Discrete-Time Signals: Frequency-Domain Representation - I

CHAPTER 3

These lecture slides are based on "Digital Signal Processing: A Computer-Based Approach, 4th ed." textbook by S.K. Mitra and its instructor materials. U.Sezen

Contents

Frequency-Domain Representation

- The frequency-domain representation of a discrete-time sequence is the **discrete-time Fourier transform** (**DTFT**)
- This transform maps a time-domain sequence into a continuous function of the frequency variable ω
- We first review briefly the **continuous-time Fourier transform** (**CTFT**)

Continuous-Time Fourier Transform

- **Definition:** The CTFT of a continuous-time signal $x_a(t)$ is given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the **Fourier spectrum** or simply the **spectrum** of the continuous-time signal
- **Definition:** The inverse CTFT of a Fourier transform $X(j\Omega)$ is given by

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

- Often referred to as the **Fourier integral**

- A CTFT pair will be denoted as

$$x_a(t) \stackrel{\mathsf{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

- Ω is real and denotes the continuous-time angular frequency variable in **radians/sec** if the unit of the independent variable t is in **seconds**
- In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$

  It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)}$$

  ehere

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

- The quantity $|X_a(j\Omega)|$ is called the **magnitude spectrum** and the quantity $\theta_a(\Omega)$ is called the **phase spectrum**
- Both spectrums are real functions of Ω
- In general, the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the **Dirichlet conditions** given on the next slide

Dirichlet Conditions:

(a) The signal $x_a(t)$ has a finite number of discontinuities and a finite number of maxima and minima in any finite interval

(b) The signal is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x_a(t)| dt < \infty$$

- If the **Dirichlet conditions** are satisfied, then

  $$\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

  converges to $x_a(t)$ at all values of t except at values of t where $x_a(t)$ has discontinuities

- It can be shown that $x_a(t)$ is **absolutely integrable**, then $|X_a(j\Omega)| < \infty$ provig the existence of CTFT

Energy Density Spectrum

- The total energy $\mathcal{E}_x$ of a finite energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt$$

The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

- Interchanging the order of the integration we get

$$\begin{aligned}\mathcal{E}_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \right] d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega\end{aligned}$$

- Hence,

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

- The above relation is more commonly known as the **Parseval's theorem** for finite-energy continuous-time signals

- The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and usually denoted as

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2$$

- The energy over a specified range of $\Omega_a \leq \Omega \leq \Omega_b$ frequencies can be computed using

$$\mathcal{E}_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)d\Omega$$

Band-limited Continuous-Time Signals

- A **full-band**, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$
- A **band-limited** continuous-time signal has a spectrum that is limited to a **portion** of the frequency range $-\infty < \Omega < \infty$
- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$, that is

$$X_a(j\Omega) \neq 0, \quad \Omega_a \leq |\Omega| \leq \Omega_b \text{ and}$$
$$X_a(j\Omega) = 0, \quad 0 \leq |\Omega| < \Omega_a \text{ or } \Omega_b < |\Omega| < \infty$$

- However, an ideal band-limited signal cannot be generated in practice

- **Band-limited signals** are classified according to the **frequency range** where most of the **signal's energy** is concentrated
- A **lowpass** continuous-time signal has a spectrum occupying the frequency range $|\Omega| \leq \Omega_p$ where $\Omega_p < \infty$. Here, Ω_p is called the **bandwidth** of the signal
- A **highpass** continuous-time signal has a spectrum occupying the frequency range $\Omega_p \leq |\Omega|$ where $0 < \Omega_p < \infty$. Here the **bandwidth** of the signal is from Ω_p to ∞
- A **bandpass** continuous-time signal has a spectrum occupying the frequency range $\Omega_L \leq |\Omega| \leq \Omega_H$ where $0 < \Omega_L \leq \Omega_H < \infty$. Here $\Omega_H - \Omega_L$ is the **bandwidth**

