Solutions – Chapter 6

Structures for Discrete-Time Systems

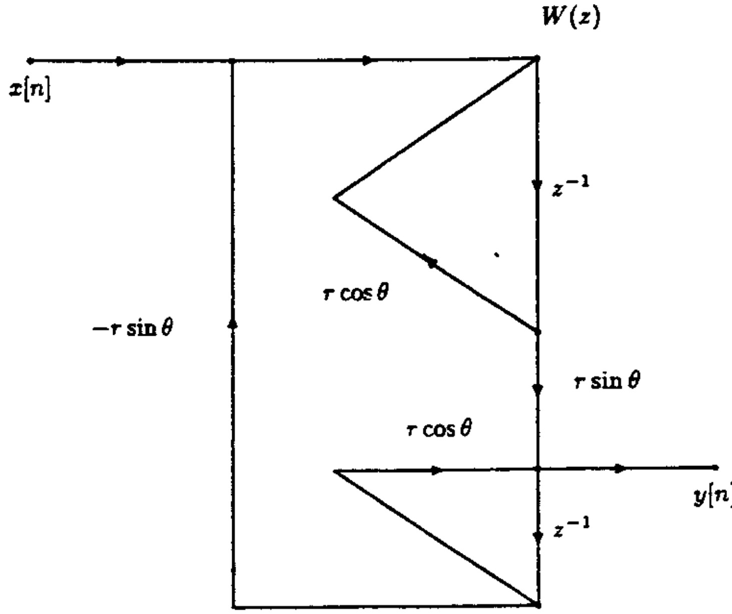
6.1. We proceed by obtaining the transfer functions for each of the networks. For network 1,

$$Y(z) = 2r\cos\theta z^{-1}Y(z) - r^2z^{-2}Y(z) + X(z)$$

or

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2z^{-2}}$$

For network 2, define $W(z)$ as in the figure below:



then

$$W(z) = X(z) - r\sin\theta z^{-1}Y(z) + r\cos\theta z^{-1}W(z)$$

and

$$Y(z) = r\sin\theta z^{-1}W(z) + r\cos\theta z^{-1}Y(z)$$

Eliminate $W(z)$ to get

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2z^{-2}}$$

Hence the two networks have the same poles.

6.2. The only input to the $y[n]$ node is a unity branch connection from the $x[n]$ node. The rest of the network does not affect the input-output relationship. The difference equation is $y[n] = x[n]$.

6.3.

$$H(z) = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

System (d) is recognizable as a transposed direct form II implementation of $H(z)$.

6.4. (a) From the flow graph, we have:

$$Y(z) = 2X(z) + (\frac{1}{4}X(z) - \frac{1}{4}Y(z) + \frac{3}{8}Y(z)z^{-1})z^{-1}.$$

That is:

$$Y(z)(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}) = X(z)(2+\frac{1}{4}z^{-1}).$$

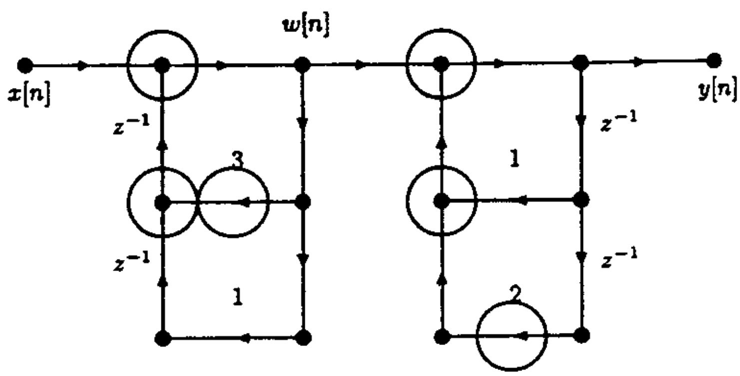
The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2+\frac{1}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

(b) To get the difference equation, we just inverse Z-transform the equation in a. We get:

$$y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2] = 2x[n]+\frac{1}{4}x[n-1].$$

6.5. The flow graph for this system is drawn below.



(a)

$$w[n] = x[n]+3w[n-1]+w[n-2]$$

$$y[n] = w[n]+y[n-1]+2y[n-2]$$

(b)

$$W(z) = X(z)+3z^{-1}W(z)+z^{-2}W(z)$$

$$Y(z) = W(z)+z^{-1}Y(z)+2z^{-2}Y(z)$$

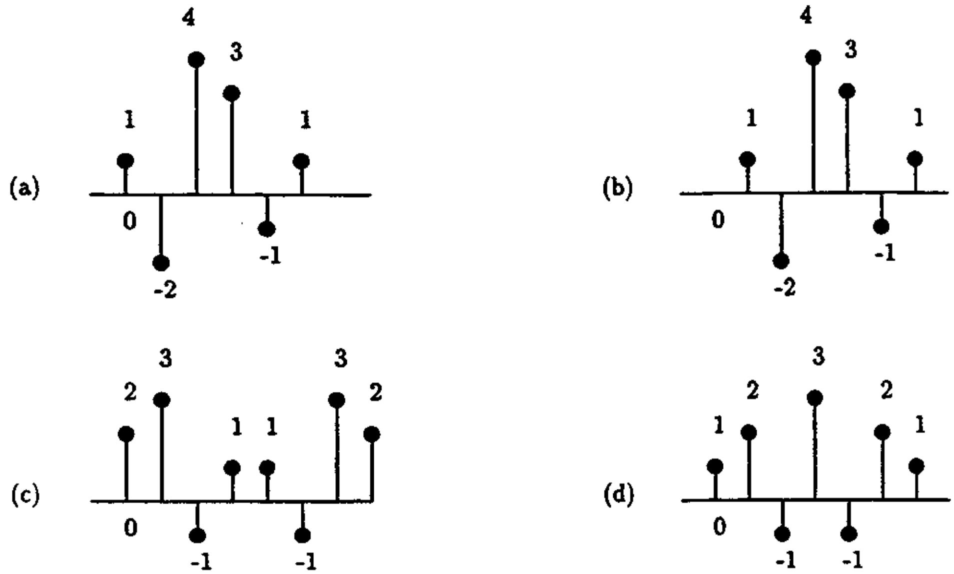
So

$$\begin{aligned}\frac{Y(z)}{X(z)} &= H(z)\\ &= \frac{1}{(1-z^{-1}-2z^{-2})(1-3z^{-1}-z^{-2})}\\ &= \frac{1}{1-4z^{-1}+7z^{-3}+2z^{-4}}.\end{aligned}$$

(c) Adds and multiplies are circled above: 4 real adds and 2 real multiplies per output point.

(d) It is not possible to reduce the number of storage registers. Note that implementing $H(z)$ above in the canonical direct form II (minimum storage registers) also requires 4 registers.

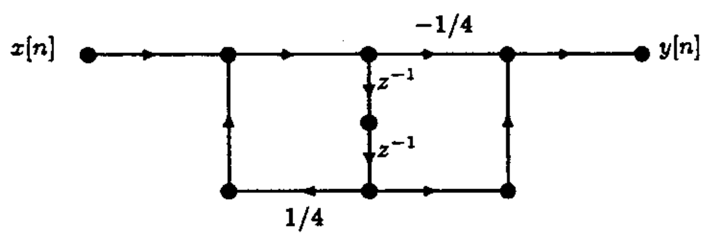
6.6. The impulse responses of each system are shown below.



6.7. We have

$$H(z) = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4}z^{-2}}.$$

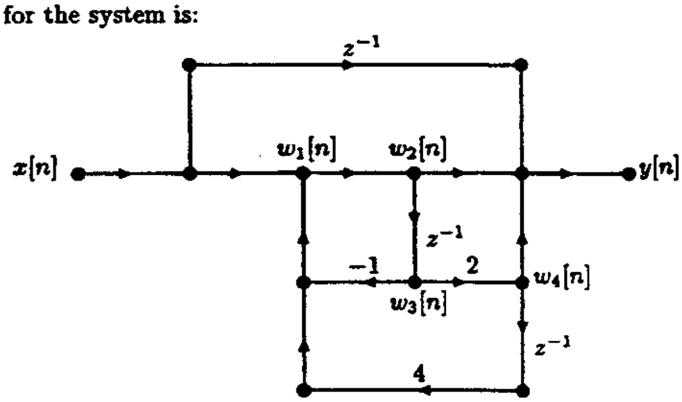
Therefore the direct form II is given by:



6.8. By looking at the graph, we get:

$$y[n] = 2y[n-2] + 3x[n-1] + x[n-2].$$

6.9. The signal flow graph for the system is:



(a) First we need to determine the transfer function. We have

$$\begin{aligned} w_1[n] &= x[n] - w_3[n] + 4w_4[n-1] \\ w_2[n] &= w_1[n] \\ w_3[n] &= w_2[n-1] \\ w_4[n] &= 2w_3[n] \\ y[n] &= w_2[n] + x[n-1] + w_4[n]. \end{aligned}$$

Taking the Z-transform of the above equations, rearranging and substituting terms, we get:

$$H(z) = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}}.$$

The difference equation is thus given by:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

The impulse response is the response to an impulse, therefore:

$$h[n] + h[n-1] - 8h[n-2] = \delta[n] + 3\delta[n-1] + \delta[n-2] - 8\delta[n-3].$$

From the above equation, we have:

$$\begin{aligned} &h[0] = 1 \\ &h[1] = 3 - h[0] = 2. \end{aligned}$$

(b) From part (a) we have:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

6.10. (a)

$$\begin{aligned} w[n] &= \frac{1}{2}y[n] + x[n] \\ v[n] &= \frac{1}{2}y[n] + 2x[n] + w[n-1] \\ y[n] &= v[n-1] + x[n]. \end{aligned}$$

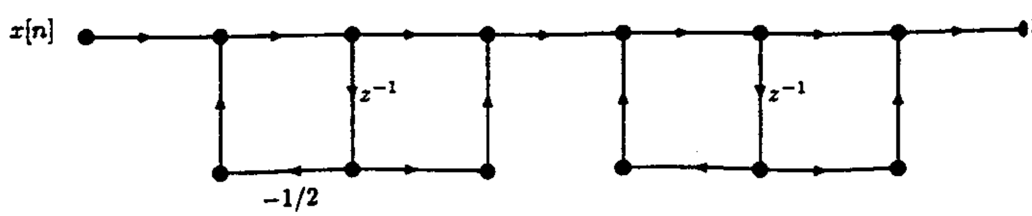
(b) Using the Z-transform of the difference equations in part (a), we get the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

We can rewrite it as :

$$H(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

We thus get the following cascade form:

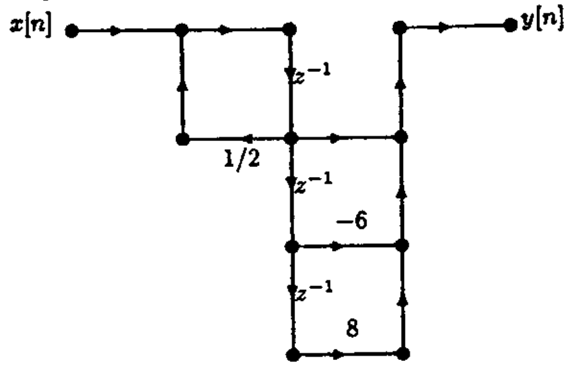


(c) The system function has poles at $z = -\frac{1}{2}$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.

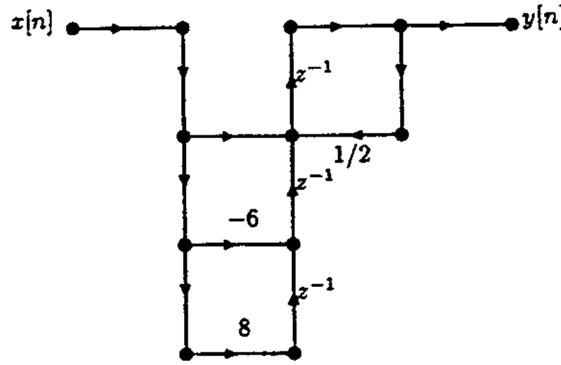
6.11. (a) $H(z)$ can be rewritten as:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}.$$

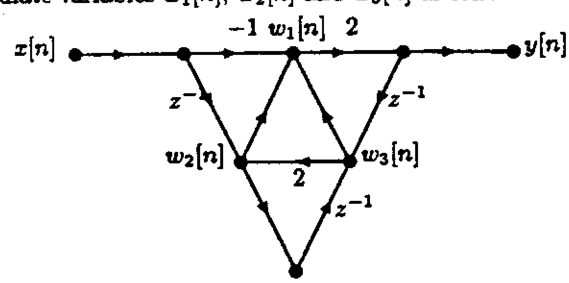
We thus get the following direct from II flow graph :



(b) To get the transposed form, we just reverse the arrows and exchange the input and the ouput. The graph can then be redrawn as:



6.12. We define the intermediate variables $w_1[n]$, $w_2[n]$ and $w_3[n]$ as follows:



We thus have the following relationships:

$$\begin{aligned} w_1[n] &= -x[n] + w_2[n] + w_3[n] \\ w_2[n] &= x[n-1] + 2w_3[n] \\ w_3[n] &= w_2[n-1] + y[n-1] \\ y[n] &= 2w_1[n]. \end{aligned}$$

Z-transforming the above equations and rearranging and grouping terms, we get:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-2 + 6z^{-1} + 2z^{-2}}{1 - 8z^{-1}}.$$

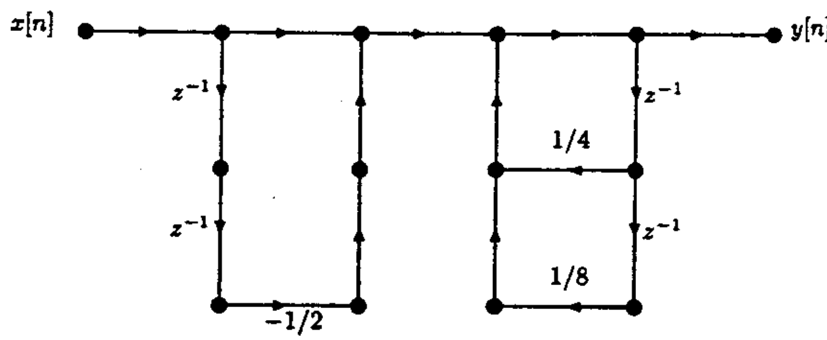
Taking the inverse Z-transform, we get the following difference equation:

$$y[n] - 8y[n-1] = -2x[n] + 6x[n-1] + 2x[n-2].$$

6.13.

