Solutions – Chapter 8

The Discrete-Time Fourier Transform

8.1. We sample a periodic continuous-time signal with a sampling rate:

$$F_s = \frac{\Omega s}{2\pi} = \frac{1}{T} = \frac{6}{10^{-3}}\text{Hz}$$

(a) The sampled signal is given by:

$$x[n] = x_c(nT)$$

Expressed as a Discrete Fourier Series:

$$x[n] = \sum_{k=-9}^{9} a_k e^{j\frac{2\pi}{6}kn}$$

We note that, in accordance with the discussion of Section 8.1, the sampled signal is represented by the summation of harmonically-related complex exponentials. The fundamental frequency of this set of exponentials is $2\pi/N$, where $N = 6$.
Therefore, the sequence $x[n]$ is periodic with period 6.

(b) For any bandlimited continuous-time signal, the Nyquist Criterion may be stated from Eq. (4.14b) as:

$$F_s \geq 2F_N,$$

where F_s is the sampling rate (Hz), and F_N corresponds to the highest frequency component in the signal (also Hz).
As evident by the finite Fourier series representation of $x_c(t)$, this continuous-time signal is, indeed, bandlimited with a maximum frequency of $F_n = \frac{9}{10^{-3}}$ Hz.
Therfore, by sampling at a rate of $F_s = \frac{6}{10^{-3}}$ Hz, the Nyquist Criterion is violated, and aliasing results.

(c) We use the analysis equation of Eq. (8.11):

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$

From part (a), $\tilde{x}[n]$ is periodic with $N = 6$.
Substitution yields:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{5}\left(\sum_{m=-9}^{9} a_m e^{j\frac{2\pi}{6}mn}\right)e^{-j\frac{2\pi}{6}kn} \\ &= \sum_{n=0}^{5}\sum_{m=-9}^{9} a_m e^{j(2\pi/6)(m-k)n}\end{aligned}$$

We reverse the order of the summations, and use the orthogonality relationship from Example 8.1:

$$\tilde{X}[k] = 6\sum_{m=-9}^{9} a_m \sum_{r=-\infty}^{\infty} \delta[m-k+rN]$$

Taking the infinite summation to the outside, we recognize the convolution between a_m and shifted impulses (Recall $a_m = 0$ for $|m| > 9$). Thus,

$$\tilde{X}[k] = 6\sum_{r=-\infty}^{\infty} a_{k-6r}$$

Note that from $\tilde{X}[k]$, the aliasing is apparent.

8.2. (a) Using the analysis equation of Eq. (8.11)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Since $\tilde{x}[n]$ is also periodic with period $3N$,

$$\begin{aligned} \tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \end{aligned}$$

Performing a change of variables in the second and third summations of $\tilde{X_3}[k]$,

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + W_{3N}^{kN} \sum_{n=0}^{N-1} \tilde{x}[n+N] W_{3N}^{kn} + W_{3N}^{2kN} \sum_{n=0}^{N-1} \tilde{x}[n+2N] W_{3N}^{kn}$$

Since $\tilde{x}[n]$ is periodic with period N, and $W_{3N}^{kn} = W_N^{(\frac{k}{3})n}$,

$$\begin{aligned} \tilde{X}_3[k] &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(\frac{k}{3}n)} \\ &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \tilde{X}[k] \\ &= \begin{cases} 3\tilde{X}[k/3], & k = 3\ell \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(b) Using $N = 2$ and $\tilde{x}[n]$ as in Fig P8.2-1:

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= \sum_{n=0}^{1} \tilde{x}[n] e^{-j\frac{2\pi}{2}kn} \\ &= \tilde{x}[0] + \tilde{x}[1] e^{-j\pi k} \\ &= 1 + 2(-1)^k \\ &= \begin{cases} 3, & k = 0 \\ -1, & k = 1 \end{cases} \end{aligned}$$

Observe, from Fig. P8.2-1, that $\tilde{x}[n]$ is also periodic with period $3N = 6$:

$$\begin{aligned} \tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{5} \tilde{x}[n] e^{-j\frac{\pi}{3}kn} \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})(1 + 2(-1)^{\frac{k}{3}}) \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})\tilde{X}[k/3] \\ &= \begin{cases} 9, & k = 0 \\ -3, & k = 3 \\ 0, & k = 1, 2, 4, 5. \end{cases} \end{aligned}$$

8.3. (a) The DFS coefficients will be real if $\tilde{x}[n]$ is even. Only signal B can be even (i.e., $\tilde{x}_B[n] = \tilde{x}_B[-n]$; if the origin is selected as the midpoint of either the nonzero block, or the zero block).

(b) The DFS coefficients will be imaginary if $\tilde{x}[n]$ is even. None of the sequences in Fig P8.3-1 can be odd.

(c) We use the analysis equation, Eq. (8.11) and the closed form expression for a geometric series. Assuming unit amplitudes and discarding DFS points which are zero:

$$\begin{aligned}
\tilde{X}_A[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1-e^{j\frac{\pi}{4}4k}}{1-e^{j\frac{\pi}{4}k}} \\
&= \frac{1-(-1)^k}{1-e^{j\frac{\pi}{4}k}} = 0, k = \pm 2, \pm 4, \ldots \\
\tilde{X}_B[k] &= \sum_{n=0}^{2} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1-e^{j\frac{\pi}{4}3k}}{1-e^{j\frac{\pi}{4}k}} \\
\tilde{X}_C[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} - \sum_{n=4} 7e^{j\frac{2\pi}{8}kn} \\
&= \sum_{n=0}^{3} \left(e^{j\frac{\pi}{4}kn} - e^{j\frac{\pi}{4}k(n+4)}\right) \\
&= \left(1-e^{j\pi k}\right)\frac{1-e^{j\pi k}}{1-e^{j\frac{\pi}{4}k}} \\
&= 0, \quad k = \pm 2, \pm 4, \ldots
\end{aligned}$$

8.4. A periodic sequence is constructed from the sequence:

$$x[n] = \alpha^n u[n] \, , \, |\alpha| < 1$$

as follows:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN] \, , \, |\alpha| < 1$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\
&= \frac{1}{1-\alpha e^{-j\omega}}, \quad |\alpha| < 1
\end{aligned}$$

(b) The DFS of $\tilde{x}[n]$:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x[n+rN] W_N^{kn} \\
&= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} \alpha^{n+rN} u[n+rN] W_N^{kn} \\
&= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \alpha^{n+rN} W_N^{kn}
\end{aligned}
$$

Rearranging the summations gives:

$$
\begin{aligned}
\tilde{X}[k] &= \sum_{r=0}^{\infty} \alpha^{rN} \sum_{n=0}^{N-1} \alpha^n W_N^{kn} \\
&= \sum_{r=0}^{\infty} \alpha^{rN} \left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}} \right), \ |\alpha| < 1 \\
&= \frac{1}{1-\alpha^N} \left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}} \right), \ |\alpha| < 1 \\
\tilde{X}[k] &= \frac{1}{1-\alpha e^{-j(2\pi k/N)}}, \ |\alpha| < 1
\end{aligned}
$$

(c) Comparing the results of part (a) and part (b):

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N}.$$

8.5. (a)

$$
\begin{aligned}
x[n] &= \delta[n] \\
X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
x[n] &= \delta[n-n_0], \quad 0 \le n_0 \le (N-1) \\
X[k] &= \sum_{n=0}^{N-1} \delta[n-n_0] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= W_N^{kn_0}
\end{aligned}
$$

(c)

$$
\begin{aligned}
x[n] &= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{(N/2)-1} W_N^{2kn} \\
&= \frac{1-e^{-j2\pi k}}{1-e^{-j(\pi k/N)}} \\
X[k] &= \begin{cases} N/2, & k = 0, N/2 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

(d)

$$\begin{aligned}
x[n] &= \begin{cases} 1, & 0 \le n \le ((N/2)-1) \\ 0, & N/2 \le n \le (N-1) \end{cases} \\
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{(N/2)-1} W_N^{kn} \\
&= \frac{1-e^{-j\pi k}}{1-e^{-j(2\pi k)/N}} \\
X[k] &= \begin{cases} N/2, & k=0 \\ \dfrac{2}{1-e^{-j(2\pi k/N)}}, & k \text{ odd} \\ 0, & k \text{ even}, 0 \le k \le (N-1) \end{cases}
\end{aligned}$$

(e)

$$\begin{aligned}
x[n] &= \begin{cases} a^n, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases} \\
X[k] &= \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j(2\pi k)/N}} \\
X[k] &= \frac{1-a^N}{1-ae^{-j(2\pi k)/N}}
\end{aligned}$$

8.6. Consider the finite-length sequence

$$x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} e^{j\omega_0 n} e^{-j\omega n} \\
X(e^{j\omega}) &= \frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}} \\
&= \frac{e^{-j(\omega-\omega_0)(N/2)}}{e^{-j(\omega-\omega_0)/2}} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right) \\
X(e^{j\omega}) &= e^{-j(\omega-\omega_0)((N-1)/2)} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right)
\end{aligned}$$

(b) N-point DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1)$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} e^{j\omega_0 n} W_N^{kn} \\
&= \frac{1 - e^{-j((2\pi k/N)-\omega_0)N}}{1 - e^{-j((2\pi k/N)-\omega_0)}} \\
&= e^{-j\left(\frac{2\pi k}{N}-\omega_0\right)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)/2\right]}
\end{aligned}
$$

Note that $X[k] = X(e^{j\omega})|_{\omega=(2\pi k)/N}$

(c) Suppose $\omega_0 = (2\pi k_0)/N$, where k_0 is an integer:

$$
\begin{aligned}
X[k] &= \frac{1 - e^{-j(k-k_0)2\pi}}{1 - e^{-j(k-k_0)(2\pi)/N}} \\
&= e^{-j(2\pi/N)(k-k_0)((N-1)/2)} \frac{\sin \pi(k-k_0)}{\sin(\pi(k-k_0)/N)}
\end{aligned}
$$

8.7. We have a six-point uniform sequence, $x[n]$, which is nonzero for $0 \le n \le 5$. We sample the Z-transform of $x[n]$ at four equally-spaced points on the unit circle.

$$X[k] = X(z)|_{z=e^{(2\pi k/4)}}$$

We seek the sequence $x_1[n]$ which is the inverse DFT of $X[k]$. Recall the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is zero for all n outside $0 \le n \le 5$, we may replace the infinite summation with a finite summation. Furthermore, after substituting $z = e^{j(2\pi k/4)}$, we obtain

$$X[k] = \sum_{n=0}^{5} x[n]W_4^{kn}, \quad 0 \le k \le 4$$

Note that we have taken a 4-point DFT, as specified by the sampling of the Z-transform; however, the original sequence was of length 6. As a result, we can expect some aliasing when we return to the time domain via the inverse DFT.

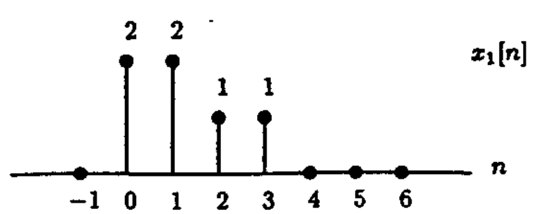
Performing the DFT,

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}, \quad 0 \le k \le 4$$

Taking the inverse DFT by inspection, we note that there are six impulses (one for each value of n above). However,

$$W_4^{4k} = W_4^{0k} \text{ and } W_4^{5k} = W_4^{k},$$

so two points are aliased. The resulting time-domain signal is



8.8. Fourier transform of $x[n] = (1/2)^n u[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Now, sample the frequency spectra of $x[n]$:

$$Y[k] = X(e^{j\omega})|_{\omega=2\pi k/10}, \quad 0 \le k \le 9$$

We have the 10-pt DFT:

$$\begin{aligned} Y[k] &= \frac{1}{1-\frac{1}{2}e^{-j(2\pi k/10)}}, \quad 0 \le k \le 9 \\ &= \sum_{n=0}^{9} y[n]W_{10}^{kn} \end{aligned}$$

Recall:

$$\left(\frac{1}{2}\right)^n \xrightarrow[\text{N-pt}]{\text{DFT}} \frac{1-(\frac{1}{2})^N}{1-\frac{1}{2}e^{-j(2\pi k/N)}}$$

So, we may infer:

$$y[n] = \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})^{10}}, \quad 0 \le n \le 9$$

8.9. Given a 20-pt finite-duration sequence $x[n]$:

(a) We wish to obtain $X(e^{j\omega})|_{\omega=4\pi/5}$ using the smallest DFT possible. A possible size of the DFT is evident by the periodicity of $e^{j\omega}|_{\omega=4\pi/5}$. Suppose we choose the size of the DFT to be $M = 5$. The data sequence is 20 points long, so we use the time-aliasing technique derived in the previous problem. Specifically, we alias $x[n]$ as:

$$x_1[n] = \sum_{r=-\infty}^{\infty} x[n+5r]$$

This aliased version of $x[n]$ is periodic with period 5 now. The 5-pt DFT is computed. The desired value occurs at a frequency corresponding to:

$$\frac{2\pi k}{N} = \frac{4\pi}{5}$$

For $N = 5$, $k = 2$, so the desired value may be obtained as $X[k]|_{k=2}$.

(b) Next, we wish to obtain $X(e^{j\omega})|_{\omega=10\pi/27}$.
The smallest DFT is of size $L = 27$. Since the DFT is larger than the data block size, we pad $x[n]$ with 7 zeros as follows:

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le 19 \\ 0, & 20 \le n \le 26 \end{cases}$$

We take the 27-pt DFT, and the desired value corresponds to $X[k]$ evaluated at $k = 5$.