Discrete-Time Fourier Transform

- **Definition:** The **discrete-time Fourier transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

where ω is a continuous variable in the range $-\infty < \omega < \infty$

- The infinite series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ may or may not converge
- If it converges for all values of ω, then the DTFT $X\left(e^{j\omega}\right)$ exists
- In general, $X\left(e^{j\omega}\right)$ is a **complex** function of the **real** variable ω and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

where $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω

Discrete-Time Fourier Transform Discrete-Time Fourier Transform

- $X\left(e^{j\omega}\right)$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\left\{X(e^{j\omega})\right\}$$

- $\left|X(e^{j\omega})\right|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**

  Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

- For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{re}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{im}(e^{j\omega})$ are odd functions of ω
- Note that

$$\begin{aligned} X(e^{j\omega}) &= \left|X(e^{j\omega})\right| e^{j\theta(\omega+2\pi k)} \\ &= \left|X(e^{j\omega})\right| e^{j\theta(\omega)} \end{aligned}$$

  for any integer k

  Thus, the phase function $\theta(\omega)$ **cannot be uniquely specified** for any DTFT
- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

  called the **principal value**

- The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses

  An alternate type of phase function that is a continuous function of ω is often used

  The process of removing the discontinuities is called **unwrapping**

  The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$

  In some cases, discontinuities of π may be present after unwrapping

- **Example:** The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\begin{aligned}\Delta(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} \\ &= \delta[0] \\ &= 1\end{aligned}$$

- **Example:** Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

Its DTFT is given by

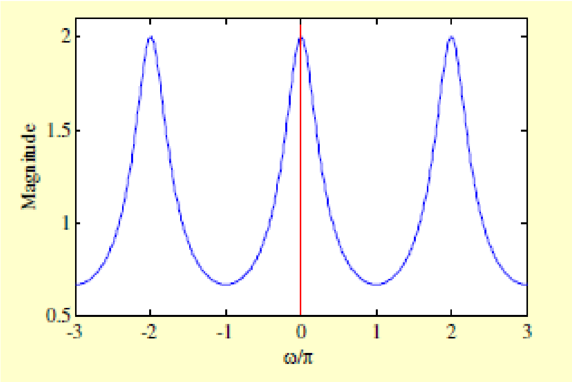
$$\begin{aligned}X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}\end{aligned}$$

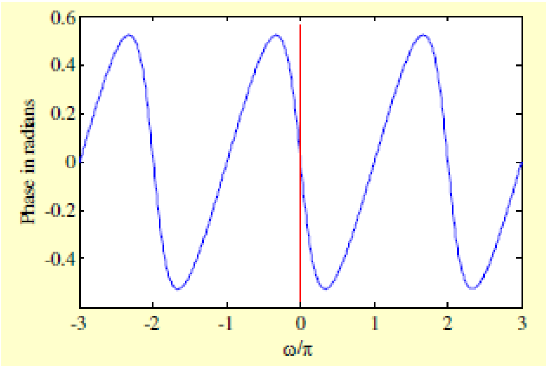
as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

The magnitude and phase of the DTFT

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

are shown below





$$\left|X(e^{j\omega})\right| = \left|X(e^{-jw})\right| \qquad \theta(\omega) = -\theta(-\omega)$$

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π. Consider $\omega = \omega_0 + 2\pi k$

$$\begin{aligned} X(e^{j(\omega_0+2\pi k)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega_0+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_0 n}e^{-j2\pi kn} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_0 n} \\ &= X(e^{j\omega_0}) \end{aligned}$$

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete-Time Fourier Transform Discrete-Time Fourier Transform

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$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Inverse Discrete-time Fourier transform

- Inverse discrete-time Fourier transform is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Proof:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega &= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right)\\ &= \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)}\end{aligned}$$

where

$$\begin{aligned}\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)} &= \begin{cases}1, & n=\ell\\ 0, & n\neq\ell\end{cases}\\ &= \delta[n-\ell]\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\left(\pi(n-\ell)\right)}{\pi(n-\ell)} &= \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell]\\ &= x[n]\end{aligned}$$

Convergence Condition

- **Convergence Condition:** An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

may or may not converge

Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Then, for uniform convergence of $X(e^{j\omega})$

$$\lim_{K\to\infty} \left| X(e^{j\omega}) - X_K\left(e^{j\omega}\right) \right| = 0$$