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

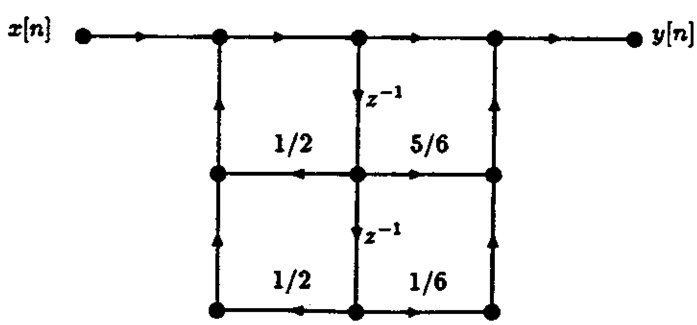
The direct form I implementation is:



6.14.

$$H(z) = \frac{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

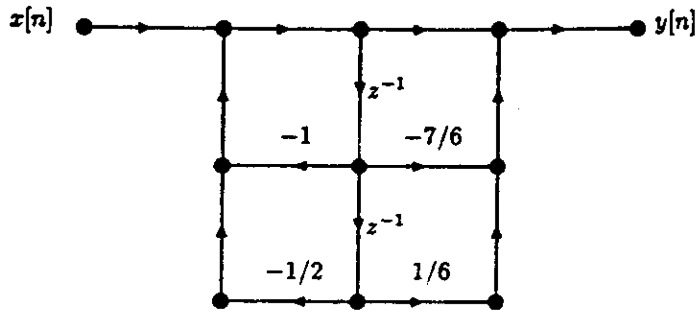
The direct form II implementation is:



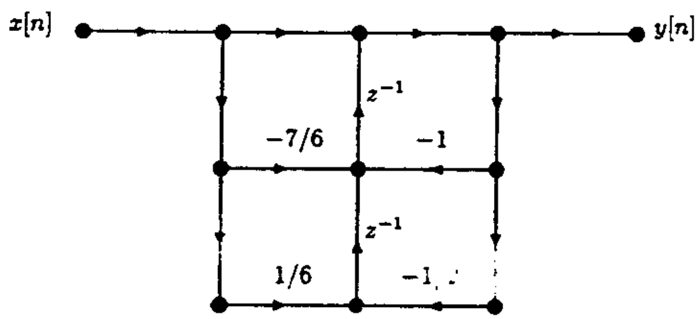
6.15.

$$H(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}.$$

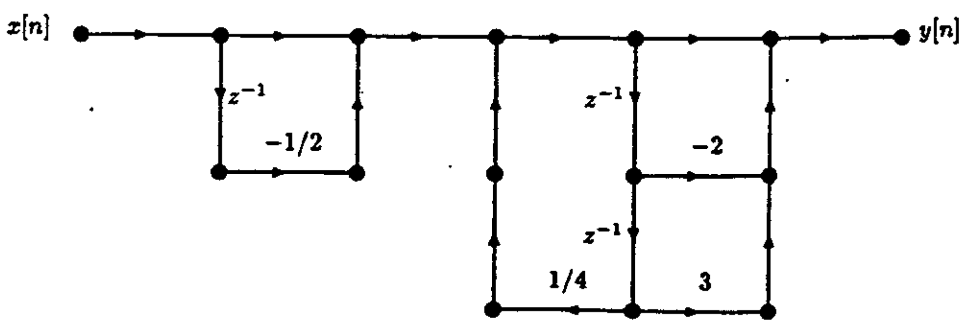
To get the transposed direct form II implementation, we first get the direct form II:



Now, we reverse the arrows and exchange the role of the input and the ouput to get the transposed direct form II:



6.16. (a) We just reverse the arrows and reverse the role of the input and the output, we get:



(b) The original system is the cascade of two transposed direct form II structures, therefore the system function is given by:

$$H(z) = \left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right)\left(1 - \frac{1}{2}z^{-1}\right).$$

The transposed graph, on the other hand, is the cascade of two direct form II structures, therefore the system function is given by:

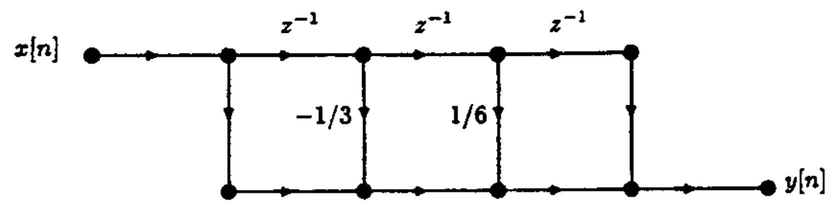
$$H(z) = \left(1 - \frac{1}{2}z^{-1}\right)\left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right).$$

This confirms that both graphs have the same system function $H(z)$.

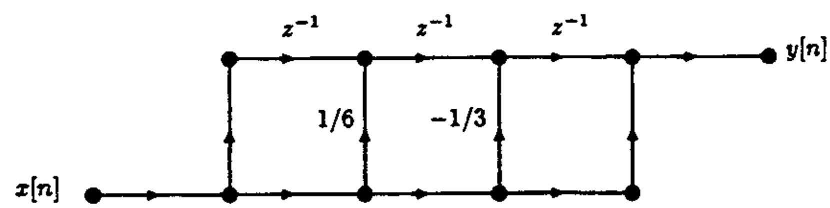
6.17.

$$H(z) = 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}.$$

(a) Direct form implementation of this system:



(b) Transposed direct form implementation of the system:



6.18. The flow graph is just a cascade of two transposed direct form II structures, the system function is thus given by:

$$H(z) = \left(\frac{1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}\right)\left(\frac{1}{1 - az^{-1}}\right).$$

Which can be rewritten as:

$$H(z) = \frac{(1+2z^{-1})(1-\frac{2}{3}z^{-1})}{(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})(1-az^{-1})}.$$

In order to implement this system function with a second-order direct form II signal flow graph, a pole-zero cancellation has to occur, this happens if $a = \frac{2}{3}$, $a = -2$ or $a = 0$. If $a = \frac{2}{3}$, the overall system function is:

$$H(z) = \frac{1+2z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is:

$$H(z) = \frac{1-\frac{2}{3}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

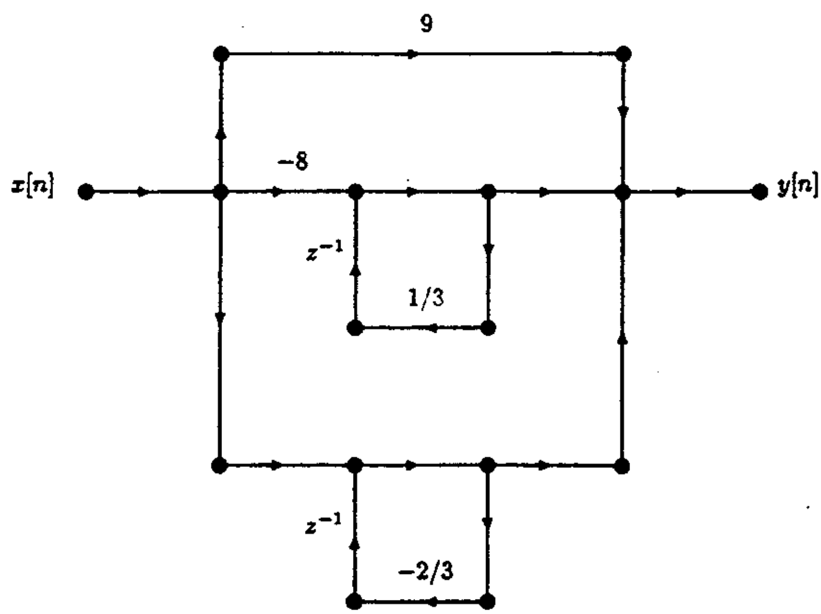
And finally if $a = 0$, the overall system function is:

$$H(z) = \frac{(1+2z^{-1})(1-\frac{2}{3}z^{-1})}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

6.19. Using partial fraction expansion, the system function can be rewritten as:

$$H(z) = \frac{-8}{1-\frac{1}{3}z^{-1}} + \frac{1}{1+\frac{2}{3}z^{-1}} + 9.$$

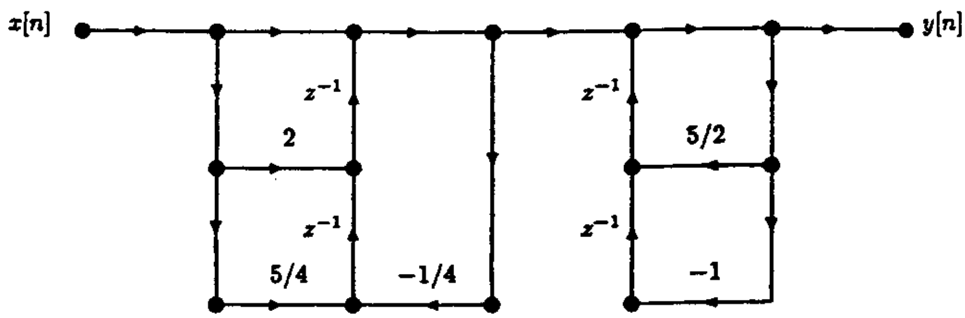
Now we can draw the flow graph that implements this system as a parallel combination of first-order transposed direct form II sections:



6.20. The transfer function can be rewritten as:

$$H(z) = \frac{(1+2z^{-1}+\frac{5}{4}z^{-2})}{(1+\frac{1}{4}z^{-2})(1-\frac{5}{2}z^{-1}+z^{-2})}$$

which can be implemented as the following cascade of second-order transposed direct form II sections:

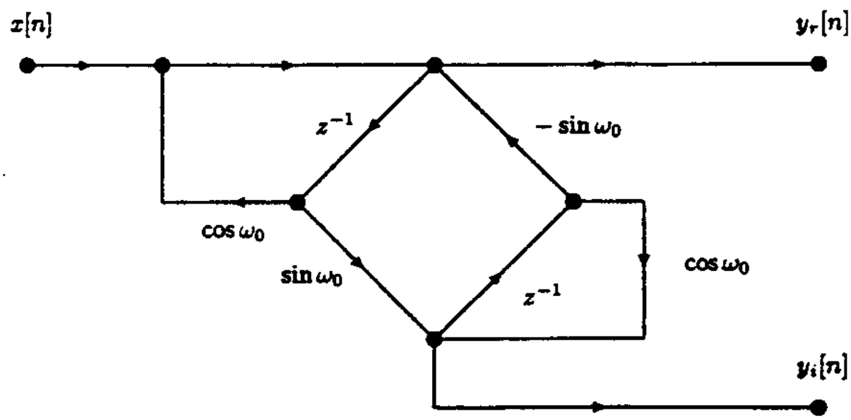


6.21.

$$h[n] = e^{j\omega_0 n}u[n] \longleftrightarrow H(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}} = \frac{Y(z)}{X(z)}.$$

So $y[n] = e^{j\omega_0}y[n-1] + x[n]$. Let $y[n] = y_r[n] + jy_i[n]$. Then $y_r[n] + jy_i[n] = (\cos\omega_0 + j\sin\omega_0)(y_r[n-1] + jy_i[n-1]) + x[n]$. Separate the real and imaginary parts:

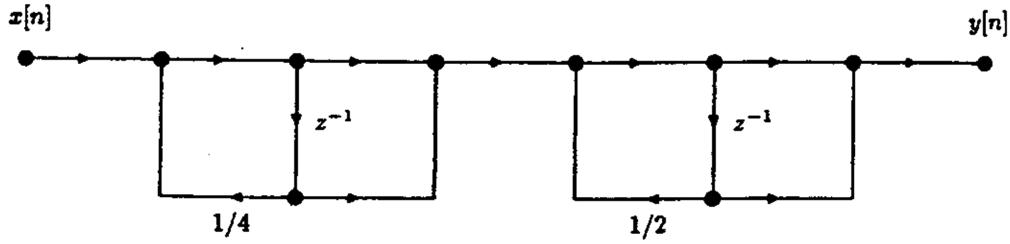
$$\begin{aligned} y_r[n] &= x[n] + \cos\omega_0 y_r[n-1] - \sin\omega_0 y_i[n-1] \\ y_i[n] &= \sin\omega_0 y_r[n-1] + \cos\omega_0 y_i[n-1]. \end{aligned}$$



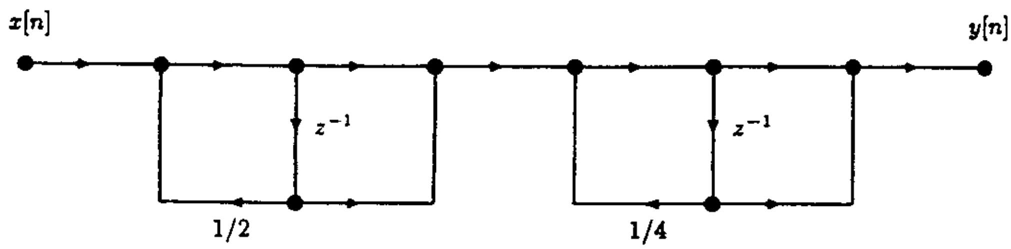
6.22.

