Solutions – Chapter 7

Filter Design Techniques

7.1. Using the partial fraction technique, we see

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{0.5}{s+a+jb} + \frac{0.5}{s+a-jb}$$

Now we can use the Laplace transform pair

$$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{s+\alpha}$$

to get

$$h_c(t) = \frac{1}{2}\left(e^{-(a+jb)t} + e^{-(a-jb)t}\right)u(t).$$

(a) Therefore,

$$\begin{aligned} h_1[n] &= h_c(nT) = \frac{1}{2}\left[e^{-(a+jb)nT} + e^{-(a-jb)nT}\right]u[n] \\ H_1(z) &= \frac{0.5}{1-e^{-(a+jb)T}z^{-1}} + \frac{0.5}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

(b) Since

$$s_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau \longleftrightarrow \frac{H_c(s)}{s} = S_c(s)$$

we get

$$S_c(s) = \frac{s+a}{s(s+a+jb)(s+a-jb)} = \frac{A_1}{s} + \frac{A_2}{s+a+jb} + \frac{A_2^*}{s+a-jb}$$

where

$$A_1 = \frac{a}{a^2+b^2}, \qquad A_2 = -\frac{0.5}{a+jb}$$

Though the system $h_2[n]$ is related by step invariance to $h_c(t)$, the signal $s_2[n]$ is related to $s_c(t)$ by impulse invariance. Therefore, we know the poles of the partial fraction expansion of $S_c(s)$ above must transform as $z_k = e^{s_k T}$, and we can find

$$S_2(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{A_2^*}{1-e^{-(a-jb)T}z^{-1}}$$

Now, since the relationship between the step response and the impulse response is

$$s_2[n] = \sum_{k=-\infty}^{n} h_2[k] = \sum_{k=-\infty}^{\infty} h_2[k]u[n-k] = h_2[n] * u[n]$$

$$S_2(z) = \frac{H_2(z)}{1-z^{-1}}$$

We can finally solve for $H_2(z)$

$$\begin{aligned} H_2(z) &= S_2(z)(1-z^{-1}) \\ &= A_1 + A_2\frac{1-z^{-1}}{1-e^{-(a+jb)T}z^{-1}} + A_2^*\frac{1-z^{-1}}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

where A_1 and A_2 are as given above.

(c)

$$\begin{aligned}
s_1[n] &= \sum_{k=-\infty}^{n} h_1[k] = \frac{1}{2}\sum_{k=0}^{n}\left(e^{-(a+jb)kT} + e^{-(a-jb)kT}\right) \\
&= \frac{1}{2}\left[\frac{1-e^{-(a+jb)(n+1)T}}{1-e^{-(a+jb)T}} + \frac{1-e^{-(a-jb)(n+1)T}}{1-e^{-(a-jb)T}}\right]u[n] \\
&= \left[B_1 + B_2 e^{-(a+jb)Tn} + B_2^* e^{-(a-jb)Tn}\right]u[n]
\end{aligned}$$

where

$$B_1 = \frac{1-e^{-aT}\cos bT}{1-2e^{-aT}\cos bT + e^{-2aT}}, \qquad B_2 = -\frac{e^{-(a+jb)T}}{1-e^{-(a+jb)T}}$$

¿From this we can see that

$$\begin{aligned}
S_1(z) &= \frac{B_1}{1-z^{-1}} + \frac{B_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{B_2^*}{1-e^{-(a-jb)T}z^{-1}} \\
&\neq S_2(z)
\end{aligned}$$

since the partial fraction constants are different. Therefore, $s_1[n] \neq s_2[n]$, the two step responses are not equal.

Taking the inverse z-transform of $H_2(z)$

$$\begin{aligned}
h_2[n] &= A_1\delta[n] + A_2\left[e^{-(a+jb)Tn}u[n] - e^{-(a+jb)T(n-1)}u[n-1]\right] \\
&\quad + A_2^*\left[e^{-(a-jb)Tn}u[n] - e^{-(a-jb)T(n-1)}u[n-1]\right]
\end{aligned}$$

where A_1 and A_2 are as defined earlier. By comparing $h_1[n]$ and $h_2[n]$ one sees that $h_1[n] \neq h_2[n]$.

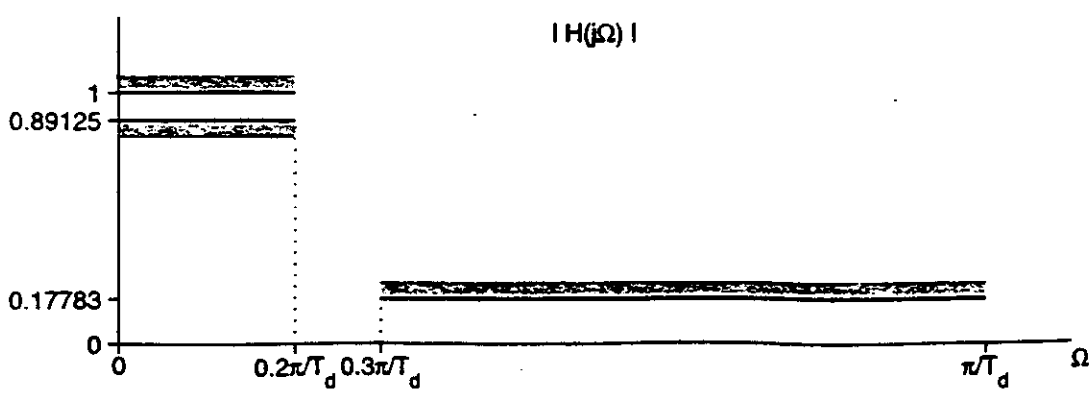
The overall idea this problem illustrates is that a filter designed with impulse invariance is different from a filter designed with step invariance.

7.2. Recall that $\Omega = \omega/T_d$.

(a) Then

$$\begin{aligned}
&0.89125 \leq |H(j\Omega)| \leq 1, && 0 \leq |\Omega| \leq 0.2\pi/T_d \\
&|H(j\Omega)| \leq 0.17783, && 0.3\pi/T_d \leq |\Omega| \leq \pi/T_d
\end{aligned}$$

The plot of the tolerance scheme is



(b) As in the book's example, since the Butterworth frequency response is monotonic, we can solve

$$|H_c(j0.2\pi/T_d)|^2 = \frac{1}{1+\left(\dfrac{0.2\pi}{\Omega_c T_d}\right)^{2N}} = (0.89125)^2$$

$$|H_c(j0.3\pi/T_d)|^2 = \frac{1}{1+\left(\dfrac{0.3\pi}{\Omega_c T_d}\right)^{2N}} = (0.17783)^2$$

to get $\Omega_c T_d = 0.70474$ and $N = 5.8858$. Rounding up to $N = 6$ yields $\Omega_c T_d = 0.7032$ to meet the specifications.

(c) We see that the poles of the magnitude-squared function are again evenly distributed around a circle of radius 0.7032. Therefore, $H_c(s)$ is formed from the left half-plane poles of the magnitude-squared function, and the result is the same for any value of T_d. Correspondingly, $H(z)$ does not depend on T_d.

7.3. We are given the digital filter constraints

$$\begin{array}{ll} 1-\delta_1 \le |H(e^{j\omega})| \le 1+\delta_1, & 0 \le |\omega| \le \omega_p \\ |H(e^{j\omega})| \le \delta_2, & \omega_s \le |\omega| \le \pi \end{array}$$

and the analog filter constraints

$$\begin{array}{ll} 1-\hat{\delta}_1 \le |H_c(j\Omega)| \le 1, & 0 \le |\Omega| \le \Omega_p \\ |H_c(j\Omega)| \le \hat{\delta}_2, & \Omega_s \le |\Omega| \end{array}$$

(a) If we divide the digital frequency specifications by $(1+\delta_1)$ we get

$$\begin{aligned} 1-\hat{\delta}_1 &= \frac{1-\delta_1}{1+\delta_1} \\ \hat{\delta}_1 &= \frac{2\delta_1}{1+\delta_1} \\ \hat{\delta}_2 &= \frac{\delta_2}{1+\delta_1} \end{aligned}$$

(b) Solving the equations in Part (a) for δ_1 and δ_2, we find

$$\begin{aligned} \delta_1 &= \frac{\hat{\delta}_1}{2-\hat{\delta}_1} \\ \delta_2 &= \frac{2\hat{\delta}_2}{2-\hat{\delta}_1} \end{aligned}$$

In the example, we were given

$$\begin{aligned} \hat{\delta}_1 &= 1-0.89125 = 0.10875 \\ \hat{\delta}_2 &= 0.17783 \end{aligned}$$

Plugging in these values into the equations for δ_1 and δ_2, we find

$$\begin{aligned} \delta_1 &= 0.0575 \\ \delta_2 &= 0.1881 \end{aligned}$$

The filter $H'(z)$ satisfies the discrete-time filter specifications where $H'(z) = (1+\delta_1)H(z)$ and $H(z)$ is the filter designed in the example. Thus,

$$
\begin{aligned}
H'(z) &= 1.0575\left[\frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}}\right. \\
&\quad \left. + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}\right] \\
&= \frac{0.3036 - 0.4723z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.2660 + 1.2114z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} \\
&\quad + \frac{1.9624 - 0.6665z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}
\end{aligned}
$$

(c) Following the same procedure used in part (b) we find

$$
\begin{aligned}
H'(z) &= 1.0575\left[\frac{0.0007378(1+z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})}\right. \\
&\quad \left. \times \frac{1}{1 - 0.9044z^{-1} + 0.2155z^{-2}}\right] \\
&= \frac{0.0007802(1+z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})} \\
&\quad \times \frac{1}{1 - 0.9044z^{-1} + 0.2155z^{-2}}
\end{aligned}
$$

7.4. (a) In the impulse invariance design, the poles transform as $z_k = e^{s_k T_d}$ and we have the relationship

$$\frac{1}{s+a} \longleftrightarrow \frac{T_d}{1 - e^{-aT_d}z^{-1}}$$

Therefore,

$$
\begin{aligned}
H_c(s) &= \frac{2/T_d}{s+0.1} - \frac{1/T_d}{s+0.2} \\
&= \frac{1}{s+0.1} - \frac{0.5}{s+0.2}
\end{aligned}
$$

The above solution is not unique due to the periodicity of $z = e^{j\omega}$. A more general answer is

$$H_c(s) = \frac{2/T_d}{s + \left(0.1 + j\frac{2\pi k}{T_d}\right)} - \frac{1/T_d}{s + \left(0.2 + j\frac{2\pi l}{T_d}\right)}$$

where k and l are integers.

(b) Using the inverse relationship for the bilinear transform,

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}$$

we get

$$
\begin{aligned}
H_c(s) &= \frac{2}{1 - e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1 - e^{-0.4}\left(\frac{1-s}{1+s}\right)} \\
&= \frac{2(s+1)}{s(1+e^{-0.2}) + (1-e^{-0.2})} - \frac{(s+1)}{s(1+e^{-0.4}) + (1-e^{-0.4})} \\
&= \left(\frac{2}{1+e^{-0.2}}\right)\left(\frac{s+1}{s + \frac{1-e^{-0.2}}{1+e^{-0.2}}}\right) - \left(\frac{1}{1+e^{-0.4}}\right)\left(\frac{s+1}{s + \frac{1-e^{-0.4}}{1+e^{-0.4}}}\right)
\end{aligned}
$$

Since the bilinear transform does not introduce any ambiguity, the representation is unique.

7.5. (a) We must use the minimum specifications!

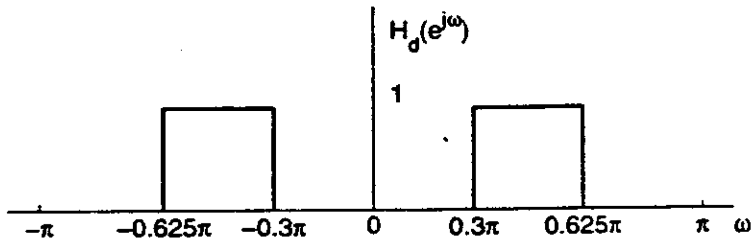
$$\begin{aligned}\delta &= 0.01\\ \Delta\omega &= 0.05\pi\\ \mathrm{A} &= -20\log_{10}\delta = 40\end{aligned}$$

$$M+1 = \frac{\mathrm{A}-8}{2.285\Delta\omega}+1 = 90.2 \rightarrow 91$$

$$\beta = 0.5842(\mathrm{A}-21)^{0.4} + 0.07886(\mathrm{A}-21) = 3.395$$

(b) Since it is a linear phase filter with order 90, it has a delay of $90/2 = 45$ samples.

(c)



$$h_d[n] = \frac{\sin(.625\pi(n-45)) - \sin(.3\pi(n-45))}{\pi(n-45)}$$

7.6. (a) The Kaiser formulas say that a discontinuity of height 1 produces a peak error of δ. If a filter has a discontinuity of a different height the peak error should be scaled appropriately. This filter can be thought of as the sum of two filters. This first is a lowpass filter with a discontinuity of 1 and a peak error of δ. The second is a highpass filter with a discontinuity of 2 and a peak error of 2δ. In the region $0.3\pi \le |\omega| \le 0.475\pi$, the two peak errors add but must be less or equal to than 0.06.

$$\begin{aligned}\delta + 2\delta &\le 0.06\\ \delta_{\max} &= 0.02\end{aligned}$$

$$A = -20\log(0.02) = 33.9794$$

$$\beta = 0.5842(33.9794-21)^{0.4} + 0.07886(33.9794-21) = 2.65$$

(b) The transition width can be

$$\begin{aligned}\Delta\omega &= 0.3\pi - 0.2\pi\\ &= 0.1\pi \text{ rad}\end{aligned} \quad \text{or} \quad \begin{aligned}\Delta\omega &= 0.525\pi - 0.475\pi\\ &= 0.05\pi \text{ rad}\end{aligned}$$

We must choose the smallest transition width so $\Delta\omega_{\max} = 0.05\pi$ rad. The corresponding value of M is

$$M = \frac{33.9794-8}{2.285(0.05\pi)} = 72.38 \rightarrow 73$$

7.7. Using the relation $\omega = \Omega T$, the passband cutoff frequency, ω_p, and the stopband cutoff frequency, ω_s, are found to be

$$\begin{aligned}\omega_p &= 2\pi(1000)10^{-4}\\ &= 0.2\pi \text{ rad}\\ \omega_s &= 2\pi(1100)10^{-4}\\ &= 0.22\pi \text{ rad}\end{aligned}$$

Therefore, the specifications for the discrete-time frequency response $H_d(e^{j\omega})$ are

$$\begin{aligned} 0.99 \le \left|H_d(e^{j\omega})\right| \le 1.01, &\qquad 0 \le |\omega| \le 0.20\pi \\ \left|H_d(e^{j\omega})\right| \le 0.01, &\qquad 0.22\pi \le |\omega| \le \pi \end{aligned}$$

7.8. Optimal Type I filters must have either $L+2$ or $L+3$ alternations. The filter is 9 samples long so its order is 8 and $L = M/2 = 4$. Thus, to be optimal, the filter must have either 6 or 7 alternations.

Filter 1:	6 alternations Meets optimal conditions	Filter 2:	7 alternations Meets optimal conditions

7.9. Using the relation $\omega = \Omega T$, the cutoff frequency ω_c for the resulting discrete-time filter is

$$\begin{aligned} \omega_c &= \Omega_c T \\ &= [2\pi(1000)][0.0002] \\ &= 0.4\pi \text{ rad} \end{aligned}$$

7.10. Using the bilinear transform frequency mapping equation,

$$\begin{aligned} \omega_c &= 2\tan^{-1}\left(\frac{\Omega_c T}{2}\right) \\ &= 2\tan^{-1}\left(\frac{2\pi(2000)(0.4\times 10^{-3})}{2}\right) \\ &= 0.7589\pi \text{ rad} \end{aligned}$$

7.11. Using the relation $\omega = \Omega T$,

$$\begin{aligned} \Omega_c &= \frac{\omega_c}{T} \\ &= \frac{\pi/4}{0.0001} \\ &= 2500\pi \\ &= 2\pi(1250)\ \frac{\text{rad}}{\text{s}} \end{aligned}$$

7.12. Using the bilinear transform frequency mapping equation,

$$\begin{aligned} \Omega_c &= \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) \\ &= \frac{2}{0.001}\tan\left(\frac{\pi/2}{2}\right) \\ &= 2000\ \frac{\text{rad}}{\text{s}} \\ &= 2\pi(318.3)\ \frac{\text{rad}}{\text{s}} \end{aligned}$$

7.13. Using the relation $\omega = \Omega T$,

$$\begin{aligned} T &= \frac{\omega_c}{\Omega_c} \\ &= \frac{2\pi/5}{2\pi(4000)} \\ &= 50\ \mu\text{s} \end{aligned}$$

This value of T is unique. Although one can find other values of T that will alias the continuous-time frequency $\Omega_c = 2\pi(4000)$ rad/s to the discrete-time frequency $\omega_c = 2\pi/5$ rad, the resulting aliased filter will not be the ideal lowpass filter.

7.14. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}
\Omega_c &= \frac{2}{T}\tan\left(\frac{\omega_c + 2\pi k}{2}\right), \quad \text{k an integer} \\
&= \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)
\end{aligned}$$

$$T = \frac{2}{2\pi(300)}\tan\left(\frac{3\pi/5}{2}\right) = 1.46 \text{ ms}$$

The only ambiguity in the above is the periodicity in ω. However, the periodicity of the tangent function "cancels" the ambiguity and so T is unique.

7.15. This filter requires a maximal passband error of $\delta_p = 0.05$, and a maximal stopband error of $\delta_s = 0.1$. Converting these values to dB gives

$$\delta p = -26 \text{ dB}$$
$$\delta s = -20 \text{ dB}$$

This requires a window with a peak approximation error less than -26 dB. Looking in Table 7.1, the Hanning, Hamming, and Blackman windows meet this criterion.

Next, the minimum length L required for each of these filters can be found using the "approximate width of mainlobe" column in the table since the mainlobe width is about equal to the transition width. Note that the actual length of the filter is $L = M + 1$.

Hanning:

$$\begin{aligned}
0.1\pi &= \frac{8\pi}{M} \\
M &= 80
\end{aligned}$$

Hamming:

$$\begin{aligned}
0.1\pi &= \frac{8\pi}{M} \\
M &= 80
\end{aligned}$$

Blackman:

$$\begin{aligned}
0.1\pi &= \frac{12\pi}{M} \\
M &= 120
\end{aligned}$$

7.16. Since filters designed by the window method inherently have $\delta_1 = \delta_2$ we must use the smaller value for δ.

$$\begin{aligned}
\delta &= 0.02 \\
A &= -20\log_{10}(0.02) = 33.9794
\end{aligned}$$

$$\beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65$$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{33.9794 - 8}{2.285(0.65\pi - 0.63\pi)} = 180.95 \rightarrow 181$$

7.17. Using the relation $\omega = \Omega T$, the specifications which should be used to design the prototype continuous-time filter are

$$\begin{aligned}
-0.02 < H(j\Omega) < 0.02, &\quad 0 \le |\Omega| \le 2\pi(20) \\
0.95 < H(j\Omega) < 1.05, &\quad 2\pi(30) \le |\Omega| \le 2\pi(70) \\
-0.001 < H(j\Omega) < 0.001, &\quad 2\pi(75) \le |\Omega| \le 2\pi(100)
\end{aligned}$$

Note: Typically, a continuous-time filter's passband tolerance is between 1 and $1-\delta_1$ since historically most continuous-time filter approximation methods were developed for passive systems which have a gain less than one. If necessary, specifications using this convention can be obtained from the above specifications by scaling the magnitude response by $\frac{1}{1.05}$.

