Solutions – Chapter 11

Discrete Hilbert Transforms

11.1. Using the fact that $x_e[n]$ is the inverse transform of $\mathcal{R}e$ we get

$$\begin{aligned}\mathcal{R}e\{X(e^{j\omega})\} &= 2 - ae^{j\omega} - ae^{-j\omega} \\ x_e[n] &= 2\delta[n] - a\delta[n+1] - a\delta[n-1]\end{aligned}$$

Since $x[n]$ is causal, we can recover it from $x_e[n]$

$$x[n] = 2x_e[n]u[n] - x_e[0]\delta[n] = 2\delta[n] - 2a\delta[n-1]$$

This implies that

$$x_o[n] = \frac{x[n] - x[-n]}{2} = a\delta[n+1] - a\delta[n-1]$$

and since $j\mathcal{I}m\{X(e^{j\omega})\}$ is the transform of $x_o[n]$ we find

$$\mathcal{I}m\{X(e^{j\omega})\} = 2a\sin\omega$$

11.2. Taking the inverse transform of $\mathcal{R}e\{X(e^{j\omega})\} = 5/4 - \cos\omega$, we get

$$x_e[n] = \frac{5}{4}\delta[n] - \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1]$$

Since $x[n]$ is causal, we can recover it from $x_e[n]$

$$x[n] = 2x_e[n]u[n] - x_e[0]\delta[n] = \frac{5}{4}\delta[n] - \delta[n-1],$$

11.3. Note that

$$\begin{aligned}|X(e^{j\omega})|^2 &= \frac{5}{4} - \cos\omega \\ &= \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{2}e^{j\omega}\right) \\ &= X(e^{j\omega})X^*(e^{j\omega})\end{aligned}$$

If $X(e^{j\omega}) = (1 - \frac{1}{2}e^{-j\omega})$ we get

$$x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

but this does not satisfy the conditions on $x[n]$ given in the problem statement.
However, if we let $X(e^{j\omega}) = (1 - \frac{1}{2}e^{-j\omega})e^{-j\omega}$ we get

$$x[n] = \delta[n-1] - \frac{1}{2}\delta[n-2]$$

which satisfies all the constraints. The idea behind this choice is that cascading a signal with an allpass system does not change the magnitude squared response.
Another choice that works is $X(e^{j\omega}) = \frac{1}{2}(1 - 2e^{-j\omega})e^{-j\omega}$ for which we get

$$x[n] = \frac{1}{2}\delta[n-1] - \delta[n-2]$$

The idea behind this choice was to flip the zero to its reciprocal location outside the unit circle. This has the same magnitude squared response up to a scaling factor; hence, the $\frac{1}{2}$ term.

11.4. Take the DTFT of $x_r[n]$ to get

$$x_r[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-2]$$

$$X_r(e^{j\omega}) = \frac{1}{2} - \frac{1}{2}\cos 2\omega.$$

where $X_r(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{j\omega})]$ is the *conjugate symmetric* part of $X(e^{j\omega})$. Since $X(e^{j\omega}) = 0$ for $-\pi \le \omega < 0$ we have

$$\begin{aligned} X(e^{j\omega}) &= \begin{cases} 2X_r(e^{j\omega}), & 0 \le \omega < \pi \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - \cos 2\omega, & 0 \le \omega < \pi \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$\mathcal{R}e\{X(e^{j\omega})\} = \begin{cases} 1 - \cos 2\omega, & 0 \le \omega < \pi \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathcal{I}m\{X(e^{j\omega})\} = 0.$$

About Notation: $X_R(e^{j\omega})$ with a capital R is the real part of $X(e^{j\omega})$. $X_r(e^{j\omega})$ with a small r is the conjugate symmetric part of $X(e^{j\omega})$ which is complex-valued in general.

11.5. The Hilbert transform can be viewed as a filter with frequency response

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0. \end{cases}$$

(a) First, take the transform of $x_r[n]$

$$X_r(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).$$

Now, filter with $H(e^{j\omega})$ and take the inverse transform to get $x_i[n]$

$$\begin{aligned} X_i(e^{j\omega}) &= H(e^{j\omega})X_r(e^{j\omega}) \\ &= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0) \\ x_i[n] &= \sin\omega_0 n \end{aligned}$$

(b) Similarly, $x_i[n] = -\cos\omega_0 n$.

(c) $x_r[n]$ is the ideal low pass filter

$$x_r[n] = \frac{\sin(\omega_c n)}{\pi n} \longleftrightarrow \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

After filtering with the Hilbert tranformer we get

$$X_i(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega \le \omega_c \\ j, & -\omega_c \le \omega \le 0 \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

Taking the inverse transform yields

$$x_i[n] = \frac{1}{2\pi}\int_{-\omega_c}^{0} je^{j\omega n}\,d\omega - \frac{1}{2\pi}\int_{0}^{\omega_c} je^{j\omega n}\,d\omega = \frac{1 - \cos\omega_c n}{\pi n}$$

11.6. Using Euler's identity,

$$\begin{aligned} jX_I(e^{j\omega}) &= j(2\sin\omega - 3\sin 4\omega) \\ &= 2\left(\frac{e^{j\omega} - e^{-j\omega}}{2}\right) - 3\left(\frac{e^{j4\omega} - e^{-j4\omega}}{2}\right) \\ &= -\frac{3}{2}e^{j4\omega} + e^{j\omega} - e^{-j\omega} + \frac{3}{2}e^{-j4\omega} \end{aligned}$$

Since $x_o[n]$ is the inverse transform of $jX_I(e^{j\omega})$ we get

$$x_o[n] = -\frac{3}{2}\delta[n+4] + \delta[n+1] - \delta[n-1] + \frac{3}{2}\delta[n-4]$$

Because $x[n]$ is real and causal we can recover most of x[n], i.e.,

$$\begin{aligned} x[n] &= 2x_o[n]u[n] + x[0]\delta[n] \\ &= x[0]\delta[n] - 2\delta[n-1] + 3\delta[n-4] \end{aligned}$$

The extra information given to us allows us to find $x[0]$,

$$\begin{aligned} 6 &= X(e^{j\omega})\big|_{\omega=0} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega(0)} \\ &= x[0] - 2 + 3 \end{aligned}$$

Plugging this into our equation for $x[n]$ we find

$$x[n] = 5\delta[n] - 2\delta[n-1] + 3\delta[n-4]$$

11.7. (a) Given the imaginary part of $X(e^{j\omega})$, we can take the inverse DTFT to find the odd part of $x[n]$, denoted $x_o[n]$.

$$\begin{aligned} \mathcal{Im}\left\{X(e^{j\omega})\right\} &= \sin\omega + 2\sin 2\omega \\ &= \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega} + \frac{1}{j}e^{j2\omega} - \frac{1}{j}e^{-j2\omega} \\ &= \frac{1}{j}e^{j2\omega} + \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega} - \frac{1}{j}e^{-j2\omega} \end{aligned}$$

$$\begin{aligned} x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\left\{X(e^{j\omega})\right\}\right] \\ &= \mathcal{DFT}^{-1}\left[e^{j2\omega} + \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega} - e^{-j2\omega}\right] \\ &= \delta[n+2] + \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1] - \delta[n-2] \end{aligned}$$

Using the formula $x[n] = 2x_o[n]u[n] + x[0]\delta[n]$, we find

$$x[n] = -\delta[n-1] - 2\delta[n-2] + x[0]\delta[n]$$

Any $x[0]$ will result in a correct solution to this problem. Setting $x[0] = 0$ gives the result

$$x[n] = -\delta[n-1] - 2\delta[n-2]$$

(b) No, the answer to part (a) is not unique, since any choice for $x[0]$ will result in a correct solution.

11.8. Using Euler's identity and the fact that $x_o[n]$ is the inverse transform of $jX_I(e^{j\omega})$ we find

$$\begin{aligned} jX_I(e^{j\omega}) &= 3j\sin 2\omega \\ &= 3\left(\frac{e^{j2\omega}-e^{-j2\omega}}{2}\right) \\ x_o[n] &= \frac{3}{2}(\delta[n+2]-\delta[n-2]) \end{aligned}$$

Because $x[n]$ is real and causal we can recover all of $x[n]$ except at $n=0$,

$$\begin{aligned} x[n] &= 2x_o[n]u[n]+x[0]\delta[n] \\ &= -3\delta[n-2]+x[0]\delta[n] \end{aligned}$$

Therefore,

$$\begin{aligned} x_e[n] &= \frac{x[n]+x[-n]}{2} \\ &= \frac{(-3\delta[n-2]+x[0]\delta[n])+(-3\delta[n+2]+x[0]\delta[n])}{2} \\ &= -\frac{3}{2}\delta[n+2]+x[0]\delta[n]-\frac{3}{2}\delta[n-2] \end{aligned}$$

Using the fact that $X_R(e^{j\omega})$ is the transform of $x_e[n]$ we find

$$\begin{aligned} X_R(e^{j\omega}) &= -\frac{3}{2}e^{j2\omega}+x[0]-\frac{3}{2}e^{-j2\omega} \\ &= x[0]-3\cos 2\omega \end{aligned}$$

Thus, $X_{R2}(e^{j\omega})$ and $X_{R3}(e^{j\omega})$ are possible if $x[0]=-1$ and $x[0]=0$ respectively.