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}.$$

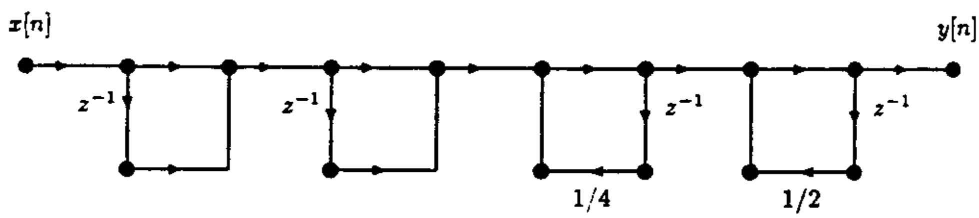
$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right).$$



$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right).$$



Plus 12 systems of this form:



with the three types of 1st-order systems taken in various orders.

6.23. Causal LTI system with system function:

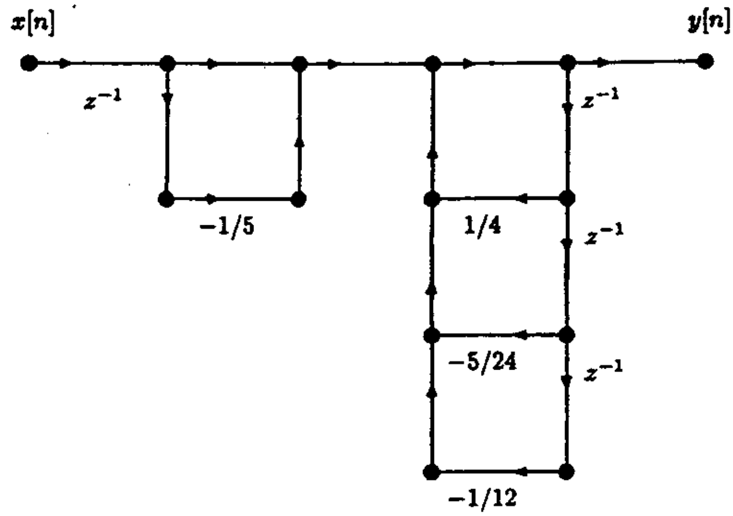
$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2})(1+\frac{1}{4}z^{-1})}.$$

(a) (i) Direct form I.

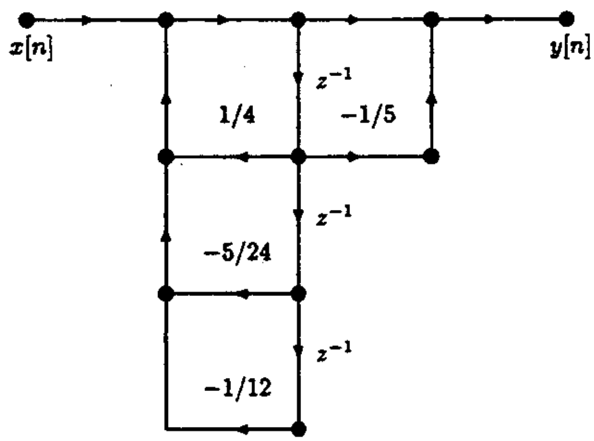
$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z(-3)}$$

so

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5} \text{ and } a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$



(ii) Direct form II.

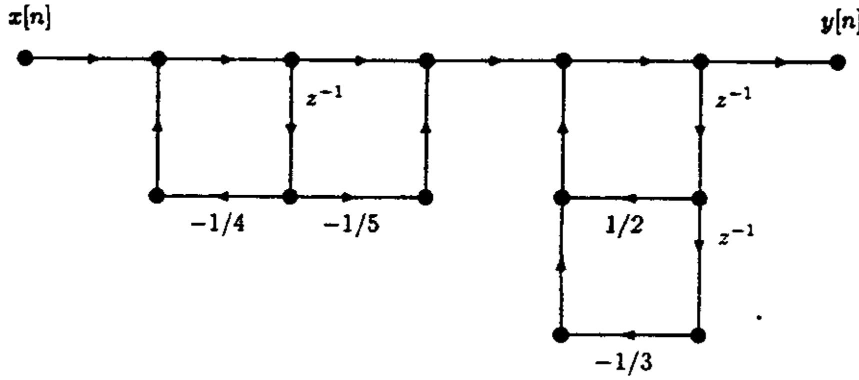


(iii) Cascade form using first and second order direct form II sections.

$$H(z) = \left(\frac{1 - \frac{1}{5}z^{-1}}{1 + \frac{1}{4}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}\right).$$

So

$$b_{01} = 1\ , b_{11} = -\tfrac{1}{5}\ , b_{21} = 0\ ,$$
$$b_{02} = 1\ , b_{12} = 0\ , b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ , a_{21} = 0\ , a_{12} = \tfrac{1}{2}\ , a_{22} = -\tfrac{1}{3}.$$

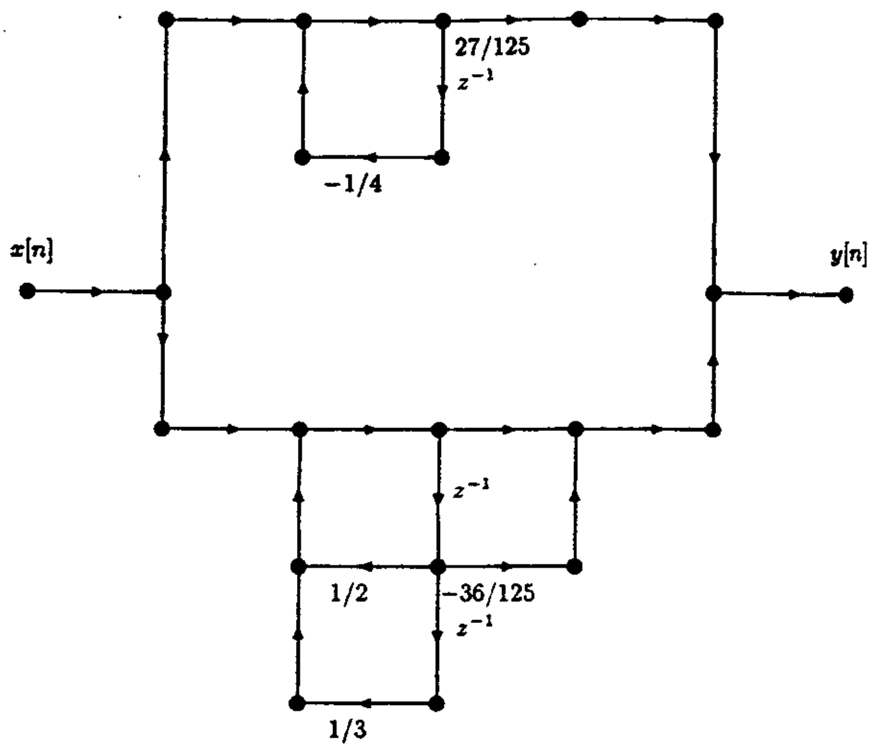


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

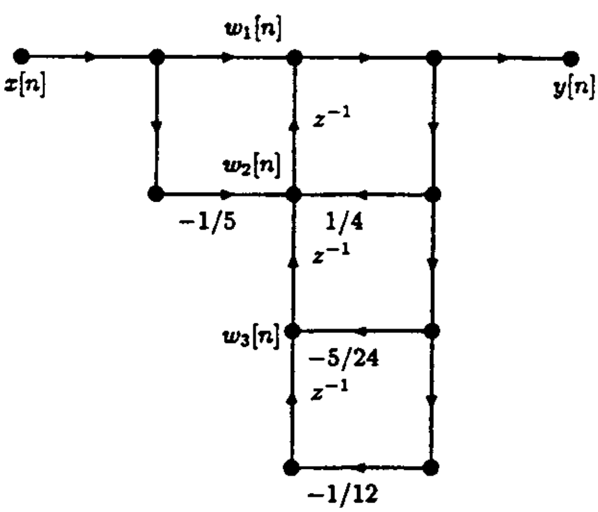
$$H(z) = \frac{\frac{27}{125}}{1 + \frac{1}{4}z^{-1}} + \frac{\frac{98}{125} - \frac{36}{125}z^{-1}}{1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}}.$$

So

$$e_{01} = \tfrac{27}{125}\ , e_{11} = 0\ ,$$
$$e_{02} = \tfrac{98}{125}\ , e_{12} = -\tfrac{36}{125}\ , \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ , a_{21} = 0\ , a_{12} = \tfrac{1}{2}\ , a_{22} = -\tfrac{1}{3}.$$



(v) Transposed direct form II
We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}$$

Combining the above equations, we get:

$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

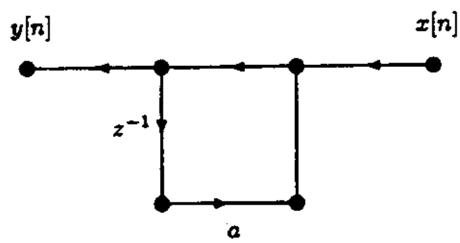
Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.24. (a)

$$H(z) = \frac{1}{1 - az^{-1}}$$

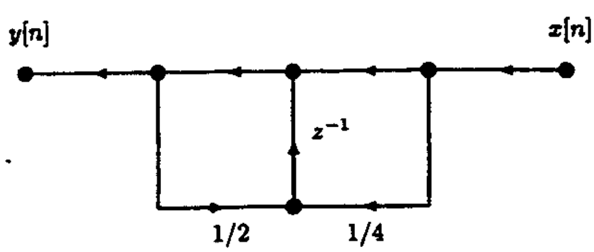


$$y[n] = x[n] + ay[n-1]$$

$$H_T(z) = \frac{1}{1 - az^{-1}} = H(z)$$

(b)

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

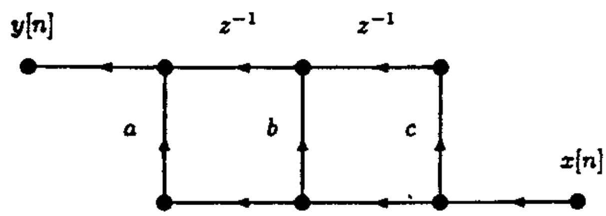


$$y[n] = x[n] + \frac{1}{4}x[n-1] + \frac{1}{2}y[n-1]$$

$$H_T(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} = H(z)$$

(c)

$$H(z) = a + bz^{-1} + cz^{-2}$$

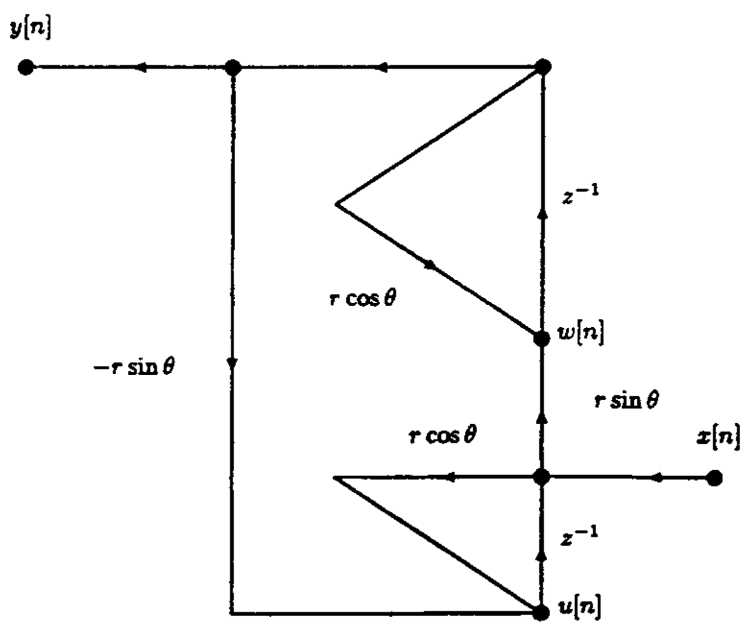


$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

$$H_T(z) = a + bz^{-1} + cz^{-2} = H(z)$$

(d)

$$H(z) = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$



$$
\begin{aligned}
V &= X + z^{-1}U \\
U &= r\cos\theta V - r\sin\theta Y \\
W &= r\sin\theta V + r\cos\theta z^{-1}W \\
Y &= z^{-1}W \\
\Longrightarrow \frac{Y}{X} &= H_T(z) \\
&= \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \\
&= H(z).
\end{aligned}
$$

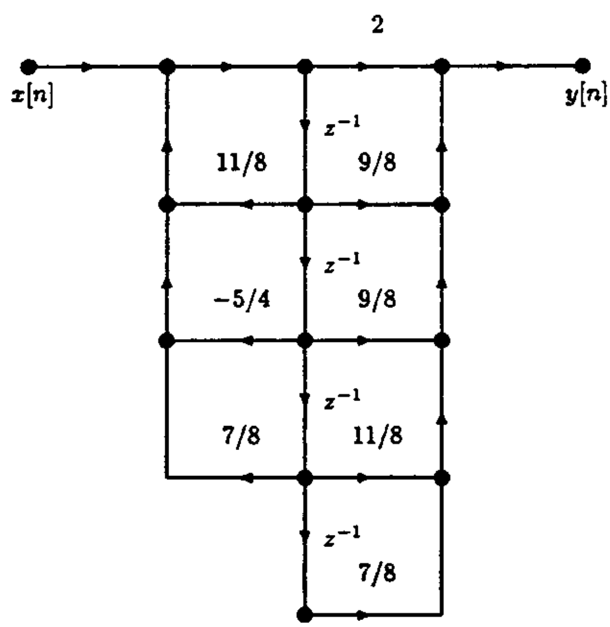
6.25. (a)

$$
\begin{aligned}
H(z) &= \frac{1}{1-z^{-1}}\left[\frac{1-\frac{1}{2}z^{-1}}{1-\frac{3}{8}z^{-1}+\frac{7}{8}z^{-2}} + 1 + 2z^{-1} + z^{-2}\right] \\
&= \frac{2+\frac{9}{8}z^{-1}+\frac{9}{8}z^{-2}+\frac{11}{8}z^{-3}+\frac{7}{8}z^{-4}}{1-\frac{11}{8}z^{-1}+\frac{5}{4}z^{-2}-\frac{7}{8}z^{-3}}.
\end{aligned}
$$

(b)