7.18. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}\Omega_s &= \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.2\pi}{2}\right) = 2\pi(51.7126)\ \frac{\text{rad}}{\text{s}} \\ \Omega_p &= \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.3\pi}{2}\right) = 2\pi(81.0935)\ \frac{\text{rad}}{\text{s}}\end{aligned}$$

Thus, the specifications which should be used to design the prototype continuous-time filter are

$$\begin{aligned}|H_c(j\Omega)| < 0.04, &\quad |\Omega| \le 2\pi(51.7126) \\ 0.995 < |H_c(j\Omega)| < 1.005, &\quad |\Omega| \ge 2\pi(81.0935)\end{aligned}$$

Note: Typically, a continuous-time filter's passband tolerance is between 1 and $1-\delta_1$ since historically most continuous-time filter approximation methods were developed for passive systems which have a gain less than one. If necessary, specifications using this convention can be obtained from the above specifications by scaling the magnitude response by $\frac{1}{1.005}$.

7.19. Using the relation $\omega = \Omega T$,

$$\begin{aligned}T &= \frac{\omega}{\Omega} \\ &= \frac{\pi/4}{2\pi(300)} \\ &= 417\ \mu s\end{aligned}$$

This choice of T is unique. It is possible to find other values of T that alias one of the given continuous-time band edges to its corresponding discrete-time band edge. However, this is the only value of T that maps both band edges correctly.

7.20. *True*. The bilinear transform is a frequency mapping. The value of $H(s)$ for a particular value of s gets mapped to $H(e^{j\omega})$ at a particular value of ω according to the mapping

$$s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right).$$

The continuous frequency axis gets warped onto the discrete-time frequency axis, but the magnitude values do not change. If $H(s)$ is constant for all s, then $H(e^{j\omega})$ must also be constant.

7.21. (a) Using the bilinear transform frequency mapping equation,

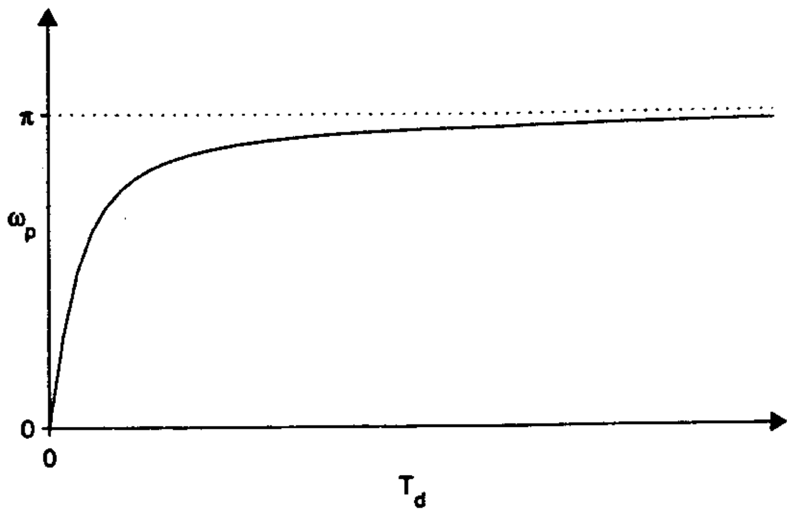
$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)$$

we have

$$\begin{aligned}T_d &= \frac{2}{\Omega_p}\tan\left(\frac{\pi}{4}\right) \\ &= \frac{2}{\Omega_p}\end{aligned}$$

(b)

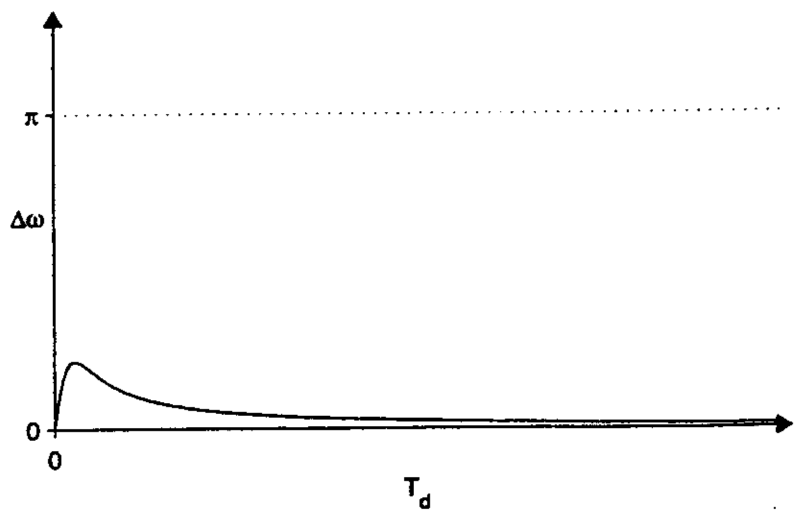
$$\omega_p = 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)$$



(c)

$$\begin{aligned} \omega_s &= 2\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) \\ \omega_p &= 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right) \end{aligned}$$

$$\Delta\omega = \omega_s - \omega_p = 2\left[\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) - \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)\right]$$



7.22. (a) Applying the bilinear transform yields

$$\begin{aligned} H(z) &= H_c(s)\,\big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{T_d}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right), \qquad |z| > 1 \end{aligned}$$

which has the impulse response

$$h[n] = \frac{T_d}{2}\left(u[n] + u[n-1]\right)$$

(b) The difference equation is

$$y[n] = \frac{T_d}{2}\left(x[n] + x[n-1]\right) + y[n-1]$$

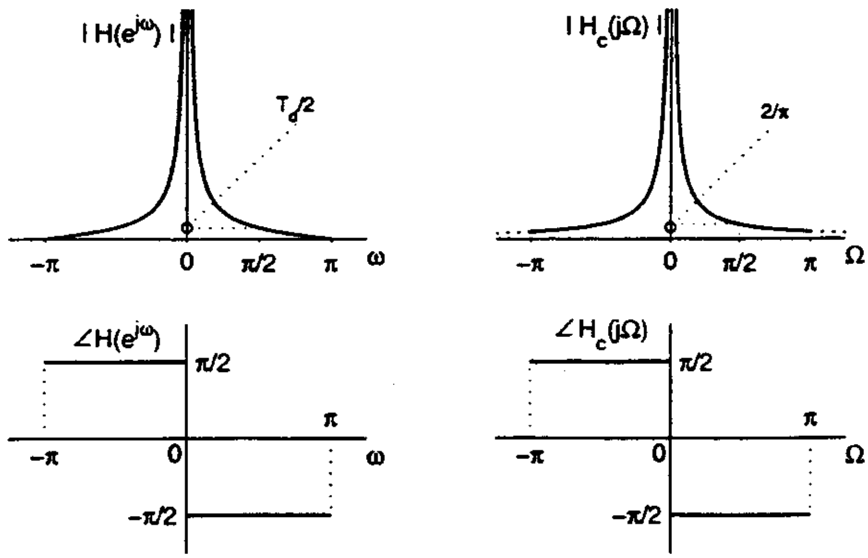
This system is not implementable since it has a pole on the unit circle and is therefore not stable.

(c) Since this system is not stable, it does not strictly have a frequency response. However, if we ignore this mathematical subtlety we get

$$\begin{aligned} H(e^{j\omega}) &= \frac{T_d}{2}\left(\frac{1+e^{-j\omega}}{1-e^{-j\omega}}\right) \\ &= \frac{T_d}{2}\left(\frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\ &= \frac{T_d}{2j}\cot(\omega/2) \end{aligned}$$

and since the Laplace transform evaluated along the $j\Omega$ axis is the continous-time Fourier transform we also have

$$H_c(j\Omega) = \frac{1}{j\Omega}$$



In general, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(d) Applying the bilinear transform yields

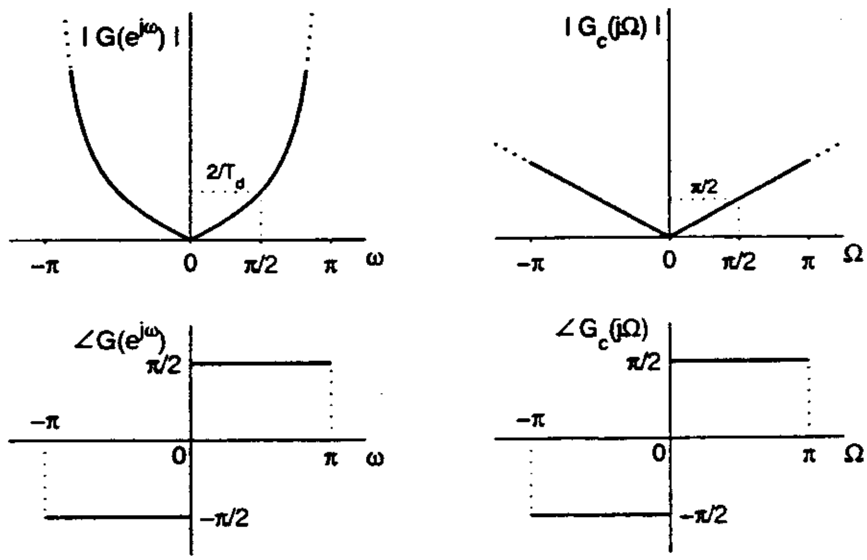
$$\begin{aligned} G(z) &= H_c(s)\,|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right], \qquad |z| > 1 \end{aligned}$$

which has the impulse response

$$\begin{aligned} g[n] &= \frac{2}{T_d}\left[(-1)^n u[n] - (-1)^{n-1}u[n-1]\right] \\ &= \frac{2}{T_d}\left[2(-1)^n u[n] - \delta[n]\right] \end{aligned}$$

(e) This system does not strictly have a frequency response either, due to the pole on the unit circle. However, ignoring this fact again we get

$$\begin{aligned} G(e^{j\omega}) &= \frac{2}{T_d}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] \\ &= \frac{2}{T_d}\left(\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right) \\ &= \frac{2j}{T_d}\tan(\omega/2) \\ G(j\Omega) &= j\Omega \end{aligned}$$

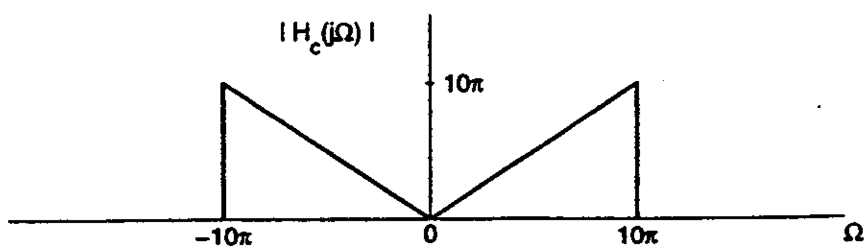


Again, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(f) If the same value of T_d is used for each bilinear transform, then the two systems are inverses of each other, since then

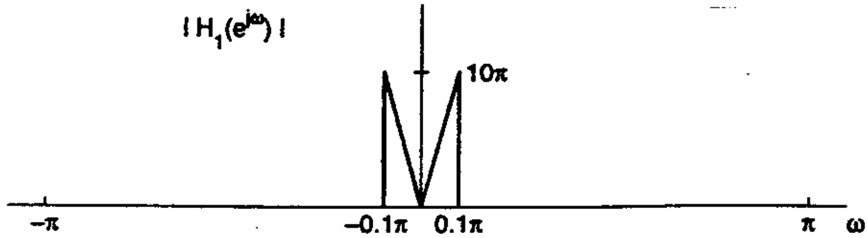
$$H(e^{j\omega})G(e^{j\omega}) = 1$$

7.23. We start with $|H_c(j\Omega)|$,



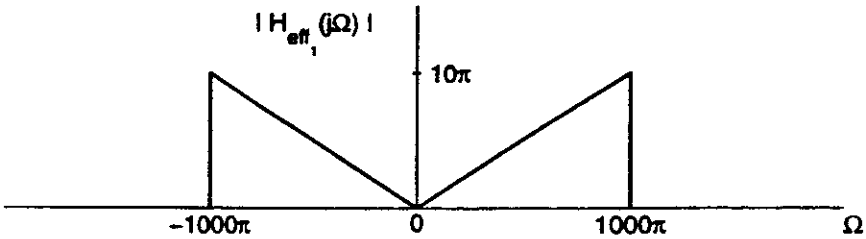
(a) By impulse invariance we scale the frequency axis by T_d to get

$$|H_1(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right)\right|$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_1}(j\Omega)| = \begin{cases} |H_1(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

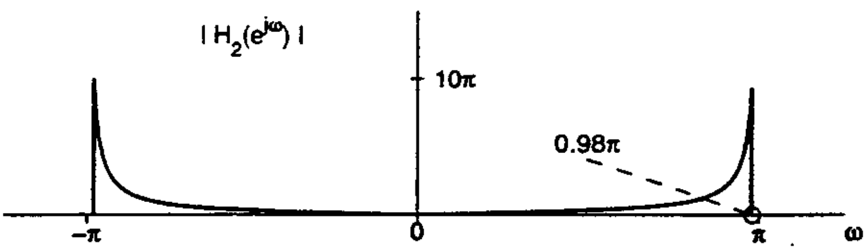


(b) Using the frequency mapping relationships of the bilinear transform,

$$\begin{aligned} \Omega &= \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right), \\ \omega &= 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right), \end{aligned}$$

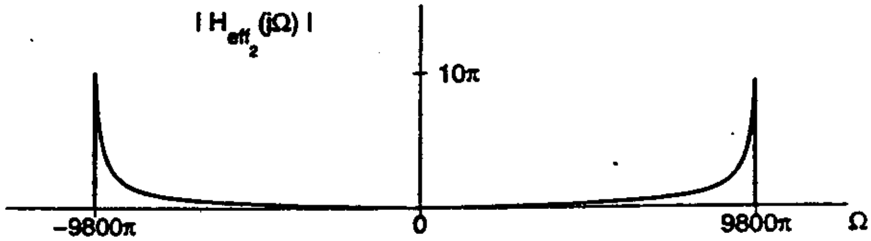
we get

$$|H_2(e^{j\omega})| = \begin{cases} |\tan\left(\frac{\omega}{2}\right)|, & |\omega| < 2\tan^{-1}(10\pi) = 0.98\pi \\ 0, & \text{otherwise} \end{cases}$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_2}(j\Omega)| = \begin{cases} |H_2(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$



7.24. (a) Expanding the sum to see things more clearly, we get

$$\begin{aligned} H_c(s) &= \sum_{k=1}^{r} \frac{A_k}{(s-s_0)^k} + G_c(s) \\ &= \frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_r}{(s-s_0)^r} + G_c(s) \end{aligned}$$

Now multiplying both sides by $(s-s_0)^r$ we get

$$(s-s_0)^r H_c(s) = A_1(s-s_0)^{r-1} + A_2(s-s_0)^{r-2} + \cdots + A_r + (s-s_0)^r G_c(s)$$

Evaluating both sides of the equal sign at $s = s_0$ gives us

$$A_r = (s-s_0)^r H_c(s)\,|_{s=s_0}$$

Note that $(s-s_0)^r G_c(s) = 0$ when $s = s_0$ because $G_c(s)$ has at most one pole at $s = s_0$.

Similarly, by taking the first derivative and evaluating at $s = s_0$ we get

$$\begin{aligned} \frac{d}{ds}\left[(s-s_0)^r H_c(s)\right] &= \sum_{k=1}^{r}(r-k)A_k(s-s_0)^{(r-k-1)} + \frac{d}{ds}\left[(s-s_0)^r G_c(s)\right] \\ &= (r-1)A_1(s-s_0)^{r-2} + (r-2)A_2(s-s_0)^{r-3} + \cdots + A_{r-1} + 0 + \frac{d}{ds}\left[(s-s_0)^r G_c(s)\right] \\ A_{r-1} &= \frac{d}{ds}\left[(s-s_0)^r H_c(s)\right]|_{s=s_0} \end{aligned}$$

This idea can be continued. By taking the $(r-k)$-th derivative and evaluating at $s = s_0$ we get the the general form

$$A_k = \frac{1}{(r-k)!}\left(\frac{d^{r-k}}{ds^{r-k}}\left[(s-s_0)^r H_c(s)\right]|_{s=s_0}\right)$$

(b) Using the following transform pair from a lookup table,

$$\frac{t^{k-1}}{(k-1)!}e^{-\alpha t}u(t) \longrightarrow \frac{1}{(s+\alpha)^k}, \quad \mathcal{R}e\{s\} > -\alpha$$

we get

$$\begin{aligned} h_c(t) &= \mathcal{L}^{-1}\{H_c(s)\} \\ &= \mathcal{L}^{-1}\left\{\sum_{k=1}^{r}\frac{A_k}{(s-s_0)^k} + G_c(s)\right\} \\ &= \sum_{k=1}^{r} A_k \frac{t^{k-1}}{(k-1)!}e^{s_0 t}u(t) + g_c(t) \end{aligned}$$

7.25. (a) **Answer:** Only the bilinear transform design will guarantee that a minimum phase discrete-time filter is created from a minimum phase continuous-time filter. For the following explanations remember that a discrete-time minimum phase system has all its poles and zeros inside the unit circle.

Impulse Invariance: Impulse invariance maps left-half s-plane poles to the interior of the z-plane unit circle. However, left-half s-plane zeros will *not necessarily* be mapped inside the z-plane unit circle. Consider:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} = \frac{\sum_{k=1}^{N} A_k \prod_{\substack{j=1 \\ j\neq k}}^{N} (s - s_j)}{\prod_{\ell=1}^{N} (s - s_\ell)}$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} = \frac{\sum_{k=1}^{N} T_d A_k \prod_{\substack{j=1 \\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})}{\prod_{\ell=1}^{N} (1 - e^{s_\ell T_d} z^{-1})}$$

If we define $\mathrm{Poly}_k(z) = \prod_{\substack{j=1 \\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})$, we can note that all the roots of $\mathrm{Poly}_k(z)$ are inside the unit circle. Since the numerator of $H(z)$ is a sum of $A_k \mathrm{Poly}_k(z)$ terms, we see that there are *no guarantees* that the roots of the numerator polynomial are inside the unit circle. In other words, the sum of minimum phase filters is not necessarily minimum phase. By considering the specific example of

$$H_c(s) = \frac{s + 10}{(s+1)(s+2)},$$

and using $T = 1$, we can show that a minimum phase filter is transformed into a non-minimum phase discrete time filter.

Bilinear Transform: The bilinear transform maps a pole or zero at $s = s_0$ to a pole or zero (respectively) at $z_0 = \frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}$. Thus,

$$|z_0| = \left| \frac{1 + \frac{T}{2}s_0}{1 - \frac{T}{2}s_0} \right|$$

Since $H_c(s)$ is minimum phase, all the poles of $H_c(s)$ are located in the left half of the s-plane. Therefore, a pole $s_0 = \sigma + j\Omega$ must have $\sigma < 0$. Using the relation for s_0, we get

$$\begin{aligned} |z_0| &= \sqrt{\frac{(1+\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1-\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} \\ &< 1 \end{aligned}$$

Thus, all poles and zeros will be inside the z-plane unit circle and the discrete-time filter will be minimum phase as well.