8.10. From Fig P8.10-1, the two 8-pt sequences are related through a circular shift. Specifically,

$$x_2[n] = x_1[((n-4))_8]$$

From property 5 in Table 8.2,

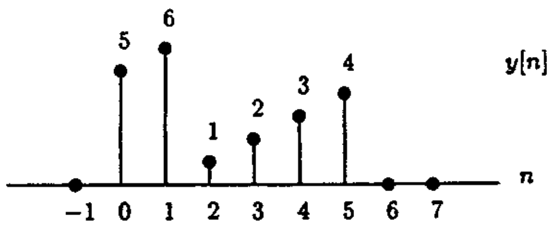
$$\mathrm{DFT}\{x_1[((n-4))_8]\} = W_8^{4k} X_1[k]$$

Thus,

$$\begin{aligned} X_2[k] &= W_8^{4k} X_1[k] \\ &= e^{-j\pi k} X_1[k] \\ X_2[k] &= (-1)^k X_1[k] \end{aligned}$$

8.11. We wish to perform the circular convolution between two 6-pt sequences. Since $x_2[n]$ is just a shifted impulse, the circular-convolution coincides with a circular shift of $x_1[n]$ by two points.

$$\begin{aligned} y[n] &= x_1[n] ⑥ x_2[n] \\ &= x_1[n] ⑥ \delta[n-2] \\ &= x_1[((n-2))_6] \end{aligned}$$



8.12. (a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \le n \le 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \le k \le 3$$

The cosine term contributes only two non-zero values to the summation, giving:

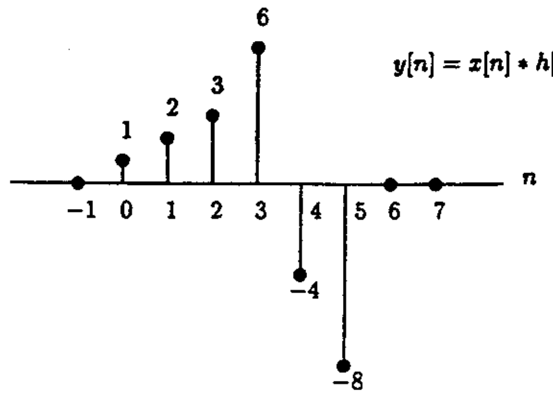
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \le k \le 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

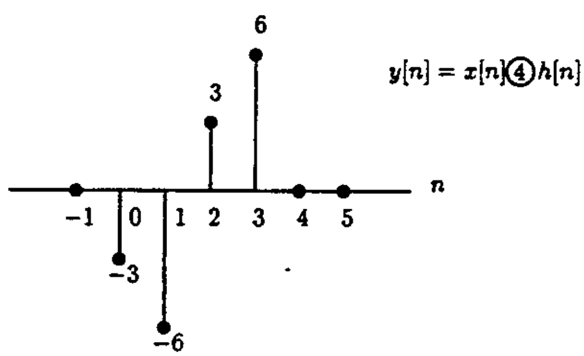
$$h[n] = 2^n, \quad 0 \le n \le 3$$

$$\begin{aligned} H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} \end{aligned}$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \geq 3+4-1=6$ to avoid aliasing. Since $N=4$,we expect to get aliasing here. First, find $y[n]=x[n]*h[n]$:



For this problem, aliasing means the last three points ($n=4,5,6$) will wrap-around on top of the first three points, giving $y[n]=x[n] \circledast_4 h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= 1+2W_4^k+4W_4^{2k}+8W_4^{3k}-W_4^{2k}-2W_4^{3k}-4W_4^{4k}-8W_4^{5k} \end{aligned}$$

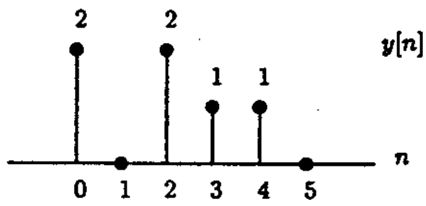
Since $W_4^{4k}=W_4^{0k}$ and $W_4^{5k}=W_4^k$

$$Y[k] = -3-6W_4^k+3W_4^{2k}+6W_4^{3k}, \quad 0 \leq k \leq 3$$

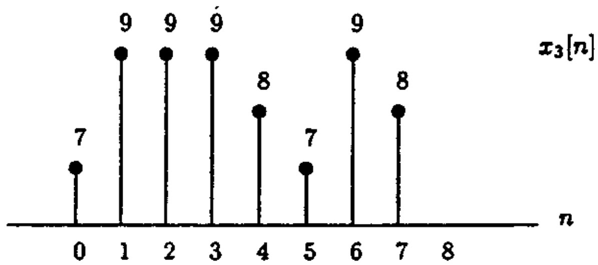
Taking the inverse DFT:

$$y[n]=-3\delta[n]-6\delta[n-1]+3\delta[n-2]+6\delta[n-3], \quad 0 \leq n \leq 3$$

8.13. Using the properties of the DFT, we get $y[n] = x[((n-2))_5]$, that is $y[n]$ is equal to $x[n]$ circularly shifted by 2. We get:

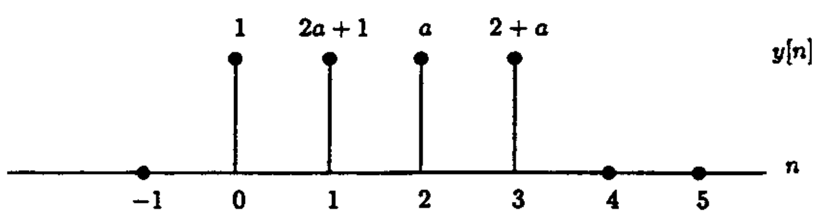


8.14. $x_3[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 8$. Carrying out the 8-point circular convolution, we get:



We thus conclude $x_3[2] = 9$.

8.15. $y[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 4$. Carrying out the 4-point circular convolution, we get:



Matching the above sequence to the one given, we get $a = -1$, which is unique.

8.16. $X_1[k]$ is the 4-point DFT of $x[n]$ and $x_1[n]$ is the 4-point inverse DFT of $X_1[k]$, therefore $x_1[n]$ is $x[n]$ time aliased to $N = 4$. In other words, $x_1[n]$ is one period of $\tilde{x}[n] = x[((n))_4]$. We thus have:

$$4 = b + 1.$$

Therefore, $b = 3$. This is clearly unique.

8.17. Looking at the sequences, we see that $x_1[n] * x_2[n]$ is non-zero for $1 \leq n \leq 8$. The smallest N such that $x_1[n] \circledN x_2[n] = x_1[n] * x_2[n]$ is therefore $N = 9$.

8.18. Taking the inverse DFT of $X_1[k]$ and using the properties of the DFT, we get:

$$x_1[n] = x[((n+3))_5].$$

Therefore:

$$x_1[0] = x[3] = c.$$

We thus conclude that $c = 2$.

8.19. $x_1[n]$ and $x[n]$ are related by a circular shift as can be seen from the plots. Using the properties of the DFT and the relationship between $X_1[k]$ and $X[k]$, we have:

$$x_1[n] = x[((n-m))_6].$$

$m = 2$ works, clearly this choice is not unique, any $m = 2 + 6l$, where l is an integer, would work.

8.20.

$$X_1[k] = X[k]e^{+j(2\pi k2/N)}.$$

Using the properties of the DFT, we get:

$$x_1[n] = x[((n+2))_N].$$

From the figures, we conclude that:

$$N = 5.$$

This choice of N is unique.

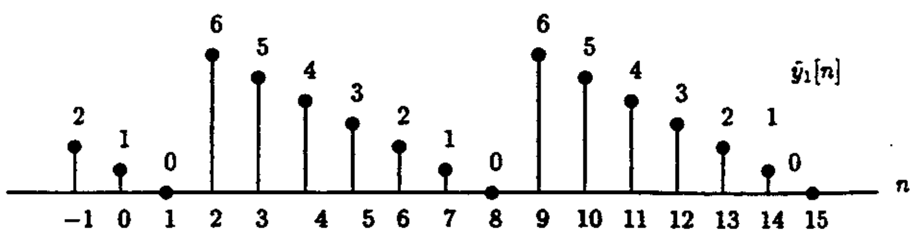
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\tilde{X}_1[k] = \sum_{n=0}^{6} \tilde{x}_1[n]W_7^{kn}$$

$$
\begin{aligned}
&= 6 + 5W_7^{k} + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n] W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}
$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$
\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n] W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}
$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

8.22. For a finite-length sequence $x[n]$, with length equal to N, the periodic repetition of $x[-n]$ is represented by

$$x[((-n))_N] = x[((-n+\ell N))_N], \qquad \ell\text{: integer}$$

where the right side is justified since $x[n]$ (and $x[-n]$) is periodic with period N.

The above statement holds true for any choice of ℓ. Therefore, for $\ell = 1$:

$$x[((-n))_n] = x[((-n+N))_N]$$

8.23. We have $x[n]$ for $0 \le n \le P$.

We desire to compute $X(z)|_{z=e^{-j(2\pi k/N)}}$ using one N-pt DFT.

(a) Suppose $N > P$ (the DFT size is larger than the data segment). The technique used in this case is often referred to as zero-padding. By appending zeros to a small data block, a larger DFT may be used. Thus the frequency spectra may be more finely sampled. It is a common misconception to believe that zero-padding enhances spectral resolution. The addition of a larger block of data to a larger DFT would enhance this quality.

So, we append $N_z = N - P$ zeros to the end of the sequence as follows:

$$x'[n] = \begin{cases} x[n], & 0 \le n \le (P-1) \\ 0, & P \le n \le N \end{cases}$$

$$x[n] \longrightarrow \boxed{N\text{-pt DFT}} \longrightarrow X[k] \quad (0 \le k \le N)$$

(b) Suppose $N > P$, consider taking a DFT which is smaller than the data block. Of course, some aliasing is expected. Perhaps we could introduce time aliasing to offset the effects.
Consider the N-pt inverse DFT of $X[k]$,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \le n \le (N-1)$$

Suppose $X[k]$ was obtained as the result of an infinite summation of complex exponents:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{\infty} x[m]e^{-j(2\pi k/N)m}\right) W_N^{-kn}$$

Rearrange to get:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(2\pi/N)(m-n)k}\right)$$

Using the orthogonality relationship of Example 8.1:

$$\begin{aligned}
x[n] &= \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} \delta[m-n+rN] \\
x[n] &= \sum_{r=-\infty}^{\infty} x[n-rN]
\end{aligned}$$

So, we should alias $x[n]$ as above. Then we take the N-pt DFT to get $X[k]$.

8.24. No. Recall that the DFT merely samples the frequency spectra. Therefore, the fact the $Im\{X[k]\} = 0$ for $0 \le k \le (N-1)$ does not guarantee that the imaginary part of the continuous frequency spectra is also zero.

For example, consider a signal which consists of an impulse centered at $n = 1$.

$$x[n] = \delta[n-1], \quad 0 \le n \le 1$$

The Fourier transform is:

$$\begin{aligned}
X(e^{j\omega}) &= e^{-j\omega} \\
Re\{X(e^{j\omega})\} &= \cos(\omega) \\
Im\{X(e^{j\omega})\} &= -\sin(\omega)
\end{aligned}$$

Note that neither is zero for all $0 \le \omega \le 2$. Now, suppose we take the 2-pt DFT:

$$\begin{aligned}
X[k] &= W_2^k, \quad 0 \le k \le 1 \\
&= \begin{cases} 1, & k = 0 \\ -1, & k = 1 \end{cases}
\end{aligned}$$

So, $Im\{X[k]\} = 0, \quad \forall k$. However, $Im\{X(e^{j\omega})\} \ne 0$.

Note also that the size of the DFT plays a large role. For instance, consider taking the 3-pt DFT of

$$\begin{aligned}
x[n] &= \delta[n-1], \quad 0 \le n \le 2 \\
X[k] &= W_3^k, \quad 0 \le k \le 2 \\
&= \begin{cases} 1, & k = 0 \\ e^{-j(2\pi/3)}, & k = 1 \\ e^{-j(4\pi/3)}, & k = 2 \end{cases}
\end{aligned}$$

Now, $Im\{X[k]\} \neq 0$, for $k = 1$ or $k = 2$.

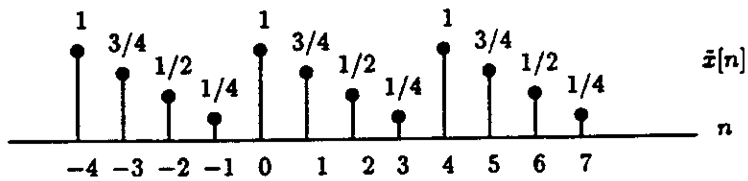
8.25. Both sequences $x[n]$ and $y[n]$ are of finite-length ($N = 4$).

Hence, no aliasing takes place. From Section 8.6.2, multiplication of the DFT of a sequence by a complex exponential corresponds to a circular shift of the time-domain sequence.

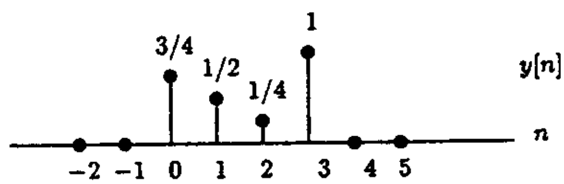
Given $Y[k] = W_4^{3k} X[k]$, we have

$$y[n] = x[((n-3))_4]$$

We use the technique suggested in problem 8.28. That is, we temporarily extend the sequence such that a periodic sequence with period 4 is formed.



Now, we shift by three (to the right), and set all values outside $0 \leq n \leq 3$ to zero.



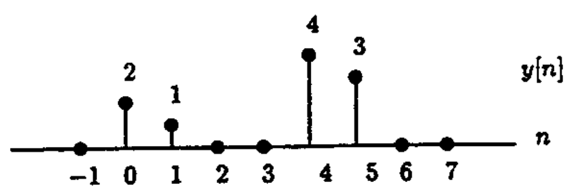
8.26. (a) When multiplying the DFT of a sequence by a complex exponential, the time-domain signal undergoes a circular shift.

For this case,

$$Y[k] = W_6^{4k} X[k], \quad 0 \leq k \leq 5$$

Therefore,

$$y[n] = x[((n-4))_6], \quad 0 \leq n \leq 5$$



(b) There are two ways to approach this problem. First, we attempt a solution by brute force.

$$\begin{aligned}
X[k] &= 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \quad W_6^k = e^{-j(2\pi k/6)} \text{ and } 0 \leq k \leq 5 \\
W[k] &= \mathcal{R}e\{X[k]\} \\
&= \frac{1}{2}(X[k] + X^*[k]) \\
&= \frac{1}{2}(4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 4 + 3W_6^{-k} + 2W_6^{-2k} + W_6^{-3k})
\end{aligned}$$

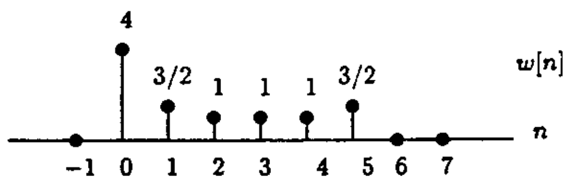
Notice that

$$
\begin{aligned}
W_N^k &= e^{-j(2\pi k/N)} \\
W_N^{-k} &= e^{j(2\pi k/N)} = e^{-j(2\pi/N)(N-k)} = W_N^{N-k} \\
W[k] &= 4 + \frac{3}{2}\left[W_6^k + W_6^{6-k}\right] + \left[W_6^{2k} + W_6^{6-2k}\right] + \frac{1}{2}\left[W_6^{3k} + W_6^{6-3k}\right], \quad 0 \le k \le 5
\end{aligned}
$$

So,

$$
\begin{aligned}
w[n] &= 4\delta[n] + \frac{3}{2}\Big(\delta[n-1] + \delta[n-5]\Big) + \delta[n-2] + \delta[n-4] \\
&\quad + \frac{1}{2}\Big(\delta[n-3] + \delta[n-3]\Big) \\
w[n] &= 4\delta[n] + \frac{3}{2}\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \frac{3}{2}\delta[n-5], \quad 0 \le n \le 5
\end{aligned}
$$

Sketching $w[n]$:

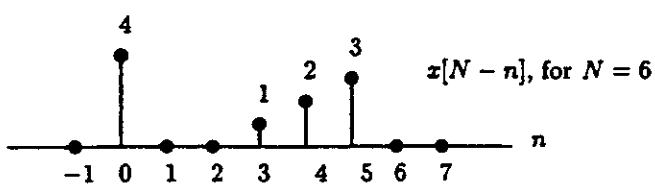


As an alternate approach, suppose we use the properties of the DFT as listed in Table 8.2.

$$
\begin{aligned}
W[k] &= \mathcal{R}e\{X[k]\} \\
&= \frac{X[k] + X^*[k]}{2} \\
w[n] &= \frac{1}{2}\,\text{IDFT}\{X[k]\} + \frac{1}{2}\,\text{IDFT}\{X^*[k]\} \\
&= \frac{1}{2}\Big(x[n] + x^*[((-n))_N]\Big)
\end{aligned}
$$

For $0 \le n \le N-1$ and $x[n]$ real:

$$
w[n] = \frac{1}{2}\Big(x[n] + x[N-n]\Big)
$$



So, we observe that $w[n]$ results as above.

