Discrete-Time Systems - I

Time-Domain Representation

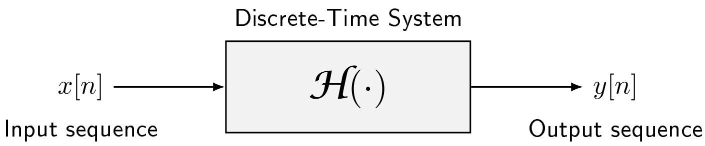
CHAPTER 4

These lecture slides are based on "Digital Signal Processing: A Computer-Based Approach, 4th ed." textbook by S.K. Mitra and its instructor materials. U.Sezen

Contents

Introduction

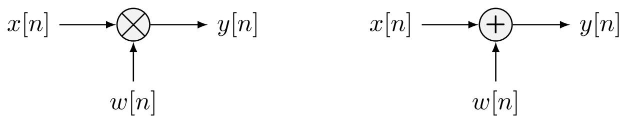
- A discrete-time system processes a given **input sequence** $x[n]$ to generates an **output sequence** $y[n]$ with more desirable properties
- In most applications, the discrete-time system is a **single-input**, **single-output system**



- Mathematically, the discrete-time system is characterized by an operator $\mathcal{H}(\cdot)$ that transforms the input sequence $x[n]$ into another sequence $y[n]$ at the output
- The discrete-time system may also have more than one input and/or more than one output

Examples

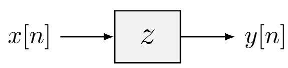
- 2-input, 1-output discrete-time systems:



Modulator Adder

- 1-input, 1-output discrete-time systems:

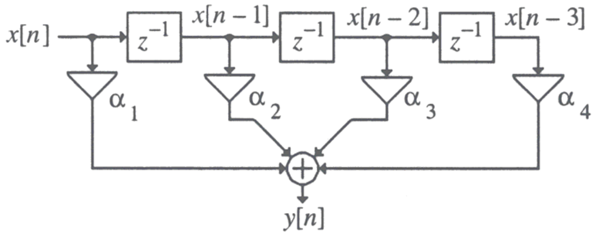
Multiplier Unit delay



Unit advance

Examples

- A more complex example of an one-input, one-output discrete-time system is shown below



- **Accumulator:**

$$
\begin{aligned}
y[n] &= \sum_{\ell=-\infty}^{n} x[\ell] \\
&= \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n] \\
&= y[n-1] + x[n]
\end{aligned}
$$

- The output $y[n]$ at time instant n is the sum of the input sample $x[n]$ at time instant n and the previous output $y[n-1]$ at time instant $n-1$ which is the sum of all previous input sample values from $-\infty$ to $n-1$

- The system cumulatively adds, i.e., it accumulates all input sample values

- Input-output relation can also be written in the form

$$
\begin{aligned}
y[n] &= \sum_{\ell=-\infty}^{-1} x[\ell] + \sum_{\ell=0}^{n} x[\ell] \\
&= y[-1] + \sum_{\ell=0}^{n} x[\ell], \quad n \geq 0
\end{aligned}
$$

- The second form is used for a causal input sequence, in which case $y[-1]$ is called the **initial condition**

- **M-point Moving-Average System:**

$$
y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]
$$

- Used in smoothing random variations in data
- In most applications, the data $x[n]$ is a bounded sequence, so
- M-point average $y[n]$ is also a bounded sequence
- If there is no bias in the measurements, an improved estimate of the noisy data is obtained by simply increasing M
- A direct implementation of the M-point moving average system requires $M-1$ additions, 1 division, and storage of $M-1$ past input data samples

- A more efficient implementation is developed next

$$y[n] = \frac{1}{M}\left(\sum_{k=1}^{M} x[n-k] + x[n] - x[n-M]\right)$$
$$= \frac{1}{M}\left(\sum_{k=0}^{M-1} x[n-1-k] + x[n] - x[n-M]\right)$$
$$= \frac{1}{M}\left(\sum_{k=0}^{M-1} x[n-1-k]\right) + \frac{1}{M}(x[n] - x[n-M])$$

- Hence

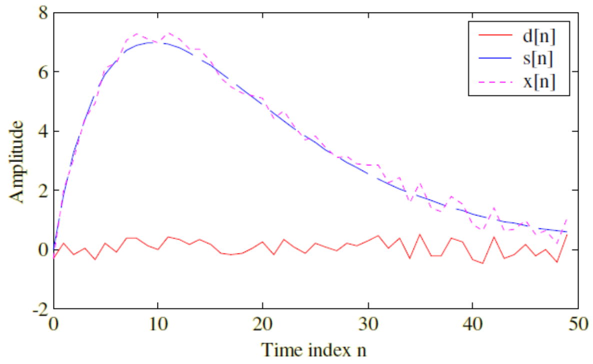
$$y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$$

