Solutions – Chapter 2

Discrete-Time Signals and Systems

2.1. (a) $T(x[n]) = g[n]x[n]$

- Stable: Let $|x[n]| \le M$ then $|T[x[n]| \le |g[n]|M$. So, it is stable if $|g[n]|$ is bounded.
- Causal: $y_1[n] = g[n]x_1[n]$ and $y_2[n] = g[n]x_2[n]$, so if $x_1[n] = x_2[n]$ for all $n < n_0$, then $y_1[n] = y_2[n]$ for all $n < n_0$, and the system is causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= g[n](ax_1[n] + bx_2[n] \\ &= ag[n]x_1[n] + bg[n]x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  So this is linear.
- Not time-invariant:

$$\begin{aligned} T(x[n-n_0]) &= g[n]x[n-n_0] \\ &\ne y[n-n_0] = g[n-n_0]x[n-n_0] \end{aligned}$$

  which is not TI.
- Memoryless: $y[n] = T(x[n])$ depends only on the n^{th} value of x, so it is memoryless.

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

- Not Stable: $|x[n]| \le M \to |T(x[n])| \le \sum_{k=n_0}^{n} |x[k]| \le |n-n_0|M$. As $n \to \infty$, $T \to \infty$, so not stable.
- Not Causal: $T(x[n])$ depends on the future values of $x[n]$ when $n < n_0$, so this is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n_0}^{n} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  The system is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= \sum_{k=n_0}^{n} x[k-n_0] \\ &= \sum_{k=0}^{n-n_0} x[k] \\ &\ne y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k] \end{aligned}$$

  The system is not TI.
- Not Memoryless: Values of $y[n]$ depend on past values for $n > n_0$, so this is not memoryless.

(c) $T(x[n]) \sum_{k=n-n_0}^{n+n_0} x[k]$

- Stable: $|T(x[n])| \le \sum_{k=n-n_0}^{n+n_0} |x[k]| \le \sum_{k=n-n_0}^{n+n_0} x[k]M \le |2n_0+1|M$ for $|x[n]| \le M$, so it is stable.
- Not Causal: $T(x[n])$ depends on future values of $x[n]$, so it is not causal.

- Linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= \sum_{k=n-n_0}^{n+n_0} ax_1[k]+bx_2[k] \\ &= a\sum_{k=n-n_0}^{n+n_0} x_1[k]+b\sum_{k=n-n_0}^{n+n_0} x_2[k] = aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

This is linear.

- TI:

$$\begin{aligned} T(x[n-n_0] &= \sum_{k=n-n_0}^{n+n_0} x[k-n_0] \\ &= \sum_{k=n-n_0}^{n} x[k] \\ &= y[n-n_0] \end{aligned}$$

This is TI.

- Not memoryless: The values of $y[n]$ depend on $2n_0$ other values of x, not memoryless.

(d) $T(x[n]) = x[n-n_0]$

- Stable: $|T(x[n])| = |x[n-n_0]| \leq M$ if $|x[n] \leq M$, so stable.
- Causality: If $n_0 \geq 0$, this is causal, otherwise it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= ax_1[n-n_0]+bx_2[n-n_0] \\ &= aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

This is linear.

- TI: $T(x[n-n_d] = x[n-n_0-n_d] = y[n-n_d]$. This is TI.
- Not memoryless: Unless $n_0 = 0$, this is not memoryless.

(e) $T(x[n]) = e^{x[n]}$

- Stable: $|x[n]| \leq M$, $|T(x[n])| = |e^{x[n]}| \leq e^{|x[n]|} \leq e^M$, this is stable.
- Causal: It doesn't use future values of $x[n]$, so it causal.
- Not linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= e^{ax_1[n]+bx_2[n]} \\ &= e^{ax_1[n]}e^{bx_2[n]} \\ &\neq aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

This is not linear.

- TI: $T(x[n-n_0]) = e^{x[n-n_0]} = y[n-n_0]$, so this is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so it is memoryless.

(f) $T(x[n]) = ax[n]+b$

- Stable: $|T(x[n])| = |ax[n]+b| \leq a|M|+|b|$, which is stable for finite a and b.
- Causal: This doesn't use future values of $x[n]$, so it is causal.
- Not linear:

$$\begin{aligned} T(cx_1[n]+dx_2[n]) &= acx_1[n]+adx_2[n]+b \\ &\neq cT(x_1[n])+dT(x_2[n]) \end{aligned}$$

This is not linear.

- TI: $T(x[n-n_0]) = ax[n-n_0] + b = y[n-n_0]$. It is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of $x[n]$ only, so it is memoryless.

(g) $T(x[n]) = x[-n]$

- Stable: $|T(x[n])| \leq |x[-n]| \leq M$, so it is stable.
- Not causal: For $n < 0$, it depends on the future value of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[-n] + bx_2x[-n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= x[-n-n_0] \\ &\neq y[n-n_0] = x[-n+n_0] \end{aligned}$$

  This is not TI.
- Not memoryless: For $n \neq 0$, it depends on a value of x other than the n^{th} value, so it is not memoryless.

(h) $T(x[n]) = x[n] + u[n+1]$

- Stable: $|T(x[n])| \leq M + 3$ for $n \geq -1$ and $|T(x[n])| \leq M$ for $n < -1$, so it is stable.
- Causal: Since it doesn't use future values of $x[n]$, it is causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] + 3u[n+1] \\ &\neq aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is not linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0] &= x[n-n_0] + 3u[n+1] \\ &= y[n-n_0] \\ &= x[n-n_0] + 3u[n-n_0+1] \end{aligned}$$

  This is not TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so this is memoryless.

2.2. For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(a) Since $h[k] \neq 0$, for $(N_0 \leq n \leq N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input, $x[n] \neq 0$, for $(N_2 \leq n \leq N_3)$, so

$$x[n-k] \neq 0, \text{ for } N_2 \leq (n-k) \leq N_3$$

Note that the minimum value of $(n-k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \leq n \leq (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_o \leq n \leq (n_o + N - 1)$, and $h[n] \neq 0$, for some $n_1 \leq n \leq (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_o + n_1) \leq n \leq (n_o + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2.3. We desire the step response to a system whose impulse response is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The step response results when the input is the unit step:

$$x[n] = u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

For $n \leq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} a^{-k} \\ &= \sum_{k=-n}^{\infty} a^{k} \\ &= \frac{a^{-n}}{1-a} \end{aligned}$$

For $n > 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} a^{-k} \\ &= \sum_{k=0}^{\infty} a^{k} \\ &= \frac{1}{1-a} \end{aligned}$$

2.4. The difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]$$

To solve, we take the Fourier transform of both sides.

$$Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega})e^{-j\omega} + \frac{1}{8}Y(e^{j\omega})e^{-j2\omega} = 2 \cdot X(e^{j\omega})e^{-j\omega}$$

The system function is given by:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{2e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} \end{aligned}$$

The impulse response (for $x[n] = \delta[n]$) is the inverse Fourier transform of $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{-8}{1 + \frac{1}{4}e^{-j\omega}} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}}$$

Thus,

$$h[n] = -8(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n]. \quad = y[n]$$

2.5. (a) The homogeneous difference equation:

$$y[n] - 5y[n-1] + 6y[n-2] = 0$$

Taking the Z-transform,

$$1 - 5z^{-1} + 6z^{-2} = 0$$
$$(1 - 2z^{-1})(1 - 3z^{-1}) = 0.$$

The homogeneous solution is of the form

$$y_h[n] = A_1(2)^n + A_2(3)^n.$$

(b) We take the z-transform of both sides:

$$Y(z)[1 - 5z^{-1} + 6z^{-2}] = 2z^{-1}X(z)$$

Thus, the system function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\ &= \frac{-2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}, \end{aligned}$$

where the region of convergence is outside the outermost pole, because the system is causal. Hence the ROC is $|z| > 3$. Taking the inverse z-transform, the impulse response is

$$h[n] = -2(2)^n u[n] + 2(3)^n u[n].$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

2.6. (a) The difference equation:

$$y[n] - \frac{1}{2} y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2} e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2} e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2} e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2} e^{-j\omega} + \frac{3}{4} e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2} e^{-j\omega} + \frac{3}{4} e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2} e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2} y[n-1] + \frac{3}{4} y[n-2] = x[n] - \frac{1}{2} x[n-1] + x[n-3].$$

2.7. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j(\frac{\pi}{6} n)} = e^{j(\frac{\pi}{6})(n+N)} = e^{j(\frac{\pi}{6} n + 2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{6} N, \text{for integers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j(\frac{3\pi}{4}n)} = e^{j(\frac{3\pi}{4})(n+N)} = e^{j(\frac{3\pi}{4}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{3\pi}{4}N, \text{for integers } k, N$$

$$\Longrightarrow N = \frac{8}{3}k, \text{for integers } k, N$$

The smallest k for which both k and N are integers are is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j(\frac{\pi}{\sqrt{2}}n)} = e^{j(\frac{\pi}{\sqrt{2}})(n+N)} = e^{j(\frac{\pi}{\sqrt{2}}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{\sqrt{2}}N, \text{for integers } k, N$$

$$\Longrightarrow N = 2\sqrt{2}k, \text{for some integers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

2.8. We take the Fourier transform of both $h[n]$ and $x[n]$, and then use the fact that convolution in the time domain is the same as multiplication in the frequency domain.

$$\begin{aligned} H(e^{j\omega}) &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \\ Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}} \\ &= \frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{2}{1-\frac{1}{3}e^{-j\omega}} \\ y[n] &= 2(\frac{1}{3})^n u[n] + 3(-\frac{1}{2})^n u[n] \end{aligned}$$

2.9. (a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{2}{1-\frac{1}{2}e^{-j\omega}} \\ h[n] &= -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

For the step response $s[n]$:

$$\begin{aligned} s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k] \\ &= -2\frac{1-(1/3)^{n+1}}{1-1/3}u[n] + 2\frac{1-(1/2)^{n+1}}{1-1/2}u[n] \\ &= (1+(\frac{1}{3})^n - 2(\frac{1}{2})^n)u[n] \end{aligned}$$

(b) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n]=0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{5}{6}c + \frac{1}{6} = 0$$

So for $c = 1/2$ and $c = 1/3$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n$$

(c) The total solution is the sum of the homogeneous and particular solutions, with the particular solution being the impulse response found in part (a):

$$\begin{aligned} y[n] &= y_h[n] + y_p[n] \\ &= A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

Now we use the constraint $y[0] = y[1] = 1$ to solve for A_1 and A_2:

$$\begin{aligned} y[0] &= A_1 + A_2 - 2 + 2 = 1 \\ y[1] &= A_1/2 + A_2/3 - 2/3 + 1 = 1 \\ A_1 + A_2 &= 1 \\ A_1/2 + A_2/3 &= 2/3 \end{aligned}$$

With $A_1 = 2$ and $A_2 = -1$ solving the simultaneous equations, we find that the impulse response is

$$y[n] = 2(\frac{1}{2})^n - (\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n]$$

2.10. (a)

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k] \\ &= \begin{cases} \sum_{k=-\infty}^{n} a^k, & n \le -1 \\ \sum_{k=-\infty}^{-1} a^k, & n > -1 \end{cases} \\ &= \begin{cases} \frac{a^n}{1-1/a}, & n \le -1 \\ \frac{1/a}{1-1/a}, & n > -1 \end{cases} \end{aligned}$$

(b) First, let us define $v[n] = 2^n u[-n-1]$. Then, from part (a), we know that

$$w[n] = u[n] * v[n] = \begin{cases} 2^{n+1}, & n \leq -1 \\ 1, & n > -1 \end{cases}$$

Now,

$$\begin{aligned} y[n] &= u[n-4] * v[n] \\ &= w[n-4] \\ &= \begin{cases} 2^{n-3}, & n \leq 3 \\ 1, & n > 3 \end{cases} \end{aligned}$$

(c) Given the same definitions for $v[n]$ and $w[n]$ from part(b), we use the fact that $h[n] = 2^{n-1}u[-(n-1)-1] = v[n-1]$ to reduce our work:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * v[n-1] \\ &= w[n-1] \\ &= \begin{cases} 2^n, & n \leq 0 \\ 1, & n > 0 \end{cases} \end{aligned}$$

(d) Again, we use $v[n]$ and $w[n]$ to help us.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (u[n] - u[n-10]) * v[n] \\ &= w[n] - w[n-10] \\ &= (2^{n+1}u[-(n+1)] + u[n]) - (2^{n-9}u[-(n-9)] + u[n-10]) \\ &= \begin{cases} 2^{(n+1)} - 2^{(n-9)}, & n \leq -2 \\ 1 - 2^{(n-9)}, & -1 \leq n \leq 8 \\ 0, & n \geq 9 \end{cases} \end{aligned}$$

2.11. First we re-write $x[n]$ as a sum of complex exponentials:

$$x[n] = \sin(\frac{\pi n}{4}) = \frac{e^{j\pi n/4} - e^{-j\pi n/4}}{2j}.$$

Since complex exponentials are eigenfunctions of LTI systems,

$$y[n] = \frac{H(e^{j\pi/4})e^{j\pi n/4} - H(e^{-j\pi/4})e^{-j\pi n/4}}{2j}$$

Evaluating the frequency response at $\omega = \pm\pi/4$:

$$\begin{aligned} H(e^{j\frac{\pi}{4}}) &= \frac{1 - e^{-j\pi/2}}{1 + 1/2e^{-j\pi}} = 2(1-j) = 2\sqrt{2}e^{-j\pi/4} \\ H(e^{-j\frac{\pi}{4}}) &= \frac{1 - e^{j\pi/2}}{1 + 1/2e^{j\pi}} = 2(1+j) = 2\sqrt{2}e^{j\pi/4} \end{aligned}$$

We get:

$$\begin{aligned} y[n] &= \frac{2\sqrt{2}e^{-j\pi/4}e^{j\pi n/4} - 2\sqrt{2}e^{j\pi/4}e^{-j\pi n/4}}{2j} \\ &= 2\sqrt{2}\sin(\pi n/4 - \pi/4). \end{aligned}$$

2.12. The difference equation:

$$y[n] = ny[n-1] + x[n]$$

Since the system is causal and satisfies initial-rest conditions, we may recursively find the response to any input.

