Solutions – Chapter 3

The z-Transform

3.1. (a)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[n]\right] = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(b)

$$\begin{aligned}\mathcal{Z}\left[-\left(\frac{1}{2}\right)^n u(-n-1)\right] &= -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{n=1}^{\infty}(2z)^n \\ &= -\frac{2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}\end{aligned}$$

(c)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[-n]\right] = \sum_{n=-\infty}^{0}(2z)^n = \frac{1}{1-2z} \qquad |z| < \frac{1}{2}$$

(d)

$$\mathcal{Z}[\delta[n]] = z^0 = 1 \qquad \text{all } z$$

(e)

$$\mathcal{Z}[\delta[n-1]] = z^{-1} \qquad |z| > 0$$

(f)

$$\mathcal{Z}[\delta[n+1]] = z^{+1} \qquad 0 \le |z| < \infty$$

(g)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right] = \sum_{n=0}^{9}\left(\frac{1}{2z}\right)^n = \frac{1-(2z)^{-10}}{1-(2z)^{-1}} \qquad |z| > 0$$

3.2.

$$x[n] = \begin{cases} n, & 0 \le n \le N-1 \\ N, & N \le n \end{cases} = n\,u[n] - (n-N)u[n-N]$$

$$\begin{aligned} n\,x[n] &\Leftrightarrow -z\frac{d}{dz}X(z) \Rightarrow n\,u[n] \Leftrightarrow -z\frac{d}{dz}\frac{1}{1-z^{-1}} \quad |z| > 1 \\ n\,u[n] &\Leftrightarrow \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \\ x[n-n_0] &\Leftrightarrow X(z)\cdot z^{-n_0} \Rightarrow (n-N)u[n-N] \Leftrightarrow \frac{z^{-N-1}}{(1-z^{-1})^2} \quad |z| > 1 \end{aligned}$$

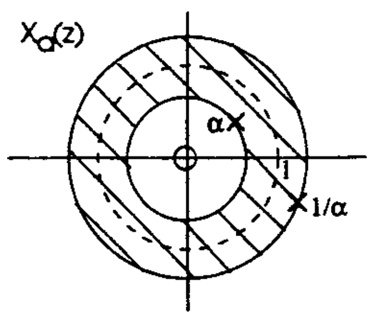
therefore

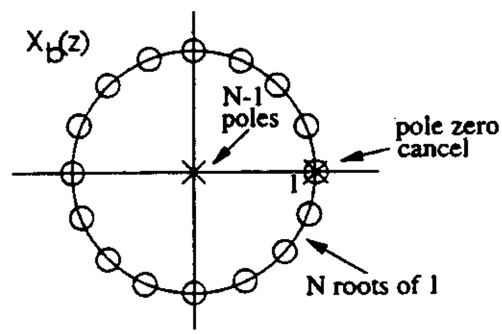
$$X(z) = \frac{z^{-1}-z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

3.3. (a)

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned} X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|} \end{aligned}$$





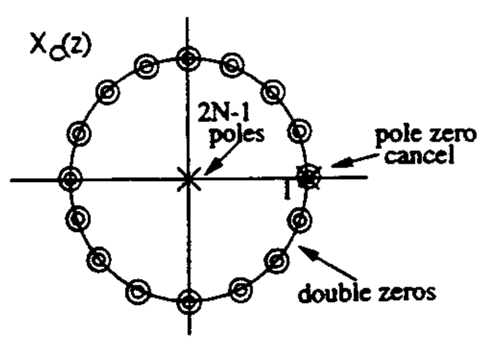
(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \quad z \neq 0$$

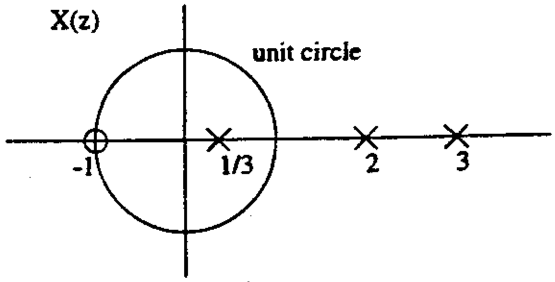
(c)

$$x_c[n] = x_b[n-1] * x_b[n] \Leftrightarrow X_c(z) = z^{-1} X_b(z) \cdot X_b(z)$$

$$X_c(z) = z^{-1} \left(\frac{z^N - 1}{z^{N-1}(z-1)} \right)^2 = \frac{1}{z^{2N-1}} \left(\frac{z^N - 1}{z-1} \right)^2 \qquad z \neq 0, 1$$



3.4. The pole-zero plot of $X(z)$ appears below.



(a) For the Fourier transform of $x[n]$ to exist, the z-transform of $x[n]$ must have an ROC which includes the unit circle, therefore, $|\frac{1}{3}| < |z| < |2|$.
Since this ROC lies outside $\frac{1}{3}$, this pole contributes a right-sided sequence. Since the ROC lies inside 2 and 3, these poles contribute left-sided sequences. The overall $x[n]$ is therefore two-sided.

(b) Two-sided sequences have ROC's which look like washers. There are two possibilities. The ROC's corresponding to these are: $|\frac{1}{3}| < |z| < |2|$ and $|2| < |z| < |3|$.

(c) The ROC must be a connected region. For stability, the ROC must contain the unit circle. For causality the ROC must be outside the outermost pole. These conditions cannot be met by any of the possible ROC's of this pole-zero plot.

3.5.

$$\begin{aligned} X(z) &= (1+2z)(1+3z^{-1})(1-z^{-1}) \\ &= 2z+5-4z^{-1}-3z^{-2} \\ &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \end{aligned}$$

Therefore,

$$x[n] = 2\delta[n+1] + 5\delta[n] - 4\delta[n-1] - 3\delta[n-2]$$

3.6. (a)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

Partial fractions: one pole $\rightarrow$ inspection, $x[n] = (-\frac{1}{2})^n u[n]$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{1}{2}z^{-1} & +\frac{1}{4}z^{-2} & + \dots \\ \hline 1+\frac{1}{2}z^{-1} & 1 & & & \\ & 1 & +\frac{1}{2}z^{-1} & & \\ \hline & & -\frac{1}{2}z^{-1} & & \\ & & -\frac{1}{2}z^{-1} & -\frac{1}{4}z^{-2} & \\ \hline & & & +\frac{1}{4}z^{-2} & \\ & & & +\frac{1}{4}z^{-2} & +\frac{1}{8}z^{-3} \\ \hline \end{array}$$

$$\Longrightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(b)

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}$$

Partial Fractions: one pole $\rightarrow$ inspection, $x[n] = -(-\frac{1}{2})^n u[-n-1]$

Long division:

$$\begin{array}{r|lllll}
 & 2z & -4z^2 & +8z^3 & + \dots \\ \hline
\frac{1}{2}z^{-1} + 1 & 1 & & & \\
 & 1 & +2z & & \\ \hline
 & & -2z & & \\
 & & -2z & -4z^2 & \\ \hline
 & & & +4z^2 & \\
 & & & +4z^2 & +8z^3 \\ \hline
\end{array}$$

$$\Longrightarrow x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

(c)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned}
X(z) &= \frac{-3}{1 + \frac{1}{4}z^{-1}} + \frac{4}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\
x[n] &= \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right] u[n]
\end{aligned}$$

Long division:

$$\begin{array}{r|llll}
 & 1 & +(-\frac{3}{4} - \frac{1}{2})z^{-1} & +(-\frac{3}{16} + 1)z^{-2} & + \dots \\ \hline
1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} & 1 & -\frac{1}{2}z^{-1} & & \\
 & 1 & +\frac{3}{4}z^{-1} & +\frac{1}{8}z^{-2} & \\ \hline
 & & (-\frac{3}{4} - \frac{1}{2})z^{-1} & -\frac{1}{8}z^{-2} & \\
 & & (-\frac{3}{4} - \frac{1}{2})z^{-1} & +\frac{3}{4}(-\frac{3}{4} - \frac{1}{2})z^{-2} & +\frac{1}{8}(-\frac{3}{4} - \frac{1}{2})z^{-3} \\ \hline
 & & & [-\frac{1}{8} + \frac{3}{4}(\frac{3}{4} + \frac{1}{2})]z^{-2} & +\frac{1}{8}(\frac{3}{4} + \frac{1}{2})z^{-3}
\end{array}$$

$$\Longrightarrow x[n] = \left[-3(\left(-\frac{1}{4}\right)^n + \left(-\frac{1}{2}\right)^{n-2}\right] u[n]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned}
X(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\
x[n] &= \left(-\frac{1}{2}\right)^n u[n]
\end{aligned}$$

Long division: see part (i) above.