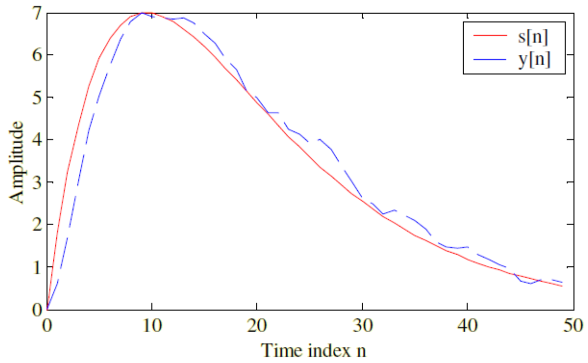
- Computation of the modified M-**point moving average system** using the recursive equation now requires 2 additions and 1 division
- **An application:** Consider

$$x[n] = s[n] + d[n]$$

where $s[n]$ is the signal corrupted by a noise $d[n]$

- **Example:** $s[n] = 2[n(0.9)^n]$ and $d[n]$ is a random signal





- **Exponentially Weighted Running Average Filter:**

$$y[n] = \alpha y[n-1] + x[n], \quad 0 < \alpha < 1$$

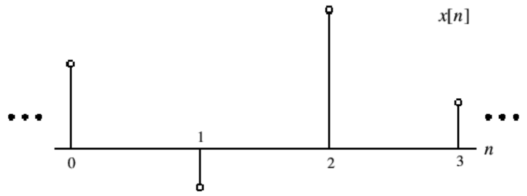
- Computation of the running average requires only 1 addition, 1 multiplication and storage of the previous running average

  Does not require storage of past input data samples

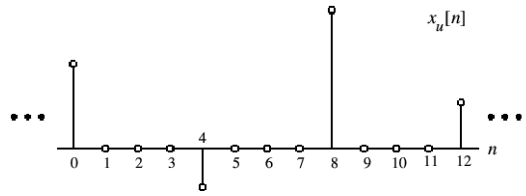
- For $0 < \alpha < 1$, the exponentially weighted average filter places more emphasis on current data samples and less emphasis on past data samples as illustrated below

$$\begin{aligned} y[n] &= \alpha(\alpha y[n-2] + x[n-1]) + x[n] \\ &= \alpha^2 y[n-2] + \alpha x[n-1] + x[n] \\ &= \alpha^2(\alpha y[n-3] + x[n-2] + \alpha x[n-1]) + x[n] \\ &= \alpha^3 y[n-3] + \alpha^2 x[n-2] + \alpha x[n-1] + x[n] \end{aligned}$$

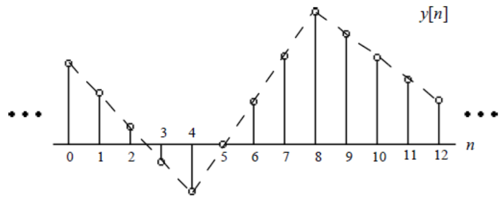
- **Linear interpolation:** Employed to estimate sample values between pairs of adjacent sample values of a discrete-time sequence
- Factor-of-4 interpolation



(a) $x[n]$



(b) $x_u[n]$



(c) $y[n]$

- Factor-of-2 linear interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

- Factor-of-3 linear interpolator

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) + \frac{2}{3}(x_u[n-2] + x_u[n+1])$$

- Factor-of-2 linear 2-D interpolator

Original (512×512)

Downsampled (256×256)

Interpolated (512×512)

- **Median Filter:** The **median** of a set of $(2K+1)$ numbers is the number such that K numbers from the set have values greater than this number and the other K numbers have values smaller
- Median can be determined by rank-ordering the numbers in the set by their values and choosing the number at the middle
- **Example:** Consider the set of numbers