(a) Suppose $x[n] = \delta[n]$:

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 1 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

$$y[n] = h[n] = n!u[n]$$

(b) To determine if the system is linear, consider the input:

$$x[n] = a\delta[n] + b\delta[n]$$

performing the recursion,

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= a + b \\ y[1] &= a + b \\ y[2] &= 2(a + b) \\ y[3] &= 6(a + b) \\ y[4] &= 24(a + b) \end{aligned}$$

Because the output of the superposition of two input signals is equivalent to the superposition of the individual outputs, the system is LINEAR.

(c) To determine if the system is time-invariant, consider the input:

$$x[n] = \delta[n-1]$$

the recursion yields

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 0 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

Using $h[n]$ from part (a),

$$h[n-1] = (n-1)!u[n-1] \neq y[n]|_{x[n]=\delta[n-1]}$$

Conclude: NOT TIME INVARIANT.

2.13. Eigenfunctions of LTI systems are of the form α^n, so functions (a), (b), and (e) are eigenfunctions. Notice that part (d), $\cos(\omega_0 n) = .5(e^{j\omega_0 n} + e^{-j\omega_0 n})$ is a sum of two α^n functions, and is therefore not an eigenfunction itself.

2.14. (a) The information given shows that the system satisfies the eigenfunction property of exponential sequences for LTI systems for one particular eigenfunction input. However, we do not know the system response for any other eigenfunction. Hence, we can say that the system may be LTI, but we cannot uniquely determine it. $\Longrightarrow$ (iv).

(b) If the system were LTI, the output should be in the form of $A(1/2)^n$, since $(1/2)^n$ would have been an eigenfunction of the system. Since this is not true, the system cannot be LTI. $\Longrightarrow$ (i).

(c) Given the information, the system *may* be LTI, but does not have to be. For example, for any input other than the given one, the system may output 0, making this system non-LTI. $\Longrightarrow$ (iii). If it were LTI, its system function can be found by using the DTFT:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \\ h[n] &= (\frac{1}{2})^n u[n] \end{aligned}$$

2.15. (a) No. Consider the following input/outputs:

$$x_1[n] = \delta[n] \quad \Longrightarrow \quad y_1[n] = (\frac{1}{4})^n u[n]$$

$$x_2[n] = \delta[n-1] \quad \Longrightarrow \quad y_2[n] = (\frac{1}{4})^{n-1} u[n]$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_1[n-1] = (\frac{1}{4})^{n-1}u[n-1]$

(b) No. Consider the input/output pair $x_2[n]$ and $y_2[n]$ above. $x_2[n] = 0$ for $n < 1$, but $y_2[0] \neq 0$.

(c) Yes. Since $h[n]$ is stable and multiplication with $u[n]$ will not cause any sequences to become unbounded, the entire system is stable.

2.16. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{1}{4}c + \frac{1}{8} = 0$$

So for $c = 1/2$ and $c = -1/4$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(-\frac{1}{4})^n$$

(b) Taking the z-transform of both sides, we find that

$$Y(z)(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}) = 3X(z)$$

and therefore

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{3}{1 - 1/4z^{-1} - 1/8z^{-2}} \\ &= \frac{3}{(1 + 1/4z^{-1})(1 - 1/2z^{-1})} \\ &= \frac{1}{1 + 1/4z^{-1}} + \frac{2}{1 - 1/2z^{-1}} \end{aligned}$$

The causal impulse response corresponds to assuming that the region of convergence extends outside the outermost pole, making

$$h_c[n] = ((-1/4)^n + 2(1/2)^n)u[n]$$

The anti-causal impulse response corresponds to assuming that the region of convergence is inside the innermost pole, making

$$h_{ac}[n] = -((-1/4)^n + 2(1/2)^n)u[-n-1]$$

(c) $h_c[n]$ is absolutely summable, while $h_{ac}[n]$ grows without bounds.

(d)

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{1/3}{1+1/4z^{-1}} + \frac{2}{1-1/2z^{-1}} + \frac{2/3}{1-1/2z^{-1}} \\ y[n] &= \frac{1}{3}(\frac{1}{4})^n u[n] + 4(n+1)(\frac{1}{2})^{n+1}u[n+1] + \frac{2}{3}(\frac{1}{2})^n u[n] \end{aligned}$$

2.17. (a) We have

$$r[n] = \begin{cases} 1, & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Taking the Fourier transform

$$\begin{aligned} R(e^{j\omega}) &= \sum_{n=0}^{M} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{e^{j\frac{M+1}{2}\omega} - e^{-j\frac{M+1}{2}\omega}}{e^{j\omega} - e^{-j\omega}}\right) \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{\sin(\frac{M+1}{2}\omega)}{\sin(\omega/2)}\right). \end{aligned}$$

(b) We have

$$w[n] = \begin{cases} \frac{1}{2}(1+\cos(\frac{2\pi n}{M}), & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

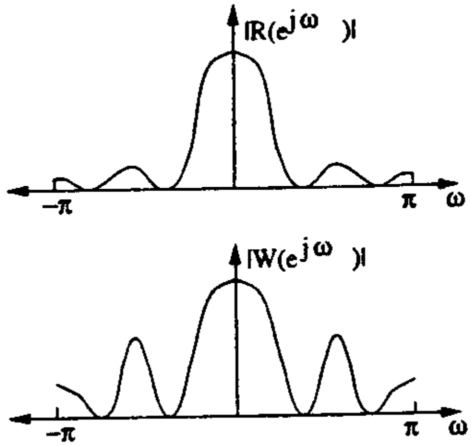
We note that,

$$w[n] = r[n] \cdot \frac{1}{2}[1 + \cos(\frac{2\pi n}{M})].$$

Thus,

$$\begin{aligned} W(e^{j\omega}) &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1+\cos(\frac{2\pi n}{M}))e^{-j\omega n} \\ &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1+\frac{1}{2}e^{j\frac{2\pi n}{M}} + \frac{1}{2}e^{-j\frac{2\pi n}{M}})e^{-j\omega} \\ &= R(e^{j\omega}) * (\frac{1}{2}\delta(\omega) + \frac{1}{4}\delta(\omega + \frac{2\pi}{M}) + \frac{1}{4}\delta(\omega - \frac{2\pi}{M})) \end{aligned}$$

(c)



2.18. $h[n]$ is causal if $h[n] = 0$ for $n < 0$. Hence, (a) and (b) are causal, while (c), (d), and (e) are not.

2.19. $h[n]$ is stable if it is absolutely summable.

(a) Not stable because $h[n]$ goes to ∞ as n goes to ∞.

(b) Stable, because $h[n]$ is non-zero only for $0 \leq n \leq 9$.

(c) Stable.

$$\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} (1/3)^n = 1/2 < \infty$$

(d) Not stable. Notice that

$$\sum_{n=0}^{5} |\sin(\pi n/3)| = 2\sqrt{3}$$

and summing $|h[n]|$ over all positive n therefore grows to ∞.

(e) Stable. Notice that $|h[n]|$ is upperbounded by $(3/4)^{|n|}$, which is absolutely summable.

(f) Stable.

$$h[n] = \begin{cases} 2, & , -5 \leq n \leq -1 \\ 1, & , 0 \leq n \leq 4 \\ 0, & , \text{otherwise} \end{cases}$$

So $\sum |h[n]| = 15$.

2.20. (a) Taking the difference equation $y[n] = (1/a)y[n-1] + x[n-1]$ and assuming $h[0] = 0$ for $n < 0$:

$$\begin{aligned} h[0] &= 0 \\ h[1] &= 1 \\ h[2] &= 1/a \\ h[3] &= (1/a)^2 \\ \vdots & \quad \vdots \\ h[n] &= (1/a)^{n-1}u[n-1] \end{aligned}$$

(b) $h[n]$ is absolutely summable if $|1/a| < 1$ or if $|a| > 1$

2.21. For an arbitrary linear system, we have

$$y[n] = T\{x[n]\},$$

Let $x[n] = 0$ for all n.

$$y[n] = T\{x[n]\}$$

For some arbitrary $x_1[n]$, we have

$$y_1[n] = T\{x_1[n]\}$$

Using the linearity of the system:

$$\begin{aligned} T\{x[n] + x_1[n]\} &= T\{x[n]\} + T\{x_1[n]\} \\ &= y[n] + y_1[n] \end{aligned}$$

Since $x[n]$ is zero for all n,

$$T\{x[n] + x_1[n]\} = T\{x_1[n]\} = y_1[n]$$

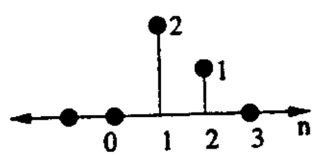
Hence, $y[n]$ must also be zero for all n.

2.22. We use the graphical approach to compute the convolution:

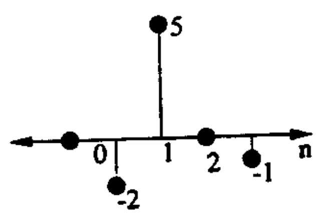
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

(a) $y[n] = x[n] * h[n]$

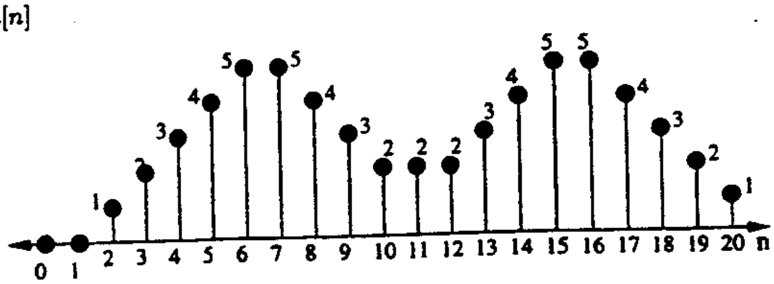
$$y[n] = \delta[n-1] * h[n] = h[n-1]$$



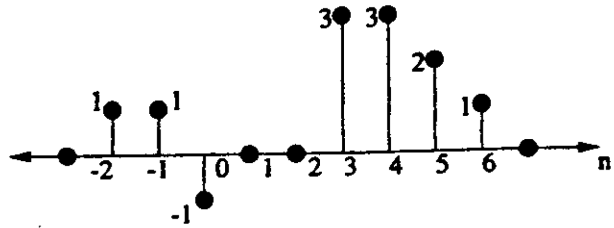
(b) $y[n] = x[n] * h[n]$



(c) $y[n] = x[n] * h[n]$



(d) $y[n] = x[n] * h[n]$



2.23. The ideal delay system:

$$y[n] = T\{x[n]\} = x[n - n_o]$$

Using the definition of linearity:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= ax_1[n - n_o] + bx_2[n - n_o] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

So, the ideal delay system is LINEAR.

The moving average system:

$$y[n] = Tx[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]$$

by linearity:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} (ax_1[n] + bx_2[n]) \\ &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} ax_1[n] + \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} bx_2[n] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Conclude, the moving average is LINEAR.

2.24. The response of the system to a delayed step:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n - k] \\ &= \sum_{k=-\infty}^{\infty} u[k - 4]h[n - k] \end{aligned}$$

$$y[n] = \sum_{k=4}^{\infty} h[n - k]$$

Evaluating the above summation:

For $n < 4$: $y[n] = 0$
For $n = 4$: $y[n] = h[0] = 1$
For $n = 5$: $y[n] = h[1] + h[0] = 2$
For $n = 6$: $y[n] = h[2] + h[1] + h[0] = 3$
For $n = 7$: $y[n] = h[3] + h[2] + h[1] + h[0] = 4$
For $n = 8$: $y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2$
For $n \geq 9$: $y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0$

2.25. The output is obtained from the convolution sum:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

The convolution may be broken into five regions over the range of n:

$$y[n] = 0, \text{for } n < 0$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} a^k \\ &= \frac{1-a^{(n+1)}}{1-a}, \text{for } 0 \le n \le N_1 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k \\ &= \frac{1-a^{(N_1+1)}}{1-a}, \text{ for } N_1 < n < N_2 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{n} a^{(k-N_2)} \\ &= \frac{1-a^{(N_1+1)}}{1-a} + \frac{1-a^{(n+1)}}{1-a} \\ &= \frac{2-a^{(N_1+1)}-a^{(n+1)}}{1-a}, \text{ for } N_2 \le n \le (N_1+N_2) \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{N_1+N_2} a^{(k-N_2)} \\ &= \sum_{k=0}^{N_1} a^k + \sum_{m=0} N_1 a^m \\ &= 2\sum_{k=0}^{N_1} a^k \\ &= 2 \cdot \left(\frac{1-a^{(N_1+1)}}{1-a}\right), \text{ for } n > (N_1+N_2) \end{aligned}$$

2.26. Recall that an eigenfunction of a system is an input signal which appears at the output of the syste[m] scaled by a complex constant.

(a) $x[n] = 5^n u[n]$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)}u[n-k] \\ &= 5^n \sum_{k=-\infty}^{n} h[k]5^{-k} \end{aligned}$$

Becuase the summation depends on n, $x[n]$ is NOT AN EIGENFUNCTION.

(b) $x[n] = e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\ &= e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\ &= e^{j2\omega n} \cdot H(e^{j2\omega}) \end{aligned}$$

YES, EIGENFUNCTION.