(e)

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1-a^2)}{1 - a^{-1}z^{-1}} \qquad |z| > |a^{-1}|$$

$$x[n] = -a\delta[n] - (1-a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|llll} & -\frac{1}{a} & -(\frac{a^{-1}-a}{a})z^{-1} & -(\frac{a^{-1}-a}{a^2})z^{-2} & + \dots \\ \hline -a + z^{-1} & 1 & -az^{-1} & & \\ & 1 & -az^{-1} & & \\ \cline{2-3} & & (a^{-1}-a)z^{-1} & & \dots \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1-a^2)a^{-(n+1)}u[n]$$

3.7. (a)

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow \qquad X(z) = \frac{-1}{1 - z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + z^{-1})} \cdot \frac{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$H(z)$ causal $\Rightarrow$ ROC $|z| > 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the z-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)

$$Y(z) = \frac{-\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 + z^{-1}} \qquad |z| > 1$$

Therefore,

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

3.8. The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})} \qquad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ causal $\Rightarrow$

$$h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$$

(b)

$$\begin{aligned} Y(z) &= X(z)H(z) = \frac{-\frac{2}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1+\frac{3}{4}z^{-1})} \qquad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1+\frac{3}{4}z^{-1}} \end{aligned}$$

Therefore the output is

$$y[n] = -\frac{8}{13}\left(\frac{1}{3}\right)^n u[n] + \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n]$$

(c) For $h[n]$ to be causal the ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

3.9.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} - \frac{1}{(1+\frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow \ |z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1 \ \Rightarrow$ stable.

(c)

$$\begin{aligned} y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\ Y(z) &= \frac{-\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} \\ &= \frac{1+z^{-1}}{(1+\frac{1}{4}z^{-1})(1-2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\ X(z) &= \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-2z^{-1})} \qquad |z| < 2 \\ x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1}u[-n] \end{aligned}$$

(d)

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

3.10. (a)

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^n u[n-10] + \left(\frac{3}{4}\right)^n u[n-10] \\ &= \left(\frac{1}{2}\right)^n u[n] + \left(\frac{3}{4}\right)^n u[n] \\ &\quad - \left[\left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right)(u[n] - u[n-11])\right] \end{aligned}$$

The last term is finite length and converges everywhere except at $z = 0$. Therefore, ROC outside largest pole $\frac{3}{4} < |z|$.

(b)

$$x[n] = \begin{cases} 1, & -10 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Finite length but has positive and negative powers at z in its $X(z)$. Therefore the ROC is $0 < |z| < \infty$.

(c)

$$\begin{aligned} x[n] &= 2^n u[-n] = \left(\frac{1}{2}\right)^{-n} u[-n] \\ x[-n] &\leftrightarrow X(1/z) \\ \left(\frac{1}{2}\right)^{n} u[n] &\Rightarrow \text{ROC is} \quad |z| > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{-n} u[-n] &\Rightarrow \text{ROC is} \quad |z| < 2 \end{aligned}$$

(d)

$$x[n] = \left[\left(\frac{1}{4}\right)^{n+4} - (e^{j\pi/3})^n\right] u[n-1]$$

$x[n]$ is right-sided, so its ROC extends outward from the outermost pole $e^{j\pi/3}$. But since it is non-zero at $n = -1$, the ROC does not include ∞. So the ROC is $1 < |z| < \infty$.

(e)

$$\begin{aligned} x[n] &= u[n+10] - u[n+5] \\ &= \begin{cases} 1, & -10 \le n \le -6 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$x[n]$ is finite-length and has only positive powers of z in its $X(z)$. So the ROC is $|z| < \infty$.

(f)

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n] + (2+3j)^{n-2} u[-n-1]$$

$x[n]$ is two-sided, with two poles. Its ROC is the ring between the two poles: $\frac{1}{2} < |z| < \left|\frac{1}{2+3j}\right|$, or $\frac{1}{2} < |z| < \frac{1}{\sqrt{13}}$.

3.11.

$$x[n] \text{ causal } \Rightarrow X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

which means this summation will include *no* positive powers of z. This means that the closed form of $X(z)$ must converge at $z = \infty$, i.e., $z = \infty$ must be in the ROC of $X(z)$, or $\lim_{z\to\infty} X(z) \neq \infty$.

(a)

$$\lim_{z\to\infty} \frac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})} = 1 \qquad \text{could be causal}$$

(b)

$$\lim_{z\to\infty}\frac{(z-1)^2}{(z-\frac{1}{2})}=\infty \qquad \text{could not be causal}$$

(c)

$$\lim_{z\to\infty}\frac{(z-\frac{1}{4})^5}{(z-\frac{1}{2})^6}=0 \qquad \text{could be causal}$$

(d)

$$\lim_{z\to\infty}\frac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5}=\infty \qquad \text{could not be causal}$$

3.12. (a)

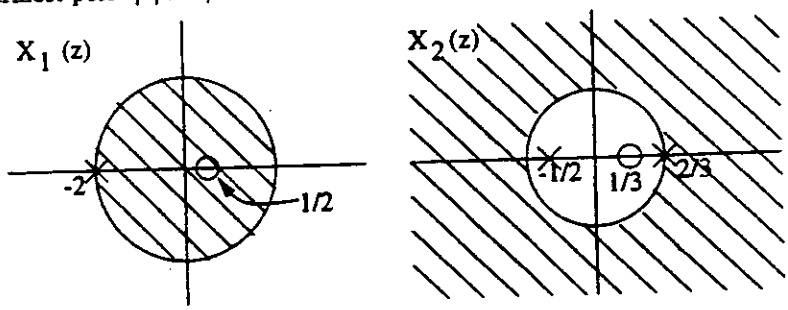
$$X_1(z)=\frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}}$$

The pole is at -2, and the zero is at 1/2.

(b)

$$X_2(z)=\frac{1-\frac{1}{3}z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$$

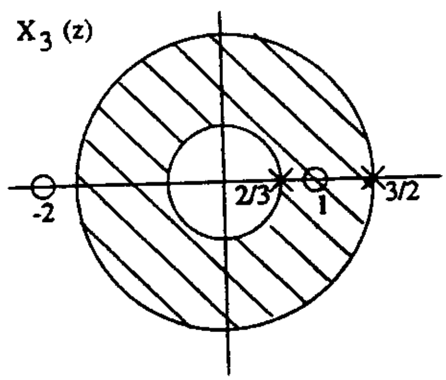
The poles are at -1/2 and 2/3, and the zero is at 1/3. Since $x_2[n]$ is causal, the ROC is extends from the outermost pole: $|z|>2/3$.



(c)

$$X_3(z)=\frac{1+z^{-1}-2z^{-2}}{1-\frac{13}{6}z^{-1}+z^{-2}}$$

The poles are at 3/2 and 2/3, and the zeros are at 1 and -2. Since $x_3[n]$ is absolutely summable, the ROC must include the unit circle: $2/3<|z|<3/2$.



3.13.

$$\begin{aligned} G(z) &= \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}) \\ &= (z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!})(1 + 3z^{-2} + 2z^{-4}) \\ &= \sum_n g[n]z^{-n} \end{aligned}$$

$g[11]$ is simply the coefficient in front of z^{-11} in this power series expansion of $G(z)$:

$$g[11] = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{11!}.$$

3.14.

$$\begin{aligned} H(z) &= \frac{1}{1 - \frac{1}{4}z^{-2}} \\ &= \frac{1}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \\ &= \frac{0.5}{1 - \frac{1}{2}z^{-1}} + \frac{0.5}{1 + \frac{1}{2}z^{-1}} \end{aligned}$$

Taking the inverse z-Transform:

$$h[n] = \frac{1}{2}(\frac{1}{2})^n u[n] + \frac{1}{2}(-\frac{1}{2})^n u[n]$$

So,

$$A_1 = \frac{1}{2}; \quad \alpha_1 = \frac{1}{2}; \quad A_2 = \frac{1}{2}; \quad \alpha_2 = -\frac{1}{2};$$

3.15. Using long division, we get

$$\begin{aligned} H(z) &= \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}} \\ &= \sum_{n=0}^{n=9} (\frac{1}{2})^n z^{-n} \end{aligned}$$

Taking the inverse z-transform,

$$h[n] = \begin{cases} (\frac{1}{2})^n, & n = 0, 1, 2, \ldots, 9 \\ 0, & \text{otherwise} \end{cases}$$

Since $h[n]$ is 0 for $n < 0$, the system is causal.