(b) **Answer:** Only the bilinear transform design will result in an allpass filter.

Impulse Invariance: In the impulse invariance design we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

The aliasing terms can destroy the allpass nature of the continuous-time filter.

Bilinear Transform: The bilinear transform only warps the frequency axis. The magnitude response is not affected. Therefore, an allpass filter will map to an allpass filter.

(c) **Answer:** Only the bilinear transform will guarantee

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Impulse Invariance: Since impulse invariance may result in aliasing, we see that

$$H(e^{j0}) = H_c(j0)$$

if and only if

$$H(e^{j0}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = H_c(j0)$$

or equivalently

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0$$

which is generally not the case.

Bilinear Transform: Since, under the bilinear transformation, $\Omega = 0$ maps to $\omega = 0$,

$$H(e^{j0}) = H_c(j0)$$

for all $H_c(s)$.

(d) **Answer:** Only the bilinear transform design is guaranteed to create a bandstop filter from a bandstop filter.

If $H_c(s)$ is a bandstop filter, the bilinear transform will preserve this because it just warps the frequency axis; however aliasing (in the impulse invariance technique) can fill in the stop band.

(e) **Answer:** The property holds under the bilinear transform, but not under impulse invariance.

Impulse Invariance: Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_{a_1}(s) = H_{a_2}(s) = e^{-sT/2}$.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned}
H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&\equiv H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_1(z)H_2(z)
\end{aligned}$$

(f) **Answer:** The property holds for both impulse invariance and the bilinear transform.

Impulse Invariance:

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\
&= \sum_{k=-\infty}^{\infty} H_{c1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) + \sum_{k=-\infty}^{\infty} H_{c2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\
&= H_1(e^{j\omega}) + H_2(e^{j\omega})
\end{aligned}$$

Bilinear Transform:

$$\begin{aligned}
H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\
&= H_1(z) + H_2(z)
\end{aligned}$$

(g) **Answer:** Only the bilinear transform will result in the desired relationship.

Impulse Invariance: By impulse invariance,

$$\begin{aligned}
H_1\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_1}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right) \\
H_2\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_2}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)
\end{aligned}$$

We can clearly see that due to the aliasing, the phase relationship is not guaranteed to be maintained.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned}
H_1(e^{j\omega}) &= H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \\
H_2(e^{j\omega}) &= H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)
\end{aligned}$$

therefore,

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \frac{H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right)}{H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)} = \begin{cases} e^{-j\pi/2}, & 0<\omega<\pi \\ e^{j\pi/2}, & -\pi<\omega<0 \end{cases}$$

7.26. (a) Since

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)$$

and we desire

$$H(e^{j\omega})\,|_{\omega=0} = H_c(j\Omega)\,|_{\Omega=0},$$

we see that

$$H(e^{j\omega})|_{\omega=0} = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0.$$

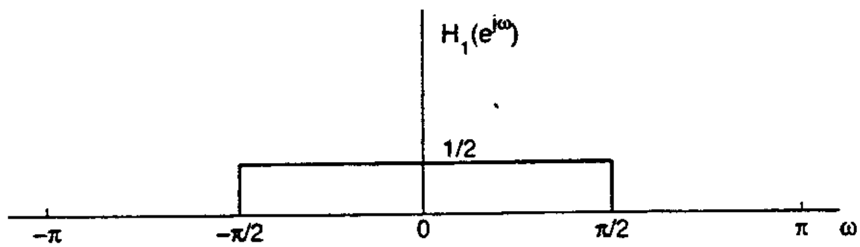
(b) Since the bilinear transform maps $\Omega=0$ to $\omega=0$, the condition will hold for any choice of $H_c(j\Omega)$.

7.27.

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega|<\frac{\pi}{4} \\ 0, & \frac{\pi}{4}<|\omega|\leq\pi \end{cases}$$

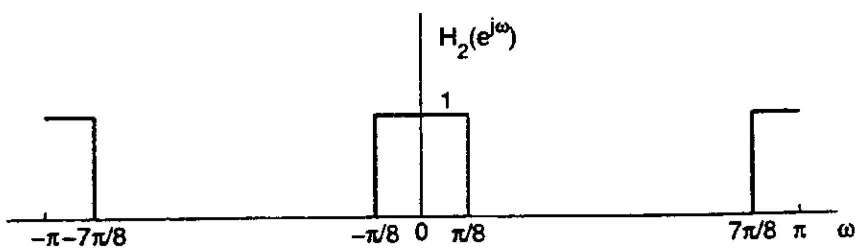
(a)

$$
\begin{aligned}
h_1[n] &= h[2n] \\
H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[2n]e^{j\omega n} \\
&= \sum_{n \text{ even}} h[n]e^{\frac{j\omega n}{2}} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[h[n] + (-1)^n h[n]\right] e^{j\frac{\omega n}{2}} \\
&= \frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H\left(e^{j\frac{\omega+2\pi}{2}}\right)
\end{aligned}
$$



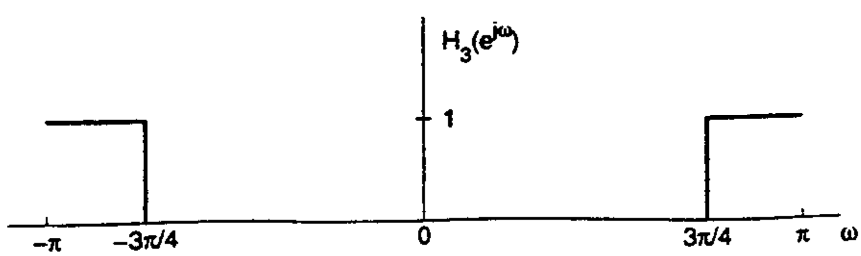
(b)

$$
\begin{aligned}
H_2(e^{j\omega}) &= \sum_{n \text{ even}} h[n/2]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega 2n} \\
&= H\left(e^{j2\omega}\right)
\end{aligned}
$$



(c)

$$
H_3(e^{j\omega}) = H\left(e^{j(\omega+\pi)}\right)
$$



7.28. (a) We have

$$\begin{aligned} s &= \frac{1-z^{-1}}{1+z^{-1}} \\ j\Omega &= \frac{1-e^{-j\omega}}{1+e^{-j\omega}} \\ &= \frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} \\ \Omega &= \tan\left(\frac{\omega}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_1}}{2}\right) \longleftrightarrow \omega_{p_1} = 2\tan^{-1}(\Omega_p)$$

(b)

$$\begin{aligned} s &= \frac{1+z^{-1}}{1-z^{-1}} \\ j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} \\ \Omega &= -\cot\left(\frac{\omega}{2}\right) \\ &= \tan\left(\frac{\omega-\pi}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_2}-\pi}{2}\right) \longleftrightarrow \omega_{p_2} = \pi + 2\tan^{-1}(\Omega_p)$$

(c)

$$\tan\left(\frac{\omega_{p_2}-\pi}{2}\right) = \tan\left(\frac{\omega_{p_1}}{2}\right)$$

$$\Rightarrow \omega_{p_2} = \omega_{p_1} + \pi$$

(d)

$$H_2(z) = H_1(z)|_{z=-z}$$

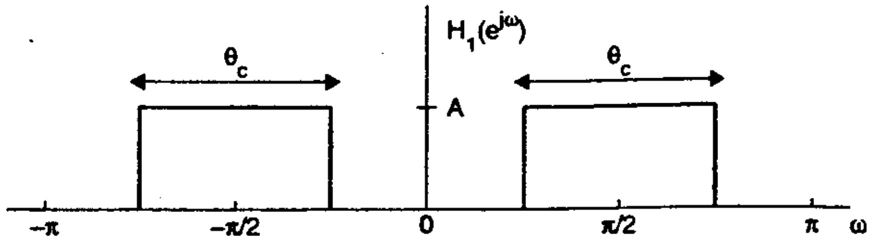
The even powers of z do not get changed by this transformation, while the coefficients of the odd powers of z change sign.
Thus, replace $A, C, 2$ with $-A, -C, -2$.

7.29. (a) Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$ we get,

$$\begin{aligned} e^{j\theta} &= -e^{j2\omega} \\ &= e^{j(2\omega+\pi)} \end{aligned}$$

$$\theta = 2\omega + \pi \longleftrightarrow \omega = \frac{\theta-\pi}{2}$$

(b)



(c)

$$
\begin{aligned}
h[n] &\longleftrightarrow H(e^{j\theta}) \\
h_1[n] &\longleftrightarrow H\left(e^{j(2\omega+\pi)}\right)
\end{aligned}
$$

In the frequency domain, we first shift by π and then we upsample by 2. In the time domain, we can write that as

$$h_1[n] = \begin{cases} (-1)^{n/2}h[n/2], & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

(d) In general, a filter

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{M-1}z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{N-1}z^{N-1} + a_N z^{-N}}$$

will transform under $H_1(z) = H(-z^2)$ to

$$H_1(z) = \frac{b_0 - b_1 z^{-2} + b_2 z^{-4} + \cdots - b_{M-1}z^{2M-2} + b_M z^{-2M}}{a_0 - a_1 z^{-2} + a_2 z^{-4} + \cdots - a_{M-1}z^{2N-2} + a_N z^{-2N}}$$

where we are assuming here that M and N are even. All the delay terms increase by a factor of two, and the sign of the coefficient in front of any odd delay term is negated.

The given difference equations therefore become

$$
\begin{aligned}
g[n] &= x[n] + a_1 g[n-2] - b_1 f[n-4] \\
f[n] &= -a_2 g[n-2] - b_2 f[n-2] \\
y[n] &= c_1 f[n] + c_2 g[n-2]
\end{aligned}
$$

To avoid any possible confusion please note that the b_k and a_k in these difference equations are not the same b_k and a_k shown above for the general case.

7.30. We are given

$$H(z) = H_c(s)\,|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]}$$

where α is a nonzero integer and β is a real number.

(a) *It is true for $\beta > 0$.*

Proof:

$$
\begin{aligned}
s &= \beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right] \\
s + sz^{-\alpha} &= \beta - \beta z^{-\alpha} \\
s - \beta &= -\beta z^{-\alpha} - sz^{-\alpha} \\
\beta - s &= z^{-\alpha}(\beta + s) \\
z^{-\alpha} &= \frac{\beta - s}{\beta + s} \\
z^{\alpha} &= \frac{\beta + s}{\beta - s}
\end{aligned}
$$

The poles s_k of a stable, causal, continuous-time filter satisfy the condition $\mathcal{R}e\{s\} < 0$. We want these poles to map to the points z_k in the z-plane such that $|z_k| < 1$. With $\alpha > 0$ it is also true that if $|z_k| < 1$ then $|z_k^\alpha| < 1$. Letting $s_k = \sigma + j\omega$ we see that

$$\begin{aligned}
|z_k| &< 1 \\
|z_k^\alpha| &< 1 \\
|\beta + \sigma + j\Omega| &< |\beta - \sigma - j\Omega| \\
(\beta + \sigma)^2 + \Omega^2 &< (\beta - \sigma)^2 + \Omega^2 \\
2\sigma\beta &< -2\sigma\beta
\end{aligned}$$

But since the continuous-time filter is stable we have $\mathcal{R}e\{s_k\} < 0$ or $\sigma < 0$. That leads to

$$-\beta < \beta$$

This can only be true if $\beta > 0$.

(b) *It is true for* $\beta < 0$. The proof is similar to the last proof except now we have $|z^\alpha| > 1$.

(c) We have

$$\begin{aligned}
z^2 &= \left.\frac{1+s}{1-s}\right|_{s=j\Omega} \\
|z^2| &= 1 \\
|z| &= 1
\end{aligned}$$

Hence, the $j\Omega$ axis of the s-plane is mapped to the unit circle of z-plane.

(d) First, find the mapping between Ω and ω.

$$\begin{aligned}
j\Omega &= \frac{1 - e^{-j2\omega}}{1 + e^{-j2\omega}} \\
&= \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}} \\
\Omega &= \tan(\omega) \\
\omega &= \tan^{-1}(\Omega)
\end{aligned}$$

Therefore,

$$1 - \delta_1 \le |H(e^{j\omega})| \le 1 + \delta_1, \qquad \left\{|\omega| \le \frac{\pi}{4}\right\} \cup \left\{\frac{3\pi}{4} < |\omega| < \pi\right\}$$

Note that the highpass region $3\pi/4 \le |w| \le \pi$ is included because $\tan(\omega)$ is periodic with period π.

7.31. (a)

$$s = \frac{1 + z^{-1}}{1 - z^{-1}} \longleftrightarrow z = \frac{s+1}{s-1}$$

Now, we evaluate the above expressions along the $j\Omega$ axis of the s-plane

$$\begin{aligned}
z &= \frac{j\Omega + 1}{j\Omega - 1} \\
|z| &= 1
\end{aligned}$$

(b) We want to show $|z| < 1$ if $\mathcal{R}e\{s\} < 0$.

$$\begin{aligned} z &= \frac{\sigma + j\Omega + 1}{\sigma + j\Omega - 1} \\ |z| &= \frac{\sqrt{(\sigma+1)^2 + \Omega^2}}{\sqrt{(\sigma-1)^2 + \Omega^2}} \end{aligned}$$

Therefore, if $|z| < 1$

$$\begin{aligned} (\sigma+1)^2 + \Omega^2 &< (\sigma-1)^2 + \Omega^2 \\ \sigma &< -\sigma \end{aligned}$$

it must also be true that $\sigma < 0$. We have just shown that the left-half s-plane maps to the interior of the z-plane unit circle. Thus, any pole of $H_c(s)$ inside the left-half s-plane will get mapped to a pole inside the z-plane unit circle.

(c) We have the relationship

$$\begin{aligned} j\Omega &= \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} \\ &= \frac{e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\ \Omega &= -\cot(\omega/2) \end{aligned}$$

$$\begin{aligned} |\Omega_s| &= |\cot(\pi/6)| = \sqrt{3} \\ |\Omega_{p_1}| &= |\cot(\pi/2)| = 0 \\ |\Omega_{p_2}| &= |\cot(\pi/4)| = 1 \end{aligned}$$

Therefore, the constraints are

$$\begin{aligned} 0.95 \le |H_c(j\Omega)| \le 1.05, &\quad 0 \le |\Omega| \le 1 \\ |H_c(j\Omega)| \le 0.01, &\quad \sqrt{3} \le |\Omega| \end{aligned}$$

7.32. (a) By using Parseval's theorem,

$$\begin{aligned} \epsilon^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega \\ &= \sum_{n=-\infty}^{\infty} |e[n]|^2 \end{aligned}$$

where

$$e[n] = \begin{cases} h_d[n], & n < 0, \\ h_d[n] - h[n], & 0 \le n \le M, \\ h_d[n], & n > M \end{cases}$$

(b) Since we only have control over $e[n]$ for $0 \le m \le M$, we get that ϵ^2 is minimized if $h[n] = h_d[n]$ for $0 \le n \le M$.

(c)

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

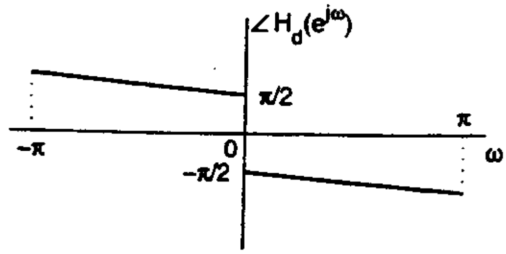
which is a rectangular window.

7.33. (a)

$$H_d(e^{j\omega}) = [1-2u(\omega)]e^{j(\pi/2-\tau\omega)} \quad \text{for} -\pi<\omega<\pi$$

$$|H_d(e^{j\omega})| = 1, \quad \forall\omega$$

$$\angle H_d(e^{j\omega}) = \begin{cases} \frac{\pi}{2}-\tau\omega, & -\pi<\omega<0 \\ -\frac{\pi}{2}-\tau\omega, & 0<\omega<\pi \end{cases}$$



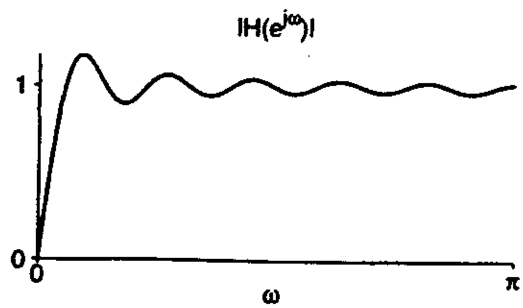
(b) A Hilbert transformer of this nature requires the filter to have a zero at $z=0$ which introduces the 180° phase difference at that point. A zero at $z=0$ means that the sum of the filter coefficients equals zero. Thus, only Types III and IV fulfill the requirements.

(c)

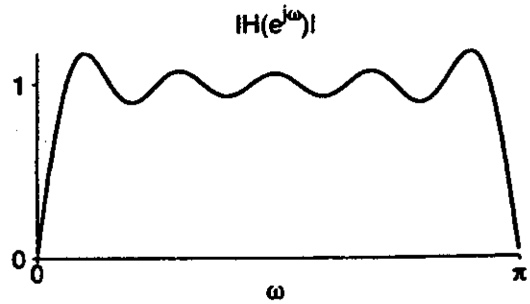
$$\begin{aligned} H_d(e^{j\omega}) &= [1-2u(\omega)]e^{j(\pi/2-\omega\tau)} \\ h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} e^{j(\pi/2-\omega\tau)}e^{j\omega n}d\omega - \frac{1}{2\pi}\int_{0}^{\pi} e^{j(\pi/2-\omega\tau)}e^{j\omega n}d\omega \\ &= \frac{e^{j\frac{\pi}{2}}}{2\pi}\int_{-\pi}^{0} e^{j\omega(n-\tau)}d\omega - \frac{e^{j\frac{\pi}{2}}}{2\pi}\int_{0}^{\pi} e^{j\omega(n-\tau)}d\omega \\ &= \begin{cases} \frac{1-\cos[\pi(n-\tau)]}{\pi(n-\tau)}, & n\neq\tau \\ 0, & n=\tau \end{cases} \\ &= \begin{cases} \frac{2}{\pi}\frac{\sin^2[\pi(n-\tau)/2]}{(n-\tau)}, & n\neq\tau \\ 0, & n=\tau \end{cases} \end{aligned}$$

For the windowed FIR system to be linear phase it must be antisymmetric about $\frac{M}{2}$. Since the ideal Hilbert transformer $h_d[n]$ is symmetric about $n=\tau$ we should choose $\tau=\frac{M}{2}$.

(d) The delay is $M/2 = 21/2 = 10.5$ samples. It is therefore a Type IV system. Notice the mandatory zero at $\omega=0$.



(e) The delay is $M/2 = 20/2 = 10$ samples. It is therefore a Type III system. Notice the mandatory zeros at $\omega = 0$ and π.



7.34. (a) It is well known that convolving two rectangular windows results in a triangular window. Specifically, to get the $(M+1)$ point Bartlett window for M even, we can convolve the following rectangular windows.

$$\begin{aligned} r_1[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 0, \ldots, \frac{M}{2} - 1 \\ 0, & \text{otherwise} \end{cases} \\ r_2[n] &= r_1[n-1] \end{aligned}$$

Using the known transform of a rectangular window we have

$$\begin{aligned} W_{R_1}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega(\frac{M}{4} - \frac{1}{2})} \\ W_{R_2}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega(\frac{M}{4} + \frac{1}{2})} \\ W_B(e^{j\omega}) &= W_{R_1}(e^{j\omega}) W_{R_2}(e^{j\omega}) \\ &= \frac{2}{M} \left(\frac{\sin(\omega M/4)}{\sin(\omega/2)} \right)^2 e^{-j\omega M/2} \end{aligned}$$

Note: The Bartlett window as defined in the text is zero at $n = 0$ and $n = M$. These points are included in the $M + 1$ points.