(c) $e^{j\omega n} + e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} + \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} + e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\ &= e^{j\omega n} \cdot H(e^{j\omega}) + e^{j2\omega n} \cdot H(e^{j2\omega}) \end{aligned}$$

Since the input cannot be extracted from the above expression, the sum of complex exponentials is NOT AN EIGENFUNCTION. (Although, separately the inputs are eigenfunctions. In general, complex exponential signals are always eigenfunctions of LTI systems.)

(d) $x[n] = 5^n$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)} \\ &= 5^n \sum_{k=-\infty}^{\infty} h[k]5^{-k} \end{aligned}$$

YES, EIGENFUNCTION.

(e) $x[n] = 5^n e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)}e^{j2\omega(n-k)} \\ &= 5^n e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]5^{-k}e^{-j2\omega k} \end{aligned}$$

YES, EIGENFUNCTION.

2.27. • System A:

$$x[n] = (\frac{1}{2})^n$$

This input is an eigenfunction of an LTI system. That is, if the system is linear, the output will be a replica of the input, scaled by a complex constant.
Since $y[n] = (\frac{1}{4})^n$, System A is NOT LTI.

• System B:

$$x[n] = e^{jn/8}u[n]$$

The Fourier transform of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} e^{jn/8}u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-j(\omega-\frac{1}{8})n} \\ &= \frac{1}{1-e^{-j(\omega-\frac{1}{8})}}. \end{aligned}$$

The output is $y[n] = 2x[n]$, thus

$$Y(e^{j\omega}) = \frac{2}{1-e^{-j(\omega-\frac{1}{8})}}.$$

Therefore, the frequency response of the system is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= 2. \end{aligned}$$

Hence, the system is a linear amplifier. We conclude that System B is LTI, and unique.

• System C: Since $x[n] = e^{jn/8}$ is an eigenfunction of an LTI system, we would expect the output to be given by

$$y[n] = \gamma e^{jn/8},$$

where γ is some complex constant, if System C were indeed LTI. The given output, $y[n] = 2e^{jn/8}$, indicates that this is so.
Hence, System C is LTI. However, it is not unique, since the only constraint is that

$$H(e^{j\omega})|_{\omega=1/8} = 2.$$

2.28. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 5:

$$e^{j(\frac{2\pi}{5}n)} = e^{j(\frac{2\pi}{5})(n+N)} = e^{j(\frac{2\pi}{5}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{2\pi}{5}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 5$ shows that $x[n]$ has period 5.

(b) $x[n]$ is periodic with period 38. Since the sin function has period of 2π:

$$x[n+38] = \sin(\pi(n+38)/19) = \sin(\pi n/19 + 2\pi) = x[n]$$

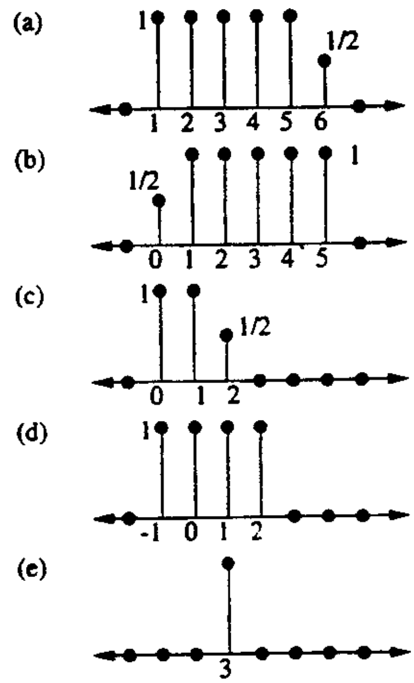
(c) This is not periodic because the linear term n is not periodic.

(d) This is again not periodic. $e^{j\omega}$ is periodic over period 2π, so we have to find k, N such that

$$x[n+N] = e^{j(n+N)} = e^{j(n+2\pi k)}$$

Since we can make k and N integers at the same time, $x[n]$ is not periodic.

2.29.



2.30. (a) Since $\cos(\pi n)$ only takes on values of +1 or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.
- It is not causal. For example, $Tx[4] = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1[n^2] + bx_2[n^2]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2 - 1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.

To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

- This is not stable. For example, $T(u[n]) = \infty$ for all n ¿ 1.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a \sum_{k=n-1}^{\infty} x_1[k] + b \sum_{k=n-1}^{\infty} x_2[k]$$

- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0]$$

2.31. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 + \frac{1}{15}c - \frac{2}{5} = 0$$

So for $c = 1/3$ and $c = -2/5$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{3})^n + A_2(-\frac{2}{5})^n$$

(b) We use the z-transform, and use different ROCs to generate the causal and anti-causal impulses responses:

$$H(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} = \frac{5/11}{1-\frac{1}{3}z^{-1}} + \frac{6/11}{1+\frac{2}{5}z^{-1}}$$

$$h_c[n] = \frac{5}{11}(\frac{1}{3})^n u[n] + \frac{6}{11}(-\frac{2}{5})^n u[n]$$

$$h_{ac}[n] = -\frac{5}{11}(\frac{1}{3})^n u[-n-1] - \frac{6}{11}(-\frac{2}{5})^n u[-n-1]$$

(c) Since $h_c[n]$ is causal, and the two exponential bases in $h_c[n]$ are both less than 1, it is absolutely summable. $h_{ac}[n]$ grows without bounds as n approaches $-\infty$.

(d)

$$\begin{aligned}
Y(z) &= X(z)H(z) \\
&= \frac{1}{1-\frac{3}{5}z^{-1}} \cdot \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} \\
&= \frac{-25/44}{1-1/3z^{-1}} + \frac{55/12}{1+2/5z^{-1}} + \frac{27/20}{1-3/5z^{-1}} \\
y[n] &= \frac{-25}{44}(\frac{1}{3})^n u[n] + \frac{55}{12}(-\frac{2}{5})^n u[n] + \frac{27}{20}(\frac{3}{5})^n u[n]
\end{aligned}$$

2.32. We first re-write the system function $H(e^{j\omega})$:

$$\begin{aligned} H(e^{j\omega}) &= e^{j\pi/4} \cdot e^{-j\omega} \left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}} \right) \\ &= e^{j\pi/4} G(e^{j\omega}) \end{aligned}$$

Let $y_1[n] = x[n] * g[n]$, then

$$\begin{aligned} x[n] &= \cos(\frac{\pi n}{2}) = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2} \\ y_1[n] &= \frac{G(e^{j\pi/2})e^{j\pi n/2} + G(e^{-j\pi/2})e^{-j\pi n/2}}{2} \end{aligned}$$

Evaluating the frequency response at $\omega = \pm\pi/2$:

$$\begin{aligned} G(e^{j\frac{\pi}{2}}) &= e^{-j\frac{\pi}{2}} \left(\frac{1 + e^{-j\pi} + 4e^{-j2\pi}}{1 + \frac{1}{2}e^{-j\pi}} \right) = 8e^{-j\pi/2} \\ G(e^{-j\frac{\pi}{2}}) &= 8e^{j\pi/2} \end{aligned}$$

Therefore,

$$y_1[n] = (8e^{j(\pi n/2 - \pi/2)} + 8e^{j(-\pi n/2 + \pi/2)})/2 = 8\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

and

$$y[n] = e^{j\pi/4} y_1[n] = 8e^{j\pi/4}\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

2.33. Since $H(e^{-j\omega}) = H^*(e^{j\omega})$, we can apply the results of Example 2.13 from the text,

$$y[n] = |H(e^{j\frac{3\pi}{2}})| \cos(\frac{3\pi}{2}n + \frac{\pi}{4} + \angle H(e^{j\frac{3\pi}{2}}))$$

To find $H(e^{j\frac{3\pi}{2}})$, we use the fact that $H(e^{j\omega})$ is periodic over 2π, so

$$H(e^{j\frac{3\pi}{2}}) = H(e^{-j\frac{\pi}{2}}) = e^{j\frac{2\pi}{3}}$$

Therefore,

$$y[n] = \cos(\frac{3\pi}{2}n + \frac{\pi}{4} + \frac{2\pi}{3}) = \cos(\frac{3\pi}{2}n + \frac{11\pi}{12})$$

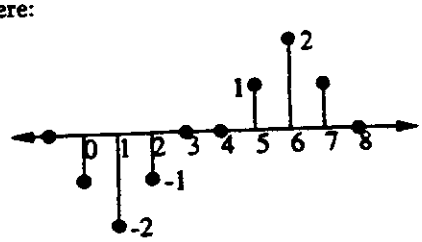
2.34. (a) Notice that

$$x[n] = x_0[n-2] + 2x_0[n-4] + x_0[n-6]$$

Since the system is LTI,

$$y[n] = y_0[n-2] + 2y_0[n-4] + y_0[n-6],$$

and we get sequence shown here:



(b) Since

$$y_0[n] = -1x_0[n+1] + x_0[n-1] = x_0[n] * (-\delta[n+1] + \delta[n-1]),$$
$$h[n] = -\delta[n+1] + \delta[n-1]$$

2.35. (a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.
(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2}x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

2.36. (a) Suppose we form the impulse:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

Since the system is linear,

$$L\{\delta[n]\} = \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n]$$

A shifted impulse results when:

$$\delta[n-1] = -\frac{1}{2}x_1[n] + \frac{1}{2}x_2[n]$$

The response to the shifted impulse

$$L\{\delta[n-1]\} = -\frac{1}{2}y_1[n] + \frac{1}{2}y_2[n]$$

Since,

$$L\{\delta[n]\} \neq L\{\delta[n-1]\}$$

The system is NOT TIME INVARIANT.

(b) An impulse may be formed:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

since the system is linear,

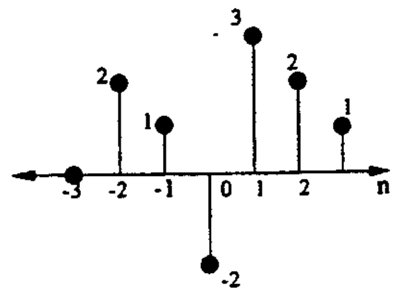
$$\begin{aligned} L\{\delta[n]\} &= \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n] \\ &= h[n] \end{aligned}$$

from the figure,

$$y_1[n] = -\delta[n+1] + 3\delta[n] + 3\delta[n-1] + \delta[n-3]$$
$$y_2[n] = -\delta[n+1] + \delta[n] - 3\delta[n-1] - \delta[n-3]$$
$$y_3[n] = 2\delta[n+2] + \delta[n+1] - 3\delta[n] + 2\delta[n-2]$$

Combining:

$$\begin{aligned} h[n] &= 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] \\ &\quad +2\delta[n-2] + \delta[n-3] \end{aligned}$$



2.37. For an LTI system, we use the convolution equation to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Let $n = m + N$:

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m+N-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k] \end{aligned}$$

Since $x[n]$ is periodic, $x[n] = x[n+rN]$ for any integer r. Hence,

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\ &= y[m] \end{aligned}$$

So, the output must also be periodic with period N.

2.38. (a) The homogeneous solution to the second order difference equation,

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1],$$

is obtained by setting the input (forcing term) to zero.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 0$$

Solving,

$$1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = 0,$$

$$(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})=0,$$

and the homogeneous solution takes the form

$$y_h[n]=A_1(\frac{1}{2})^n+A_2(\frac{1}{4})^n,$$

for the constants A_1 and A_2.

(b) Substituting the intial conditions,

$$y_h[-1]=A_1(\frac{1}{2})^{-1}+A_2(\frac{1}{4})^{-1}=1,$$

and

$$y_h[0]=A_1+A_2=0.$$

We have

$$\begin{aligned}2A_1+4A_2&=1\\A_1+A_2&=0\end{aligned}$$

Solving,

$$A_1=-1/2$$

and

$$A_2=1/2.$$

(c) Homogeneous equation:

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=0$$

Solving,

$$1-z^{-1}+\frac{1}{4}z^{-2}=0,$$

$$(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})=0,$$

and the homogeneous solution takes the form

$$y_h[n]=A_1(\frac{1}{2})^n.$$

Invoking the intial conditions, we have

$$\begin{aligned}y_h[-1]&=2A_1=1\\y_h[0]&=A_1=0\end{aligned}$$

Evident from the above contradiction, the initial conditions cannot be met.

(d) The homogeneous difference equation:

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=0$$

Suppose the homogeneous solution is of the form

$$y_h[n]=A_1(\frac{1}{2})^n+nB_1(\frac{1}{2})^n,$$

substituting into the difference equation:

$$y_h[n]-y_h[n-1]+\frac{1}{4}y_h[n-2]=0$$

$$\begin{aligned}A_1(\frac{1}{2})^n+nB_1(\frac{1}{2})^n-A_1(\frac{1}{2})^{n-1}-(n-1)B_1(\frac{1}{2})^{n-1}&\\ \frac{1}{4}A_1(\frac{1}{2})^{n-2}+\frac{1}{4}(n-2)B_1(\frac{1}{2})^{n-2}&=0.\end{aligned}$$

(e) Using the solution from part (d):

$$y_h[n] = A_1(\frac{1}{2})^n + nB_1(\frac{1}{2})^n,$$

and the initial conditions

$$y_h[-1] = 1$$

and

$$y_h[0] = 0,$$

we solve for A_1 and B_1:

$$A_1 = 0$$
$$B_1 = -1/2.$$

2.39. (a) For $x_1[n] = \delta[n]$,

$$\begin{aligned} y_1[0] &= 1 \\ y_1[1] &= ay[0] = a \end{aligned}$$

For $x_2[n] = \delta[n-1]$,

$$\begin{aligned} y_2[0] &= 1 \\ y_2[1] &= ay[0] + x_2[1] = a + 1 \neq y_1[0] \end{aligned}$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_2[n-1]$. Hence the system is NOT TIME INVARIANT.

(b) A linear system has the property that

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}\}$$

Hence, if the input is doubled, the output must also double at each value of n. Because $y[0] = 1$, always, the system is NOT LINEAR.