$$\{2, -3, 10, 5, 1\}$$

Rank-ordered set is given by

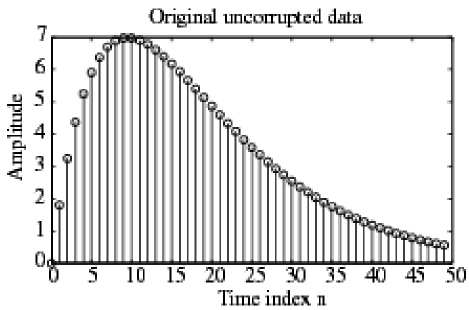
$$\{-3, -1, 2, 5, 10\}$$

Hence,

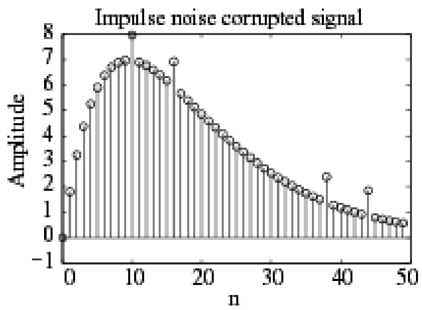
$$\text{med}\,\{2, -3, 10, 5, 1\} = 2$$

- Implemented by sliding a window of odd length over the input sequence $\{x[n]\}$ one sample at a time
- Output $y[n]$ at instant n is the median value of the samples inside the window centered at n
- Finds applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal
- Usually used for the smoothing of signals corrupted by impulse noise

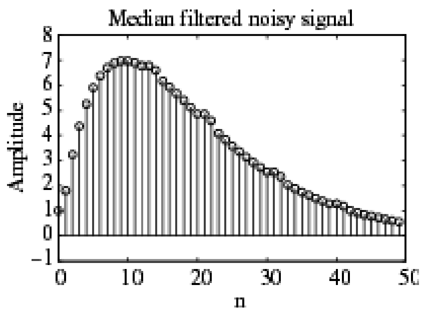
- **Example:**



(a) Original data



(b) Impulse noise corrupted data



(c) Median filtered noisy data

Classification

- Linear System
- Shift-Invariant System
- Causal System
- Stable System
- Passive and Lossless Systems

Linear System

- **Definition:** If $y_1[n]$ is the output due to one input $x_1[n]$ and $y_2[n]$ is the output due to another input $x_2[n]$ then for an input

  $$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

  the output is given by

  $$y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$$

- Above property must hold for any arbitrary constants α and β and for all possible inputs $x_1[n]$ and $x_2[n]$

- **Example:** Consider two accumulators with

$$y_1[n] = \sum_{\ell=-\infty}^{n} x_1[\ell] \quad \text{and} \quad y_2[n] = \sum_{\ell=-\infty}^{n} x_2[\ell]$$

For an input

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

the output is

$$\begin{aligned} y[n] &= \sum_{\ell=-\infty}^{n} (\alpha\, x_1[\ell] + \beta\, x_2[\ell]) \\ &= \alpha \sum_{\ell=-\infty}^{n} x_1[\ell] + \beta \sum_{\ell=-\infty}^{n} x_2[\ell] \\ &= \alpha\, y_1[n] + \beta\, y_2[n] \end{aligned}$$

- Hence, the above system is **linear**

- **Example:** The median filter described earlier is a **nonlinear** discrete-time system.

- To show this, consider a median filter with a window of length 3

Output $y_1[n]$ of the filter for an input $x_1[n]$,

$$\{x_1[n]\} = \{3, 4, 5\}\,, \quad 0 \leq n \leq 2$$

is

$$\{y_1[n]\} = \{3, 4, 4\}\,, \quad 0 \leq n \leq 2$$

Output $y_2[n]$ of the filter for another input $x_2[n]$,

$$\{x_2[n]\} = \{2, -1, -1\}\,, \quad 0 \leq n \leq 2$$

is

$$\{y_1[n]\} = \{0, -1, -1\}\,, \quad 0 \leq n \leq 2$$

However, the output $y[n]$ for the input, $x[n] = x_1[n] + x_2[n]$,

$$\{x[n]\} = \{5, 3, 4\}\,, \quad 0 \le n \le 2$$

is

$$\{y[n]\} = \{3, 4, 3\}\,, \quad 0 \le n \le 2$$

Note:

$$\{y_1[n] + y_2[n]\} = \{3, 3, 3\} \neq \{y[n]\}$$

- Hence, the median filter is a **nonlinear** discrete-time system

Shift-Invariant System

- For a shift-invariant system, if $y_1[n]$ is the response to an input $x_1[n]$, then the response to an input

$$x[n] = x_1[n - n_0]$$

is simply

$$y[n] = y_1[n - n_0]$$

where n_0 is any positive or negative integer

- The above relation must hold for any arbitrary input and its corresponding output

- In the case of sequences and systems with indices n related to discrete instants of time, the above property is called **time-invariance** property
- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied

Example:

- Consider the upsampler

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

with an input-output relation given by

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- For an input $x_1[n] = x[n - n_0]$ the output $x_{1,u}[n]$ is given by

$$\begin{aligned} x_{1,u}[n] &= \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} x[n/L - n_0], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

However from the definition of the up-sampler

$$x_u[n-n_0] = \begin{cases} x_1[(n-n_0)/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$
$$\neq x_{1,u}[n]$$

- Hence, the upsampler is a **time-varying** system

Linear Time-Invariant (LTI) System

- **Linear Time-Invariant** (**LTI**) system is a system satisfying both the linearity and the time-invariance property
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design
- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades

Causal System

- In a **causal system**, the n_0-th output sample $y[n_0]$ depends only on input samples $x[n]$ for $n \leq n_0$ and does not depend on input samples for $n > n_0$
- Let $y_1[n]$ and $y_2[n]$ be the responses of a causal discrete-time system to the inputs $x_1[n]$ and $x_2[n]$, respectively

  Then
  $$x_1[n] = x_2[n] \quad \text{for } n < N$$
  implies also that
  $$y_1[n] = y_2[n] \quad \text{for } n < N$$
- For a causal system, changes in output samples do not precede changes in the input samples

Examples:

- Examples of causal systems:

$$y[n] = \alpha_1\, x[n] + \alpha_2\, x[n-1] + \alpha_3\, x[n-2] + \alpha_4\, x[n-3]$$
$$y[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + a_1\, y[n-1] + a_2\, y[n-2]$$
$$y[n] = y[n-1] + x[n]$$

- Examples of noncausal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$
$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) + \frac{2}{3}(x_u[n-2] + x_u[n+1])$$

- A noncausal system can be implemented as a causal system by delaying the output by an appropriate number of samples
- For example a causal implementation of the factor-of-2 interpolator is given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

Stable System

- There are various definitions of stability
- We consider here the **bounded-input**, **bounded-output** (**BIBO**) stability, i.e.,

  If $y[n]$ is the response to an input $x[n]$ and if

  $$|x| \leq B_x \quad \text{for all values of } n$$

  then

  $$|y| \leq B_y \quad \text{for all values of } n$$

  where $B_x < \infty$ and $B_y < \infty$

- **Example:** The M-point moving average filter is **BIBO stable**

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

For a bounded input we have

$$\begin{aligned} |y[n]| &= \left| \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \right| \\ &\leq \frac{1}{M} \sum_{k=0}^{M-1} |x[n-k]| \\ &\leq \frac{1}{M} (M\, B_x) \\ &\leq B_x \end{aligned}$$

Passive and Lossless Systems

- A discrete-time system is defined to be **passive** if, for every finite-energy input $x[n]$, the output $y[n]$ has, at most, the same energy, i.e.

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- For a **lossless** system, the above inequality is satisfied with an equal sign for every input

- **Example:** Consider the discrete-time system defined by $y[n] = \alpha\, x[n-N]$ with N a positive integer

  Its output energy is given by

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2$$

  Hence, it is a **passive** system if $|\alpha| \leq 1$ and is a **lossless** system if $|\alpha| = 1$

Impulse and Step Responses

- The response of a discrete-time system to a unit sample sequence $\{\delta[n]\}$ is called the **unit sample response** or simply, the **impulse response**, and is denoted by $\{h[n]\}$
- The response of a discrete-time system to a unit step sequence $\{\mu[n]\}$ is called the **unit step response** or simply, the **step response**, and is denoted by $\{s[n]\}$

- **Example:** The **impulse response** $h[n]$ of the discrete-time **accumulator**

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

is obtained by setting

$$x[n] = \delta[n]$$

resulting in

$$\begin{aligned} h[n] &= \sum_{\ell=-\infty}^{n} \delta[\ell] \\ &= \mu[n] \end{aligned}$$

- **Example:** The **impulse response** $h[n]$ of the factor-of-2 interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

is obtained by setting

$$x_u[n] = \delta[n]$$

and is given by

$$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

The impulse response $\{h[n]\}$ is thus a finite-length sequence of length 3:

$$\{h[n]\} = \{0.5, 1, 0.5\}, \quad -1 \le n \le 1$$

Input-Output Relationship

- A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely characterized by its impulse response
- Thus, knowing the impulse response one can compute the output of the system for any arbitrary input

Example:

- Let $h[n]$ denote the impulse response of a LTI discrete-time system,

  and compute its output $y[n]$ for the input:

$$x[n] = 0.5\,\delta[n+2] + 1.5\,\delta[n-1] - \delta[n-2] + 0.75\,\delta[n-5]$$

  As the system is linear, we can compute its outputs for each member of the input separately and add the individual outputs to determine $y[n]$

