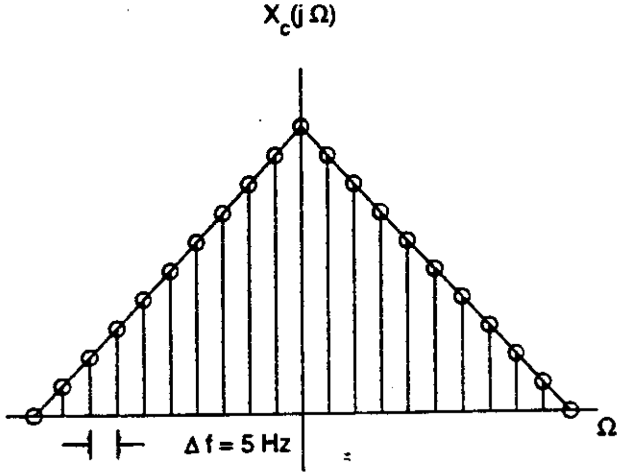
Therefore, we see this does not provide the desired samples. A sketch is provided below, for a triangular-shaped $X_c(j\Omega)$.



Method 2: This time we plug in values of $N = 4000$, and $T = 50\mu s$ to find

$$X_2[k] = \begin{cases} 20,000 X_c\left(j2\pi \cdot 5 \cdot k\right), & \text{for } 0 \leq k \leq 1999 \\ 20,000 X_c\left(j2\pi \cdot 5 \cdot (k-4000)\right), & \text{for } 2000 \leq k \leq 3999 \end{cases}$$

Therefore, we see this *does* provide the desired samples. A sketch is provided below.



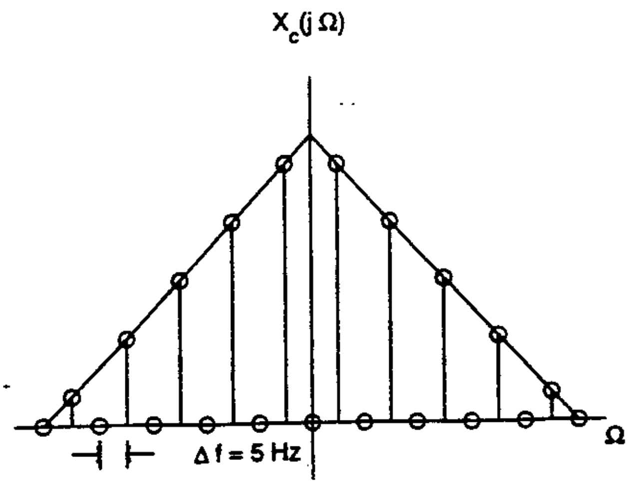
Method 3: Noting that $x_3[n] = x_2[n] + x_2\left[n - \frac{N}{2}\right]$, we get

$$X_3[k] = X_2[k] + (-1)^k X_2[k]$$

$$X_3[k] = \begin{cases} 2X_2[k], & \text{for } k \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

$$X_3[k] = \begin{cases} 40,000 X_c\left(j2\pi \cdot 5 \cdot k\right), & \text{for } k \text{ even, and } 0 \leq k \leq 1999 \\ 40,000 X_c\left(j2\pi \cdot 5 \cdot (k-4000)\right), & \text{for } k \text{ even, and } 2000 \leq k \leq 3999 \\ 0, & \text{otherwise} \end{cases}$$

This system provides the desired samples only for k an even integer. A sketch is provided below.



10.24. (a) In this problem, we relate the DFT $X[k]$ of a discrete-time signal $x[n]$ to the continuous-time Fourier tranform $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$. Since $x[n]$ is obtained by sampling $x_c(t)$,

$$\begin{aligned} x[n] &= x_c(nT) \\ X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right) \end{aligned}$$

Over one period, assuming no aliasing, this is

$$X(e^{j\omega}) = \frac{1}{T}X_c\left(j\frac{\omega}{T}\right) \qquad \text{for } -\pi \le \omega \le \pi$$

which is equivalent to

$$X(e^{j\omega}) = \begin{cases} \frac{1}{T}X_c\left(j\frac{\omega}{T}\right), & \text{for } 0 \le \omega < \pi \\ \frac{1}{T}X_c\left(j\frac{\omega-2\pi}{T}\right), & \text{for } \pi \le \omega < 2\pi \end{cases}$$

Since the DFT is a sampled version of $X(e^{j\omega})$,

$$X[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N} \qquad \text{for } 0 \le k \le N-1$$

we find

$$X[k] = \begin{cases} \frac{1}{T}X_c\left(j\frac{2\pi k}{NT}\right), & \text{for } 0 \le k < \frac{N}{2} \\ \frac{1}{T}X_c\left(j\frac{2\pi(k-N)}{NT}\right), & \text{for } \frac{N}{2} \le k \le N-1 \end{cases}$$

Breaking up the DFT into two terms like this is necessary to relate the negative frequencies of $X_c(j\Omega)$ to the proper indicies $\frac{N}{2} \le k \le N-1$ in $X[k]$.

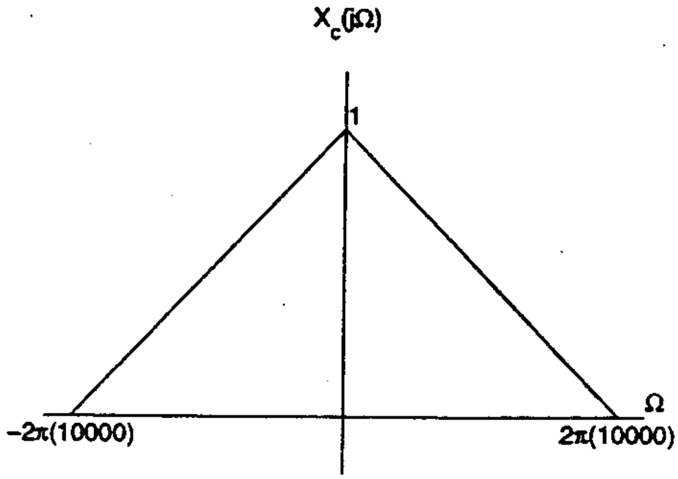
The effective frequency spacing is

$$\begin{aligned} \Delta\Omega &= \frac{2\pi}{NT} \\ &= \frac{2\pi}{(1000)(1/20,000)} \\ &= 2\pi(20) \text{ rad/s} \end{aligned}$$

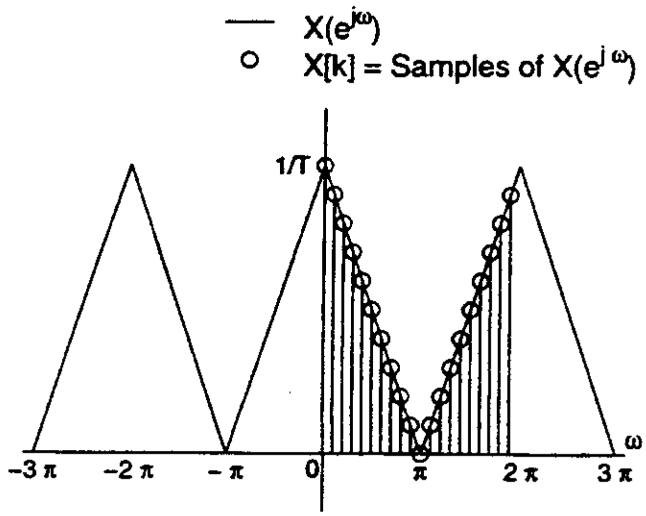
(b) Next, we determine if the designer's assertion that

$$Y[k] = \alpha X_c(j2\pi \cdot 10 \cdot k)$$

is correct. To understand the effect of each step in the procedure, it helps to draw some frequency domain plots. Assume the spectrum of the original signal $x_c(t)$ looks like



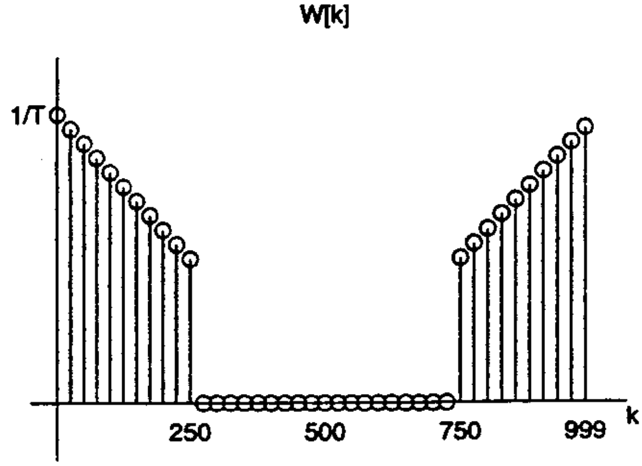
Sampling this continuous-time signal will produce the discrete-time signal $x[n]$, with a spectrum



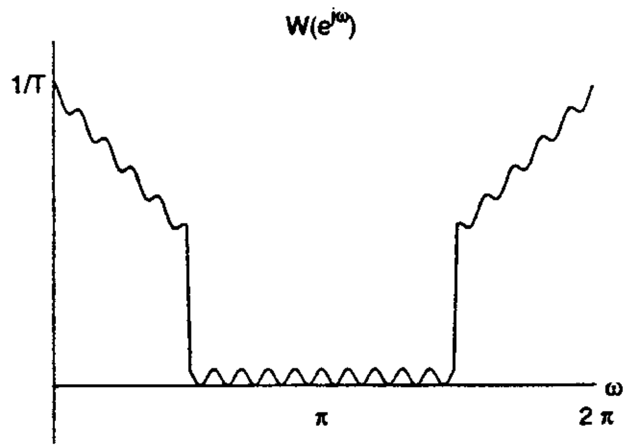
Next, we form

$$W[k] = \begin{cases} X[k], & 0 \leq k \leq 250 \\ 0, & 251 \leq k \leq 749 \\ X[k], & 750 \leq k \leq 999 \end{cases}$$

and find $w[n]$ as the inverse DFT of $W[k]$.



Before going on, we should plot the Fourier transform, $W(e^{j\omega})$, of $w[n]$. It will look like



$W(e^{j\omega})$ goes through the DFT points and therefore is equal to samples of $X_c(j\Omega)$ at these points for $0 \le k \le 250$ and $750 \le k \le 999$, but it is *not equal* to $X_c(j\Omega)$ between those frequencies. Furthermore, $W(e^{j\omega}) = 0$ at the DFT frequencies for $251 \le k \le 749$, but it is not zero between those frequencies; i.e. we can not do ideal lowpass filtering using the DFT.

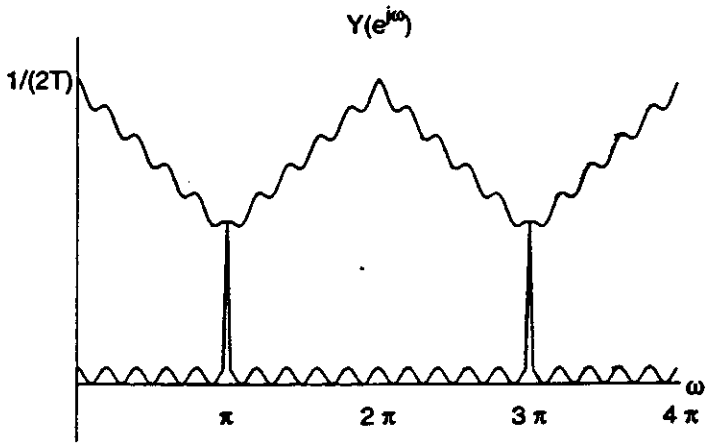
Now we define

$$y[n] = \begin{cases} w[2n], & 0 \le n \le 499 \\ 0, & 500 \le n \le 999 \end{cases}$$

and let $Y[k]$ be the DFT of $y[n]$. First note that $Y(e^{j\omega})$ is

$$Y(e^{j\omega}) = \frac{1}{2}W(e^{j\omega/2}) + \frac{1}{2}W(e^{j(\omega-2\pi)/2})$$

which looks like



$Y[k]$ is equal to samples of the $Y(e^{j\omega})$

$$\begin{aligned} Y[k] &= Y(e^{j\omega})|_{\omega=2\pi k/N} \\ &= \frac{1}{2}W\left(e^{j\frac{2\pi}{N}\frac{k}{2}}\right) + \frac{1}{2}W\left(e^{j\frac{2\pi}{N}\left(\frac{k-N}{2}\right)}\right) \end{aligned}$$

Now putting all that we know together, we see that for $k = 0, 1, \ldots, 500$, $Y[k]$ is related to $X_c(j\Omega)$ as follows.

$$Y[k] = \begin{cases} \frac{1}{2T}X_c(j2\pi \cdot 10 \cdot k), & k \text{ even, } k \neq 500 \\ \frac{1}{T}X_c(j2\pi \cdot 10 \cdot k), & k = 500 \\ \frac{1}{2T}W(e^{j\pi k/N}) + \frac{1}{2T}W(e^{j\pi(k-N)/N}) & k \text{ odd} \end{cases}$$

In other words, the even indexed DFT samples are not aliased, but the odd indexed values (and $k = 500$) are aliased. The designer's assertion is not correct.

10.25. (a) Starting with definition of the time-dependent Fourier transform,

$$Y[n,\lambda) = \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m}$$

we plug in

$$y[n+m] = \sum_{k=0}^{M} h[k]x[n+m-k]$$

to get

$$\begin{aligned} Y[n,\lambda) &= \sum_{m=-\infty}^{\infty}\sum_{k=0}^{M} h[k]x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k] \sum_{m=-\infty}^{\infty} x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k]X[n-k,\lambda) \\ &= h[n] * X[n,\lambda) \end{aligned}$$

where the convolution is for the variable n.

(b) Starting with

$$\check{Y}[n,\lambda) = e^{-j\lambda n} Y[n,\lambda)$$

we find

$$\begin{aligned}
\check{Y}[n,\lambda) &= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]X[n-k,\lambda)\right] \\
&= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]e^{j(n-k)\lambda}\check{X}[n-k,\lambda)\right] \\
&= \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n-k,\lambda)
\end{aligned}$$

If the window is long compared to M, then a small time shift in $\check{X}[n,\lambda)$ won't radically alter the spectrum, and

$$\check{X}[n-k,\lambda) \simeq \check{X}[n,\lambda)$$

Consequently,

$$\begin{aligned}
\check{Y}[n,\lambda) &\simeq \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n,\lambda) \\
&\simeq H(e^{j\lambda})\check{X}[n,\lambda)
\end{aligned}$$

10.26. Plugging in the relation for $c_{vv}[m]$ into the equation for $I(\omega)$ gives

$$\begin{aligned}
I(\omega) &= \frac{1}{LU}\sum_{m=-(L-1)}^{L-1}\left[\sum_{n=0}^{L-1} v[n]v[n+m]\right]e^{-j\omega m} \\
&= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]\sum_{m=-(L-1)}^{L-1} v[n+m]e^{-j\omega m}
\end{aligned}$$

Let $\ell = n+m$ in the second summation. This gives

$$\begin{aligned}
I(\omega) &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]\sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega(\ell-n)} \\
&= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]e^{j\omega n}\sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega \ell}
\end{aligned}$$

Note that for all values of $0 \leq n \leq L-1$, the second summation will be over *all* non-zero values of $v[\ell]$ in the range $0 \leq \ell \leq L-1$. As a result,

$$\begin{aligned}
I(\omega) &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]e^{j\omega n}\sum_{\ell=0}^{L-1} v[\ell]e^{-j\omega \ell} \\
&= \frac{1}{LU}V^*(e^{j\omega})V(e^{j\omega}) \\
&= \frac{1}{LU}|V(e^{j\omega})|^2
\end{aligned}$$

Note that in this analysis, we have assumed that $v[n]$ is a real sequence.