(c) The DFT is decimated by two. By taking alternate points of the DFT output, we have half as many points. The influence of this action in the time domain is, as expected, the appearance of aliasing. For the case of decimation by two, we shall find that an additional replica of x[n] surfaces, since the sequence is now periodic with period 3.

From part (b):

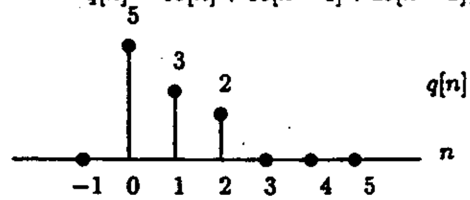
$$X[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \quad 0 \le k \le 5$$

Let $Q[k] = X[2k]$,

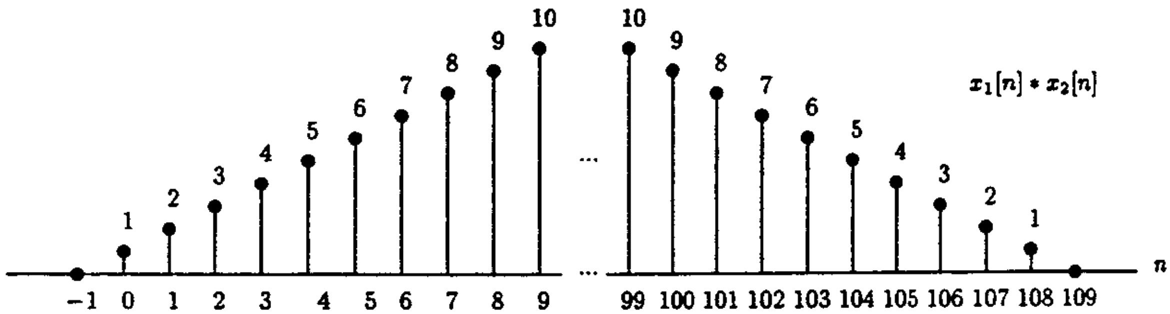
$$Q[k] = 4 + 3W_3^k + 2W_3^{2k} + W_3^{3k}, \quad 0 \le k \le 2$$

Noting that $W_3^{3k} = W_3^{0k}$

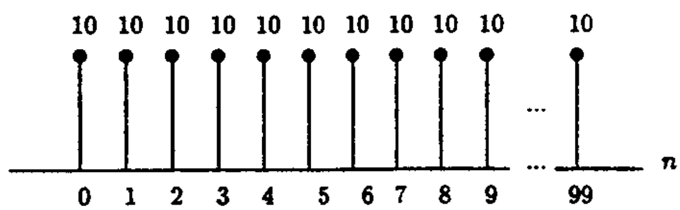
$$q[n] = 5\delta[n] + 3\delta[n-1] + 2\delta[n-2], \quad 0 \le n \le 2$$



8.27. (a) The linear convolution, $x_1[n] * x_2[n]$ is a sequence of length $100 + 10 - 1 = 109$.

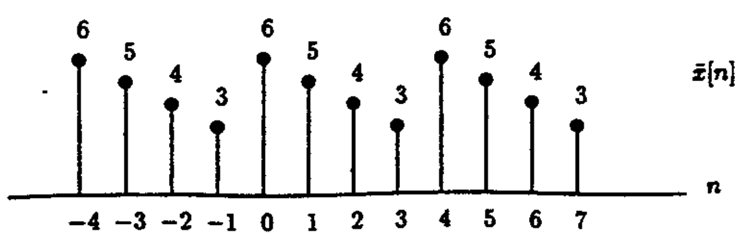


(b) The circular convolution, $x_1[n]$ (100) $x_2[n]$, can be obtained by aliasing the first 9 points of the linear convolution above:

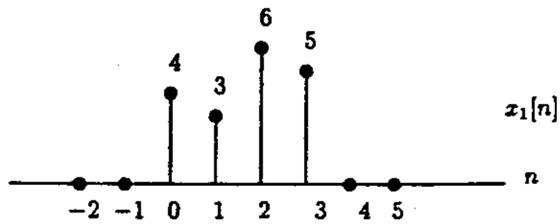


(c) Since $N \ge 109$, the circular convolution $x_1[n]$ (110) $x_2[n]$ will be equivalent to the linear convolution of part (a).

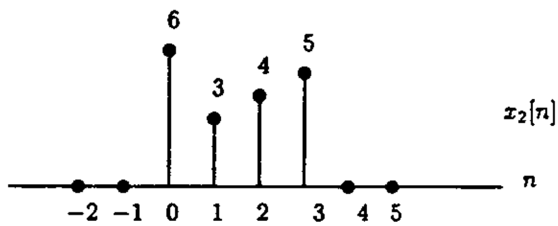
8.28. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \le n \le 3$):



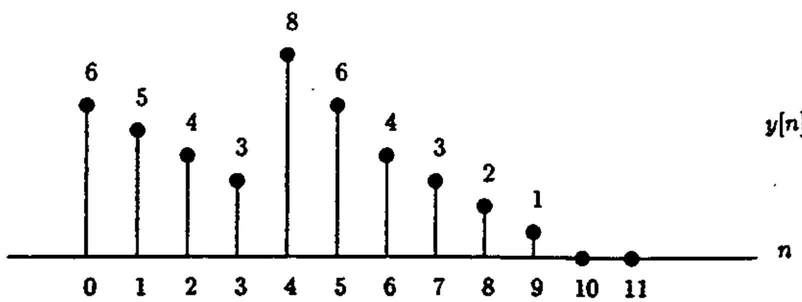
To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \le n \le 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

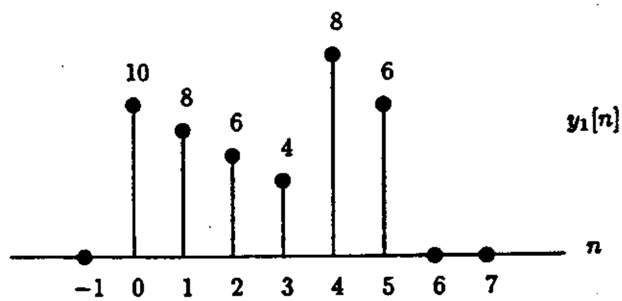
In the second approach, we work with the given signal. The signal is confined to $0 \le n \le 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.29. Circular convolution equals linear convolution plus aliasing. First, we find $y[n] = x_1[n] * x_2[n]$:



Note that $y[n]$ is a ten point sequence ($N = 6 + 5 - 1$).

(a) For $N = 6$, the last four non-zero point ($6 \le n \le 9$) will alias to the first four points, giving us $y_1[n] = x_1[n]⑥x_2[n]$



(b) For $N = 10, N \geq 6 + 5 - 1$, so no aliasing occurs, and circular convolution is identical to linear convolution.

8.30. We have a finite length sequence, whose 64-pt DFT contains only one nonzero point (for $k = 32$).

(a) Using the synthesis equation Eq. (8.68):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \leq n \leq (N-1)$$

Substitution yields:

$$\begin{aligned}
x[n] &= \frac{1}{64}X[32]W_{64}^{-32n} \\
&= \frac{1}{64}e^{j\frac{2\pi}{64}(32)n} \\
&= \frac{1}{64}e^{j\pi n} \\
x[n] &= \frac{1}{64}(-1)^n, \quad 0 \leq n \leq (N-1)
\end{aligned}$$

The answer is unique because we have taken the 64-pt DFT of a 64-pt sequence.

(b) The sequence length is now $N = 192$.

$$\begin{aligned}
x[n] &= \frac{1}{192}\sum_{k=0}^{191} X[k]W_{192}^{-kn}, \; 0 \leq n \leq 191 \\
x[n] &= \begin{cases} \frac{1}{64}(-1)^n & 0 \leq n \leq 63 \\ 0 & 64 \leq n \leq 191 \end{cases}
\end{aligned}$$

This solution is not unique. By taking only 64 spectral samples, $x[n]$ will be aliased in time. As an alternate sequence, consider

$$x'[n] = \frac{1}{64}\left(\frac{1}{3}\right)(-1)^n, \quad 0 \leq n \leq 191$$

8.31. We have a 10-point sequence, $x[n]$. We want a modified sequence, $x_1[n]$, such that the 10-pt. DFT of $x_1[n]$ corresponds to

$$X_1[k] = X(z)|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}}$$

Recall the definition of the Z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is of finite duration ($N = 10$), we assume:

$$x[n] = \begin{cases} \text{nonzero}, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$X(z) = \sum_{n=0}^{9} x[n] z^{-n}$$

Substituting in $z = \frac{1}{2} e^{j[(2\pi k/10)+(\pi/10)]}$:

$$X(z)\big|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}} = \sum_{n=0}^{9} x[n] \left(\frac{1}{2} e^{j[(2\pi k/10)+(\pi/10)]}\right)^{-n}$$

We seek the signal $x_1[n]$, whose 10-pt. DFT is equivalent to the above expression. Recall the analysis equation for the DFT:

$$X_1[k] = \sum_{n=0}^{9} x_1[n] W_{10}^{kn}, \quad 0 \le k \le 9$$

Since $W_{10}^{kn} = e^{-j(2\pi/10)kn}$, by comparison

$$x_1[n] = x[n] \left(\frac{1}{2} e^{j(\pi/10)}\right)^{-n}$$

8.32. We have a finite-length sequence, $x[n]$ with $N = 8$. Suppose we interpolate by a factor of two. That is, we wish to double the size of $x[n]$ by inserting zeros at all odd values of n for $0 \le n \le 15$.

Mathematically,

$$y[n] = \begin{cases} x[n/2], & n \text{ even}, \quad 0 \le n \le 15 \\ 0, & n \text{ odd}, \end{cases}$$

The 16-pt. DFT of $y[n]$:

$$\begin{aligned} Y[k] &= \sum_{n=0}^{15} y[n] W_{16}^{kn}, \quad 0 \le k \le 15 \\ &= \sum_{n=0}^{7} x[n] W_{16}^{2kn} \end{aligned}$$

Recall, $W_{16}^{2kn} = e^{j(2\pi/16)(2k)n} = e^{-j(2\pi/8)kn} = W_8^{kn}$,

$$Y[k] = \sum_{n=0}^{7} x[n] W_8^{kn}, \quad 0 \le k \le 15$$

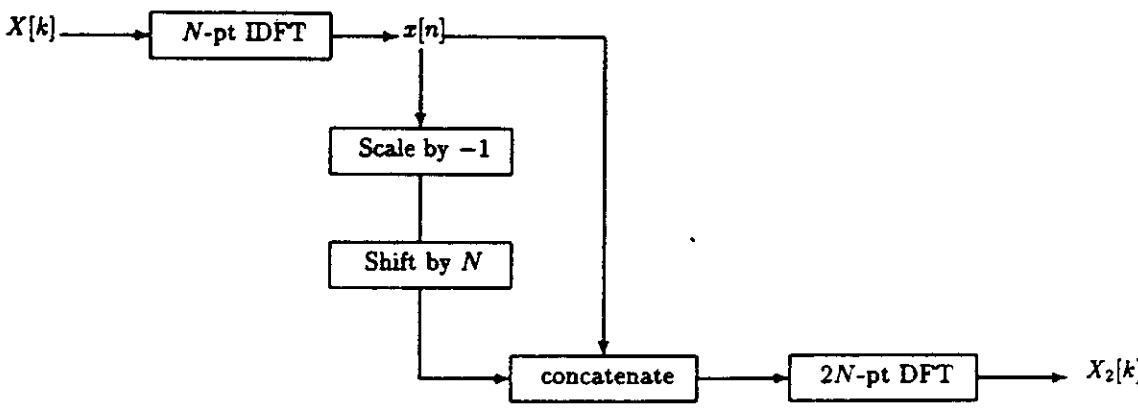
Therefore, the 16-pt. DFT of the interpolated signal contains two copies of the 8-pt. DFT of $x[n]$. This is expected since $Y[k]$ is now periodic with period 8 (see problem 8.1). Therefore, the correct choice is C.

As a quick check, $Y[0] = X[0]$.

8.33. (a) Since

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ -x[n-N], & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

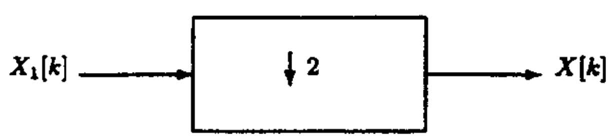
If $X[k]$ is known, $x_2[n]$ can be constructed by :



(b) To obtain $X[k]$ from $X_1[k]$, we might try to take the inverse DFT (2N-pt) of $X_1[k]$, then take the N-pt DFT of $x_1[n]$ to get $X[k]$.
However, the above approach is highly inefficient. A more reasonable approach may be achieved if we examine the DFT analysis equations involved. First,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{2N-1} x_1[n] W_{2N}^{kn}, \qquad 0 \le k \le (2N-1) \\ &= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{(k/2)n}, \quad 0 \le k \le (N-1) \\ X_1[k] &= X[k/2], \quad 0 \le k \le (N-1) \end{aligned}$$

Thus, an easier way to obtain $X[k]$ from $X_1[k]$ is simply to decimate $X_1[k]$ by two.