11.9. (a) Given the imaginary part of $X(e^{j\omega})$, we can take the inverse DTFT to find the odd part of $x[n]$, denoted $x_o[n]$.

$$\begin{aligned} \mathcal{Im}\{X(e^{j\omega})\} &= 3\sin\omega+\sin 3\omega \\ &= \frac{3}{2j}e^{j\omega}-\frac{3}{2j}e^{-j\omega}+\frac{1}{2j}e^{j3\omega}-\frac{1}{2j}e^{-j3\omega} \\ &= \frac{1}{2j}e^{j3\omega}+\frac{3}{2j}e^{j\omega}-\frac{3}{2j}e^{-j\omega}-\frac{1}{2j}e^{-j3\omega} \end{aligned}$$

$$\begin{aligned} x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\{X(e^{j\omega})\}\right] \\ &= \mathcal{DFT}^{-1}\left[\frac{1}{2}e^{j3\omega}+\frac{3}{2}e^{j\omega}-\frac{3}{2}e^{-j\omega}-\frac{1}{2}e^{-j3\omega}\right] \\ &= \frac{1}{2}\delta[n+3]+\frac{3}{2}\delta[n+1]-\frac{3}{2}\delta[n-1]-\frac{1}{2}\delta[n-3] \end{aligned}$$

Using the formula $x[n]=2x_o[n]u[n]+x[0]\delta[n]$, we find

$$x[n]=-3\delta[n-1]-\delta[n-3]+x[0]\delta[n]$$

Taking the DTFT of $x[n]$ gives

$$X(e^{j\omega})=-3e^{-j\omega}-e^{-j3\omega}+x[0]$$

Evaluating this at $\omega = \pi$ gives

$$X(e^{j\omega})\big|_{\omega=\pi} = -3e^{-j\pi} - e^{-j3\pi} + x[0] = 3$$

$$\begin{aligned} 3 + 1 + x[0] &= 3 \\ x[0] &= -1 \end{aligned}$$

Therefore,

$$x[n] = -3\delta[n-1] - \delta[n-3] - \delta[n]$$

(b) Yes, the answer to part (a) is unique. The specification of $X(e^{j\omega})$ at $\omega = \pi$ allowed us to find a unique $x[n]$.

11.10. Factoring the magnitude squared response we get

$$|H(e^{j\omega})|^2 = \frac{\frac{5}{4} - \cos\omega}{5 + 4\cos\omega} = \frac{1 - \cos\omega + \frac{1}{4}}{1 + 4\cos\omega + 4} = \frac{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{2}e^{j\omega})}{(1 + 2e^{-j\omega})(1 + 2e^{j\omega})}$$

$$\begin{aligned} |H(z)|^2 &= \frac{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}{(1 + 2z^{-1})(1 + 2z)} \\ &= H(z)H^*(1/z^*) \end{aligned}$$

Since $h[n]$ is stable and causal and has a stable and causal inverse, it must be a minimum phase system. It therefore has all its poles and zeros inside the unit circle which allows us to uniquely identify $H(z)$ from $|H(z)|^2$.

$$\begin{aligned} H(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 + 2z} \\ &= \frac{1}{2}z^{-1}\left(\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right) \\ &= \frac{1}{2}z^{-1}\left(-1 + \frac{2}{1 + \frac{1}{2}z^{-1}}\right), \qquad |z| > \tfrac{1}{2} \end{aligned}$$

$$h[n] = -\frac{1}{2}\delta[n-1] + \left(-\frac{1}{2}\right)^{(n-1)} u[n-1]$$

11.11. Note that $x_i[n]$ can be written as

$$x_i[n] = -4\delta[n+3] + 4\delta[n-3]$$

Taking the DTFT of $x_i[n]$ gives

$$\begin{aligned} X_i(e^{jw}) &= -4e^{j3\omega} + 4e^{-j3\omega} \\ &= -4(2j\sin 3\omega) \\ &= -8j\sin 3\omega \end{aligned}$$

Since $X(e^{jw}) = 0$ for $-\pi \le \omega < 0$, we can find $X(e^{jw})$ using the relation

$$X(e^{jw}) = \begin{cases} 2jX_i(e^{jw}), & 0 < \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

Thus,

$$X(e^{j\omega}) = \begin{cases} 16\sin 3\omega, & 0 < \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

Therefore, the real part of $X(e^{j\omega})$ is

$$\begin{aligned} X_r(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right] \\ &= \begin{cases} 8\sin 3\omega, & 0 < \omega < \pi \\ -8\sin 3\omega, & -\pi \le \omega < 0 \end{cases} \end{aligned}$$

11.12. (a) Factoring the magnitude squared response we get

$$\begin{aligned} |H(e^{j\omega})|^2 &= \frac{10}{9} - \frac{2}{3}\cos\omega = 1 - \frac{2}{3}\cos\omega + \frac{1}{9} = \left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{j\omega}\right) \\ &= H(e^{j\omega})H^*(e^{j\omega}) \end{aligned}$$

Thus, one choice for $H(e^{j\omega})$ and $h[n]$ is

$$H(e^{j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

$$h[n] = \delta[n] - \frac{1}{3}\delta[n-1]$$

(b) *No.* We can find a new system by taking the zero from the original system and flipping it to its reciprocal location. This only changes the magnitude squared response by a scaling factor. If we compensate for the scaling factor the two magnitude squared responses will be the same. Thus, we find

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{3}(1 - 3e^{-j\omega}) \\ h[n] &= \frac{1}{3}\delta[n] - 3\delta[n-1] \end{aligned}$$

satisifies the given conditions.

11.13. Expressing $X_R(e^{j\omega})$ in terms of complex exponentials gives

$$\begin{aligned} X_R(e^{j\omega}) &= 1 + \cos\omega + \sin\omega - \sin 2\omega \\ &= 1 + \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} + \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega} - \frac{1}{2j}e^{j2\omega} + \frac{1}{2j}e^{-j2\omega} \\ &= -\frac{1}{2j}e^{j2\omega} + \frac{1}{2}e^{j\omega} + \frac{1}{2j}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} - \frac{1}{2j}e^{-j\omega} + \frac{1}{2j}e^{-j2\omega} \end{aligned}$$

Taking the inverse DTFT of $X_R(e^{j\omega})$ gives the conjugate-symmetric part of $x[n]$, denoted as $x_e[n]$.

$$x_e[n] = -\frac{1}{2j}\delta[n+2] + \frac{1}{2}\delta[n+1] + \frac{1}{2j}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1] - \frac{1}{2j}\delta[n-1] + \frac{1}{2j}\delta[n-2]$$

Using the relation $x[n] = 2x_e[n]u[n] - x_e[0]\delta[n]$,

$$x[n] = \delta[n] + \delta[n-1] + j\delta[n-1] - j\delta[n-2]$$

We then find the conjugate-antisymmetric part, $x_o[n]$ as

$$\begin{aligned}
x_o[n] &= \frac{1}{2}\left(x[n] - x^*[-n]\right)\\
&= \frac{1}{2}\left(\delta[n] + \delta[n-1] + j\delta[n-1] - j\delta[n-2] - \delta[n] - \delta[-n-1] + j\delta[-n-1] - j\delta[-n-2]\right)\\
&= \frac{1}{2}\left(\delta[n-1] + j\delta[n-1] - j\delta[n-2] - \delta[n+1] + j\delta[n+1] - j\delta[n+2]\right)\\
&= -\frac{1}{2}\left(\delta[n+1] - \delta[n-1]\right) + \frac{j}{2}\left(\delta[n+1] + \delta[n-1]\right) - \frac{j}{2}\left(\delta[n+2] + \delta[n-2]\right)
\end{aligned}$$

Taking the DTFT of $x_o[n]$ gives $jX_I(e^{j\omega})$.

$$\begin{aligned}
jX_I(e^{j\omega}) &= -\frac{1}{2}\left(e^{j\omega} - e^{-j\omega}\right) + \frac{j}{2}\left(e^{j\omega} + e^{-j\omega}\right) - \frac{j}{2}\left(e^{j2\omega} + e^{-j2\omega}\right)\\
&= -j\sin\omega + j\cos\omega - j\cos 2\omega
\end{aligned}$$

So

$$X_I(e^{j\omega}) = -\sin\omega + \cos\omega - \cos 2\omega$$

11.14. First note that,

(a) The inverse transform of $X_R(e^{j\omega})$ is $x_e[n]$, the even part of $x[n]$. This is true for any sequence whether it is causal, anticausal, or neither.