For M odd, the Bartlett window is the convolution of

$$\begin{aligned} r_3[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 0, \ldots, \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \\ r_4[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 1, \ldots, \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

In the frequency domain we have

$$\begin{aligned} W_{R_3}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega(M+1)/4)}{\sin(\omega/2)} e^{-j\omega(\frac{M-1}{4})} \\ W_{R_4}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega(M-1)/4)}{\sin(\omega/2)} e^{-j\omega(\frac{M-3}{4}+1)} \\ W_B(e^{j\omega}) &= W_{R_3}(e^{j\omega}) W_{R_4}(e^{j\omega}) \\ &= \frac{2}{M} \left(\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} \right) \left(\frac{\sin[\omega(M-1)/2]}{\sin(\omega/2)} \right) e^{-j\omega M/2} \end{aligned}$$

(b)

$$w[n] = \left[A + B\cos\left(\frac{2\pi n}{M}\right) + C\cos\left(\frac{4\pi n}{M}\right)\right] w_R[n]$$

$$\begin{aligned} W(e^{j\omega}) &= \left\{2\pi A\delta(\omega) + B\pi\left[\delta\left(\omega + \frac{2\pi}{M}\right) + \delta\left(\omega - \frac{2\pi}{M}\right)\right] + C\pi\left[\delta\left(\omega + \frac{4\pi}{M}\right) + \delta\left(\omega - \frac{4\pi}{M}\right)\right]\right\} \\ &\quad \frac{\otimes}{2\pi}\left\{\frac{\sin(\omega(M+1)/2))}{\sin(\omega/2)} e^{-j\omega M/2}\right\} \end{aligned}$$

where $\otimes$ denotes periodic convolution.

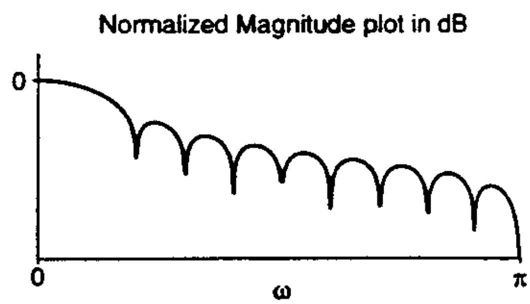
(c) For the Hanning window $A = 0.5$, $B = -0.5$, and $C = 0$.

$$\begin{aligned} w_{\text{Hanning}}[n] &= \left[0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right)\right] w_r[n] \\ W_{\text{Hanning}}(e^{j\omega}) &= 0.5W_R(e^{j\omega}) - 0.25W_R(e^{j\omega}) \otimes \left[\delta\left(\omega + \frac{2\pi}{M}\right) + \delta\left(\omega - \frac{2\pi}{M}\right)\right] \\ &= 0.5W_R(e^{j\omega}) - 0.25\left[W_R(e^{j(\omega + \frac{2\pi}{M})}) + W_R(e^{j(\omega - \frac{2\pi}{M})})\right] \end{aligned}$$

where

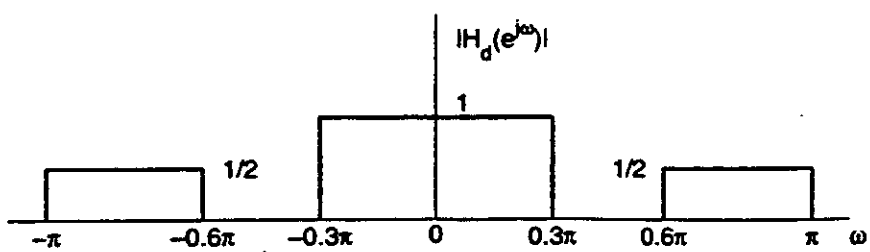
$$W_R(e^{j\omega}) = \frac{\sin(\omega(M+1)/2))}{\sin(\omega/2)} e^{-j\omega M/2}$$

Below is a normalized sketch of the magnitude response in dB.



7.35. (a) The delay is $\frac{M}{2} = 24$.

(b)



This can be viewed as the sum of two lowpass filters, one of which has been shifted in frequency (modulation in time-domain) to $\omega = \pi$. The linear phase factor adds a delay.

$$h_d[n] = \frac{\sin(0.3\pi(n-24))}{\pi(n-24)} + \frac{1}{2}(-1)^{(n-24)}\frac{\sin(0.4\pi(n-24))}{\pi(n-24)}$$

(c) To find the ripple values, which are all the same in this case since it is a Kaiser window design, we first need to determine A. Since we know β and A are related by

$$\beta = 3.68 = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \leq A \leq 50 \\ 0, & A < 21 \end{cases}$$

we can solve for A in the following manner:

1. We know $\beta = 3.68$. Therefore, from the formulas above, we see that $A \geq 21$.
2. If we assume $A > 50$ we find,

$$\begin{aligned} 3.68 &= 0.1102(A-8.7) \\ A &= 42.1 \end{aligned}$$

But, this contradicts our assumption that $A > 50$. Thus, $21 \leq A \leq 50$.

3. With $21 \leq A \leq 50$ we find,

$$\begin{aligned} 3.68 &= 0.5842(A-21)^{0.4} + 0.07886(A-21) \\ A &= 42.4256 \end{aligned}$$

With A, we can now calculate δ.

$$\begin{aligned} \delta &= 10^{-A/20} \\ &= 10^{-42.4256/20} \\ &= 0.0076 \end{aligned}$$

The discontinuity of 1 in the first passband creates a ripple of δ. The discontinuity of 1/2 in the second passband creates a ripple of $\delta/2$. The total ripple is $3\delta/2 = 0.0114$ and we therefore have

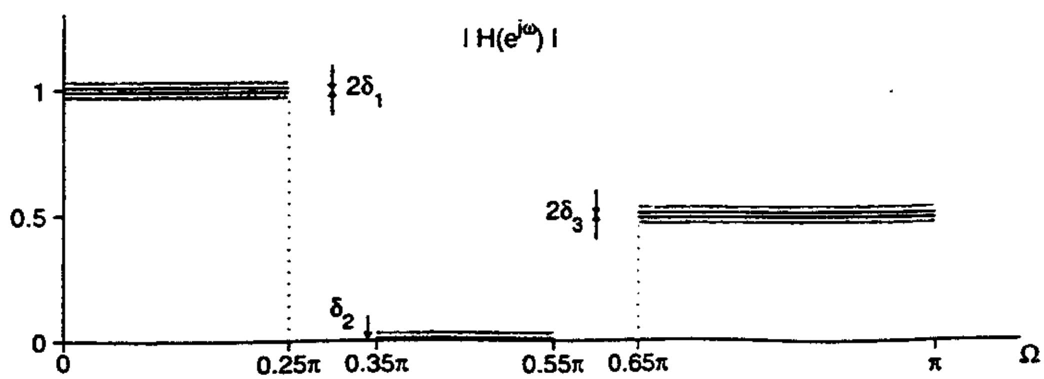
$$\delta_1 = \delta_2 = \delta_3 = 0.0114$$

Now using the relationship between M, A, and $\Delta\omega$

$$\begin{aligned} M &= \frac{A-8}{2.285\Delta\omega} \\ \Delta\omega &= \frac{42.4256-8}{2.285(48)} = 0.3139 \approx 0.1\pi \end{aligned}$$

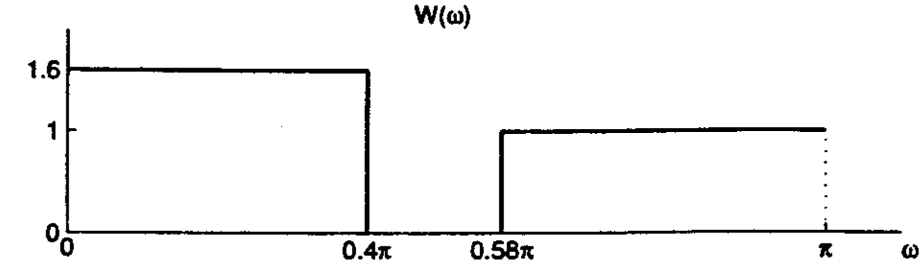
Putting it all together with the information about $H_d(e^{j\omega})$ we arrive at our final answer.

$$\begin{aligned} 0.9886 \leq |H(e^{j\omega})| &\leq 1.0114, & 0 \leq \omega \leq 0.25\pi \\ |H(e^{j\omega})| &\leq 0.0114, & 0.35\pi \leq \omega \leq 0.55\pi \\ 0.4886 \leq |H(e^{j\omega})| &\leq 0.5114, & 0.65\pi \leq \omega \leq \pi \end{aligned}$$

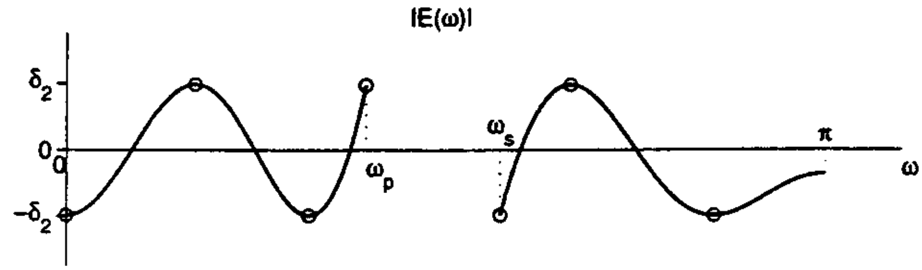


7.36. (a) Since $H(e^{j0}) \neq 0$ and $H(e^{j\pi}) \neq 0$, this must be a Type I filter.

(b) With the weighting in the stopband equal to 1, the weighting in the passband is $\frac{\delta_2}{\delta_1}$.



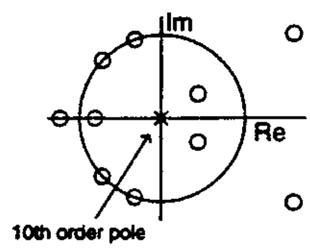
(c)



(d) An optimal (in the Parks-McClellan sense) Type I lowpass filter can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = \pi$ it should only have $L+2$ alternations. From the figure, we see that there are 7 alternations so $L = 5$. Thus, the filter length is $2L + 1 = 11$ samples long.

(e) Since the filter is 11 samples long, it has a delay of 5 samples.

(f) Note the zeroes off the unit circle are implied by the dips in the frequency response at the indicated frequencies.



7.37. (a) The most straightforward way to find $h_d[n]$ is to recognize that $H_d(e^{j\omega})$ is simply the (periodic) convolution of two ideal lowpass filters with cutoff frequency $\omega_c = \pi/4$. That is,

$$H_d(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lpf}(e^{j\theta})H_{lpf}(e^{j(\omega-\theta)})d\theta$$

where

$$H_{lpf}(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, in the time domain, $h_d[n]$ is $(h_{lpf}[n])^2$, or

$$\begin{aligned} h_d[n] &= \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2 \\ &= \frac{\sin^2(\pi n/4)}{\pi^2 n^2} \end{aligned}$$

(b) $h[n]$ must have even symmetry around $(N-1)/2$. $h[n]$ is a type-I FIR generalized linear phase system, since N is an odd integer, and $H(e^{j\omega}) \neq 0$ for $\omega = 0$. Type-I FIR generalized linear phase systems have even symmetry around $(N-1)/2$.

(c) Shifting the filter $h_d[n]$ by $(N-1)/2$ and applying a rectangular window will result in a causal $h[n]$ that minimizes the integral squared error ϵ. Consequently,

$$h[n] = \frac{\sin^2\left[\frac{\pi}{4}(n - \frac{N-1}{2})\right]}{\pi^2(n - \frac{N-1}{2})^2} w[n]$$

where

$$w[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

(d) The integral squared error ϵ

$$\epsilon = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|A(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

can be reformulated, using Parseval's theorem, to

$$\epsilon = \sum_{-\infty}^{\infty} |a[n] - h_d[n]|^2$$

Since

$$a[n] = \begin{cases} h_d[n], & -\frac{N-1}{2} \leq n \leq \frac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \epsilon &= \sum_{-\infty}^{-(N-1)/2-1} |a[n] - h_d[n]|^2 + \sum_{-(N-1)/2}^{(N-1)/2} |a[n] - h_d[n]|^2 + \sum_{(N-1)/2+1}^{\infty} |a[n] - h_d[n]|^2 \\ &= \sum_{-\infty}^{-(N-1)/2-1} |h_d[n]|^2 + 0 + \sum_{(N-1)/2+1}^{\infty} |h_d[n]|^2 \end{aligned}$$

By symmetry,

$$\epsilon = 2\sum_{(N-1)/2+1}^{\infty} |h_d[n]|^2$$

7.38. (a) A Type-I lowpass filter that is optimal in the Parks-McClellan can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = 0$ it only has $L+2$ alternations. From the figure we see there are 9 alternations so $L = 7$. Thus, $M = 2L = 2(7) = 14$.

(b) We have

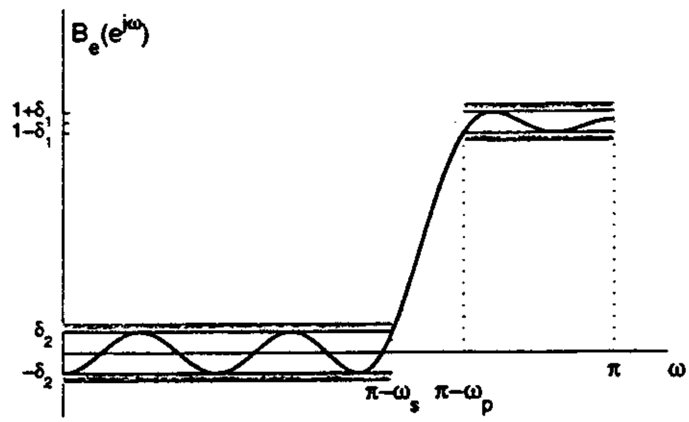
$$\begin{aligned} h_{HP}[n] &= -e^{j\pi n} h_{LP}[n] \\ H_{HP}(e^{j\omega}) &= -H_{LP}(e^{j(\omega-\pi)}) \\ &= -A_e(e^{j(\omega-\pi)})e^{-j(\omega-\pi)\frac{M}{2}} \\ &= A_e(e^{j(\omega-\pi)})e^{-j\omega\frac{M}{2}} \\ &= B_e(e^{j\omega})e^{-j\omega\frac{M}{2}} \end{aligned}$$

where

$$B_e(e^{j\omega}) = A_e(e^{j(\omega-\pi)})$$

The fact that $M = 14$ is used to simplify the exponential term in the third line above.

(c)



(d) *The assertion is correct.* The original amplitude function was optimal in the Parks-McClellan sense. The method used to create the new filter did not change the filter length, transition width, or relative ripple sizes. All it did was slide the frequency response along the frequency axis creating a new error function $E'(\omega) = E(\omega - \pi)$. Since translation does not change the Chebyshev error $(\max|E(\omega)|)$ the new filter is still optimal.

7.39. For this filter, $N = 3$, so the polynomial order L is

$$L = \frac{N-1}{2} = 1$$

Note that $h[n]$ must be a type-I FIR generalized linear phase filter, since it consists of three samples, and $H(e^{j\omega}) \neq 0$ for $\omega = 0$. $h[n]$ can therefore be written in the form

$$h[n] = a\delta[n] + b\delta[n-1] + a\delta[n-2]$$

Taking the DTFT of both sides gives

$$\begin{aligned} H(e^{j\omega}) &= a + be^{-j\omega} + ae^{-j2\omega} \\ &= e^{-j\omega}(ae^{j\omega} + b + ae^{-j\omega}) \\ &= e^{-j\omega}(b + 2a\cos w) \\ A(e^{j\omega}) &= b + 2a\cos w \end{aligned}$$

The filter must have at least $L+2=3$ alternations, but no more than $L+3=4$ alternations to satisfy the alternation theorem, and therefore be optimal in the minimax sense. Four alternations can be obtained if all four band edges are alternation frequencies such that the frequency response overshoots at $\omega=0$, undershoots at $\omega=\frac{\pi}{3}$, overshoots at $\omega=\frac{\pi}{2}$, and undershoots at $\omega=\pi$.

Let the error in the passband and the stopband be δ_p and δ_s. Then,

$$\begin{aligned}
A(e^{j\omega})\,|_{\omega=0} &= 1+\delta_p \\
A(e^{j\omega})\,|_{\omega=\pi/3} &= 1-\delta_p \\
A(e^{j\omega})\,|_{\omega=\pi/2} &= \delta_s \\
A(e^{j\omega})\,|_{\omega=\pi} &= -\delta_s
\end{aligned}$$

Using $A(e^{j\omega})=b+2a\cos w$,

$$\begin{aligned}
A(e^{j\omega})\,|_{\omega=0} &= b+2a \\
A(e^{j\omega})\,|_{\omega=\pi/3} &= b+a \\
A(e^{j\omega})\,|_{\omega=\pi/2} &= b \\
A(e^{j\omega})\,|_{\omega=\pi} &= b-2a
\end{aligned}$$

Solving these systems of equations for a and b gives

$$\begin{aligned}
a &= \frac{2}{5} \\
b &= \frac{2}{5}
\end{aligned}$$

Thus, the optimal (in the minimax sense) causal 3-point lowpass filter with the desired passband and stopband edge frequencies is

$$h[n]=\frac{2}{5}\delta[n]+\frac{2}{5}\delta[n-1]+\frac{2}{5}\delta[n-2]$$

7.40. *True.* Since filter C is a stable IIR filter it has poles in the left half plane. The bilinear transform maps the left half plane to the inside of the unit circle. Thus, the discrete filter B has to have poles and is therefore an IIR filter.

7.41. *No.* The resulting discrete-time filter would not have a constant group delay. The bilinear transformation maps the entire $j\Omega$ axis in the s-plane to one revolution of the unit circle in the z-plane. Consequently, the linear phase of the continuous-time filter will get nonlinearly warped via the bilinar transform, resulting in a nonlinear phase for the discrete-time filter. Thus, the group delay of the discrete-time filter will not be a constant.