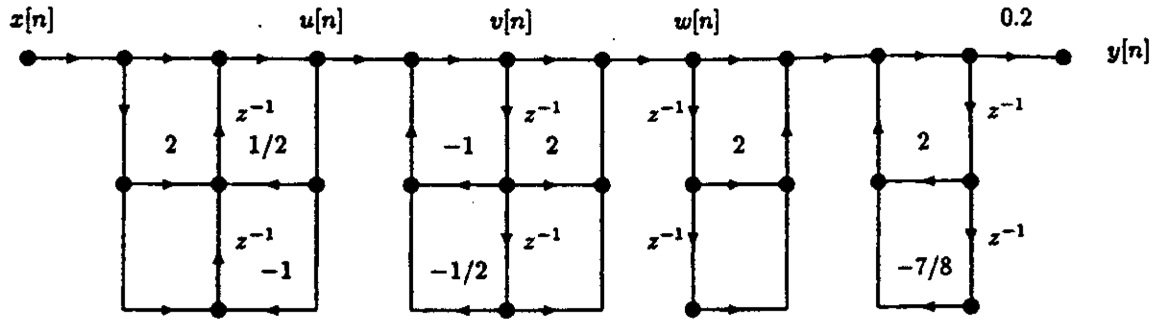
$$
\begin{aligned}
y[n] &= 2x[n] + \frac{9}{8}x[n-1] + \frac{9}{8}x[n-2] + \frac{11}{8}x[n-3] + \frac{7}{8}x[n-4] \\
&+ \frac{11}{8}y[n-1] - \frac{5}{4}y[n-2] + \frac{7}{8}y[n-3].
\end{aligned}
$$

(c) Use Direct Form II:



6.26. (a) We can rearrange $H(z)$ this way:

$$
H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2
$$

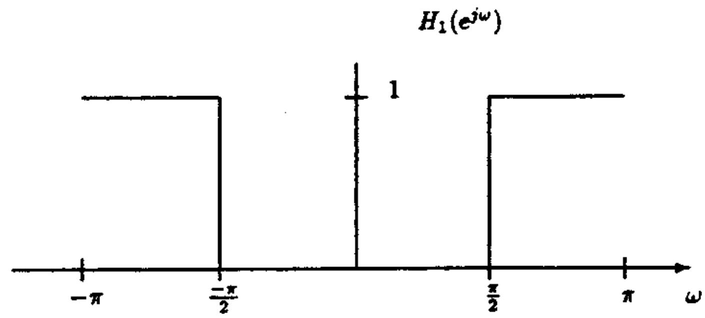


The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

(b)

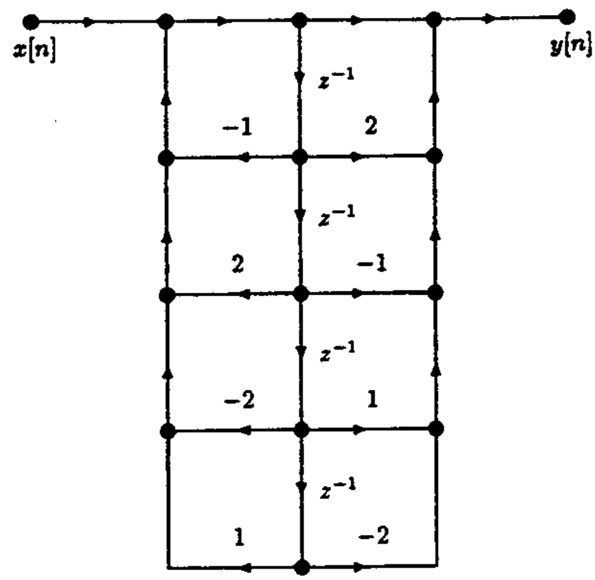
$$
\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}
$$

6.27. (a) $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$.

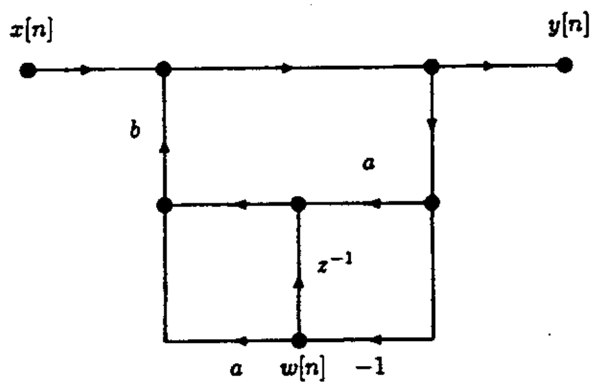


(b) For $H_1(z) = H(-z)$, replace each z^{-1} by $-z^{-1}$. Alternatively, replace each coefficient of an odd-delayed variable by its negative.

(c)



6.28.



(a)

$$\begin{aligned} y[n] &= x[n] + abw[n] + bw[n-1] + aby[n] \\ w[n] &= -y[n]. \end{aligned}$$

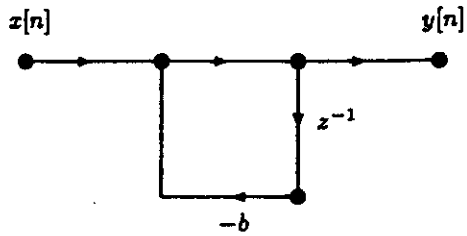
Eliminate w[n]:

$$\begin{aligned} y[n] &= x[n] - aby[n] - by[n-1] + aby[n] \\ y[n] &= x[n] - by[n-1] \end{aligned}$$

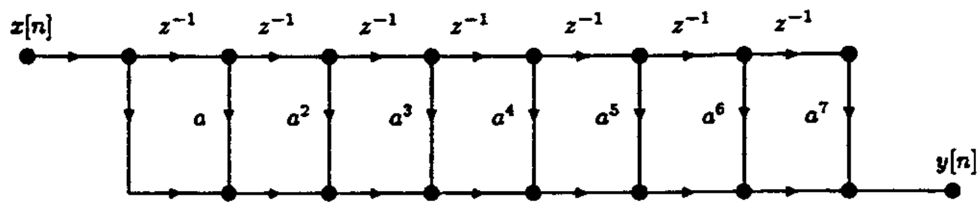
So:

$$H(z) = \frac{1}{1 + bz^{-1}}.$$

(b)



6.29. (a)



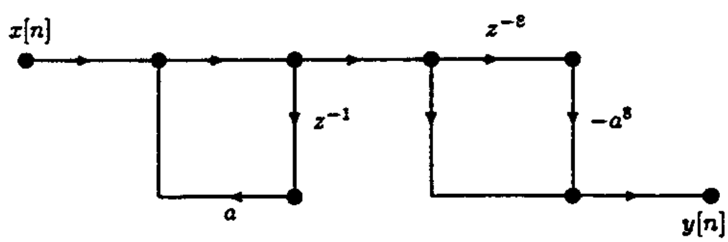
(b) From

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

it follows that

$$\sum_{n=0}^{7} a^n z^{-n} = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

(c)

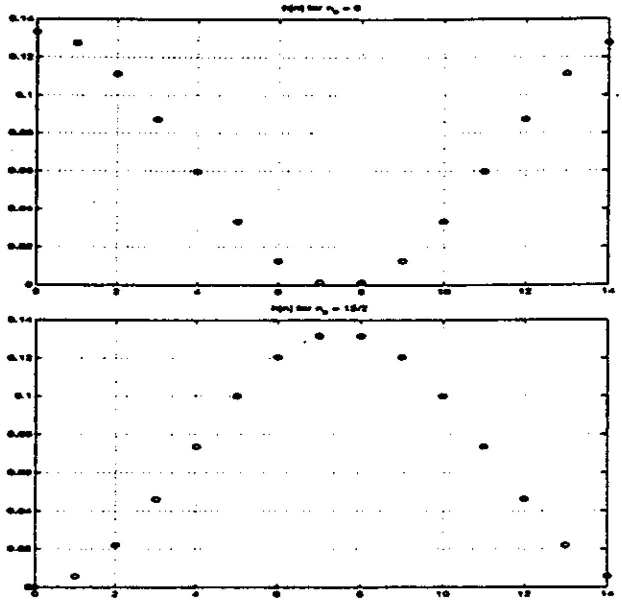


(d) (i) (c) has the most storage: 9 vs. 7.

(ii) (a) has the most arithmetic: 7 adds + 7 multiplies per sample, vs. 2 multiplies + 2 adds per sample.

6.30.

(a)



(b)

$$
\begin{aligned}
H(z) &= \frac{1}{15}\sum_{n=0}^{14}\left[1+\cos\left(\frac{2\pi}{15}(n-n_0)\right)\right]z^{-n} \\
&= \frac{1}{15}\sum_{n=0}^{14}z^{-n}+\frac{1}{15}\sum_{n=0}^{14}\frac{1}{2}\left[e^{j\frac{2\pi}{15}(n-n_0)}+e^{-j\frac{2\pi}{15}(n-n_0)}\right]z^{-n} \\
&= \frac{1}{15}\frac{1-z^{-15}}{1-z^{-1}}+\frac{1}{15}\frac{1}{2}\frac{e^{-j\frac{2\pi}{15}n_0}[1-(e^{j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{j\frac{2\pi}{15}}z^{-1}} \\
&\quad +\frac{1}{15}\frac{1}{2}\frac{e^{j\frac{2\pi}{15}n_0}[1-(e^{-j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{-j\frac{2\pi}{15}}z^{-1}} \\
&= \frac{1}{15}(1-z^{-15})\left[\frac{1}{1-z^{-1}}+\frac{\frac{1}{2}e^{-j\frac{2\pi}{15}n_0}}{1-e^{j\frac{2\pi}{15}}z^{-1}}+\right. \\
&\quad \left.\frac{\frac{1}{2}e^{j\frac{2\pi}{15}n_0}}{1-e^{-j\frac{2\pi}{15}}z^{-1}}\right].
\end{aligned}
$$

(c)

$$
H(e^{j\omega})=\frac{1}{15}e^{-j7\omega}\left[\frac{\sin((15\omega)/2)}{\sin(\omega/2)}-\frac{1}{2}\frac{e^{-j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega-\frac{2\pi}{15})/2)}-\frac{1}{2}\frac{e^{j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega+(2\pi)/15)/2)}\right].
$$

$$
H(e^{j\omega})=\frac{1-e^{-j15\omega}}{15}\left[\frac{1}{1-e^{-j\omega}}+\frac{\frac{1}{2}e^{-j\frac{2\pi n_0}{15}}}{1-e^{j\frac{2\pi}{15}}e^{-j\omega}}+\frac{\frac{1}{2}e^{j\frac{2\pi n_0}{15}}}{1-e^{-j\frac{2\pi}{15}}e^{-j\omega}}\right]
$$

When $n_0 = 15/2$, .

$$
H(e^{j\omega}) = \frac{1}{15}\left[\frac{e^{j\frac{\omega}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}-\frac{\frac{1}{2}e^{j\frac{\omega-(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega-(2\pi/15)}{2}}-e^{-j\frac{\omega-(2\pi/15)}{2}}}-\right.
$$

$$\left.\frac{\frac{1}{2}e^{j\frac{\omega+(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega+(2\pi/15)}{2}}-e^{-j\frac{\omega+(2\pi/15)}{2}}}\right]$$

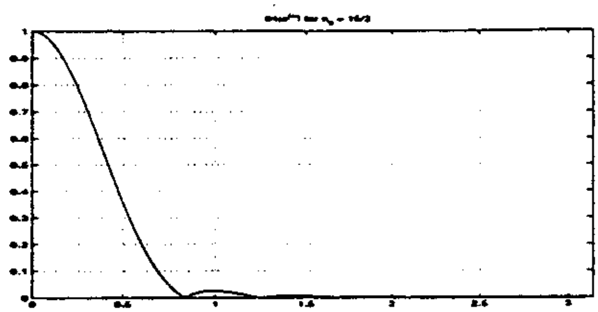
$$= \frac{1}{15}\left[\frac{e^{-j\omega 7}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\frac{\omega}{2}} - \right.$$

$$\frac{\frac{1}{2}e^{-j\omega 7}e^{-j\frac{\pi}{15}}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} -$$

$$\left.\frac{\frac{1}{2}e^{-j\omega 7}e^{j\frac{\pi}{15}}(e^{j\frac{15}{2}\omega}-e^{-j\frac{15}{2}\omega})}{2j\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$

$$= \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} - \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \right.$$

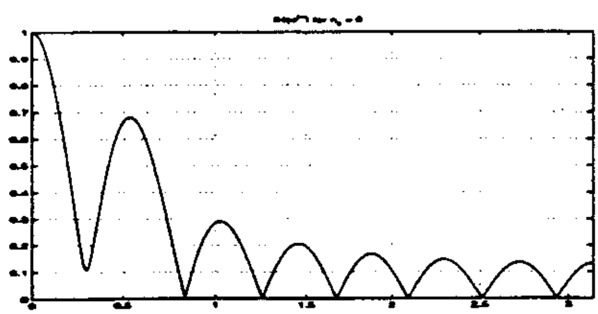
$$\left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



When $n_0 = 0$,

$$H(e^{j\omega}) = \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} + \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} + \right.$$

$$\left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



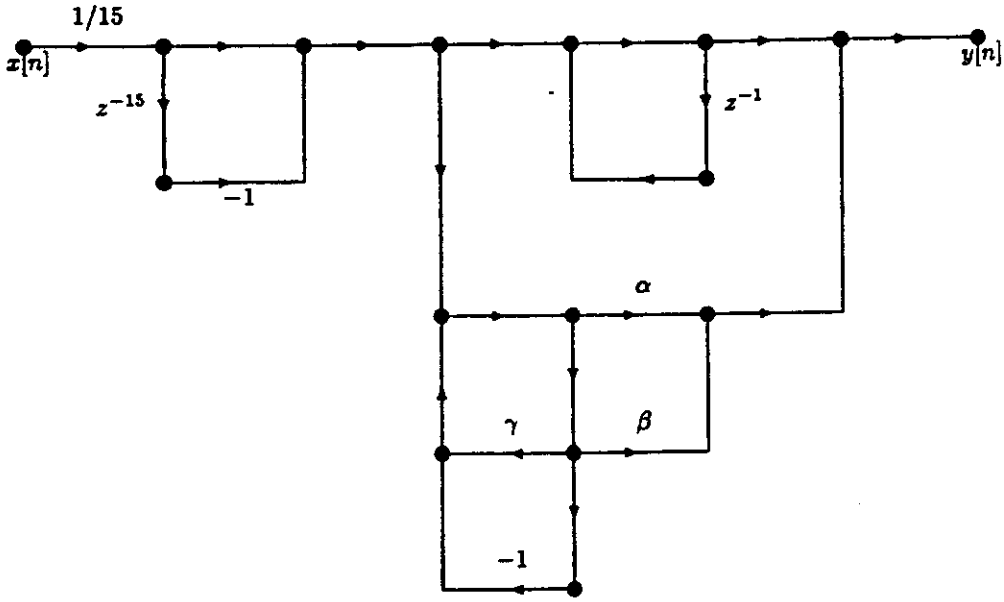
The system will have generalized linear phase if the impulse response has even symmetry (note it cannot have odd symmetry), or alternatively, if the frequency response can be expressed as:

$$H(e^{j\omega}) = e^{-j\omega 7}A_e(e^{j\omega})$$

where $A_e(e^{j\omega})$ is a real, even, periodic function in ω. We thus conclude that the system will have generalized linear phase for $n_0 = \frac{15}{2}k$, where k is an odd integer.