(c) Let $x_3 = \alpha x_1[n] + \beta x_2[n]$.
For $n \geq 0$:

$$\begin{aligned} y_3[n] &= x_3[n] + ay_3[n-1] \\ &= \alpha x_1[n] + \beta x_2[n] + a(x_3[n-1] + y_3[n-2]) \\ &= \alpha \sum_{k=0}^{n-1} a^k x_1[n-k] + \beta \sum_{k=0}^{n-1} a^k x_2[n-k] \\ &= \alpha(h[n] * x_1[n]) + \beta(h[n] * x_2[n]) \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n < 0$:

$$\begin{aligned} y_3[n] &= a^{-1}(y_3[n+1] - x_3[n]) \\ &= -\alpha \sum_{k=-1}^{n} a^k x_1[n-k] - \beta \sum_{k=-1}^{n} a^k x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n = 0$:

$$y_3[n] = y_1[n] = y_2[n] = 0.$$

Conclude,

$$y_3[n] = \alpha y_1[n] + \beta y_2[n], \text{ for all } n.$$

Therefore, the system is LINEAR. The system is still NOT TIME INVARIANT.

2.40. For the input

$$x[n] = \cos(\pi n)u[n] = (-1)^n u[n],$$

the output is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} (j/2)^k u[k](-1)^{(n-k)} u[n-k] \\ &= (-1)^n \sum_{k=0}^{n} (j/2)^k (-1)^{-k} \\ &= (-1)^n \sum_{k=0}^{n} (-j/2)^k \\ &= (-1)^n \left(\frac{1-(-j/2)^{(n+1)}}{1+j/2} \right) \end{aligned}$$

For large n, $(-j/2)^{(n+1)} \to 0$. Thus, the steady-state response becomes

$$\begin{aligned} y[n] &= \frac{(-1)^n}{1+j/2} \\ &= \frac{\cos(\pi n)}{1+j/2}. \end{aligned}$$

2.41. The input sequence,

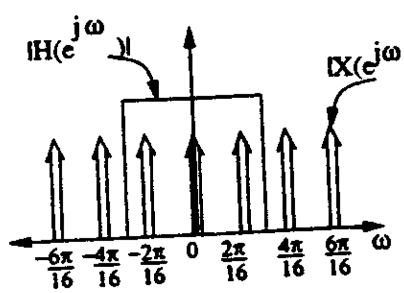
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n+16k],$$

has the Fourier representation

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta[n+16k] e^{-j\omega n} \\ &= \frac{1}{16} \sum_{k=-\infty}^{\infty} \delta(\omega + \frac{2\pi k}{16}). \end{aligned}$$

Therefore, the frequency representation of the input is also a periodic impulse train. There are 16 frequency impulses in the range $-\pi \le \omega \le \pi$.

We sketch the magnitudes of $X(e^{j\omega})$ and $H(e^{j\omega})$:



From the sketch, we observe that the LTI system is a lowpass filter which removes all but three of the frequency impulses. To these, it multiplies a phase factor $e^{-j3\omega}$.

The Fourier transform of the output is

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{16}\delta(\omega) + \frac{1}{16}e^{-j\frac{6\pi}{16}}\delta(\omega - \frac{2\pi}{16}) \\ &\quad + \frac{1}{16}e^{j\frac{6\pi}{16}}\delta(\omega + \frac{2\pi}{16}) \end{aligned}$$

Thus the output sequence is

$$y[n] = \frac{1}{16} + \frac{1}{8}\cos(\frac{2\pi n}{16} + \frac{3\pi}{8}).$$

2.42. (a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta^{(n-1)} u[n-1]. \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \alpha^n u[n]e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)} u[n-1]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell}, \end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1. \end{aligned}$$

Note that the Fourier transform of $\alpha^n u[n]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^n u[n]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1-\alpha e^{-j\omega}] = X(e^{j\omega})[1+\beta e^{-j\omega}]$$

taking the inverse Fourier transform, we have

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

(d) From part (a):

$$h[n] = 0, \text{ for } n < 0.$$

This implies that the system is CAUSAL.
If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.

2.43. For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

(a) real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2} \cdot [X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(b) imaginary part:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j} \cdot [X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(c) magnitude:

$$\begin{aligned} |X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\ &= \left(\frac{1}{1 - 2a\cos(\omega) + a^2}\right)^{\frac{1}{2}} \end{aligned}$$

(d) phase:

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

2.44. (a)

$$\begin{aligned} X(e^{j\omega})|_{\omega=0} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}|_{\omega=0} \\ &= \sum_{n=-\infty}^{\infty} x[n] \\ &= 6 \end{aligned}$$

(b)

$$\begin{aligned} X(e^{j\omega})|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\ &= 2 \end{aligned}$$

(c) Because $x[n]$ is symmetric about $n = 2$ this signal has linear phase.

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$A(\omega)$ is a zero phase (real) function of ω. Hence,

$$\angle X(e^{j\omega}) = -2\omega, \quad -\pi \le \omega \le \pi$$

(d)

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega n}d\omega = 2\pi x[n]$$

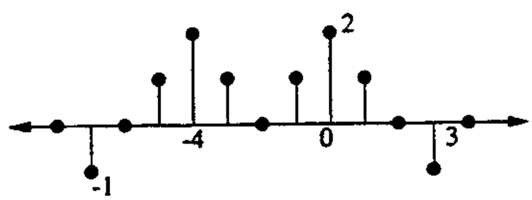
for $n = 0$:

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = 4\pi$$

(e) Let $y[n]$ be the unknown sequence. Then

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{-j\omega}) \\ &= \sum_n x[n]e^{j\omega n} \\ &= \sum_n x[-n]e^{-j\omega n} \\ &= \sum_n y[n]e^{-j\omega n} \end{aligned}$$

Hence $y[n] = x[-n]$.



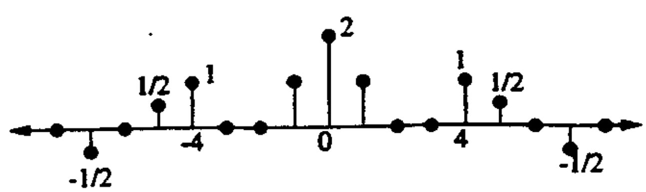
(f) We have determined that:

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$$\begin{aligned} X_R(e^{j\omega}) &= \mathcal{R}e\{X(e^{j\omega})\} \\ &= A(\omega)\cos(2\omega) \\ &= \frac{1}{2}A(\omega)\left(e^{j2\omega} + e^{-j2\omega}\right) \end{aligned}$$

Taking the inverse transform, we have

$$\frac{1}{2}a[n+2] + \frac{1}{2}a[n-2] = \frac{1}{2}x[n+4] + \frac{1}{2}x[n]$$



2.45. Let $x[n] = \delta[n]$, then

$$X(e^{j\omega}) = 1$$

The output of the ideal lowpass filter:

$$W(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})$$

The multiplier:

$$(-1)^n w[n] = e^{-j\pi n} w[n]$$

causes a shift in the frequency domain:

$$W(e^{j(\omega-\pi)}) = H(e^{j(\omega-\pi)})$$

The overall output:

$$y[n] = e^{-j\pi n} w[n] + w[n]$$
$$Y(e^{j\omega}) = H(e^{j(\omega-\pi)}) + H(e^{j\omega})$$

Noting that:

$$H(e^{j(\omega-\pi)}) = \begin{cases} 1, & \frac{\pi}{2} \le |\omega| \le \pi \\ 0, & |\omega| < \frac{\pi}{2} \end{cases}$$

$Y(e^{j\omega}) = 1$, thus $y[n] = \delta[n]$.

2.46. (a) We first perform a partial-fraction expansion of $X(e^{j\omega})$:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})} \\ &= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} \\ x[n] &= a^n u[n] + a^{-n} u[-n-1] \\ &= a^{|n|} \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cos(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\frac{e^{j\omega}+e^{-j\omega}}{2}d\omega \\ &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega}d\omega\right) \\ &= \frac{1}{2}(x[n-1] + x[n+1]) \\ &= \frac{1}{2}(a^{|n-1|} + a^{|n+1|}) \end{aligned}$$

2.47. (a)

$$\begin{aligned} y[n] &= x[n] + 2x[n-1] + x[n-2] \\ &= x[n] * h[n] \\ &= x[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2]) \\ h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

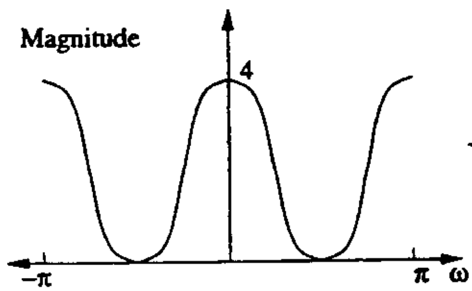
(b) Yes. $h[n]$ is finite-length and absolutely summable.

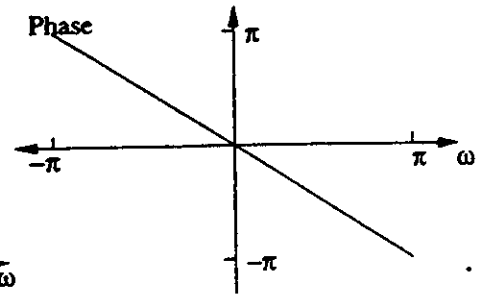
(c)

$$
\begin{aligned}
H(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-2j\omega} \\
&= 2e^{-j\omega}(\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}) \\
&= 2e^{-j\omega}(\cos(\omega) + 1)
\end{aligned}
$$

(d)

$$
\begin{aligned}
|H(e^{j\omega})| &= 2(\cos(\omega) + 1) \\
\angle H(e^{j\omega}) &= -\omega
\end{aligned}
$$





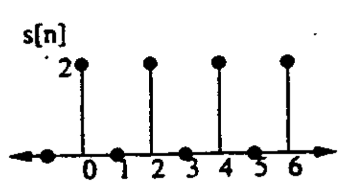
(e)

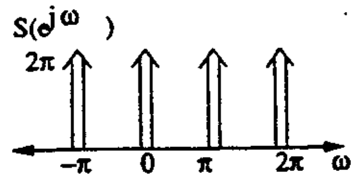
$$
\begin{aligned}
h_1[n] &= \frac{1}{2\pi}\int_{<2\pi>} H_1(e^{j\omega})e^{j\omega}d\omega \\
&= \frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega+\pi)})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega)})e^{j(\omega-\pi)n}d\omega \\
&= e^{-j\pi n}\frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega)})e^{j\omega n}d\omega \\
&= -1^n h[n] \\
&= \delta[n] - 2\delta[n-1] + \delta[n-2]
\end{aligned}
$$

2.48. (a) Notice that

$$
s[n] = 1 + \cos(\pi n) = 1 + (-1)^n
$$

$$
S(e^{j\omega}) = 2\pi\sum_k \delta(\omega - k\pi)
$$





(b) Since $y[n] = x[n]s[n]$,

$$
Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})X(e^{j(\omega-\theta)})d\omega
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})X(e^{j(\omega-\theta)})d\omega \\
&= X(e^{j\omega}) + X(e^{j(\omega-\pi)})
\end{aligned}
$$

$Y(e^{j\omega})$ contains copies of $X(e^{j\omega})$ replicated at intervals of π.

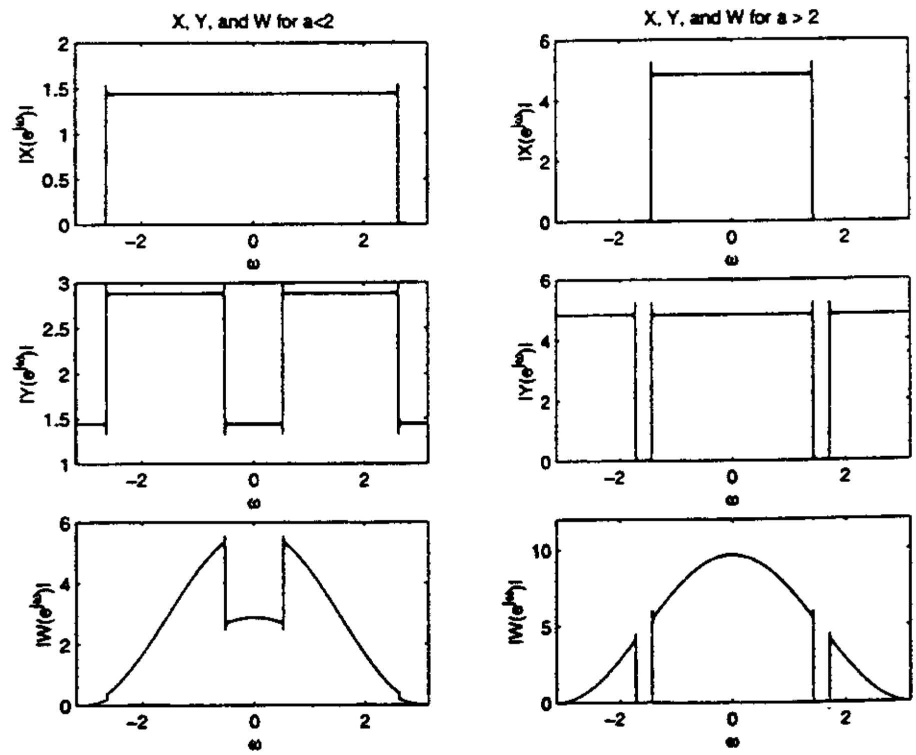
(c) Since $w[n] = y[n] + (1/2)(y[n+1] + y[n-1])$,

$$
\begin{aligned}
W(e^{j\omega}) &= Y(e^{j\omega}) + \frac{1}{2}\left(e^{j\omega}Y(e^{j\omega}) + e^{-j\omega}Y(e^{j\omega})\right) \\
&= Y(e^{j\omega})(1+\cos(\omega))
\end{aligned}
$$

(d) The following figure shows $X(e^{j\omega})$, $Y(e^{j\omega})$, and $W(e^{j\omega})$ for $a < 2$ and $a > 2$. Notice that

$$
X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/a, \\ 0, & \pi/a \le |\omega| \ge \pi \end{cases}
$$

So, for $a > 2$, $Y(e^{j\omega})$ contains two non-overlapping replications of $X(e^{j\omega})$, whereas for $a < 2$, "aliasing" occurs. When there is aliasing, $W(e^{j\omega})$ is not at all close to $X(e^{j\omega})$. Hence, a must be greater than 2 for $w[n]$ to be "close" to $x[n]$.



