EEM 401 Digital Signal Processing

The Inverse z-Transform

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The Inverse Z-Transform

- Formal inverse z-transform is based on a Cauchy integral

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

- Less formal ways sufficient most of the time
  - Inspection method
  - Partial fraction expansion
  - Power series expansion
- Inspection Method
  - Make use of known z-transform pairs such as

$$a^n u[n] \xleftrightarrow{\;z\;} \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

  - Example: The inverse z-transform of

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \qquad |z| > \frac{1}{2} \qquad \rightarrow \qquad x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse Z-Transform by Partial Fraction Expansion

- Assume that a given z-transform can be expressed as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Apply partial fractional expansion

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - d_i z^{-1}\right)^m}$$

- First term exist only if M>N
  - B_r is obtained by long division
- Second term represents all first order poles
- Third term represents an order s pole
  - There will be a similar term for every high-order pole
- Each term can be inverse transformed by inspection

Partial Fractional Expression

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - d_i z^{-1}\right)^m}$$

- Coefficients are given as

$$A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k}$$

$$C_m = \frac{1}{(s-m)!\,(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[\left(1 - d_i w\right)^s X\left(w^{-1}\right) \right] \right\}_{w=d_i^{-1}}$$

- Easier to understand with examples

Example: 2nd Order Z-Transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \qquad \text{ROC}: |z| > \frac{1}{2}$$

- Order of nominator is smaller than denominator (in terms of z^{-1})
- No higher order pole

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

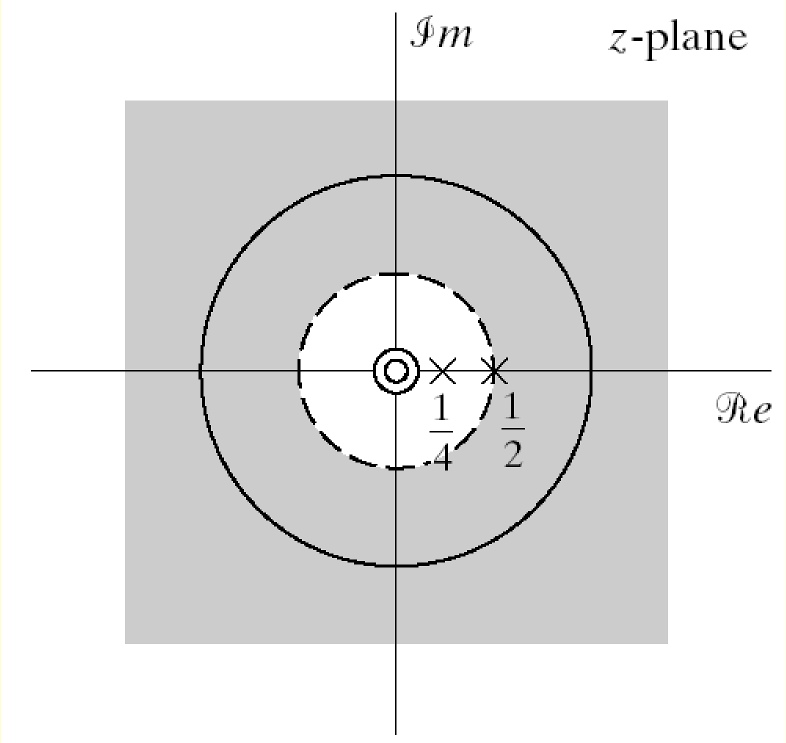
$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}} = \frac{1}{\left(1 - \frac{1}{2}\left(\frac{1}{4}\right)^{-1}\right)} = -1$$

$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = \frac{1}{\left(1 - \frac{1}{4}\left(\frac{1}{2}\right)^{-1}\right)} = 2$$

Example Continued

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

- ROC extends to infinity
  - Indicates right sided sequence



$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Example #2

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} \qquad |z| > 1$$

- Long division to obtain B_o

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \overline{\big) z^{-2} + 2z^{-1} + 1} \\ \underline{z^{-2} - 3z^{-1} + 2} \\ 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$X(z) = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