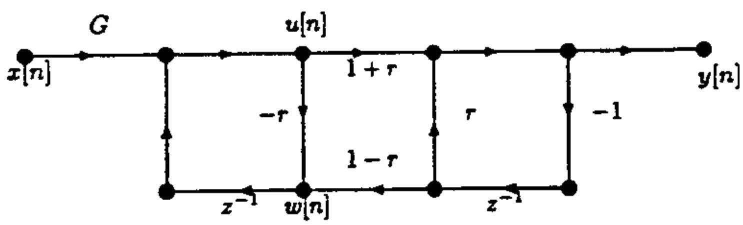
(d) Rewrite $H(z)$ as

$$H(z) = \frac{1 - z^{-15}}{15}\left[\frac{1}{1 - z^{-1}} + \frac{\cos\frac{2\pi n_0}{15} - \cos\left(\frac{2\pi}{15} + \frac{2\pi n_0}{15}\right)z^{-1}}{1 - 2\cos\frac{2\pi}{15}z^{-1} + z^{-2}}\right].$$



where $\alpha = \cos(2\pi n_0/15)$, $\beta = -\cos(2\pi(n_0 + 1)/15)$, and $\gamma = 2\cos(2\pi/15)$.

6.31. (a)



$$\begin{aligned} u[n] &= Gx[n] + w[n-1] \\ w[n] &= -ru[n] - (1-r)y[n-1] \\ y[n] &= (1+r)u[n] - ry[n-1]. \end{aligned}$$

(b)

$$\begin{aligned} U(z) &= GX(z) + z^{-1}W(z) \\ W(z) &= -ru(z) - (1-r)z^{-1}Y(z) \\ Y(z) &= (1+r)U(z) - rz^{-1}Y(z). \end{aligned}$$

Solve for $U(z)$ in terms of $X(z)$ and $Y(z)$:

$$U(z) = \frac{GX(z) - (1-r)z^{-2}Y(z)}{1+rz^{-1}}$$

Then

$$Y(z) = (1+r)\left\{\frac{GX(z) - (1-r)z^{-2}Y(z)}{1+rz^{-1}}\right\} - rz^{-1}Y(z)$$

$$Y(z)(1+rz^{-1}) = G(1+r)X(z) - (1-r^2)z^{-2}Y(z) - rz^{-1}Y(z) - r^2z^{-2}Y(z)$$

$$Y(z)(1+2rz^{-1}+z^{-2}) = G(1+r)X(z)$$

$$H_1(z) = \frac{G(1+r)}{1+2rz^{-1}+z^{-2}}.$$

From the quadratic formula, the poles are at $(-r+j\sqrt{1-r^2})^{-1}$ and $(-r-j\sqrt{1-r^2})^{-1}$. The magnitude of each pole is 1. The angles are

$$-\tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right),$$

respectively.

(c) $U(z) = z^{-1}(GX(z)+W(z))$, $W(z) = -rU(z) - (1-r)Y(z)$, and $Y(z) = z^{-1}((1+r)U(z) - rY(z))$ lead to

$$H_2(z) = \frac{G(1+r)z^{-2}}{1+2rz^{-1}+z^{-2}} = z^{-2}H_1(z).$$

6.32. (a)

$$\begin{aligned} y_1[n] &= (1+r)x_1[n] + rx_2[n] \\ y_2[n] &= -rx_1[n] + (1-r)x_2[n]. \end{aligned}$$

(b)

$$\begin{aligned} y_1[n] &= (1+a)x_1[n] + dx_2[n] \quad (a=r=d) \\ y_2[n] &= (1+cd)x_2[n] + abx_1[n] \quad (c=d=-1). \end{aligned}$$

(c)

$$\begin{aligned} y_1[n] &= (1+e)x_1[n] + ex_2[n] \quad (e=r) \\ y_2[n] &= efx_1[n] + (1+ef)x_2[n] \quad (f=-1). \end{aligned}$$

(d) B and C preferred over A:

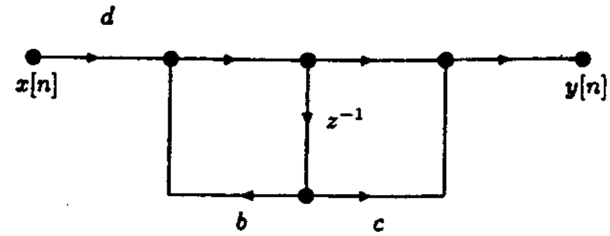
(i) coefficient quantization. If r is small, $1+r$ may not be precisely representable even in floating point. Also, network A has 4 multipliers that must be quantized, while B and C have only 1.

(ii) computational complexity. Networks B and C require fewer multiplications per output sample.

6.33.

$$H(z) = \frac{z^{-1}-0.54}{1-0.54z^{-1}}.$$

(a)

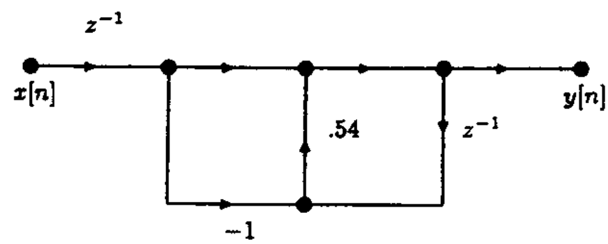


$$H(z) = \frac{cdz^{-1} + d}{1 - bz^{-1}}$$

so set $b = 0.54$, $c = -1.852$, and $d = -0.54$.

(b) With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{c}\hat{d} \neq 1$ and $\hat{d} \neq -\hat{b}$ in general, so the resulting system would not be allpass.

(c)

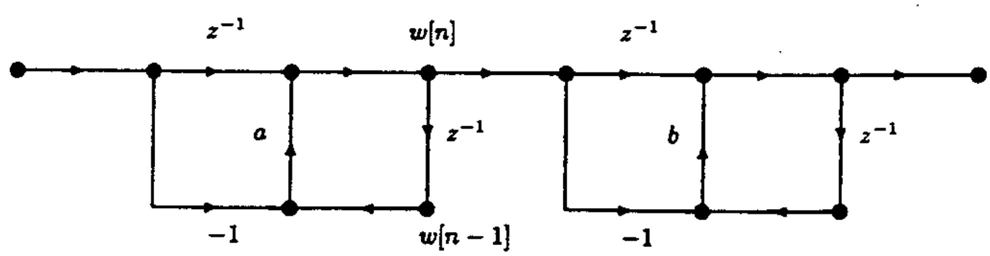


(d) Yes, since there is only one "0.54" to quantize.

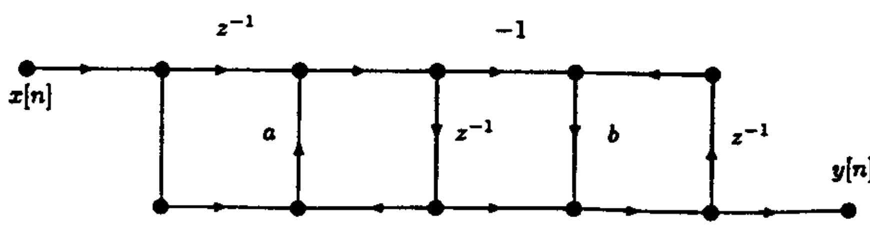
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections like (c) gives

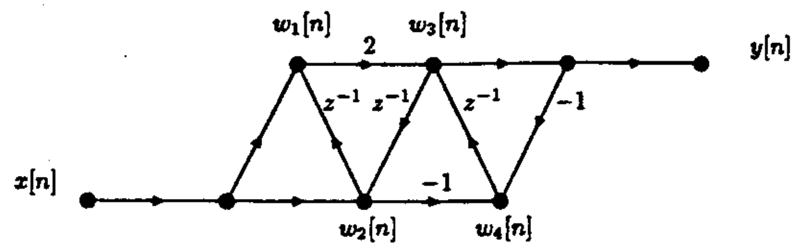


The first delay in the second section has output $w[n-1]$ so we can combine with the second delay of the first section.



(f) Yes, same reason as part (d).

6.34. (a) We have:



First, we find the system function, we have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] + w_4[n-1] \\
(4)\quad y[n] &= w_3[n] \\
(5)\ w_4[n] &= -y[n] - w_2[n]
\end{aligned}$$

Taking the Z-transform of the above equations and combining terms, we get:

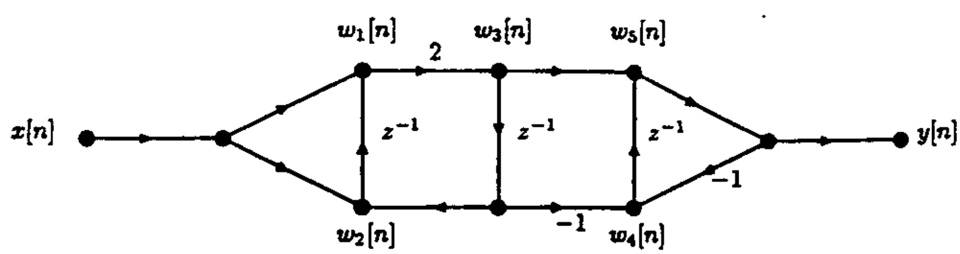
$$(1 - z^{-1})Y(z) + z^{-1}Y(z) = (2 + z^{-1})X(z).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + z^{-1}}{1 + z^{-1} - z^{-2}}$$

Since the system function is second order (highest order term is z^{-2}), we should be able to implement this system using only 2 delays, this can be done with a direct form II implementation. Therefore, the minimum number of delays required to implement an equivalent system is 2.

(b) Now we have:



Let's find the transfer function, we have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] \\
(4)\ w_4[n] &= -w_3[n-1] - y[n] \\
(5)\ w_5[n] &= w_3[n] + w_4[n-1] \\
(6)\quad y[n] &= w_5[n]
\end{aligned}$$

Taking the Z-transform of the above equations and combining terms, we get:

$$(1+z^{-1})Y(z) = \frac{(1-z^{-2})(2+2z^{-1})}{1-2z^{-2}}X(z).$$

The system function is thus given by:

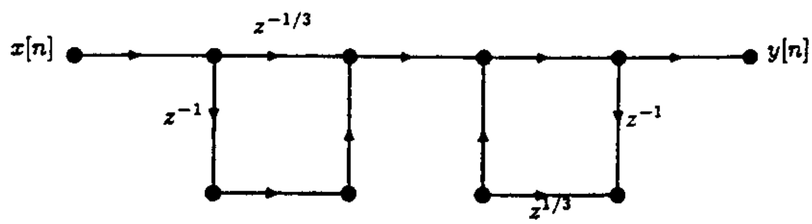
$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1+z^{-1})(1-z^{-1})}{1-2z^{-2}}.$$

Since the transfer function is not the same as the one in part a, we conclude that system B does not represent the same input-output relationship as system A. This should not be surprising since in system B we added two unidirectional wires and therefore changed the input-output relationship.

6.35.

$$H(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}.$$

(a) Direct form I:



From the graph above, it is clear that 2 delays and 2 multipliers are needed.