  Since the system is time-invariant

input	**output**
$\delta[n+2] \rightarrow$	$h[n+2]$
$\delta[n-1] \rightarrow$	$h[n-1]$
$\delta[n-2] \rightarrow$	$h[n-2]$
$\delta[n-5] \rightarrow$	$h[n-5]$

Likewise, as the system is linear

$$\begin{array}{rcl} \textbf{input} & & \textbf{output} \\ 0.5\,\delta[n+2] & \to & 0.5\,h[n+2] \\ 1.5\,\delta[n-1] & \to & 1.5\,h[n-1] \\ -\delta[n-2] & \to & -h[n-2] \\ 0.75\,\delta[n-5] & \to & 0.75\,h[n-5] \end{array}$$

Hence, because of the linearity property we get

$$y[n] = 0.5\,h[n+2] + 1.5\,h[n-1] - h[n-2] + 0.75\,h[n-5]$$

- Now, any arbitrary input sequence $x[n]$ can be expressed as a linear combination of delayed and advanced unit sample sequences in the form

$$x[n] = x[n] \circledast \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The response of the LTI system to an input $x[k]\delta[n-k]$ will be $x[k]h[n-k]$
- Hence, the response $y[n]$ to the input above is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which can be alternately written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

- The summation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

is thus the **convolution sum** of the sequences $x[n]$ and $h[n]$ and represented compactly as

$$\boxed{y[n] = x[n] \circledast h[n]}$$

Example:

- Consider an LTI discrete-time system with an impulse response $h[n]$ generating an output $y[n]$ for a input $x[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \circledast h[n]$$

Let us determine the output $y_1[n]$ of an LTI discrete-time system with an impulse response $h[n-N_0]$ for the same input $x[n]$

- Now

$$y_1[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-N_0-k] = x[n] \circledast h[n-N_0]$$

Hence,

$$y_1[n] = y[n-N_0]$$

Convolution Sum Properties

- **Commutative property:**

$$x[n] \circledast h[n] = h[n] \circledast x[n]$$

Associative property:

$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

Distributive property:

$$x[n] \circledast (h[n] + y[n]) = x[n] \circledast h[n] + x[n] \circledast y[n]$$

- In practice, if either the input or the impulse response is of finite length, the convolution sum can be used to compute the output sample as it involves a finite sum of products
- If both the input sequence and the impulse response sequence are of finite length, the output sequence is also of finite length
- If both the input sequence and the impulse response sequence are of infinite length, convolution sum cannot be used to compute the output
- For systems characterized by an infinite impulse response sequence, an alternate time-domain description involving a finite sum of products will be considered

Tabular Method of Convolution Sum Computation

- Can be used to convolve two finite-length sequences
- Consider the convolution of $\{g[n]\}, 0 \leq n \leq 3$, with $\{h[n]\}, 0 \leq n \leq 2$, generating the sequence $y[n] = g[n] \circledast h[n]$
- Samples of $\{g[n]\}$ and $\{h[n]\}$ are then multiplied using the conventional multiplication method without any carry operation as shown on the next slide.

n	0	1	2	3	4	5
$g[n]$	$g[0]$	$g[1]$	$g[2]$	$g[3]$		
$h[n]$	$h[0]$	$h[1]$	$h[2]$			
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$		
$+$		$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$g[3]h[1]$	
$+$			$g[0]h[2]$	$g[1]h[2]$	$g[2]h[2]$	$g[3]h[2]$
$y[n]$	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$	$y[5]$

- The samples $y[n]$ generated by the convolution sum are obtained by adding the entries in the column above each sample
- The samples of $\{y[n]\}$ are given by

$$\begin{aligned}
y[0] &= g[0]h[0] \\
y[1] &= g[1]h[0] + g[0]h[1] \\
y[2] &= g[2]h[0] + g[1]h[1] + g[0]h[2] \\
y[3] &= g[3]h[0] + g[2]h[1] + g[1]h[2] \\
y[4] &= g[3]h[1] + g[2]h[2] \\
y[5] &= g[3]h[2]
\end{aligned}$$

- The method can also be applied to convolve any two finite-length two-sided sequences as explained below

Consider two sequences $\{x_1[n]\}$ with $N_1 \leq n \leq N_2$ and $\{x_2[n]\}$ with $N_a \leq n \leq N_b$ and we are asked to compute the convolution $y_1[n] = x_1[n] \circledast x_2[n]$ of these two sequences of size $N_2 - N_1 + N_b - N_a + 1$