7.42. (a) Using the fact that $H_c(s)=\frac{Y_c(s)}{X_c(s)}$ and cross multiplying we get

$$\begin{aligned}
H_c(s)=\frac{Y_c(s)}{X_c(s)} &= \frac{A}{s+c} \\
(s+c)Y_c(s) &= AX_c(s) \\
\frac{dy_c(t)}{dt}+cy_c(t) &= Ax_c(t)
\end{aligned}$$

(b)

$$\begin{aligned}
\left.\frac{dy_c(t)}{dt}\right|_{t=nT} &= [Ax_c(t)-cy_c(t)]\,|_{t=nT} \\
&= Ax_c(nT)-cy_c(nT) \\
\frac{y_c(nT)-y_c(nT-T)}{T} &\approx Ax_c(nT)-cy_c(nT)
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{y[n]-y[n-1]}{T} &= Ax[n]-cy[n] \\
Ax[n] &= \left(c+\frac{1}{T}\right)y[n]-\frac{1}{T}y[n-1] \\
AX(z) &= \left(c+\frac{1}{T}\right)Y(z)-\frac{1}{T}Y(z)z^{-1} \\
H(z)=\frac{Y(z)}{X(z)} &= \frac{A}{c+\frac{1}{T}-\frac{1}{T}z^{-1}}
\end{aligned}$$

(d)

$$\begin{aligned}
H_c(s)\,\Big|_{s=\frac{1-z^{-1}}{T}} &= \left.\frac{A}{s+c}\right|_{s=\frac{1-z^{-1}}{T}} \\
&= \frac{A}{\frac{1-z^{-1}}{T}+c} \\
&= H(z)
\end{aligned}$$

(e) First solve for z

$$\begin{aligned}
s &= \frac{1-z^{-1}}{T} \\
z &= \frac{1}{1-sT}
\end{aligned}$$

and then substitute $s=\sigma+j\Omega$ to get

$$\begin{aligned}
z &= \frac{1}{1-(\sigma+j\Omega)T} \\
&= \frac{1}{\sqrt{(1-\sigma)^2+(\Omega T)^2}}e^{j\tan^{-1}\left(\frac{\Omega T}{1-\sigma}\right)}
\end{aligned}$$

If we let $\theta=\tan^{-1}\left(\frac{\Omega T}{1-\sigma}\right)$ we see that

$$\begin{aligned}
\frac{1}{\sqrt{(1-\sigma)^2+(\Omega T)^2}} &= \frac{\cos(\theta)}{1-\sigma} \\
&= \frac{1}{2(1-\sigma)}\left(e^{j\theta}+e^{-j\theta}\right)
\end{aligned}$$

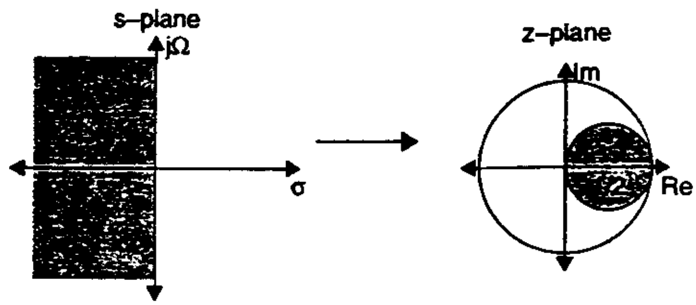
and thus the s-plane maps to the z-plane in the following manner

$$\begin{aligned}
z &= \left[\frac{1}{2(1-\sigma)}\left(e^{j\theta}+e^{-j\theta}\right)\right]e^{j\theta} \\
&= \frac{1}{2(1-\sigma)}+\frac{1}{2(1-\sigma)}e^{j2\theta} \\
&= \frac{1}{2(1-\sigma)}+\frac{1}{2(1-\sigma)}e^{j2\tan^{-1}\left(\frac{\Omega T}{1-\sigma}\right)}
\end{aligned}$$

To find where the $j\Omega$ axis of the s-plane maps, we let $s=j\Omega$, i.e., $\sigma=0$ and find

$$z=\frac{1}{2}+\frac{1}{2}e^{j2\tan^{-1}(\Omega T)}$$

Therefore, the $j\Omega$-axis maps to a circle of radius $1/2$ that is centered at $1/2$ in the z-plane. We also see that the region $\sigma < 0$, i.e., the left half of the s-plane, maps to the interior of this circle.



If the continuous-time system is stable, its poles are in the left half s-plane. As shown above, these poles map to the interior of the unit circle and so the discrete-time system will also be stable. The stability is independent of T.

Since the $j\Omega$-axis does not map to the unit circle, the discrete-time frequency response will not be a faithful reproduction of the continuous-time frequency response. As T gets smaller, i.e., as we oversample more, a larger portion of the $j\Omega$-axis gets mapped to the region close to the unit circle at $\omega = 0$. Although the frequency range becomes more compressed the shape of the two responses will look more similar. Thus, as T decreases we improve our approximation.

(f) Substituting for the first derivative in the differential equation obtained in part (a) we get

$$
\begin{aligned}
\frac{y_c(nT+T) - y_c(nT)}{T} + cy_c(nT) &= Ax_c(nT) \\
\frac{y[n+1] - y[n]}{T} + cy[n] &= Ax[n]
\end{aligned}
$$

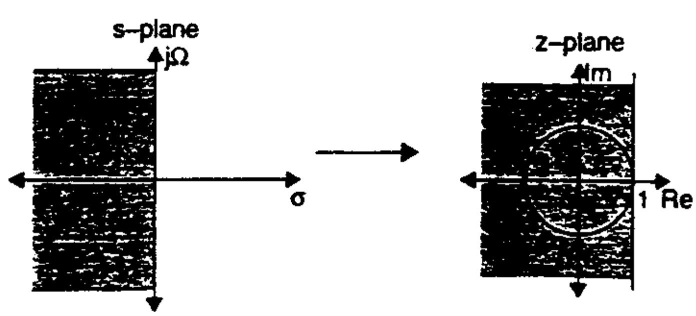
$$
H(z) = \frac{Y(z)}{X(z)} = \frac{A}{\frac{z-1}{T} + c} = H_c(s)\,|_{s=\frac{z-1}{T}}
$$

$$
\begin{aligned}
s &= \frac{z-1}{T} \\
z &= 1 + sT \\
&= 1 + \sigma + j\Omega T
\end{aligned}
$$

To find where the $j\Omega$ axis of the s-plane maps, we let $s = j\Omega$, i.e., $\sigma = 0$ and find

$$
z = 1 + j\Omega T
$$

Therefore, the $j\Omega$-axis lies on the line $\mathcal{R}e\{z\} = 1$. We also see that the region $\sigma < 0$, i.e., the left half of the s=plane, maps to the left of this line.



If the continuous-time system is stable, its poles are in the left half s-plane. As shown above, these poles can map to a point outside the unit circle and so the discrete-time system will not necessarily be stable. There are cases where varying T can turn an unstable system into a stable system, but it is not true for the general case.

Since the $j\Omega$-axis does not map to the unit circle, the discrete-time frequency response will not be a faithful reproduction of the continuous-time frequency response. However, as T gets smaller our approximation gets better for the same reasons outlined for the first backward difference.

7.43. (a) Just doing the integration reveals

$$\int_{nT-T}^{nT} \dot{y}_c(\tau)d\tau + y_c(nT-T) = y_c(\tau)|_{nT-T}^{nT} + y_c(nT-T) = y_c(nT)$$

Using the area in the trapezoidal region to replace the integral above, we get

$$\begin{aligned} y_c(nT) &= \int_{nT-T}^{nT} \dot{y}_c(\tau)d\tau + y_c(nT-T) \\ &\approx \left[\dot{y}_c(nT) + \dot{y}_c(nT-T)\right]\frac{T}{2} + y_c(nT-T) \end{aligned}$$

(b) Solving for $\dot{y}_c(nT)$ in the differential equation we get

$$\dot{y}_c(nT) = Ax_c(nT) - cy_c(nT)$$

Substituting this into the answer from part (a) yields

$$y_c(nT) = [Ax_c(nT) - cy_c(nT) + Ax_c(nT-T) - cy_c(nT-T)]\frac{T}{2} + y_c(nT-T)$$

(c) The difference equation is

$$y[n] = (Ax[n] - cy[n] + Ax[n-1] - cy[n-1])\frac{T}{2} + y[n-1]$$

$$y[n]\left(1 + c\frac{T}{2}\right) - y[n-1]\left(1 - c\frac{T}{2}\right) = A\frac{T}{2}(x[n] + x[n-1])$$

Therefore,

$$Y(z)\left[1 + c\frac{T}{2}\right] - Y(z)z^{-1}\left[1 - c\frac{T}{2}\right] = A\frac{T}{2}X(z)\left[1 + z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{A\frac{T}{2}(1+z^{-1})}{1 + c\frac{T}{2} - z^{-1} + z^{-1}c\frac{T}{2}}$$

(d)

$$\begin{aligned} H_c(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} &= \left.\frac{A}{s+c}\right|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \\ &= \frac{\frac{T}{2}A(1+z^{-1})}{1 - z^{-1} + c\frac{T}{2}(1+z^{-1})} \\ &= H(z) \end{aligned}$$

7.44.

$$\begin{aligned} \Phi_c(j\Omega) &= H_c(j\Omega)H_c(-j\Omega) \\ \Phi(z) &= H(z)H(z^{-1}) \end{aligned}$$

(a) (i) Since $H_c(s)$ has poles at s_k, $H_c(-s)$ has poles at $-s_k$.

(ii) The material in this chapter shows that under impulse invariance

$$\frac{A_k}{s-s_k} \longleftrightarrow \frac{T_d A_k}{1-e^{s_k T_d} z^{-1}}.$$

Thus, going from step 1 to step 2 means that the autocorrelation of the discrete-time system is a sampled version of the autocorrelation of the continuous-time system.

(iii) Since $\Phi(z) = H(z)H(z^{-1})$ we can choose the poles and zeros of $H(z)$ to be all the poles inside the unit circle, and that choice leaves all the poles and zeros outside the unit circle for $H(z^{-1})$.

Consider the following example using $h_c(t) = e^{-\alpha t}u(t)$.

$$H_c(s) = \frac{1}{s+\alpha} \quad \text{and} \quad H_c(-s) = \frac{1}{-s+\alpha}$$

$$\begin{aligned}
\Phi_c(s) &= H_c(s)H_c(-s) \\
&= \left[\frac{1}{s+\alpha}\right]\left[\frac{1}{-s+\alpha}\right] \\
&= \frac{1/2\alpha}{s+\alpha} - \frac{1/2\alpha}{s-\alpha}
\end{aligned}$$

$$\begin{aligned}
\Phi(z) &= \frac{T_d/2\alpha}{1-e^{-\alpha T_d}z^{-1}} - \frac{T_d/2\alpha}{1-e^{\alpha T_d}z^{-1}} \\
&= \frac{T_d}{2\alpha}\frac{\left[1-e^{\alpha T_d}z^{-1} - 1 + e^{-\alpha T_d}z^{-1}\right]}{(1-e^{-\alpha T_d}z^{-1})(1-e^{\alpha T_d}z^{-1})} \\
&= \frac{T_d}{2\alpha}\frac{\left(e^{-\alpha T_d} - e^{\alpha T_d}\right)z^{-1}}{(1-e^{-\alpha T_d}z^{-1})(1-e^{\alpha T_d}z^{-1})} \\
&= \frac{T_d}{2\alpha}\frac{\left(e^{\alpha T_d} - e^{-\alpha T_d}\right)z^{-1}}{(1-e^{-\alpha T_d}z^{-1})(1-e^{-\alpha T_d}z)e^{\alpha T_d}z^{-1}} \\
&= \frac{T_d}{2\alpha}\frac{(1-e^{-2\alpha T_d})}{(1-e^{-\alpha T_d}z^{-1})(1-e^{-\alpha T_d}z)} \\
&= \left[\sqrt{\frac{T_d}{2\alpha}(1-e^{-2\alpha T_d})}\frac{1}{(1-e^{-\alpha T_d}z^{-1})}\right]\left[\sqrt{\frac{T_d}{2\alpha}(1-e^{-2\alpha T_d})}\frac{1}{(1-e^{-\alpha T_d}z)}\right]
\end{aligned}$$

if $\alpha > 0$, then

$$h[n] = \sqrt{\frac{T_d}{2\alpha}(1-e^{-2\alpha T_d})}\left(e^{-\alpha T_d}\right)^n u[n]$$

(b) Since $|H_c(j\Omega)|^2 = \Phi_c(j\Omega)$ and $\Phi(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) = |H(e^{j\omega})|^2$, we see that since $\phi[m] = T_d\phi_c(mT_d)$,

$$\Phi(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \Phi_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right).$$

Therefore, if $\Phi_c(j\Omega) \simeq 0$ for $|\Omega| \geq \frac{\pi}{T_d}$, then $\Phi(e^{j\omega}) \simeq \Phi_c\left(j\frac{\omega}{T_d}\right)$ and $|H(e^{j\omega})|^2 \simeq \left|H_c\left(j\frac{\omega}{T_d}\right)\right|^2$.

(c) *No.* We could always cascade $H(z)$ with an allpass filter. The new filter is different, but has the same autocorrelation.

7.45. (a) Since the two flow diagrams are equivalent we have

$$\begin{aligned} Z^{-1} &= \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} = \frac{1-\alpha z}{z-\alpha} \\ Z &= \frac{z-\alpha}{1-\alpha z} \end{aligned}$$

$$H(z) = H_{lp}(Z)|_{Z=\frac{z-\alpha}{1-\alpha z}} = H_{lp}\left(\frac{z-\alpha}{1-\alpha z}\right)$$

(b) Let $Z = e^{j\theta}$ and $z = e^{j\omega}$. Then

$$\begin{aligned} Z^{-1} &= \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \\ e^{-j\theta} &= \frac{e^{-j\omega}-\alpha}{1-\alpha e^{-j\omega}} \\ e^{-j\theta} - \alpha e^{-j\theta}e^{-j\omega} &= e^{-j\omega}-\alpha \\ e^{-j\omega}(1+\alpha e^{-j\theta}) &= e^{-j\theta}+\alpha \\ e^{-j\omega} &= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}} \\ &= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}} \cdot \frac{1+\alpha e^{j\theta}}{1+\alpha e^{j\theta}} \\ &= \frac{e^{-j\theta}+2\alpha+\alpha^2 e^{j\theta}}{1+2\alpha\cos\theta+\alpha^2} \end{aligned}$$

Using Euler's formula,

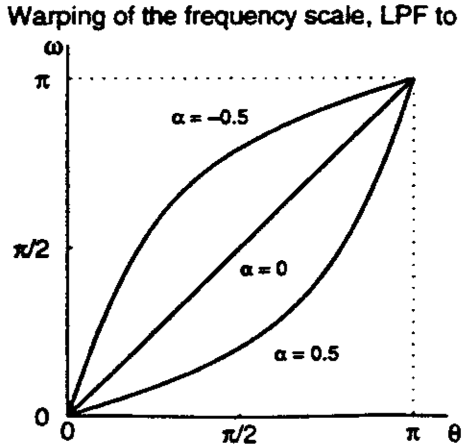
$$\begin{aligned} e^{-j\omega} &= \frac{\cos\theta - j\sin\theta + 2\alpha + \alpha^2\cos\theta + j\alpha^2\sin\theta}{1+2\alpha\cos\theta+\alpha^2} \\ &= \frac{2\alpha + (1+\alpha^2)\cos\theta + j[(\alpha^2-1)\sin\theta]}{1+2\alpha\cos\theta+\alpha^2} \end{aligned}$$

Noting that $-\omega = \tan^{-1}\left[\frac{Im\{\cdot\}}{Re\{\cdot\}}\right]$,

$$\begin{aligned} -\omega &= \tan^{-1}\left[\frac{(\alpha^2-1)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right] \\ \omega &= \tan^{-1}\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right] \end{aligned}$$

This relationship is plotted in the figure below for $\alpha = 0, \pm 0.5$.

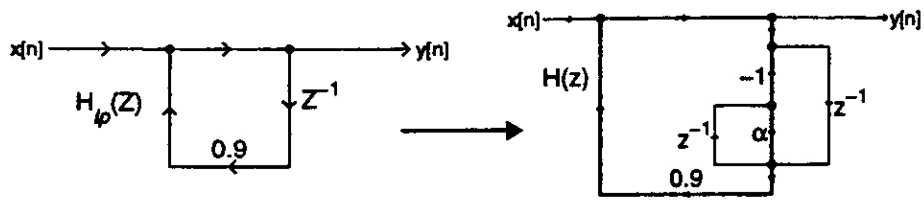
Warping of the frequency scale, LPF to LPF



Although a warping of the frequency scale is evident in the figure, (except when $\alpha = 0$, which corresponds to $Z^{-1} = z^{-1}$), if the original system has a piecewise-constant lowpass frequency response with cutoff frequency θ_p, then the transformed system will likewise have a similar lowpass response with cutoff frequency ω_p determined by the choice of α.

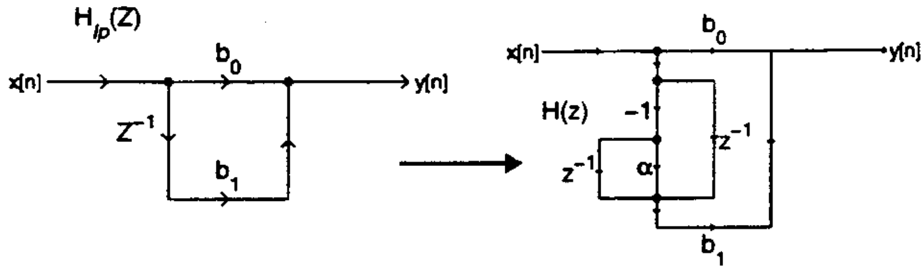
$$\omega_p = \tan^{-1}\left[\frac{(1-\alpha^2)\sin(\theta_p)}{2\alpha + (1+\alpha^2)\cos(\theta_p)}\right]$$

(c)



Looking at the flow graph for $H(z)$ we see a feedback loop with no delay. This effectively makes the current output, $y[n]$, a function of itself. Hence, there is no computable difference equation.

(d) *Yes*, the flow graph manipulation would lead to a computable difference equation. The flowgraph of an FIR filter has a path without delays leading from input to output, but this does not present any problems in terms of computation. Below is an example.



The transformation would destroy the linear phase of the FIR filter since the mapping between θ and ω is nonlinear. The only exception is the special case when $\alpha = 0$, i.e., when $\theta = \omega$.
Since there are feedback terms in the transformed filter, it must be an IIR filter. It therefore has an infinitely long impulse response.