(b)

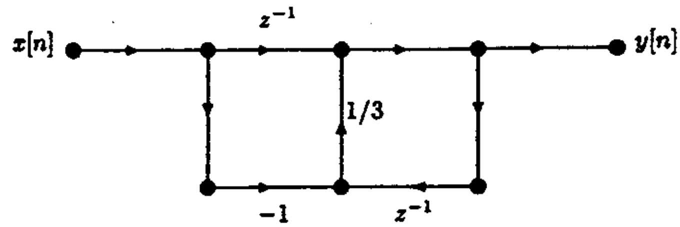
$$(1 - \frac{1}{3}z^{-1})Y(z) = (-\frac{1}{3} + z^{-1})X(z)$$

Inverse Z−transforming, we get:

$$y[n] - \frac{1}{3}y[n-1] = -\frac{1}{3}x[n] + x[n-1]$$

$$y[n] = \frac{1}{3}(y[n-1] - x[n]) + x[n-1]$$

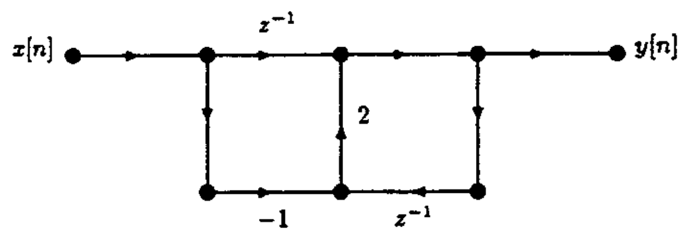
Which can be implemented with the following flow diagram:



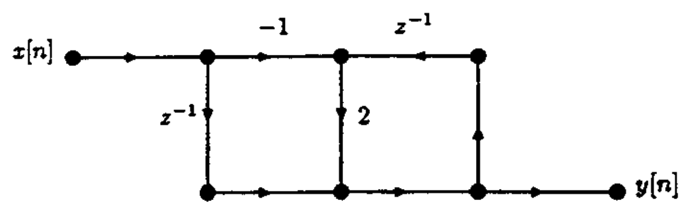
(c)

$$H(z) = \left(\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}\right)\left(\frac{z^{-1} - 2}{1 - 2z^{-1}}\right).$$

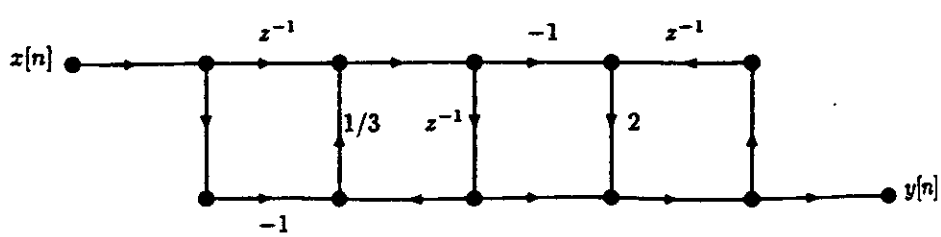
This can be implemented as the cascade of the flow graph in part (b) with the following flow graph:



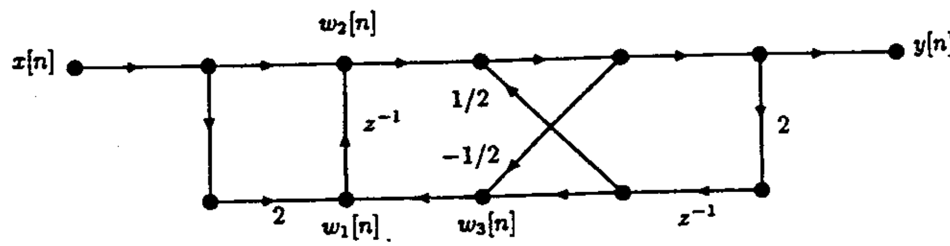
However the above flow graph can be redrawn as:



Now cascading the above flow graph with the one from part (b) and grouping the delay element we get the following system with two multipliers and three delays:



6.36. (a) Transpose = reverse arrows direction and reverse the input/output, we get:



(b) From part (a), we have:

$$(1)\ w_1[n] = 2x[n] + w_3[n]$$
$$(2)\ w_2[n] = x[n] + w_1[n-1]$$
$$(3)\ w_3[n] = -\frac{1}{2}y[n] + 2y[n-1]$$
$$(4)\ y[n] = w_2[n] + y[n-1]$$

Taking the $Z-$transform of the above equations, substituting and rearranging terms, we get:

$$(1 - \frac{1}{2}z^{-1} - 2z^{-2})Y(z) = (2z^{-1} + 1)X(z).$$

Finally, inverse $Z-$transforming, we get the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] - 2y[n-2] = x[n] + 2x[n-1].$$

(c) From part (b), the system function is given by:

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1} - 2z^{-2}}.$$

It has poles at

$$z = -\frac{8}{1-\sqrt{33}} \text{ and } z = -\frac{8}{1+\sqrt{33}}$$

which are outside the unit circle, therefore the system is NOT BIBO stable.

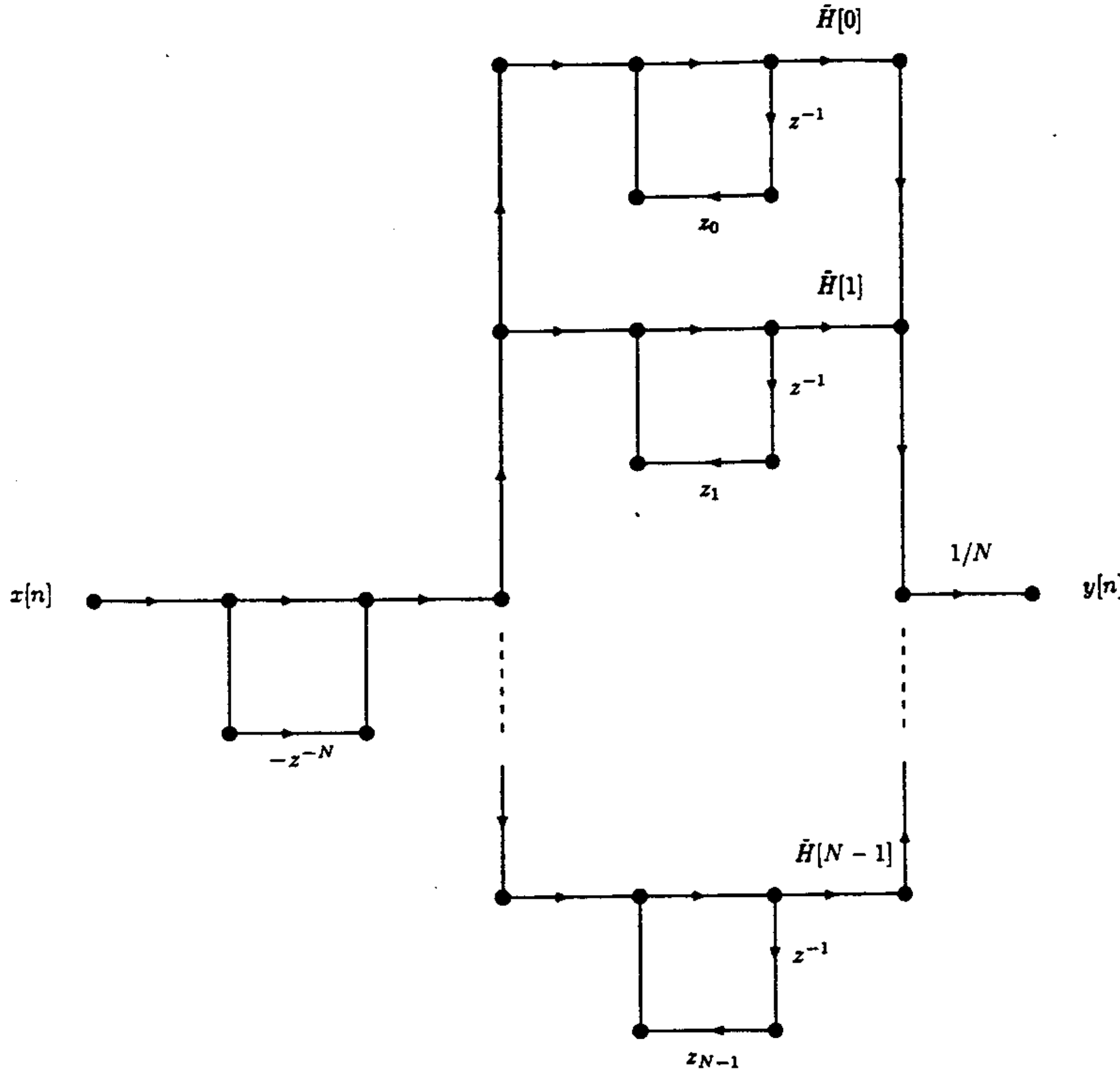
(d)

$$\begin{aligned} &y[2] = x[2] + 2x[1] + \tfrac{1}{2}y[1] + 2y[0] \\ &y[0] = x[0] = 1 \\ &y[1] = x[1] + 2x[0] + \tfrac{1}{2}y[0] = \tfrac{1}{2} + 2 + \tfrac{1}{2} = 3 \end{aligned}$$

Therefore,

$$y[2] = \frac{1}{4} + 1 + \frac{3}{2} + 2 = \frac{19}{4}.$$

6.37. (a)



(b) Note that the z_k's are the zeros of $(1 - z^{-N})$. Then write $H(z)$ over a common denominator:

$$\begin{aligned} H(z) &= \frac{\prod_{\ell=0}^{N-1}(1 - z_\ell z^{-1}) \sum_{k=0}^{N-1} \frac{\tilde{H}[k]}{N} \prod_{\substack{i=0 \\ i\neq k}}^{N-1}(1 - z_i z^{-1})}{\prod_{k=0}^{N-1}(1 - z_k z^{-1})} \\ &= \sum_{k=0}^{N-1} \frac{\tilde{H}[k]}{N} \prod_{\substack{i=0 \\ i\neq k}}^{N-1} (1 - z_i z^{-1}). \end{aligned}$$

Therefore, $H(z)$ is the sum of polynomials in z^{-1} with degree $\leq N-1$. Hence, the system impulse response has length $\leq N$.

(c)

$$\mathcal{Z}^{-1}\left[(1 - z^{-N})\frac{\tilde{H}[k]/N}{1 - z_k z^{-1}}\right] = \frac{\tilde{H}[k]}{N}\mathcal{Z}^{-1}[1 - z^{-N}] * \mathcal{Z}^{-1}\left[\frac{1}{1 - z_k z^{-1}}\right]$$

$$
\begin{aligned}
&= \frac{\tilde{H}[k]}{N}(\delta[n]-\delta[n-N]) * (z_k^n u[n]) \\
&= \frac{\tilde{H}[k]}{N}[z_k^n u[n] - z_k^{n-N} u[n-N]] \\
&= \frac{\tilde{H}[k]}{N} z_k^n \{u[n]-u[n-N]\}.
\end{aligned}
$$

So

$$
h[n] = \left(\frac{1}{N}\sum_{k=0}^{N-1} \tilde{H}[k] e^{j\frac{2\pi}{N}kn}\right)(u[n]-u[n-N]).
$$

(d) Note that, since $(1-z_m^{-N})=0$,

$$
\begin{aligned}
H(z_m) &= \left.\frac{(1-z^{-N})\tilde{H}[m]/N}{1-z_m z^{-1}}\right|_{z=z_m} \\
&= \frac{\frac{d}{dz}\{(1-z^{-N})\tilde{H}[m]/N\}|_{z=z_m}}{\frac{d}{dz}\{1-z_m z^{-1}\}|_{z=z_m}} \\
&= \frac{N z_m^{-N-1}\tilde{H}[m]/N}{z_m z_m^{-2}} \\
&= \tilde{H}[m] z_m^{-N} \\
&= \tilde{H}[m].
\end{aligned}
$$

(e) If $h[n]$ is real, $|H(e^{j\omega})| = |H(e^{j(2\pi-\omega)})|$, and $\angle H(e^{j\omega}) = -\angle H(e^{j(2\pi-\omega)})$. $H(e^{j2\pi k/N}) = \tilde{H}[k] = |\tilde{H}[k]|e^{j\tilde{\theta}[k]}$, so $|\tilde{H}[k]| = |\tilde{H}[N-k]|$ and $\tilde{\theta}[k] = -\tilde{\theta}[N-k]$, $k=0,1,\ldots,N-1$.

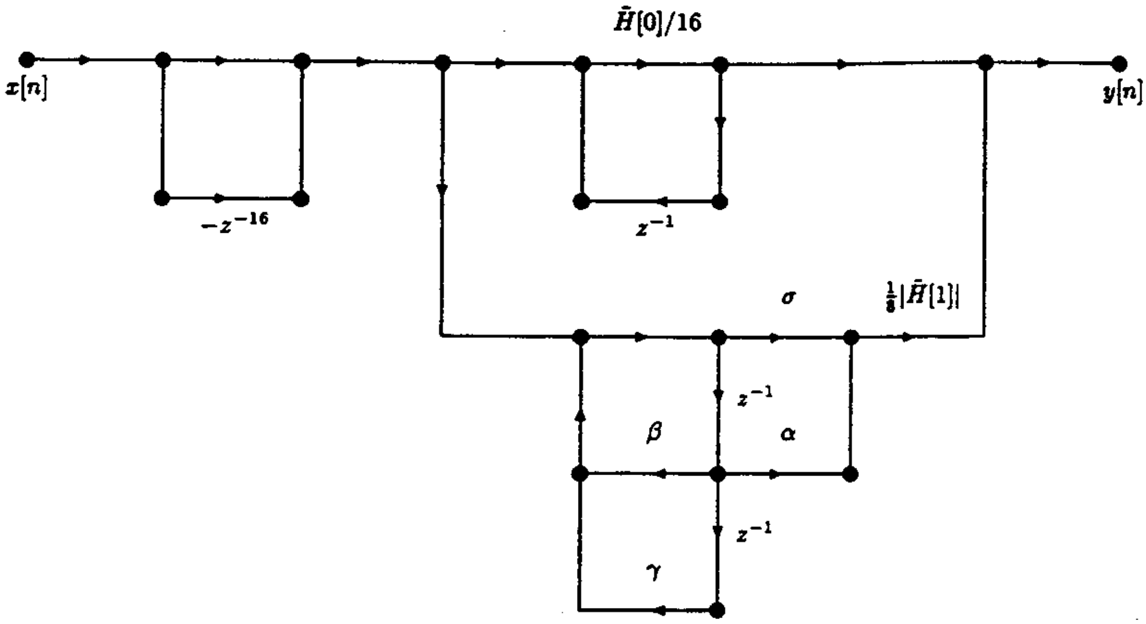
$$
\begin{aligned}
H(z) &= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \frac{\tilde{H}[N/2]/N}{1-z_{N/2}z^{-1}} + \sum_{\ell=\frac{N}{2}+1}^{N-1}\frac{\tilde{H}[\ell]/N}{1-z_\ell z^{-1}}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \sum_{p=1}^{\frac{N}{2}-1}\frac{\tilde{H}[N-p]/N}{1-z_{N-p}z^{-1}}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\left(\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \frac{\tilde{H}[N-k]/N}{1-z_{-k}z^{-1}}\right)\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \right. \\
&\qquad \left. \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k](1-z_{-k}z^{-1}) + \tilde{H}[N-k](1-z_k z^{-1})}{(1-z_k z^{-1})(1-z_{-k}z^{-1})}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \right. \\
&\qquad \left. \sum_{k=1}^{\frac{N}{2}-1}\frac{2|\tilde{H}[k]|}{N}\cdot\frac{\cos(\tilde{\theta}[k]) - z^{-1}\cos(\tilde{\theta}[k]-2\pi k/N)}{1-2\cos\frac{2\pi k}{N}z^{-1}+z^{-2}}\right]
\end{aligned}
$$

And since $\tilde{H}[0]=H(1)$, $\tilde{H}[N/2]=H(-1)$,

$$
H(z) = (1-z^{-N})\left[\frac{H(1)/N}{1-z^{-1}} + \frac{H(-1)/N}{1+z^{-1}} + \right.
$$

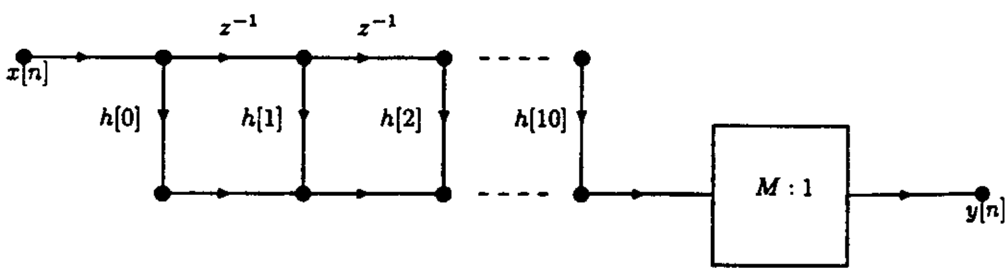
$$\left. \sum_{k=1}^{\frac{N}{2}-1} \frac{2|H(e^{j\frac{2\pi}{N}k})|}{N} \cdot \frac{\cos[\theta(2\pi k/N)] - z^{-1}\cos[\theta(2\pi k/N) - 2\pi k/N]}{1 - 2\cos(2\pi k/N)z^{-1} + z^{-2}} \right] .$$

If $\tilde{H}[14] = 0$,then $\tilde{H}[16-14] = \tilde{H}[2] = 0$ also.