10.27. (a) Since $x[n]$ has length L, the aperiodic function, $c_{xx}[m]$, will be $2L-1$ points long. Therefore, in order for the aperiodic correlation function to equal the periodic correlation fuction, $\tilde{c}_{xx}[m]$, for $0 \leq m \leq L-1$, we require that the inverse DFT is not time aliased. So, the minimum inverse DFT length N_{min} is

$$N_{\text{min}} = 2L-1$$

(b) If we require M points to be unaliased, we can have $L-M$ aliased points. Therefore, for $\tilde{c}_{xx}[m] = c_{xx}[m]$ for $0 \leq m \leq M-1$, the minimum inverse DFT length N_{min} is

$$\begin{aligned} N_{\text{min}} &= 2L-1-(L-M) \\ &= L+M-1 \end{aligned}$$

10.28. (a) Let

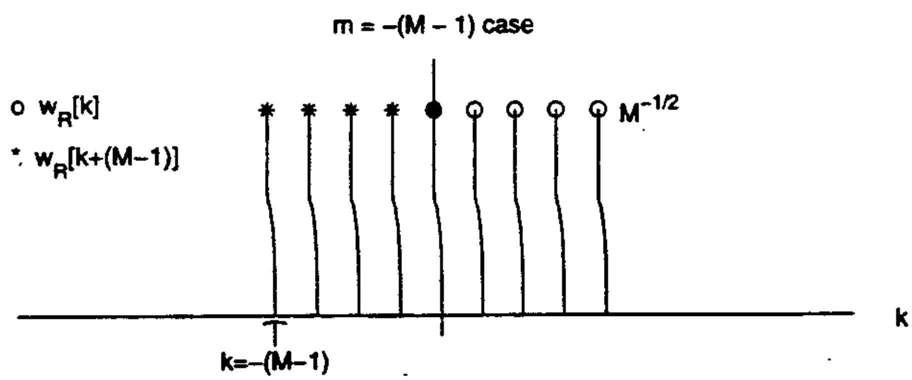
$$w_R[m] = \frac{1}{\sqrt{M}}(u[n]-u[n-M])$$

be a scaled rectangular pulse. Then we can write the aperiodic autocorrelation as,

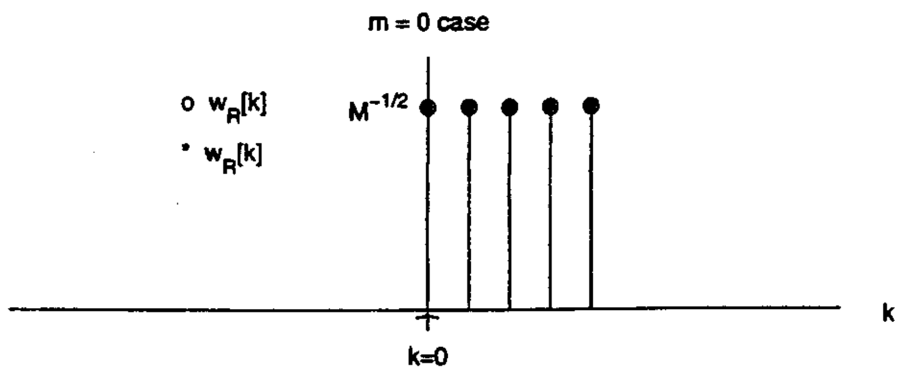
$$\begin{aligned} w_B[m] &= \sum_{n=-\infty}^{\infty} w_R[n]w_R[n+m] \\ &= \sum_{k=-\infty}^{\infty} w_R[k-m]w_R[k] \\ &= \sum_{k=-\infty}^{\infty} w_R[k]w_R[-(m-k)] \\ &= w_R[m] * w_R[-m] \end{aligned}$$

The convolution above is the triangular signal described by the symmetric Bartlett window formula. This is shown graphically below for a few critical cases of m.

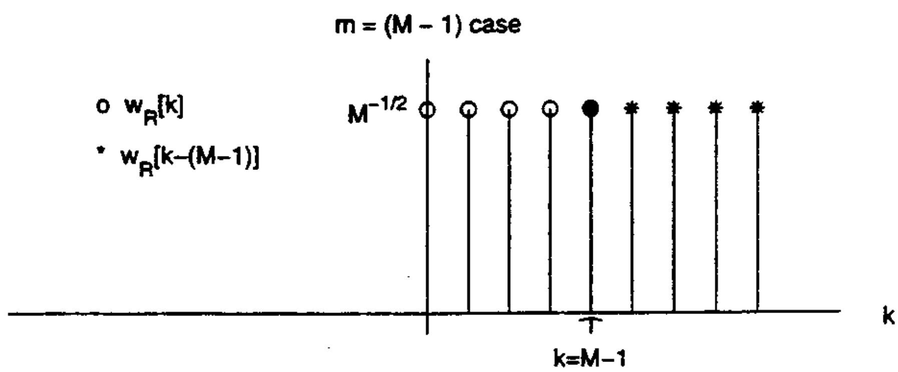
Consider $m = -(M-1)$. This is first value of m for which the two signals overlap.



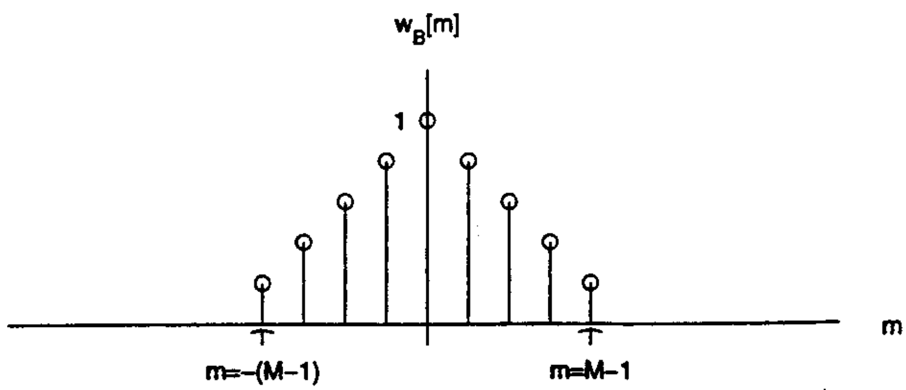
At $m = 0$, all non-zero samples overlap.



Consider $m = (M-1)$. This is last value of m for which the two signals overlap.



The final result of the aperiodic autocorrelation is



Stated mathematically, this is

$$w_B[m] = \begin{cases} 1 - |m|/M, & |m| \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

(b) The transform of the causal scaled rectangular pulse $w_R[n]$ is

$$W_R(e^{j\omega}) = \frac{1}{\sqrt{M}} \frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j\omega(M-1)/2}$$

From part (a), we know that the Bartlett window can be found by convolving $w_R[m]$ with $w_R[-m]$. In the frequency domain, we therefore have,

$$\begin{aligned} W_B(e^{j\omega}) &= W_R(e^{j\omega})W_R(e^{-j\omega}) \\ &= \left[\frac{1}{\sqrt{M}}\frac{\sin(\omega M/2)}{\sin(\omega/2)}e^{-j\omega(M-1)/2}\right]\left[\frac{1}{\sqrt{M}}\frac{\sin(-\omega M/2)}{\sin(-\omega/2)}e^{j\omega(M-1)/2}\right] \\ &= \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

(c) The power spectrum, defined as the Fourier transform of the aperiodic autocorrelation sequence, is always nonnegative. Thus, any window that can be represented as an aperiodic autocorrelation sequence will have a nonnegative Fourier transform. So to generate other finite-length window sequences, $w[n]$, that have nonnegative Fourier transforms, simply take the aperiodic autocorrelation of an input sequence, $x[n]$.

$$w[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m]$$

The signal $w[n]$ will have a nonnegative Fourier transform.

10.29. (a) **Rectangular:** The Fourier transform of the rectangular window is given by

$$W_R(e^{j\omega}) = \sum_{m=-(M-1)}^{M-1} (1)e^{-j\omega m}$$

Let $n = m + (M-1)$. Then, $m = n - (M-1)$, and

$$\begin{aligned} W_R(e^{j\omega}) &= \sum_{n=0}^{2(M-1)} e^{-j\omega[n-(M-1)]} \\ &= e^{j\omega(M-1)} \sum_{n=0}^{2(M-1)} e^{-j\omega n} \end{aligned}$$

Using the relation

$$\sum_{n=0}^{M-1} a^n = \frac{1-a^M}{1-a}$$

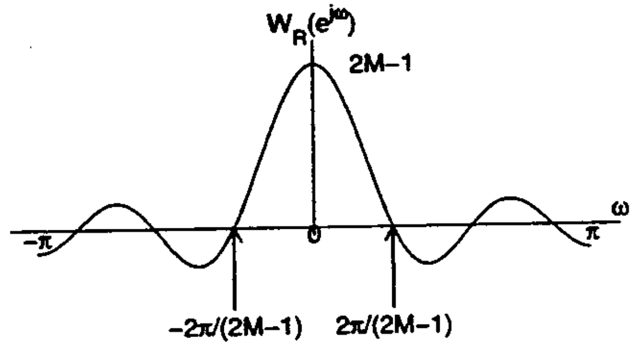
we find

$$\begin{aligned} W_R(e^{j\omega}) &= e^{j\omega(M-1)}\frac{1-e^{-j\omega[2(M-1)+1]}}{1-e^{-j\omega}} \\ &= e^{j\omega(M-1)}\frac{1-e^{-j\omega(2M-1)}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega(M-1)}-e^{-j\omega M}}{1-e^{-j\omega}} \\ &= \frac{e^{-j\omega/2}[e^{j\omega(M-1/2)}-e^{-j\omega(M-1/2)}]}{e^{-j\omega/2}[e^{j\omega/2}-e^{-j\omega/2}]} \\ &= \frac{e^{j\omega(M-1/2)}-e^{-j\omega(M-1/2)}}{e^{j\omega/2}-e^{-j\omega/2}} \\ &= \frac{2j\sin[\omega(M-\frac{1}{2})]}{2j\sin(\omega/2)} \\ &= \frac{\sin[\omega(M-\frac{1}{2})]}{\sin(\omega/2)} \end{aligned}$$

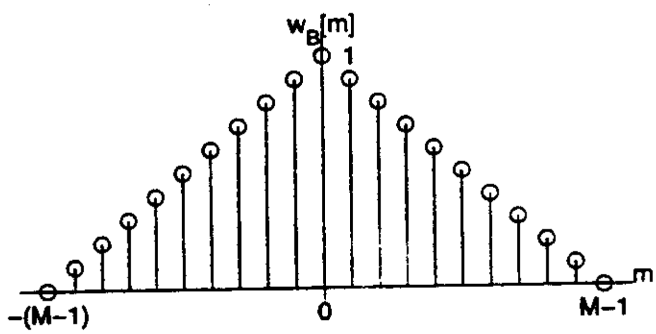
or

$$W_R(e^{j\omega}) = \frac{\sin[\omega\frac{2M-1}{2}]}{\sin(\omega/2)}$$

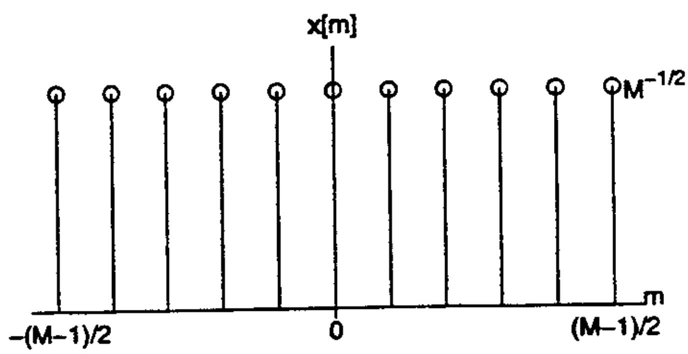
where $2M-1$ is the window length. A sketch of $W_R(e^{j\omega})$ appears below.



Bartlett (triangular): $W_B(e^{j\omega})$ is the Fourier transform of a triangular signal,



which is the convolution of a rectangular signal,



with itself. That is, $w_B[m] = x[m] * x[m]$.
Above, we found the Fourier tranform of a rectangular window, as

$$W_R(e^{j\omega}) = \frac{\sin[\omega\frac{2M-1}{2}]}{\sin(\omega/2)}$$

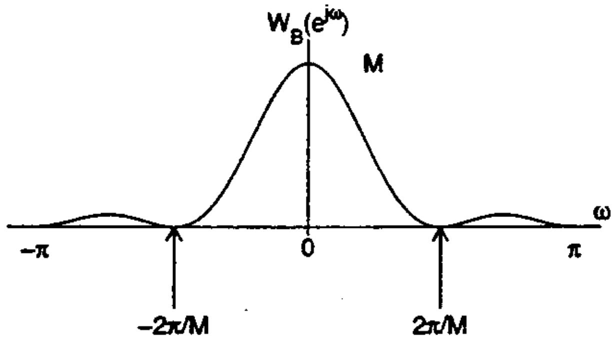
where $2M-1$ was the length of the window. We can use this result to find the Fourier transform of $x[m]$. The signal $x[m]$ is similar to the rectangular window, the difference being that it is scaled by $\frac{1}{\sqrt{M}}$ and has a length $2\frac{M-1}{2}+1=M$. Therefore,

$$X(e^{j\omega}) = \frac{1}{\sqrt{M}}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

The time domain convolution, $w_B[m] = x[m] * x[m]$ corresponds to a multiplication, $W_B(e^{j\omega}) = [X(e^{j\omega})]^2$ in the frequency domain. As a result,

$$\begin{aligned} W_B(e^{j\omega}) &= \left[X(e^{j\omega})\right]^2 \\ &= \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

A sketch of $W_B(e^{j\omega})$ appears below.