(b) $jX_I(e^{j\omega})$ is the transform of $x_o[n]$, the odd part of $x[n]$. This is true for any sequence whether it is causal, anticausal, or neither.

(c) For an anticausal sequence

$$x[n] = 2x_e[n]u[-n] - x_e[0]\delta[n]$$

Using Euler's identity and (a),

$$\begin{aligned}
X_R(e^{j\omega}) &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cos(k\omega)\\
&= 1 + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \left(e^{jk\omega} + e^{-jk\omega}\right)\\
x_e[n] &= \delta[n] + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \left(\delta[n+k] + \delta[n-k]\right)
\end{aligned}$$

Using (c) and then taking the odd part we get,

$$\begin{aligned}
x[n] &= 2x_e[n]u[-n] - x_e[0]\delta[n]\\
&= \delta[n] + \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \delta[n+k]\\
x_o[n] &= \frac{x[n] - x[-n]}{2}\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \left(\delta[n+k] - \delta[n-k]\right)
\end{aligned}$$

Now taking the DTFT and using (b),

$$
\begin{aligned}
jX_I(e^{j\omega}) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k (e^{jk\omega} - e^{-jk\omega}) \\
&= j\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega) \\
&= j\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega)
\end{aligned}
$$

Thus,

$$
X_I(e^{j\omega}) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega)
$$

11.15. Given $X_i(e^{j\omega})$, we can take the inverse DTFT of $jX_i(e^{j\omega})$ to find the odd part of $x[n]$, denoted $x_o[n]$.

$$
\begin{aligned}
\mathcal{Im}\left\{X(e^{j\omega})\right\} &= \sin\omega \\
&= \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega}
\end{aligned}
$$

$$
\begin{aligned}
x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\left\{X(e^{j\omega})\right\}\right] \\
&= \mathcal{DFT}^{-1}\left[\frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega}\right] \\
&= \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1]
\end{aligned}
$$

Using the formula $x[n] = 2x_o[n]u[n] + x[0]\delta[n]$,

$$
x[n] = -\delta[n-1] + x[0]\delta[n]
$$

Since

$$
\sum_{n=-\infty}^{\infty} x[n] = 3
$$

$$
\begin{aligned}
-1 + x[0] &= 3 \\
x[0] &= 4
\end{aligned}
$$

Therefore,

$$
x[n] = 4\delta[n] - \delta[n-1]
$$

11.16. Using Euler's identity and the fact that $x_e[n]$ is the inverse transform of $X_R(e^{j\omega})$ we have

$$
\begin{aligned}
X_R(e^{j\omega}) &= 2 - 4\cos(3\omega) \\
&= 2 - 2(e^{j\omega} + e^{-j\omega})
\end{aligned}
$$

$$
x_e[n] = -2\delta[n+3] + 2\delta[n] - 2\delta[n-3]
$$

Since $x[n]$ is real and causal, it is fully determined by its even part $x_e[n]$,

$$
\begin{aligned}
x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\
&= 4\delta[n] - 4\delta[n-3] - 2\delta[n] \\
&= 2\delta[n] - 4\delta[n-3]
\end{aligned}
$$

Using this information in the second condition we find

$$\begin{aligned} X(e^{j\omega})|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} x[n]e^{j\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\ &= 2+4 \\ &\neq 7 \end{aligned}$$

Thus, there is no real, causal sequence that satisfies both conditions.

11.17. There is more than one way to solve this problem. Two solutions are presented below.

Solution 1: Yes, it is possible to determine $x[n]$ uniquely. Note that $X[k]$, the 2 point DFT of a real signal $x[n]$, is also real, as demonstrated below.

$$\begin{aligned} X[k] &= \sum_{n=0}^{1} x[n]e^{-j2\pi nk/2} \\ X[k] &= \sum_{n=0}^{1} x[n](-1)^{nk} \end{aligned}$$

Thus,

$$\begin{aligned} X[0] &= x[0]+x[1] \\ X[1] &= x[0]-x[1] \end{aligned}$$

Clearly, if $x[n]$ is real, then $X[k]$ is real. Therefore, we can conclude that the imaginary part $X_I[k]$ is zero.

Therefore, the inverse DFT of $X_R[k]$ is $x[n]$, computed below.

$$\begin{aligned} x[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k]e^{j2\pi nk/2} \\ x[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k](-1)^{nk} \end{aligned}$$

$$\begin{aligned} x[0] &= \frac{1}{2}(X_R[0]+X_R[1]) \\ &= -1 \\ x[1] &= \frac{1}{2}(X_R[0]-X_R[1]) \\ &= 3 \end{aligned}$$

Thus,

$$x[n] = -\delta[n] + 3\delta[n-1]$$

Solution 2: Start by making the assumption that $X[k]$ is complex, i.e., $X_I[k]$ is nonzero and $X_R[k] = 2\delta[k] - 4\delta[k-1]$. Then, because $x_{ep}[n]$ is the inverse DFT of $X_R[k]$ we find

$$\begin{aligned} x_{ep}[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k]e^{j2\pi nk/2} \\ &= \frac{1}{2}\sum_{k=0}^{1} X_R[k](-1)^{nk} \end{aligned}$$

and

$$\begin{aligned} x_{ep}[0] &= \frac{1}{2}(X_R[0] + X_R[1]) \\ &= -1 \\ x_{ep}[1] &= \frac{1}{2}(X_R[0] - X_R[1]) \\ &= 3 \end{aligned}$$

$$x_{ep}[n] = -\delta[n] + 3\delta[n-1]$$

Because $x[n]$ is real and causal, we can determine it from $x_{ep}[n]$

$$x[n] = \begin{cases} x_{ep}[n], & n = 0 \\ 2x_{ep}[n], & 0 < n < N/2 \\ x_{ep}[N/2], & n = N/2 \\ 0, & \text{otherwise} \end{cases}$$

With $N = 2$ we have

$$x[n] = -\delta[n] + 3\delta[n-1]$$

If we began by making the assumption that $X[k]$ was real, i.e., $X_I[k] = 0$ and $X[k] = X_R[k] = 2\delta[k] - 4\delta[k-1]$ than by taking the inverse transform we find that

$$x[n] = x_{ep}[n] = -\delta[k] + 3\delta[k-1]$$

This is the same answer we got before. Since there was no ambiguities in our determination of $x[n]$, we conclude that $x[n]$ can be uniquely determined.

The next problem shows that when $N > 2$, we cannot necessarily uniquely determine $x[n]$ from $X_R[k]$ unless we make additional assumptions about $x[n]$ such as periodic causality. When $N > 2$ the two assumptions we used above leads to two different sequences with the same $X_R[k]$.

11.18. **Sequence 1:** For $k = 0, 1, 2$ we have

$$X_R[k] = 9\delta[k] + 6\delta[k-1] + 6\delta[((k+1))_3]$$

and $X_R[k] = 0$ for any other k. Using the DFT properties and taking the inverse DFT we find for $n = 1, 2, 3$

$$\begin{aligned} x_{ep}[n] &= 3 + 2\left(e^{j(2\pi/3)n} + e^{-j(2\pi/3)n}\right) \\ &= 3 + 4\cos(2\pi n/3) \\ &= 7\delta[n] + \delta[n-1] + \delta[n-2] \end{aligned}$$

If we let $x[n] = x_{ep}[n]$ we have the desired sequence.

Sequence 2: If we assume $x[n]$ is periodically causal, we can use the following property to solve for $x[n]$ from $x_{ep}[n]$:

$$x[n] = \begin{cases} x_{ep}[0], & n = 0 \\ 2x_{ep}[n], & 0 < n < \frac{N}{2} \\ 0, & \text{otherwise} \end{cases}$$

Note that this is only true for odd N. For even N, we would also need to handle the $n = N/2$ point as shown in the chapter. We have

$$\begin{aligned} x[n] &= \begin{cases} x_{ep}[0], & n = 0 \\ 2x_{ep}[n], & n = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= 7\delta[n] + 2\delta[n-1] \end{aligned}$$

11.19. Given the real part of $X[k]$, we can take the inverse DFT to find the even periodic part of $x[n]$, denoted $x_{ep}[n]$.