Now, if $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Then

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

for all values of ω

Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$

- **Example:** The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT $X(e^{j\omega})$ therefore converges to $\dfrac{1}{1-\alpha e^{-j\omega}}$ uniformly

- Since

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left| \sum_{n=-\infty}^{\infty} |x[n]| \right|^2 < \infty$$

an **absolutely summable sequence** has always a **finite energy**

However, a finite-energy sequence is not necessarily absolutely summable

- **Example:** The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

But, $x[n]$ is **not absolutely summable**

- To represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a **mean-square convergence** of $X(e^{j\omega})$:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K\left(e^{j\omega}\right)\right|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

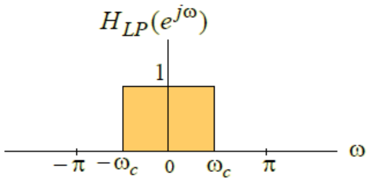
must approach zero at each value of ω as K goes to ∞

- In such a case, the absolute value of the error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ may not go to zero as K goes to ∞ and the DTFT is no longer bounded

- **Example:** Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

shown below



The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$\begin{aligned} h_{LP}[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) \\ &= \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty \end{aligned}$$

The energy of $h_{LP}[n]$ is given by ω_c/π

Hence, $h_{LP}[n]$ **is a finite-energy sequence, but it is not absolutely summable**

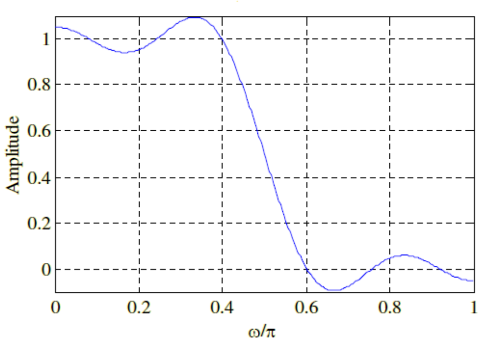
As a result

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$

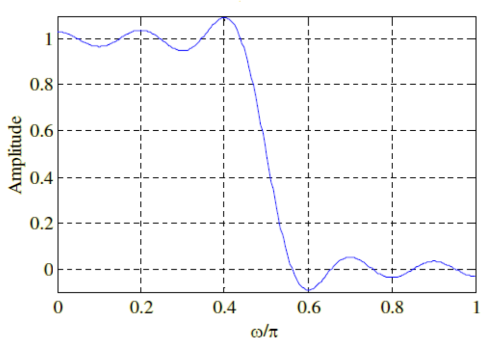
does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense

The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

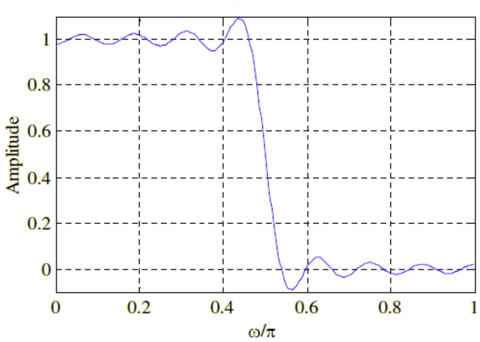
$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$



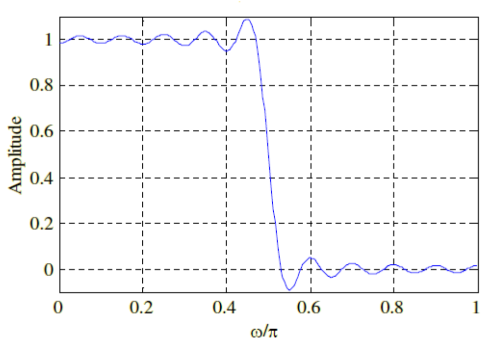
(a) $K = 10$



(b) $K = 20$



(c) $K = 30$



(d) $K = 40$

Gibbs phenomenon

As can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$

The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K

As K goes to infinity, the condition

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) - H_{LP,K}\left(e^{j\omega}\right) \right|^2 d\omega = 0$$

holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$

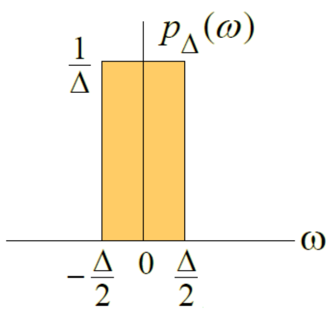
The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**