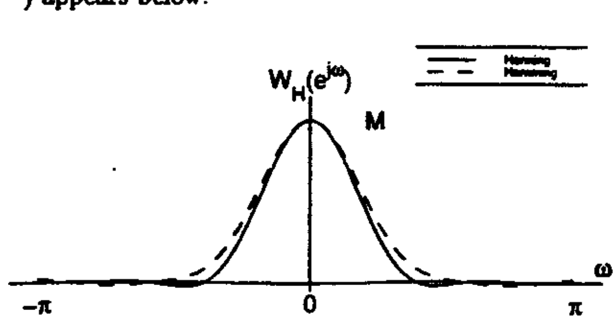
Hanning/Hamming: Starting with

$$\begin{aligned} w_H[m] &= (\alpha + \beta\cos[\pi m/(M-1)])\, w_R[m] \\ w_H[m] &= \left(\alpha + \frac{\beta}{2}e^{j\pi m/(M-1)} + \frac{\beta}{2}e^{-j\omega m/(M-1)}\right) w_R[m] \end{aligned}$$

We take the Fourier transform to find

$$\begin{aligned} W_H(e^{j\omega}) &= \alpha W_R(e^{j\omega}) + \frac{\beta}{2}\left(W_R(e^{j[\omega-\pi/(M-1)]}) + W_R(e^{j[\omega+\pi/(M-1)]})\right) \\ &= \alpha\frac{\sin[\omega\,(M-\frac{1}{2})]}{\sin(\omega/2)} + \frac{\beta}{2}\left[\frac{\sin[(\omega-\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega-\frac{\pi}{M-1})/2]}\right] \\ &+ \frac{\beta}{2}\left[\frac{\sin[(\omega+\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega+\frac{\pi}{M-1})/2]}\right] \end{aligned}$$

A sketch of $W_H(e^{j\omega})$ appears below.



(b) **Rectangular:** The approximate mainlobe width, and the approximate variance ratio, F, for the rectangular window are found below for large M.

In part (a), we found the Fourier transform of the rectangular window as

$$W_R(e^{j\omega}) = \frac{\sin[\omega(M-\frac{1}{2})]}{\sin(\omega/2)}$$

The numerator becomes zero when the argument of its sine term equals πn.

$$\begin{aligned}\frac{(2M-1)\omega}{2} &= \pi n\\ \omega &= \frac{2\pi n}{2M-1}\end{aligned}$$

Plugging in $n = 1$ gives us half the mainlobe bandwidth.

$$\begin{aligned}\frac{1}{2}\text{Mainlobe bandwidth} &= \frac{2\pi}{2M-1}\\ \text{Mainlobe bandwidth} &= \frac{4\pi}{2M-1}\\ \text{Mainlobe bandwidth} &\simeq \frac{2\pi}{M}\end{aligned}$$

$$\begin{aligned}\text{F} &= \frac{1}{Q}\sum_{m=-(M-1)}^{(M-1)} w^2[m]\\ &= \frac{1}{Q}(2M-1)\\ &\simeq \frac{2M}{Q}\end{aligned}$$

Bartlett (triangular): The approximate mainlobe width, and the approximate variance ratio, F, for the Bartlett window are found below for large M.

In part (a), we found the Fourier transform of the Bartlett window as

$$W_B(e^{j\omega}) = \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2$$

The numerator becomes zero when the argument of its sine term equals πn.

$$\begin{aligned}\frac{\omega M}{2} &= \pi n\\ \omega &= \frac{2\pi n}{M}\end{aligned}$$

Plugging in $n = 1$ gives us half the mainlobe bandwidth.

$$\begin{aligned}\frac{1}{2}\text{Mainlobe bandwidth} &= \frac{2\pi}{M}\\ \text{Mainlobe bandwidth} &= \frac{4\pi}{M}\end{aligned}$$

To compute F, we use the relations

$$\begin{aligned}\sum_{m=0}^{M-1} m &= \frac{M(M-1)}{2}\\ \sum_{m=0}^{M-1} m^2 &= \frac{M(M-1)(2M-1)}{6}\end{aligned}$$

$$
\begin{aligned}
\mathrm{F} &= \frac{1}{Q}\sum_{m=-(M-1)}^{(M-1)}\left(1-\frac{|m|}{M}\right)^2 \\
&= \frac{1}{Q}\left[2\sum_{m=0}^{M-1}\left(1-\frac{m}{M}\right)^2-1\right] \\
&= \frac{1}{Q}\left[2\sum_{m=0}^{M-1}1-\frac{4}{M}\sum_{m=0}^{M-1}m+\frac{2}{M^2}\sum_{m=0}^{M-1}m^2-1\right] \\
&= \frac{1}{Q}\left[2M-\frac{4(M-1)M}{2M}+\frac{2(M-1)M(2M-1)}{6M^2}-1\right] \\
&\simeq \frac{1}{Q}\left[2M-2M+\frac{2M}{3}\right] \\
&\simeq \frac{2M}{3Q}
\end{aligned}
$$

Hanning/Hamming: We can approximate the mainlobe bandwidth by analyzing the Fourier transform derived in Part (a). Looking at one of the terms from this expression,

$$\frac{\beta}{2}\left[\frac{\sin[(\omega-\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega-\frac{\pi}{M-1})/2]}\right]$$

we note that the numerator is zero whenever the its argument equals πn, or

$$
\begin{aligned}
\left(\omega-\frac{\pi}{M-1}\right)\left(M-\frac{1}{2}\right) &= \pi n \\
\omega &= \frac{n\pi}{M-(1/2)}+\frac{\pi}{M-1} \\
&\simeq \frac{n\pi}{M}+\frac{\pi}{M} \\
&\simeq \frac{\pi(n+1)}{M}
\end{aligned}
$$

So the mainlobe bandwidth for this term is

$$
\begin{aligned}
\frac{1}{2}\text{Mainlobe bandwidth} &\simeq \frac{\pi}{M} \\
\text{Mainlobe bandwidth} &\simeq \frac{2\pi}{M}
\end{aligned}
$$

Note that the peak value for this term occurs at a frequency $\omega \simeq \pi/M$.

A similar analysis can be applied to the other terms in Fourier transform derived in Part (a). The mainlobe bandwidth for the term

$$\frac{\beta}{2}\left[\frac{\sin[(\omega+\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega+\frac{\pi}{M-1})/2]}\right]$$

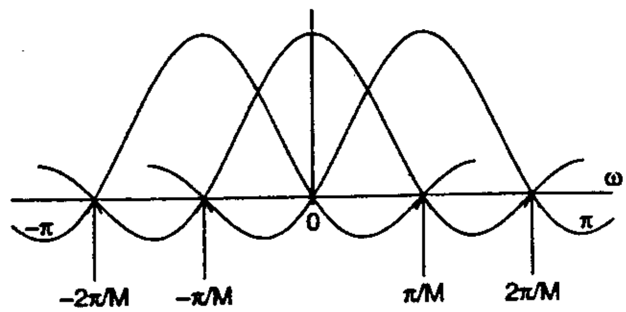
is also $2\pi/M$. Note that the peak value for this term occurs at a frequency $\omega \simeq -\pi/M$.

Finally, the mainlobe bandwidth for the term

$$\alpha\frac{\sin[\omega\,(M-\frac{1}{2})]}{\sin(\omega/2)}$$

is also $2\pi/M$. Note that the peak value for this term occurs at a frequency $\omega = 0$.

A sample plot of these three terms, for $\beta = 2\alpha$ and large M is shown below.



Thus, for large M, the mainlobe bandwidth is bounded by

$$\frac{2\pi}{M} < \text{Mainlobe bandwidth} < \frac{4\pi}{M}$$

Therefore, a reasonable approximation for the mainlobe bandwidth is

$$\text{Mainlobe bandwidth} \simeq \frac{3\pi}{M}$$

$$\begin{aligned} F &= \frac{1}{Q} \sum_{m=-(M-1)}^{M-1} \left(\alpha + \beta \cos\left(\frac{\pi m}{M-1}\right)\right)^2 \\ &= \frac{1}{Q}\left[\sum_{m=-(M-1)}^{M-1} \alpha^2 + 2\alpha\beta \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1}\right) + \beta^2 \sum_{m=-(M-1)}^{M-1} \cos^2\left(\frac{\pi m}{M-1}\right)\right] \end{aligned}$$

Using the relation

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we get

$$\begin{aligned} F = \frac{1}{Q}\Bigg[&\sum_{m=-(M-1)}^{M-1} \alpha^2 + 2\alpha\beta \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1}\right) \\ &+ \frac{\beta^2}{2} \sum_{m=-(M-1)}^{M-1} (1) + \frac{\beta^2}{2} \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{2\pi m}{M-1}\right)\Bigg]. \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1}\right) &= -1 \\ \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{2\pi m}{M-1}\right) &= 1 \end{aligned}$$

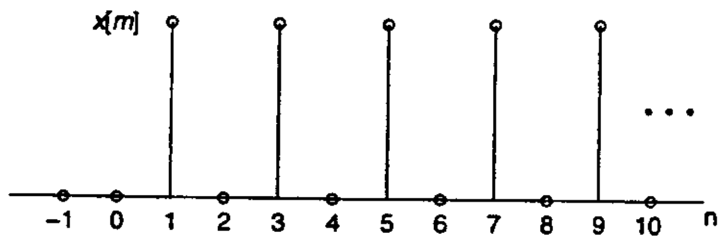
we conclude

$$\begin{aligned} \mathrm{F} &= \frac{1}{Q}\left[(2M-1)\alpha^2 - 2\alpha\beta + \frac{\beta^2}{2}(2M-1) + \frac{\beta^2}{2}\right] \\ &\simeq \frac{2M}{Q}\left(\alpha^2 + \frac{\beta^2}{2}\right) \end{aligned}$$

10.30. (a) Using the definition of the time-dependent Fourier transform we find

$$\begin{aligned} X[0,k] &= \sum_{m=0}^{13} x[m]e^{-j(2\pi/7)km} \\ &= \sum_{m=0}^{6} x[m]e^{-j(2\pi/7)km} + \sum_{l=7}^{13} x[l]e^{-j(2\pi/7)kl} \\ &= \sum_{m=0}^{6} x[m]e^{-j(2\pi/7)km} + \sum_{m=0}^{6} x[m+7]e^{-j(2\pi/7)km}e^{-j2\pi k} \\ &= \sum_{m=0}^{6} (x[m] + x[m+7])e^{-j(2\pi/7)km} \end{aligned}$$

By plotting $x[m]$



we see that $x[m] + x[m+7] = 1$ for $0 \leq m \leq 6$. Thus,

$$\begin{aligned} X[0,k] &= \sum_{m=0}^{6} (1)e^{-j(2\pi/7)km} \\ &= \mathcal{DFT}\{1\} \\ &= 7\delta[k] \end{aligned}$$

(b) If we follow the same procedure we used in part (a) we find

$$\begin{aligned} X[n,k] &= \sum_{m=0}^{13} x[n+m]e^{-j(2\pi/7)km} \\ &= \sum_{m=0}^{6} x[n+m]e^{-j(2\pi/7)km} + \sum_{l=7}^{13} x[n+l]e^{-j(2\pi/7)kl} \\ &= \sum_{m=0}^{6} (x[n+m] + x[n+m+7])e^{-j(2\pi/7)km} \end{aligned}$$

With $n \geq 0$ we have $x[n+m] + x[n+m+7] = 1$ for $0 \leq m \leq 6$, and so

$$\begin{aligned} X[n,k] &= \mathcal{DFT}\{1\} \\ &= 7\delta[k] \end{aligned}$$

Therefore, for $0 \le n \le \infty$ we have

$$\begin{aligned}\sum_{k=0}^{6} X[n,k] &= \sum_{k=0}^{6} 7\delta[k] \\ &= 7\end{aligned}$$

10.31. (a) Sampling the continuous-time input signal

$$x(t) = e^{j(3\pi/8)10^4 t}$$

with a sampling period $T = 10^{-4}$ yields a discrete-time signal

$$x[n] = x(nT) = e^{j3\pi n/8}$$

In order for $X_w[k]$ to be nonzero at exactly one value of k, it is necessary for the frequency of the complex exponential of $x[n]$ to correspond to that of a DFT coefficent, $w_k = 2\pi k/N$. Thus,

$$\begin{aligned}\frac{3\pi}{8} &= \frac{2\pi k}{N} \\ N &= \frac{16k}{3}\end{aligned}$$

The smallest value of k for which N is an integer is $k = 3$. Thus, the smallest value of N such that $X_w[k]$ is nonzero at exactly one value of k is

$$N = 16$$

(b) The rectangular windows, $w_1[n]$ and $w_2[n]$, differ only in their lengths. $w_1[n]$ has length 32, and $w_2[n]$ has length 8. Recall that compared to that of a longer window, the Fourier transform of a shorter window has a larger mainlobe width and higher sidelobes. Since the DFT is a sampled version of the Fourier transform, we might try to look for these features in the two plots. We notice that the second plot, Figure P10.31-3, appears to have a larger mainlobe width and higher sidelobes. As a result, we conclude that Figure P10.31-2 corresponds to $w_1[n]$, and P10.31-3 corresponds to $w_2[n]$.

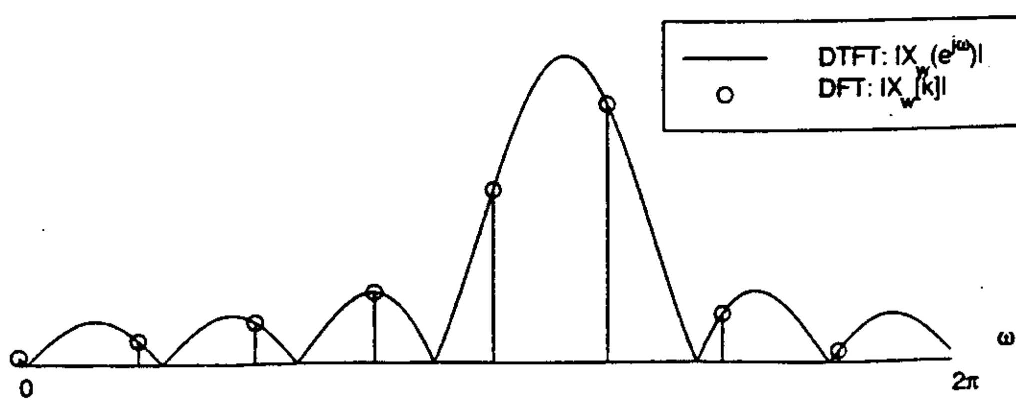
(c) A simple technique to estimate the value of ω_0 is to find the value of k at which the peak of $|X_w[k]|$ occurs. Then, the estimate, is

$$\hat{\omega}_0 = \frac{2\pi k}{N}$$

The corresponding value of $\hat{\Omega}_0$ is

$$\hat{\Omega}_0 = \frac{2\pi k}{NT}$$

This estimate is not exact, since the peak of the Fourier transform magnitude $|X_w(e^{j\omega})|$ might occur between two values of the DFT magnitude $|X_w[k]|$, as shown below.