Using the inverse DFT relation,

$$x_{ep}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_R[k]W^{-nk}$$

we find

$$\begin{aligned}
x_{ep}[0] &= \frac{1}{4}(4+1+2+1) = 2 \\
x_{ep}[1] &= \frac{1}{4}(4+j-2-j) = \frac{1}{2} \\
x_{ep}[2] &= \frac{1}{4}(4-1+2-1) = 1 \\
x_{ep}[3] &= \frac{1}{4}(4-j-2+j) = \frac{1}{2}
\end{aligned}$$

Thus,

$$x_{ep}[n] = 2\delta[n] + \frac{1}{2}\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3]$$

Next, we can relate the odd periodic and even periodic parts of $x[n]$ using

$$x_{op}[n] = \begin{cases} x_{ep}[n], & 0 < n < N/2 \\ -x_{ep}[n], & N/2 < n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Performing this operation gives

$$x_{op}[n] = \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n-3]$$

Taking the DFT of $x_{op}[n]$ yields $jX_I[k]$. Using the DFT relation,

$$jX_I[k] = \sum_{n=0}^{N-1} x_{op}[n]W^{nk}$$

we find

$$\begin{aligned}
jX_I[0] &= \left(0+\frac{1}{2}+0-\frac{1}{2}\right) = 0 \\
jX_I[1] &= \left(0-\frac{j}{2}+0-\frac{j}{2}\right) = -j \\
jX_I[2] &= \left(0+\frac{1}{2}+0-\frac{1}{2}\right) = 0 \\
jX_I[3] &= \left(0+\frac{j}{2}+0+\frac{j}{2}\right) = j
\end{aligned}$$

Thus,

$$jX_I[k] = -j\delta[k-1] + j\delta[k-3]$$

11.20. As the following shows, the second condition implies $x[0] = 1$.

$$\begin{aligned}
x[0] &= \left.\frac{1}{6}\sum_{k=0}^{5} X[k]e^{j(2\pi/6)kn}\right|_{n=0} \\
&= \frac{1}{6}\sum_{k=0}^{5} X[k] \\
&= 1
\end{aligned}$$

This condition eliminates all choices except $x_2[n]$ and $x_3[n]$.

The odd periodic parts of $x_2[n]$ and $x_3[n]$ for $n = 0, \ldots, 5$ are

$$\begin{aligned} x_{op_2}[n] &= \frac{x_2[n] - x_2^*[((-n))_6]}{2} \\ &= \frac{1}{3}\left(\delta[n-4] - \delta[((n+4))_6]\right) - \frac{1}{3}\left(\delta[n-5] - \delta[((n+5))_6]\right) \end{aligned}$$

$$\begin{aligned} x_{op_3}[n] &= \frac{x_3[n] - x_3^*[((-n))_6]}{2} \\ &= \frac{1}{3}\left(\delta[n-1] - \delta[((n+1))_6]\right) - \frac{1}{3}\left(\delta[n-2] - \delta[((n+2))_6]\right) \end{aligned}$$

For $n < 0$ or $n > 5$, these sequences are zero. Since the transform of $x_{op}[n]$ is $jX_I[k]$ we find for $k = 0, \ldots, 5$

$$\begin{aligned} jX_{I_2}[k] &= \frac{1}{3}\left(e^{-j(2\pi/6)4k} - e^{j(2\pi/6)4k}\right) - \frac{1}{3}\left(e^{-j(2\pi/6)5k} - e^{j(2\pi/6)5k}\right) \\ &= -\frac{2}{3}j\sin(4\pi k/3) + \frac{2}{3}j\sin(5\pi k/3) \\ &= j\frac{2}{\sqrt{3}}\left(-\delta[k-2] + \delta[k-4]\right) \end{aligned}$$

$$\begin{aligned} jX_{I_3}[k] &= \frac{1}{3}\left(e^{-j(2\pi/6)k} - e^{j(2\pi/6)k}\right) - \frac{1}{3}\left(e^{-j(2\pi/6)2k} - e^{j(2\pi/6)2k}\right) \\ &= -\frac{2}{3}j\sin(\pi k/3) + \frac{2}{3}j\sin(2\pi k/3) \\ &= j\frac{2}{\sqrt{3}}\left(-\delta[k-2] + \delta[k-4]\right) \end{aligned}$$

Thus, both $x_2[n]$ and $x_3[n]$ are consistent with the information given.

11.21. (a) *Method 1:*

We are given

$$\begin{aligned} X_R(\rho e^{j\omega}) &= U(\rho, \omega) \\ &= 1 + \rho^{-1}\alpha\cos\omega \end{aligned}$$

Since $\frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial V}{\partial \omega}$ we have,

$$\begin{aligned} \frac{\partial V}{\partial \omega} &= -\alpha\rho^{-1}\cos\omega \\ V &= -\alpha\rho^{-1}\sin\omega + K(\rho) \end{aligned}$$

Since $\frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{\partial U}{\partial \omega}$ we have,

$$\overbrace{\alpha\rho^{-2}\sin\omega + K'(\rho)}^{\frac{\partial V}{\partial \rho}} = \overbrace{\alpha\rho^{-2}\sin\omega}^{-\frac{1}{\rho}\frac{\partial U}{\partial \omega}}$$

Thus,

$$\begin{aligned} K'(\rho) &= 0 \\ K(\rho) &= C \end{aligned}$$

Since $x[n]$ is real $V(\rho,\omega)$ is an odd function of ω. Hence, $V(\rho,0) = 0$, implying that $C = 0$. Therefore,

$$\begin{aligned}
X(\rho e^{j\omega}) &= U(\rho,\omega) + jV(\rho,\omega) \\
&= 1 + \rho^{-1}\alpha\cos\omega - j\rho^{-1}\alpha\sin\omega \\
&= 1 + \alpha\rho^{-1}(\cos\omega - j\sin\omega) \\
&= 1 + \alpha\rho^{-1}e^{-j\omega} \\
X(z) &= 1 + \alpha z^{-1}
\end{aligned}$$

(b) *Method 2*: Since $X_R(e^{j\omega})$ is the transform of $x_e[n]$ we have

$$\begin{aligned}
X_R(e^{j\omega}) &= 1 + \alpha\cos\omega \\
&= 1 + \frac{\alpha}{2}e^{j\omega} + \frac{\alpha}{2}e^{-j\omega} \\
x_e[n] &= \delta[n] + \frac{\alpha}{2}\delta[n+1] + \frac{\alpha}{2}\delta[n-1]
\end{aligned}$$

Because $x[n]$ is real and causal, we can recover $x_o[n]$ from $x_e[n]$ as follows

$$\begin{aligned}
x_o[n] &= \begin{cases} x_e[n], & n > 0 \\ 0, & n = 0 \\ -x_e[n], & n < 0 \end{cases} \\
&= -\frac{\alpha}{2}\delta[n+1] + \frac{\alpha}{2}\delta[n-1]
\end{aligned}$$

Thus,

$$\begin{aligned}
x[n] &= x_e[n] + x_o[n] \\
&= \delta[n] + \alpha\delta[n-1] \\
X(z) &= 1 + \alpha z^{-1}
\end{aligned}$$

Note that we could have obtained $x[n]$ directly from $x_e[n]$ as follows

$$\begin{aligned}
x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\
&= (2\delta[n] + \alpha\delta[n+1] + \alpha\delta[n-1])u[n] - \delta[n] \\
&= \delta[n] + \alpha\delta[n-1]
\end{aligned}$$

11.22. Taking the z-transform of $u_N[n]$ we get

$$\begin{aligned}
U_N(z) &= \frac{2}{1-z^{-1}} - \frac{2z^{-N/2}}{1-z^{-1}} - 1 + z^{-N/2} \\
&= \frac{1 - z^{-N/2} + z^{-1} - z^{-1-N/2}}{1-z^{-1}}, \qquad |z| \neq 0
\end{aligned}$$

Sampling this we find

$$\begin{aligned}
\tilde{U}_N[k] &= U_N(e^{2\pi k/N}) \\
&= \frac{1 - (-1)^k + e^{-j2\pi k/N} - e^{-j2\pi k/N}(-1)^k}{1 - e^{-j2\pi k/N}}
\end{aligned}$$

When k is even but $k \neq 0$ we see that $\tilde{U}_N[k] = 0$. For k odd, we get

$$\begin{aligned}
\tilde{U}_N[k] &= \frac{2 + 2e^{-j2\pi k/N}}{1 - e^{-j2\pi k/N}} \\
&= \frac{2e^{-j\pi k/N}(e^{j\pi k/N} + e^{-j\pi k/N})}{e^{-j\pi k/N}(e^{j\pi k/N} - e^{-j\pi k/N})} \\
&= -2j\cot(\pi k/N)
\end{aligned}$$