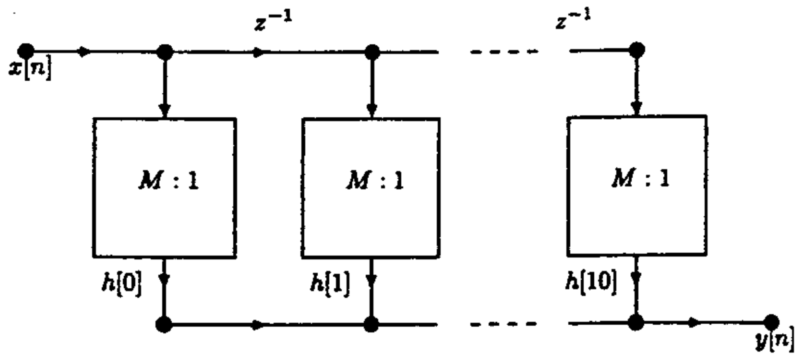
where $\sigma = \cos(\tilde{\theta}[1])$, $\alpha = -\cos(\tilde{\theta}[1] - (2\pi/16))$, $\beta = 2\cos(2\pi/16)$, and $\gamma = -1$.

6.38. (a)



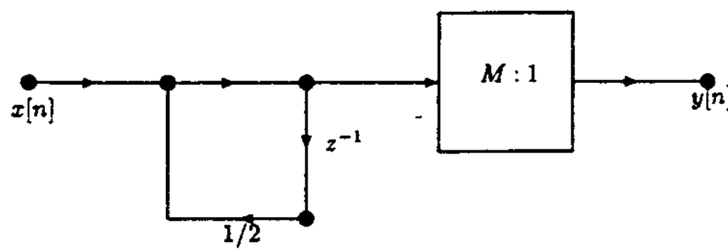
$M(N+1)$ multiplies per output sample; MN adds per output sample.

(b)



$N+1$ multiplies per output sample; N adds per output sample. The number of computations has been reduced by a factor of M in both adds and multiplies.

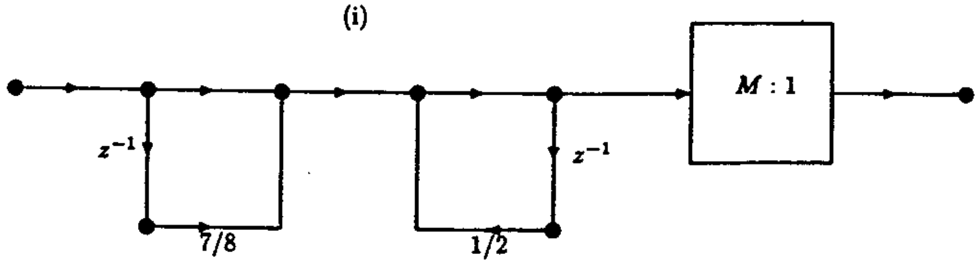
(c)



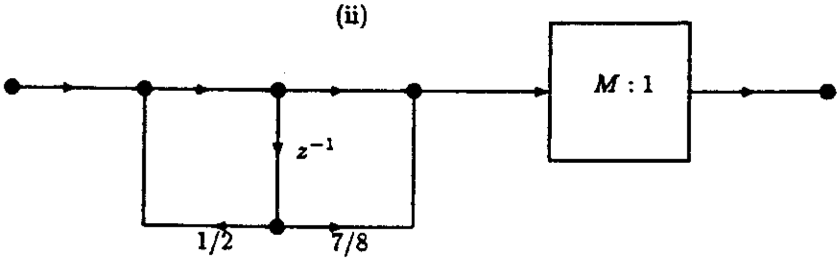
The total computation can not be reduced because to compute the value of any given output sample, the previous output value must be known.

(d)

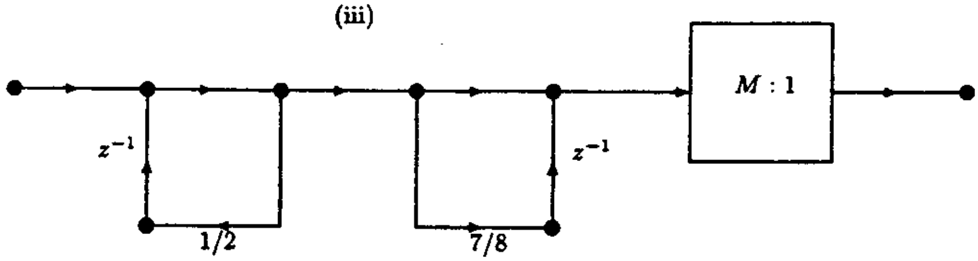
(i)



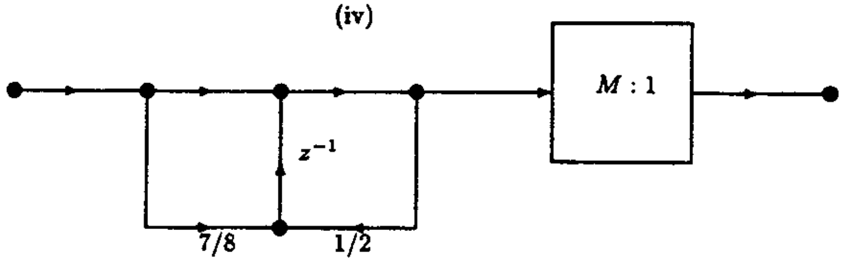
(ii)



(iii)



(iv)



Only direct form II (ii) can be implemented more efficiently by commuting operations with the downsamplers.

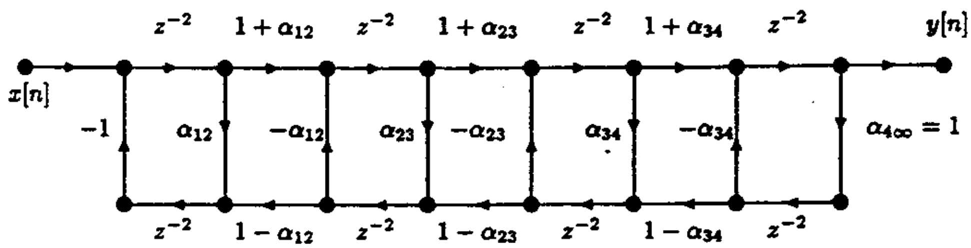
6.39. Since each section is 3.4cm long, it takes

$$\frac{3.4\text{cm}}{3.4\frac{\text{cm}}{\text{sec}}\cdot 10^4} = 10^{-4}\text{sec}$$

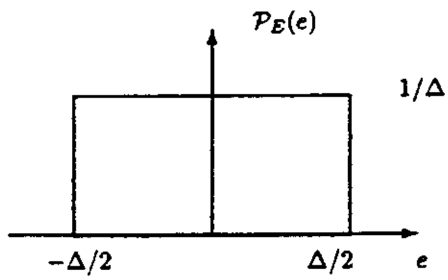
to traverse one section. Since the sampling rate is 20kHz ($T_s = 0.5\cdot 10^{-4}$sec), it takes two sampling intervals to traverse a section. The entire system is linear and so the forward going and backward going waves add at a boundary. Let

$$\alpha_{kn} = \frac{A_n - A_k}{A_n + A_k}$$

(from A_k into A_n); then $\alpha_{kn} = -\alpha_{nk}$ and we get:



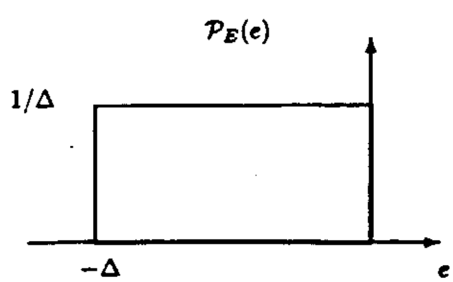
6.40. (a) For rounding:



$$m_e = \frac{1}{\Delta}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} e\,de = \frac{1}{\Delta}\frac{e^2}{2}\Big]_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} = 0$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} e^2\,de = \frac{e^3}{3\Delta}\Big]_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}.$$

(b) For truncation:



$$m_e = \frac{1}{\Delta}\int_{-\Delta}^{0} e\,de = \frac{1}{\Delta}\frac{e^2}{2}\bigg]_{-\Delta}^{0} = \frac{-\Delta}{2}$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{-\Delta}^{0} e^2\,de - \frac{\Delta^2}{4} = \frac{e^3}{3\Delta}\bigg]_{-\Delta}^{0} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12}.$$

6.41. Since the system is linear, $y[n]$ is the sum of the outputs due to $x_1[n]$ and $x_2[n]$. Therefore

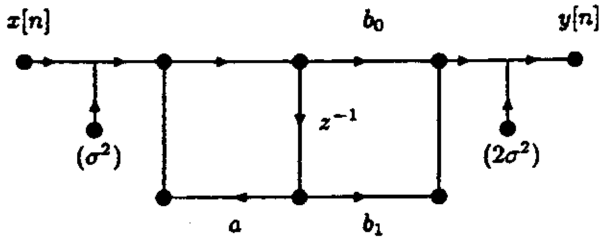
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h_1[k]x_1[n-k] + \sum_{k=-\infty}^{\infty} h_2[k]x_2[n-k] \\ &= y_1[n] + y_2[n]. \end{aligned}$$

The correlation between $y_1[n]$ and $y_2[n]$ is

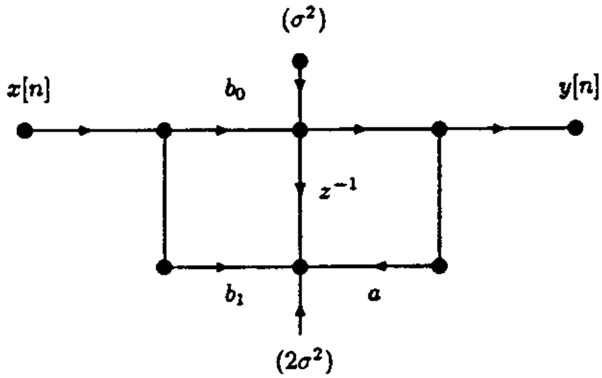
$$\begin{aligned} E\{y_1[m]y_2[n]\} &= E\left\{\sum_{\ell=-\infty}^{\infty} h_1[\ell]x_1[m-\ell]\cdot\sum_{k=-\infty}^{\infty} h_2[k]x_2[n-k]\right\} \\ &= \sum_{\ell=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_1[\ell]h_2[k]E\{x_1[m-\ell]x_2[n-k]\} \end{aligned}$$

If $x_1[n]$ and $x_2[n]$ are uncorrelated, $E\{x_1[m-\ell]x_2[n-k]\} = 0$; hence, $E\{y_1[m]y_2[n]\} = 0$. Therefore, $y_1[n]$ and $y_2[n]$ are uncorrelated.

6.42. (a) The linear noise model for each system is drawn below.