Dirac delta function

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function** $\delta(\omega)$

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_\Delta(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$



- **Example:** Consider the complex exponential sequence

$$x[n] = e^{j\omega_0 n}$$

Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and $-\pi \le \omega_0 \le \pi$

The function $X(e^{j\omega})$ above is a periodic function of ω with a period 2π and is called a **periodic impulse train**

To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$ we compute the inverse DTFT of $X(e^{j\omega})$

Thus

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} d\omega \\ &= e^{j\omega_0 n} \end{aligned}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Table : Commonly Used DTFT Pairs

Sequence		**DTFT**
$\delta[n]$	$\longleftrightarrow$	1
1	$\longleftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\longleftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\mu[n]$	$\longleftrightarrow$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\alpha^n \mu[n]$ $(\lvert\alpha\rvert < 1)$	$\longleftrightarrow$	$\frac{1}{1-\alpha e^{-j\omega}}$

DTFT Properties and Theorems

- There are a number of important properties and theorems of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table : DTFT Symmetry Relations for a **complex sequence** $x[n]$

Sequence	**Discrete-Time Fourier Transform**
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-jw})$
$x^*[-n]$	$X^*(e^{-jw})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$j\,\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{cs}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Table : DTFT Symmetry Relations for a **real sequence** $x[n]$

Sequence	**Discrete-Time Fourier Transform**
$x[n]$	$X(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetric relations	$X(e^{j\omega}) = X^*(e^{-jw})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-jw})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-jw})$ $\left\|X(e^{j\omega})\right\| = \left\|X(e^{-jw})\right\|$ $\arg\left\{X(e^{j\omega})\right\} = -\arg\left\{X(e^{-jw})\right\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha\, g[n] + \beta\, h[n]$	$\alpha\, G(e^{j\omega}) + \beta\, H(e^{j\omega})$
Time-shifting	$g[n - n_0]$	$e^{-j\omega n_0}\, G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_0 n}\, g[n]$	$G(e^{j(\omega-\omega_0)})$
Differentiation in frequency	$n\, g[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$	

- **Example:** Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\,\alpha^n \mu[n], \quad |\alpha| < 1$$

Let $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$

We can therefore write

$$y[n] = n\,x[n] + x[n]$$

Using the **linearity** and **differentiation theorems** in Table 3.4, the $Y(e^{j\omega})$ is given by

$$Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

From Table 3.1, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1-\alpha\, e^{-j\omega}}$$

Using the **differentiation theorem** of the DTFT given in Table 3.4, we observe that the DTFT of $n\,x[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha\,e^{-j\omega}}\right) = \frac{\alpha\,e^{-j\omega}}{(1-\alpha\,e^{-j\omega})^2}$$

Next using the **linearity theorem** of the DTFT given in Table 3.4, we arrive at

$$Y(e^{j\omega}) = \frac{\alpha\,e^{-j\omega}}{(1-\alpha\,e^{-j\omega})^2} + \frac{1}{1-\alpha\,e^{-j\omega}}$$

- **Example:** Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0\,v[n] + d_1\,v[n-1] = p_0\,\delta[n] + p_1\,\delta[n-1]$$

From Table 3.1, the DTFT of $\delta[n]$ is 1

Using the **time-shifting theorem** of the DTFT, given in Table 3.4, we observe that the DTFT $\delta[n]$ of is e^{-jw} and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

Using the **linearity theorem** in Table 3.4, we then obtain the frequency-domain representation of the difference equation above as

$$d_0\,V(e^{j\omega}) + d_1\,e^{-j\omega}\,V(e^{j\omega}) = p_0 + p_1\,e^{-j\omega}$$

Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1\,e^{-j\omega}}{d_0 + d_1\,e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a **finite-energy** sequence $g[n]$ is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$$