8.34. (a) The DFT of the even part of a real sequence:

If $x[n]$ is of length N, then $x_e[n]$ is of length $2N-1$:

$$
\begin{aligned}
x_e[n] &= \begin{cases} \dfrac{x[n]+x[-n]}{2}, & (-N+1) \le n \le (N-1) \\ 0 & \text{otherwise} \end{cases} \\
X_e[k] &= \sum_{n=-N+1}^{N-1} \left(\frac{x[n]+x[-n]}{2}\right) W_{2N-1}^{kn}, \qquad (-N+1) \le k \le (N-1) \\
&= \sum_{n=-N+1}^{0} \frac{x[-n]}{2} W_{2N-1}^{kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn}
\end{aligned}
$$

Let $m=-n$,

$$
\begin{aligned}
X_e[k] &= \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{-kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn} \\
X_e[k] &= \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{2N-1}\right)
\end{aligned}
$$

Recall

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le (N-1)$$

and

$$Re\{X[k]\} = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{N}\right)$$

So: $\text{DFT}\{x_e[n]\} \neq Re\{X[k]\}$

(b)

$$
\begin{aligned}
Re\{X[k]\} &= \frac{X[k]+X^*[k]}{2} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} x[n] W_N^{kn} + \frac{1}{2}\sum_{n=0}^{N-1} x[n] W_N^{-kn} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} (x[n] + x[N-n]) W_N^{kn}
\end{aligned}
$$

So,

$$Re\{X[k]\} = \text{DFT}\left\{\frac{1}{2}(x[n] + x[N-n])\right\}$$

8.35. From condition 1, we can determine that the sequence is of finite length ($N=5$). Given:

$$
\begin{aligned}
X(e^{j\omega}) &= 1 + A_1 \cos\omega + A_2 \cos 2\omega \\
&= 1 + \frac{A_1}{2}(e^{j\omega} + e^{-j\omega}) + \frac{A_2}{2}(e^{j2\omega} + e^{-j2\omega})
\end{aligned}
$$

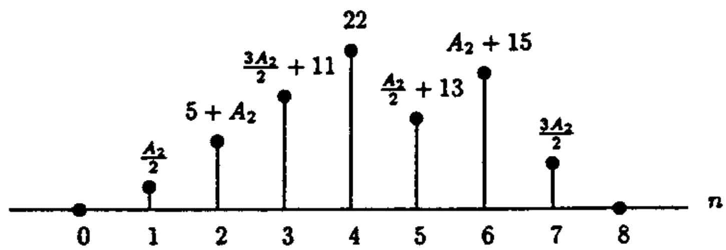
From the Fourier analysis equation, we can see by matching terms that:

$$x[n] = \delta[n] + \frac{A_1}{2}(\delta[n-1] + \delta[n+1]) + \frac{A_2}{2}(\delta[n-2] + \delta[n+2])$$

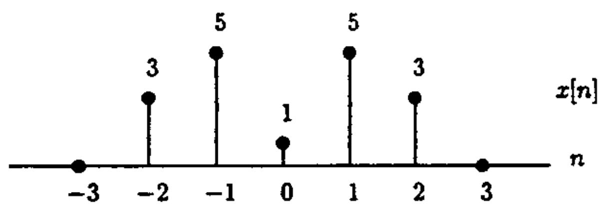
Condition 2 yields one of the values for the amplitude constants of condition 1. Since $x[n] * \delta[n-3] = x[n-3] = 5$ for $n = 2$, we know $x[-1] = 5$, and also that $x[1] = x[-1] = 5$. Knowing both these values tells us that $A_1 = 10$.

For condition 3, we perform a circular convolution between $x[((n-3))_8]$ and $w[n]$, a three-point sequence. For this case, linear convolution is the same as circular convolution since $N = 8 \geq 6 + 3 - 1$.

We know $x[((n-3))_8] = x[n-3]$, and convolving this with $w[n]$ from Fig P8.35-1 gives:



For $n = 2, w[n] * x[n-3] = 11$ so $A_2 = 6$. Thus, $x[2] = x[-2] = 3$, and we have fully specified $x[n]$:



8.36. We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-pt DFT:

$$X[k] = 2 + W_5^k + W_5^{3k}, \quad 0 \leq k \leq 5$$

where $W_5^k = e^{-j(\frac{2\pi}{5})k}$.

(ii) Now, we square the DFT of $x[n]$:

$$\begin{aligned} Y[k] &= X^2[k] \\ &= 2 + 2W_5^k + 2W_5^{3k} \\ &\quad + 2W_5^k + W_5^{2k} + W_5^{5k} \\ &\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \quad 0 \leq k \leq 5 \end{aligned}$$

Using the fact $W_5^{5k} = W_5^0 = 1$ and $W_5^{6k} = W_5^k$

$$Y[k] = 3 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + W_5^{4k}, \quad 0 \leq k \leq 5$$

(a) By inspection,

$$y[n] = 3\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + \delta[n-4], \quad 0 \le n \le 5$$

(b) This procedure performs the autocorrelation of a real sequence. Using the properties of the DFT, an alternative method may be achieved with convolution:

$$y[n] = \text{IDFT}\{X^2[k]\} = x[n] * x[n]$$

The IDFT and DFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \ge 2M - 1$ where M is the length of $x[n]$. Since $M = 3, N \ge 5$.

8.37. (a)

$$g_1[n] = x[N-1-n], \quad 0 \le n \le (N-1)$$

$$G_1[k] = \sum_{n=0}^{N-1} x[N-1-n]W_N^{kn}, \quad 0 \le k \le (N-1)$$

Let $m = N - 1 - n$,

$$\begin{aligned} G_1[k] &= \sum_{m=0}^{N-1} x[m]W_N^{k(N-1-m)} \\ &= W_N^{k(N-1)} \sum_{m=0}^{N-1} x[m]W_N^{-km} \end{aligned}$$

Using $W_N^k = e^{-j(2\pi k/N)}, W_N^{k(N-1)} = W_N^{-k} = e^{j(2\pi k/N)}$

$$\begin{aligned} G_1[k] &= e^{j(2\pi k/N)} \sum_{m=0}^{N-1} x[m]e^{j(2\pi km/N)} \\ &= e^{j(2\pi k/N} X(e^{j\omega})|_{\omega=(2\pi k/N)} \\ G_1[k] &= H_7[k] \end{aligned}$$

(b)

$$g_2[n] = (-1)^n x[n], \quad 0 \le n \le (N-1)$$

$$\begin{aligned} G_2[k] &= \sum_{n=0}^{N-1} (-1)^n x[n]W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} x[n]W_N^{(\frac{N}{2})n}W_N^{kn} \\ &= \sum_{n=0}^{N-1} x[n]W_N^{(k+\frac{N}{2})n} \\ &= X(e^{j\omega})|_{\omega=2\pi(k+\frac{N}{2})/N} \\ G_2[k] &= H_8[k] \end{aligned}$$

(c)

$$g_3[n] = \begin{cases} x[n], & 0 \le n \le (N-1) \\ x[n-N], & N \le n \le (2N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
G_3[k] &= \sum_{n=0}^{2N-1} x[n] W_{2N}^{kn}, \quad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{n=N}^{2N-1} x[n-N] W_{2N}^{kn} \\
&= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{m=0}^{N-1} x[m] W_{2N}^{k(m+N)} \\
&= \sum_{n=0}^{N-1} x[n] \left(1 + W_{2N}^{kN}\right) W_{2N}^{kn} \\
&= \left(1 + W_N^{(kN/2)}\right) \sum_{n=0}^{N-1} x[n] W_N^{(kn/2)} \\
&= \left(1 + (-1)^k\right) X(e^{j\omega})|_{\omega=(\pi k/N)} \\
G_3[k] &= H_3[k]
\end{aligned}$$

(d)

$$g_4[n] = \begin{cases} x[n] + x[n+N/2], & 0 \le n \le (N/2-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
G_4[k] &= \sum_{n=0}^{N/2-1} \left(x[n] + x[n+\frac{N}{2}]\right) W_{N/2}^{kn}, \quad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{kn} + \sum_{n=0}^{N/2-1} x[n+N/2] W_{N/2}^{kn} \\
&= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{kn} + \sum_{m=N/2}^{N-1} x[m] W_{N/2}^{k(m-N/2)} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{2kn} \\
&= X(e^{j\omega})|_{\omega=(4\pi k/N)} \\
G_4[k] &= H_6[k]
\end{aligned}$$

(e)

$$g_5[n] = \begin{cases} x[n], & 0 \le n \le (N-1) \\ 0, & N \le n \le (2N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} G_5[k] &= \sum_{n=0}^{2N-1} x[n]W_{2N}^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} x[n]W_{2N}^{kn} \\ &= X(e^{j\omega})|_{\omega=(\pi k/N)} \\ G_5[k] &= H_2[k] \end{aligned}$$

(f)

$$g_6[n] = \begin{cases} x[n/2], & n \text{ even}, \quad 0 \le n \le (2N-1) \\ 0, & n \text{ odd} \end{cases}$$

$$\begin{aligned} G_6[k] &= \sum_{n=0}^{2N-1} x[n/2]W_{2N}^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} x[n]W_N^{kn} \\ &= X(e^{j\omega})|_{\omega=(2\pi k/N)} \\ G_6[k] &= H_1[k] \end{aligned}$$

(g)

$$g_7[n] = x[2n], \quad 0 \le n \le (N/2-1)$$

$$\begin{aligned} G_7[k] &= \sum_{n=0}^{\frac{N}{2}-1} x[2n]W_{N/2}^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} x[n]\left(\frac{1+(-1)^n}{2}\right)W_N^{kn} \\ &= \sum_{n=0}^{N-1} x[n]\left(\frac{1+W_N^{(N/2)n}}{2}\right)W_N^{kn} \\ &= \frac{1}{2}\sum_{n=0}^{N-1} x[n]\left(W_N^{nk} + W_N^{n(k+N/2)}\right) \\ &= \frac{1}{2}\left[X(e^{j(2\pi/N)}) + X(e^{j(2\pi/N)(k+N/2)})\right] \\ G_7[k] &= H_5[k] \end{aligned}$$

8.38. From Table 8.2, the N-pt DFT of an N-pt sequence will be real-valued if

$$x[n] = x[((-n))_N].$$

For $0 \le n \le (N-1)$, this may be stated as,

$$x[n] = x[N-n], \quad 0 \le n \le (N-1)$$

For this case, N = 10, and

$$\begin{aligned} x[1] &= x[9] \\ x[2] &= x[8] \\ &\vdots \end{aligned}$$

The Fourier transform of $x[n]$ displays generalized linear phase (see Section 5.7.2). This implies that for $x[n] \neq 0$, $0 \leq n \leq (N-1)$:

$$x[n] = x[N-1-n]$$

For $N = 10$,

$$\begin{aligned} x[0] &= x[9] \\ x[1] &= x[8] \\ x[2] &= x[7] \\ &\vdots \end{aligned}$$

To satify both conditions, $x[n]$ must be a constant for $0 \leq n \leq 9$.

8.39. We have two 100-pt sequences which are nonzero for the interval $0 \leq n \leq 99$.

If $x_1[n]$ is nonzero for $10 \leq n \leq 39$ only, the linear convolution

$$x_1[n] * x_2[n]$$

is a sequence of length $40 + 100 - 1 = 139$, which is nonzero for the range $10 \leq n \leq 139$.

A 100-pt circular convolution is equivalent to the linear convolution with the first 40 points aliased by the values in the range $100 \leq n \leq 139$.

Therefore, the 100-pt circular convolution will be equivalent to the linear convolution only in the range $40 \leq n \leq 99$.

8.40. (a) Since $x[n]$ is 50 points long, and $h[n]$ is 10 points long, the linear convolution $y[n] = x[n] * h[n]$ must be $50 + 10 - 1 = 59$ pts long.

(b) Circular convolution = linear convolutin + aliasing.
If we let $y[n] = x[n] * h[n]$, a more mathematical statement of the above is given by

$$x[n] \, \textcircled{N} \, h[n] = \sum_{r=-\infty}^{\infty} y[n + rN], \quad 0 \leq n \leq (N-1)$$

For $N = 50$,

$$x[n] \, \textcircled{50} \, h[n] = y[n] + y[n + 50], \quad 0 \leq n \leq 49$$

We are given: $x[n] \, \textcircled{50} \, h[n] = 10$
Hence,

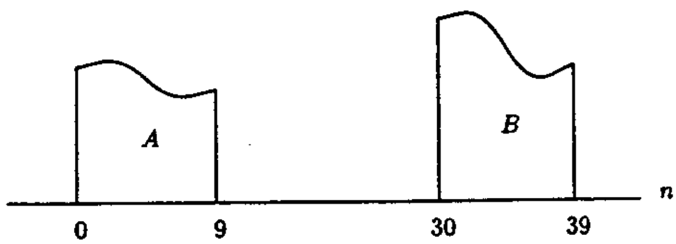
$$y[n] + y[n + 50] = 10, \quad 0 \leq n \leq 49$$

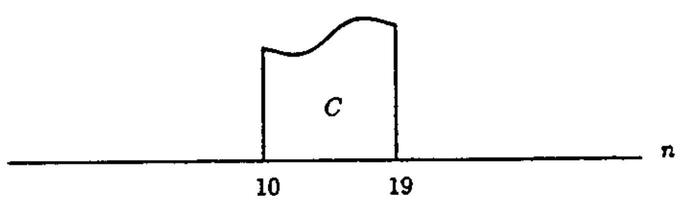
Also, $y[n] = 5, \quad 0 \leq n \leq 4$.
Using the above information:

$$
\begin{aligned}
n = 0 \quad & y[0] + y[50] &=& \ 10 \\
\vdots \quad & y[50] &=& \ 5 \\
n = 4 \quad & y[4] + y[54] &=& \ 10 \\
& y[54] &=& \ 5 \\
n = 5 \quad & y[5] + y[55] &=& \ 10 \\
\vdots \quad & y[55] &=& \ ? \\
n = 8 \quad & y[8] + y[58] &=& \ 10 \\
& y[58] &=& \ ? \\
n = 9 \quad & y[9] &=& \ 10 \\
\vdots \quad & & & \\
n = 49 \quad & y[49] &=& \ 10
\end{aligned}
$$

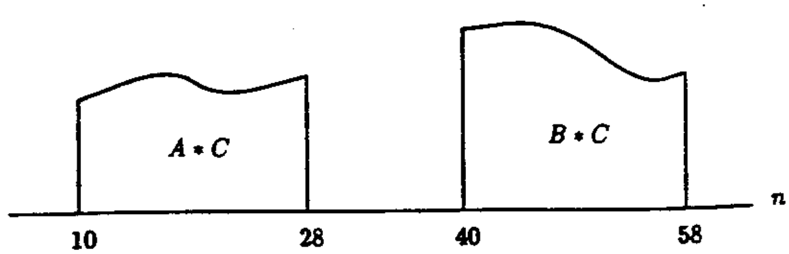
To conclude, we can determine $y[n]$ for $9 \leq n \leq 55$ only. (Note that $y[n]$ for $0 \leq n \leq 4$ is given.)

8.41. We have





(a) The linear convolution $x[n] * y[n]$ is a $40 + 20 - 1 = 59$ point sequence:



Thus, $x[n] * y[n] = w[n]$ is nonzero for $10 \leq n \leq 28$ and $40 \leq n \leq 58$.

(b) The 40-pt circluar convolution can be obtained by aliasing the linear convolution. Specifically, we alias the points in the range $40 \le n \le 58$ to the range $0 \le n \le 18$.
Since $w[n] = x[n] * y[n]$ is zero for $0 \le n \le 9$, the circular convolution $g[n] = x[n] \textcircled{40} y[n]$ consists of only the (aliased) values:

$$w[n] = x[n] * y[n], \quad 40 \le n \le 49$$

Also, the points of $g[n]$ for $18 \le n \le 39$ will be equivalent to the points of $w[n]$ in this range.
To conclude,

$$\begin{aligned} w[n] &= g[n], \quad 18 \le n \le 39 \\ w[n+40] &= g[n], \quad 0 \le n \le 9 \end{aligned}$$

8.42. (a) The two sequences are related by the circular shift:

$$h_2[n] = h_1[((n+4))_8]$$

Thus,

$$H_2[k] = W_8^{-4k} H_1[k]$$

and

$$|H_2[k]| = |W_8^{-4k} H_1[k]| = |H_1[k]|$$

So, yes the magnitudes of the 8-pt DFTs are equal.

(b) $h_1[n]$ is nearly like $(\sin x)/x$.
Since $H_2[k] = e^{j\pi k} H_1[k]$, $h_1[n]$ is a better lowpass filter.