1. Create two causal sequences $g[n] = x_1[n + N_1]$ with $0 \leq n \leq N_2 - N_1$ and $h[n] = x_2[n + N_a]$ with $0 \leq n \leq N_b - N_a$, where both have their first elements at $n = 0$
2. Compute the convolution $y[n] = g[n] \circledast h[n]$ using the tabular method explained on the previous slide. Here $\{y[n]\}$ is defined for $0 \leq n \leq N_2 - N_1 + N_b - N_a$.
3. Then, obtain the real convolution $y_1[n]$ as $y_1[n] = y[n - N_1 - N_b]$

Convolution Using MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences
- If

$$a = \begin{bmatrix} -2 & 0 & 1 & -1 & 3 \end{bmatrix}$$
$$n = \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}$$

then `conv(a,b)` yields

$$\begin{bmatrix} -2 & -4 & 1 & 3 & 1 & 5 & 1 & -3 \end{bmatrix}$$

Stability Condition of an LTI Discrete-Time System

- **BIBO Stability Condition:** A discrete-time is BIBO stable **if and only if** the output sequence $\{y[n]\}$ remains bounded for all bounded input sequence $\{x[n]\}$
- An LTI discrete-time system is BIBO stable **if and only if** its impulse response sequence $\{h[n]\}$ is absolutely summable, i.e.

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Proof can be found in the textbook.

Example:

- Consider an LTI discrete-time system with an impulse response

$$h[n] = \alpha^n \mu[n]$$

- For this system

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} |\alpha|^n \mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}$$

- Therefore $\mathcal{S} < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable
- If $|\alpha| = 1$, the system is not BIBO stable

Causality Condition of an LTI Discrete-Time System

- Let $x_1[n]$ and $x_2[n]$ be two input sequences with

$$\begin{aligned} x_1[n] &= x_2[n] \quad \text{for } n \leq n_0 \\ x_1[n] &\neq x_2[n] \quad \text{for } n > n_0 \end{aligned}$$

  then the system is causal if the corresponding outputs $y_1[n]$ and $y_2[n]$ are also given by

$$\begin{aligned} y_1[n] &= y_2[n] \quad \text{for } n \leq n_0 \\ y_1[n] &\neq y_2[n] \quad \text{for } n > n_0 \end{aligned}$$

- An LTI discrete-time system is causal **if and only if** its impulse response $\{h[n]\}$ is a causal sequence.
- Proof can be found in the textbook.

Examples:

- The discrete-time system defined by

$$y[n] = \alpha_1\, x[n] + \alpha_2\, x[n-1] + \alpha_3\, x[n-2] + \alpha_4\, x[n-3]$$

  is a causal system as it has a causal impulse response

$$\{h[n]\} = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix} \right\}, \quad 0 \leq n \leq 3$$

  then the system is causal if the corresponding outputs $y_1[n]$ and $y_2[n]$ are also given by

- The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

  is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

Examples:

- The factor-of-2 interpolator defined by

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

is **noncausal** as it has a noncausal impulse response given by

$$\{h[n]\} = \left\{ \begin{bmatrix} 0.5 & 1 & 0.5 \end{bmatrix} \right\}, \quad -1 \le n \le 1$$

- **Note:** A **noncausal** LTI discrete-time system with a finite-length impulse response can often be realized as a **causal** system by inserting an appropriate amount of delay

  For example, a causal version of the factor-of-2 interpolator is obtained by delaying the input by one sample period

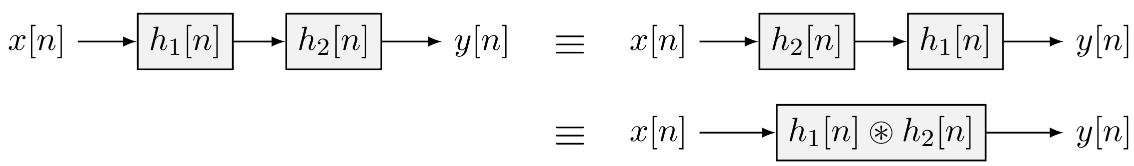
$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

Simple Interconnection Schemes

Two simple interconnection schemes are:

- Cascade Connection
- Parallel Connection

Cascade Connection



- Impulse response $h[n]$ of the cascade of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] \circledast h_2[n]$$

- **Note:** The ordering of the systems in the cascade has no effect on the overall impulse response because of the commutative property of convolution
- A cascade connection of two stable systems is stable
- A cascade connection of two passive (lossless) systems is also passive (lossless)