(e) Since the two flow diagrams are equivalent we have

$$\begin{aligned} Z^{-1} &= z^{-1}\frac{z^{-1}-\alpha}{1-\alpha z^{-1}} = z^{-1}\frac{1-\alpha z}{z-\alpha} \\ Z &= z\frac{z-\alpha}{1-\alpha z} \end{aligned}$$

$$H(z) = H_{lp}(Z)|_{Z=z\frac{z-\alpha}{1-\alpha z}} = H_{lp}\left(z\frac{z-\alpha}{1-\alpha z}\right)$$

Letting $Z = e^{j\theta}$ and $z = e^{j\omega}$ we have,

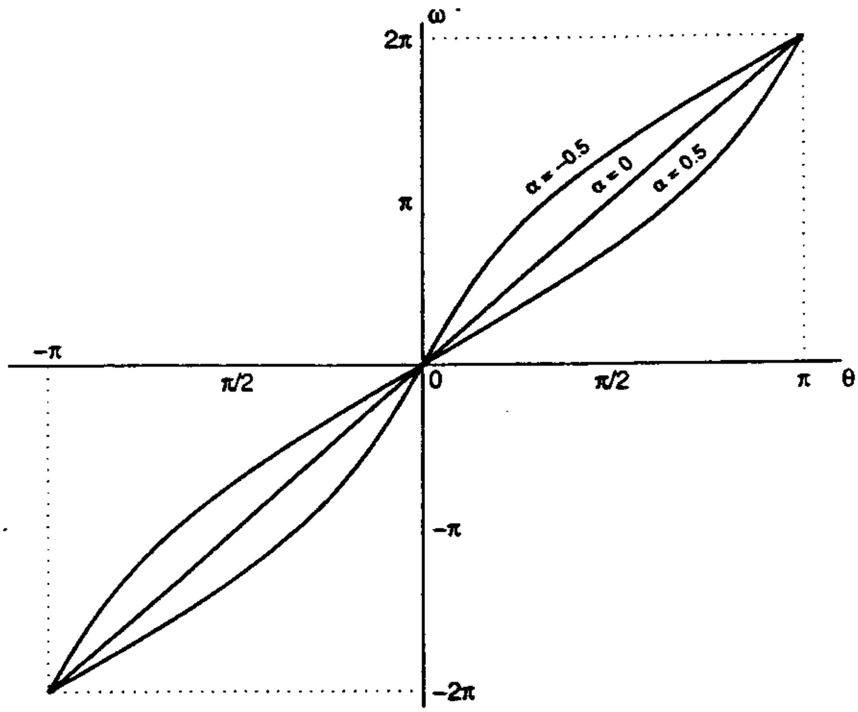
$$\begin{aligned} e^{j\theta} &= e^{j\omega}\frac{e^{j\omega}-\alpha}{1-\alpha e^{j\omega}} \\ &= e^{j\omega}\frac{e^{j\omega}-\alpha}{1-\alpha e^{j\omega}}\cdot\frac{1-\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= e^{j\omega}\frac{e^{j\omega}-2\alpha+\alpha^2 e^{-j\omega}}{1-2\alpha\cos\omega+\alpha^2} \end{aligned}$$

Using Euler's formula,

$$e^{j\theta} = e^{j\omega} \frac{(1+\alpha^2)\cos\omega - 2\alpha + j(1-\alpha^2)\sin\omega}{1 - 2\alpha\cos\omega + \alpha^2}$$

Noting that $\theta = \omega + \tan^{-1}\left[\frac{Im\{\cdot\}}{Re\{\cdot\}}\right]$,

$$\theta = \omega + \tan^{-1}\left[\frac{(1-\alpha^2)\sin\omega}{(1+\alpha^2)\cos\omega - 2\alpha}\right]$$



We see from the plot of ω versus θ that a lowpass filter will not always transform into a lowpass filter. Take, for example, the case when the original lowpass filter has a cutoff of $\theta = \pi/2$. With $\alpha = 0$ it would transform into an allpass filter.

7.46. (a) Since

$$\begin{aligned} y[n] &= (2x[n] - h[n] * x[n]) * h[n] \\ &= (2h[n] - h[n] * h[n]) * x[n] \end{aligned}$$

the new transfer function is

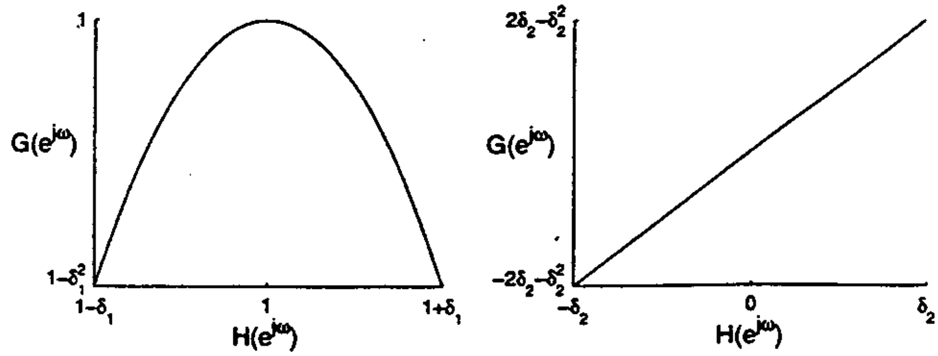
$$g[n] = 2h[n] - h[n] * h[n]$$

(i) It is FIR since the convolution of two finite length sequences results in a finite length sequence.

(ii) Note that the term $h[n] * h[n]$ is symmetric since it is the convolution of two symmetric sequences. Therefore, g[n] must be symmetric since it is the difference of two symmetric sequences.

(b) The frequency response for $G(e^{j\omega})$ is

$$G(e^{j\omega}) = 2H(e^{j\omega}) - H(e^{j\omega})H(e^{j\omega})$$



As shown above, if the passband of $H(e^{j\omega})$ is the region $[1-\delta_1, 1+\delta_1]$, then the passband of $G(e^{j\omega})$ is in the region $[1-\delta_1^2, 1]$ which is a smaller band. However, the stop band gets bigger since it maps to $[-2\delta_2 - \delta_2^2, 2\delta_2 - \delta_2^2]$.
Thus,

$$\begin{aligned} A &= (1-\delta_1^2) \\ B &= 1 \\ C &= -2\delta_2 - \delta_2^2 \\ D &= 2\delta_2 - \delta_2^2 \end{aligned}$$

If $\delta_1 \ll 1$ and $\delta_2 \ll 1$ then,

$$\begin{aligned} \text{Maximum passband approximation error} &\approx 0 \\ \text{Maximum stopband approximation error} &\approx 2\delta_2 \end{aligned}$$

(c) Since

$$\begin{aligned} y[n] &= (3x[n] - 2x[n] * h[n]) * h[n] * h[n] \\ &= (3h[n] * h[n] - 2h[n] * h[n] * h[n]) * x[n] \end{aligned}$$

the new transfer function is

$$h_{\text{sharp}}[n] = 3h[n] * h[n] - 2h[n] * h[n] * h[n]$$

and so

$$H_{\text{sharp}}(e^{j\omega}) = 3H(e^{j\omega})^2 - 2H(e^{j\omega})^3$$

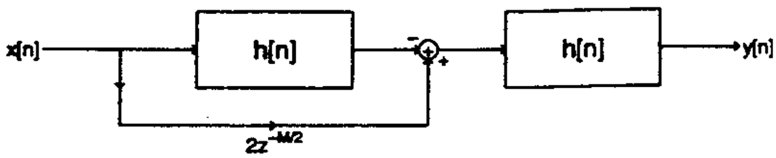
The new tolerance specifications can be found in a similar manner to the last section. We get,

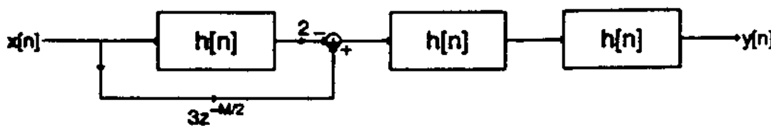
$$\begin{aligned} A &= 1 - 3\delta_1^2 - 2\delta_1^3 \\ B &= 1 \\ C &= 0 \\ D &= 3\delta_2^2 + 2\delta_2^3 \end{aligned}$$

If $\delta_1 \ll 1$ and $\delta_2 \ll 1$ then,

$$\begin{aligned} \text{Maximum passband approximation error} &\approx 0 \\ \text{Maximum stopband approximation error} &\approx 0 \end{aligned}$$

(d) The order of the impulse response $h[n]$ is M. Since it is linear phase it must therefore have a delay of $\frac{M}{2}$ samples. To convert the two systems we must add a delay in the lower leg of each network to match the delay that was added by the first filter.





The restrictions on the filter that carry over from part (a) are that it have

(i) Even symmetry

(ii) Odd Length

Hence, Type I FIR filters can be used.

The length of $h[n]$ is $2L+1$. Since the term that is longest in the twicing system's impulse response is the $h[n] * h[n]$ term, the length of $g[n]$ is $4L+1$. Since the term that is longest in the sharpening system's impulse response is the $h[n] * h[n] * h[n]$ term, the length of $h_{\text{sharp}}[n]$ is $6L+1$.

7.47. We know that *any* system whose frequency response is of the form

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos(\omega))^k$$

can have at most $L-1$ local maxima and minima in the open interval $0 < \omega < \pi$ since it is in the form of a polynomial of degree L.

If we include all endpoints of the approximation region

$$\{0 \leq |\omega| \leq \omega_p\} \bigcup \{\omega_s \leq |\omega| \leq \pi\}$$

then we see we can have at most $L+3$ alternation frequencies.

If the transition band has two of the local minima or maxima of $A_e(e^{j\omega})$, then only $L-3$ can be in the approximation bands. Even with all four endpoints of the approximation region as alternation points, we can only have a maximum of $L+1$ alternation points. This does not satisfy the optimality condition of the Alternation Theorem which requires at least $L+2$ alternation points. It follows that the transition band cannot have more than two local minima or maxima of $A_e(e^{j\omega})$ either.

If the transition band only has one of the local minima or maxima of $A_e(e^{j\omega})$, then the error will not alternate between ω_p and ω_s and they cannot both be alternation frequencies. In this case, only $L-2$ of the local minima or maxima of $A_e(e^{j\omega})$ are in the approximation bands. If we add the maximum of three band edges to the total count of alternation frequencies we get $L+1$, which is again too low.

Therefore, the transition band cannot have any local minima or maxima and must be monotonic.

7.48. (a) $A_e(e^{j\omega})$ has 7 alternations of the error. If the approximation bands are of equal length and the weighting function is unity in both bands, why would the stopband have 1 extra alternation than the passband? The answer is that, if it were an optimal filter, it would not. The optimal filter for this set of specifications should have the *same* number of alternations in each band and therefore requires an *even* number of alternations. Since the optimal approximation is unique, the one shown in the figure cannot be optimal.

(b) A polynomial of degree L can have at most $L-1$ local minima or maxima in an open interval. Since $A_e(e^{j\omega})$ has three local extrema in the interval from $0 < \omega < \pi$, we know $L \geq 4$.

Note that the *optimal* filter is half wave anti-symmetric if you lower its frequency response by one half, i.e.,

$$A_{hw}(e^{j\omega}) = -A_{hw}\left(e^{j(\pi-\omega)}\right)$$

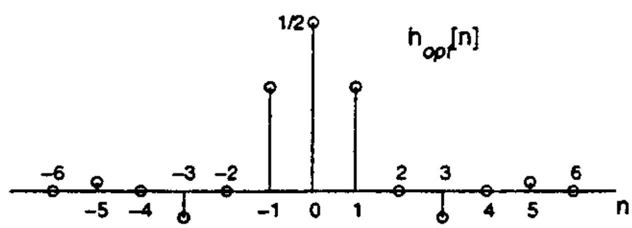
where $A_{hw}(e^{j\omega}) = H_{opt}(e^{j\omega}) - 1/2$. Another way of saying this is to say that the optimal filter is anti-symmetric around $\omega = \pi/2$ after lowering the response by $1/2$. This property holds because the optimal filter has symmetric bands with the same number of alternations. Plugging in $A_{hw}(e^{j\omega}) = H_{opt}(e^{j\omega}) - 1/2$ into the above expression gives

$$\begin{aligned}
H_{opt}(e^{j\omega}) - 1/2 &= -\left[H_{opt}\left(e^{j(\pi-\omega)}\right) - 1/2\right] \\
H_{opt}(e^{j\omega}) &= -H_{opt}\left(e^{j(\pi-\omega)}\right) + 1 \\
h_{opt}[n] &= -(-1)^n h_{opt}[-n] + \delta[n]
\end{aligned}$$

This condition implies that

$$h_{opt}[n] = \begin{cases} h_{opt}[-n], & n \text{ odd} \\ 0, & n \text{ even, n} \neq 0 \\ 0.5, & n = 0 \end{cases}$$

A sample plot of $h_{opt}[n]$ appears below, for $L = 6$.



Note that because $h_{opt}[n] = 0$ for n even, $n \neq 0$, a plot of $h_{opt}[n]$ for $L = 5$ would have the same nonzero samples, and therefore be equivalent. So the optimal filter with $L = 6$ is really the same filter as the case of $L = 5$, just as the optimal filter with $L = 4$ is the same filter as the case with $L = 3$.

We know the filter non-optimal filter has 7 alternations. The optimal filter should be able to meet the same specifications, but with a lower order. From part (a), we know the number of alternations must be even. Thus, the optimal filter for these specifications will have 6 alternations.

An optimal lowpass filter has either $L + 2$ or $L + 3$ alternations which means $L = 4$ or $L = 3$. However, we showed above that these are really the same filter. Since the optimal filter has $L = 4$, the filter shown in the problem cannot have $L = 4$.

Putting it all together we find $L > 4$ for the filter shown in the figure.

7.49. (a)

$$\begin{aligned}
H_{\text{eff}}(j\Omega) &= \frac{1}{T} H(e^{j\Omega T}) H_0(j\Omega) H_r(j\Omega) \\
&= \begin{cases} \dfrac{2\sin\left(\frac{\Omega T}{2}\right)}{\Omega T} H\left(e^{j\Omega T}\right) e^{-j\frac{\Omega T}{2}}, & |\Omega| < \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b) The delay of the linear phase system is $51/2 = 25.5$ samples since it is a linear phase system of order 51. Therefore, the total delay is

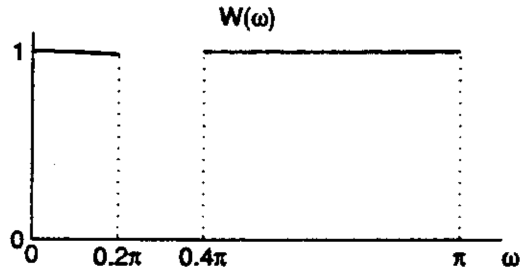
$$\begin{aligned}\text{Delay} &= \overbrace{25.5T}^{H(e^{j\Omega T})} + \overbrace{0.5T}^{H_0(j\Omega)} \\ &= 26T \\ &= 2.6 \text{ ms}\end{aligned}$$

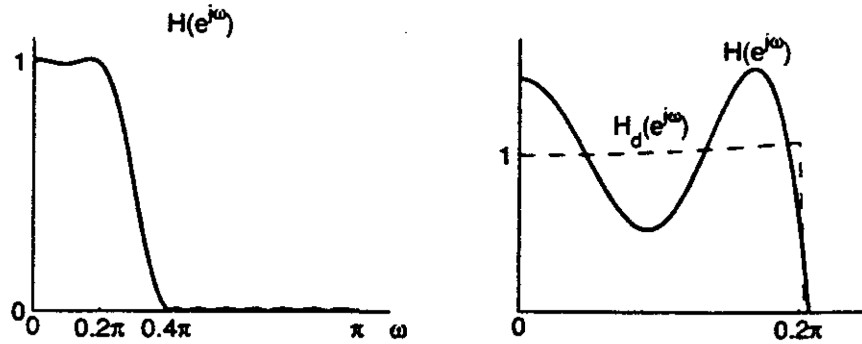
(c) $H(e^{j\Omega T})$ should cancel the effects of $H_0(j\Omega)$. However, to cancel the effects of the delay introduced by $H_0(j\Omega)$ would require a noncausal filter which is not practical in this situation. Using the relation $\omega = \Omega T$,

$$H_d(e^{j\omega}) = \begin{cases} \dfrac{\frac{\omega}{2}}{\sin\left(\frac{\omega}{2}\right)}, & |\omega| \le 0.2\pi \\ 0, & 0.4\pi \le |\omega| \le \pi \end{cases}$$

To obtain equiripple behavior in $H_{\text{eff}}(j\Omega)$, we need to weight the error so that the ripples grow with $H_d(e^{j\omega})$. Then when we multiply by $H_0(j\Omega)$ the ripples will be decreased to an equal size. Therefore, we need

$$W(\omega) = \begin{cases} \dfrac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}, & |\omega| \le 0.2\pi \\ 1, & 0.4\pi \le |\omega| \le \pi \end{cases}$$





(d) If $H_r(j\Omega)$ is also sloping across the band, $|\Omega| < \pi/T$, we would combine its effects with those of $H_0(j\Omega)$ and compensate as in part (c), i.e.,

$$H_d(e^{j\omega}) = \begin{cases} \dfrac{\frac{\omega}{2}}{\sin\left(\frac{\omega}{2}\right)} \dfrac{1}{\left|H_r\left(j\frac{\omega}{T}\right)\right|}, & |\omega| < 0.2\pi \\ 0, & 0.4\pi \le |\omega| \le \pi \end{cases}$$

This would take care of the distortion due to $|H_r(j\Omega)|$ but not of any phase distortion. The weighting function will change in a similar manner.

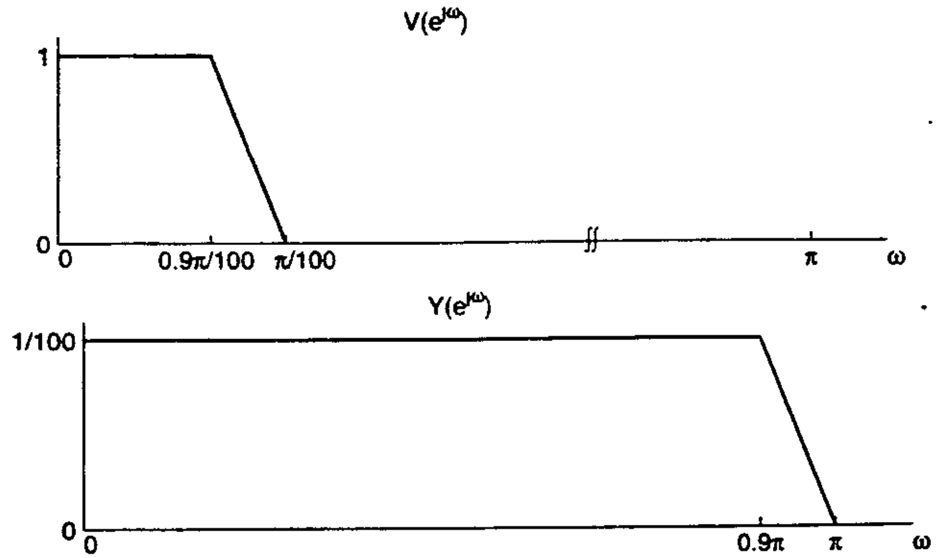
7.50. (a) To avoid aliasing, we require

$$\begin{aligned} M\omega_s &\leq \pi \\ M &\leq \frac{\pi}{\omega_s} \end{aligned}$$

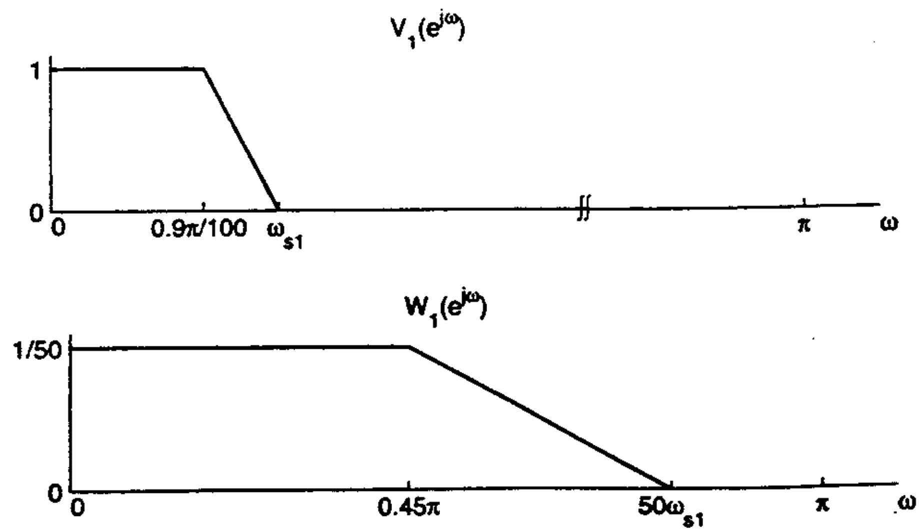
So the maximum allowable decimation factor is