8.43. (a) Overlap add:
If we divide the input into sections of length L, each section will have an output length:

$$L + 100 - 1 = L + 99$$

Thus, the required length is

$$L = 256 - 99 = 157$$

If we had 63 sections, $63 \times 157 = 9891$, there will be a remainder of 109 points. Hence, we must pad the remaining data to 256 and use another DFT.
Therefore, we require 64 DFTs and 64 IDFTs. Since $h[n]$ also requires a DFT, the total:

65 DFTs and 64 IDFTs

(b) Overlap save:
We require 99 zeros to be padded in from of the sequence. The first 99 points of the output of each section will be discarded. Thus the length after padding is 10099 points. The length of each section overlap is $256 - 99 = 157 = L$.
We require $65 \times 157 = 10205$ to get all 10099 points. Because $h[n]$ also requires a DFT:

66 DFTs and 65 IDFTs

(c) Ignoring the transients at the beginning and end of the direct convolution, each output point requires 100 multiplies and 99 adds.
overlap add:

$$\begin{aligned} \#\,\text{mult} &= 129(1024) &= 132096 \\ \#\,\text{add} &= 129(2048) &= 264192 \end{aligned}$$

overlap save:

$$\begin{aligned} \#\,\text{mult} &= 131(1024) &&= 134144 \\ \#\,\text{add} &= 131(2048) &&= 268288 \end{aligned}$$

direct convolution:

$$\begin{aligned} \#\,\text{mult} &= 100(10000) &&= 1000000 \\ \#\,\text{add} &= 99(10000) &&= 990000 \end{aligned}$$

8.44. First we need to compute the values $Q[0]$ and $Q[3]$:

$$\begin{aligned} Q[0] &= X_1(1) = X_1(e^{jw})|_{w=0} \\ &= \sum_{n=0}^{\infty} x_1[n] = \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3} \\ Q[3] &= X_1(-1) = X_1(e^{jw})|_{w=\pi} \\ &= \sum_{n=0}^{\infty} x_1[n](-1)^n \\ &= \frac{4}{3} \end{aligned}$$

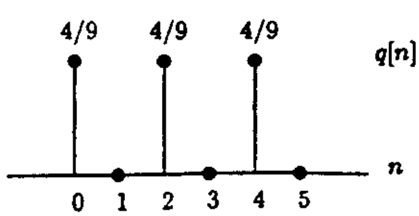
One possibility for $Q[k]$, the six-point DFT, is:

$$Q[k] = \frac{4}{3}\delta[k] + \frac{4}{3}\delta[k-3].$$

We then find $q[n]$, for $0 \leq n < 6$:

$$\begin{aligned} q[n] &= \frac{1}{6}\sum_{k=0}^{5} Q[k]e^{\frac{2\pi}{6}kn} \\ &= \frac{1}{6}\left(\frac{4}{3} + \frac{4}{3}(-1)^n\right) \\ &= \frac{2}{9}(1 + (-1)^n) \end{aligned}$$

otherwise it's 0. Here's a sketch of q[n]:



8.45. We have:

$$DFT_7\{x_2[n]\} = X_2[k] = \sum_0^4 x_2[n]e^{-j\frac{2\pi}{7}kn}.$$

Then:

$$\begin{aligned}
x_2[0] &= \frac{1}{7}\sum_{k=0}^{6} X_2[k] \\
&= \frac{1}{7}\sum_{k=0}^{6}(Re\{X_2[k]\} + jIm\{X_2[k]\}) \\
&= \frac{1}{7}\sum_{k=0}^{6} Re\{X_2[k]\} \text{ , since } x_2[0] \text{ is real.} \\
&= g[0].
\end{aligned}$$

To determine the relationship between $x_2[1]$ and g[1], we first note that since $x_2[n]$ is real:

$$X(e^{jw}) = X^*(e^{-jw}).$$

Therefore:

$$X[k] = X^*[N-k] \text{ , } k = 0, ..., 6.$$

We thus have:

$$\begin{aligned}
g[1] &= \frac{1}{7}\sum_{k=0}^{6} Re\{X_2[k]\}W_7^{-k} \\
&= \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[k] + X_2^*[k]}{2} W_7^{-k} \\
&= \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[k]}{2} W_7^{-k} + \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[N-k]}{2} W_7^{-k} \\
&= \frac{1}{2}x_2[1] + \frac{1}{14}\sum_{k=0}^{6} X_2[k]W_7^{k} \\
&= \frac{1}{2}x_2[1] + \frac{1}{14}\sum_{k=0}^{6} X_2[k]W_7^{-6k} \\
&= \frac{1}{2}(x_2[1] + x_2[6]) \\
&= \frac{1}{2}(x_2[1] + 0) \\
&= \frac{1}{2}x_2[1].
\end{aligned}$$

8.46. (i) This corresponds to $x_i[n] = x_i^*[((-n))_N]$, where $N = 5$. Note that this is only true for $x_2[n]$.

(ii) $X_i(e^{jw})$ has linear phase corresponds to $x_i[n]$ having some internal symmetry, this is only true for $x_1[n]$.

(iii) The DFT has linear phase corresponds to $\tilde{x}_i[n]$ (the periodic sequence obtained from $x_i[n]$) being symmetric, this is true for $x_1[n]$ and $x_2[n]$ only.

8.47. (a)

$$\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k9} = \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k} = x[1].$$

(b)

$$\begin{aligned}
V[k] &= X(z)\Big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=-\infty}^{n=\infty} x[n]z^{-n}\Big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n]z^{-n}\Big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n](2e^{j\frac{\pi}{8}})^{-n}e^{-j\frac{2\pi k}{8}n} \\
&= \sum_{n=0}^{n=8} v[n]e^{-j\frac{2\pi k}{8}n}.
\end{aligned}$$

We thus conclude that

$$v[n] = x[n](2e^{j\frac{\pi}{8}})^{-n}.$$

(c)

$$\begin{aligned}
w[n] &= \frac{1}{4}\sum_{k=0}^{3} W[k]W_4^{-kn} \\
&= \frac{1}{4}\sum_{k=0}^{3}(X[k]+X[k+4])e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=0}^{3} X[k+4]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=4}^{7} X[k]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{7} X[k]e^{+j\frac{2\pi}{8}k2n} \\
&= 2x[2n].
\end{aligned}$$

We thus conclude that

$$w[n] = 2x[2n].$$

(d) Note that $Y[k]$ can be written as:

$$\begin{aligned}
Y[k] &= X[k] + (-1)^k X[k] \\
&= X[k] + W_8^{4k}X[k].
\end{aligned}$$

Using the DFT properties, we thus conclude that

$$y[n] = x[n] + x[((n-4))_8].$$

8.48. (a) No. x[n] only has N degrees of freedom and we have $M \geq N$ constraints which can only be satisfied if $x[n] = 0$. Specifically, we want

$$X(e^{j\frac{2\pi k}{M}}) = DFT_M\{x[n]\} = 0.$$

Since $M \geq N$, there is no aliasing and x[n] can be expressed as:

$$x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]W_M^{kn}\ ,\ n = 0, ..., M-1.$$

Where $X[k]$ is the M-point DFT of $x[n]$, since $X[k] = 0$, we thus conclude that $x[n] = 0$, and therefore the answer is NO.

(b) Here, we only need to make sure that when time-aliased to M samples, $x[n]$ is all zeros. For example, let

$$x[n] = \delta[n] - \delta[n-2]$$

then,

$$X(e^{jw}) = 1 - e^{-2jw}.$$

Let $M = 2$, then we have

$$\begin{aligned} X(e^{j\frac{2\pi}{2}0}) &= 1 - 1 = 0 \\ X(e^{j\frac{2\pi}{2}1}) &= 1 - 1 = 0 \end{aligned}$$

8.49. $x_2[n]$ is $x_1[n]$ time aliased to have only N samples. Since

$$x_1[n] = (\frac{1}{3})^n u[n],$$

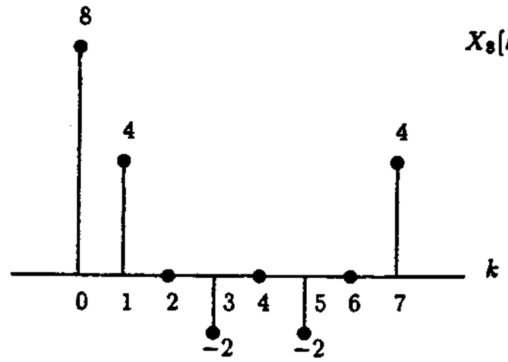
We get:

$$x_2[n] = \begin{cases} \frac{(\frac{1}{3})^n}{1-(\frac{1}{3})^N} & ,\ n = 0, ..., N-1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

8.50. (a) Let $n = 0, ..., 7$, we can write $x[n]$ as:

$$\begin{aligned} x[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\ &= 1 + \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{j\frac{2\pi}{8}n7} - \frac{1}{4}e^{j\frac{2\pi}{8}n3} - \frac{1}{4}e^{j\frac{2\pi}{8}n5} \\ &= \frac{1}{8}(8 + 4e^{j\frac{2\pi}{8}n} + 4e^{j\frac{2\pi}{8}n7} - 2e^{j\frac{2\pi}{8}n3} - 2e^{j\frac{2\pi}{8}n5}) \\ &= \frac{1}{8}\sum_{k=0}^{7} X_8[k]e^{j\frac{2\pi k}{8}n} \end{aligned}$$

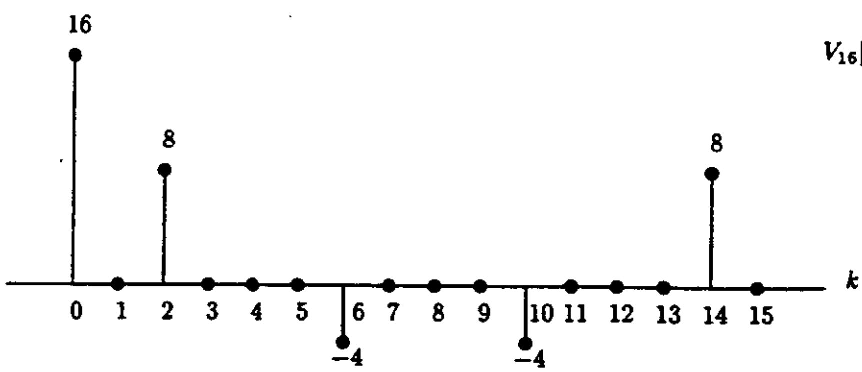
We thus get the following plot for $X_8[k]$:



(b) Now let $n = 0, ..., 15$, we can write $v[n]$ as:

$$\begin{aligned}
v[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\
&= 1 + \frac{1}{2}e^{j\frac{2\pi}{16}n2} + \frac{1}{2}e^{j\frac{2\pi}{16}n14} - \frac{1}{4}e^{j\frac{2\pi}{16}n6} - \frac{1}{4}e^{j\frac{2\pi}{16}n10} \\
&= \frac{1}{16}(16 + 8e^{j\frac{2\pi}{16}n2} + 8e^{j\frac{2\pi}{16}n14} - 4e^{j\frac{2\pi}{16}n6} - 4e^{j\frac{2\pi}{16}n10}) \\
&= \frac{1}{16}\sum_{k=0}^{15} V_{16}[k]e^{j\frac{2\pi k}{16}n}
\end{aligned}$$

We thus get the following plot for $V_{16}[k]$:



(c)

$$|X_{16}[k]| = X(e^{j\omega})|_{\omega=\frac{2\pi}{16}k} \qquad 0 \leq k \leq 15$$

where $X(e^{j\omega})$ is the Fourier transform of $x[n]$.
Note that $x[n]$ can be expressed as:

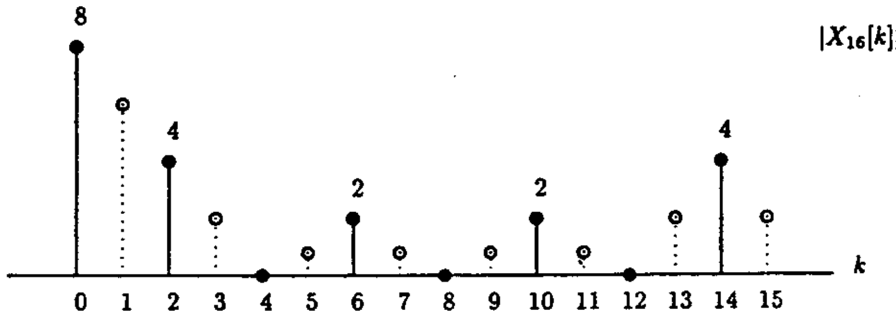
$$x[n] = y[n]w[n]$$

where:

$$y[n] = 1 + \cos(\frac{\pi n}{4}) - \frac{1}{2}\cos(\frac{3\pi n}{4})$$

and $w[n]$ is an eight-point rectangular window.
$|X_{16}[k]|$ will therefore have as its even points the sequence $|X_8[k]|$ .The odd points will correspond to the bandlimited interpolation between the even-point samples. The values that we can find exactly by inspection are thus:

$$|X_{16}[k]| = |X_8[k/2]| \qquad k = 0, 2, 4, ..., 14.$$



8.51. We wish to verify the identity of Eq. (8.7):

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & k-r = mN,\ m:\ \text{integer} \\ 0, & \text{otherwise} \end{cases}$$

(a) For $k - r = mN$,

$$\begin{aligned} e^{j\frac{2\pi}{N}(k-r)n} &= e^{j\frac{2\pi}{N}(mN)n} \\ & e^{j2\pi mn} \\ & (1)^{mn} \end{aligned}$$

Since m and n are integers;
$$e^{j\frac{2\pi}{N}(k-r)n} = 1 \text{ , for } k - r = mN$$

So,

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} &= \frac{1}{N}\sum_{n=0}^{N-1} 1 \\ &= 1 \text{ , for } k - r = mN \end{aligned}$$

(b)

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln} = \frac{1 - e^{j\frac{2\pi}{N}lN}}{1 - e^{j\frac{2\pi}{N}l}}$$

This closed form solution is indeterminate for $l = mN$ only.
For the case when $l = mN$, we use L'Hôpital's Rule to find:

$$\begin{aligned} \lim_{l \to mN} \frac{1 - e^{j2\pi l}}{1 - e^{j\frac{2\pi}{N}l}} &= \left[\frac{-j2\pi e^{j2\pi l}}{-j\frac{2\pi}{N}e^{j\frac{2\pi}{N}l}}\right]_{l=mN} \\ &= N \end{aligned}$$

(c) For the case when $k - r \neq mN$:

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} &= \frac{1-e^{j2\pi(k-r)}}{1-e^{j\frac{2\pi}{N}(k-r)}} \\
&= 0
\end{aligned}$$

Note that the denominator is nonzero, while the numerator will always be zero for $k - r \neq mN$.