3.16. (a) To determine $H(z)$, we first find $X(z)$ and $Y(z)$:

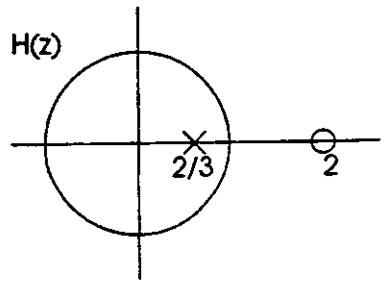
$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - 2z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}, \qquad \frac{1}{3} < |z| < 2 \end{aligned}$$

$$
\begin{aligned}
Y(z) &= \frac{5}{1-\frac{1}{3}z^{-1}} - \frac{5}{1-\frac{2}{3}z^{-1}} \\
&= \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})}, \qquad |z| > \frac{2}{3}
\end{aligned}
$$

Now

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} \qquad |z| > \frac{2}{3}
\end{aligned}
$$

The pole-zero plot of $H(z)$ is plotted below.



(b) Taking the inverse z-transform of $H(z)$, we get

$$
\begin{aligned}
h[n] &= (\frac{2}{3})^n u[n] - 2(\frac{2}{3})^{n-1} u[n-1] \\
&= (\frac{2}{3})^n (u[n] - 3u[n-1])
\end{aligned}
$$

(c) Since

$$
H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}},
$$

we can write

$$
Y(z)(1-\frac{2}{3}z^{-1}) = X(z)(1-2z^{-1}),
$$

whose inverse z-transform leads to

$$
y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]
$$

(d) The system is stable because the ROC includes the unit circle. It is also causal since the impulse response $h[n] = 0$ for $n < 0$.

3.17. We solve this problem by finding the system function $H(z)$ of the system, and then looking at the different impulse responses which can result from our choice of the ROC.

Taking the z-transform of the difference equation, we get

$$
Y(z)(1-\frac{5}{2}z^{-1}+z^{-2}) = X(z)(1-z^{-1}),
$$

and thus

$$
H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}
$$

$$= \frac{1-z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$
$$= \frac{2/3}{1-2z^{-1}} + \frac{1/3}{1-\frac{1}{2}z^{-1}}$$

If the ROC is

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^n u[-n-1]$$
$$\Longrightarrow \qquad h[0] = 0.$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \qquad h[0] = \frac{1}{3}.$$

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}2^n u[n] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \qquad h[0] = 1.$$

(d) $|z| > 2$ or $|z| < \frac{1}{2}$:

$$h[n] = \frac{2}{3}2^n u[n] - \frac{1}{3}(\frac{1}{2})^n u[n-1]$$
$$\Longrightarrow \qquad h[0] = \frac{2}{3}.$$

3.18. (a)

$$H(z) = \frac{1+2^z-1+z^{-2}}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$
$$= -2 + \frac{\frac{1}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{8}{3}}{1-z^{-1}}$$

Taking the inverse z-transform:

$$h[n] = -2\delta[n] + \frac{1}{3}(-\frac{1}{2})^n u[n] + \frac{8}{3}u[n].$$

(b) We use the eigenfunction property of the input:

$$y[n] = H(e^{j\pi/2})x[n],$$

where

$$H(e^{j\pi/2}) = -2 + \frac{\frac{1}{3}}{1+\frac{1}{2}e^{-j\pi/2}} + \frac{\frac{8}{3}}{1-e^{-j\pi/2}}$$
$$= -2 + \frac{\frac{1}{3}}{1-\frac{1}{2}j} + \frac{\frac{8}{3}}{1+j}$$
$$= \frac{-2j}{\frac{3}{2}+\frac{j}{2}}.$$

Putting it together,

$$y[n] = \frac{-2j}{\frac{3}{2}+\frac{j}{2}}e^{j(\pi/2)n}.$$

3.19. The ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$Y(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

The intersection of ROCs of $H(z)$ and $X(z)$ is $|z| > \frac{1}{2}$. So the ROC of $Y(z)$ is $|z| > \frac{1}{2}$.

(b) The ROC of $Y(z)$ is exactly the intersection of ROCs of $H(z)$ and $X(z)$: $\frac{1}{3} < |z| < 2$.

(c)

$$Y(z) = \frac{1}{(1-\frac{1}{6}z^{-1})(1+\frac{1}{3}z^{-1})}$$

The ROC is $|z| > \frac{1}{3}$.

3.20. In both cases, the ROC of $H(z)$ has to be chosen such that ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$H(z) = \frac{1-\frac{3}{4}z^{-1}}{1+\frac{2}{3}z^{-1}}$$

The ROC is $|z| > \frac{2}{3}$.

(b)

$$H(z) = \frac{1}{1-\frac{1}{6}z^{-1}}$$

The ROC is $|z| > \frac{1}{6}$.

3.21. (a)

$$\begin{aligned}
y[n] &= 0 \qquad n<0 \\
y[n] &= \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} a^{n-k} = a^n \frac{1-a^{-(n+1)}}{1-a^{-1}} = \frac{1-a^{n+1}}{1-a} \qquad 0 \le n < N-1 \\
y[n] &= \sum_{k=0}^{N-1} x[k]h[n-k] = \sum_{k=0}^{N-1} a^{n-k} = a^n \frac{1-a^{-N}}{1-a^{-1}} = a^{n+1}\frac{1-a^{-N}}{a-1}, \qquad n \ge N
\end{aligned}$$

(b)

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \qquad |z| > |a| \\
X(z) &= \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} \qquad |z| > 0
\end{aligned}$$

Therefore,

$$Y(z) = \frac{1-z^{-N}}{(1-az^{-1})(1-z^{-1})} = \frac{1}{(1-az^{-1})(1-z^{-1})} - \frac{z^{-N}}{(1-az^{-1})(1-z^{-1})} \qquad |z| > |a|$$

Now,

$$\frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{\frac{1}{1-a^{-1}}}{1-az^{-1}} + \frac{\frac{1}{1-a}}{1-z^{-1}} = \left(\frac{1}{1-a}\right)\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right)$$

So

$$\begin{aligned}
y[n] &= \left(\frac{1}{1-a}\right)\left[u[n] - a^{n+1}u[n] - u[n-N] - a^{n-N+1}u[n-N]\right] \\
&= \frac{1-a^{n+1}}{1-a}u[n] - \frac{1-a^{n-N+1}}{1-a}u[n-N] \\
y[n] &= \begin{cases} 0 & n<0 \\ \frac{1-a^{n+1}}{1-a} & 0 \le n \le N-1 \\ a^{n+1}\left(\frac{1-a^{-N}}{a-1}\right) & n \ge N \end{cases}
\end{aligned}$$

3.22. (a)

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty}\left(3\left(-\frac{1}{3}\right)^k u[k]\right)u[n-k] \\
&= \sum_{k=0}^{n} 3\left(-\frac{1}{3}\right)^k \\
&= \begin{cases} \frac{9}{4}\left(1-\left(-\frac{1}{3}\right)^{n+1}\right), & n \ge 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
Y(z) &= H(z)X(z) \\
&= \frac{3}{1+\frac{1}{3}z^{-1}}\frac{1}{1-z^{-1}} \\
&= \frac{\frac{3}{4}}{1+\frac{1}{3}z^{-1}} + \frac{\frac{9}{4}}{1-z^{-1}} \\
y[n] &= \frac{3}{4}\left(-\frac{1}{3}\right)^n u[n] + \frac{9}{4}u[n] \\
&= \frac{9}{4}\left(1+\frac{1}{3}\left(-\frac{1}{3}\right)^n\right)u[n] \\
&= \frac{9}{4}\left(1-\left(-\frac{1}{3}\right)^{n+1}\right)u[n]
\end{aligned}$$