The maximum possible error, $\Omega_{\text{max error}}$, of the frequency estimate is one half of the frequency resolution of the DFT.

$$\begin{aligned}\Omega_{\text{max error}} &= \frac{1}{2}\frac{2\pi}{NT} \\ &= \frac{\pi}{NT}\end{aligned}$$

For the system parameters of $N = 32$, and $T = 10^{-4}$, this is

$$\Omega_{\text{max error}} = 982 \text{ rad/s}$$

(d) To develop a procedure to get an exact estimate of Ω_0, it helps to derive $X_w[k]$. First, let's find the Fourier transform of $x_w[n] = x[n]w[n]$, where $w[n]$ is an N-point rectangular window.

$$\begin{aligned}X_w(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{j\omega_0 n}e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n}\end{aligned}$$

Let $\omega' = \omega - \omega_0$. Then,

$$\begin{aligned}X_w(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega' n} \\ &= \frac{1-e^{-j\omega' N}}{1-e^{-j\omega'}} \\ &= \frac{(e^{j\omega' N/2} - e^{-j\omega' N/2})e^{-j\omega' N/2}}{(e^{-j\omega'/2} - e^{-j\omega'/2})e^{-j\omega'/2}} \\ &= \frac{\sin(\omega' N/2)}{\sin(\omega'/2)}e^{-j\omega'(N-1)/2} \\ &= \frac{\sin[(\omega-\omega_0)N/2]}{\sin[(\omega-\omega_0)/2]}e^{-j(\omega-\omega_0)(N-1)/2}\end{aligned}$$

Note that $X_w(e^{j\omega})$ has generalized linear phase. Having established this equation for $X_w(e^{j\omega})$, we now find $X_w[k]$. Recall that $X_w[k]$ is simply the Fourier transform $X_w(e^{j\omega})$ evaluated at the frequencies $\omega = 2\pi k/N$, for $k = 0, \ldots, N-1$. Thus,

$$X_w[k] = \frac{\sin[(2\pi k/N - \omega_0)N/2]}{\sin[(2\pi k/N - \omega_0)/2]}e^{-j(2\pi k/N-\omega_0)(N-1)/2}$$

Note that the phase of $X_w[k]$, using the above equation, is

$$\angle X_w[k] = -\frac{(2\pi k/N - \omega_0)(N-1)}{2} + m\pi$$

where the $m\pi$ term comes from the fact that the term

$$\frac{\sin[(2\pi k/N - \omega_0)N/2]}{\sin[(2\pi k/N - \omega_0)/2]}$$

can change sign (i.e. become negative or positive), and thereby offset the phase by π radians. In addition, this term accounts for wrapping the phase, so that the phase stays in the range $[-\pi, \pi]$.

Re-expressing the equation for $\angle X_w[k]$, we find

$$\omega_0 = \frac{2(\angle X_w[k] - m\pi)}{N-1} + \frac{2\pi k}{N}$$

Let $X_{1w}[k]$ be the DFT of the 32-point sequence $x_{1w}[n] = x[n]w_1[n]$, and let $X_{2w}[k]$ be the DFT of the 8-point sequence $x_{2w}[n] = x[n]w_2[n]$. Note that the kth DFT coefficient of $X_{2w}[k]$ corresponds to $X_{1w}[4k]$. Thus, we can relate the 8 DFT coeffients of $X_{2w}[k]$ to 8 of the DFT coefficients in $X_{1w}[k]$. Using the $k=0$th DFT coefficient for simplicity, we find

$$\begin{aligned}
\omega_0 &= \frac{2(\angle X_{w1}[0] - m\pi)}{32-1} = \frac{2(\angle X_{w2}[0] - p\pi)}{8-1} \\
&= \frac{\angle X_{w1}[0] - m\pi}{15.5} = \frac{\angle X_{w2}[0] - p\pi}{3.5}
\end{aligned}$$

A solution that satisfies these equations, with m and p integers, will yield a precise estimate of ω_0. We can accelerate solving these equations by determining which values of m and p to check. This is done by looking at the peak of $|X_w[k]|$ in a procedure similar to Part (c). Suppose that the indices for two largest values of $|X_w[k]|$ are k_{min} and k_{max}. Then, we know that the peak of $|X(e^{j\omega})|$ will occur in the range

$$\frac{2\pi k_{min}}{N} \leq \omega_0 \leq \frac{2\pi k_{max}}{N}$$

By re-expressing the equation for $\angle X_w[k]$, we see that

$$\begin{aligned}
m_{min} &= \frac{1}{\pi}\left[2\angle X_{w1}[k_{min}] + \left(\frac{2\pi k_{min}}{N} - \hat{\omega}_0\right)(N-1)\right] \\
m_{max} &= \frac{1}{\pi}\left[2\angle X_{w1}[k_{max}] + \left(\frac{2\pi k_{max}}{N} - \hat{\omega}_0\right)(N-1)\right]
\end{aligned}$$

In these equations, $\hat{\omega}_0$ is the estimate found in Part (c). So we would look for values of m in the range $[\lfloor m_{min}\rfloor, \lceil m_{max}\rceil]$. Similar expressions hold for p.

Once ω_0 is known, we can find Ω_0 using the relation $\Omega_0 = \omega_0/T$.

10.32. For each part, we use the definition of the time-dependent Fourier transform,

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}.$$

(a) *Linearity*: using $x[n] = ax_1[n] + bx_2[n]$,

$$\begin{aligned}
X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} (ax_1[n+m] + bx_2[n+m])\, w[m]e^{-j\lambda m} \\
&= a\sum_{m=-\infty}^{\infty} x_1[n+m]w[m]e^{-j\lambda m} + b\sum_{m=-\infty}^{\infty} x_2[n+m]w[m]e^{-j\lambda m} \\
&= aX_1[n,\lambda) + bX_2[n,\lambda)
\end{aligned}$$

(b) *Shifting*: using $y[n] = x[n-n_0]$,

$$Y[n,\lambda) = \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m}$$

$$
\begin{aligned}
&= \sum_{m=-\infty}^{\infty} x[n-n_0+m]w[m]e^{-j\lambda m} \\
&= X[n-n_0,\lambda)
\end{aligned}
$$

(c) *Modulation:* using $y[n] = e^{j\omega_0 n}x[n]$,

$$
\begin{aligned}
y[n+m] &= e^{j\omega_0(n+m)}x[n+m] \\
Y[n,\lambda) &= \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} e^{j\omega_0(n+m)}x[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} e^{j\omega_0 n}x[n+m]w[m]e^{-j(\lambda-\omega_0)m} \\
&= e^{j\omega_0 n}X[n,\lambda-\omega_0)
\end{aligned}
$$

(d) *Conjugate Symmetry:* for $x[n]$ and $w[n]$ real,

$$
\begin{aligned}
X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \\
&= \left[\sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{j\lambda m}\right]^* \\
&= [X[n,-\lambda)]^* \\
&= X^*[n,-\lambda)
\end{aligned}
$$

10.33. (a) We are given that $\phi_c(\tau) = \mathcal{E}\{x_c(t)x_c(t+\tau)\}$. Since $x[n] = x_c(nT)$,

$$
\begin{aligned}
\phi[m] &= \mathcal{E}\{x[n]x[n+m]\} \\
&= \mathcal{E}\{x_c(nT)x_c(nT+mT)\} \\
&= \phi_c(mT)
\end{aligned}
$$

(b) $P(\omega)$ and $P_c(\Omega)$ are the transforms of $\phi[m]$ and $\phi_c(\tau)$ respectively. Since $\phi[m]$ is a sampled version of $\phi_c(\tau)$, $P(\omega)$ and $P_c(\Omega)$ are related by

$$
P(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} P_c\left(\frac{\omega-2\pi k}{T}\right)
$$

(c) The condition is that no aliasing occurs when sampling. Thus, we require that $P_c(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ so that

$$
P(\omega) = \frac{1}{T}P_c\left(\frac{\omega}{T}\right), \qquad |\omega| < \pi
$$

10.34. In this problem, we are given

- $x[n] = A\cos(\omega_0 n + \theta) + e[n]$
- θ is a uniform random variable on 0 to 2π
- $e[n]$ is an independent, zero mean random variable

(a) Computing the autocorrelation function,

$$\begin{aligned}
\phi_{xx}[m] &= \mathcal{E}\{x[n]x[n+m]\} \\
&= \mathcal{E}\{(A\cos(\omega_0 n+\theta)+e[n])(A\cos(\omega_0(n+m)+\theta)+e[n+m])\} \\
&= \mathcal{E}\{A^2\cos(\omega_0 n+\theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad +\mathcal{E}\{Ae[n]\cos(\omega_0(n+m)+\theta)\}+\mathcal{E}\{Ae[n+m]\cos(\omega_0 n+\theta)\} \\
&\quad +\mathcal{E}\{e[n]e[n+m]\} \\
&= A^2\mathcal{E}\{\cos(\omega_0 n+\theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad +A\mathcal{E}\{e[n]\}\mathcal{E}\{\cos(\omega_0(n+m)+\theta)\}+A\mathcal{E}\{e[n+m]\}\mathcal{E}\{\cos(\omega_0 n+\theta)\} \\
&\quad +\mathcal{E}\{e[n]e[n+m]\}
\end{aligned}$$

First, note that

$$\cos(a)\cos(b)=\frac{1}{2}\cos(a+b)+\frac{1}{2}\cos(a-b)$$

Therefore, the first term can be re-expressed as

$$A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n+\omega_0 m+2\theta)+\frac{1}{2}\cos(\omega_0 m)\right\}$$

Next, note that

$$\mathcal{E}\{e[n]\}=0$$

As a result, the two middle terms drop out. Finally, note that since $e[n]$ is a sequence of zero-mean variables that are uncorrelated with each other,

$$\mathcal{E}\{e[n]e[n+m]\}=\sigma_e^2\delta[m], \qquad \text{where } \sigma_e^2=\mathcal{E}\{e^2[n]\}$$

Putting this together, we get

$$\phi_{xx}[m]=A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n+\omega_0 m+2\theta)+\frac{1}{2}\cos(\omega_0 m)\right\}+\sigma_e^2\delta[m]$$

Since $\frac{1}{2\pi}\int_0^{2\pi}\cos(2\omega_0 n+\omega_0 m+2\theta)d\theta=0$, we have

$$\phi_{xx}[m]=\frac{A^2}{2}\cos(\omega_0 m)+\sigma_e^2\delta[m]$$

(b) Since the Fourier transform of $\cos(\omega_0 m)$ is $\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$ for $|\omega|\leq\pi$,

$$\Phi_{xx}(e^{j\omega})=P_{xx}(\omega)=\frac{A^2\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\sigma_e^2$$

10.35. (a) Plugging in the equation

$$I[k]=I(\omega_k)=\frac{1}{L}|V[k]|^2$$

into the relation

$$\text{var}[I(\omega)]\simeq P_{xx}^2(\omega)$$

we find that

$$\begin{aligned}
\text{var}\left[\frac{1}{L}|V[k]|^2\right] &\simeq P_{xx}^2(\omega) \\
\text{var}\left[|V[k]|^2\right] &\simeq L^2P_{xx}^2(\omega)
\end{aligned}$$

This equation can be used to find the approximate variance of $|X[k]|^2$. We substitute the signal $X[k]$ for $V[k]$, the DFT length N for L, and use the power spectrum

$$P_{xx}(w) = \sigma_x^2$$

This gives

$$\text{var}\left[|X[k]|^2\right] = N^2\sigma_x^4$$

(b) The cross-correlation is found below.

$$\begin{aligned}
\mathcal{E}\{X[k]X^*[r]\} &= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\mathcal{E}\{x[n_1]x[n_2]\}W_N^{kn_1}W_N^{-rn_2} \\
&= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sigma_x^2\delta[n_1-n_2]W_N^{kn_1}W_N^{-rn_2} \\
&= \sum_{n=0}^{N-1}\sigma_x^2 W_N^{(k-r)n} \\
&= \sigma_x^2\left[\frac{1-W_N^{N(k-r)}}{1-W_N^{(k-r)}}\right] \\
&= N\sigma_x^2\delta[k-r]
\end{aligned}$$

Note that the cross-correlation is zero everywhere except when $k = r$. This is what one would expect for white noise, since samples for which $k \neq r$ are completely uncorrelated.

10.36. (a) The length of the data record is

$$\begin{aligned}
Q &= 10 \text{ seconds} \cdot \frac{20,000 \text{ samples}}{\text{second}} \\
Q &= 200,000 \text{ samples}
\end{aligned}$$

(b) To achieve a 10 Hz or less spacing between samples of the power spectrum, we require

$$\begin{aligned}
\frac{1}{NT} &\leq 10 \text{ Hz} \\
N &\geq \frac{1}{10T} \\
&\geq \frac{20,000}{10} \\
&\geq 2,000 \text{ samples}
\end{aligned}$$

Since N must also be a power of 2, we choose $N = 2048$.

(c)

$$\begin{aligned}
K &= \frac{Q}{L} \\
&= \frac{200,000}{2048} \\
&= 97.66 \text{ segments}
\end{aligned}$$

If we zero-pad the last segment so that it contains 2048 samples, we will have $K = 98$ segments.

(d) The key to reducing the variance is to use more segments. Two methods are discussed below. Note that in both methods, we want the segments to be length $L = 2048$ so that we maintain the frequency spacing.

(i) Decreasing the length of the segments to $\frac{1}{10}$th their length, and then zero-padding them to $L = 2048$ samples will increase K by a factor of 10. Accordingly, the variance will decrease by a factor of 10. However, the frequency resolution will be reduced.

(ii) If we increase the data record to 2,000,000 samples, we can keep the window length the same and increase K by a factor of 10.

10.37. (a) Taking the expected value of

$$\overline{\phi}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{I}(\omega)e^{j\omega m}d\omega$$

gives

$$\begin{aligned}\mathcal{E}\{\overline{\phi}[m]\} &= \mathcal{E}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{I}(\omega)e^{j\omega m}d\omega\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{E}\{\overline{I}(\omega)\}\, e^{j\omega m}d\omega\end{aligned}$$

Using the relation

$$\mathcal{E}\{\overline{I}(w)\} = \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)C_{ww}(e^{j(\omega-\theta)})d\theta$$

we find

$$\begin{aligned}\mathcal{E}\{\overline{\phi}[m]\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)C_{ww}(e^{j(\omega-\theta)})d\theta\right]e^{j\omega m}d\omega \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{ww}(e^{j(\omega-\theta)})e^{j\omega m}d\omega\right]d\theta\end{aligned}$$

Substituting $\omega' = \omega - \theta$ in the inner integral yields

$$\begin{aligned}\mathcal{E}\{\overline{\phi}[m]\} &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)\left[\frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta} C_{ww}(e^{j\omega'})e^{j(\omega'+\theta)m}d\omega'\right]d\theta \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\left[\frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta} C_{ww}(e^{j\omega'})e^{j\omega' m}d\omega'\right]d\theta\end{aligned}$$

Note we can change the limits of integration of the inner integral to be $[-\pi, \pi]$ because we are integrating over the whole period. Doing this gives

$$\begin{aligned}\mathcal{E}\{\overline{\phi}[m]\} &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{ww}(e^{j\omega'})e^{j\omega' m}d\omega'\right]d\theta \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\{c_{ww}[m]\}\,d\theta \\ &= \frac{1}{LU}c_{ww}[m]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}d\theta\right] \\ &= \frac{1}{LU}c_{ww}[m]\phi_{xx}[m]\end{aligned}$$

(b)

$$\overline{\phi}_p[m] = \frac{1}{N}\sum_{k=0}^{N-1} \overline{I}[k]e^{j2\pi km/N}$$

By applying the sampling theorem to Fourier transforms, we see that

$$\begin{aligned}
\overline{\phi}_p[m] &= \sum_{r=-\infty}^{\infty} \overline{\phi}_{xx}[m+rN] \\
\mathcal{E}\left\{\overline{\phi}_p[m]\right\} &= \sum_{r=-\infty}^{\infty} \mathcal{E}\left\{\overline{\phi}_{xx}[m+rN]\right\} \\
&= \frac{1}{LU}\sum_{r=-\infty}^{\infty} c_{ww}[m+rN]\phi_{xx}[m+rN]
\end{aligned}$$

which is a time aliased version of $\mathcal{E}\left\{\overline{\phi}_{xx}[m]\right\}$.