When $k = 0$ we get $0/0$ which, if the function was continuous, you would use l'Hôpital's rule. In this case the function is discrete so that is not available to us. One route to the answer is to use the definition of the DFS

$$\begin{aligned}
\tilde{U}_N[0] &= \sum_{k=0}^{N} \tilde{u}_N[n] e^{-j\frac{2\pi}{N}kn}\Big|_{k=0} \\
&= \sum_{k=0}^{N} \tilde{u}_N[n] \\
&= N
\end{aligned}$$

Putting it all together gives us the desired answer

$$\tilde{U}_N[k] = \begin{cases} N, & k = 0, \\ -2j\cot(\pi k/N), & k \text{ odd}, \\ 0, & k \text{ even}, k \neq 0 \end{cases}$$

11.23. (a) Because $x_{ep}[n]$ is the inverse DFT of $X_R[k]$ we have for $n = 0, \ldots, N-1$ and $k = 0, \ldots, N-1$

$$\begin{aligned}
X_R[k] &= \frac{X[k] + X^*[k]}{2} \\
x_{ep}[n] &= \frac{x[n] + x^*[((-n))_N]}{2}
\end{aligned}$$

or equivalently, if we periodically extend these sequences with period N

$$\tilde{x}_e[n] = \frac{\tilde{x}[n] + \tilde{x}[-n]}{2}$$

Note that since the signal is real $\tilde{x}^*[-n] = \tilde{x}[-n]$.
The first period of $\tilde{x}[n]$ is zero from $n = M$ to $n = N-1$. If $N = 2(M-1)$ there is no overlap of $\tilde{x}[n]$ and $\tilde{x}[-n]$ except at $n = 0$ and $n = N/2$. We can therefore recover $\tilde{x}[n]$ from $\tilde{x}_e[n]$ with the following:

$$\tilde{x}[n] = \begin{cases} 2\tilde{x}_e[n], & n = 1, \ldots, N/2 - 1 \\ \tilde{x}_e[n], & n = 0, N/2 \\ 0, & n = M, \ldots, N-1 \end{cases}$$

If we tried to make N any smaller, the overlap of $\tilde{x}[n]$ and $\tilde{x}[-n]$ would prevent the recovery of $x[n]$. Consequently, the smallest value of N we can use to recover $X[k]$ from $X_R[k]$ is $N = 2(M-1)$.

(b) If $N = 2(M-1)$,

$$x[n] = x_{ep}[n]u_N[n] = \begin{cases} 2x_{ep}[n] & n = 1, \ldots, N/2 - 1 \\ x_{ep}[n], & n = 0, N/2 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} u_N[n] &= \begin{cases} 2, & n = 1, 2, \ldots, N/2 - 1 \\ 1, & n = 0, N/2 \\ 0, & \text{otherwise} \end{cases} \\ &= 2u[n] - 2u[n - N/2] - \delta[n] + \delta[n - N/2] \end{aligned}$$

Taking the DFT of $x[n]$ we find

$$X[k] = X_R[k] \circledast U_N[k]$$

where

$$\begin{aligned} U_N[k] &= DFT\{2u[n] - 2u[n - N/2] - \delta[n] + \delta[n - N/2]\} \\ &= \frac{1 - (-1)^k + e^{-j2\pi k/N} - e^{-j2\pi k/N}(-1)^k}{1 - e^{-j2\pi k/N}}, \qquad k = 0, \ldots, N-1 \\ U_N[k] &= \begin{cases} N, & k = 0, \\ -2j\cot(\pi k/N), & 0 < k < N-1,\ k \text{ odd} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

11.24. We are given

$$\begin{aligned} H_R(e^{j\omega}) &= H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega}) \\ H_I(e^{j\omega}) &= H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega}) \end{aligned}$$

$$\begin{aligned} h_r[n] &\longleftrightarrow H_r(e^{j\omega}) = H_A(e^{j\omega}) + jH_B(e^{j\omega}) \\ h_i[n] &\longleftrightarrow H_i(e^{j\omega}) = H_C(e^{j\omega}) + jH_D(e^{j\omega}) \end{aligned}$$

where $h_r[n]$, $h_i[n]$, $H_R(e^{j\omega})$, $H_I(e^{j\omega})$, $H_{ER}(e^{j\omega})$, $H_{OR}(e^{j\omega})$, $H_{EI}(e^{j\omega})$, and $H_{OI}(e^{j\omega})$ are real. Begin by breaking $H(e^{j\omega})$ into its real and imaginary parts $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$

$$\begin{aligned} H(e^{j\omega}) &= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\ &= [H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})] \end{aligned}$$

Now solve for the conjugate symmetric and conjugate antisymmetric parts of $H(e^{j\omega})$

$$\begin{aligned} H_r(e^{j\omega}) &= \frac{H(e^{j\omega}) + H^*(e^{-j\omega})}{2} \\ &= \frac{[H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})]}{2} \\ &\quad + \frac{[H_{ER}(e^{j\omega}) - H_{OR}(e^{j\omega})] - j[H_{EI}(e^{j\omega}) - H_{OI}(e^{j\omega})]}{2} \\ &= H_{ER}(e^{j\omega}) + jH_{OI}(e^{j\omega}) \end{aligned}$$

$$\begin{aligned} H_i(e^{j\omega}) &= \frac{H(e^{j\omega}) - H^*(e^{-j\omega})}{2j} \\ &= \frac{[H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})]}{2j} \\ &\quad - \frac{[H_{ER}(e^{j\omega}) - H_{OR}(e^{j\omega})] - j[H_{EI}(e^{j\omega}) - H_{OI}(e^{j\omega})]}{2j} \\ &= H_{EI}(e^{j\omega}) - jH_{OR}(e^{j\omega}) \end{aligned}$$

Thus,

$$\begin{aligned} H_A(e^{j\omega}) &= H_{ER}(e^{j\omega}) \qquad & H_C(e^{j\omega}) &= H_{EI}(e^{j\omega}) \\ H_B(e^{j\omega}) &= H_{OI}(e^{j\omega}) \qquad & H_D(e^{j\omega}) &= -H_{OR}(e^{j\omega}) \end{aligned}$$

11.25. (a) By inspection,

$$H(e^{j\omega}) = j(2H_{\text{lp}}(e^{j(\omega+\frac{\pi}{2})}) - 1)$$

$$H_{\text{lp}}(e^{j\omega}) = \frac{1 - jH(e^{j(\omega-\frac{\pi}{2})})}{2}$$

(b) **Find h[n]:**
Taking the inverse DTFT of $H(e^{j\omega})$ yields

$$\begin{aligned}
h[n] &= j\left[2e^{-j(\pi/2)n}\, h_{\text{lp}}[n] - \delta[n]\right] \\
&= j\left[2\cos(\pi n/2)h_{\text{lp}}[n] - j2\sin(\pi n/2)h_{\text{lp}}[n] - \delta[n]\right] \\
&= 2\sin(\pi n/2)h_{\text{lp}}[n]
\end{aligned}$$

The simplification in the last step used the fact that $h_{\text{lp}}[n] = \frac{\sin(\pi n/2)}{\pi n}$ is zero for even n and equals $1/2$ for $n = 0$.

Find $\mathbf{h_{lp}[n]}$:
Taking the inverse DTFT of $H_{\text{lp}}(e^{j\omega})$ yields

$$\begin{aligned}
h_{\text{lp}}[n] &= \frac{\delta[n] - je^{j(\pi/2)n}\, h[n]}{2} \\
&= \frac{1}{2}\delta[n] - \frac{1}{2}(j)^{n+1}h[n]
\end{aligned}$$

Using the fact that $h[n]$ is zero for $n = 0$ and n even we can reduce this to

$$h_{\text{lp}}[n] = \frac{\sin(\pi n/2)}{2}h[n] + \frac{1}{2}\delta[n]$$

(c) The linear phase causes a delay of $n_d = M/2$ in the responses. If n_d is not an integer, then we interpret $h_{\text{lp}}[n]$ and $h[n]$ as

$$\begin{aligned}
h_{\text{lp}}[n - n_d] &= \frac{\sin(\pi(n - n_d)/2)}{\pi(n - n_d)} \\
h[n - n_d] &= \frac{2}{\pi}\frac{\sin^2(\pi(n - n_d)/2)}{(n - n_d)}
\end{aligned}$$