3.23. (a)

$$\begin{aligned}
H(z) &= \frac{1-\frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \\
&= -4 + \frac{5+\frac{7}{2}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \\
&= -4 - \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{7}{1-\frac{1}{4}z^{-1}} \\
h[n] &= -4\delta[n] - 2\left(\frac{1}{2}\right)^n u[n] + 7\left(\frac{1}{4}\right)^n u[n]
\end{aligned}$$

(b)

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-2]$$

3.24. The plots of the sequences are shown below.

(a) Let

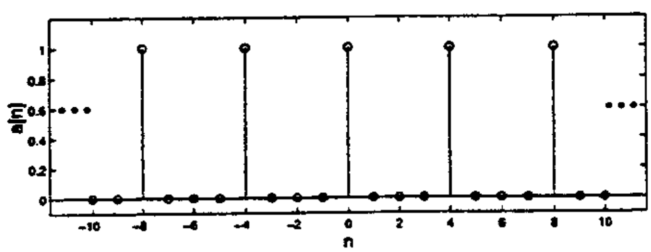
$$a[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k],$$

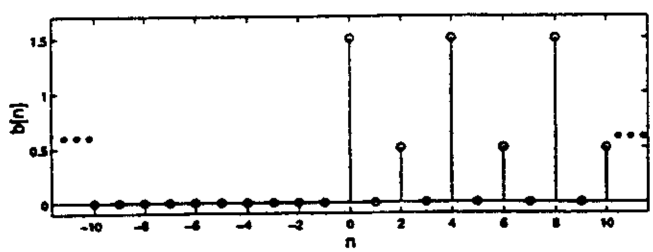
Then

$$A(z) = \sum_{k=-\infty}^{\infty} z^{-4n}$$

(b)

$$\begin{aligned}
b[n] &= \frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right]u[n] \\
&= \frac{1}{2}\left[(-1)^n + \cos\left(\frac{\pi}{2}n\right) + 1\right]u[n] \\
&= \begin{cases} \frac{3}{2}, & n = 4k,\ k \geq 0 \\ \frac{1}{2}, & n = 4k+2,\ k \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
B(z) &= \sum_{n=0}^{\infty} \frac{3}{2} z^{-4n} + \sum_{n=0}^{\infty} \frac{1}{2} z^{-(4n+2)} \\
&= \frac{3/2 + 1/2z^{-2}}{1 - z^{-4}}, \qquad |z| > 1
\end{aligned}$$





3.25.

$$X(z) = \frac{z^2}{(z-a)(z-b)} = \frac{z^2}{z^2 - (a+b)z + ab}$$

Obtain a proper fraction:

$$
\begin{array}{r|l}
 & \quad 1 \\
\cline{2-2}
z^2-(a+b)z+ab & z^2 \\
 & \underline{z^2 \quad -(a+b)z \quad +ab} \\
 & \qquad (a+b)z \quad -ab
\end{array}
$$

$$
\begin{aligned}
X(z) &= 1+\frac{(a+b)z-ab}{(z-a)(z-b)} = 1+\frac{\frac{(a+b)a-ab}{a-b}}{z-a}+\frac{\frac{(a+b)b-ab}{b-a}}{z-b} \\
&= 1+\frac{\frac{a^2}{a-b}}{z-a}-\frac{\frac{b^2}{a-b}}{z-b} = 1+\frac{1}{a-b}\left(\frac{a^2z^{-1}}{1-az^{-1}}-\frac{b^2z^{-1}}{1-bz^{-1}}\right) \\
x[n] &= \delta[n]+\frac{a^2}{a-b}a^{n-1}u[n-1]-\frac{b^2}{a-b}b^{n-1}u[n-1] \\
&= \delta[n]+\left(\frac{1}{a-b}\right)(a^{n+1}-b^{n+1})u[n-1]
\end{aligned}
$$

3.26. (a) $x[n]$ is right-sided and

$$X(z)=\frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$$

Long division:

$$
\begin{array}{r|llll}
 & 1 & -\frac{2}{3}z^{-1} & +\frac{2}{9}z^{-2} & +\ \cdots \\
\cline{2-5}
1+\frac{1}{3}z^{-1} & 1 & -\frac{1}{3}z^{-1} & & \\
 & 1 & -\frac{1}{3}z^{-1} & & \\
\cline{2-3}
 & & -\frac{2}{3}z^{-1} & & \\
 & & -\frac{2}{3}z^{-1} & -\frac{2}{9}z^{-2} & \\
\cline{3-4}
 & & & +\frac{2}{9}z^{-2} &
\end{array}
$$

Therefore, $x[n]=2(-\frac{1}{3})^n u[n]-\delta[n]$

(b)

$$X(z)=\frac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}}=\frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}=\frac{4}{1-\frac{1}{2}z^{-1}}-\frac{4}{1-\frac{1}{4}z^{-1}}$$

Poles at $\frac{1}{2}$, and $-\frac{1}{4}$. $x[n]$ stable, $\Rightarrow |z|>\frac{1}{2}$ $\Rightarrow$ causal.
Therefore,

$$x[n]=4\left(\frac{1}{2}\right)^n u[n]-4\left(-\frac{1}{4}\right)^n u[n]$$

(c)

$$
\begin{aligned}
X(z) &= \ln(1-4z) \qquad |z|<\frac{1}{4} \\
&= -\sum_{i=1}^{\infty}\frac{(4z)^i}{i} = -\sum_{\ell=-\infty}^{-1}\frac{1}{\ell}(4z)^{-\ell}
\end{aligned}
$$

Therefore,

$$x[n]=\frac{1}{n}(4)^{-n}u[-n-1]$$

(d)

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-3}} \qquad |z| > (3)^{-\frac{1}{3}} \Rightarrow \text{ causal}$$

By long division:

$$\begin{array}{r|llll}
 & 1 & +\frac{1}{3}z^{-3} & +\frac{1}{9}z^{-6} & + \ldots \\ \hline
1 - \frac{1}{3}z^{-3} & 1 & & & \\
 & 1 & -\frac{1}{3}z^{-3} & & \\ \hline
 & & +\frac{1}{3}z^{-3} & & \\
 & & +\frac{1}{3}z^{-3} & -\frac{1}{9}z^{-6} & \\ \hline
 & & & +\frac{1}{9}z^{-6} &
\end{array}$$

$$\Longrightarrow \qquad x[n] = \begin{cases} \left(\frac{1}{3}\right)^{\frac{n}{3}}, & n = 0, 3, 6, \ldots \\ 0, & \text{otherwise} \end{cases}$$

3.27. (a)

$$\begin{aligned} X(z) &= \frac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})} \qquad \frac{1}{2} < |z| < 2 \\ &= \frac{\frac{1}{35}}{(1+\frac{1}{2}z^{-2})^2} + \frac{\frac{58}{1225}}{(1+\frac{1}{2}z^{-1})} - \frac{\frac{1568}{1225}}{(1-2z^{-1})} + \frac{\frac{2700}{1225}}{(1-3z^{-1})} \end{aligned}$$

Therefore,

$$x[n] = \frac{1}{35}(n+1)\left(\frac{-1}{2}\right)^{n+1}u[n+1] + \frac{58}{(35)^2}\left(\frac{-1}{2}\right)^{n}u[n] + \frac{1568}{(35)^2}(2)^n u[-n-1] - \frac{2700}{(35)^2}(3)^n u[-n-1]$$

(b)

$$X(z) = e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \ldots$$

Therefore, $x[n] = \frac{1}{n!}u[n]$.

(c)

$$X(z) = \frac{z^3 - 2z}{z - 2} = z^2 + 2z + \frac{2}{1 - 2z^{-1}} \qquad |z| < 2$$

Therefore,

$$x[n] = \delta[n+2] + 2\delta[n+1] - 2(2)^n u[-n-1]$$

3.28. (a)

$$nx[n] \Leftrightarrow -z\frac{d}{dz}X(z)$$

$$x[n - n_0] \Leftrightarrow z^{-n_0}X(z)$$

$$X(z) = \frac{3z^{-3}}{(1 - \frac{1}{4}z^{-1})^2} = 12z^{-2}\left[-z\frac{d}{dz}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)\right]$$

$x[n]$ is left-sided. Therefore, $X(z)$ corresponds to:

$$x[n] = -12(n-2)\left(\frac{1}{4}\right)^{n-2}u[-n+1]$$

(b)

$$X(z) = \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \qquad \text{ROC includes } |z| = 1$$

Therefore,

$$x[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \delta[n + 2k + 1]$$

Which is stable.