(c) N should be chosen so that no time aliasing occurs. Since $\overline{\phi}_{xx}[m]$ is $2L-1$ points long, we should choose $N \geq 2L$.

10.38. (a) For $0 \leq m \leq M$,

$$\begin{aligned}
\hat{\phi}_{xx}[m] &= \frac{1}{Q}\sum_{n=0}^{Q-m-1} x[n]x[n+m] \\
&= \frac{1}{Q}\left[\sum_{n=0}^{M-1} x[n]x[n+m] + \sum_{n=M}^{2M-1} x[n]x[n+m] + \ldots + \sum_{n=(K-1)M}^{KM-1} x[n]x[n+m]\right] \\
&= \frac{1}{Q}\left[\sum_{n=0}^{M-1} x[n]x[n+m]\right. \\
&\qquad \left. + \sum_{n=0}^{M-1} x[n+M]x[n+M+m] + \ldots + \sum_{n=0}^{M-1} x[n+(K-1)M]x[n+(K-1)M+m]\right] \\
&= \frac{1}{Q}\sum_{i=0}^{K-1}\sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \\
&= \frac{1}{Q}\sum_{i=0}^{K-1} c_i[m]
\end{aligned}$$

where

$$c_i[m] = \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \qquad \text{for } 0 \leq m \leq M-1$$

(b) We can rewrite the expression for $c_i[m]$ from part (a) as

$$\begin{aligned}
c_i[m] &= \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \\
&= \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] + \sum_{n=M}^{N-1} 0 \cdot x[n+iM+m] \\
&= \sum_{n=0}^{N-1} x_i[n]y_i[n+m]
\end{aligned}$$

where

$$x_i[n] = \begin{cases} x[n+iM], & 0 \leq n \leq M-1 \\ 0, & M \leq n \leq N-1 \end{cases}$$

and

$$y_i[n] = x[n + iM] \qquad \text{for } 0 \le n \le N-1$$

Thus, the correlations $c_i[m]$ can be obtained by computing N-point *linear* correlations. Next, we show that for $N \ge 2M-1$, circular correlation is equivalent to linear correlation.

Note that the circular correlation of $x_i[n]$ with $y_i[n]$,

$$\tilde{c}_{yx}[m] = \sum_{n=0}^{N-1} x_i[n] y_i[((n+m))_N]$$

can be expressed as

$$\begin{aligned} \tilde{c}_{yx}[m] &= \tilde{c}_{xy}[-m] \\ &= \sum_{n=0}^{N-1} x_i[((n-m))_N] y_i[n] \\ &= \sum_{n=0}^{N-1} x_i'[((m-n))_N] y_i[n] \end{aligned}$$

where $x'[n] = x[-n]$. Note that this is a circular *convolution* of $x_i[-n]$ with $y_i[n]$. Thus, we have expressed the circular correlation of $x_i[n]$ with $y_i[n]$ as a circular convolution of $x_i[-n]$ with $y_i[n]$. Now recall from chapter 8 that the circular convolution of two M point signals is equivalent to their linear convolution when $N \ge 2M-1$. Since we can express the circular correlation in terms of a circular convolution, this result applies to circular correlation as well. Therefore, we see that if $N \ge 2M-1$,

$$c_i[m] = \tilde{c}_i[m] \qquad \text{for } 0 \le m \le M-1$$

Thus, the minimum value of N is $2M-1$.

(c) A procedure for computing $\hat{\phi}_{xx}[m]$ is described below.

step 1: Compute $X_i[k]$ and $Y_i[k]$, which are the $N \ge 2M-1$ point DFTs of $x_i[n]$ and $y_i[n]$.

step 2: Multiply $X_i[k]$ and $Y_i^*[k]$ point by point, yielding $C_i[k] = \tilde{C}_i[k] = X_i[k]Y_i^*[k]$.

step 3: Repeat the above two steps for all data (K times), then compute

$$\hat{\Phi}_{xx}[k] = \frac{1}{Q} \sum_{i=0}^{K-1} C_i[k] \qquad \text{for } 0 \le k \le N-1$$

step 4: Take the N point inverse DFT of $\hat{\Phi}_{xx}[k]$ to get $\hat{\phi}_{xx}[m]$.

Assuming that a radix-2 FFT, requiring $\frac{N}{2}\log_2 N$ complex multiplications is used to compute the forward and inverse DFTS, the number of complex multiplications is

$2 \cdot \frac{N}{2}\log_2 N \cdot K = KN\log_2 N,$	for step 1
$KN,$	for step 2
$N,$	for divide by Q operation in step 3
$\frac{N}{2}\log_2 N$	for step 4

So the total number of complex multiplications is $(K + \frac{1}{2})N\log_2 N + (K+1)N$.

(d) The procedure developed in part (c) would compute the cross-correlation estimate $\hat{\phi}_{xy}$ without any major modifications. All we need to do is redefine $y_i[n]$ as

$$y_i[n] = y[n + iM], \qquad 0 \le n \le N-1$$

and $x_i[n]$ is the same as it was before, namely

$$x_i[n] = \begin{cases} x[n+iM], & 0 \le n \le M-1 \\ 0, & M \le n \le N-1 \end{cases}$$

Note that for $m < 0$, $\hat{\phi}_{xy}[m] = \hat{\phi}_{yx}[-m]$.

(e) For $N = 2M$,

$$\begin{aligned} y_i[n] &= x[n+iM], \qquad \text{for } 0 \le n \le 2M-1 \\ &= x[n+iM](u[n]-u[n-M]) + x[n+iM](u[n-M]-u[n-2M]) \\ &= x[n+iM](u[n]-u[n-M]) + x[n-M+(i+1)M](u[n-M]-u[n-2M]) \\ &= x_i[n] + x_{i+1}[n-M] \end{aligned}$$

Taking the DFT of this expression yields

$$Y_i[k] = X_i[k] + (-1)^k X_{i+1}[k]$$

A procedure for computing $\hat{\phi}_{xx}$ for $0 \le m \le M-1$ is described below.

step 1: Compute the N point DFT $X_i[k]$ for $i = 0, 1, \ldots, K$.

step 2: Compute $Y_i[k] = X_i[k] + (-1)^k X_{i+1}[k]$ for $i = 0, 1, \ldots, K-1$.

step 3: Let $A_0[k] = 0$ and compute

$$A_i[k] = A_{i-1}[k] + X_i[k]Y_i^*[k], \quad i = 1, \ldots, K-1$$

step 4: Define $V[k] = A_{K-1}[k]$. Compute $v[m]$, the N point inverse DFT of $V[k]$.

step 5: Compute

$$\hat{\phi}_{xx}[m] = \frac{1}{Q}v[m]$$

Assuming that a radix-2 FFT, requiring $\frac{N}{2}\log_2 N$ complex multiplications is used to compute the forward and inverse DFTs, the number of complex multiplications is

$(K+1)\frac{N}{2}\log_2 N$,	for step 1
0,	for step 2
$(K-1)N$,	for step 3
$\frac{N}{2}\log_2 N$,	for step 4
N,	for the divide by Q in step 5

So the total number of complex multiplications is $\frac{K+2}{2}N\log_2 N + KN$. Note that for large N and K, this procedure requires roughly half the number of complex multiplications as the procedure described in part (c).

10.39. (a) Using the relations,

$$\begin{aligned} c[n,m] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda \\ X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} d\lambda \end{aligned}$$

we find

$$
\begin{aligned}
c[n,m] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X[n,\lambda)X[n,-\lambda)e^{j\lambda m} d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{l=-\infty}^{\infty} x[n+l]w[l]e^{-j\lambda l}\right)\left(\sum_{r=-\infty}^{\infty} x[n+r]w[r]e^{j\lambda r}\right) e^{j\lambda m} d\lambda \\
&= \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\lambda l}e^{j\lambda r}e^{j\lambda m} d\lambda\right) \\
&= \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\lambda(-l+r)}e^{j\lambda m} d\lambda\right)
\end{aligned}
$$

Using the Fourier transform relation,

$$\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$$

we find

$$c[n,m] = \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\delta[m-l+r]$$

The $\delta[m-l+r]$ term is zero everwhere except when $m-l+r=0$. Therefore, we can replace the two sums of l and r with one sum over r, by substituting $l=m+r$.

$$
\begin{aligned}
c[n,m] &= \sum_{r=-\infty}^{\infty} x[n+m+r]w[m+r]x[n+r]w[r] \\
&= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[n+m+r]w[m+r]
\end{aligned}
$$

(b) First, note that

$$
\begin{aligned}
|X[n,\lambda)|^2 &= X[n,-\lambda)X[n,\lambda) \\
&= |X[n,-\lambda)|^2
\end{aligned}
$$

Starting with the definition of $c[n,m]$,

$$c[n,m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda$$

$$c[n,-m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{-j\lambda m} d\lambda$$

we substitute $\lambda' = -\lambda$ to get

$$
\begin{aligned}
c[n,-m] &= -\frac{1}{2\pi}\int_{\pi}^{-\pi} |X[n,-\lambda')|^2 e^{j\lambda' m} d\lambda' \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,-\lambda')|^2 e^{j\lambda' m} d\lambda' \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda')|^2 e^{j\lambda' m} d\lambda' \\
&= c[n,m]
\end{aligned}
$$

Thus, the time-dependent autocorrelation function is an even function of m for n fixed. Next, we use this fact to obtain the equivalent expression for $c[n,m]$.

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[m+n+r]w[m+r] \\ &= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[-m+n+r]w[-m+r] \end{aligned}$$

Substituting $r' = n + r$ gives

$$\begin{aligned} &= \sum_{r'=-\infty}^{\infty} x[r']w[r'-n]x[r'-m]w[(r'-m)-n] \\ &= \sum_{r'=-\infty}^{\infty} x[r']x[r'-m]w[r'-n]w[-(m+n-r')] \\ &= \sum_{r'=-\infty}^{\infty} x[r']x[r'-m]h_m[n-r'] \end{aligned}$$

where

$$h_m[r] = w[-r]w[-(m+r)]$$

(c) To compute $c[n,m]$ by causal operations, we see that

$$h_m[r] = w[-r]w[-(m+r)]$$

requires that $w[r]$ must be zero for

$$\begin{aligned} -r &< 0 \\ r &> 0 \end{aligned}$$

and $w[r]$ must be zero for

$$\begin{aligned} -(m+r) &< 0 \\ m+r &> 0 \\ r &> -m \end{aligned}$$

Thus, $w[r]$ must be zero for $r > \min(0,-m)$. If m is positive, then $w[r]$ must be zero for $r > 0$. This is equivalent to the requirement that $w[-r]$ must be zero for $r < 0$.

(d) Plugging in

$$w[-r] = \begin{cases} a^r, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

into $h_m[r] = w[-r]w[-(m+r)]$, we find

$$h_m[r] = \begin{cases} a^{2r+m}, & r \geq 0, r \geq -m \\ 0, & \text{otherwise} \end{cases}$$

Taking the z-transform of this expression gives

$$\begin{aligned} H_m(z) &= \sum_{r=-\infty}^{\infty} h_m[r]z^{-r} \\ &= \sum_{r=0}^{\infty} a^{2r+m}z^{-r} \\ &= a^m \sum_{r=0}^{\infty} (a^2 z^{-1})^r \end{aligned}$$

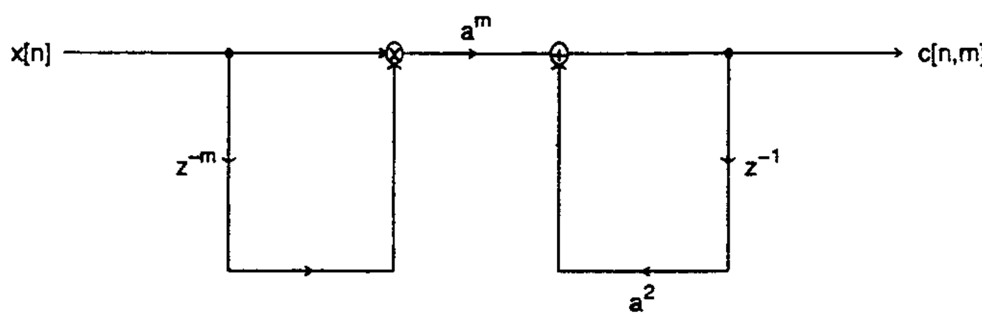
Again we have assumed that m is positive. If $|z| > a^2$, then

$$\begin{aligned} H_m(z) &= \frac{a^m}{1-a^2z^{-1}} \\ h_m[r] &= a^m\delta[r] + a^2h_m[r-1] \end{aligned}$$

Using this in the equation for $c[n,m]$ gives

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty} x[r]x[r-m]h_m[n-r] \\ &= \sum_{r=-\infty}^{\infty} x[r]x[r-m]\left(a^m\delta[n-r] + a^2h_m[n-r-1]\right) \\ &= a^m x[n]x[n-m] + a^2 \sum_{r=-\infty}^{\infty} x[r]x[r-m]h_m[n-r-1] \\ &= a^m x[n]x[n-m] + a^2c[n-1,m] \end{aligned}$$

A block diagram of this system appears below.