2.49. (a) We start by interpreting each clue.

(i) The system is causal implies

$$h[n] = 0 \text{ for } n \le 0.$$

(ii) The Fourier transform is conjugate symmetric implies $h[n]$ is real.

(iii) The DTFT of the sequence $h[n+1]$ is real implies $h[n+1]$ is even.

From the above observations, we deduce that $h[n]$ has length 3, therefore it has finite duration.

(b) From part (a) we know that $h[n]$ is length 3 with even symmetry around $h[1]$. Let $h[0] = h[2] = a$ and $h[1] = b$, from (iv) and using Parseval's theorem, we have

$$2a^2 + b^2 = 2.$$

From (v), we also have

$$2a - b = 0.$$

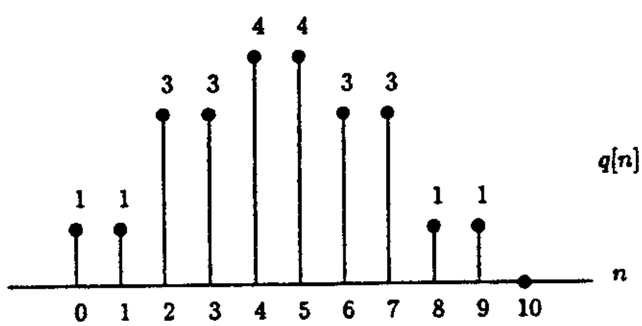
Solving the above equations, we get

$$\begin{aligned} h[0] &= \frac{1}{\sqrt{3}} \\ h[1] &= \frac{2}{\sqrt{3}} \\ h[2] &= \frac{1}{\sqrt{3}} \end{aligned}$$

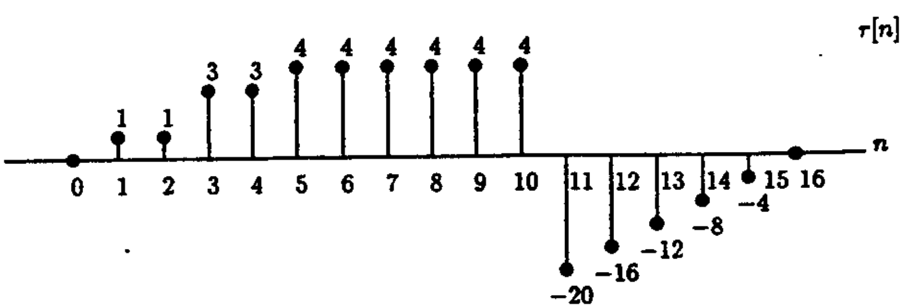
or

$$\begin{aligned} h[0] &= -\frac{1}{\sqrt{3}} \\ h[1] &= -\frac{2}{\sqrt{3}} \\ h[2] &= -\frac{1}{\sqrt{3}}. \end{aligned}$$

2.50. (a) Carrying out the convolution sum, we get the following sequence $q[n]$:



(b) Again carrying out the convolution sum, we get the following sequence $r[n]$:



(c) Let $a[n] = v[-n]$ and $b[n] = w[-n]$, then:

$$\begin{aligned} a[n] * b[n] &= \sum_{k=-\infty}^{+\infty} a[k]b[n-k] \\ &= \sum_{k=-\infty}^{+\infty} v[-k]w[k-n] \\ &= \sum_{r=-\infty}^{+\infty} v[r]w[-n-r] \text{ where } r = -k \\ &= q[-n]. \end{aligned}$$

We thus conclude that $q[-n] = v[-n] * w[-n]$.

2.51. For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

(a) real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2} \cdot [X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(b) imaginary part:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j} \cdot [X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(c) magnitude:

$$\begin{aligned} |X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\ &= \left(\frac{1}{1 - 2a cos(\omega) + a^2}\right)^{\frac{1}{2}} \end{aligned}$$

(d) phase:

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

2.52. $x[n]$ can be rewritten as:

$$\begin{aligned} x[n] &= cos(\frac{5\pi n}{2}) \\ &= cos(\frac{\pi n}{2}) \\ &= \frac{e^{j\frac{\pi n}{2}}}{2} + \frac{e^{-j\frac{\pi n}{2}}}{2}. \end{aligned}$$

We now use the fact that complex exponentials are eigenfunctions of LTI systems, we get:

$$\begin{aligned} y[n] &= e^{-j\frac{\pi}{8}}\frac{e^{j\frac{\pi n}{2}}}{2} + e^{j\frac{\pi}{8}}\frac{e^{-j\frac{\pi n}{2}}}{2} \\ &= \frac{e^{j(\frac{\pi n}{2}-\frac{\pi}{8})}}{2} + \frac{e^{-j(\frac{\pi n}{2}-\frac{\pi}{8})}}{2} \\ &= cos(\frac{\pi}{2}(n-\frac{1}{4})). \end{aligned}$$

2.53. First $x[n]$ goes through a lowpass filter with cutoff frequency 0.5π. Since the cosine has a frequency of 0.6π, it will be filtered out. The delayed impulse will be filtered to a delayed sinc and the constant will remain unchanged. We thus get:

$$w[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} + 2.$$

$y[n]$ is then given by:

$$y[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} - 3\frac{sin(0.5\pi(n-6))}{\pi(n-6)}.$$

2.54.

$$\begin{aligned} x[n] &= \cos(\frac{15\pi n}{4} - \frac{\pi}{3}) \\ &= \cos(-\frac{\pi n}{4} - \frac{\pi}{3}) \\ &= \cos(\frac{\pi n}{4} + \frac{\pi}{3}) \\ &= \frac{e^{j\frac{\pi}{3}}e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{\pi}{3}}e^{-j\frac{\pi n}{4}}}{2}. \end{aligned}$$

Using the fact that complex exponentials are eigenfunctions of LTI systems, we get:

$$\begin{aligned} y[n] &= e^{-j\frac{3\pi}{8}}\frac{e^{j\frac{\pi}{3}}e^{j\frac{\pi n}{4}}}{2} + e^{-j\frac{\pi}{8}}\frac{e^{-j\frac{\pi}{3}}e^{-j\frac{\pi n}{4}}}{2} \\ &= \frac{e^{-j\frac{\pi}{24}}e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{11\pi}{24}}e^{-j\frac{\pi n}{4}}}{2} \\ &= e^{-j\frac{\pi}{4}}(\frac{e^{j\frac{5\pi}{24}}e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{5\pi}{24}}e^{-j\frac{\pi n}{4}}}{2}) \\ &= e^{-j\frac{\pi}{4}}\cos(\frac{\pi n}{4} + \frac{5\pi}{24}). \end{aligned}$$

2.55. Since system 1 is memoryless, it is time invariant. The input, $x[n]$ is periodic in ω, therefore $w[n]$ will also be periodic in ω. As a consequence, $y[n]$ is periodic in ω and so is A.

2.56. (a)

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}x[k]h[n-k]. \end{aligned}$$

Let $x[n] = ax_1[n] + bx_2[n]$, then:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}(ax_1[n] + bx_2[n])) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}(ax_1[k] + bx_2[k])h[n-k] \\ &= a \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_1[k]h[n-k] + b \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_2[k]h[n-k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

where $y_1[n]$ and $y_2[n]$ are the responses to $x_1[n]$ and $x_2[n]$ respectively. We thus conclude that system S is linear.

(b) Let $x_2[n] = x[n-n_0]$, then:

$$\begin{aligned} y_2[n] &= h[n] * (e^{-j\omega_0 n} x_2[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x_2[n-k]h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n-n_0-k]h[k] \\ &\neq y[n-n_0]. \end{aligned}$$

We thus conclude that system S is not time invariant.

(c) Since the magnitude of $e^{-j\omega_0 n}$ is always bounded by 1 and $h[n]$ is stable, a bounded input $x[n]$ will always produce a bounded input to the stable LTI system and therefore the output $y[n]$ will be bounded. We thus conclude that system S is stable.

(d) We can rewrite $y[n]$ as:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n} x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 n} e^{j\omega_0 k} x[n-k]h[k] \\ &= e^{-j\omega_0 n} \sum_{k=-\infty}^{+\infty} e^{j\omega_0 k} x[n-k]h[k]. \end{aligned}$$

System C should therefore be a multiplication by $e^{-j\omega_0 n}$.

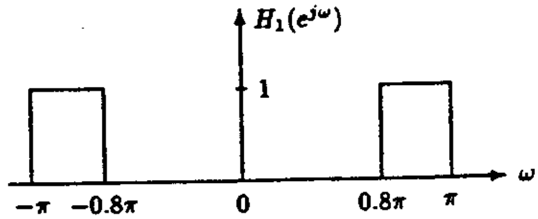
2.57. (a) $H_1(e^{j\omega})$ corresponds to a frequency shifted version of $H(e^{j\omega})$, specifically:

$$H_1(e^{j\omega}) = H(e^{j(\omega-\pi)}).$$

We thus have:

$$H_1(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.8\pi \\ 1 & , \quad 0.8\pi \le |\omega| \le \pi. \end{cases}$$

This is a highpass filter.



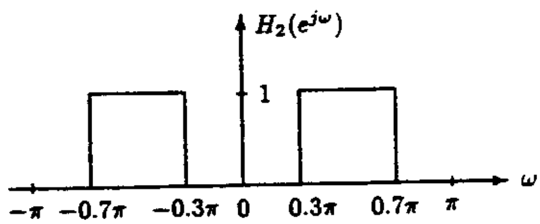
(b) $H_2(e^{j\omega})$ corresponds to a frequency modulated version of $H(e^{j\omega})$, specifically:

$$H_2(e^{j\omega}) = H(e^{j\omega}) * (\delta(\omega - 0.5\pi) + \delta(\omega + 0.5\pi)) \qquad \text{where } |\omega| \le \pi.$$

We thus have:

$$H_2(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.3\pi \\ 1 & , \quad 0.3\pi \le |\omega| \le 0.7\pi \\ 0 & , \quad 0.7\pi < |\omega| \le \pi. \end{cases}$$

This is a bandpass filter.



(c) $H_3(e^{j\omega})$ corresponds to a periodic convolution of $H_{lp}(e^{j\omega})$ with another lowpass filter, specifically:

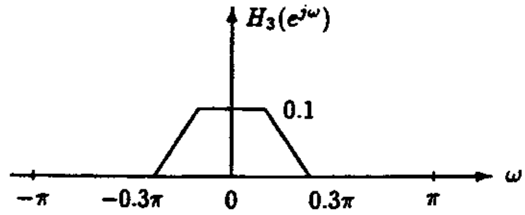
$$H_3(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) H_{lp}(e^{j\omega - \theta})\, d\theta$$

where $H(e^{j\omega})$ is given by:

$$H(e^{j\omega}) = \begin{cases} 1 & , \quad |\omega| < 0.1\pi \\ 0 & , \quad 0.1\pi \le |\omega| \le \pi \end{cases}$$

Carrying out the convolution, we get:

$$H_3(e^{j\omega}) = \begin{cases} 0.1 & , \quad |\omega| < 0.1\pi \\ -\frac{|\omega|}{2\pi} + 0.15 & , \quad 0.1\pi \le |\omega| \le 0.3\pi \\ 0 & , \quad 0.3\pi < |\omega| \le \pi. \end{cases}$$



2.58. Note that $X(e^{j\omega})$ is real, and $Y(e^{j\omega})$ is given by:

$$Y(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}) & , \quad 0 < \omega < \pi \\ +jX(e^{j\omega}) & , \quad -\pi < \omega < 0. \end{cases}$$

$w[n] = x[n] + jy[n]$, therefore:

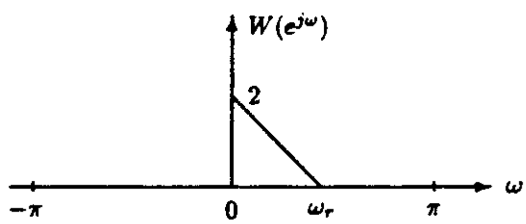
$$W(e^{j\omega}) = X(e^{j\omega}) + jY(e^{j\omega}).$$

Using the above, we get:

$$jY(e^{j\omega}) = \begin{cases} X(e^{j\omega}) & , \quad 0 < \omega < \pi \\ -X(e^{j\omega}) & , \quad -\pi < \omega < 0. \end{cases}$$

We thus conclude:

$$W(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}) & , \quad 0 < \omega < \pi \\ 0 & , \quad -\pi < \omega < 0. \end{cases}$$



2.59. (a) Using the change of variable: $r = -k$, we can rewrite $R_x[n]$ as:

$$R_x[n] = \sum_{r=-\infty}^{\infty} x^*[-r]x[n-r] = x^*[-n] * x[n].$$

We therefore have:

$$g[n] = x^*[-n].$$

(b) The Fourier transform of $x^*[-n]$ is $X^*(e^{j\omega})$, therefore:

$$R_x(e^{j\omega}) = X^*(e^{j\omega})X(e^{j\omega}) = |X(e^{j\omega})|^2.$$

2.60. (a) Note that $x_2[n] = -\sum_{k=0}^{k=4} x[n-k]$. Since the system is LTI, we have:

$$y_2[n] = -\sum_{k=0}^{k=4} y[n-k].$$

(b) By carrying out the convolution, we get:

$$h[n] = \begin{cases} -1 & , \quad n = 0, n = 2 \\ -2 & , \quad n = 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

2.61. The system is not stable, any bounded input that excites the zero input response will result in an unbounded output.