(c)

$$X(z) = \frac{z^7 - 2}{1 - z^{-7}} = z^7 - \frac{1}{1 - z^{-7}} \qquad |z| > 1$$

$$X(z) = z^7 - \sum_{n=0}^{\infty} z^{-7n}$$

Therefore,

$$x[n] = \delta[n+7] - \sum_{n=0}^{\infty} \delta[n - 7k]$$

3.29.

$$X(z) = e^z + e^{1/z} \quad z \neq 0$$

$$X(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \Longrightarrow x[n] = \frac{1}{|n|!} + \delta[n]$$

3.30.

$$X(z) = \log_2(\frac{1}{2} - z) \qquad |z| < \frac{1}{2}$$

(a)

$$X(z) = \log(1 - 2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i} = -\sum_{\ell=-\infty}^{-1} \frac{1}{-\ell}(2z)^{-\ell} = \sum_{\ell=-\infty}^{1} \frac{1}{\ell} \left(\frac{1}{2}\right)^{\ell} z^{-\ell}$$

Therefore,

$$x[n] = \frac{1}{n} \left(\frac{1}{2}\right)^n u[-n-1]$$

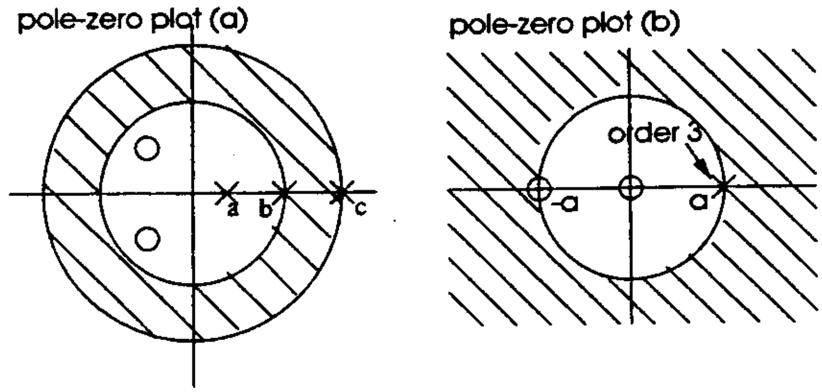
(b)

$$\begin{aligned} nx[n] &\Leftrightarrow -z\frac{d}{dz}\log(1-2z) = -z\left(\frac{1}{1-2z}\right)(-2) = z^{-1}\left(\frac{-1}{1-\frac{1}{2}z^{-1}}\right), \qquad |z| < \frac{1}{2} \\ nx[n] &= \left(\frac{1}{2}\right)^n u[-n-1] \\ x[n] &= \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1] \end{aligned}$$

3.31. (a)

$$\begin{aligned} x[n] &= a^n u[n] + b^n u[n] + c^n u[-n-1] & |a| < |b| < |c| \\ X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}} & |b| < |z| < |c| \\ X(z) &= \frac{1 - 2cz^{-1} + (bc+ac-ab)z^{-2}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})} & |b| < |z| < |c| \end{aligned}$$

Poles: a, b, c,
Zeros: z_1, z_2, ∞ where z_1 and z_2 are roots of numerator quadratic.



(b)

$$\begin{aligned} x[n] &= n^2 a^n u[n] \\ x_1[n] &= a^n u[n] \Leftrightarrow X_1(z) = \frac{1}{1-az^{-1}} \quad |z| > a \\ x_2[n] = nx_1[n] &= na^n u[n] \Leftrightarrow X_2(z) = -z\frac{d}{dz}X_1(z) = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > a \\ x[n] = nx_2[n] &= n^2 a^n u[n] \Leftrightarrow -z\frac{d}{dz}X_2(z) = -z\frac{d}{dz}\left(\frac{az^{-1}}{(1-az^{-1})^2}\right) \quad |z| > a \\ X(z) &= \frac{-az^{-1}(1+az^{-1})}{(1-az^{-1})^3} \quad |z| > a \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n] - e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n-1] \\ &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)(u[n] - u[n-1]) = \delta[n] \end{aligned}$$

Therefore, $X(z) = 1$ for all $|z|$.

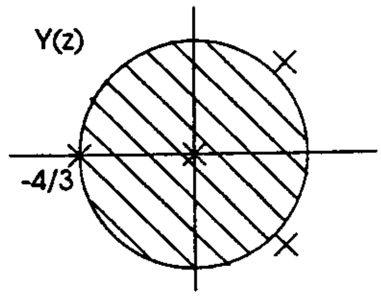
3.32. From the pole-zero diagram

$$X(z) = \frac{z}{(z^2 - z + \frac{1}{2})(z + \frac{3}{4})} \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} y[n] &= x[-n+3] = x[-(n-3)] \\ &\Rightarrow Y(z) = z^{-3}X(z^{-1}) = \frac{z^{-3}z^{-1}}{(z^{-2} - z^{-1} + \frac{1}{2})(z^{-1} + \frac{3}{4})} \\ &= \frac{8/3}{z(2 - 2z + z^2)(\frac{4}{3} + z)} \end{aligned}$$

Poles at $0, -\frac{4}{3}, 1 \pm j$, zeros at ∞

$x[n]$ causal $\Rightarrow x[-n+3]$ is left-sided $\Rightarrow$ ROC is $0 < |z| < 4/3$.



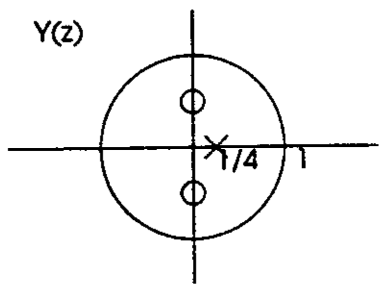
3.33. From pole-zero diagram

$$X(z) = \frac{z^2+1}{z-\frac{1}{2}}$$

(a)

$$y[n] = \left(\frac{1}{2}\right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2+1}{2z-\frac{1}{2}}$$

zeros $\pm\frac{1}{2}j$
poles $\frac{1}{4}, \infty$



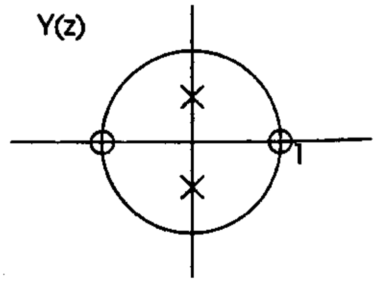
(b)

$$w[n] = \cos\left(\frac{\pi n}{2}\right) x[n] = \frac{1}{2}(e^{j\pi n/2} + e^{-j\pi n/2})x[n]$$

$$W(z) = \frac{1}{2}X(e^{-j\pi/2}z) + \frac{1}{2}X(e^{j\pi/2}z) = \frac{1}{2}X(-jz) + \frac{1}{2}X(jz)$$

$$W(z) = \frac{1}{2}\left(\frac{-z^2+1}{-jz-\frac{1}{2}}\right) + \frac{1}{2}\left(\frac{-z^2+1}{jz-\frac{1}{2}}\right) = \frac{z^2-1}{2(z^2+\frac{1}{4})}$$

poles at $\pm\frac{1}{2}j$
zeros at ± 1



3.34.

$$H(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = 5 + \frac{1}{1 - 2z^{-1}} - \frac{3}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] \text{ stable } \Rightarrow h[n] = 5\delta[n] - 2^n u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]$$

(a)

$$\begin{aligned}
y[n] &= h[n] * x[n] = \sum_{k=-\infty}^{n} h[k] \\
&= \begin{cases} -\sum_{k=-\infty}^{n} 2^k = -2^{n+1} & n < 0 \\ -\sum_{k-\infty}^{-1} 2^k + 5 - \sum_{k=0}^{n} 3\left(\frac{1}{2}\right)^k = 4 - 3\frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = -2 + 3\left(\frac{1}{2}\right)^n & n \geq 0 \end{cases} \\
&= -2u[n] + 3\left(\frac{1}{2}\right)^n u[n] - 2^{n+1}u[-n-1]
\end{aligned}$$