(e) Next, consider the system

$$w[-r] = \begin{cases} ra^r, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

$$\begin{aligned} h_m[r] &= \{ra^r u[r]\}\{(r+m)a^{r+m}u[r+m]\} \\ &= a^m r^2 a^{2r} + a^m m r a^{2r} \qquad\qquad r \geq 0; r \geq -m \end{aligned}$$

To get the z-transform $H_m(z)$, recall the z-transform property: $rx[r] \leftrightarrow -z\frac{dX(z)}{dz}$. Using this property, we find

$$\begin{aligned} ra^{2r}u[r] &\Longleftrightarrow \frac{a^2z^{-1}}{(1-a^2z^{-1})^2} \\ r^2a^{2r}u[r] &\Longleftrightarrow \frac{a^2z^{-1}(1+a^2z^{-1})}{(1-a^2z^{-1})^3} \end{aligned}$$

Again we have assumed that m is positive. Thus,

$$\begin{aligned} H_m(z) &= a^m\left[\frac{a^2z^{-1}(1+a^2z^{-1})}{(1-a^2z^{-1})^3}\right] + ma^m\left[\frac{a^2z^{-1}}{(1-a^2z^{-1})^2}\right] \\ &= \frac{a^{m+2}z^{-1}(1+a^2z^{-1}) + ma^m(a^2z^{-1})(1-a^2z^{-1})}{(1-a^2z^{-1})^3} \end{aligned}$$

$$= \frac{a^{m+2}z^{-1}(1+a^2z^{-1}+m-ma^2z^{-1})}{(1-a^2z^{-1})^3}$$
$$= \frac{a^{m+2}(1+m)z^{-1}+a^{m+4}(1-m)z^{-2}}{1-3a^2z^{-1}+3a^4z^{-2}-a^6z^{-3}}$$

Cross-multiplying and taking the inverse z-transform gives

$$h_m[r]-3a^2h_m[r-1]+3a^4h_m[r-2]-a^6h_m[r-3]=a^{m+2}(1+m)\delta[r-1]+a^{m+4}(1-m)\delta[r-2]$$
$$h_m[r]=3a^2h_m[r-1]-3a^4h_m[r-2]+a^6h_m[r-3]+a^{m+2}(1+m)\delta[r-1]+a^{m+4}(1-m)\delta[r-2]$$

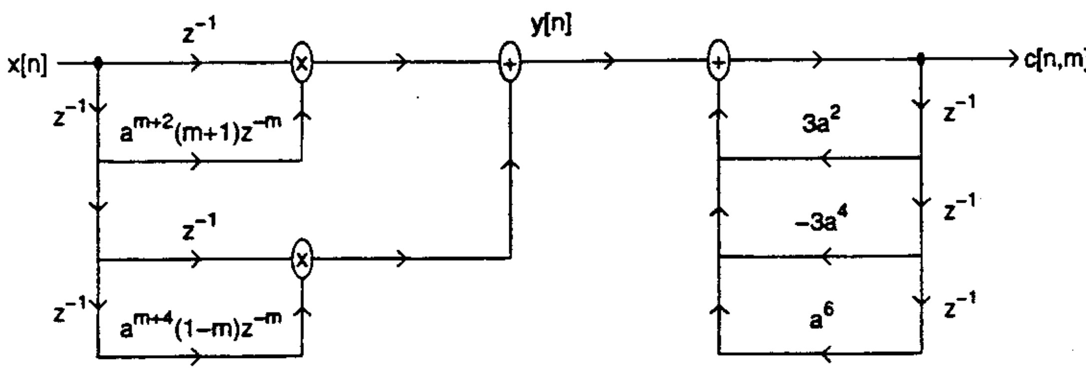
Using this relation for $h_m[r]$ in

$$c[n,m]=\sum_{r=-\infty}^{\infty}x[r]x[r-m]h_m[n-r]$$

we get

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty}x[r]x[r-m]\left(3a^2h_m[n-r-1]-3a^4h_m[n-r-2]+a^6h_m[n-r-3]\right) \\ &\quad+\sum_{r=-\infty}^{\infty}x[r]x[r-m]\left(a^{m+2}(1+m)\delta[n-r-1]+a^{m+4}(1-m)\delta[n-r-2]\right) \\ &= 3a^2c[n-1,m]-3a^4c[n-2,m]+a^6c[n-3,m] \\ &\quad+a^{m+2}(1+m)x[n-1]x[n-1-m]+a^{m+4}(1-m)x[n-2]x[n-2-m] \end{aligned}$$

A block diagram of this system appears below.



10.40. (a) Looking at the figure, we see that

$$\begin{aligned} X[n,\lambda] &= \left\{\left(x[n]e^{-j\lambda n}\right)*h_0[n]\right\}e^{j\lambda n} \\ &= \left[\sum_{m=-\infty}^{\infty}x[n-m]e^{-j\lambda(n-m)}h_0[m]\right]e^{j\lambda n} \\ &= \sum_{m=-\infty}^{\infty}x[n-m]h_0[m]e^{j\lambda m} \end{aligned}$$

Let $m'=-m$. Then,

$$X[n,\lambda] = \sum_{m'=\infty}^{-\infty}x[n+m']h_0[-m']e^{-j\lambda m'}$$

$$\begin{aligned} &= \sum_{m'=-\infty}^{\infty} x[n+m']h_0[-m']e^{-j\lambda m'} \\ &= X[n,\lambda) \end{aligned}$$

if $h_0[-m] = w[m]$. Next, we show that for λ fixed, $X[n,\lambda)$ behaves as a linear, time-invariant system.

Linear: Inputting the signal $ax_1[n] + bx_2[n]$ into the system yields

$$\sum_{m=-\infty}^{\infty} (ax_1[n+m] + bx_2[n+m])\, h_0[-m]e^{-j\lambda m} =$$

$$\sum_{m=-\infty}^{\infty} ax_1[n+m]h_0[-m]e^{-j\lambda m} + \sum_{m=-\infty}^{\infty} bx_2[n+m]h_0[-m]e^{-j\lambda m} = aX_1[n,\lambda) + bX_2[n,\lambda)$$

The system is linear.

Time invariant: Shifting the input $x[n]$ by an amount l yields

$$\sum_{m=-\infty}^{\infty} x[n+m+l]h_0[-m]e^{-j\lambda m} = X[n+l,\lambda)$$

which is the output shifted by l samples. The system is time-invariant.

Next, we find the impulse response and frequency response of the system. To find the impulse response, denoted as $h[n]$, we let $x[n] = \delta[n]$.

$$\begin{aligned} h[n] &= \sum_{m=-\infty}^{\infty} \delta[n+m]w[m]e^{-j\lambda m} \\ &= w[-n]e^{j\lambda n} \\ &= h_0[n]e^{j\lambda n} \end{aligned}$$

Taking the DTFT gives the frequency response, denoted as $H(e^{j\omega})$.

$$H(e^{j\omega}) = H_0(e^{j(\omega-\lambda)})$$

(b) We find $S(e^{j\omega})$ to be

$$\begin{aligned} s[n] &= (x[n]e^{-j\lambda n}) * w[-n] \\ S(e^{j\omega}) &= X\left(e^{j(\omega+\lambda)}\right) W(e^{-j\omega}) \\ S(e^{j\omega}) &= X\left(e^{j(\omega+\lambda)}\right) H_0(e^{j\omega}) \end{aligned}$$

Note that most typical window sequences are lowpass in nature, and are centered around a frequency of $\omega = 0$. Since $H_0(e^{j\omega}) = W(e^{-j\omega})$ is the Fourier transform of a window which is lowpass in nature, the signal $S(e^{j\omega})$ is also lowpass.

The signal $s[n] = \check{X}[n,\lambda)$ is multiplied by a complex exponential $e^{j\lambda n}$. This modulation shifts the frequency response of $S(e^{j\omega})$ so that it is centered at $\omega = \lambda$.

$$\begin{aligned} h[n] &= s[n]e^{j\lambda n} \\ H(e^{j\omega}) &= S\left(e^{j(\omega-\lambda)}\right) \end{aligned}$$

Since $S(e^{j\omega})$ is lowpass filter centered at $\omega = 0$, the overall system is a bandpass filter centered at $\omega = \lambda$.

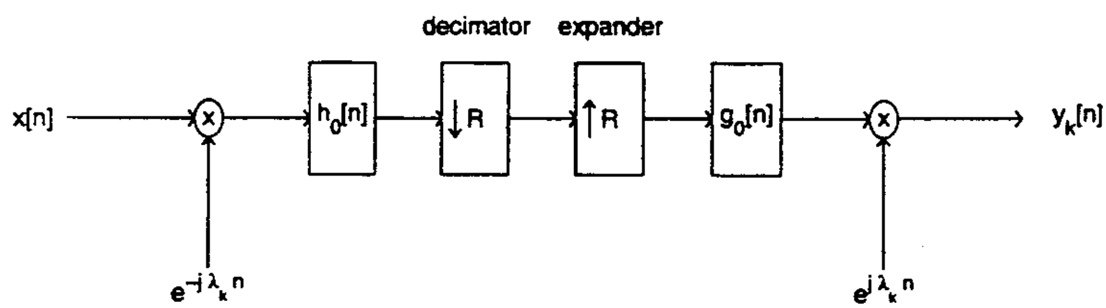
(c) First, it is shown that the individual outputs $y_k[n]$ are samples (in the λ dimension) of the time-dependent Fourier transform.

$$\begin{aligned} y_k[n] &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda_k m} \\ &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j2\pi km/N} \\ &= X[n,\lambda)|_{\lambda=2\pi k/N} \end{aligned}$$

Next, it is shown that the overall output is $y[n] = Nw[0]x[n]$.

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} y_k[n] \\ &= \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j2\pi km/N} \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} x[n+m]w[m]e^{-j2\pi km/N} \\ &= \sum_{m=-\infty}^{\infty} x[n+m]w[m] \underbrace{\sum_{k=0}^{N-1} e^{-j2\pi km/N}}_{N\delta[m]} \\ &= Nw[0]x[n] \end{aligned}$$

(d) Consider a single channel,



In the frequency domain, the input to the decimator is

$$X\left(e^{j(\omega+\lambda_k)}\right) H_0(e^{j\omega})$$

so the output of the decimator is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j((\omega-2\pi l)/R+\lambda_k)}\right) H_0\left(e^{j(\omega-2\pi l)/R}\right)$$

The output of the expander is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j(\omega+\lambda_k-2\pi l/R)}\right) H_0\left(e^{j(\omega-2\pi l/R)}\right)$$

The output $Y_k(e^{j\omega})$ is then

$$Y_k(e^{j\omega}) = \frac{1}{R}\sum_{l=0}^{R-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)$$

The overall system output is formed by summing these terms over k.

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{k=0}^{N-1} Y_k(e^{j\omega}) \\ &= \frac{1}{R}\sum_{l=0}^{R-1}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) \end{aligned}$$

To cancel the aliasing, we rewrite the equation as follows:

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})\frac{1}{R}\sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) \\ &+ \underbrace{\sum_{l=1}^{R-1} X\left(e^{j(\omega-2\pi l/R)}\right)\frac{1}{R}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)}_{\text{Aliasing Component}} \end{aligned}$$

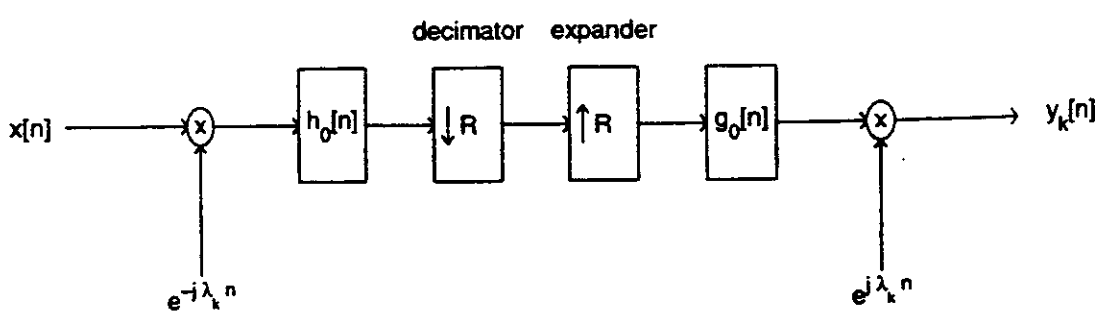
Therefore, we require the following relations to be satisfied so that $y[n] = x[n]$:

$$\begin{aligned} \sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) &= 0, \qquad \forall\,\omega, \text{ and } l = 1,\ldots,R-1 \\ \sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) &= R, \qquad \forall\,\omega \end{aligned}$$

(e) *Yes*, it is possible. $G_o(e^{j\omega}) = NH_0(e^{j\omega})$ will yield exact reconstruction.

(f) See chapter 7 in "Multirate Digital Signal Processing" by Crochiere and Rabiner, 1983.

(g) Once again, we consider a single channel,



From Part (a), we know that the output of the filter $h_0[n]$ is

$$\check{X}[n,\lambda_k] = \sum_{m=-\infty}^{\infty} x[m]h_0[n-m]e^{-j\lambda_k m}$$

or, using $\lambda_k = 2\pi k/N$,

$$\tilde{X}[n,k] = \sum_{m=-\infty}^{\infty} x[m]h_0[n-m]e^{-j2\pi km/N}$$

Therefore, the output of the decimator is

$$\tilde{X}[Rn,k] = \sum_{m=-\infty}^{\infty} x[m]h_0[Rn-m]e^{-j2\pi km/N}$$

Recall that in general, the output of an expander with expansion factor R is

$$x_e[n] = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell R]$$

This relation is given in chapter 3. Therefore, the output of the expander is

$$\sum_{\ell=-\infty}^{\infty} \tilde{X}[R\ell,k]\delta[n-\ell R]$$

This signal is then convolved with $g_0[n]$, giving

$$\sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \tilde{X}[R\ell,k]\delta[m-\ell R]g_0[n-m] = \sum_{\ell=-\infty}^{\infty} \tilde{X}[R\ell,k]g_0[n-\ell R]$$

Therefore,

$$\begin{aligned}
y_k[n] &= \sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\left(\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi km/N}\right)e^{j2\pi kn/N} \\
y[n] &= \sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\left(\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi km/N}\right)e^{j2\pi kn/N} \\
&= \sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi k(m-n)/N} \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_0[n-\ell R]h_0[Rl-m]x[m]\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N}
\end{aligned}$$

Now recall that $\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N} = N\delta[((n-m))_N]$, by considering it as a Fourier series expansion, or as an inverse DFT of $Ne^{-j2\pi mk/N}$. Thus,

$$\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N} = N\sum_{r=-\infty}^{\infty}\delta[n-m-rN]$$

where r is an integer. Therefore,

$$\begin{aligned}
y[n] &= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R-m]x[m]N\sum_{r=-\infty}^{\infty}\delta[n-m-rN] \\
&= N\sum_{\ell=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R-n+rN]x[n-rN] \\
&= N\sum_{r=-\infty}^{\infty} x[n-rN]\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R+rN-n]
\end{aligned}$$

Therefore, if we want $y[n] = x[n]$, we require

$$\sum_{\ell=-\infty}^{\infty} g_0[n - \ell R] h_0[\ell R + rN - n] = \delta[r]$$

for all values n.

(h) Intuitively, we see that it is possible since we are keeping the necessary number of samples. If $g_0[n] = \delta[n]$ find that

$$\begin{aligned} \sum_{\ell=-\infty}^{\infty} \delta[n - \ell R] h_0[\ell R + rN - n] &= h_0[rN] \\ &= \delta[r] \end{aligned}$$

since $h_0[rN]$ is zero for all values of r, except $r = 0$, where it is equal to 1. Thus, the condition derived in Part (g) is satisfied.