The solution to the difference equation is given by:

$$y[n] = y_{zir}[n] + y_{zsr}[n]$$

where $y_{zir}[n]$ is the zero input response and $y_{zsr}[n]$ is the zero state response, the response to zero initial conditions:

$$\begin{aligned} y_{zir}[n] &= a(\frac{1}{2})^n \qquad \text{where } a \text{ is a constant determined by the initial condition.} \\ y_{zsr}[n] &= (\frac{1}{2})^n u[n] * x[n]. \end{aligned}$$

An example of a bounded input that results in an unbounded output is:

$$x[n] = \delta[n+1].$$

The output is unbounded and given by:

$$y[n] = (\frac{1}{2})^{n+1} u[n+1] - \frac{1}{2}(\frac{1}{2})^n.$$

2.62. The definition of causality implies that the output of a causal LTI system may only be derived from past and present inputs.

The convolution sum:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k] \end{aligned}$$

Note that the first summation represents a weighted sum of future values of the input. Thus, if the system is causal,

$$\sum_{k=-\infty}^{-1} h[k]x[n-k] = 0.$$

This can only be guaranteed if $h[k] = 0$ for $n < 0$.

Using reverse logic, we can show that if $h[n] = 0$ for $n < 0$,

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

Since the convolution sum specifies that the input is formed from past and present input values, the system is, by definition, causal.

2.63. The system could be LTI. A possible impulse response is:

$$\begin{aligned} h[n] &= (\delta[n] - \frac{1}{4}\delta[n-1]) * (\frac{1}{2})^n \\ &= (\frac{1}{2})^n - \frac{1}{4}(\frac{1}{2})^{n-1}. \end{aligned}$$

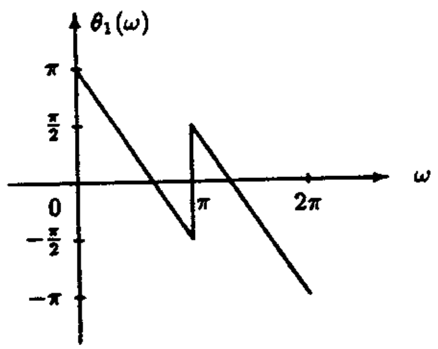
2.64. Let the input be $x[n] = \delta[n-1]$, if the system is causal then the output, $y[n]$, should be zero for $n < 1$. Let's evaluate $y[0]$:

$$\begin{aligned} y[0] &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} Y(e^{j\omega})\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-j\omega}e^{-j\omega/2}\,d\omega \\ &= -\frac{2}{3\pi} \\ &\neq 0. \end{aligned}$$

This proves that the system is not causal.

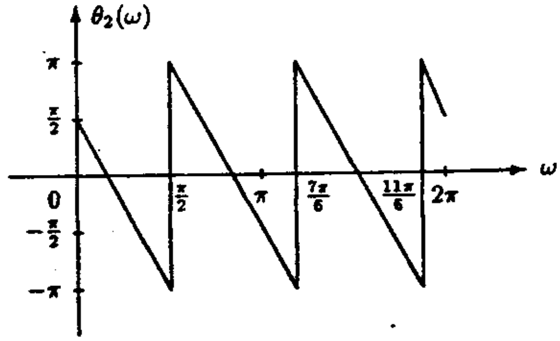
2.65. $x_1[n]$ is even-symmetric around $n = 1.5$, furthermore since $\sum x_1[n] < 0$ and we want $A_1(0) \geq 0$, we need to include a π factor in the phase. An appropriate choice for $\theta_1(\omega)$ is therefore:

$$\theta_1(\omega) = -\frac{3}{2}\omega + \pi \qquad |\omega| < \pi.$$



$x_2[n]$ is odd-symmetric around $n = 3$, therefore:

$$\theta_2(\omega) = -3\omega + \frac{\pi}{2} \qquad |\omega| < \pi.$$



2.66. (a)

$$\begin{aligned}
E(e^{j\omega}) &= H_1(e^{j\omega})X(e^{j\omega}) \\
F(e^{j\omega}) &= E(e^{-j\omega}) \\
&= H_1(e^{-j\omega})X(e^{-j\omega}) \\
G(e^{j\omega}) &= H_1(e^{j\omega})F(e^{j\omega}) \\
&= H_1(e^{j\omega})H_1(e^{-j\omega})X(e^{-j\omega}) \\
Y(e^{j\omega}) &= G(e^{-j\omega}) \\
&= H_1(e^{-j\omega})H_1(e^{j\omega})X(e^{j\omega}).
\end{aligned}$$

(b) Since:

$$Y(e^{j\omega}) = H_1(e^{-j\omega})H_1(e^{j\omega})X(e^{j\omega}),$$

We get:

$$H(e^{j\omega}) = H_1(e^{-j\omega})H_1(e^{j\omega}).$$

(c) Taking the inverse transform of $H(e^{j\omega})$, we get:

$$h[n] = h_1[-n] * h_1[n].$$

2.67. (a) Using the properties of the Fourier transform and the fact that $(-1)^n = e^{j\pi n}$, we get:

$$\begin{aligned}
V(e^{j\omega}) &= X(e^{j(\omega+\pi)}) \\
W(e^{j\omega}) &= H_1(e^{j\omega})V(e^{j\omega}) \\
&= H_1(e^{j\omega})X(e^{j(\omega+\pi})) \\
Y(e^{j\omega}) &= W(e^{j(\omega-\pi)}) \\
&= H_1(e^{j(\omega-\pi)})X(e^{j\omega})
\end{aligned}$$

$H(e^{j\omega})$ is thus given by:

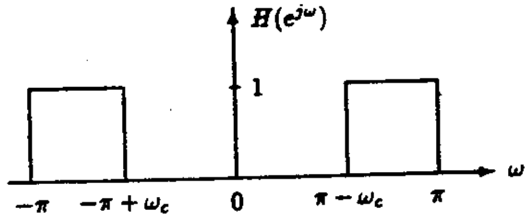
$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

(b)

$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

With the given choice of $H_1(e^{j\omega})$,

$$H(e^{j\omega}) = \begin{cases} 0 , & |\omega| < \pi - \omega_c \\ 1 , & \pi - \omega_c < |\omega| \leq \pi. \end{cases}$$



2.68. If $x_1[n] = x_2[n]$, $w_1[n]$ and $w_2[n]$ will not be necessarily equal.

$$\begin{aligned} w_1[n] &= x_1[-n-2] \\ w_2[n] &= x_2[-n+2] \\ &\neq x_2[-n-2] \end{aligned}$$

A simple counterexample is $x_1[n] = x_2[n] = \delta[n]$. Then:

$$\begin{aligned} w_1[n] &= \delta[n+2] \\ w_2[n] &= \delta[n-2]. \end{aligned}$$

2.69. (a) The overall system is not guaranteed to be an LTI system. A simple counterexample is:

$$\begin{aligned} y_1[n] &= x[n] \\ y_2[n] &= x[n] \\ y[n] &= y_1[n]y_2[n] = x^2[n] \end{aligned}$$

which is not a linear system, therefore the system is not LTI.

(b)

$$\begin{aligned} Y_1(e^{j\omega}) &= H_1(e^{j\omega})X(e^{j\omega}) \\ Y_2(e^{j\omega}) &= H_2(e^{j\omega})X(e^{j\omega}) \\ Y(e^{j\omega}) &= Y_1(e^{j\omega}) * Y_2(e^{j\omega}). \end{aligned}$$

Using the above relationships, we get:

$$Y(e^{j\omega}) = \begin{cases} \text{unspecified} , & 0 < |\omega| < 0.6\pi \\ 0 , & 0.6\pi \leq |\omega| \leq \pi. \end{cases}$$

2.70. The first difference:

$$y[n] = \nabla(x[n]) = x[n] - x[n-1].$$

(a) To determine if the system is linear:

$$\begin{aligned}\nabla(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] - ax_1[n-1] - bx_2[n-1] \\ &= a(x_1[n] - x_1[n-1]) + b(x_2[n] - x_2[n-1]) \\ &= \nabla(ax_1[n]) + \nabla(ax_2[n]).\end{aligned}$$

Therefore, the system is LINEAR.
To determine if the first difference is time invariant:

$$\begin{aligned}\nabla(x_1[n-1]) &= x[n-1] - x[n-2] \\ &= y[n-1].\end{aligned}$$

The system is TIME INVARIANT.

(b) The impulse response is obtained by setting the input to $x[n] = \delta[n]$,

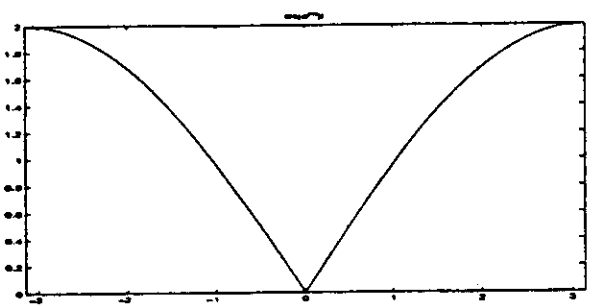
$$y[n] = h[n] = \delta[n] - \delta[n-1]$$

(c) Taking the Fourier transform of the result of part (b), we find that the system function is

$$H(e^{j\omega}) = 1 - e^{-j\omega}.$$

Thus the magnitude of the frequency response is

$$\begin{aligned}|H(e^{j\omega})| &= \sqrt{(1 - e^{-j\omega})(1 - e^{-j\omega})} \\ &= \sqrt{2 - 2\cos(\omega)}.\end{aligned}$$

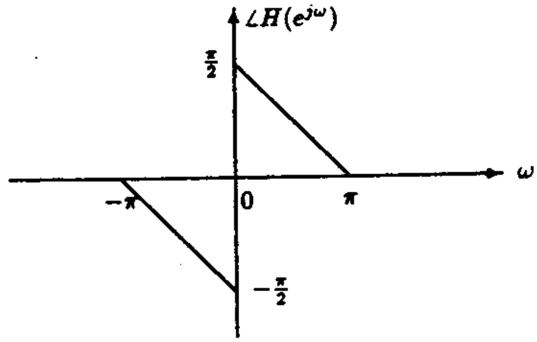


We calculate the phase of the frequency response:

$$H(e^{j\omega}) = (1 - \cos(\omega)) + j\sin(\omega)$$

Thus,

$$\angle H(e^{j\omega}) = \arctan\left(\frac{\sin(\omega)}{1 - \cos(\omega)}\right)$$



(d) In general,

$$\begin{aligned}\nabla(x[n]) &= x[n] * (\delta[n] - \delta[n-1]) \\ &= x[n] - x[n-1].\end{aligned}$$

So, for $x[n] = f[n] * g[n]$,

$$\begin{aligned}\nabla(x[n]) &= f[n] * g[n] * (\delta[n] - \delta[n-1]) \\ &= f[n] * \nabla(g[n]) \\ &= \nabla(f[n]) * g[n].\end{aligned}$$

Where we have used the commutivity of the convolution operator to obtain the last two equalities.

(e) We desire the inverse system, $h_i[n]$, such that

$$h_i[n] * \nabla(x[n]) = x[n]$$

The inverse system must satisfy:

$$h_i[n] * h[n] = \delta[n],$$

in the frequency domain,

$$H_i(e^{j\omega})H(e^{j\omega}) = 1.$$

Recall from part (c),

$$H(e^{j\omega}) = 1 - e^{-j\omega}.$$

So,

$$H_i(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}},$$

and

$$h_i[n] = u[n].$$

Hence, the unit step is the inverse system for the first difference.

2.71. For impulse response $h[n]$, the frequency response of an LTI system is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

(a) Suppose the impulse response is $h^*[n]$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} h^*[n]e^{-j\omega n} &= \left(\sum_{n=-\infty}^{\infty} h[n]e^{j\omega n}\right)^* \\ &= H^*(e^{-j\omega}).\end{aligned}$$

(b) We have

$$\begin{aligned} H^*(e^{j\omega}) &= \left(\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}\right)^* \\ &= \sum_{n=-\infty}^{\infty} h^*[n]e^{j\omega n}. \end{aligned}$$

If $h[n]$ is real,

$$\begin{aligned} H^*(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{j\omega n} \\ &= H(e^{-j\omega}). \end{aligned}$$

Hence, the frequency response is conjugate symmetric.

2.72. The analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

(a) The Fourier transform of $x^*[n]$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* \\ &= X^*(e^{-j\omega n}). \end{aligned}$$

(b) The Fourier transform of $x^*[-n]$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} &= \sum_{l=-\infty}^{\infty} x^*[l]e^{j\omega l} \\ &= \left(\sum_{l=-\infty}^{\infty} x[l]e^{-j\omega l}\right)^* \\ &= X^*(e^{j\omega}). \end{aligned}$$

2.73. From property 1:

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n}$$

for $x[n]$ real, $x[n] = x^*[n]$, so

$$\begin{aligned} X^*(e^{-j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= X(e^{j\omega}). \end{aligned}$$

Thus, the Fourier transform of a real input is conjugate symmetric.