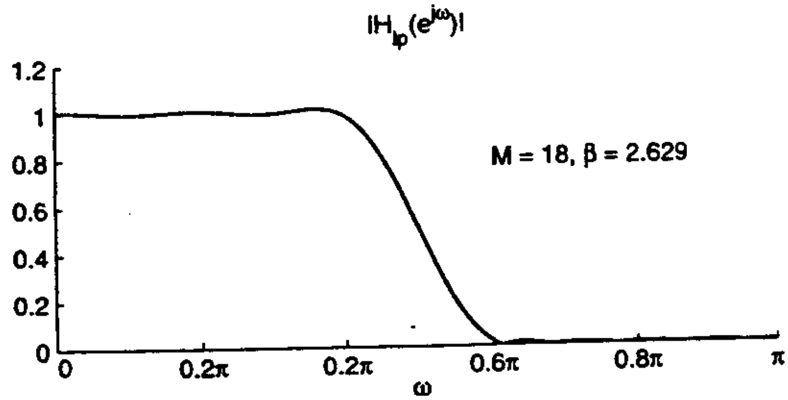
Then,

$$\begin{aligned}
\hat{h}[n] &= h[n - n_d]w[n] \\
&= 2\sin(\pi(n - n_d)/2)\, h_{\text{lp}}[n - n_d]w[n] \\
&= 2\sin(\pi(n - n_d)/2)\, \hat{h}_{\text{lp}}[n]
\end{aligned}$$

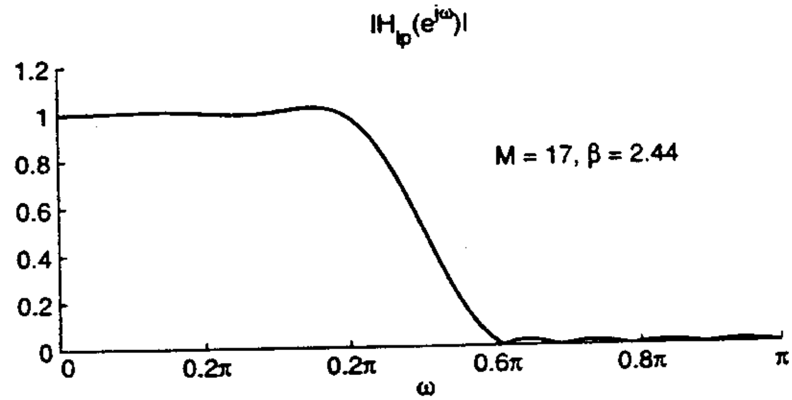
where $\hat{h}[n]$ and $\hat{h}_{\text{lp}}[n]$ are the causal FIR approximations to $h[n]$ and $h_{\text{lp}}[n]$. Similarly,

$$\hat{h}_{\text{lp}}[n] = \begin{cases} \dfrac{\sin(\pi(n - n_d)/2)}{2}\hat{h}[n] + \dfrac{1}{2}\delta[n - n_d]w[n], & M \text{ even} \\ \sin(\pi(n - n_d)/2)\hat{h}[n], & M \text{ odd} \end{cases}$$

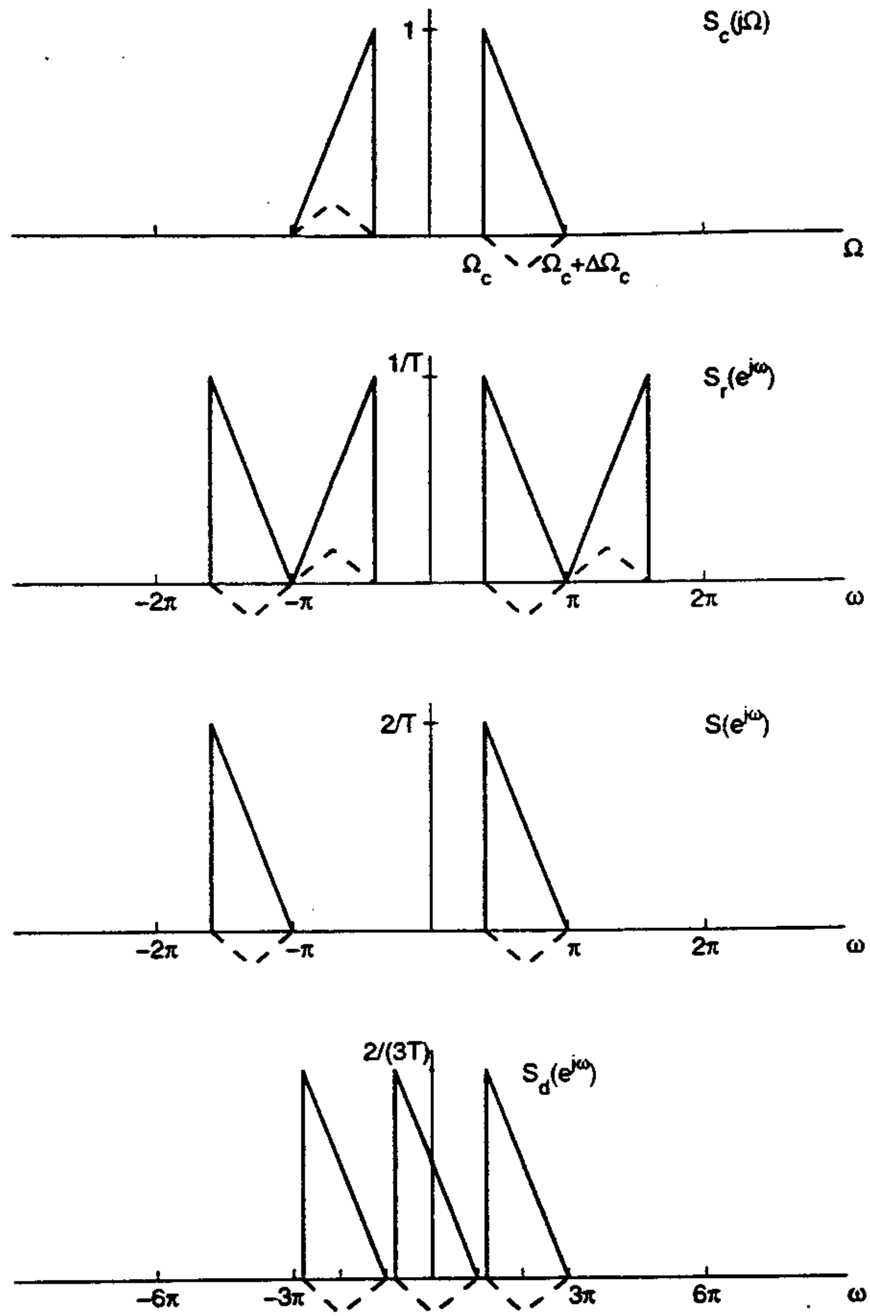
(d) The lowpass filter corresponding to the first filter in the example looks like



The lowpass filter corresponding to the second filter in the example looks like



11.26. (a) The example shown here samples at the Nyquist rate of $T = \pi/(\Omega_c + \Delta\Omega)$ as in the chapter's example, but the bandpass signal is such that $\Delta\Omega/(\Omega_c + \Delta\Omega) = 3/5$. Then, $2\pi/(\Delta\Omega T) = 10/3$.



(b) If $2\pi/(\Delta\Omega T) = M + e$, where M is an integer and e some fraction, then using the Nyquist rate of $2\pi/T = 2(\Omega_c + \Delta\Omega)$ will force decimation by M. As just shown, this choice for T causes $S_d(e^{j\omega})$ to have intervals of zero. Instead, choose T such that $2\pi/(\Delta\Omega T)$ is the next highest integer

$$\frac{2\pi}{\Delta\Omega T} = M + 1.$$

Then decimating by $(M + 1)$ produces the desired result.

11.27. *Yes*, it is possible to always uniquely recover the system input from the system output. Although

$Y(e^{j\omega})$ contains roughly half the frequency spectrum as $X(e^{j\omega})$, we can reconstruct $X(e^{j\omega})$ from $Y(e^{j\omega})$. We can accomplish this by recognizing that since $x[n]$ is real, $X(e^{j\omega})$ must be conjugate symmetric.