(i) See Rabiner and Crochiere or Portnoff. (Hint: consider an overlap and add FFT algorithm.)

10.41. Note that $h[n]$ is real in this problem.

(a) First, we express $y[n]$ as the convolution of $h[n]$ and $x[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

The autocorrelation of $y[n]$ is then

$$\begin{aligned} \phi_{yy}[m] &= \mathcal{E}\{y[n+m]y[n]\} \\ &= \mathcal{E}\left\{ \sum_{k=-\infty}^{\infty} h[k]x[n+m-k] \sum_{l=-\infty}^{\infty} h[l]x[n-l] \right\} \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h[k]h[l]\mathcal{E}\{x[n+m-k]x[n-l]\} \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h[k]h[l]\phi_{xx}[l+m-k] \end{aligned}$$

Since $x[n]$ is white noise, it has the autocorrelation function

$$\phi_{xx}[l+m-k] = \sigma_x^2 \delta[l+m-k]$$

Substituting this into the expression for $\phi_{yy}[m]$ gives

$$\begin{aligned} \phi_{yy}[m] &= \sigma_x^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h[k]h[l]\delta[l+m-k] \\ &= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l+m]h[l] \end{aligned}$$

Note that

$$\phi_{yy}[m] = \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l-m]h[l]$$

is also a correct answer, since $\phi_{yy}[m] = \phi_{yy}[-m]$.

(b) Taking the DTFT of $\phi_{yy}[m]$ will give the power density spectrum $\Phi_{yy}(\omega)$.

$$\begin{aligned}
\Phi_{yy}(\omega) &= \sum_{m=-\infty}^{\infty} \left\{ \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l+m]h[l] \right\} e^{-j\omega m} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l] \sum_{m=-\infty}^{\infty} h[l+m]e^{-j\omega m}
\end{aligned}$$

Substituting $k = l + m$ into the second summation gives

$$\begin{aligned}
\Phi_{yy}(\omega) &= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l] \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega(k-l)} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l]e^{j\omega l} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[-l]e^{-j\omega l} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\
&= \sigma_x^2 H^*(e^{j\omega})H(e^{j\omega}) \\
&= \sigma_x^2 \left|H(e^{j\omega})\right|^2
\end{aligned}$$

(c) This problem can be approached either in the time domain or the z-transform domain.

Time domain: Since all the a_k's are zero for a MA process,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

so $y[n]$ is nonzero for $0 \le n \le M$. Note that the autocorrelation sequence,

$$\phi_{yy}[m] = \sum_{n=-\infty}^{\infty} y[n+m]y[n]$$

can be re-written as a convolution

$$\phi_{yy}[m] = \sum_{n=-\infty}^{\infty} g[m-n]y[n]$$

where $g[n] = y[-n]$. Therefore,

$$\phi_{yy}[n] = y[-n] * y[n]$$

Since $y[-n]$ is nonzero for $-M \le n \le 0$, and $y[n]$ is nonzero for $0 \le n \le M$, we see that their convolution $\phi_{yy}[m]$ is nonzero only in the interval $|m| \le M$.

Z-transform domain: Note that

$$\Phi_{yy}(z) = \sigma_x^2 H(z)H^*(z)$$

If all the a_k's $= 0$, then

$$\begin{aligned}
H(z) &= \sum_{k=0}^{M} b_k z^{-k} \\
\Phi_{yy}(z) &= \sum_{k=0}^{M} b_k z^{-k} \sum_{\ell=0}^{M} b_\ell^* z^{\ell}
\end{aligned}$$

The relation for $\Phi_{yy}(z)$ above is found by multiplying two polynomials in z. The highest power of z in $\Phi_{yy}(z)$ is z^M which arises from the multiplication of the $k=0$ and $l=M$ coefficients. The smallest power of z in $\Phi_{yy}(z)$ is z^{-M} which arises from the multiplication of the $k=M$ and $l=0$ coefficents. Thus, $\phi_{yy}[m]$ is nonzero only in the interval $|m| \leq M$.

(d) For an AR process,

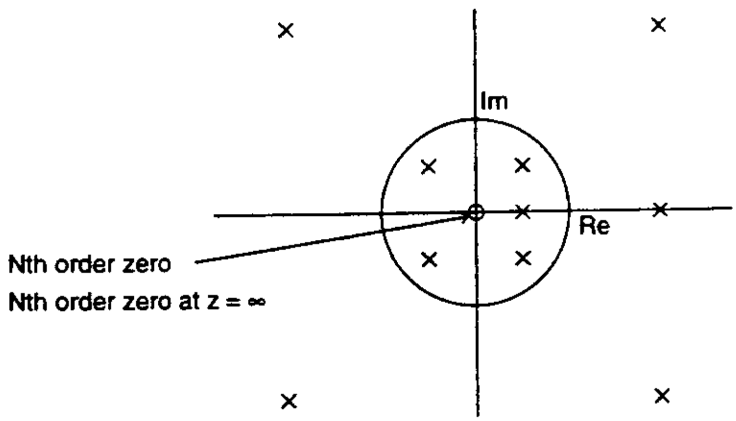
$$
\begin{aligned}
H(z) &= \frac{b_0}{1-\sum_{k=1}^{N} a_k z^{-k}} \\
&= \frac{b_0}{\prod_{k=1}^{N}(1-\alpha_k z^{-1})}
\end{aligned}
$$

Since

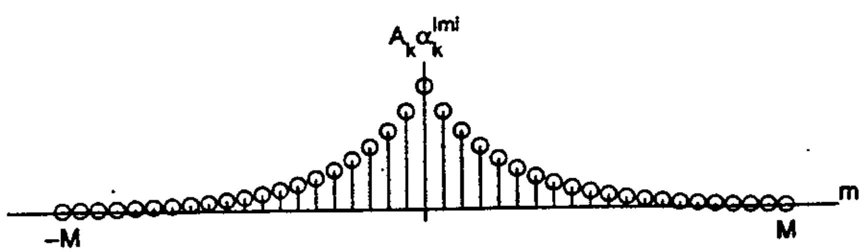
$$\Phi_{yy}(z) = \sigma_x^2 H(z) H^*(z)$$

$$\Phi_{yy}(z) = \frac{b_0^2}{\prod_{k=1}^{N}(1-\alpha_k z^{-1})(1-\alpha_k^* z)}$$

Thus, the poles for $\Phi_{yy}(z)$ come in conjugate reciprocal pairs. A sample pole-zero diagram appears below.



By performing a partial fraction expansion on $\Phi_{yy}(z)$ we find that each pole pair contributes a sequence of the form $A_k \alpha_k^{|m|}$



and therefore

$$\phi_{yy}[m] = \sum_{k=1}^{N} A_k \alpha_k^{|m|}$$

(e) For an AR process, with $b_0 = 1$,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

which means that

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + x[n]$$

The autocorrelation function is then

$$\begin{aligned}
\phi_{yy}[m] &= \phi_{yy}[-m] \\
&= \mathcal{E}\left\{y[n-m]y[n]\right\} \\
&= \mathcal{E}\left\{y[n-m]\left(\sum_{k=1}^{N} a_k y[n-k] + x[n]\right)\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{y[n-m]y[n-k]\right\} + \mathcal{E}\left\{y[n-m]x[n]\right\} \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{yx}[-m] \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{xy}[m]
\end{aligned}$$

For $m = 0$,

$$\phi_{yy}[0] = \sum_{k=1}^{N} a_k \phi_{yy}[-k] + \phi_{xy}[0]$$

The $\phi_{xy}[0]$ term is

$$\begin{aligned}
\phi_{xy}[0] &= \mathcal{E}\left\{x[n]y[n]\right\} \\
&= \mathcal{E}\left\{x[n]\left(\sum_{k=1}^{N} a_k y[n-k] + x[n]\right)\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{x[n]y[n-k]\right\} + \mathcal{E}\left\{x[n]x[n]\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{x[n]y[n-k]\right\} + \sigma_x^2
\end{aligned}$$

Note that $x[n]$ is uncorrelated with the $y[n-k]$, for $k = 1, \ldots, N$. Therefore,

$$\phi_{xy}[0] = \sigma_x^2$$

Thus,

$$\begin{aligned}
\phi_{yy}[0] &= \sum_{k=1}^{N} a_k \phi_{yy}[-k] + \sigma_x^2 \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[k] + \sigma_x^2
\end{aligned}$$

since $\phi_{yy}[k] = \phi_{yy}[-k]$. For $m \geq 1$,

$$\begin{aligned}\phi_{yy}[m] &= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{xy}[m] \\ &= \sum_{k=1}^{N} a_k \phi_{yy}[m-k]\end{aligned}$$

since $\phi_{xy}[m]$ is zero for all $m \geq 1$.

(f) By symmetry of the autocorrelation sequence, we know that

$$\begin{aligned}\phi_{yy}[m-k] &= \phi_{yy}[k-m] \\ &= \phi_{yy}[|m-k|]\end{aligned}$$

Thus,

$$\sum_{k=1}^{N} a_k \phi_{yy}[|m-k|] = \sum_{k=1}^{N} a_k \phi_{yy}[m-k]$$

Using the result from part (e), we get

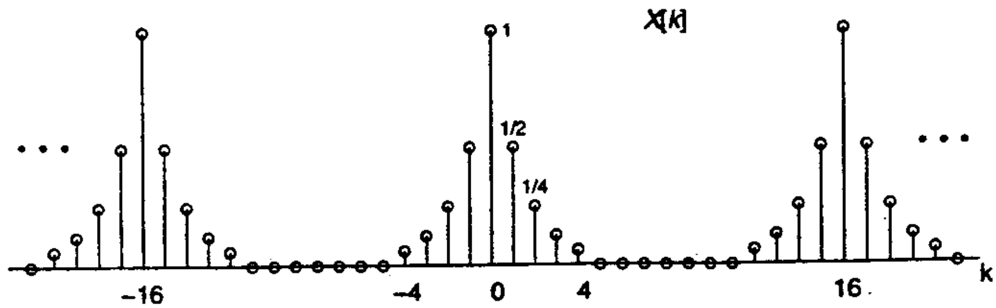
$$\sum_{k=1}^{N} a_k \phi_{yy}[|m-k|] = \phi_{yy}[m]$$

for $m = 1, 2, \ldots, N$.

10.42. (a) Sampling $x_c(t)$ we get

$$\begin{aligned}x[n] &= x_c(nT) \\ &= \frac{1}{16} \sum_{k=-4}^{4} \left(\frac{1}{2}\right)^{|k|} e^{j(2\pi/16)kn}\end{aligned}$$

Define the periodic sequence $X[k]$ to be



Then we see that we can write $x[n]$ in terms of $X[k]$:

$$\begin{aligned}x[n] &= \frac{1}{16} \sum_{k=-4}^{4} X[k] e^{j(2\pi/16)kn} \\ &= \frac{1}{16} \sum_{k=-8}^{7} X[k] e^{j(2\pi/16)kn} \\ &= \text{IDFS}\{X[k]\}\end{aligned}$$

However, since the period we use in the sum of the IDFS is unimportant we can also write

$$\begin{aligned} x[n] &= \frac{1}{16}\sum_{k=0}^{15} X[k]e^{j(2\pi/16)kn} \\ &= \text{IDFS}\{X[k]\} \\ &= \text{IDFT}\{X_0[k]\} \end{aligned}$$

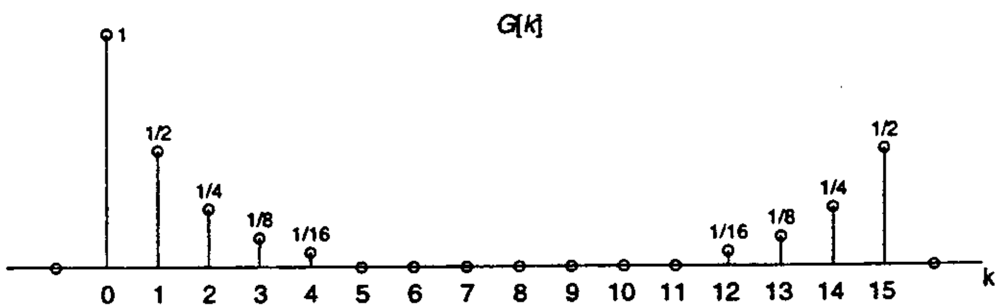
where $X_0[k]$ is the period of $X[k]$ starting at zero, i.e.,

$$X_0[k] = \begin{cases} X[k], & k = 0,\ldots,15 \\ 0, & \text{otherwise} \end{cases}$$

Using this information we can now find $G[k]$

$$\begin{aligned} G[k] &= \text{DFT}\{g[n]\} \\ &= \text{DFT}\{x[n](u[n]-u[n-16])\} \\ &= \text{DFT}\{x[n]\} \\ &= \text{DFT}\{\text{IDFT}\{X_0[k]\}\} \\ &= X_0[k] \end{aligned}$$

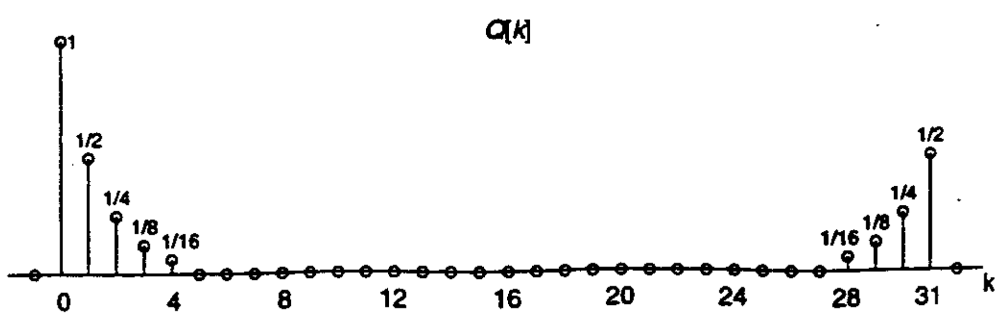
Thus, $G[k]$ looks like



(b) We want to find a sequence $Q[k]$ such that

$$\begin{aligned} q[n] &= \alpha x_c\left(\frac{n2\pi}{32}\right) \\ &= \frac{\alpha}{16}\sum_{k=-4}^{4}\left(\frac{1}{2}\right)^{|k|} e^{j(2\pi/32)kn} \end{aligned}$$

We can apply the same idea as we did in part (a), except now the DFS and DFT size should be 32 instead of 16. Going through the same steps will lead us to the sequence $Q[k]$ that looks like:



(Here we have assumed $\alpha = 1$). We see that we can interpolate in the time domain by zero padding in the *middle* of the DFT samples.