8.52. (a) We know from Eq. (8.11) that if $\tilde{x}_1[n] = \tilde{x}[n-m]$, we have:

$$\tilde{X}_1[k] = \sum_{n=0}^{N-1} \tilde{x}[n-m]W_N^{kn}$$

If we substitute $r = n - m$ into this equation, we get:

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{r=-m}^{N-1-m} \tilde{x}[r]W_N^{k(r+m)} \\
&= W_N^{km}\sum_{r=-m}^{N-1-m} \tilde{x}[r]W_N^{kr}
\end{aligned}$$

(b) We can decompose the summation from part (a) into

$$\tilde{X}_1[k] = W_N^{km}\left[\sum_{r=-m}^{-1} \tilde{x}[r]W_N^{kr} + \sum_{r=0}^{N-1-m} \tilde{x}[r]W_N^{kr}\right]$$

Using the fact that $\tilde{x}[r]$ and W_N^{kr} are periodic with period N:

$$\sum_{r=-m}^{-1} \tilde{x}[r]W_N^{kr} = \sum_{r=-m}^{-1} \tilde{x}[r+N]W_N^{k(r+N)}$$

Substituting $\ell = r + N$

$$\sum_{r=-m}^{-1} \tilde{x}[r]W_N^{kr} = \sum_{\ell=N-m}^{N-1} \tilde{x}[\ell]W_N^{k\ell}$$

(c) Using the result from part (b):

$$\begin{aligned}
\tilde{X}_1[k] &= W_N^{km}\left[\sum_{r=N-m}^{N-1} \tilde{x}[r]W_N^{kr} + \sum_{r=0}^{N-m-1} \tilde{x}[r]W_N^{kr}\right] \\
&= W_N^{km}\sum_{r=0}^{N-1} \tilde{x}[r]W_N^{kr} \\
&= W_N^{km}\tilde{X}[k]
\end{aligned}$$

Hence, if $\tilde{x}_1[n] = \tilde{x}[n-m]$, then $\tilde{X}_1[k] = W_N^{km}\tilde{X}[k]$.

8.53. (a) 1. The DFS of $\tilde{x}^*[n]$ is given by:

$$\begin{aligned}
\sum_{n=0}^{N-1} \tilde{x}^*[n]W_N^{kn} &= \left(\sum_{n=0}^{N-1} \tilde{x}[n]W_N^{-kn}\right)^* \\
&= \tilde{X}^*[-k]
\end{aligned}$$

2. The DFS of $\tilde{x}^*[-n]$:

$$\begin{aligned}\sum_{n=0}^{N-1} \tilde{x}^*[-n]W_N^{kn} &= \left(\sum_{l=-N+1}^{0} \tilde{x}[l]W_N^{kl}\right)^* \\ &= X^*[k]\end{aligned}$$

3. The DFS of $Re\{\tilde{x}[n]\}$:

$$\begin{aligned}\sum_{n=0}^{N-1} \frac{\tilde{x}[n]+\tilde{x}^*[n]}{2} W_N^{kn} &= \frac{1}{2}\left(\tilde{X}[k]+\tilde{X}^*[-k]\right) \\ &\doteq \tilde{X}_e[k]\end{aligned}$$

4. The DFS of $jIm\{\tilde{x}[n]\}$:

$$\begin{aligned}\sum_{n=0}^{N-1} \frac{\tilde{x}[n]-\tilde{x}^*[n]}{2} W_N^{kn} &= \frac{1}{2}\left(\tilde{X}[k]-\tilde{X}^*[-k]\right) \\ &= \tilde{X}_0[k]\end{aligned}$$

(b) Consider $\tilde{x}[n]$ real:

1.

$$Re\{\tilde{X}[k]\} = \frac{\tilde{X}[k]+\tilde{X}^*[k]}{2}$$

From part (a), if $\tilde{x}[n]$ is real,

$$\begin{aligned}\text{DFS }\{\tilde{x}[n]\} &= \text{DFS }\{\tilde{x}^*[n]\} \\ \text{DFS }\{\tilde{x}[-n]\} &= \text{DFS }\{\tilde{x}^*[-n]\}\end{aligned}$$

So,

$$\begin{aligned}\tilde{X}[k] &= \tilde{X}^*[-k] \\ \tilde{X}[-k] &= \tilde{X}^*[k] \\ Re\{\tilde{X}[k]\} &= \frac{X^*[k]+X[-k]}{2} \\ &= Re\{\tilde{X}[-k]\}\end{aligned}$$

(i.e. the real part of $\tilde{X}[k]$ is even.)

2.

$$\begin{aligned}Im\{\tilde{X}[k]\} &= \frac{\tilde{X}[k]-\tilde{X}^*[k]}{2} \\ &= \frac{\tilde{X}^*[-k]-\tilde{X}[-k]}{2} \\ &= Im\{\tilde{X}[-k]\}\end{aligned}$$

(i.e., the imaginary part of $\tilde{X}[k]$ is odd.)

3.

$$\begin{aligned}|\tilde{X}[k]| &= \sqrt{\tilde{X}[k]\tilde{X}^*[k]} \\ &= \sqrt{\tilde{X}^*[-k]\tilde{X}[-k]} \\ &= |\tilde{X}[-k]|\end{aligned}$$

(i.e., the magnitude of $\tilde{X}[k]$ is even)

4.

$$
\begin{aligned}
\angle\tilde{X}[k] &= \arctan\left(\frac{\mathcal{I}m\{\tilde{X}[k]\}}{\mathcal{R}e\{\tilde{X}[k]\}}\right) \\
&= \arctan\left(\frac{\mathcal{I}m\{\tilde{X}[-k]\}}{\mathcal{R}e\{\tilde{X}[-k]\}}\right) \\
&= -\angle\tilde{X}[-k]
\end{aligned}
$$

(i.e., the angle of $\tilde{X}[k]$ is odd.)

8.54. 1. Let $x[n]$ $(0 \le n \le N-1)$ be one period of the periodic sequence $\tilde{x}[n]$. The Fourier transform of this periodic sequence can be expressed as:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n]e^{-j\omega n}$$

Recall the synthesis equation, Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$

Substitution yields:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}\right)e^{-j\omega n}$$

Rearranging the summations and combining terms:

$$\tilde{X}(e^{j\omega}) = \sum_{k=0}^{N-1}\tilde{X}[k]\left(\frac{1}{N}\sum_{n=-\infty}^{\infty} e^{j(\frac{2\pi k}{N}-\omega)n}\right)$$

The infinite summation is recognized as an impulse at $\omega = (2\pi k/N)$:

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

2. Since $x[n]$ corresponds to one period of $\tilde{x}[n]$, we must apply a rectangular window (unit amplitude and length N) to the periodic sequence. Thus, to extract one period from $\tilde{x}[n]$:

$$x[n] = \tilde{x}[n]w[n]$$

where,

$$w[n] = \begin{cases} 1, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}$$

The window has a Fourier transform:

$$
\begin{aligned}
W(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} w[n]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \qquad = \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\frac{\omega}{2}}}\,\frac{e^{j\frac{\omega N}{2}}-e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}} \\
&\qquad\qquad\qquad\qquad = e^{-j\omega(\frac{N-1}{2})}\frac{\sin\left(\omega\frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}
\end{aligned}
$$

3. Since $x[n] = \tilde{x}[n]w[n]$, the Fourier transform of $x[n]$ can be represented by the periodic convolution (see Eq. (8.28)).

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\delta\left(\theta - \frac{2\pi k}{N}\right)\frac{\sin\left[\frac{N}{2}(\omega-\theta)\right]}{\sin\left(\frac{\omega-\theta}{2}\right)}e^{-j\left(\frac{N-1}{2}\right)(\omega-\theta)}$$

Integration over $-\pi \leq \theta \leq \pi$ reduces to the summation (note the impulse train):

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\frac{\sin\left[(N\omega - 2\pi k)/2\right]}{\sin\left[\left(\omega - \frac{2\pi k}{N}\right)/2\right]}e^{-j\left(\frac{N-1}{2}\right)\left(\omega - \frac{2\pi k}{N}\right)}$$

Hence, the Fourier transform is obtained from the DFS via an interpolation formula.

8.55. The N-point DFT of the N-pt sequence, $x[n]$ is given by

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad 0 \leq k \leq (N-1) \\ X[0] &= \sum_{n=0}^{N-1} x[n] \end{aligned}$$

(a) Suppose $x[n] = -x[N-1-n]$. For N even, all elements of $x[n]$ will cancel with an antisymmetric component. For N odd, all elements have a counterpart with opposite sign. However, $x[(N-1)/2]$ must also be zero.
Therefore, for $x[n] = -x[N-1-n]$, $X[0] = 0$.

(b) Suppose $x[n] = x[N-1-n]$ and N even.

$$\begin{aligned} X[N/2] &= \sum_{n=0}^{N-1} x[n]W_N^{(N/2)n} \\ &= \sum_{n=0}^{N-1} x[n](-1)^n \\ &= x[0] - x[1] + x[2] - x[3] + \cdots + x[N-2] - x[N-1] \end{aligned}$$

Since $x[n] = x[N-1-n]$, then

$$\begin{aligned} x[0] &= x[N-1] \\ x[1] &= x[N-2] \\ &\vdots \end{aligned}$$

Therefore, $X[N/2] = 0$.

8.56. (a) The conjugate-symmetric part of a sequence:

$$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$$

The periodic conjugate-symmetric part:

$$x_{ep}[n] = \frac{1}{2}\left(x[((n))_N] + x^*[((-n))_N]\right), \quad 0 \leq n \leq (N-1)$$

Note that:

$$\begin{aligned} x[((n))_N] &= x[n], \quad 0 \le n \le (N-1) \\ x^*[((-n))_N] &= x^*[-n+N] + x^*[0]\delta[n] - x^*[0]\delta[n-N] \end{aligned}$$

Substituting into $x_{ep}[n]$:

$$x_{ep}[n] = \frac{1}{2}\left[x[n] + x^*[-n+N] + x^*[0]\delta[n] - x^*[0]\delta[n-N]\right], \quad 0 \le n \le (N-1)$$

Since,

$$\begin{aligned} x_e[n] &= \frac{1}{2}\left(x[n] + x^*[-n]\right) \\ &= \frac{1}{2}\left(x[n] + x^*[0]\delta[n]\right), \quad 0 \le n \le (N-1) \\ \text{and } x_e[n-N] &= \frac{1}{2}\left(x[n-N] + x^*[N-n]\right) \\ &= \frac{1}{2}\left(-x^*[0]\delta[n-N] + x^*[N-n]\right) \quad 0 \le n \le (N-1) \end{aligned}$$

We can combine to get:

$$x_{ep}[n] = x_e[n] + x_e[n-N] \quad 0 \le n \le (N-1)$$

The periodic conjugate-antisymmetric part is given as

$$x_{op}[n] = (x_o[n] + x_o[n-N]), \quad 0 \le n \le (N-1)$$

Recall that the odd part can be expressed as

$$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$$

So,

$$x_o[n-N] = \frac{1}{2}\left(x[n-N] - x^*[N-n]\right)$$

For $0 \le n \le (N-1)$:

$$\begin{aligned} x_o[n] &= \frac{1}{2}\left(x[n] - x^*[0]\delta[n]\right), \quad 0 \le n \le N-1 \\ x_o[n-N] &= \frac{1}{2}\left(x[0]\delta[n-N] - x^*[N-n]\right) \end{aligned}$$

From the definition of $x_{op}[n]$:

$$\begin{aligned} x_{op}[n] &= \frac{1}{2}\left(x[((n))_N] - x^*[((-n))_N]\right), \quad 0 \le n \le (N-1) \\ &= \frac{1}{2}\left(x[n] - x^*[0]\delta[n] + x^*[0]\delta[n-N] - x^*[N-n]\right), \quad 0 \le n \le (N-1) \end{aligned}$$

Recognizing the expressions for $x_o[n]$ and $x_o[n-N]$ in $x_{op}[n]$, we have

$$x_{op}[n] = x_o[n] + x_o[n-N], \quad 0 \le n \le (N-1)$$

(b) $x[n]$ is a sequence of length N; however,

$$x[n] = \begin{cases} x_1[n], & 0 \le n \le N/2 \\ 0, & N/2 \le n \le N-1 \end{cases}$$

The even part: (assume N is even)

$$x_e[n] = \begin{cases} \frac{x[n]}{2} + \frac{x^*[0]\delta[n]}{2}, & 0 \le n \le N/2 \\ \frac{x^*[-n]}{2}, & -N/2 \le n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

From part (a):

$$x_{ep}[n] = x_e[n] + x_e[n-N], \qquad 0 \le n \le (N-1)$$

Because $x[n] = 0$ for $|n| \ge N/2$,

$$x_{ep}[n] = x_e[n], \qquad 0 \le n \le (N/2-1)$$

Also, since $x_e[n] = x_e^*[-n]$,

$$x_e[n] = x_{ep}^*[-n], \qquad -N/2 < n \le -1$$

To conclude:

$$x_e[n] = \begin{cases} x_{ep}[n], & 0 \le n \le N/2 \\ \frac{x_{ep}[n]}{2}, & n = N/2 \\ x_{ep}^*[-n], & -N/2 < n \le -1 \\ \frac{x_{ep}^*[-n]}{2}, & n = -N/2 \end{cases}$$

8.57.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]x^*[n]$$

From the synthesis equation:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$

Hence,

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W_N^{kn}$$

substituting:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]\left(\frac{1}{\mathrm{N}}\sum_{k=0}^{N-1} X^*[k]W_{\mathrm{N}}^{kn}\right)$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\left(\sum_{n=0}^{N-1} x[n]W_N^{kn}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]X[k] \\
\sum_{n=0}^{N-1}|x[n]|^2 &= \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2
\end{aligned}
$$

8.58. (a) This statement is TRUE:

$$
\begin{aligned}
X[k] &= X(e^{j\omega})|_{\omega=2\pi k/N} \\
&= B\left(\frac{2\pi k}{N}\right)e^{j(2\pi/N)k\alpha} \\
A[k] &= B\left(\frac{2\pi k}{N}\right) \\
\gamma &= \frac{2\pi\alpha}{N}
\end{aligned}
$$

(b) This statement is FALSE:
Suppose $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$,

$$
\begin{aligned}
X[k] &= 1 + \frac{1}{2}e^{-j\pi k} \\
&= 1 + \frac{1}{2}(-1)^k
\end{aligned}
$$

Expressed in the form

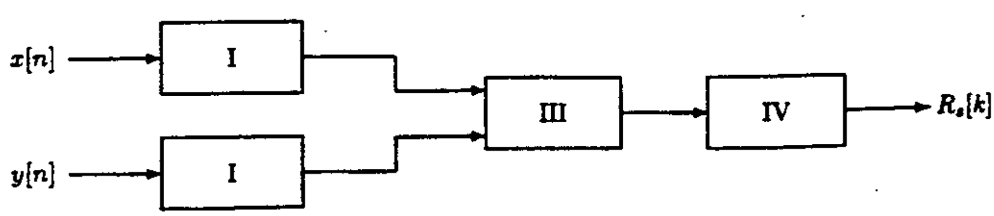
$$
\begin{aligned}
X[k] &= A[k]e^{j\gamma k}, \\
A[k] &= 1 + \frac{1}{2}(-1)^k \\
\text{and } \gamma &= 0
\end{aligned}
$$

The Fourier transform of $x[n]$ is $X(e^{j\omega}) = 1 + \frac{1}{2}e^{j\omega}$, which cannot be expressed in the form $X(e^{-j\omega}) = B(\omega)e^{j\alpha\omega}$.

8.59. We desire 128 samples of $X(e^{j\omega})Y(e^{j\omega})$.

Since $x[n]$ and $y[n]$ are 256 points long, the linear convolution, $x[n] * y[n]$, will be 512 points long.

We are given a 128-pt DFT only. Therefore, we must time-alias to get 128 samples. The most efficient implementation is:



Total cost = 110.