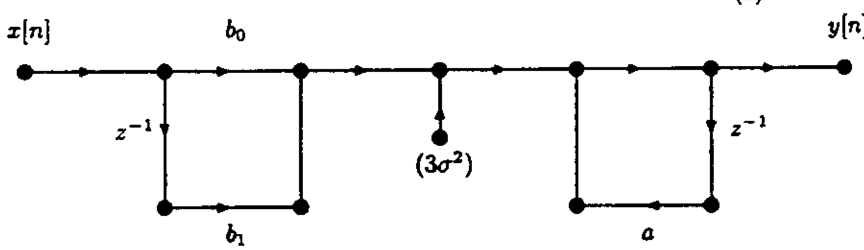


(a)

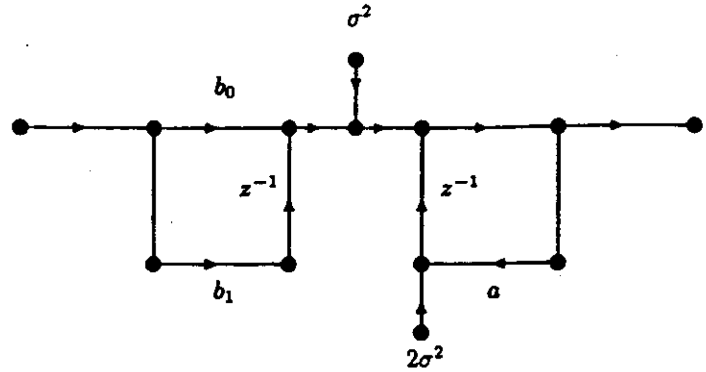


(b)

(c)



(b) Clearly (a) and (c) are different. Thus the answer is either (a) and (b) or (b) and (c). If we take (b) apart, we get



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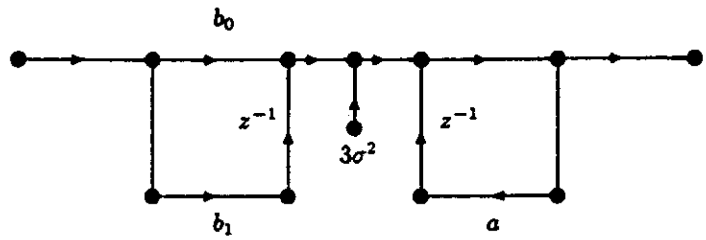


We see that the noise all goes through the poles. Note that the $1\sigma^2$ source sees a system function $(1 - az^{-1})^{-1}$ while the $2\sigma^2$ source sees $z^{-1}/(1 - az^{-1})$. However, the delay (z^{-1}) does not affect the average power. Hence, the answer is (b) and (c).

(c) For network (c),

$$\sigma_f^2 = 3\sigma^2 \sum_{n=0}^{\infty} (a^n)^2 = \frac{3\sigma^2}{1 - a^2},$$

or using the frequency domain formula,

$$\begin{aligned} \sigma_f^2 &= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - az} \frac{dz}{z} \\ &= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{dz}{(z - a)(1 - az)} \\ &= \frac{3\sigma^2}{1 - a^2}. \end{aligned}$$

For network (a),

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}}$$

$$h[n] = b_0\delta[n] + (b_0 + \frac{b_1}{a})a^n u[n]$$

- Time domain calculation:

$$\sigma_f^2 = 2\sigma^2 + \sigma^2 \sum_n h^2[n]$$

$$= 2\sigma^2 + \sigma^2 \left(b_0^2 + \left(b_0 + \frac{b_1}{a}\right)^2 \underbrace{\sum_{n=1}^{\infty} a^{2n}}_{\frac{a^2}{1-a^2}} \right)$$

$$= 2\sigma^2 + \sigma^2 \left(b_0^2 + \frac{(ab_0 + b_1)^2}{1 - a^2} \right).$$

- Frequency domain calculation:

$$\sum_n h^2[n] = \frac{1}{2\pi j} \oint H(z)H(z^{-1}) \frac{dz}{z}$$

$$= \sum \left(\text{residues of } \frac{H(z)H(z^{-1})}{z} \text{inside unit circle} \right).$$

$$\frac{H(z)H(z^{-1})}{z} = \frac{(b_0 + b_1 z^{-1})(b_0 + b_1 z)}{(z - a)(1 - az)} \frac{z}{z}$$

$$= \frac{(b_0 z + b_1)(b_0 + b_1 z)}{z(z - a)(1 - az)}.$$

$$\text{residue } (z = 0) = \frac{-b_1 b_0}{a}$$

$$\text{residue } (z = a) = \frac{(b_0 a + b_1)(b_0 + b_1 a)}{a(1 - a^2)} = \frac{b_0^2 a + b_1^2 a + b_1 b_0 + b_1 b_0 a^2}{a(1 - a^2)}.$$

$$\oint H(z)H(z^{-1}) \frac{dz}{z} = \frac{b_0^2 a + b_1^2 a + b_1 b_0 + b_1 b_0 a^2 - b_1 b_0 + b_1 b_0 a^2}{a(1 - a^2)}$$

$$= \frac{b_0^2 + b_1^2 + 2b_0 b_1 a}{1 - a^2}$$

$$= b_0^2 + \frac{(ab_0 + b_1)^2}{1 - a^2}$$

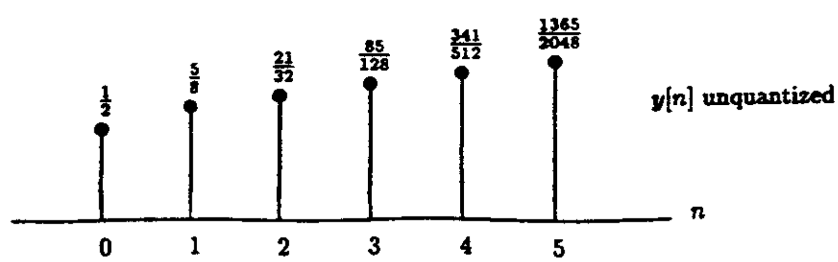
6.43. (a)

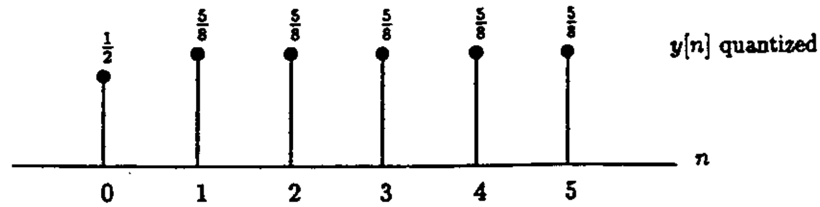
$$y[n] = \frac{1}{4} y[n-1] + \frac{1}{2}, \quad n \geq 0$$

$$y[n] = \frac{1}{2} \sum_{i=0}^{n} \left(\frac{1}{4}\right)^i = \frac{1}{2} \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{\frac{3}{4}}$$

For large n, $y[n] = (1/2)/(3/4) = 2/3$.

(b) Working from the difference equation and quantizating after multiplication, it is easy to see that, in the quantized case, $y[0] = 1/2$ and $y[n] = 5/8$ for $n \geq 1$. In the unquantized case, the output monotonically approaches $2/3$.





(c) The system diagram is direct form II:

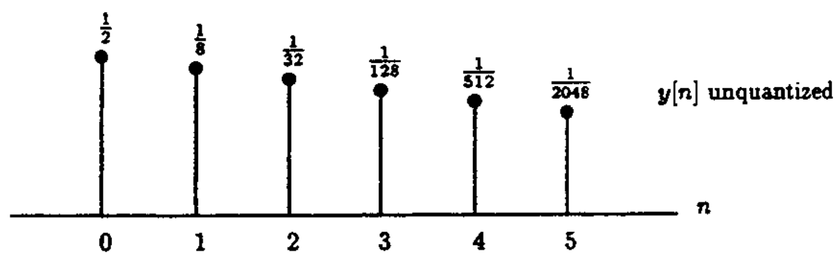
$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

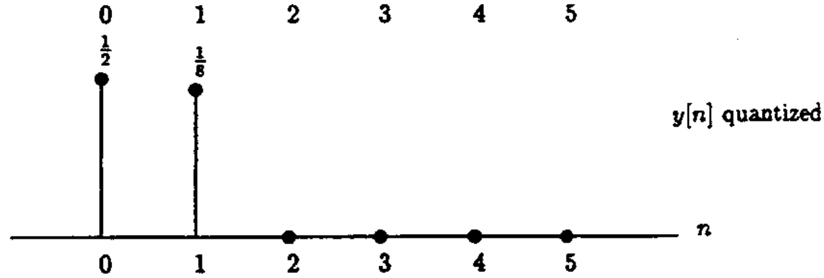
$$X(e^{j\omega}) = \frac{\frac{1}{2}}{1+e^{-j\omega}}$$

So

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{\frac{1}{2}}{1-\frac{1}{4}e^{-j\omega}}$$

which implies that $y[n] = (1/2)(1/4)^n$, which approaches 0 as n grows large.
To find the quantized output (working from the difference equation): $y[0] = 1/2$, $y[1] = 1/8$, and $y[n] = 0$ for $n \geq 2$.





6.44. (a) To check for stability, we look at the poles location. The poles are located at

$$z \approx 0.52 + 0.84j \text{ and } z \approx 0.52 - 0.84j.$$

Note that

$$|z|^2 \approx 0.976 < 1.$$

The poles are inside the unit circle, therefore the system function is stable.

(b) If the coefficients are rounded to the nearest tenth, we have

$$1.04 \to 1.0 \text{ and } 0.98 \to 1.0.$$

Now the poles are at

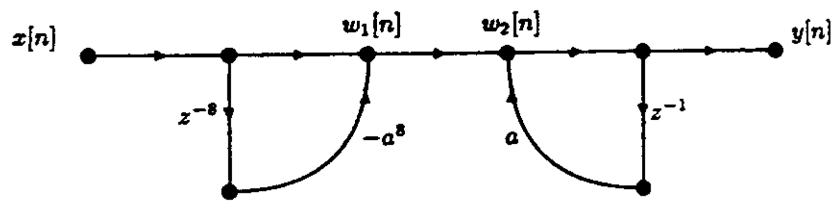
$$z = \frac{1 - j\sqrt{3}}{2} \text{ and } z = \frac{1 - j\sqrt{3}}{2}.$$

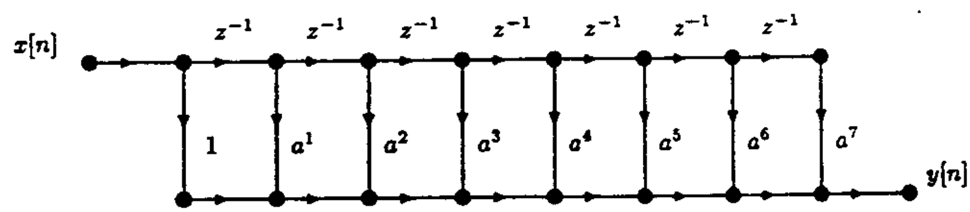
Note that now,

$$|z|^2 = 1.$$

The poles are on the unit circle, therefore the system is not stable.

6.45. The flow graphs for networks 1 and 2 respectively are:





(a) For Network 1, we have:

$$\begin{aligned} w_1[n] &= x[n] - a^8 x[n-8] \\ w_2[n] &= ay[n-1] + w_1[n] \\ y[n] &= w_2[n] \end{aligned}$$

Taking the Z−transform of the above equations and combining terms, we get:

$$Y(z)(1 - az^{-1}) = (1 - a^8 z^{-8})X(z)$$

That is:

$$H(z) = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

For Network 2, we have:

$$y[n] = x[n] + ax[n-1] + a^2 x[n-2] + \dots + a^7 x[n-7].$$

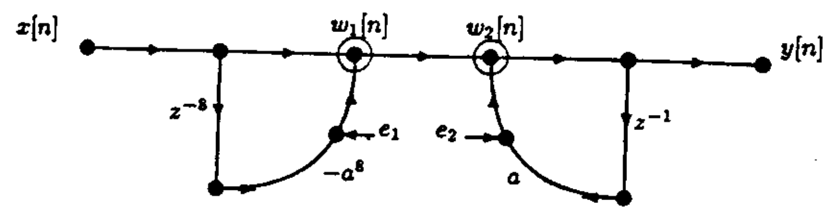
Taking the Z-transform, we get:

$$Y(z) = (1 + az^{-1} + a^2 z^{-2} + ... + a^7 z^{-7})X(z).$$

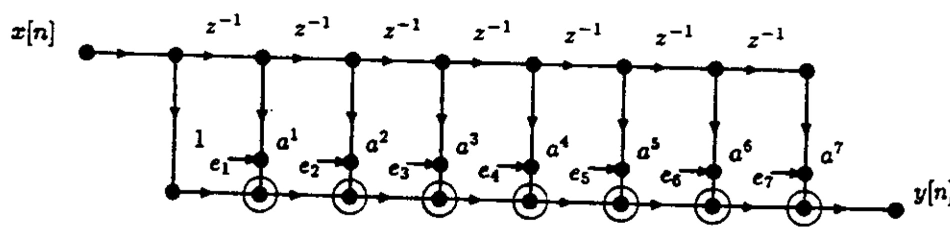
So:

$$H(z) = 1 + az^{-1} + a^2 z^{-2} + ... + a^7 z^{-7} = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

(b) Network 1:



Network 2:



(c) The nodes are circled on the figures in part (b).

(d) In order to avoid overflow in the system, each node in the network must be constrained to have a magnitude less than 1. That is if $w_k[n]$ denotes the value of the kth node variable and $h_k[n]$ denotes the impulse response from the input $x[n]$ to the node variable $w_k[n]$, a sufficient condition for $|w_k[n]| < 1$ is

$$x_{max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k[m]|}.$$

In this problem, we need to make sure overflow does not occur in each node, i.e. we need to take the tighter bound on x_{max}. For network 1, the impulse response from $w_2[n]$ to $y[n]$ is $a^n u[n]$, therefore the condition to avoid overflow from that node to the output is

$$w_{max} < 1 - |a|.$$

Where we assumed that $|a| < 1$. The transfer function from $x[n]$ to $w_1[n]$ is $1 - a^8 z^{-8}$, therefore to avoid overflow at that node we need:

$$w_1[n] < x_{max}(1 - a^8) < 1 - |a|.$$

We thus conclude that to avoid overflow in network 1, we need:

$$x_{max} < \frac{1-|a|}{1-a^8}.$$

Now, for network 2, the transfer function from input to output is given by $\delta[n] + a\delta[n-1] + a^2\delta[n-2] + ... + a^7\delta[n-7]$, therefore to avoid overflow, we need:

$$x_{max} < \frac{1}{1+|a|+a^2+...+|a|^7}.$$

(e) For network 1, the total noise power is $\frac{2\sigma_e^2}{1-|a|}$. For network 2, the total noise power is $7\sigma_e^2$. For network 1 to have less noise power than network 2, we need

$$\frac{2\sigma_e^2}{1-|a|} < 7\sigma_e^2.$$

That is:

$$|a| < \frac{5}{7}.$$

The largest value of $|a|$ such that the noise in network 1 is less than network 2 is therefore $\frac{5}{7}$.