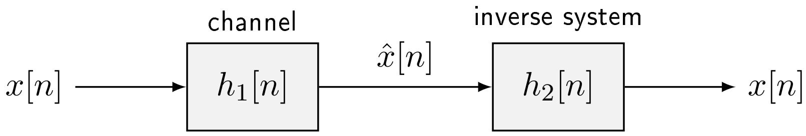
- An application is in the development of an **inverse system** as explained below:

  If the cascade connection satisfies the relation

  $$h_1[n] \circledast h_2[n] = \delta[n]$$

  then the LTI system $h_1[n]$ is said to be the inverse of $h_2[n]$ and vice-versa

- An application of the inverse system concept is in the recovery of a signal $x[n]$ from its distorted version $\hat{x}[n]$ appearing at the output of a transmission channel
- If the impulse response of the channel is known, then $x[n]$ can be recovered by designing an inverse system of the channel



$$h_1[n] \circledast h_2[n] = \delta[n]$$

Example:

- Consider the discrete-time accumulator with an impulse response $\mu[n]$

  Its inverse system satisfy the condition

$$\mu[n] \circledast h_2[n] = \delta[n]$$

- It follows from the above that $h_2[n] = 0$ for $n < 0$ and

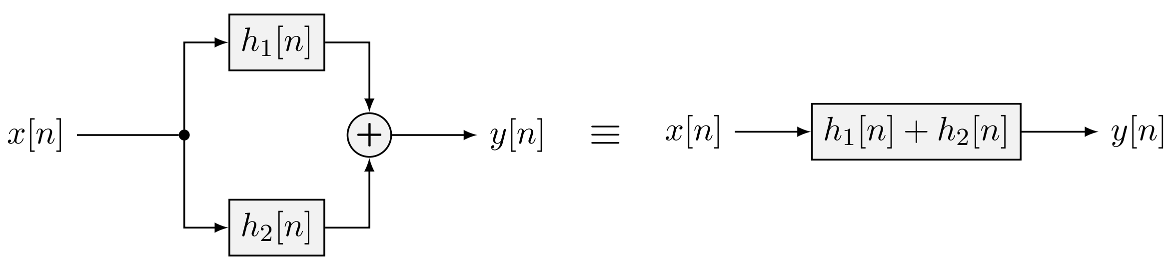
$$h_2[0] = 1$$
$$\sum_{\ell=0}^{n} h_2[\ell] = 0 \quad \text{for } n \geq 1$$

- Thus the impulse response of the inverse system of the discrete-time accumulator is given by

$$h_2[n] = \delta[n] - \delta[n-1]$$

  which is called a **backward difference system**

Parallel Connection

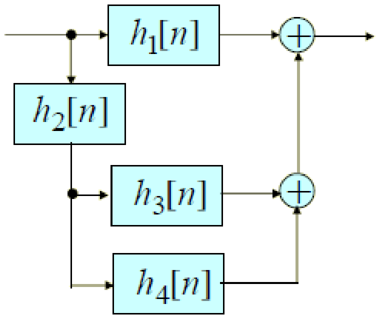


- Impulse response $h[n]$ of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_1[n]$ is given by

$$h[n] = h_1[n] + h_2[n]$$

Example:

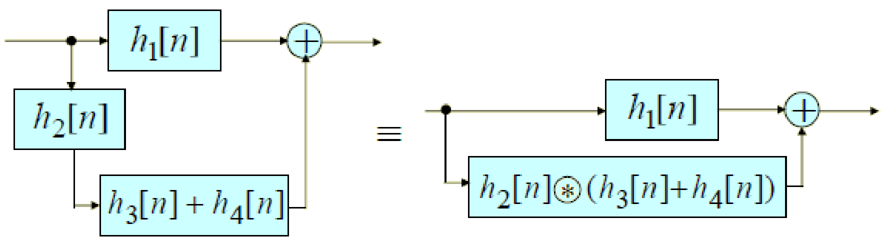
- Consider the discrete-time system shown in the figure below



where

$$
\begin{aligned}
h_1[n] &= \delta[n] + 0.5\,\delta[n-1] \\
h_2[n] &= 0.5\,\delta[n] - 0.25\,\delta[n-1] \\
h_3[n] &= 2\,\delta[n] \\
h_4[n] &= -2(0.5)^n \mu[n]
\end{aligned}
$$