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

$$X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{j\omega})$$

From property 7, $X(e^{j\omega}) = X^*(e^{-j\omega})$ for $x[n]$ real. Thus,

$$X_R(e^{j\omega}) + jX_I(e^{j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega}).$$

We may infer

$$\begin{array}{ll} \text{property 8:} & X_R(e^{j\omega}) = X_R(e^{-j\omega}) \\ \text{property 9:} & X_I(e^{j\omega}) = -X_I(e^{-j\omega}) \end{array}$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

$$X^*(e^{-j\omega}) = |X(e^{-j\omega})|e^{-j\angle X(e^{-j\omega})}$$

From property 7:

$$X(e^{j\omega}) = X^*(e^{-j\omega}).$$

So,

$$\begin{array}{ll} \text{property 10:} & |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \text{property 11:} & \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}). \end{array}$$

2.74. Theorem 1:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} ax_1[n]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2[n]e^{-j\omega n} \\ &= aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \end{aligned}$$

Theorem 2:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n-n_d]e^{-j\omega n} &= \sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega(\ell-n_d)} \\ &= e^{j\omega n_d}\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell} \\ &= e^{j\omega n_d}X(e^{j\omega}) \end{aligned}$$

Theorem 3:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\ &= X(e^{j(\omega-\omega_0)}) \end{aligned}$$

Theorem 4:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} &= \sum_{\ell=-\infty}^{\infty} x[\ell]e^{j\omega\ell} \\ &= X(e^{-j\omega}) \end{aligned}$$

Theorem 5:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n} &= -\frac{1}{j}\frac{d}{d\omega}\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right) \\ &= j\frac{d}{d\omega}(X(e^{j\omega})) \end{aligned}$$

2.75. The output of an LTI system is obtained by the convolution sum,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Taking the Fourier transform,

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right) e^{-j\omega n} \\
&= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k]e^{-j\omega n} \right) \\
&= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \left(\sum_{n=-\infty}^{\infty} h[n-k]e^{-j\omega(n-k)} \right)
\end{aligned}$$

Hence,

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

2.76. The Modulation theorem:

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})\, d\theta$$

the time-domain representation,

$$\begin{aligned}
y[n] &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\omega\, X(e^{j\theta})W(e^{j(\omega-\theta)})e^{j\omega n} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, X(e^{j\theta})w[n]e^{j\theta n} \\
&= x[n]w[n]
\end{aligned}$$

2.77. (a) The Fourier transform of $y^*[-n]$ is $Y^*(e^{j\omega})$, and $X(e^{j\omega})Y(e^{j\omega})$ forms a transform pair with $x[n] * y[n]$. So

$$G(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$$

and

$$g[n] = x[n] * y^*[-n]$$

form a transform pair.

(b)

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega &= \sum_{n=-\infty}^{\infty} \left(x[n] * y^*[-n]\right) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y^*[k-n]e^{-j\omega n}
\end{aligned}$$

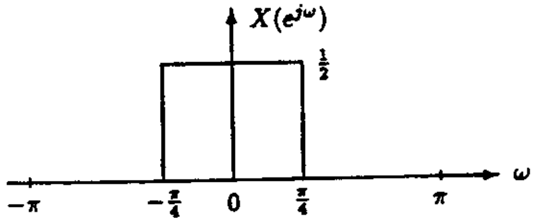
for $n = 0$:

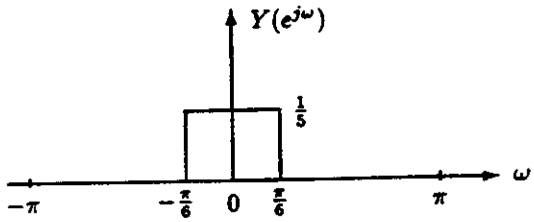
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})Y^*(e^{j\omega})d\omega = \sum_{k=-\infty}^{\infty} x[k]y^*[k]$$

(c) Using the result from part (b):

$$\begin{aligned} x[n] &= \frac{\sin(\pi n/4)}{2\pi n} \\ y^*[n] &= \frac{\sin(\pi n/6)}{5\pi n} \end{aligned}$$

We recognize each sequence to be a pulse in the frequency domain:

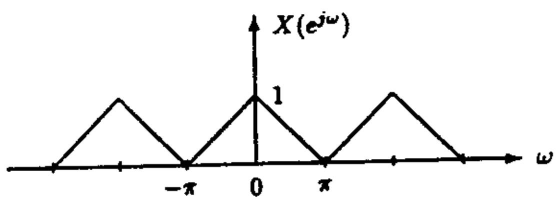




Substituting into Eq. (P2.77-1):

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]y^*[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\left[(\frac{1}{2})(\frac{1}{5})(\frac{2\pi}{6})\right] \\ &= \frac{1}{60} \end{aligned}$$

2.78. $X(e^{j\omega})$ is given by:

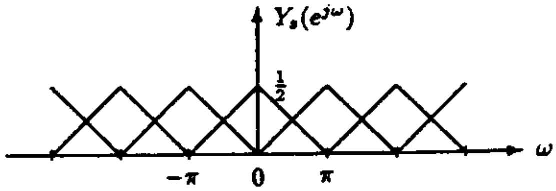


(a)

$$\begin{aligned} y_s[n] &= \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= \frac{1}{2}\left(1 + e^{j\pi n}\right) x[n] \end{aligned}$$

which transforms to

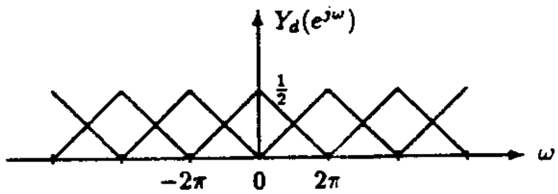
$$Y_s(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X(e^{j(\omega+\pi)})\right]$$



(b)

$$y_d[n] = x[2n]$$

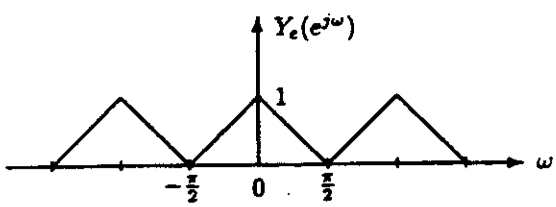
$$\begin{aligned} Y_d(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}+\pi)})\right] \\ &= Y_s(e^{j\frac{\omega}{2}}) \end{aligned}$$



(c)

$$y_e[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$Y_e(e^{j\omega}) = X(e^{j2\omega})$$



2.79. (a)

$$\begin{aligned} \Phi_x(-N,-\omega) &= \sum_{n=-\infty}^{\infty} x[n-N]x^*[n+N]e^{j\omega n}. \\ \Phi_x^*(N,\omega) &= \left(\sum_{n=-\infty}^{\infty} x[n+N]x^*[n-N]e^{-j\omega n}\right)^* \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} (x[n+N]x^*[n-N]e^{-j\omega n})^* \\
&= \sum_{n=-\infty}^{\infty} x^*[n+N]x[n-N]e^{j\omega n} \\
&= \Phi_x(-N,-\omega).
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Phi_x(N,\omega) &= \sum_{n=-\infty}^{\infty} Aa^{n+N}u[n+N]Aa^{n-N}u[n-N]e^{-j\omega n} \\
&= A^2 \sum_{n=N}^{\infty} a^{2n}e^{-j\omega n} \\
&= A^2 \sum_{n=N}^{\infty} (a^2e^{-j\omega})^n \\
&= A^2 \frac{(a^2e^{-j\omega})^N}{1-a^2e^{-j\omega}} \\
&= A^2 \frac{a^{2N}e^{-j\omega N}}{1-a^2e^{-j\omega}}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
X(e^{j(v+(\omega/2))}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(v+\omega/2)n}. \\
X^*(e^{j(v-(\omega/2))}) &= \sum_{n=-\infty}^{\infty} x^*[n]e^{j(v-\omega/2)n}.
\end{aligned}
$$

Let $S = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j(v+(\omega/2))})X^*(e^{j(v-(\omega/2))})e^{j2vN}\,dv$, then:

$$
\begin{aligned}
S &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} x[n]e^{-j(v+\omega/2)n} \sum_{k=-\infty}^{\infty} x^*[k]e^{j(v-\omega/2)k}e^{j2vN}\,dv \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[n]x^*[k]e^{-j\frac{\omega(n+k)}{2}}\int_{-\pi}^{\pi} e^{j(k-n+2N)}\,dv \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[n]x^*[k]e^{-j\frac{\omega(n+k)}{2}}\frac{2\sin(\pi(k-n+2N))}{k-n+2N} \\
&= \sum_{n=-\infty}^{\infty} x[n]x^*[n-2N]e^{-j\frac{\omega(2n-2N)}{2}} \\
&= \sum_{n=-\infty}^{\infty} x[n+N]x^*[n-N]e^{-j\omega n} \\
&= \Phi_x(N,\omega).
\end{aligned}
$$

We thus conclude that:

$$
\Phi_x(N,\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j(v+(\omega/2))})X^*(e^{j(v-(\omega/2))})e^{j2vN}\,dv.
$$

2.80.

$$w[n] = x[n] + y[n]$$

The mean of $w[n]$:

$$\begin{aligned} m_w &= E\{w[n]\} \\ &= E\{x[n] + y[n]\} \\ &= E\{x[n]\} + E\{y[n]\} \\ &= m_x + m_y \end{aligned}$$

The variance of $w[n]$:

$$\begin{aligned} \sigma_w^2 &= E\{(w[n] - m_w)^2\} \\ &= E\{w^2[n]\} - m_w^2 \\ &= E\{(x[n] + y[n])^2\} - m_w^2 \\ &= E\{x^2[n]\} + 2E\{x[n]y[n]\} + E\{y^2[n]\} - m_x^2 - 2m_x m_y - m_y^2 \end{aligned}$$

If $x[n]$ and $y[n]$ are uncorrelated:

$$\begin{aligned} \sigma_w^2 &= E\{x^2[n]\} - m_x^2 + E\{y^2[n]\} - m_y^2 \\ &= \sigma_x^2 + \sigma_y^2 \end{aligned}$$

2.81. Let $e[n]$ be a white noise sequence and $E\{s[n]e[m]\} = 0$ for all n and m.

$$\begin{aligned} E\{y[n]y[n+m]\} &= E\{s[n]e[n]s[n+m]e[n+m]\} \\ &= E\{s[n]s[n+m]e[n]e[n+m]\} \end{aligned}$$

Since $s[n]$ is uncorrelated with $e[n]$:

$$\begin{aligned} E\{y[n]y[n+m]\} &= E\{s[n]s[n+m]\}E\{e[n]e[n+m]\} \\ &= \sigma_s^2 \sigma_e^2 \delta[m] \end{aligned}$$

2.82. (a)

$$\begin{aligned} \phi_{xx}[m] &= E(x[n]x[n+m]) \\ &= E((s[n] + e[n])(s[n+m] + e[n+m])) \\ &= E(s[n]s[n+m]) + E(e[n]e[n+m]) + E(s[n]e[n+m]) + E(e[n]s[n+m]) \\ &= \phi_{ss}[m] + \phi_{ee}[m] + 2E(e[n])E(s[n]) \text{ since } s[n] \text{ and } e[n] \text{ are independent and stationary.} \\ &= \phi_{ss}[m] + \phi_{ee}[m] \qquad \text{where we assumed } e[n] \text{ has zero mean.} \end{aligned}$$

Taking the Fourier transform of the above equation, we get:

$$\Phi_{xx}(e^{j\omega}) = \Phi_{ss}(e^{j\omega}) + \Phi_{ee}(e^{j\omega}).$$

(b)

$$\begin{aligned} \phi_{xe}[m] &= E(x[n]e[n+m]) \\ &= E((s[n] + e[n])e[n+m]) \\ &= E(s[n])E(e[n]) + \phi_{ee}[m] \quad \text{since } s[n] \text{ and } e[n] \text{ are independent and stationary.} \\ &= \phi_{ee}[m] \qquad \text{where we assumed } e[n] \text{ has zero mean.} \end{aligned}$$

Taking the Fourier transform of the above equation, we get:

$$\Phi_{xe}(e^{j\omega}) = \Phi_{ee}(e^{j\omega}).$$

(c)

$$\begin{aligned}
\phi_{xs}[m] &= E(x[n]s[n+m]) \\
&= E((s[n]+e[n])s[n+m]) \\
&= \phi_{ss}[m] + E(e[n])E(s[n]) \quad \text{since } s[n] \text{ and } e[n] \text{ are independent and stationary.} \\
&= \phi_{ss}[m] \quad \text{where we assumed } e[n] \text{ has zero mean.}
\end{aligned}$$

Taking the Fourier transform of the above equation, we get:

$$\Phi_{xs}(e^{j\omega}) = \Phi_{ss}(e^{j\omega}).$$

2.83. (Throughout this problem, we will assume $|a| < 1$.)