The output of the system, $y[n]$, has a Fourier transform $Y(e^{j\omega})$ that is the product of $X(e^{j\omega})$ and $H(e^{j\omega})$. Therefore, $Y(e^{j\omega})$ will correspond to

$$Y(e^{j\omega}) = \begin{cases} X(e^{j\omega}), & 0 \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

At first glance, it may seem like $X(e^{j\omega}) = Y(e^{j\omega}) + Y^*(e^{-j\omega})$. This is close to the right answer, but it doesn't take into consideration the fact that $Y(e^{j\omega})$ is non-zero at $\omega = 0$ and $\omega = \pi$. Thus, the solution $X(e^{j\omega}) = Y(e^{j\omega}) + Y^*(e^{-j\omega})$, will be incorrect at $\omega = 0$ and $\omega = \pi$, since $Y(e^{j\omega})$ and $Y^*(e^{j\omega})$ will overlap at these frequencies. It is necessary to pay special attention to these frequencies to get the right answer. Let

$$Z(e^{j\omega}) = \begin{cases} 0, & \omega = 0, \omega = \pi \\ Y(e^{j\omega}), & \text{otherwise} \end{cases}$$

Alternatively, we can express $Z(e^{j\omega})$ with the constants a and b defined as

$$\begin{aligned} a &= Y(e^{j\omega})\big|_{\omega=0} &= \sum_{n=-\infty}^{\infty} y[n] \\ b &= Y(e^{j\omega})\big|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} y[n](-1)^n \end{aligned}$$

$$Z(e^{j\omega}) = Y(e^{j\omega}) - a\delta(\omega) - b\delta(\omega - \pi)$$

We can construct a conjugate symmetric $X(e^{j\omega})$ from $Y(e^{j\omega})$ and $Z(e^{j\omega})$ as

$$X(e^{j\omega}) = Y(e^{j\omega}) + Z^*(e^{-j\omega})$$

In the time domain, this is

$$x[n] = y[n] + z^*[n]$$

Or, since

$$z[n] = y[n] - \frac{a}{2\pi} - \frac{b(-1)^n}{2\pi}$$

$$x[n] = y[n] + y^*[n] - \frac{a}{2\pi} - \frac{b(-1)^n}{2\pi}$$

11.28. Since $H(z)$ corresponds to a real anticausal sequence $h[n]$, $F(z) = H(1/z)$ corresponds to a real, stable, causal sequence $f[n]$. We can apply the equation developed in the book for causal sequences to $F(z)$.

$$F(z) = \frac{1}{2\pi j} \oint_C F_R(v) \left(\frac{z+v}{z-v}\right) \frac{dv}{v}, \quad |z| \ge 1.$$

where $v = e^{j\theta}$ is the integration variable; i.e., the closed contour C is the unit circle of the v-plane. Now find $H(z)$

$$\begin{aligned} H(z) &= F(1/z) \\ &= \frac{1}{2\pi j} \oint_C H_R(v^{-1}) \left(\frac{z^{-1}+v}{z^{-1}-v}\right) \frac{dv}{v}, \quad |z| \le 1 \end{aligned}$$

where $H_R(v) = \mathcal{Re}\{H(e^{j\theta})\}$.

11.29. (a) We have

$$\mathcal{H}\{x[n]\} = x[n] * h[n]$$
$$\mathcal{H}\{\mathcal{H}\{x[n]\}\} = x[n] * h[n] * h[n]$$

We need to show that $h[n]*h[n] = -\delta[n]$. Alternatively, we need to show that $H(e^{j\omega})H(e^{j\omega}) = -1$, which is easily seen from

$$H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

(b) In Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} f[n]g^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})G^*(e^{j\omega})\,d\omega$$

Let $f[n] = \mathcal{H}\{x[n]\}$ and $g^*[n] = x[n]$. Then

$$\sum_{n=-\infty}^{\infty} \mathcal{H}\{x[n]\}x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})\,d\omega$$

where

$$H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

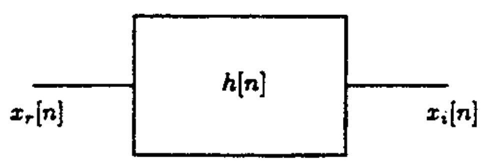
but the integral = 0 since the integrand is an odd function over the symmetric interval.

(c) Since $\mathcal{H}\{x[n]\} = x[n] * h[n]$

$$\begin{aligned} \mathcal{H}\{x[n] * y[n]\} &= (x[n] * y[n]) * h[n] \\ &= (x[n] * h[n]) * y[n] \\ &= x[n] * (y[n] * h[n]) \end{aligned}$$

by the commutativity and associativity of convolution.

11.30.



$h[n]$ is an ideal Hilbert Transformer:

$$h[n] = \begin{cases} \frac{2}{\pi}\frac{\sin^2(\pi n/2)}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases}$$

(a) In the frequency domain,

$$\begin{aligned} \Phi_{x_i x_i}(e^{j\omega}) &= |H(e^{j\omega})|^2 \Phi_{x_r x_r}(e^{j\omega}) \\ &= \Phi_{x_r x_r}(e^{j\omega}) \end{aligned}$$

Therefore, $\phi_{x_i x_i}[m] = \phi_{x_r x_r}[m]$.

(b) The cross-correlation between input and output is just the convolution of $\phi_{x_r x_r}[m]$ and $h[m]$,

$$\phi_{x_r x_i}[m] = \sum_{k=-\infty}^{\infty} h[k]\phi_{x_r x_r}[m-k]$$

The following shows that it is an odd function of m:

$$\begin{aligned}
\phi_{x_r x_i}[-m] &= \sum_{k=-\infty}^{\infty} h[k]\phi_{x_r x_r}[-m-k] \\
&= \sum_{\ell=-\infty}^{\infty} h[-\ell]\phi_{x_r x_r}[-m+\ell] \\
&= \sum_{\ell=-\infty}^{\infty} h[-\ell]\phi_{x_r x_r}[m-\ell] \\
&= -\sum_{\ell=-\infty}^{\infty} h[\ell]\phi_{x_r x_r}[m-\ell] \\
&= -\phi_{x_r x_i}[m]
\end{aligned}$$

since $h[n] = -h[-n]$ and $\phi_{x_r x_r}[m] = \phi_{x_r x_r}[-m]$.

(c) Starting from the definition of the autocorrelation and using the linearity of the expectation operator we get

$$\begin{aligned}
\phi_{xx}[m] &= \mathcal{E}[x[n]x^*[n+m]] \\
&= \mathcal{E}[(x_r[n]+jx_i[n])(x_r[n+m]-jx_i[n+m])] \\
&= \phi_{x_r x_r}[m] + \phi_{x_i x_i}[m] + j(\phi_{x_i x_r}[m] - \phi_{x_r x_i}[m]) \\
&= 2\phi_{x_r x_r}[m] - 2j\phi_{x_r x_i}[m]
\end{aligned}$$

The last line was found using the results from parts (a) and (b).

(d) Taking the transform of both sides of the equality from part (c) we find

$$\begin{aligned}
P_{xx}(\omega) &= 2\Phi_{x_r x_r}(e^{j\omega}) - 2j\Phi_{x_r x_i}(e^{j\omega}) \\
&= 2[\Phi_{x_r x_r}(e^{j\omega}) - jH(e^{j\omega})\Phi_{x_r x_r}(e^{j\omega})] \\
&= 2\Phi_{x_r x_r}(e^{j\omega})[1 - jH(e^{j\omega})]
\end{aligned}$$