Simplifying the block-diagram we obtain



Overall impulse response $h[n]$ is given by

$$
\begin{aligned}
h[n] &= h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n]) \\
&= h_1[n] + h_2[n] \circledast h_3[n] + h_2[n] \circledast h_4[n]
\end{aligned}
$$

Now,

$$
\begin{aligned}
h_2[n] \circledast h_3[n] &= \left(\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\right) \circledast 2\,\delta[n] \\
&= \delta[n] - \frac{1}{2}\delta[n-1]
\end{aligned}
$$

and

$$
\begin{aligned}
h_2[n] \circledast h_4[n] &= \left(\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\right) \circledast (-2(0.5)^n\mu[n]) \\
&= -(0.5)^n\mu[n] + \frac{1}{2}(0.5)^{n-1}\mu[n-1] \\
&= -(0.5)^n\mu[n] + (0.5)^n\mu[n-1] \\
&= -(0.5)^n(\mu[n] - \mu[n-1]) \\
&= -(0.5)^n\delta[n] \\
&= -\delta[n]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
h[n] &= \delta[n] + \frac{1}{2}\delta[n-1] + \delta[n] - \frac{1}{2}\delta[n-1] - \delta[n] \\
&= \delta[n]
\end{aligned}
$$

Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$
\sum_{k=0}^{N} d_k\, y[n-k] = \sum_{k=0}^{M} p_k\, x[n-k]
$$

  where $x[n]$ and $y[n]$ are, respectively, the input and the output of the system, and, $\{d_k\}$ and $\{p_k\}$ are constants characterizing the system
- The **order** of the system is given by $\max(N, M)$, which is the order of the difference equation
- It is possible to implement an LTI system characterized by a constant coefficient difference equation as here the computation involves two finite sums of products

- If we assume the system to be causal, then the output $y[n]$ can be recursively computed using

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0}\, y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0}\, x[n-k]$$

  where $d_0 \neq 0$
- $y[n]$ can be computed for all $n \geq n_0$, knowing $x[n]$ and the initial conditions

$$y[n_0], y[n_1], \ldots, y[n-N]$$

Finite Impulse Response (FIR) Discrete-Time Systems

Based on Impulse Response Length:

- If the impulse response $h[n]$ is of finite length, i.e.,

$$h[n] = 0 \quad \text{for } n < N_1 \text{ and } n > N_2, N_1 < N_2$$

  then it is known as a **finite impulse response** (**FIR**) discrete-time system
- The convolution sum description here is

$$y[n] = \sum_{k=N_1}^{N_2} h[k]\, x[n-k]$$

- The output $y[n]$ of an FIR LTI discrete-time system can be computed directly from the convolution sum as it is a finite sum of products
- Examples of FIR LTI discrete-time systems are the moving-average system and the linear interpolators

Infinite Impulse Response (IIR) Discrete-Time Systems

- If the impulse response is of infinite length, then it is known as an **infinite impulse response** (**IIR**) discrete-time system
- The class of IIR systems we are concerned with in this course are characterized by linear constant coefficient difference equations

Examples:

- **Example:** The discrete-time accumulator defined by

$$y[n] = y[n-1] + x[n]$$

is seen to be an IIR system

- **Example:** The familiar numerical integration formulas that are used to numerically solve integrals of the form

$$y(t) = \int_0^t x(\tau)\,d\tau$$

can be shown to be characterized by linear constant coefficient difference equations, and hence, are examples of IIR systems

If we divide the interval of integration into n equal parts of length T, then the previous integral can be rewritten as

$$y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)\,d\tau$$

where we have set $t = nT$ and used the notation

$$y(nT) = \int_0^{nT} x(\tau)\,d\tau$$

Using the trapezoidal method we can write

$$\int_{(n-1)T}^{nT} x(\tau)\,d\tau = \frac{T}{2}\left\{x((n-1)T) + x(nT)\right\}$$

Hence, a numerical representation of the definite integral is given by

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

Let $y[n] = y(nT)$ and $x[n] = x(nT)$

Then

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

reduces to

$$y[n] = y[n-1] + \frac{T}{2}\{x[n-1] + x[n]\}$$

which is recognized as the difference equation representation of a first-order IIR discrete-time system

Nonrecursive and Recursive Systems

Based on the Output Calculation Process:

- **Nonrecursive System:** Here the output can be calculated sequentially, knowing only the present and past input samples
- **Recursive System:** Here the output computation involves past output samples in addition to the present and past input samples

Real and Complex Systems

Based on the Coefficients:

- **Real Discrete-Time System:** The impulse response samples are real valued
- **Complex Discrete-Time System:** The impulse response samples are complex valued