(a)

$$\phi_{hh}[m] = h[m] * h[-m].$$

Taking the Fourier transform, we get:

$$\begin{aligned}
\Phi_{hh}(e^{j\omega}) &= H(e^{j\omega})H(e^{-j\omega}) \\
&= \frac{1}{(1-ae^{-j\omega})}\frac{1}{(1-ae^{j\omega})} \\
&= \frac{1}{1-a^2}\left(\frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}}\right).
\end{aligned}$$

Taking the Inverse Fourier transform, we get:

$$\phi_{hh}[m] = \frac{a^{|n|}}{1-a^2}.$$

(b) Using part (a), we get:

$$\begin{aligned}
|H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\
&= H(e^{j\omega})H(e^{-j\omega}) \text{ since } h[n] \text{ is real} \\
&= \Phi_{hh}(e^{j\omega}) \\
&= \frac{1}{(1-ae^{-j\omega})}\frac{1}{(1-ae^{j\omega})} \\
&= \frac{1}{1-a^2}\left(\frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}}\right).
\end{aligned}$$

(c) Using Parseval's theorem:

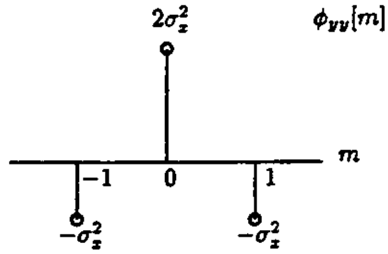
$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega &= \sum_{n=-\infty}^{+\infty} |h[n]|^2 \\
&= \sum_{n=-\infty}^{+\infty} |a|^{2n}u[n] \\
&= \sum_{n=0}^{+\infty} (|a|^2)^n \\
&= \frac{1}{1-|a|^2}.
\end{aligned}$$

2.84. The first-backward-difference system is given by:

$$y[n] = x[n] - x[n-1].$$

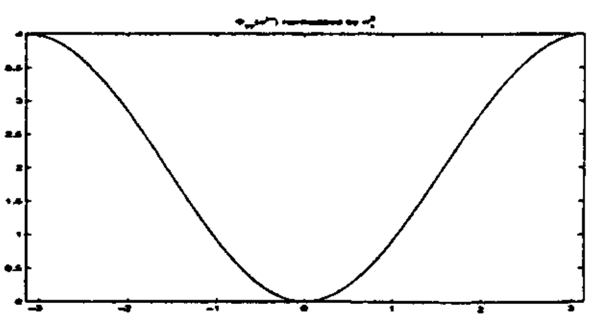
(a)

$$\begin{aligned}
\phi_{yy}[m] &= E(y[n]y[n+m]) \\
&= E((x[n]-x[n-1])(x[n+m]-x[n+m-1])) \\
&= E(x[n]x[n+m]) - E(x[n]x[n+m-1]) - E(x[n-1]x[n+m]) \\
&\quad + E(x[n-1]x[n+m-1]) \\
&= \phi_{xx}[m] - \phi_{xx}[m-1] - \phi_{xx}[m+1] + \phi_{xx}[m] \\
&= 2\phi_{xx}[m] - \phi_{xx}[m-1] - \phi_{xx}[m+1] \\
&= 2\sigma_x^2\delta[m] - \sigma_x^2\delta[m-1] - \sigma_x^2\delta[m+1].
\end{aligned}$$



To get the power spectrum, we take the Fourier transform of the autocorrelation function:

$$\begin{aligned}
\Phi_{yy}(e^{j\omega}) &= 2\sigma_x^2 - \sigma_x^2 e^{-j\omega} - \sigma_x^2 e^{j\omega} \\
&= 2\sigma_x^2 - 2\sigma_x^2 cos(\omega) \\
&= 2\sigma_x^2(1 - cos(\omega)).
\end{aligned}$$



(b) The average power of the output of the system is given by $\phi_{yy}[0]$:

$$\phi_{yy}[0] = 2\sigma_x^2.$$

(c) The noise power increased by going through the first-backward-difference system. This tells us that the first backward difference amplifies the noise of a signal.

2.85. (a)

$$\begin{aligned}
E\{x[n]y[n]\} &= E\{x[n]\sum_{k=-\infty}^{\infty} h[n]x[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} h[k]E\{x[n]x[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[k]
\end{aligned}$$

Because $x[n]$ is a real, stationary white noise process:

$$\phi_{xx}[n] = \sigma_x^2\delta[n].$$

Therefore,

$$\begin{aligned}
E\{x[n]y[n]\} &= \sigma_x^2 \sum_{k=-\infty}^{\infty} h[k]\delta[k] \\
&= \sigma_x^2 h[0].
\end{aligned}$$

(b) The variance of the output:

$$\begin{aligned}
\sigma_y^2 &= E\{(y[n]-m_y)^2\} \\
&= E\{y^2[n]\} - m_y^2.
\end{aligned}$$

When a zero-mean random process is input to a determistic LTI system, the output is also zero-mean:

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k].
\end{aligned}$$

Taking the expected value of both sides:

$$\begin{aligned}
m_y &= \sum_{k=-\infty}^{\infty} E\{x[n]\}h[n-k] \\
m_y &= 0, \quad \text{if } m_x = 0.
\end{aligned}$$

So,

$$\begin{aligned}
\sigma_y^2 &= E\{y^2[n]\} \\
&= E\left\{\sum_{m=-\infty}^{\infty} h[m]x[n-m] \sum_{k=-\infty}^{\infty} h[k]x[n-k]\right\} \\
&= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[m]h[k]E\{x[n-m]x[n-k]\} \\
&= \sigma_x^2 \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[m]h[k]\delta[m-k] \\
\sigma_y^2 &= \sigma_x^2 \sum_{m=-\infty}^{\infty} h^2[m].
\end{aligned}$$

2.86. Using the solution to problem 2.85:

(a)

$$\sigma_y^2 = \sigma_x^2 \sum_{k=-\infty}^{\infty} h_1^2[k]$$

this statement is TRUE, because $x[n]$ is a white noise sequence.

(b) Since $y[n]$ is not a white noise sequence, this statement is FALSE.

(c) Let

$$\begin{aligned} h_1[n] &= a^n u[n] \\ h_2[n] &= b^n u[n]. \end{aligned}$$

These systems are cascaded:

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] \\ &= \sum_{k=0}^{n} a^k b^{n-k}, \quad n \geq 0 \\ &= b^n \left(\frac{1-(a/b)^{n+1}}{1-(a/b)} \right) u[n] \end{aligned}$$

$$w[n] = x[n] * h[n].$$

Since $x[n]$ is zero-mean, $m_w = 0$ also.

$$\begin{aligned} \sigma_w^2 &= E\{w^2[n]\} \\ &= \sigma_x^2 \sum_{k=0}^{\infty} h^2[k]. \end{aligned}$$

2.87. (a) x[n] is a stationary white noise process.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad n \geq 0$$

$$\begin{aligned} E\{y[n]\} &= E\left\{ \sum_{k=-\infty}^{n} h[k]x[n-k] \right\}, \quad n \geq 0 \\ &= \sum_{k=-\infty}^{n} h[k]E\{x[n-k]\} \\ &= \begin{cases} m_x \sum_{k=-\infty}^{n} h[k], & n \geq 0 \\ 0, & n < 0 \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} \phi_{yy}[n_1, n_2] &= E\{y[n_1]y[n_2]\} \\ &= E\left\{ \sum_{k=-\infty}^{n_1} h[k]x[n_1-k] \sum_{m=-\infty}^{n_2} h[m]x[n_2-m] \right\}, \quad n \geq 0 \\ &= \sum_{k=-\infty}^{n_1} \sum_{m=-\infty}^{n_2} h[k]h[m]E\{x[n_1-k]x[n_2-m]\} \\ &= \sum_{k=-\infty}^{n_1} \sum_{m=-\infty}^{n_2} h[k]h[m]\phi_{xx}[n_1-k, n_2-m]. \end{aligned}$$

(c)

$$\begin{aligned}\lim_{n\to\infty} m_x \sum_{k=-\infty}^{n} h[k] &= m_x \sum_{k=-\infty}^{\infty} h[k] \\ &= m_y.\end{aligned}$$

$$\lim_{n_1,n_2\to\infty} \sum_{k=-\infty}^{n_1} \sum_{m=-\infty}^{n_2} h[k]h[m]\phi_{xx}[n_1-k, n_2-m] = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[k]h[m]\phi_{xx}[k,m].$$

(d)

$$h[n] = a^n u[n]$$

$$\begin{aligned}E\{y[n]\} &= m_x \sum_{k=-\infty}^{\infty} a^n u[n] \\ &= \frac{m_x}{1-a}.\end{aligned}$$

2.88. (a) No, the system is not linear. In the expression of $y[n]$, we have nonlinear terms such as $x^2[n]$ and divisions by $x[n]$, $x[n-1]$ and $x[n+1]$.

(b) Yes, the system is shift invariant. If we shift the input by n_0, $m_x[n]$ shifts by n_0 as well as $\sigma_x^2[n]$ and $\sigma_s^2[n]$, therefore $y[n]$ shifts by n_0 and the system is thus shift invariant.

(c) If $x[n]$ is bounded, $m_x[n]$ is bounded so is $\sigma_x^2[n]$ and $\sigma_s^2[n]$. As a result, $y[n]$ is bounded and therefore the system is stable.

(d) No, the system is not causal. Values of the output at time n depend on values of the input at time $n+1$ (through $\sigma_x^2[n]$ and $m_x[n]$). Since present values of the ouput depend of future values of the input, the system cannot be causal.

(e) When $\sigma_w^2[n]$ is very large, $\sigma_s^2[n]$ is zero, therefore:

$$\begin{aligned}y[n] &= m_x[n] \\ &= \frac{1}{3}\sum_{k=n-1}^{n+1} x[k]\end{aligned}$$

which is the average of the previous, present and next value of the input.

When $\sigma_w^2[n]$ is very small (approximately zero), then:

$$y[n] = x[n].$$

$y[n]$ makes sense for these extreme cases, because in very small noise power, the ouput is equal to the input since the noise is negligible. On the other hand, in very large noise power, the input is too noisy and so the output is an average of the input.

2.89. (a)

$$E\{x[n]x[n]\} = \phi_{xx}[0].$$

(b)

$$\begin{aligned}\Phi_{xx}(e^{j\omega}) &= X(e^{j\omega})X^*(e^{j\omega}) \\ &= W(e^{j\omega})H(e^{j\omega})W^*(e^{j\omega})H^*(e^{j\omega}) \\ &= \Phi_{ww}(e^{j\omega})|H(e^{j\omega})|^2 \\ &= \sigma_w^2 \frac{1}{1-\cos(\omega)+1/4}.\end{aligned}$$

(c)

$$\begin{aligned}
\phi_{xx}[n] &= \phi_{ww}[n] * h[n] * h[-n] \\
&= \sigma_w^2 \left(\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^{-n} u[-n] \right) \\
&= \sigma_w^2 \phi_{hh}[n].
\end{aligned}$$

2.90. (a)

$$\begin{aligned}
\phi_{yz}[n] &= E\{y[k]z[k-n]\} \\
&= E\left\{ \sum_{r=-\infty}^{\infty} h[r]x[k-r] \sum_{m=-\infty}^{\infty} h[m]v[k-n-m] \right\}
\end{aligned}$$

Note that $\phi_{xv}[n] = E\{x[p]v[p-\ell]\}$, therefore:

$$\begin{aligned}
\phi_{yz}[n] &= \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[r]h[m]E\{x[p]v[p-(n+\ell-s)]\} \\
&= h[-n] * h[n] * \phi_{xv}[n].
\end{aligned}$$

$$\Phi_{yz}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xv}(e^{j\omega}).$$

(b) No, consider $x[n]$ white and

$$v[n] = -x[n]$$
$$\phi_{xv}[n] = -\sigma_x^2 \delta[n]$$
$$\Phi_{xv}(e^{j\omega}) = -\sigma_x^2.$$

Noting that $|H(e^{j\omega})|^2$ is positive,

$$\Phi_{yz}(e^{j\omega}) = -\sigma_x^2 |H(e^{j\omega})|^2$$

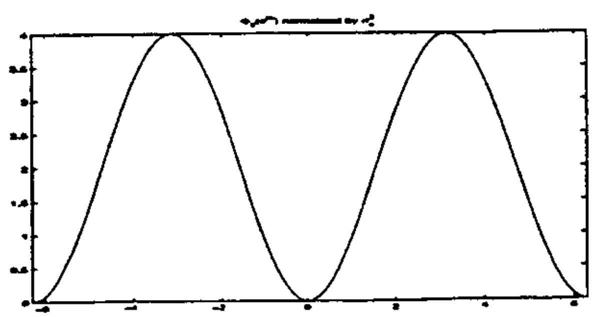
Hence, the cross power spectrum can be negative.

2.91. (a) Since $f[n] = e[n] - e[n-1]$,

$$H_1(e^{j\omega}) = 1 - e^{-j\omega}.$$

$\Phi_{ff}(e^{j\omega})$ is given by:

$$\begin{aligned}
\Phi_{ff}(e^{j\omega}) &= H_1(e^{j\omega})H_1(e^{-j\omega})\Phi_{ee}(e^{j\omega}) \\
&= (1-e^{-j\omega})(1-e^{j\omega})\sigma_e^2 \\
&= \sigma_e^2(2 - e^{j\omega} - e^{-j\omega}) \\
&= \sigma_e^2(2 - 2\cos(\omega)).
\end{aligned}$$

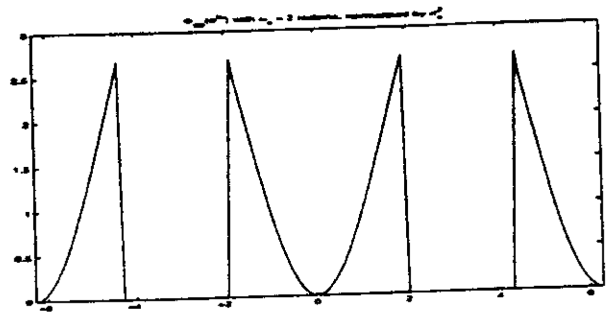


(b) $\phi_{ff}[m]$ is the inverse Fourier transform of $\Phi_{ff}(e^{j\omega})$. Using part (a), we get:

$$\phi_{ff}[m] = \sigma_e^2(2\delta[m] - \delta[m+1] - \delta[m-1]).$$

(c)

$$\begin{aligned}\Phi_{gg}(e^{j\omega}) &= H_2(e^{j\omega})H_2(e^{-j\omega})\Phi_{ff}(e^{j\omega}) \\ &= \begin{cases} \sigma_e^2(2 - 2\cos(\omega)) & , \quad |\omega| < \omega_c \\ 0 & , \quad \omega_c < |\omega| \leq \pi. \end{cases}\end{aligned}$$



(d)

$$\begin{aligned}\sigma_g^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{gg}(e^{j\omega})\,d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \sigma_e^2(2 - 2\cos(\omega))\,d\omega \\ &= \frac{2\sigma_e^2}{\pi}(\omega_c - \sin(\omega_c)).\end{aligned}$$