8.60. Ideally, the inverse system would be:

$$H_i(z) = z - bz^{-1}$$

Hence,

$$X(z) = (z - bz^{-1})Y(z)$$

and

$$x[n] = y[n+1] - by[n-1], \qquad -\infty \le n \le \infty$$

If we use an N-pt block of $y[n]$:

$$y[n]: \quad 0 \le n \le N,$$

then

$$V[k] = (W_N^{-k} - bW_N^{k})Y[k]$$

and

$$v[n] = y[((n+1))_N] - by[((n-1))_N]$$

Because the shift is circular, the points at $n = 0$ and $n = (N-1)$ will not be correct. Therefore, only the points in the range $1 \le n \le (N-2)$ are valid.

8.61. (a)

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}, \qquad 0 \le k \le (N-1) \\
X_M[k] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N+\pi/N)n} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)kn}
\end{aligned}$$

So, $x_M[n] = x[n]e^{-j(\pi n/N)}$.

(b)

$$\begin{aligned}
X_M[N-k] &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-k)n}, \qquad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{j(2\pi/N)k} \\
X_M[N-(k+1)] &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-k-1)n} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-1)n}e^{j(2\pi/N)kn} \\
&= \sum_{n=0}^{N-1} x[n]e^{j(\pi n/N)}e^{j(2\pi/N)kn} \\
&= X_M^*[k]
\end{aligned}$$

So,

$$X_M[k] = X_M^*[N-(k+1)]\ , \text{ for } 0 \le k \le (N-1) \text{ and } x[n] \text{ real.}$$

(c) (i) $N-k-1$ is odd when k is even. If $R[k] = X_M[2k]$, we may obtain $X_M[k]$ from $R[k]$ as follows:

$$G[k] = \begin{cases} R[k/2], & \text{k even} \\ R^*[(N-(k+1))/2], & \text{k odd} \end{cases}$$

where we note that

$$R^*[(N-(k+1))/2] = X_M^*[N-(k+1)]$$

for k odd.

(ii)

$$\begin{aligned}
R[k] &= X_M[2k] \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j(4\pi k/N+\pi/N)n} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} + \sum_{n=N/2}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} \\
&= \sum_{n=0}^{(N/2)-1} \left(x[n]e^{-j(\pi n/N)} + x[n+N/2]e^{-j(\pi n/N)}e^{-j(\pi/2)}\right)e^{-j\left(\frac{2\pi kn}{N/2}\right)}
\end{aligned}$$

So,

$$r[n] = (x[n] - jx[n+N/2])e^{-j(\pi n/N)}, \quad 0 \le n \le \left(\frac{N}{2}-1\right)$$

(d)

$$\begin{aligned}
X_{3M}[k] &= X_{1M}[k]X_{2M}[k] \\
X_{3M}[n] &= \sum_{r=0}^{N-1} x_{1M}[r]x_{2M}[((n-r))_N]
\end{aligned}$$

From part (a):

$$\begin{aligned}
x_{1M}[n] &= x_1[n]e^{-j(\pi n/N)} \\
x_{2M}[n] &= x_2[n]e^{-j(\pi n/N)} \\
x_{3M}[n] &= x_3[n]e^{-j(\pi n/N)}
\end{aligned}$$

So,

$$\begin{aligned}
x_3[n] &= e^{j(\pi n/N)} \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]e^{-j(\pi/N)[((n-r))_N+r]} \\
&= \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]e^{-j(\pi/N)[((n-r))_N-(n-r)]}
\end{aligned}$$

Since,

$$((n-r))_N = \begin{cases} n-r, & n \ge r \\ N+n-r, & n < r \end{cases}$$

$$((n-r))_N - (n-r) = \begin{cases} 0, & n \ge r \\ N, & n < r \end{cases}$$

then

$$e^{-j(\pi/N)[((n-r))_N-(n-r)]} = \text{sgn}\,[n-r] = \begin{cases} 1, & n \geq r \\ -1, & n < r \end{cases}$$

and

$$x_3 = \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]\text{sgn}\,[n-r]$$

(e) Suppose, that for $n \geq N/2$:

$$\begin{aligned} x_{1M}[n] &= x_1[n]e^{-j(\pi n/N)} = 0 \\ x_{2M}[n] &= x_2[n]e^{-j(\pi n/N)} = 0 \end{aligned}$$

then the modified circular convolution is equivalent to the modified linear convolution:

$$x_{1M}[n] \, N \, x_{2M}[n] = x_{1M}[n] * x_{2M}[n]$$

(i.e. no aliasing occurs.)

$$\begin{aligned} x_{3M}[n] &= x_{1M}[n] * x_{2M}[n] \\ &= \sum_{r=0}^{N-1} x_{1M}[r]x_{2M}[n-r] \end{aligned}$$

Thus,

$$\begin{aligned} x_3[n] &= e^{j(\pi n/N)} \sum_{r=0}^{N-1} x_1[n]x_2[n-r]e^{-j(\pi n/N)}e^{-j(\pi n/N)(n-r)} \\ &= \sum_{r=0}^{N-1} x_1[r]x_2[n-r]e^{-j(\pi/N)(n-r)} \end{aligned}$$

So,

$$x_3[n] = \sum_{r=0}^{N-1} x_1[r]x_2[n-r]e^{-j(\pi/N)(n-r)} = x_1[n] * x_2[n]$$

8.62. (a) We wish to compute $x[n] \, 63 \, h[n]$:

$$\begin{aligned} \text{let } x_1[n] &= x[n], & 0 \leq n \leq 31 \\ x_2[n] &= x[n+32], & 0 \leq n \leq 30 \\ h_1[n] &= h[n], & 0 \leq n \leq 31 \\ h_2[n] &= h[n+32], & 0 \leq n \leq 30 \end{aligned}$$

$$\begin{aligned} x[n] * h[n] &= x_1[n] * h_1[n] + x_1[n] * h_2[n] * \delta[n-32] + x_2[n] * h_1[n] * \delta[n-32] \\ &\quad + x_2[n] * h_2[n] * \delta[n-32] * \delta[n-32] \end{aligned}$$

Let

$$\begin{aligned} y_1[n] &= x_1[n] * h_1[n] &= x_1[n] \, 64 \, h_1[n] \\ y_2[n] &= x_1[n] * h_2[n] &= x_1[n] \, 64 \, h_2[h] \\ y_3[n] &= x_2[n] * h_1[n] &= x_2[n] \, 64 \, h_1[n] \\ y_4[n] &= x_2[n] * h_2[n] &= x_2[n] \, 64 \, h_2[n] \end{aligned}$$

We can compute each of the above circular convolutions with two 64-pt DFTs and one 64-pt inverse DFT.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= y_1[n] + y_2[n-32] + y_3[n-32] + y_4[n-64] \end{aligned}$$

So

$$x[n] \, (63) \, h[n] = y[n] + y[n+63], \qquad 0 \le n \le 62$$

The total computational cost is 12 DFTs of size $N = 64$.

(b) Using two 128-pt DFTs and one 128-pt inverse DFT:

$$y[n] = x[n] \, (128) \, h[n] = x[n] * h[n]$$

The 63-pt circular convolution:

$$x[n] \, (63) \, h[n] = y[n] + y[n+63], \qquad 0 \le n \le 62$$

(c) Using the 64-pt DFT method of part (a):

$$\#\text{mult} = 4(12)(64 \log_2(64)) = 18432$$

Using 128-pt DFTs:

$$\#\text{mult} = 4(3)(128 \log_2(128)) = 10752$$

Direct convolution:

$$\#\text{mult} = 2 \sum_{n=1}^{63} n - 63 = 3969$$

8.63. From each circular convolution, the first 49 points will be incorrect. Therefore, we get 51 good points and the input must be overlapped by $100 - 51 = 49$ points.

(a) $V = 49$

(b) $M = 51$

(c) The points extracted correspond to the range $49 \le n \le 99$.

Distorting filter: $h[n] = \delta[n] - \frac{1}{2}\delta[n - n_0]$

8.64. (a) The Z-transform of $h[n]$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \\ H(z) &= 1 - \frac{1}{2} z^{-n_0} \end{aligned}$$

The N-pt DFT of $h[n]$: $(N = 4n_0)$

$$\begin{aligned} H[k] &= \sum_{n=0}^{4n_0-1} h[n] W_{4n}^{kn_0}, \qquad 0 \le k \le (4n_0 - 1) \\ &= 1 - \frac{1}{2} W_{4n_0}^{kn_0} \\ H[k] &= 1 - \frac{1}{2} e^{-j(\pi/2)k} \end{aligned}$$

(b)

$$\begin{aligned} H_i(z) &= \frac{1}{1+1/2z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \text{ for causality} \\ h_i[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta[n-kn_0] \end{aligned}$$

The filter is IIR.

(c)

$$G[k] = \frac{1}{H[k]} = \frac{1}{1-e^{-j(\pi/2)k}}, \quad 0 \le k \le (4n_0 - 1)$$

The impulse response, $g[n]$, is just $h_i[n]$ time-aliased by $4n_0$ points:

$$\begin{aligned} g[n] &= \left(1 + \frac{1}{16} + \frac{1}{256} + \cdots\right)\delta[n] + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \cdots\right)\delta[n-n_0] \\ &+ \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \cdots\right)\delta[n-2n_0] + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \cdots\right)\delta[n-3n_0] \\ g[n] &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n-n_0] + \frac{4}{15}\delta[n-2n_0] + \frac{2}{15}\delta[n-3n_0] \end{aligned}$$

(d) Indeed,

$$G[k]H[k] = 1, \quad 0 \le k \le (4n_0 - 1)$$

However, this relationship is only true at $4n_0$ distinct frequencies. This fact does not imply that for all ω:

$$G(e^{j\omega})H(e^{j\omega}) = 1$$

(e)

$$\begin{aligned} y[n] &= g[n] * h[n] \\ &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n-n_0] + \frac{4}{15}\delta[n-2n_0] + \frac{2}{15}\delta[n-3n_0] - \frac{8}{15}\delta[n-n_0] \\ &\quad -\frac{4}{15}\delta[n-2n_0] - \frac{2}{15}\delta[n-3n_0] - \frac{1}{15}\delta[n-4n_0] \\ y[n] &= \frac{16}{15}\delta[n] - \frac{1}{15}\delta[n-4n_0] \end{aligned}$$

8.65. (a) We start by computing $\tilde{X}_H[k+N]$:

$$\begin{aligned} \tilde{X}_H[k+N] &= \sum_{n=0}^{N-1} \tilde{x}[n] H_N[n(k+N)] \\ &= \sum_{n=0}^{N-1} \tilde{x}[n](\cos(\frac{2\pi(nk+nN)}{N}) + \sin(\frac{2\pi(nk+nN)}{N}) \\ &= \sum_{n=0}^{N-1} \tilde{x}[n](\cos(\frac{2\pi nk}{N}) + \sin(\frac{2\pi nk}{N})) \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] H_N[nk] \\ &= \tilde{X}_H[k]. \end{aligned}$$

We thus conclude that the DHS coefficients form a sequence that is also periodic with period N.

(b) We have:

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}_H[k]H_N[nk] &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1}\tilde{x}[m]H_N[mk]\right)H_N[nk] \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\tilde{x}[m]\sum_{k=0}^{N-1}H_N[mk]H_N[nk] \\
&= \frac{1}{N}\tilde{x}[n]N \\
&= \tilde{x}[n].
\end{aligned}$$

Where we have used the fact that $\sum_{k=0}^{N-1} H_N[mk]H_N[nk] = N$ only if $((m))_N = ((n))_N$, otherwise it's 0.

This completes the derivation of the DHS synthesis formula.

(c) We have:

$$\begin{aligned}
H_N[a+N] &= \cos\left(\frac{2\pi(a+N)}{N}\right)+\sin\left(\frac{2\pi(a+N)}{N}\right) \\
&= \cos\left(\frac{2\pi a}{N}+2\pi\right)+\sin\left(\frac{2\pi a}{N}+2\pi\right) \\
&= \cos\left(\frac{2\pi a}{N}\right)+\sin\left(\frac{2\pi a}{N}\right) \\
&= H_N[a].
\end{aligned}$$

And:

$$\begin{aligned}
H_N[a+b] &= \cos\left(\frac{2\pi(a+b)}{N}\right)+\sin\left(\frac{2\pi(a+b)}{N}\right) \\
&= (C_N[a]C_N[b]-S_N[a]S_N[b])+(S_N[a]C_N[b]+C_N[a]S_N[b]) \\
&= C_N[b](C_N[a]+S_N[a])+S_N[b](-S_N[a]+C_N[a]) \\
&= C_N[b](C_N[a]+S_N[a])+S_N[b](S_N[-a]+C_N[-a]) \\
&= C_N[b]H_N[a]+S_N[b]H_N[-a] \\
&= C_N[a]H_N[b]+S_N[a]H_N[-b] \text{ (since } H_N[a+b]=H_N[b+a] \text{)}
\end{aligned}$$

Where we have used trigonometric properties.

(d) We have:

$$\begin{aligned}
DHS(\tilde{x}[n-n_0]) &= \sum_{n=0}^{N-1}\tilde{x}[n-n_0]H_N[nk] \\
&= \sum_{n=-n_0}^{N-1-n_0}\tilde{x}[n]H_N[(n+n_0)k]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n](H_N[nk]C_N[n_0k] + H_N[-nk]S_N[n_0k]) \\
&= C_N[n_0k] \sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n]H_N[nk] + S_N[n_0k] \sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n]H_N[-nk] \\
&= C_N[n_0k] \sum_{n=0}^{N-1} \tilde{x}[n]H_N[nk] + S_N[n_0k] \sum_{n=0}^{N-1} \tilde{x}[n]H_N[-nk] \\
&= C_N[n_0k]\tilde{X}_H[k] + S_N[n_0k]\tilde{X}_H[-k]
\end{aligned}
$$

Where we have used the periodicity of $H_N[nk]$ and $\tilde{x}[n]$.

(e) We have:

$$
\begin{aligned}
DHT\{\tilde{x}_3[n]\} &= DHT\{\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]\} \\
X_{H3}[k] &= \sum_{n=0}^{N-1}(\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N])H_N[nk] \\
&= \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[((n-m))_N]H_N[nk] \\
&= \sum_{m=0}^{N-1} x_1[m]DHT\{x_2[((n-m))_N]\} \\
&= \sum_{m=0}^{N-1} x_1[m](X_{H2}[k]C_N[mk] + X_{H2}[((-k))_N]S_N[mk]) \text{ (using P8.65-7)} \\
&= \sum_{m=0}^{N-1} x_1[m]X_{H2}[k]C_N[mk] + \sum_{m=0}^{N-1} x_1[m]X_{H2}[((-k))_N]S_N[mk] \\
&= \sum_{m=0}^{N-1} x_1[m]X_{H2}[k](\frac{H_N[mk]+H_N[-mk]}{2}) \\
&\quad + \sum_{m=0}^{N-1} x_1[m]X_{H2}[((-k))_N](\frac{H_N[mk]-H_N[-mk]}{2}) \\
&= \frac{1}{2}X_{H2}[k](X_{H1}[k] + X_{H1}[((-k))_N]) + \frac{1}{2}X_{H2}[((-k))_N](X_{H1}[k] - X_{H1}[((-k))_N]) \\
&= \frac{1}{2}X_{H1}[k](X_{H2}[k] + X_{H2}[((-k))_N]) + \frac{1}{2}X_{H1}[((-k))_N](X_{H2}[k] - X_{H2}[((-k))_N])
\end{aligned}
$$

This is the desired convolution property.