- From **Parseval's theorem** given in Table 3.4, we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G(e^{j\omega})\right|^2 d\omega$$

- The quantity

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$$

  is called the **energy density spectrum**
- The area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence

Band-limited Discrete-time Signals

- Since the spectrum of a discrete-time signal is a periodic function of ω with a period 2π, a **full-band** signal has a spectrum occupying the frequency range $-\pi \leq \omega \leq \pi$
- A **band-limited** discrete-time signal has a spectrum that is limited to a **portion** of the frequency range $-\pi \leq \omega \leq \pi$
- An **ideal band-limited** signal has a spectrum that is zero outside a frequency range $\omega_a \leq |\omega| \leq \omega_b$ with $0 \leq \omega_a \leq \omega_b < \pi$, that is

$$\begin{aligned} X_a(e^{j\omega}) &\neq 0, \quad \omega_a \leq |\omega| \leq \omega_b \text{ and} \\ X_a(e^{j\omega}) &= 0, \quad 0 \leq |\omega| < \omega_a \text{ or } \omega_b < |\omega| < \pi \end{aligned}$$

- An ideal band-limited discrete-time signal cannot be generated in practice

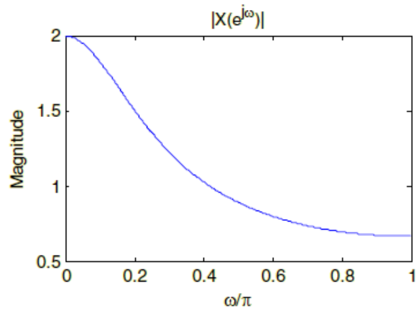
- A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal's energy is concentrated
- A **lowpass** discrete-time real signal has a spectrum occupying the frequency range $|\omega| \leq \omega_p$ where $0 \leq \omega_p < \pi$ and has a **bandwidth** of ω_p
- A **highpass** discrete-time real signal has a spectrum occupying the frequency range $\omega_p \leq |\omega|$ where $0 < \omega_p < \pi$ and has a **bandwidth** of $\pi - \omega_p$
- A **bandpass** discrete-time real signal has a spectrum occupying the frequency range $\omega_L \leq |\omega| \leq \omega_H$ where $0 \leq w_L \leq w_H < \pi$ and has a **bandwidth** of $\omega_H - \omega_L$.

- **Example:** Consider the sequence

$$x[n] = (0.5)^n \mu[n]$$

Its DTFT is given below on the left along with its magnitude spectrum shown below on the right

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



It can be shown that 80% of the energy of this **lowpass** signal is contained in the frequency range $0 \leq |\omega| \leq 0.5081\pi$

Hence, we can define the 80% **bandwidth** to be 0.5081π radians

- **Example:** Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty$$

Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega})\right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Therefore

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Hence, $h_{LP}[n]$ is a finite-energy lowpass sequence

DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1\, e^{-j\omega} + \ldots + p_M\, e^{-j\omega M}}{d_0 + d_1\, e^{-j\omega} + \ldots + d_N\, e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_\ell$

- For example, the statement

```
H = freqz(num,den,w)
```

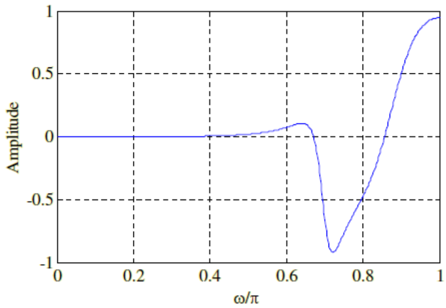
returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector `w`

- There are several other forms of the function `freqz`
- `Program 3_1.m` in the text can be used to compute the values of the DTFT of a real finite-length sequence
- It computes the real and imaginary parts, and the magnitude and phase of the DTFT