10.43. (a) Using the relation,

$$f_k = \begin{cases} \frac{k}{NT}, & 0 \le k \le N/2 \\ \frac{k-N}{NT}, & N/2 \le k \le N \end{cases}$$

where N is the DFT length and T is the sampling period, the continuous-time frequencies corresponding to the DFT indices $k = 32$ and $k = 231$ are

$$\begin{aligned} f_{32} &= \frac{32}{(256)(1/20,000)} \\ &= 2500 \text{ Hz} \\ f_{231} &= \frac{231 - 256}{(256)(1/20,000)} \\ &= -1953 \text{ Hz} \end{aligned}$$

(b) Since

$$\hat{x}[n] = x[n]w_R[n]$$

the DTFT of $\hat{x}[n]$ is simply the periodic convolution of $X(e^{j\omega})$ with $W_R(e^{j\omega})$.

$$\hat{X}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W_R(e^{j(\omega-\theta)})d\theta$$

(c) Multiplication in the time domain corresponds to periodic convolution in the frequency domain, as shown in part (b). To evaluate this periodic convolution at the frequency $\omega_{32} = 2\pi(32)/L$, (where $L = N = 256$) corresponding to the $k = 32$ DFT coefficient, we first shift the window $W_{avg}(e^{j\omega})$ to ω_{32}. Then, we multiply the shifted window with $X(e^{j\omega})$, and integrate the result. In order for

$$X_{avg}[32] = \alpha\hat{X}[31] + \hat{X}[32] + \alpha\hat{X}[33]$$

we must therefore have

$$W_{avg}(e^{j\omega}) = \begin{cases} 1, & \omega = 0 \\ \alpha, & \omega = \pm 2\pi/L \\ 0, & 2\pi k/L, \qquad \text{for } k = 2, 3, \ldots, L-2 \end{cases}$$

Note that we are only specifying $W_{avg}(e^{j\omega})$ at the DFT frequencies $\omega = 2\pi k/L$, for $k = 0, \ldots, L-1$.

(d) Note that the L point DFT of a rectangular window of length L is

$$\begin{aligned} W_R[k] &= \sum_{n=0}^{L-1}(1)e^{-j2\pi k/L} \\ &= \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi kL}} \\ &= L\delta[k] \end{aligned}$$

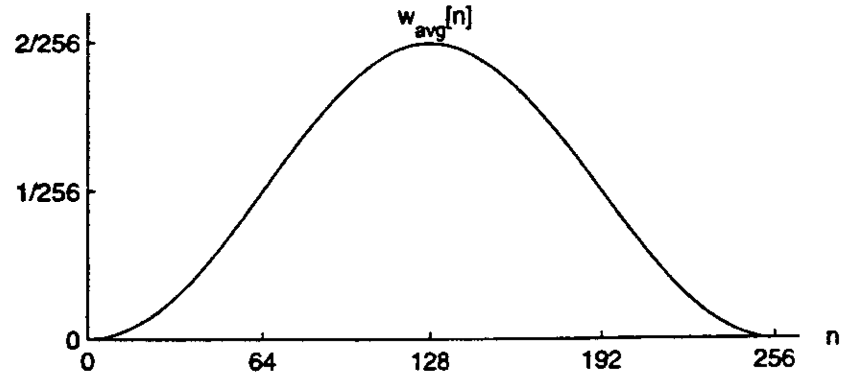
$W_{avg}(e^{j\omega})$ is only specified at DFT frequencies $\omega = 2\pi k/L$, and it can take on other values between these frequencies. Therefore, the DTFT of $W_{avg}(e^{j\omega})$ can be written in terms of $W_R(e^{j\omega})$ and two shifted versions of $W_R(e^{j\omega})$.

$$W_{avg}(e^{j\omega}) = \frac{\alpha}{L}W_R\left(e^{j(\omega+2\pi/L)}\right) + \frac{1}{L}W_R(e^{j\omega}) + \frac{\alpha}{L}W_R\left(e^{j(\omega-2\pi/L)}\right)$$

(e) Taking the inverse DTFT of $W_{avg}(e^{j\omega})$ gives $w_{avg}[n]$.

$$\begin{aligned} w_{avg}[n] &= \frac{\alpha}{L} w_R[n] e^{-j2\pi n/L} + \frac{1}{L} w_R[n] + \frac{\alpha}{L} w_R[n] e^{j2\pi n/L} \\ &= \left[\frac{1}{L} + \frac{2\alpha}{L}\cos\left(\frac{2\pi n}{L}\right)\right] w_R[n] \end{aligned}$$

A sketch of $w_{avg}[n]$ is provided below.

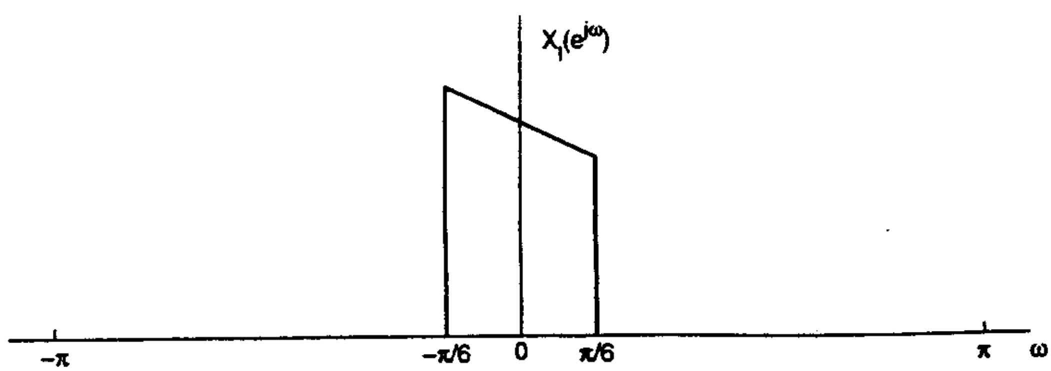


10.44. (a) After the lowpass filter, the highest frequency in the signal is $\Delta\omega$. To avoid aliasing in the downsampler we must have

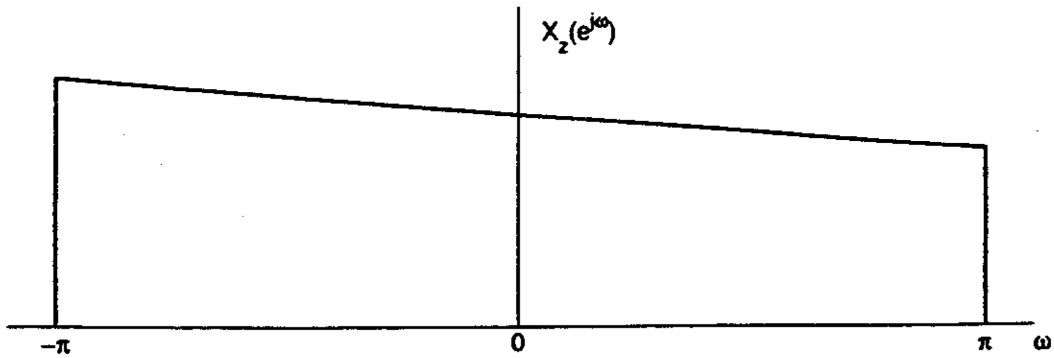
$$\begin{aligned} \Delta\omega M &\le \pi \\ M &\le \frac{\pi}{\Delta\omega} \\ &\le \frac{N}{2k_\Delta} \end{aligned}$$

$$M_{\max} = \frac{N}{2k_\Delta}$$

(b) The fourier transform of $x_l[n]$ looks like



so $M = 6$ is the largest M we can use that avoids aliasing. With this choice of M the fourier transform of $x_s[n]$ looks like



Taking the DFT of $x_z[n]$ gives us N samples of $X_z(e^{j\omega})$ spaced $2\pi/N$ apart in frequency. By examining the figures above we see that these samples correspond to the desired samples of $X(e^{j\omega})$ which will be spaced $2\Delta\omega/N$ apart inside the region $-\Delta\omega < \omega < \Delta\omega$.

Note that after downsampling the endpoints of the region alias. Therefore, we cannot trust the values our new DFT provides at those points. However, the way the problem is set up we already know the values at the endpoints from the original DFT.

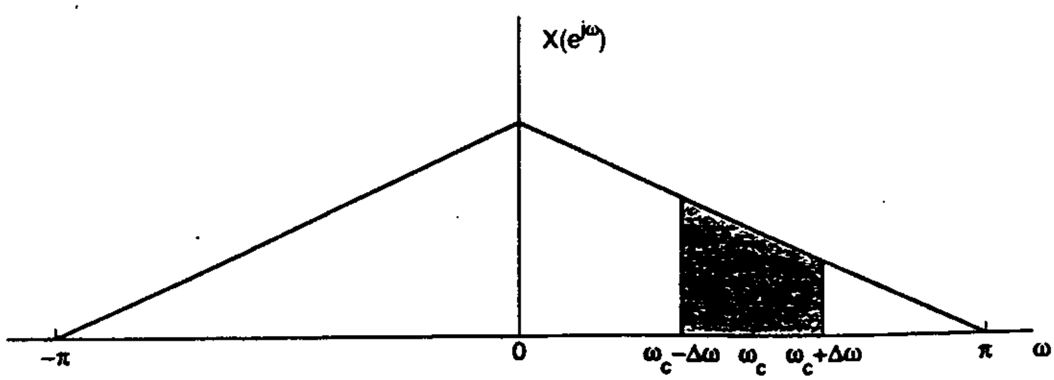
(c) The system $p[n]$ periodically replicates $X_N[n]$ to create $\tilde{X}_N[n]$. Then, the upsampler inserts $M-1$ zeros in betweeen each sample of $\tilde{X}_N[n]$. Thus, the samples $k_c - k_\Delta$ and $k_c + k_\Delta$ which border the zoom region in the original DFT map to $M(k_c - k_\Delta)$ and $M(k_c + k_\Delta)$. The system $h[n]$ then interpolates between the nonzero points filling in the "missing" samples. Since the linear phase filter is length 513 it adds a delay of $M/2 = 512/2 = 256$ samples so the desired samples of $\tilde{X}_{NM}[n]$ now lie in the region

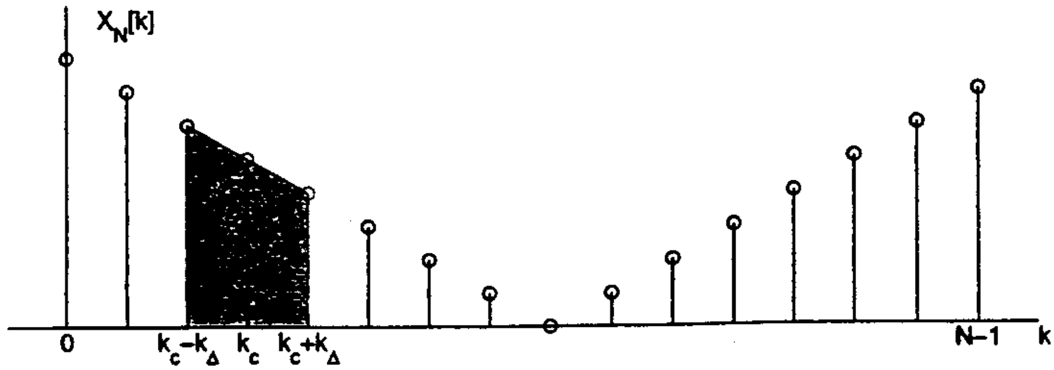
$$\begin{aligned} M(k_c - k_\Delta) + 256 &\le n \le M(k_c + k_\Delta) + 256 \\ k_c' - k_\Delta' &\le n \le k_c' + k_\Delta' \end{aligned}$$

where

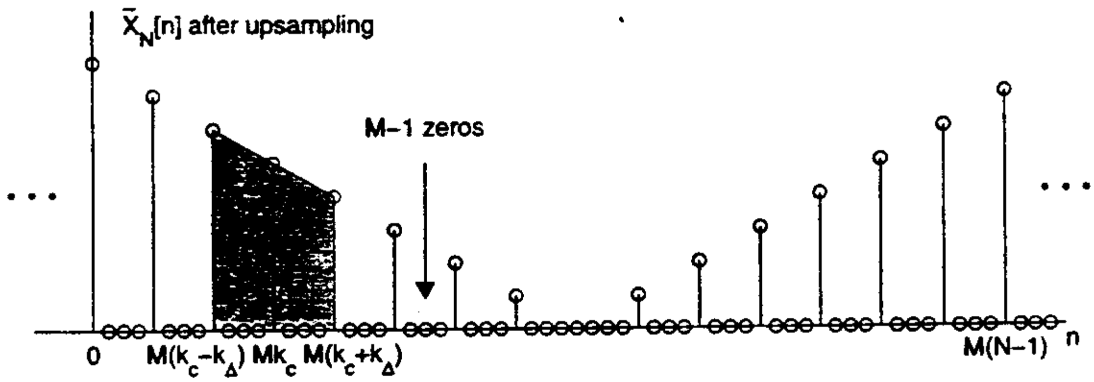
$$\begin{aligned} k_c' &= Mk_c + 256 \\ k_\Delta' &= Mk_\Delta \end{aligned}$$

(d) A typical sketch of $X(e^{j\omega})$ and $X_N[k]$ look like.

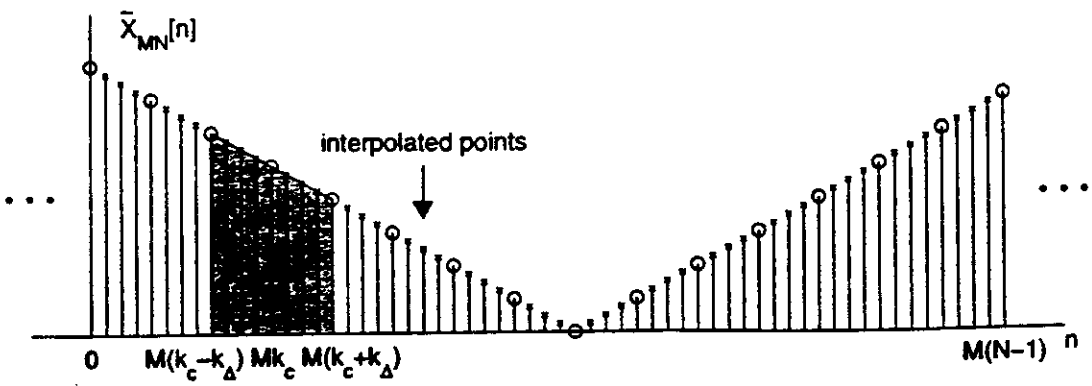




After periodically replicating and upsampling by M we have a signal that looks like



Filtering by $h[n]$ then interpolates between the samples. $\tilde{X}_{NM}[n]$ is shown below if we assume that $h[n]$ is the ideal zero phase filter. The points with an x correspond to the interpolated points.



Thus, we need to extract the points

$$\begin{aligned} M(k_c - k_\Delta) &\leq n \leq M(k_c + k_\Delta) \\ k'_c - k'_\Delta &\leq n \leq k'_c + k'_\Delta \end{aligned}$$

where

$$\begin{aligned} k'_c &= M k_c \\ k'_\Delta &= M k_\Delta \end{aligned}$$