(b)

$$\begin{aligned}
Y(z) &= \frac{1}{1-z^{-1}}H(z) = -2\frac{1}{1-z^{-1}} + 2\frac{1}{1-2z^{-1}} + 3\frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \frac{1}{2} < |z| < 2 \\
y[n] &= -2u[n] - 2(2)^n u[-n-1] + 3\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

3.35.

$$\begin{aligned}
H(z) &= \frac{1-z^3}{1-z^4} = z^{-1}\left(\frac{1-z^{-3}}{1-z^{-4}}\right) \qquad |z| > 1 \\
u[n] &\Leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1 \\
U(z)H(z) &= \frac{z^{-1} - z^{-4}}{(1-z^{-4})(1-z^{-1})} \\
&= \frac{z^{-1}}{1-z^{-1}} - \frac{z^{-4}}{1-z^{-4}} \qquad |z| > 1 \\
u[n] * h[n] &= u[n-1] - \sum_{k=0}^{\infty} \delta[n-4-4k]
\end{aligned}$$

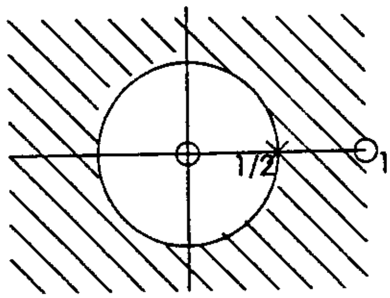
3.36.

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1] = 4\left(\frac{1}{2}\right)^{n+1} u[n+1] \Leftrightarrow Y(z) = \frac{4z}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(a)

$$H(z) = \frac{Y(z)}{X(Z)} = \frac{4z(1-z^{-1})}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



(b)

$$\begin{aligned} H(z) &= \frac{4z}{1-\frac{1}{2}z^{-1}} - \frac{4}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\ h[n] &= 4\left(\frac{1}{2}\right)^{n+1} u[n+1] - 4\left(\frac{1}{2}\right)^{n} u[n] \\ &= 4\delta[n+1] - 2\left(\frac{1}{2}\right)^{n} u[n] \end{aligned}$$

(c) The ROC of $H(z)$ includes $|z| = 1 \Rightarrow$ stable.

(d) From part (b) we see that $h[n]$ starts at $n = -1 \Rightarrow$ not causal

3.37.

$$X(z) = \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1-2z^{-1}}$$

has poles at $z = \frac{1}{2}$ and $z = 2$.

Since the unit circle is in the region of convergence $X(z)$ and $x[n]$ have both a causal and an anticausal part. The causal part is "outside" the pole at $\frac{1}{2}$. The anticausal part is "inside" the pole at 2, therefore, $x[0]$ is the sum of the two parts

$$x[0] = \lim_{z\to\infty} \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \lim_{z\to 0} \frac{\frac{1}{4}z}{z-2} = \frac{1}{3} + 0 = \frac{1}{3}$$

3.38.

$$Y(z) = \frac{z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} \cdot \frac{2}{1-z^{-1}} \qquad |z| > 1$$

Therefore using a contour C that lies outside of $|z| = 1$ we get

$$\begin{aligned}
y[1] &= \frac{1}{2\pi j}\oint_C \frac{2(z+1)z^n dz}{(z-\frac{1}{2})(z+\frac{1}{3})(z-1)} \\
&= \frac{2(\frac{1}{2}+1)(\frac{1}{2})}{(\frac{1}{2}+\frac{1}{3})(\frac{1}{2}-1)} + \frac{2(-\frac{1}{3}+1)(-\frac{1}{3})}{(-\frac{1}{3}-\frac{1}{2})(-\frac{1}{3}-1)} + \frac{2(1+1)(1)}{(1-\frac{1}{2})(1+\frac{1}{3})} \\
&= -\frac{18}{5} - \frac{2}{5} + 6 = 2
\end{aligned}$$

3.39. (a)

$$X(z) = \frac{z^{10}}{(z-\frac{1}{2})(z-\frac{3}{2})^{10}(z+\frac{3}{2})^2(z+\frac{5}{2})(z+\frac{7}{2})}$$

Stable $\Rightarrow$ ROC includes $|z| = 1$. Therefore, the ROC is $\frac{1}{2} < |z| < \frac{3}{2}$.

(b) $x[-8] = \Sigma$[residues of $X(z)z^{-9}$ inside C], where C is contour in ROC (say the unit circle).

$$x[8] = \Sigma\left[\text{residues of } \frac{z}{(z-\frac{1}{2})(z-\frac{3}{2})^{10}(z+\frac{3}{2})^2(z+\frac{5}{2})(z+\frac{7}{2})} \text{ inside unit circle}\right]$$

First order pole at $z = \frac{1}{2}$ is only one inside the unit circle. Therefore

$$x[-8] = \frac{\frac{1}{2}}{(\frac{1}{2}-\frac{3}{2})^{10}(\frac{1}{2}+\frac{3}{2})^2(\frac{1}{2}+\frac{5}{2})(\frac{1}{2}+\frac{7}{2})} = \frac{1}{96}$$

3.40. (a) After writing the following equalities:

$$\begin{aligned}
V(z) &= X(z) - W(z) \\
W(z) &= V(z)H(z) + E(z)
\end{aligned}$$

we solve for W(z):

$$W(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z)$$

(b)

$$\begin{aligned}
H_1(z) &= \frac{H(z)}{1+H(z)} = \frac{\frac{z^{-1}}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = z^{-1} \\
H_2(z) &= \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1}
\end{aligned}$$

(c) $H(z)$ is not stable due to its pole at $z = 1$, but $H_1(z)$ and $H_2(z)$ are.

3.41. (a) Yes, $h[n]$ is BIBO stable if its ROC includes the unit circle. Hence, the system is stable if $r_{min} < 1$ and $r_{max} > 1$.

(b) Let's consider the system step by step.

(i) First, $v[n] = \alpha^{-n}x[n]$. By taking the z-transform of both sides, $V(z) = X(\alpha z)$.

(ii) Second, $v[n]$ is filtered to get $w[n]$. So $W(z) = H(z)V(z) = H(z)X(\alpha z)$.

(iii) Finally, $y[n] = \alpha^n w[n]$. In the z-transform domain, $Y(z) = W(z/\alpha) = H(z/\alpha)X(z)$.

In conclusion, the system is LTI, with system function $G(z) = H(z/\alpha)$ and $g[n] = \alpha^n h[n]$.

(c) The ROC of $G(z)$ is $\alpha r_{min} < |z| < \alpha r_{max}$. We want $r_{min} < 1/\alpha$ and $r_{max} > 1/\alpha$ for the system to be stable.

3.42. (a) $h[n]$ is the response of the system when $x[n] = \delta[n]$. Hence,

$$h[n] + \sum_{k=1}^{10} \alpha_k h[n-k] = \delta[n] + \beta\delta[n-1],$$

Further, since the system is causal, $h[n] = 0$ for $n < 0$. Therefore,

$$h[0] + \sum_{k=1}^{10} \alpha_k h[-k] = h[0] = \delta[0] = 1.$$

(b) At $n = 1$,

$$h[1] + \alpha_1 h[0] = \delta[1] + \beta\delta[0] \qquad \Longrightarrow \alpha_1 = \frac{\beta - h[1]}{h[0]} = \beta - h[1]$$

(c) How can we extend h[n] for $n > 10$ and still have it compatible with the difference equation for S? Note that the difference equation can describe systems up to order 10. If we choose

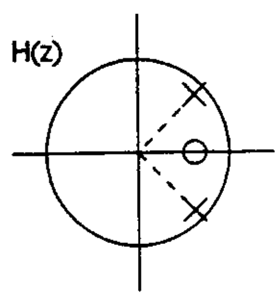
$$h[n] = (0.9)^n \cos(\frac{\pi}{4}n)u[n],$$

we only need a second order difference equation:

$$\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 0.$$

The z-transform of $h[n]$ can be found from the z-transform table:

$$H(z) = \frac{1 - \frac{0.9}{\sqrt{2}}}{(1 - 0.9e^{j\pi/4}z^{-1})(1 - 0.9e^{-j\pi/4}z^{-1})}$$