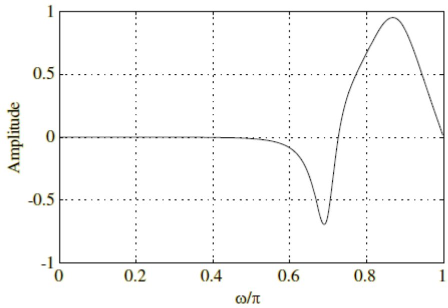
- **Example:** Plots of the real and imaginary parts, and the magnitude and phase of the DTFT as a function of the **normalized angular frequency variable** ω/π

$$X(e^{j\omega}) = \frac{0.008 - 0.033\,e^{-j\omega} + 0.05\,e^{-j2\omega} - 0.033\,e^{-j3\omega} + 0.008\,e^{-j4\omega}}{1 + 2.37\,e^{-j\omega} + 2.7\,e^{-j2\omega} + 1.6\,e^{-j3\omega} + 0.41\,e^{-j4\omega}}$$

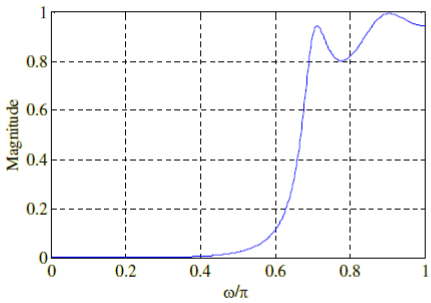
are shown on the next slide



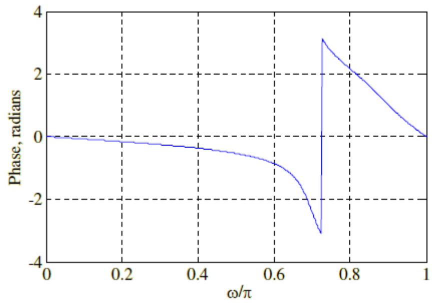
(a) Real part, $X_{re}(e^{j\omega})$



(b) Imaginary part, $X_{im}(e^{j\omega})$



(c) Magnitude spectrum, $\left|X(e^{j\omega})\right|$

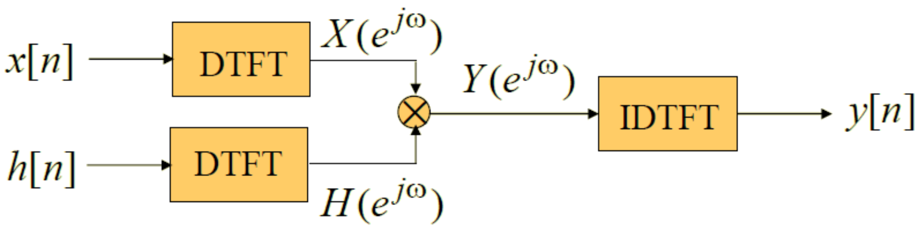


(d) Phase spectrum, $\arg\left\{X(e^{j\omega})\right\}$

Linear Convolution Using DTFT

- An important property of the DTFT is given by the convolution theorem in Table 3.4
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by
$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$
- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

1. Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
2. Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
3. Compute the IDFT $y[n]$ of $Y(e^{j\omega})$

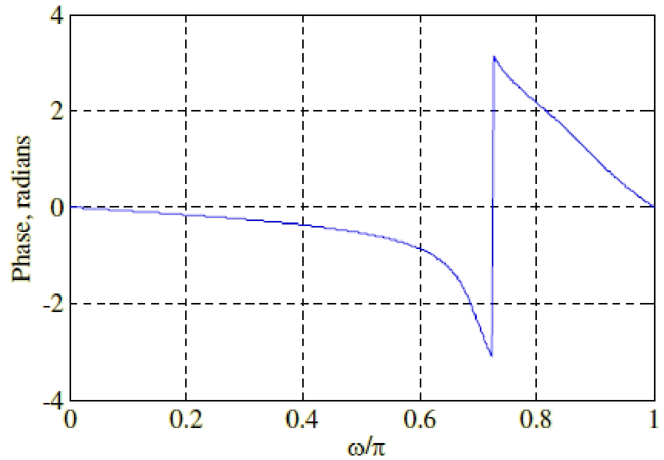


The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range
- Thus. the phase functions of some sequences exhibit discontinuities of 2π radians in the plot