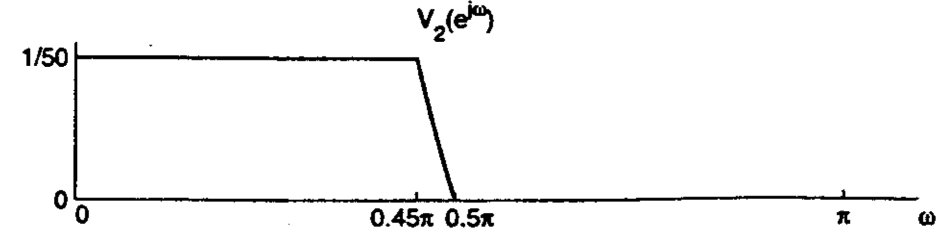
$$M_{\text{max}} = \frac{\pi}{\omega_s}$$

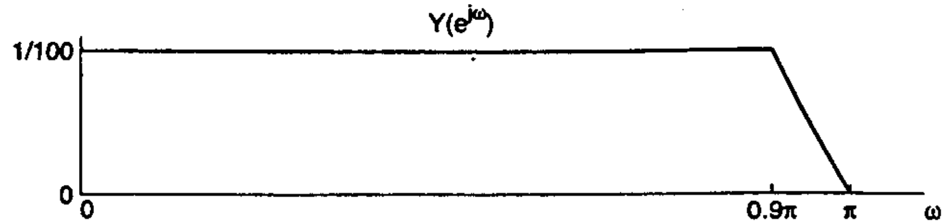
(b)



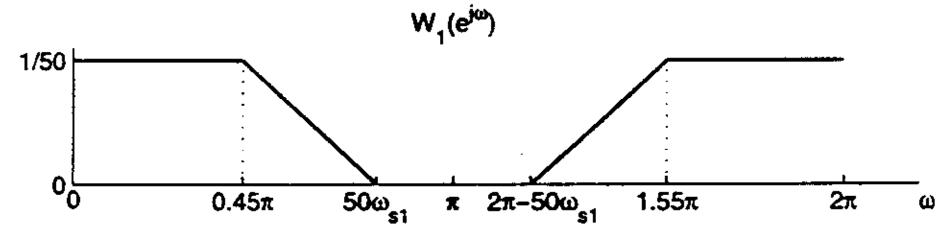
(c)







(d) After the first decimation by 50 is performed, $W_1(e^{j\omega})$ should look like the following:



Since we allow aliasing to occur in the transition bands, we have

$$\begin{aligned} 50\omega_{s_1} &\leq 1.55\pi \\ \omega_{s_1} &\leq 0.031\pi \end{aligned}$$

(e) Using $\delta_p = 0.01$, $\delta_s = 0.001$, $\Delta\omega = 0.001\pi$ we get

$$\begin{aligned} N &= \frac{-10\log_{10}(0.01 \times 0.001) - 13}{2.324(0.001\pi)} + 1 \\ &\simeq 5069 \end{aligned}$$

In general, the number of multiplies required to compute a single output sample is just N. For a linear phase filter, however, the symmetry in the coefficients allow us to cut the number of multiplies (roughly) in half if implementing the filter with a difference equation. The following is an example of how this is accomplished using the simple Type I linear phase filter $h[n] = 0.25\delta[n] + \delta[n-1] + 0.25\delta[n-2]$.

$$\begin{aligned} y[n] &= 0.25x[n] + x[n-1] + 0.25x[n-2] \quad \text{(2 multiplies)} \\ &= x[n-1] + 0.25(x[n] + x[n-2]) \quad \text{(1 multiply)} \end{aligned}$$

The procedure is similar for the other types of linear phase filters.

Thus, we need 2535 multiplies to compute each sample of the output.

(f) We have

$$N_1 = \frac{-10\log_{10}(0.01 \times 0.001) - 13}{2.324(0.031\pi - 0.009\pi)} + 1$$

$$
\begin{aligned}
&\simeq 232 \\
N_2 &= \frac{-10\log_{10}(0.01\times 0.001)-13}{2.324(0.5\pi-0.45\pi)}+1 \\
&\simeq 103
\end{aligned}
$$

If we again use linear phase filters we find

116	multiplies to get each sample of $v_1[n]$
0	multiplies to get each sample of $w_1[n]$ from $v_1[n]$
52	multiplies to get each sample of $v_2[n]$ from $w_1[n]$
0	multiplies to get each sample of $y[n]$ from $v_2[n]$

The total number of multiplies is 168.

(g) We have

$$
\begin{aligned}
N_1 &= \frac{-10\log_{10}(0.005\times 0.001)-13}{2.324(0.022\pi)}+1 \\
&\simeq 251 \\
N_2 &= \frac{-10\log_{10}(0.005\times 0.001)-13}{2.324(0.05\pi)}+1 \\
&\simeq 111
\end{aligned}
$$

Therefore, we have a total of $126+56=182$ multiplies per output point.

(h) *No.* Since $\delta_s \ll 1$ we have $\delta_s^2 < \delta_s$ which means the stopband ripple is getting smaller. Thus, we could actually increase the specifications.

(i) Performing a similar analysis on the other possibilities yields

M_1	M_2	Multiplies per output
50	2	182
25	4	156
20	5	172
10	10	291
5	20	557
4	25	693
2	50	1375

Thus, the choice $M_1 = 25$ and $M_2 = 4$ yields the minimum number of multiplications for this example.

7.51. (a)

$$
\begin{aligned}
h_1[n] &= h[n]+\delta_2\delta[n-n_0] \\
H_1(e^{j\omega}) &= H(e^{j\omega})+\delta_2 e^{-j\omega n_0} \\
&= A_e(e^{j\omega})e^{-j\omega n_0}+\delta_2 e^{-j\omega n_0} \\
&= \underbrace{\left[A_e(e^{j\omega})+\delta_2\right]}_{H_3(e^{j\omega})} e^{-j\omega n_0}
\end{aligned}
$$

$H_3(e^{j\omega})$ is real since $A_e(e^{j\omega})$ is real and δ_2 is real. It is nonnegative since $A_e(e^{j\omega}) \geq -\delta_2$. Note that $H_3(e^{j\omega})$ is an even function of ω and is a zero-phase filter.

(b) $H_3(e^{j\omega})$ is a zero-phase filter with real coefficients. Thus, a zero at z_k implies there must also be zeros at z_k^*, $1/z_k$, and $1/z_k^*$. In addition, a zero on the unit circle must be a double zero because both its value and its derivative is zero. Note that this last property is true for $H_3(e^{j\omega})$ but not for $A_e(e^{j\omega})$. We can write $H_3(z)$ as

$$H_3(z) = H_2(z)H_2(1/z)$$

where $H_2(z)$ contains all the complex conjugate zero pairs inside the unit circle and $H_2(1/z)$ contains the corresponding complex conjugate zero pairs outside the unit circle. We factor one of the double zeros on the unit circle and its complex conjugate zero into $H_2(z)$. The other pair on the unit circle goes into $H_2(1/z)$.

Since $H_2(z)$ has its zeros on or inside the unit circle it is minimum phase (we allow minimum phase systems to have zeros on the unit circle in this problem). Since the zeros occur in complex conjugate pairs, $h_2[n]$ is real.

(c)

$$\begin{aligned} |H_{min}(e^{j\omega})|^2 &= \frac{H_2(e^{j\omega})H_2^*(e^{j\omega})}{a^2} \\ &= \frac{A_e(e^{j\omega}) + \delta_2}{a^2} \end{aligned}$$

where $a = \frac{\sqrt{1+\delta_1+\delta_2}+\sqrt{1-\delta_1+\delta_2}}{2}$. Since $1-\delta_1 \leq A_e(e^{j\omega}) \leq 1+\delta_1$ in the passband and $-\delta_2 \leq A_e(e^{j\omega}) \leq \delta_2$ in the stopband, we have

$$\frac{\sqrt{1-\delta_1+\delta_2}}{a} \leq |H_{min}(e^{j\omega})| \leq \frac{\sqrt{1+\delta_1+\delta_2}}{a}, \qquad \omega \in \text{passband}$$

$$0 \leq |H_{min}(e^{j\omega})| \leq \sqrt{\frac{2\delta_2}{1+\delta_2}}, \qquad \omega \in \text{stopband}$$

Therefore,

$$\begin{aligned} \delta_1' &= \frac{1}{2}\left[\frac{\sqrt{1+\delta_1+\delta_2}}{a} - \frac{\sqrt{1-\delta_1+\delta_2}}{a}\right] \\ &= \frac{1-b}{1+b}, \quad b = \sqrt{\frac{1-\delta_1+\delta_2}{1+\delta_1+\delta_2}} \\ \delta_2' &= \sqrt{\frac{2\delta_2}{1+\delta_2}} \end{aligned}$$

The original filter $h[n]$ has order M. Therefore, $h_1[n]$ also has order M, but $h_2[n]$ has order $M/2$ due to the spectral factorization. Since $h_{min}[n]$ has the same order as $h_2[n]$ we find that the length of $h_{min}[n]$ is $M/2+1$.

(d) *No.* If we remove the linear phase constraint, then the zeros of $H_3(z)$ are not distributed in conjugate reciprocal quads. It then becomes impossible to express

$$H_3(z) = H_2(z)H_2(z^{-1})$$

where $H_2(z)$ is a minimum phase filter.

No. It will not work with a Type II linear phase filter. In this case $n_0 = M/2$ is not an integer.

7.52. (a)

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$
\begin{aligned}
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega M}e^{j\omega m} \\
&= e^{-j\omega M/2}\left[\sum_{n=0}^{(M-1)/2} h[n]e^{j\omega(M/2-n)} + \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega(M/2-n)}\right] \\
&= e^{-j\omega M/2}\sum_{n=0}^{(M-1)/2} 2h[n]\cos\omega(M/2-n) \\
&= e^{-j\omega M/2}\sum_{n=1}^{(M+1)/2} 2h[\tfrac{M+1}{2}-n]\cos\omega(n-\tfrac{1}{2})
\end{aligned}
$$

Then

$$H(e^{j\omega}) = e^{-j\omega M/2}\sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-1/2)$$

where $b[n] = 2h[(M+1)/2-n]$ for $n = 1, \ldots, (M+1)/2$.

(b) Using the trigonometric identity

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$

we get

$$
\begin{aligned}
\cos(\omega/2)\sum_{n=0}^{\frac{M-1}{2}} \tilde{b}[n]\cos\omega n &= \frac{1}{2}\sum_{n=0}^{\frac{M-1}{2}} \tilde{b}[n]\cos\omega(n+\tfrac{1}{2}) + \frac{1}{2}\sum_{n=0}^{\frac{M-1}{2}} \tilde{b}[n]\cos\omega(n-\tfrac{1}{2}) \\
&= \frac{1}{2}\sum_{n=1}^{\frac{M+1}{2}} \tilde{b}[n-1]\cos\omega(n-\tfrac{1}{2}) + \frac{1}{2}\sum_{n=1}^{\frac{M+1}{2}} \tilde{b}[n]\cos\omega(n-\tfrac{1}{2}) \\
&\quad + \frac{1}{2}\tilde{b}[0]\cos\omega/2 - \frac{1}{2}\tilde{b}[\tfrac{M+1}{2}]\cos\omega M/2 \\
&= \frac{1}{2}\sum_{n=1}^{\frac{M+1}{2}} \left(\tilde{b}[n]+\tilde{b}[n-1]\right)\cos\omega(n-\tfrac{1}{2}) + \tfrac{1}{2}\tilde{b}[0]\cos\omega/2 - \tfrac{1}{2}\tilde{b}[\tfrac{M+1}{2}]\cos\omega M/
\end{aligned}
$$

Since this last expression must equal

$$\sum_{n=1}^{\frac{M+1}{2}} b[n]\cos\omega(n-\tfrac{1}{2})$$

we can just match up the multipliers in front of the cosine terms of the two expressions. We get

$$
b[n] = \begin{cases}
\dfrac{\tilde{b}[1]+2\tilde{b}[0]}{2}, & n=1 \\
\dfrac{\tilde{b}[n]+\tilde{b}[n-1]}{2}, & 2 \le n \le \frac{M-1}{2} \\
\dfrac{\tilde{b}[\frac{M-1}{2}]}{2}, & n = \frac{M+1}{2}
\end{cases}
$$

(c) Consider

$$\begin{aligned}
W(\omega)\left[H_d(e^{j\omega}) - A(e^{j\omega})\right] &= W(\omega)\left[H_d(e^{j\omega}) - \sum_{n=1}^{\frac{M+1}{2}} b[n]\cos\omega\left(n-\frac{1}{2}\right)\right] \\
&= W(\omega)\left[H_d(e^{j\omega}) - \cos(\omega/2)\sum_{n=0}^{\frac{M-1}{2}} \tilde{b}[n]\cos\omega n\right] \\
&= \tilde{W}(\omega)\left[\tilde{H}_d(e^{j\omega}) - \sum_{n=0}^{\tilde{L}} \tilde{b}[n]\cos\omega n\right]
\end{aligned}$$

where we have defined

$$\begin{aligned}
\tilde{L} &= \frac{M-1}{2} \\
\tilde{H}_d(e^{j\omega}) &= \frac{H_d(e^{j\omega})}{\cos(\omega/2)} \\
\tilde{W}(\omega) &= W(\omega)\cos(\omega/2)
\end{aligned}$$

We also see that

$$\min_{\tilde{b}[n]}\left\{\max_{\omega\in\tilde{F}}\{\cdot\}\right\} \Longleftrightarrow \min_{b[n]}\left\{\max_{\omega\in F}\{\cdot\}\right\}$$

(d) **Type III filters:**

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + 0 + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} - \sum_{m=0}^{M/2-1} h[m]e^{-j\omega(M-m)} \\
&= e^{-j\omega M/2}\sum_{n=0}^{M/2-1} h[n]\left(e^{-j\omega(n-M/2)} - e^{j\omega(n-M/2)}\right) \\
&= e^{-j\omega M/2}\sum_{n=0}^{M/2-1} (-2j)h[n]\sin\omega(n-M/2) \\
&= e^{-j\omega M/2}\sum_{m=1}^{M/2} 2jh[M/2-m]\sin\omega m
\end{aligned}$$

Then

$$H(e^{j\omega}) = e^{-j\omega M/2}\sum_{n=1}^{M/2} c[n]\sin\omega n$$

where $c[n] = 2jh[M/2-n]$ for $n = 1,\ldots,M/2$.
If we follow a similar analysis as the one in part (b) we get

$$\sin\omega\sum_{n=0}^{\frac{M}{2}-1}\tilde{c}[n]\cos\omega n = \frac{1}{2}\sum_{n=0}^{\frac{M}{2}-1}\tilde{c}[n]\sin\omega(n+1) - \frac{1}{2}\sum_{n=0}^{\frac{M}{2}-1}\tilde{c}[n]\sin\omega(n-1)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{\frac{M}{2}} \tilde{c}[n-1]\sin\omega n - \frac{1}{2}\sum_{n=1}^{\frac{M}{2}} \tilde{c}[n]\sin\omega n \\
&\quad + \frac{1}{2}\tilde{c}[0]\sin\omega + \frac{1}{2}\tilde{c}[\tfrac{M}{2}]\sin\omega M/2 \\
&= \frac{1}{2}\sum_{n=1}^{\frac{M}{2}} (\tilde{c}[n-1] - \tilde{c}[n])\sin\omega n + \frac{1}{2}\tilde{c}[0]\sin\omega + \frac{1}{2}\tilde{c}[\tfrac{M}{2}]\sin\omega M/2
\end{aligned}
$$

Matching terms we get

$$
c[n] = \begin{cases} \dfrac{2\tilde{c}[0] - \tilde{c}[1]}{2}, & n = 1 \\ \dfrac{\tilde{c}[n-1] - \tilde{c}[n]}{2}, & 2 \le n \le \frac{M}{2} - 1 \\ \dfrac{\tilde{c}[\frac{M}{2} - 1]}{2}, & n = \frac{M}{2} \end{cases}
$$

In a manner similar to that of part (c) we can find

$$
\begin{aligned}
\tilde{L} &= \frac{M}{2} - 1 \\
\tilde{H}_d(e^{j\omega}) &= \frac{H_d(e^{j\omega})}{\sin\omega} \\
\tilde{W}(\omega) &= W(\omega)\sin\omega \\
\tilde{F} &= F
\end{aligned}
$$

Type IV filters:

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} - \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega(M-m)} \\
&= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} h[n]\left(e^{-j\omega(n-M/2)} - e^{j\omega(n-M/2)}\right) \\
&= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} (-2j)h[n]\sin\omega(n - M/2) \\
&= e^{-j\omega M/2} \sum_{m=1}^{(M+1)/2} 2jh[(M+1)/2 - m]\sin\omega(m - 1/2)
\end{aligned}
$$

Then

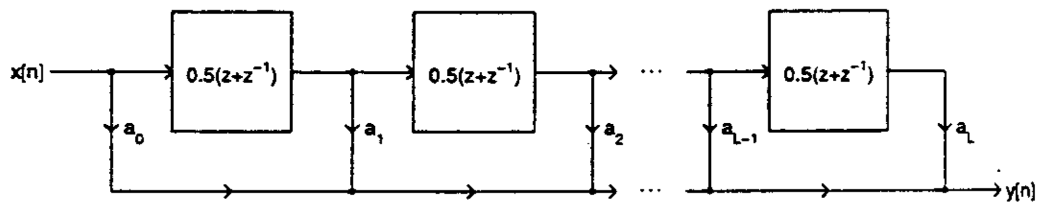
$$
H(e^{j\omega}) = e^{-j\omega M/2} \sum_{n=1}^{(M+1)/2} d[n]\sin\omega(n - 1/2)
$$

where $d[n] = 2jh[(M+1)/2 - n]$ for $n = 1, \ldots, (M+1)/2$. We can find

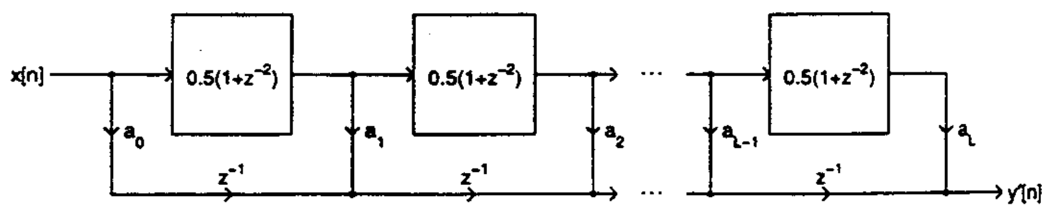
$$d[n] = \begin{cases} \dfrac{2\tilde{d}[0] - \tilde{d}[1]}{2}, & n = 1 \\ \dfrac{\tilde{d}[n-1] - \tilde{d}[n]}{2}, & 2 \le n \le \frac{M-1}{2} \\ \dfrac{\tilde{d}[\frac{M-1}{2}]}{2}, & n = \frac{M+1}{2} \end{cases}$$

$$\begin{aligned} \tilde{L} &= \frac{M-1}{2} \\ \tilde{H}_d(e^{j\omega}) &= \frac{H_d(e^{j\omega})}{\sin \omega/2} \\ \tilde{W}(\omega) &= W(\omega) \sin \omega/2 \\ \tilde{F} &= F \end{aligned}$$