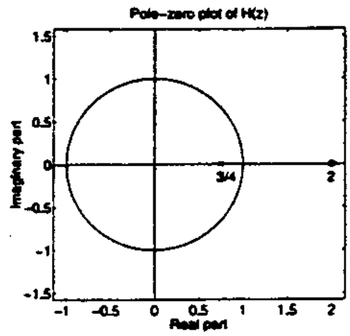


3.43. (a)

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - 2z^{-1}}, \qquad \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1 - \frac{1}{2}z^{-1}} - \frac{6}{1 - \frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}}{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}} \\ &= \frac{1 - 2z^{-1}}{1 - \frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4} \end{aligned}$$



(b)

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

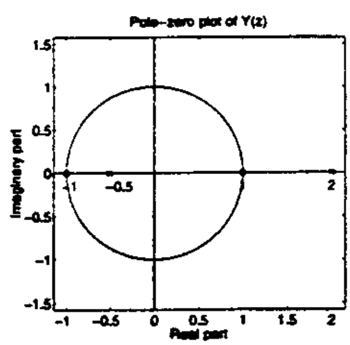
(d) The system is stable because the ROC includes the unit circle. It is also causal since $h[n] = 0$ for $n < 0$.

3.44. (a)

$$X(z) = \frac{-\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1 - 2z^{-1}}$$

The ROC is $\frac{1}{2} < |z| < 2$.

(b) The following figure shows the pole-zero plot of $Y(z)$. Since $X(z)$ has poles at 0.5 and 2, the poles at 1 and -0.5 are due to $H(z)$. Since $H(z)$ is causal, its ROC is $|z| > 1$. The ROC of $Y(z)$ must contain the intersection of the ROC of $X(z)$ and the ROC of $H(z)$. Hence the ROC of $Y(z)$ is $1 < |z| < 2$.



(c)

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}}{\frac{1}{(1-12z^{-1})(1-2z^{-1})}} \end{aligned}$$

$$\begin{aligned} &= \frac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= 1+\frac{\frac{2}{3}}{1-z^{-1}}+\frac{-\frac{2}{3}}{1+\frac{1}{2}z^{-1}} \end{aligned}$$

Taking the inverse z-transform, we find

$$h[n]=\delta[n]+\frac{2}{3}u[n]-\frac{2}{3}(-\frac{1}{2})^n u[n]$$

(d) Since $H(z)$ has a pole on the unit circle, the system is not stable.

3.45. (a)

$$\begin{aligned} ny[n] &= x[n] \\ -z\frac{dY(z)}{dz} &= X(z) \\ Y(z) &= -\int z^{-1}X(z)dz \end{aligned}$$

(b) To apply the results of part (a), we let $x[n]=u[n-1]$, and $w[n]=y[n]$.

$$\begin{aligned} W(z) &= -\int z^{-1}\frac{z^{-1}}{1-z^{-1}}dz \\ &= -\int \frac{1}{z(z-1)}dz \\ &= -\int \frac{-1}{z}+\frac{1}{z-1}dz \\ &= \ln(z)-\ln(z-1) \end{aligned}$$

3.46. (a) Since $y[n]$ is stable, its ROC contains the unit-circle. Hence, $Y(z)$ converges for $\frac{1}{2}<|z|<2$.

(b) Since the ROC is a ring on the z-plane, $y[n]$ is a two-sided sequence.

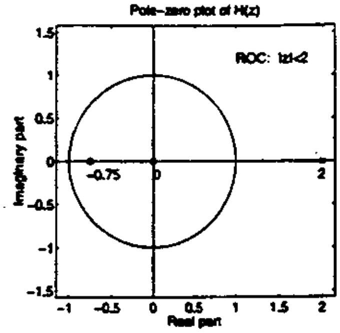
(c) $x[n]$ is stable, so its ROC contains the unit-circle. Also, it has a zero at ∞ so the ROC includes ∞. ROC: $|z|>\frac{3}{4}$.

(d) Since the ROC of $x[n]$ includes ∞, $X(z)$ contains no positive powers of z, and so $x[n]=0$ for $n<0$. Therefore $x[n]$ is causal.

(e)

$$\begin{aligned} x[0] &= X(z)|_{z=\infty} \\ &= \frac{A(1-\frac{1}{4}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{2}z^{-1})}|_{z=\infty} \\ &= 0 \end{aligned}$$

(f) $H(z)$ has zeros at -.75 and 0, and poles at 2 and ∞. Its ROC is $|z|<2$.



(g) Since the ROC of $h[n]$ includes 0, $H(z)$ contains no negative powers of z, which implies that $h[n] = 0$ for $n > 0$. Therefore $h[n]$ is anti-causal.

3.47. (a)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ X(\infty) &= \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] \end{aligned}$$

Therefore, $X(\infty) = x[0] \neq 0$ and finite by assumption. Thus, $X(z)$ has neither a pole nor a zero at $z = \infty$.

(b) Suppose $X(z)$ has finite numbers of poles and zeros in the finite z-plane. Then the most general form for $X(z)$ is

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = Kz^{L} \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{K=1}^{N}(z - d_k)}$$

where K is a constant and M and N are finite positive integers and L is a finite positive or negative integer representing the net number of poles ($L < 0$) or zeros ($L > 0$) at $z = 0$. Clearly, since $X(\infty) = x[0] \neq 0$ and $< \infty$ we must have $L + M = N$; i.e., the total number of zeros in the finite z-plane must equal the total number of poles in the finite z-plane.

3.48.

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z. Sequence is absolutely summable $\Rightarrow$ ROC contains $|z| = 1$ and roots of $Q(z)$ inside $|z| = 1$.

These conditions do not necessarily imply that $x[n]$ is causal. A shift of a causal sequence would only add more zeros at $z = 0$ to $P(z)$. For example, consider

$$\begin{aligned} X(z) &= \frac{z^2}{z - \frac{1}{2}} \quad |z| > \frac{1}{2} \\ &= \frac{z}{z - \frac{1}{2}} = z \cdot \frac{1}{1 - \frac{1}{2}z^{-1}} \\ &\Rightarrow \quad x[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1] \Rightarrow \text{ right-sided but non-causal.} \end{aligned}$$

3.49.

$$\begin{aligned} x[n] &= \delta[n] + a\delta(n-N) \quad |a|<1 \\ X(z) &= 1 + az^{-N} \\ \hat{X}(z) &= \log X(z) = \log(1+az^{-N}) = az^{-N} - \frac{a^2z^{-2N}}{2} + \frac{a^3z^{-3N}}{3} - \ldots \end{aligned}$$

Therefore,

$$\hat{x}[n] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a^k \delta[n-kN]$$

3.50. (a)

$$x[n] = x[-n] \Longrightarrow X(z) = X\left(\frac{1}{z}\right)$$

Therefore,

$$X(z_0) = 0 = X\left(\frac{1}{z_0}\right)$$

i.e., $1/z_0$ is also a zero of $X(z)$.

(b)

$$x[n] \text{ real} \Longrightarrow x[n] = x^*[n] \Longrightarrow X(z) = X^*(z^*)$$

Therefore

$$X(z_0) = 0 = X(z_0^*)$$

i.e., z_0^* is also a zero and by part (a) so is $1/z_0^*$.

3.51. (a)

$$\mathcal{Z}[x^*[n]] = \sum_{n=-\infty}^{\infty} X^*[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* = X^*(z^*)$$

(b)

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{-n} = X(z^{-1})$$

(c)

$$\mathcal{Z}[\mathcal{R}e\{x[n]\}] = \mathcal{Z}\left[\frac{x[n]+x^*[n]}{2}\right] = \frac{1}{2}[X(z)+X^*(z^*)]$$

(d)

$$\mathcal{Z}[\mathcal{I}m\{x[n]\}] = \mathcal{Z}\left[\frac{x[n]-x^*[n]}{2j}\right] = \frac{1}{2j}[X(z)-X^*(z^*)]$$

3.52.

$$x_1(n) = (-1)^n x(n) \Rightarrow X_1(z) = \sum_{n=-\infty}^{\infty} (-1)^n x(n) z^{-n} = X(-z)$$

The poles and zeros are rotated 180 degrees about the origin.