- **Example:** For example, there is a discontinuity of 2π at $\omega = 0.72$ in the phase function of $X(e^{j\omega})$ below

$$X(e^{j\omega}) = \frac{0.008 - 0.033\,e^{-j\omega} + 0.05\,e^{-j2\omega} - 0.033\,e^{-j3\omega} + 0.008\,e^{-j4\omega}}{1 + 2.37\,e^{-j\omega} + 2.7\,e^{-j2\omega} + 1.6\,e^{-j3\omega} + 0.41\,e^{-j4\omega}}$$



Phase spectrum, $\arg\left\{X(e^{j\omega})\right\}$

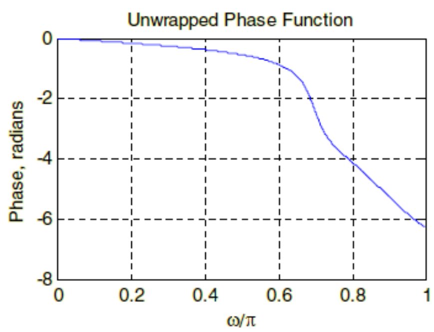
In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π

Process of discontinuity removal is called **unwrapping the phase**

The unwrapped phase function will be denoted as $\theta_c(\omega)$

In MATLAB, the unwrapping can be implemented using the `unwrap` function (or command)

The unwrapped phase function of the DTFT $X(e^{j\omega})$ is shown below



- The conditions under which the phase function $\theta(\omega)$ will be a continuous function of ω is next derived
- Now

$$\ln X(e^{j\omega}) = \left|X(e^{j\omega})\right| + j\theta(\omega)$$

where

$$\theta(\omega) = \arg\left\{X(e^{j\omega})\right\}$$

- If $\ln X(e^{j\omega})$ exits, then its derivative with respect to ÏL' also exists and is given by

$$\begin{aligned}\frac{d\ln X(e^{j\omega})}{d\omega} &= \frac{1}{X(e^{j\omega})}\frac{dX(e^{j\omega})}{d\omega} \\ &= \frac{1}{X(e^{j\omega})}\left(\frac{dX_{re}(e^{j\omega})}{d\omega} + j\frac{dX_{im}(e^{j\omega})}{d\omega}\right)\end{aligned}$$

- As $\ln X(e^{j\omega}) = \left|X(e^{j\omega})\right| + j\theta(\omega)$, then $\dfrac{d\ln X(e^{j\omega})}{d\omega}$ is also given by

$$\frac{d\ln X(e^{j\omega})}{d\omega} = \frac{d\left|X(e^{j\omega})\right|}{d\omega} + j\frac{d\theta(\omega)}{d\omega}$$

- Thus, $\dfrac{d\theta(\omega)}{d\omega}$ is given by the imaginary part of

$$\frac{1}{X(e^{j\omega})}\left(\frac{dX_{re}(e^{j\omega})}{d\omega} + j\frac{dX_{im}(e^{j\omega})}{d\omega}\right)$$

- Hence,

$$\frac{d\theta(\omega)}{d\omega} = \frac{1}{\left|X(e^{j\omega})\right|^2}\left(X_{re}(e^{j\omega})\frac{dX_{im}(e^{j\omega})}{d\omega} - X_{im}(e^{j\omega})\frac{dX_{re}(e^{j\omega})}{d\omega}\right)$$

- The phase function can thus be defined unequivocally by its derivative $\dfrac{d\theta(\omega)}{d\omega}$

$$\theta(\omega) = \int_0^w \frac{d\theta(\eta)}{d\eta} d\eta$$

with the constraint $\theta(0) = 0$

- The phase function $\theta(\omega)$ defined above is called the **unwrapped phase function** of $X(e^{j\omega})$ and it is a continuous function of ω. Thus, $\ln X(e^{j\omega})$ exists
- Moreover, the phase function will be an odd function of ω if

$$\frac{1}{\pi}\int_0^{2\pi} \frac{d\theta(\eta)}{d\eta} d\eta = 0$$

- If the above constraint is not satisfied, then the computed phase function will exhibit absolute jumps greater than π. For unwrapping the phase, these jumps should be replaced with their 2π complements, e.g. like the `unwrap` function in MATLAB.