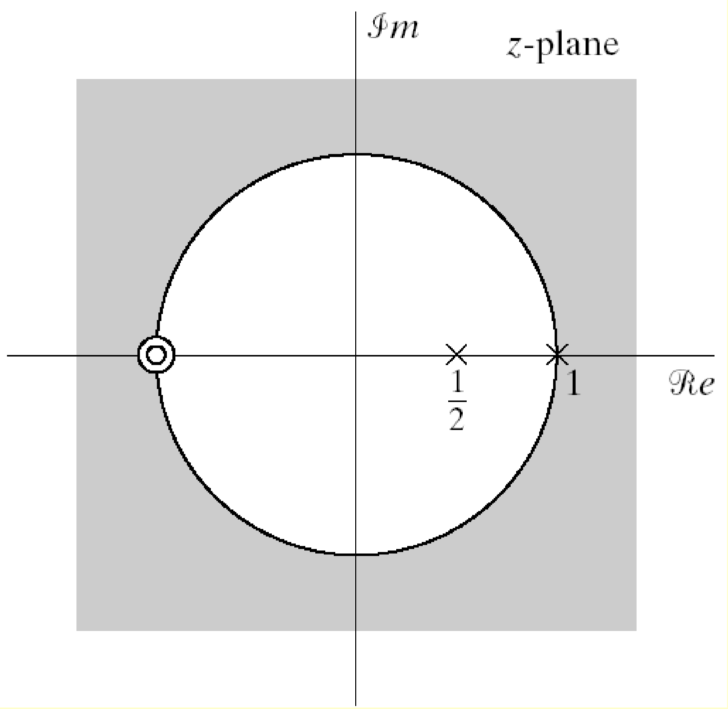
$$A_1 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = -9 \qquad A_2 = \left(1 - z^{-1}\right)X(z)\Big|_{z=1} = 8$$

Example #2 Continued

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \qquad |z| > 1$$

- ROC extends to infinity
  - Indicates right-sides sequence

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8\mathrm{u}[\mathrm{n}]$$



Example # 3

$$X(z) = \frac{z}{(z-1)(z-2)^2} \qquad |z| > 2$$

$$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{dz^k}\left[(z-p_i)^r \frac{X(z)}{z}\right]\Bigg|_{z=p_i}$$

$$c_k = (z-p_k)\frac{X(z)}{z}\Bigg|_{z=p_k}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)^2} = \frac{c_1}{z-1} + \frac{\lambda_1}{z-2} + \frac{\lambda_2}{(z-2)^2}$$

$$c_1 = \frac{1}{(z-2)^2}\Bigg|_{z=1} = 1 \qquad \lambda_2 = \frac{1}{z-1}\Bigg|_{z=2} = 1$$

Example #3 Continued

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} + \frac{\lambda_1}{z-2} + \frac{1}{(z-2)^2}$$

$z = 0$

$$-\frac{1}{4} = -1 - \frac{\lambda_1}{2} + \frac{1}{4} \longrightarrow \lambda_1 = -1$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2} \qquad |z| > 2$$

$$x[n] = (1 - 2^n + n2^{n-1})u[n]$$

Inverse Z-Transform by Power Series Expansion

- The z-transform is power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- In expanded form

$$X(z) = \cdots + x[-2]\,z^{2} + x[-1]\,z^{1} + x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + \cdots$$

- Z-transforms of this form can generally be inversed easily
- Especially useful for finite-length series
- Example

$$X(z) = z^{2}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$$= z^{2} - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & n = 2 \end{cases}$$

Z-Transform Properties: Linearity

- Notation

$$x[n] \xleftrightarrow{Z} X(z) \qquad \text{ROC} = R_x$$

- Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z) \qquad \text{ROC} = R_{x_1} \cap R_{x_2}$$

  - Note that the ROC of combined sequence may be larger than either ROC
  - This would happen if some pole/zero cancellation occurs

Z-Transform Properties: Time Shifting

$$x[n-n_o] \xleftrightarrow{Z} z^{-n_o}X(z) \qquad \text{ROC} = R_x$$

- Here n_o is an integer
  - If positive the sequence is shifted right
  - If negative the sequence is shifted left
- The ROC can change the new term may
  - Add or remove poles at z=0 or z=∞
- Example

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

Z-Transform Properties: Multiplication by Exponential

$$z_o^n x[n] \xleftrightarrow{\;Z\;} X(z/z_o) \qquad \text{ROC} = |z_o| R_x$$

- ROC is scaled by $|z_o|$
- All pole/zero locations are scaled
- If z_o is a positive real number: z-plane shrinks or expands
- If z_o is a complex number with unit magnitude it rotates
- Example: We know the z-transform pair

$$u[n] \xleftrightarrow{\;Z\;} \frac{1}{1 - z^{-1}} \qquad \text{ROC}: |z| > 1$$

- Let's find the z-transform of

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad |z| > r$$

Z-Transform Properties

Özellik	Dizi	Dönüşüm	Yakınsama Bölgesi
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Doğrusallık	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Zaman Öteleme	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\lvert z\rvert<\infty\}$
z_0^n ile çarpma	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \lvert z_0\rvert R$
$e^{j\Omega_0 n}$ ile çarpma	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Zamanda geri dönüş	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
n ile çarpma	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Birikim	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\lvert z\rvert > 1\}$
Konvolüsyon	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$