3.53. (a)

$$\theta_x(\omega) = \tan^{-1}\left(\frac{\mathcal{Im}\{X(e^{j\omega})\}}{\mathcal{Re}\{X(e^{j\omega})\}}\right) \Longrightarrow \tan\theta_x(\omega) = \frac{-\sum_{n=0}^{N-1} x[n]\sin(n\omega)}{\sum_{n=0}^{N-1} x[n]\cos(n\omega)}$$

$$\tan\theta_x(\omega)\sum_{n=0}^{N-1} x[n]\cos(\omega n) = -\sum_{n=1}^{N-1} x[n]\sin(n\omega)$$

$$\tan\theta_x(\omega)x[0] + \sum_{n=1}^{N-1} x[n](\tan\theta_x(\omega)\cos(n\omega) + \sin(n\omega)) = 0$$

$$\tan\theta_x(\omega_k) + \frac{1}{x[0]}\sum_{n=1}^{N-1} x[n](\tan\theta_x(\omega_k)\cos(n\omega_k) + \sin(n\omega_k)) = 0$$

for $N-1$ values of ω_k in the range $0 < \omega_k < \pi$.

(b) $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] \Longrightarrow X(z) = 1 + 2z^{-1} + 3z^{-2}$

$$\theta_x(\omega) = \tan^{-1}\left(\frac{-2\sin(\omega) - 3\sin(2\omega)}{1 + 2\cos(\omega) + 3\cos(2\omega)}\right)$$

Consider the values $\theta_x\left(\frac{\pi}{2}\right) = \frac{5\pi}{4}$ and $\theta_x\left(\frac{2\pi}{3}\right) = \frac{5\pi}{6}$, which give the equations

$$\begin{aligned}
\tan\theta_x\left(\frac{\pi}{2}\right) &+ \frac{1}{x[0]}\left[x[1]\left(\tan\theta_x\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)\right. \\
&+ \left. x[2]\left(\tan\theta_x\left(\frac{\pi}{2}\right)\cos\pi + \sin\pi\right)\right] = 0 \\
\tan\theta_x\left(\frac{2\pi}{3}\right) &+ \frac{1}{x_0}\left[x[1]\left(\tan\theta_x\left(\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right)\right. \\
&+ \left. x[2]\left(\tan\theta_x\left(\frac{2\pi}{3}\right)\cos\frac{4\pi}{3} + \sin\frac{4\pi}{3}\right)\right] = 0
\end{aligned}$$

$$\begin{aligned}
1 + \frac{1}{x[0]}(x[1]\cdot 1 + x[2]\cdot -1) &= 0 \\
-\frac{1}{\sqrt{3}} + \frac{1}{x[0]}\left(x[1] + \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}\right) + x[2]\left(\frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2}\right) &= 0
\end{aligned}$$

$$\left.\begin{aligned} x[0] + x[1] - x[2] &= 0 \\ -x[0] + 2x[1] - x[2] &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} x[1] &= 2x[0] \\ x[2] &= 3x[0] \end{aligned}\right.$$

Therefore

$$x[n] = x[0](\delta[n] + 2\delta[n-1] + 3\delta[n-2])$$

where $x[0]$ is undetermined.

3.54. $x[n] = 0$ for $n < 0$ implies:

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \lim_{z\to\infty}\sum_{n=1}^{\infty} x[n]z^{-n} = x[0]$$

For the case $x[n] = 0$ for $n > 0$,

$$\lim_{z\to 0} X(z) = \lim_{z\to 0} \sum_{n=-\infty}^{0} x[n]z^{-n} = x[0] + \lim_{z\to 0}\sum_{n=1}^{\infty} x[-n]z^{n} = x[0]$$

3.55. (a)

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k] = \sum_{k=-\infty}^{\infty} x[-k]x[n-k] = x[-n] * x[n]$$

$$C_{xx}(z) = X(z^{-1})X(z) = X(z)X(z^{-1})$$

$X(z)$ has ROC: $r_R < |z| < r_L$ and therefore $X(z^{-1})$ has ROC: $r_L^{-1} < |z| < r_R^{-1}$. Therefore $C_{xx}(z)$ has ROC: $\max[r_L^{-1}, r_R] < |z| < \min[r_R^{-1}, r_L]$

(b) $x[n] = a^n u[n]$ is stable if $|a| < 1$. In this case

$$X(z) = \frac{1}{1-az^{-1}} \qquad |a| < |z| \quad \text{and} \quad X(z^{-1}) = \frac{1}{1-az} \qquad |z| < |a^{-1}|$$

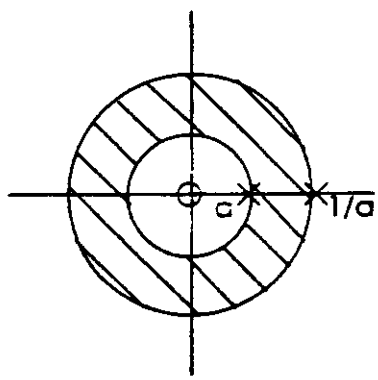
Therefore

$$\begin{aligned} C_{xx}(z) &= \frac{1}{1-az^{-1}}\frac{1}{1-az} = \frac{-az^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})} \\ &= \frac{\frac{1}{1-a^2}}{1-az^{-1}} - \frac{\frac{1}{1-a^{-2}}}{1-a^{-1}z^{-1}} \qquad |a| < |z| < |a^{-1}| \end{aligned}$$

This implies that

$$c_{xx}[n] = \frac{1}{1-a^2}\left[a^n u[n] + a^{-n}u[-n-1]\right]$$

Thus, in summary, the poles are at a and a^{-1}; the zeros are at 0 and ∞; and the ROC of $C_{xx}(z)$ is $|a| < |z| < |a^{-1}|$.



(c) Clearly, $x_1[n] = x[-n]$ will have the same autocorrelation function. For example,

$$X_1(z) = \frac{1}{1-az} \qquad |z| < |a^{-1}| \Longrightarrow C_{x_1x_1}(z) = \frac{1}{1-az}\frac{1}{1-az^{-1}} = C_{xx}(z)$$

(d) Also, any delayed version of $x[n]$ will have the same autocorrelation function; e.g., $x_2[n] = x[n-m]$ implies

$$X_2(z) = \frac{z^{-m}}{1-az^{-1}} \qquad |a| < |z| \Longrightarrow C_{x_2x_2}(z) = \frac{z^{-m}}{1-az^{-1}}\frac{z^{m}}{1-az} = C_{xx}(z)$$

3.56. In order to be a z-transform, $X(z)$ must be analytic in some annular region of the z-plane. To determine if $X(z) = z^*$ is analytic we examine the existence of $X'(z)$ by the Cauchy Riemann conditions. If

$$X(z) = X(x + jy) = u(x, y) + jv(x, y)$$

then for the derivative to exist at z, we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In our case,

$$X(x + jy) = x - jy$$

and thus,

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

unless x and y are zero. Thus, $X'(z)$ exists only at $z = 0$. $X(z)$ is not analytic anywhere. Therefore,

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \quad \text{does not exist.}$$

3.57. If $X(z)$ has a pole at $z = z_0$ then $A(z)$ can be expressed as a Taylor's series about $z = z_0$.

$$A(z) = A(z_0) + \sum_{n=1}^{\infty} \frac{A^n(z_0)}{n!}(z - z_0)^n$$

where $A(z_0) = 0$. Thus

$$\begin{aligned}
\text{Res}\,[X(z) \text{ at } z = z_0] &= X(z)(z - z_0)|_{z=z_0} = \left.\frac{B(z)}{A(z)}\right|_{z=z_0} \\
&= \left.\frac{B(z)(z - z_0)}{\sum_{n=1}^{\infty} \frac{A^n(z_0)}{n!}(z - z_0)^n}\right|_{z=z_0} \\
&= \left.\frac{B(z)}{A'(z_0) + \sum_{n=2}^{\infty} \frac{A^n(z_0)}{n!}(z - z_0)^{n-1}}\right|_{z=z_0} = \frac{B(z_0)}{A'(z_0)}
\end{aligned}$$