(f) Since the DFT of $x[n]$ is given by:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n](\cos(-\frac{2\pi kn}{N}) + j\sin(-\frac{2\pi kn}{N}))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} x[n](\cos(\frac{2\pi kn}{N}) - j\sin(\frac{2\pi kn}{N})) \\
&= \sum_{n=0}^{N-1} x[n](C_N[kn] - jS_N[kn])
\end{aligned}
$$

then:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x[n]C_N[kn] &= \frac{1}{2}(X[k] + X[((-k))_N]) \\
\sum_{n=0}^{N-1} x[n]S_N[kn] &= -\frac{1}{2j}(X[k] - X[((-k))_N])
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
X_H[k] &= \sum_{n=0}^{N-1} x[n](C_N[kn] + S_N[kn]) \\
&= \frac{1}{2}(X[k] + X[((-k))_N]) - \frac{1}{2j}(X[k] - X[((-k))_N]) \\
&= (\frac{1}{2} - \frac{1}{2j})X[k] + (\frac{1}{2} + \frac{1}{2j})X[((-k))_N]
\end{aligned}
$$

This allows us to obtain $X_H[k]$ from $X[k]$.

(g) We have:

$$
X_H[k] = \sum_{n=0}^{N-1} x[n](C_N[kn] + S_N[kn])
$$

Therefore:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x[n]C_N[kn] &= \frac{1}{2}(X_H[k] + X_H[((-k))_N]) \\
\sum_{n=0}^{N-1} x[n]S_N[kn] &= \frac{1}{2}(X_H[k] - X_H[((-k))_N])
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n](C_N[kn] - jS_N[kn])
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}(X_H[k] + X_H[((-k))_N]) - j\frac{1}{2}(X_H[k] - X_H[((-k))_N]) \\
&= (\frac{1}{2} - \frac{j}{2})X_H[k] + (\frac{1}{2} + \frac{j}{2})X_H[((-k))_N]
\end{aligned}
$$

This allows us to obtain $X[k]$ from $X_H[k]$.

8.66. (a) The DTFT is given by:

$$
\begin{aligned}
\tilde{X}(e^{jw}) &= X(e^{jw}) + X(e^{jw})e^{-jwN} \\
&= X(e^{jw})(1 + e^{-jwN})
\end{aligned}
$$

The DFT is just samples of the DTFT:

$$
\begin{aligned}
\tilde{X}[k] &= \tilde{X}(e^{jw})|_{w=\frac{2\pi k}{2N}} \\
&= X(e^{j2\pi k2N})(1 + (-1)^k)
\end{aligned}
$$

Therefore:

$$
\tilde{X}[k] = \begin{cases} 2X[\frac{k}{2}] & , \quad k \text{ even} \\ 0 & , \quad k \text{ odd} \end{cases}
$$

(b) The original system computes the following:

$$
\tilde{X}[k]H[k] = \begin{cases} 2X[\frac{k}{2}]H[k] & , \quad k \text{ even} \\ 0 & , \quad k \text{ odd} \end{cases}
$$

We thus want:

$$
\begin{aligned}
X[k]G[k] &= 2X[k]H[2k] && k = 0, ..., N-1 \\
G[k] &= 2H[2k] \\
&= 2\sum_{n=0}^{2N-1} h[n]e^{-j\frac{2\pi 2kn}{2N}} && k = 0, ..., N-1 \\
g[n] &= 2(h[n] + h[n+N])
\end{aligned}
$$

System A time aliases and multiplies by 2.

For system B, we need:

$$
Y[k] = \begin{cases} W[\frac{k}{2}] & , \quad k \text{ even} \\ 0 & , \quad k \text{ odd} \end{cases}
$$

Thus:

$$y[n] = \begin{cases} w[n] & , \quad 0 \le n \le N-1 \\ w[n-N] & , \quad N \le n \le 2N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

System B regenerates the $2N$-point sequence by repeating w[n].

8.67. (a) We have:

$$|H(e^{jw})| = \begin{cases} 1 & , \quad |w| \le \frac{\pi}{4} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Since $h[n]$ is FIR, we assume it is non-zero over $0 \le n \le N$. The phase of $H(e^{jw})$ should be set such that $h[n]$ is symmetric about the center of its range, i.e. $\frac{N}{2}$. Therefore, the phase of $H(e^{jw})$ should be $e^{j\frac{wN}{2}}$. So one possible $H[k]$ may be:

$$H[k] = \begin{cases} e^{j\frac{2\pi}{4N}\frac{N}{2}k} & , \quad 0 \le k \le \frac{1}{8}4N \\ 0 & , \quad \text{otherwise} \\ e^{j\frac{2\pi}{4N}\frac{N}{2}k} & , \quad 4N - \frac{N}{2} \le k \le 4N \end{cases}$$

that is:

$$H[k] = \begin{cases} e^{j\frac{\pi}{4}k} & , \quad 0 \le k \le \frac{N}{2} \\ 0 & , \quad \text{otherwise} \\ e^{j\frac{\pi}{4}k} & , \quad 4N - \frac{N}{2} \le k \le 4N \end{cases}$$

(b) System A needs to perform the following operations:

$$Y_2[k] = \begin{cases} X[k]H'[k] & , \quad 0 \le k \le \frac{N}{2} \\ 0 & , \quad \text{otherwise} \\ X[k-3N]H'[k-3N] & , \quad 4N - \frac{N}{2} \le k \le 4N \end{cases}$$

Where $H'[k]$ is the N-point DFT of $h[n]$.

(c) It is cheaper to implement N-point DFTs than $4N$-point DFTs, therefore the implementation in Figure P8.67-2 is usually preferable to the one in Figure P8.67-1.

8.68. Substituting the expression for $X_1[k]$ from equation (8.164) into equation (8.165), we get:

$$\begin{aligned} x_1[n] &= \frac{1}{2N-2}\sum_{k=0}^{2N-3} X_1[k]e^{j2\pi kn/(2N-2)} \\ &= \frac{1}{2N-2}\Big(\sum_{k=0}^{N-1} X^{c1}[k]e^{j2\pi kn/(2N-2)} + \sum_{k=N}^{2N-3} X^{c1}[2N-2-k]e^{j2\pi kn/(2N-2)}\Big) \end{aligned}$$

Note that:

$$\begin{aligned}
\sum_{k=N}^{2N-3} X^{c1}[2N-2-k]e^{j2\pi kn/(2N-2)} &= \sum_{r=1}^{N-2} X^{c1}[r]e^{j2\pi(2N-2-r)n/(2N-2)} \\
&= \sum_{k=1}^{N-2} X^{c1}[k]e^{-j2\pi kn/(2N-2)}
\end{aligned}$$

therefore:

$$\begin{aligned}
x_1[n] &= \frac{1}{2N-2}\Big(\sum_{k=0}^{N-1} X^{c1}[k]e^{j2\pi kn/(2N-2)} + \sum_{k=1}^{N-2} X^{c1}[k]e^{-j2\pi kn/(2N-2)}\Big) \\
&= \frac{1}{2N-2}\Big(X^{c1}[0] + X^{c1}[N-1]e^{j2\pi n} + \sum_{k=1}^{N-2} X^{c1}[k](e^{j2\pi kn/(2N-2)} + e^{-j2\pi kn/(2N-2)})\Big) \\
&= \frac{1}{2N-2}\Big(X^{c1}[0] + X^{c1}[N-1]e^{j\pi n} + \sum_{k=1}^{N-2} X^{c1}[k]2\cos(\frac{\pi kn}{N-1})\Big)
\end{aligned}$$

and:

$$\begin{aligned}
x[n] &= x_1[n] && \text{for } n = 0, 1, ..., N-1 \\
&= \frac{1}{N-1}\Big(\sum_{k=0}^{N-1} \alpha[k]X^{c1}[k]\cos(\frac{\pi kn}{N-1})\Big) && 0 \le n \le N-1
\end{aligned}$$

where $\alpha[k]$ is given by:

$$\alpha[k] = \begin{cases} \frac{1}{2} &, \quad k = 0 \text{ and } N-1 \\ 1 &, \quad 1 \le k \le N-2. \end{cases}$$

This completes the derivation.

8.69.

$$v[n] = x_2[2n]$$

therefore, for $k = 0, 1, ..., N-1$:

$$V[k] = \frac{1}{2}(X_2[k] + X_2[k+N]).$$

Using equation (8.168), we have:

$$\begin{aligned}
V[k] &= \frac{1}{2}(X_2[k] + X_2[k+N]) \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + e^{j\frac{\pi(k+N)}{2N}} Re\{X[k+N]e^{-j\frac{\pi(k+N)}{2N}}\} \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + e^{j\frac{\pi}{2}}e^{j\frac{\pi k}{2N}} Re\{X[k+N]e^{-j\frac{\pi k}{2N}}e^{-j\frac{\pi}{2}}\} \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + je^{j\frac{\pi k}{2N}} Re\{-jX[k+N]e^{-j\frac{\pi k}{2N}}\} \\
&= e^{j\frac{\pi k}{2N}}(Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + jIm\{X[k+N]e^{-j\frac{\pi k}{2N}}\}.
\end{aligned}$$

Using the above expression, we get:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}}V[k]\} &= 2Re\{e^{-j\frac{\pi k}{2N}}e^{j\frac{\pi k}{2N}}(Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + jIm\{X[k+N]e^{-j\frac{\pi k}{2N}}\}))\} \\
&= 2Re\{X[k]e^{-j\frac{\pi k}{2N}}\} \\
&= X^{c2}[k] \qquad \text{where we used equation (8.170).}
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}}V[k]\} &= 2Re\{e^{-j\frac{2\pi k}{4N}}\sum_{n=0}^{N-1} v[n]e^{-j\frac{2\pi kn}{N}}\} \\
&= 2Re\{\sum_{n=0}^{N-1} v[n]e^{-j(\frac{2\pi k}{N}(n+\frac{1}{4}))}\} \\
&= 2\sum_{n=0}^{N-1} Re\{v[n]e^{-j(\frac{2\pi k}{N}(n+\frac{1}{4}))}\} \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{2\pi k}{N}(n+\frac{1}{4})) \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{\pi k(4n+1)}{2N}).
\end{aligned}$$

and:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}}V[k]\} &= 2Re\{X[k]e^{-j\frac{\pi k}{2N}}\} \\
&= 2Re\{\sum_{n=0}^{2N-1} x[n]e^{-j\frac{2\pi kn}{2N}}e^{-j\frac{\pi k}{2N}}\} \\
&= 2Re\{\sum_{n=0}^{N-1} x[n]e^{-j\frac{\pi k}{2N}(2n+1)}\} \\
&= 2\sum_{n=0}^{N-1} Re\{x[n]e^{-j\frac{\pi k}{2N}(2n+1)}\} \\
&= 2\sum_{n=0}^{N-1} x[n]\cos(\frac{\pi k(2n+1)}{2N}).
\end{aligned}$$

From the results above, we conclude, for $k = 0, 1, ..., N-1$:

$$\begin{aligned}
X^{c2}[k] &= 2Re\{e^{-j\frac{2\pi k}{4N}}V[k]\} \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{\pi k(4n+1)}{2N}) \\
&= 2\sum_{n=0}^{N-1} x[n]\cos(\frac{\pi k(2n+1)}{2N}).
\end{aligned}$$

8.70. Substituting the expression for $X_2[k]$ from equation (8.174) into equation (8.175), we get:

$$
\begin{aligned}
x_2[n] &= \frac{1}{2N}\sum_{k=0}^{2N-1} X_2[k]e^{j2\pi kn/(2N)} \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k/(2N)}e^{j2\pi kn/(2N)} - \sum_{k=N+1}^{2N-1} X^{c2}[2N-k]e^{j\pi k/(2N)}e^{j2\pi kn/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} - \sum_{k=N+1}^{2N-1} X^{c2}[2N-k]e^{j\pi k(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} - \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi(2N-k)(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} + \sum_{k=1}^{N-1} X^{c2}[k]e^{-j\pi k(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k](e^{j\pi k(2n+1)/(2N)} + e^{-j\pi k(2n+1)/(2N)})) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]\cos(\frac{\pi k(2n+1)}{2N})).
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
x[n] &= x_2[n] && \text{for } n = 0, 1, ..., N-1 \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \beta[k]X^{c2}[k]\cos(\frac{\pi k(2n+1)}{2N}) && 0 \le n \le N-1
\end{aligned}
$$

where $\beta[k]$ is given by:

$$
\beta[k] = \begin{cases} \frac{1}{2} & , \quad k = 0 \\ 1 & , \quad 1 \le k \le N-1. \end{cases}
$$

This completes the derivation.

8.71. First we derive Parseval's theorem for the DFT.
Let $x[n]$ be an N point sequence and define $y[n]$ as follows:

$$y[n] = x[n] \circledN x^*[((-n))_N].$$

Using the properties of the DFT, we have:

$$Y[k] = X[k]X^*[k] = |X[k]|^2.$$

Note that:

$$y[0] = \sum_n |x[n]|^2$$

and using the DFT synthesis equation, we get:

$$y[0] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k].$$

Parseval's Theorem for the DFT is therefore:

$$\sum_n |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

(a) Note that:

$$\sum_{n=0}^{N-1} |X^{c1}[k]|^2 = \sum_{n=0}^{N-1} |X_1[k]|^2$$

and, using equation (8.164):

$$\sum_{n=0}^{2N-3} |X_1[k]|^2 = 2\sum_{n=0}^{N-1} |X^{c1}[k]|^2 - |X^{c1}[0]|^2 - |X^{c1}[N-1]|^2.$$

Using the DFT properties:

$$\sum_n |x_1[n]|^2 = \frac{1}{2N-2}\sum_{k=0}^{2N-3} |X_1[k]|^2$$

and, using equation (8.161):

$$\sum_{n=0}^{2N-3} |x_1[n]|^2 = 2\sum_{n=0}^{N-1} |x[n]|^2 - |x[0]|^2 - |x[N-1]|^2.$$

We thus conclude:

$$\frac{1}{2N-2}(2\sum_{n=0}^{N-1} |X^{c1}[k]|^2 - |X^{c1}[0]|^2 - |X^{c1}[N-1]|^2) = 2\sum_{n=0}^{N-1} |x[n]|^2 - |x[0]|^2 - |x[N-1]|^2.$$

(b) Using equation (8.171),

$$\sum_{n=0}^{N-1} |X^{c2}[k]|^2 = \sum_{n=0}^{N-1} |X_2[k]|^2.$$

Note that, using equation (8.167):

$$\sum_{k=0}^{2N-1} |X_2[k]|^2 = 2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2,$$

and, using equation (8.166):

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = 2\sum_{n=0}^{N-1} |x[n]|^2.$$

Using the DFT properties:

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = \frac{1}{2N}\sum_{k=0}^{2N-1} |X_2[k]|^2.$$

We thus conclude:

$$\frac{1}{2N}(2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2) = 2\sum_{n=0}^{N-1} |x[n]|^2.$$