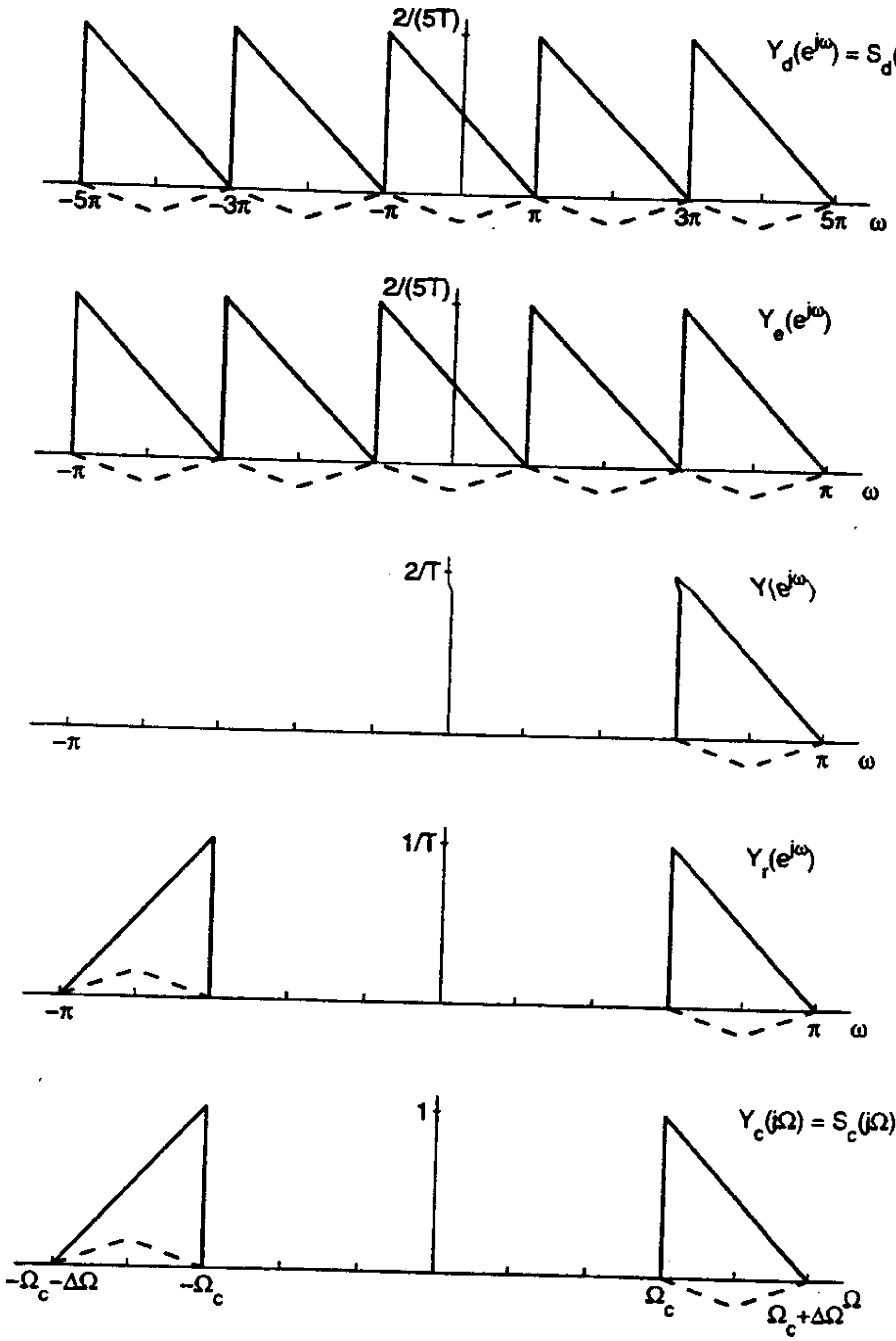
Since

$$1 - jH(e^{j\omega}) = \begin{cases} 0, & 0 < \omega < \pi \\ 2, & -\pi < \omega < 0 \end{cases}$$

we get

$$P_{xx}(\omega) = \begin{cases} 0, & 0 < \omega < \pi \\ 4\Phi_{x_r x_r}(e^{j\omega}), & -\pi < \omega < 0 \end{cases}$$

11.31. (a) As shown in the figure below, the system reconstructs the original bandpass signal. As in the example, $T = \pi/(\Omega_c + \Delta\Omega)$ and $M = 5$.



(b) In the frequency domain

$$H_i(e^{j\omega}) = \begin{cases} 5, & \frac{3\pi}{5} < \omega < \pi \\ 0, & \text{otherwise} \end{cases}$$

Note that $H_i(e^{j\omega}) = 5G(e^{j(\omega-4\pi/5)})$, where

$$G(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \text{otherwise} \end{cases}$$

$$g[n] = \frac{\sin(\pi n/5)}{\pi n}$$

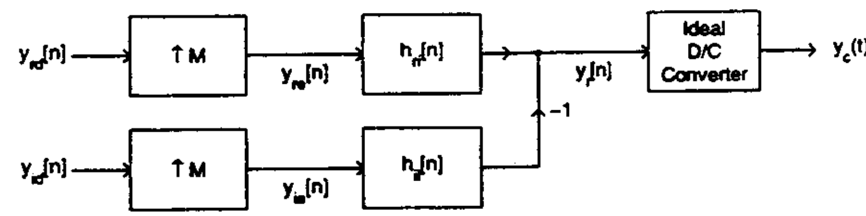
We can therefore write $h_i[n]$ as

$$\begin{aligned} h_i[n] &= 5\frac{\sin(\frac{\pi}{5}n)}{\pi n}e^{j\frac{4\pi}{5}n} \\ &= \underbrace{\frac{5\cos(\frac{4\pi}{5}n)\sin(\frac{\pi}{5}n)}{\pi n}}_{h_{ri}[n]} + j\underbrace{\frac{5\sin(\frac{4\pi}{5}n)\sin(\frac{\pi}{5}n)}{\pi n}}_{h_{ii}[n]} \end{aligned}$$

(c) Using the information from part (b) we find

$$\begin{aligned} y[n] &= y_e[n] * h_i[n] \\ &= (y_{re}[n] + jy_{ie}[n]) * (h_{ri}[n] + jh_{ii}[n]) \\ &= \underbrace{(y_{re}[n] * h_{ri}[n] - y_{ie}[n] * h_{ii}[n])}_{y_r[n]} + j(y_{ie}[n] * h_{ri}[n] + y_{re}[n] * h_{ii}[n]) \end{aligned}$$

We can now redraw the figure using only real operations:



(d) From comparing the top and bottom figures in the answer to part (a), it is evident that the desired complex system response is given by:

$$H(e^{j\omega}) = \begin{cases} 1, & -\pi < \omega < 0 \\ 0, & 0 \le \omega \le \pi \end{cases}$$

11.32. (a) We know

$$\hat{X}(z) = \log[X(z)]$$

When $X(z)$ has a zero or a pole, the term $\log[X(z)]$ goes to negative infinity or infinity respectively. Therefore, $\hat{X}(z)$ has a pole at these locations.

If $\hat{x}[n]$ is causal, $\hat{X}(z)$ has a region of convergence that is the outside of a circle corresponding to its largest pole. However, we require the region of convergence to include the unit circle, i.e., $\hat{X}(e^{j\omega})$ is defined. These two conditions imply that the poles of $\hat{X}(z)$ must be inside the unit circle.

But the pole locations for $\hat{X}(z)$ correspond to the pole and zero locations for $X(z)$. We conclude the poles and zeros of $X(z)$ must be inside the unit circle, i.e., $x[n]$ must be minimum phase.

(b) This argument is similar to the last, but in reverse. Start with the fact that $x[n]$ is minimum phase and must have its poles and zeros inside the unit circle. Then, as shown in part (a), the poles of $\hat{X}(z)$ must also be inside the unit circle. Because the region of convergence must include the unit circle, we know it lies outside the circle defined by its largest pole. Thus, $\hat{x}[n]$ must be causal.

(c)

$$\hat{X}(z) = \log\left[A\frac{\prod_{k=1}^{M_i}(1 - a_k z^{-1})\prod_{k=1}^{M_o}(1 - b_k z)}{\prod_{k=1}^{N_i}(1 - c_k z^{-1})\prod_{k=1}^{N_o}(1 - d_k z)}\right]$$

$$
\begin{aligned}
&= \log(A) + \sum_{k=1}^{M_i} \log(1 - a_k z^{-1}) + \sum_{k=1}^{M_o} \log(1 - b_k z) \\
&\quad - \sum_{k=1}^{N_i} \log(1 - c_k z^{-1}) - \sum_{k=1}^{N_o} \log(1 - d_k z)
\end{aligned}
$$

(d) Using the power series expansion

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

we find

$$
\begin{aligned}
\log(1 - \alpha z^{-1}) &= -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}, \qquad |z| > |\alpha| \\
\log(1 - \beta z) &= -\sum_{n=1}^{\infty} \frac{\beta^n}{n} z^n, \qquad |z| > |\beta^{-1}| \\
&= \sum_{n=-\infty}^{-1} \frac{\beta^{-n}}{n} z^{-n}, \qquad |z| > |\beta^{-1}|
\end{aligned}
$$

From the equations above we can identify the following z-transform pairs

$$
\begin{aligned}
-\frac{\alpha^n}{n} u[n-1] &\longleftrightarrow \log(1 - \alpha z^{-1}), \qquad |z| > |\alpha| \\
\frac{\beta^{-n}}{n} u[-n-1] &\longleftrightarrow \log(1 - \beta z), \qquad |z| > |\beta^{-1}|
\end{aligned}
$$

We can now take the inverse transform of $\hat{X}(z)$.

$$
\hat{x}[n] = \begin{cases} \log(A), & n = 0 \\ -\sum_{k=1}^{M_i} \frac{a_k^n}{n} + \sum_{k=1}^{N_i} \frac{c_k^n}{n}, & n > 0 \\ \sum_{k=1}^{M_o} \frac{b_k^{-n}}{n} - \sum_{k=1}^{N_o} \frac{d_k^{-n}}{n}, & n < 0 \end{cases}
$$

(e) From the results of part (d), we see if $\hat{x}[n]$ is causal, all the b_k and d_k terms must be zero. But the expression for $X(z)$ shows these terms correspond to the zeros and poles outside the unit circle. We conclude that all the zeros and poles of $X(z)$ are inside the unit circle, i.e., $x[n]$ is a minimum phase sequence.