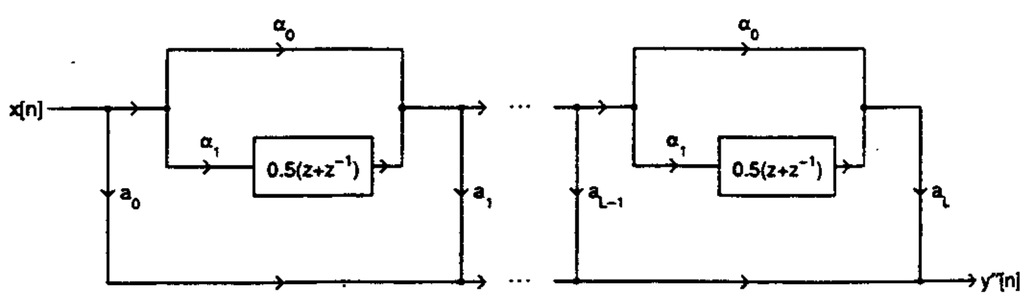
7.53. (a) The flow graph for $A_e(z)$ looks like



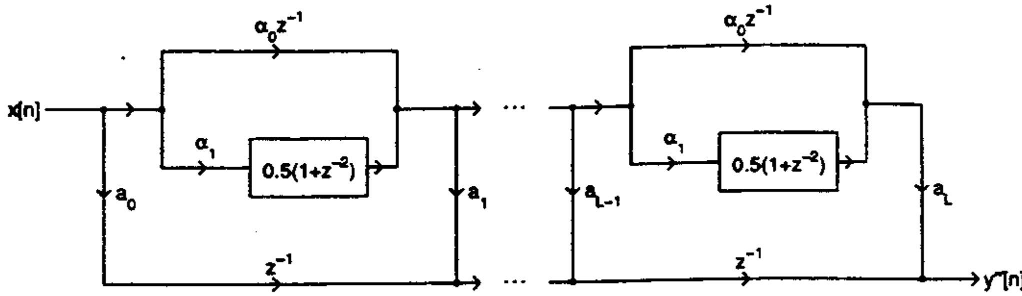
(b) The filter length is $2L+1$. The causal version of the flow graph looks like



(c) The flow graph for $B_e(z)$ looks like



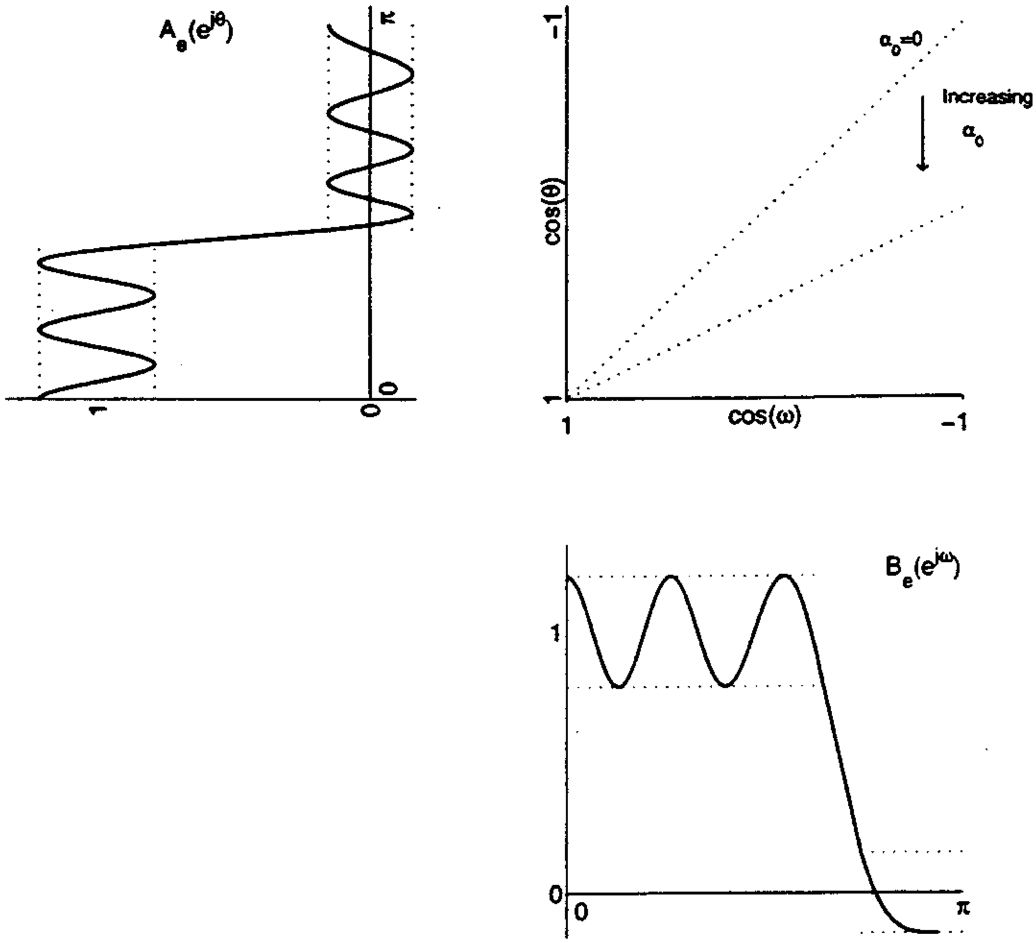
The filter length is still $2L+1$. The modified flow graph looks like



(d) Because $Z = e^{j\theta}$ and $z = e^{j\omega}$ we have

$$\begin{aligned}
\frac{e^{j\theta} + e^{-j\theta}}{2} &= \alpha_0 + \alpha_1 \left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right] \\
\cos\theta &= \alpha_0 + \alpha_1 \cos\omega \\
\cos\omega &= \frac{\cos\theta - \alpha_0}{\alpha_1} \\
\omega &= \cos^{-1}\left(\frac{\cos\theta - \alpha_0}{\alpha_1}\right), \quad \text{for } \left|\frac{\cos\theta - \alpha_0}{\alpha_1}\right| \leq 1
\end{aligned}$$

(e)



The picture above shows the mapping for α_0 somewhere between 0 and 1. The top right plot is the mapping of

$$\cos\theta = \alpha_0 + (1-\alpha_0)\cos\omega$$

We see that as α_0 increases, the transformation pushes the new passband further towards π. The new filter is not generally an optimal filter since we lose ripples or alternations while keeping L fixed. (Note that some of the original filter does not map anywhere in the new filter).

(f) In a similar manner, this choice of α_0 will cause the new passband to decrease with decreasing α_0.

7.54. (a) Let $D_k(z)$ be the z-transform of $\Delta^{(k)}\{x[n]\}$. Then

$$\begin{aligned}
D_0(z) &= \mathcal{Z}\left\{\Delta^0\{x[n]\}\right\} = X(z) \\
D_1(z) &= \mathcal{Z}\left\{\Delta^1\{x[n]\}\right\} = (z-z^{-1})X(z) \\
D_2(z) &= \mathcal{Z}\left\{\Delta^2\{x[n]\}\right\} = (z-z^{-1})^2X(z) \\
&\ \vdots \\
D_k(z) &= \mathcal{Z}\left\{\Delta^k\{x[n]\}\right\} = (z-z^{-1})^kX(z)
\end{aligned}$$

(b) By taking the transform of both sides of the continuous-time differential equation one gets (assuming initial rest conditions)

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{r=0}^{M} b_r s^r X(s)$$

Solving for $H_c(s)$

$$H_c(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{r=0}^{M} b_r s^r}{\sum_{k=0}^{N} a_k s^k}$$

Similarly,

$$\sum_{k=0}^{N} a_k (z - z^{-1})^k Y(z) = \sum_{r=0}^{M} b_r (z - z^{-1})^r X(z)$$

$$\begin{aligned} H_d(z) &= \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r (z - z^{-1})^r}{\sum_{k=0}^{N} a_k (z - z^{-1})^k} \\ &= H_c(s)\big|_{s=z-z^{-1}} \end{aligned}$$

$$\Rightarrow m(z) = z - z^{-1}$$

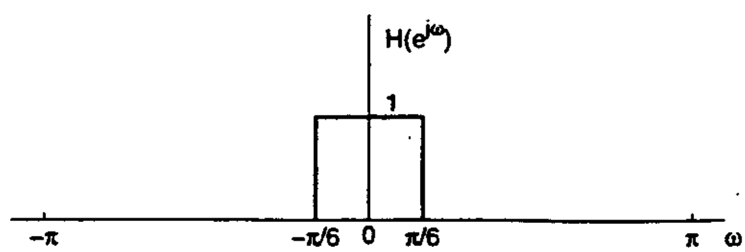
(c) First, map the continuous-time cutoff frequency into discrete-time and then make the sketch.

$$s = z - z^{-1}$$

$$j\Omega = e^{j\omega} - e^{-j\omega}$$

$$\Omega = \frac{e^{j\omega} - e^{-j\omega}}{j} = 2\sin(\omega) = 1$$

$$\omega = \frac{\pi}{6}$$



7.55. (a) Using DTFT properties,

$$\begin{aligned} h_1[n] &= h[-n] \\ H_1(e^{j\omega}) &= H(e^{-j\omega}) \end{aligned}$$

Since $H(e^{j\omega})$ is symmetric about $\omega = 0$, $H(e^{-j\omega}) = H(e^{j\omega})$. Thus, $H_1(e^{j\omega}) = H(e^{-j\omega}) = H(e^{j\omega})$. $H(e^{j\omega})$ is optimal in the minimax sense, so $H_1(e^{j\omega})$ is optimal in minimax sense as well.

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

$$W(e^{j\omega}) = \begin{cases} \delta_2/\delta_1, & 0 \le \omega \le \omega_p \\ 1, & \omega_s \le \omega \le \pi \end{cases}$$

(b) Using DTFT properties,

$$\begin{aligned} h_2[n] &= (-1)^n h[n] \\ &= (e^{-j\pi})^n h[n] \\ H_2(e^{j\omega}) &= H(e^{j(\omega+\pi)}) \end{aligned}$$

$H_2(e^{j\omega})$ is a high pass filter obtained by shifting $H(e^{j\omega})$ by π along the frequency axis. $H_2(e^{j\omega})$ satisfies the alternation thereom, and is therefore optimal in the minimax sense.

$$H_d(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega \le \pi - \omega_s \\ 1, & \pi - \omega_p \le \omega \le \pi \end{cases}$$

$$W(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \pi - \omega_s \\ \delta_2/\delta_1, & \pi - \omega_p \le \omega \le \pi \end{cases}$$

(c) Using DTFT properties,

$$\begin{aligned} h_3[n] &= h[n] * h[n] \\ H_3(e^{j\omega}) &= H(e^{j\omega})H(e^{j\omega}) \end{aligned}$$

In the passband, $H_3(e^{j\omega})$ alternates about $1 + \delta_1^2$ with a maximal error of $2\delta_1^2$. In the stopband, $H_3(e^{j\omega})$ alternates about $\delta_2^2/2$ with a maximal error of $\delta_2^2/2$. At first glance, it may appear that $H_3(e^{j\omega})$ is optimal. However, this is not the case. Counting alternations, we find that the original filter $H(e^{j\omega})$ has 8 alternations.

We know that since $H(e^{j\omega})$ is optimal, it must have at least $L+2$ alternations. It is also possible that $H(e^{j\omega})$ has $L+3$ alternations, if it corresponds to the extraripple case. So L is either 5 or 6 for this filter. Consequently, the filter length of $h[n]$, denoted as N, is either 11 or 13.

The filter $h_3[n]$ is the convolution of two length N sequences. Therefore, the length of $h_3[n]$, denoted as N', is $2N-1$. Since N is either 11 or 13, N' must be either 21 or 25. It follows that the polynomial order for $h_3[n]$, denoted as L', is either 10 or 12. For $h_3[n]$ to be optimal in the minimax sense, it must have at least $L'+2$ alternations. Thus, $h_3[n]$ must exhibit at least 12 alternations, for the non-extraripple case, or at least 14 alternations in the extraripple case to be optimal.

A simple counting of the alternations in $H_3(e^{j\omega})$ reveals that there are 11 alternations, consisting of the 8 alternations that were in $H(e^{j\omega})$ plus 3 where $H(e^{j\omega}) = 0$. These are too few to satisfy either the non-extraripple case or the extraripple case. As a result, this filter is not optimal in the minimax sense.

(d)

$$\begin{aligned} h_4[n] &= h[n] - K\delta[n] \\ H_4(e^{j\omega}) &= H(e^{j\omega}) - K \end{aligned}$$

This filter is simply $H(e^{j\omega})$ shifted down by K along the $H_4(e^{j\omega})$ axis. Consequently, this filter satisfies the alternation theorem, and is optimal in the minimax sense.

$$H_d(e^{j\omega}) = \begin{cases} 1-K, & 0 \le \omega \le \omega_p \\ -K, & \omega_s \le \omega \le \pi \end{cases}$$

$$W(e^{j\omega}) = \begin{cases} \delta_2/\delta_1, & 0 \le \omega \le \omega_p \\ 1, & \omega_s \le \omega \le \pi \end{cases}$$

(e) $h_5[n]$ is $h[n]$ upsampled by a factor of 2. In the frequency domain, upsampling by a factor of 2 will cause the frequency axis to get scaled by a factor of $1/2$. Consequently, $H_5(e^{j\omega})$ will be a bandstop filter that satisfies the alternation theorem, with twice as many alternations as $H(e^{j\omega})$. This filter is optimal in the minimax sense.

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \omega_p/2 \\ 0, & \omega_s/2 \le \omega \le \pi - \omega_s/2 \\ 1, & \pi - \omega_p/2 \le \omega \le \pi \end{cases}$$

$$W(e^{j\omega}) = \begin{cases} \delta_2/\delta_1, & 0 \le \omega \le \omega_p/2 \\ 1, & \omega_s/2 \le \omega \le \pi - \omega_s/2 \\ \delta_2/\delta_1, & \pi - \omega_p/2 \le \omega \le \pi \end{cases}$$

7.56. We have an odd length causal linear phase filter with values from $n = 0, \ldots, 24$. It must therefore be either a Type I or Type III filter.

(a) *True.* We know either

$$\begin{array}{ccc} \text{Type I} & & \text{Type III} \\ h[m] = h[24-m] & \text{or} & h[m] = -h[24-m] \end{array}$$

for $-\infty < m < \infty$ since the filter has linear phase. Substituting $m = n + 12$ we get

$$h[n+12] = h[12-n] \quad \text{or} \quad h[n+12] = -h[12-n]$$

(b) *False.* Since the filter is linear phase it either has zeros both inside and outside the unit circle or it has zeros only on the unit circle.
If the filter has zeros both inside and outside the unit circle, its inverse has poles both inside and outside the unit circle. The only region of convergence that would correspond to a stable inverse would be the ring that includes the unit circle. The inverse would therefore be two-sided and not causal.
If the filter only has zeros on the unit circle, its inverse has poles on the unit circle and is therefore unstable.

(c) *Insufficient Information.* If it is a Type III filter it would have a zero at $z = -1$ but if it is a Type I filter this is not necessarily true.

(d) *True.* To minimize the maximum weighted approximation error is the goal of the Parks-McClellan algorithm.

(e) *True.* The filter is FIR so there are no feedback paths in the signal flow graph.

(f) *True.* The filter has linear phase and

$$\arg\left[H(e^{j\omega})\right] = \beta - 12\omega$$

where $\beta = 0, \pi$ for a Type I filter or $\beta = \pi/2, 3\pi/2$ for a Type III filter. The group delay is

$$\begin{aligned} \text{grd}\left[H(e^{j\omega})\right] &= -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\} \\ &= 12 \\ &> 0 \end{aligned}$$

7.57. (a) The desired tolerance scheme is

$$H_d(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| \le \omega_1 \\ 1, & \omega_2 \le |\omega| \le \omega_3 \\ 0, & \omega_4 \le |\omega| \le \pi \end{cases}$$

(b)

$$W(\omega) = \begin{cases} 1 & \left(\text{or } \frac{\delta_2}{\delta_1}\right) & \left(\text{or } \frac{\delta_3}{\delta_1}\right) & 0 \le |\omega| \le \omega_1 \\ \frac{\delta_1}{\delta_2} & (\text{or } 1) & \left(\text{or } \frac{\delta_3}{\delta_2}\right) & \omega_2 \le |\omega| \le \omega_3 \\ \frac{\delta_1}{\delta_3} & \left(\text{or } \frac{\delta_2}{\delta_3}\right) & (\text{or } 1) & \omega_4 \le |\omega| \le \pi \end{cases}$$

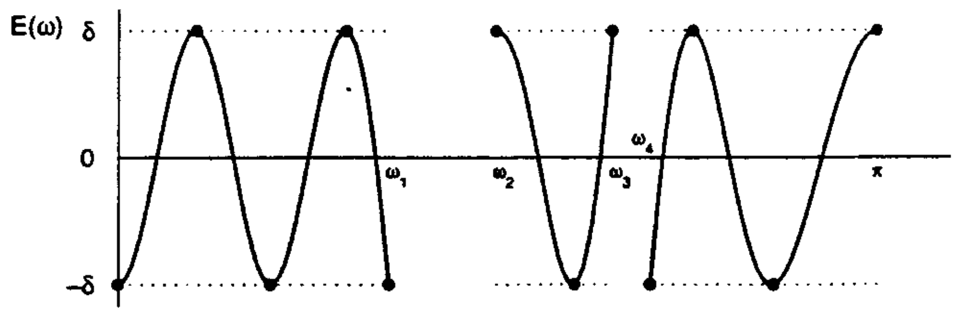
(c) From the Alternation Theorem, the minimum number of alternations is $L + 2$.

(d) The trigonometric polynomial (of degree L) can have at most $L - 1$ points of local minima or maxima in the open interval between 0 and π. If these are all alternation points and, in addition, all the band edges are alternation points, we find the maximum number of alternations is

$$L - 1 + 6 = L + 5$$

(e) If $M = 14$, then $L = M/2 = 7$. The maximum number of alternations is therefore $7 + 5 = 12$.

Typical $E(\omega)$ looks like :



(f) As will be shown in part (g), the 3 band case can have maxima and minima in the transition regions. It follows that we do not have to have an extremal frequency at ω_4. Therefore, if we started with an optimal maximal ripple filter and just slid ω_4 over we may move a local minimum or maximum into the transition region, but there will still be enough alternations left to satisfy the alternation theorem. Thus, the maximum approximation error does not have to decrease.

(g) (i) If a point in the transition region has a local minimum or maximum then there is the possibility that the surrounding points of maximum error do not alternate. Thus, we might lower the number of alternations by two. However, if we started with $L + 5$ alternations this reduction does not drop the number of alternations below the lower limit of $L + 2$ set by the Alternation Theorem. Therefore, local maxima and minima of $A_e(e^{j\omega})$ can occur in the transition regions. Note that this is not true in the 2 band case.

(ii) If a point in the approximation bands is a local minimum or maximum, the surrounding points of maximum error do not alternate. Thus, a local minimum or maximum in the approximation bands implies that the total number of alternations is reduced by two. However, if we started with $L + 5$ alternations this reduction does not drop the number of alternations below the lower limit of $L + 2$ set by the Alternation Theorem. Therefore, we can have a local maximum or minimum in the approximation bands. Note that in the 2-band case we drop from $L + 3$ to $L + 1$ which violates the Alternation Theorem.

7.58. (a) In order for condition 3 to hold, $G(z^{-1})$ must be an allpass system, since

$$\begin{aligned} Z^{-1} &= G(z^{-1}) \\ e^{-j\theta} &= G(e^{-j\omega}) \\ &= \left|G(e^{-j\omega})\right| e^{j\angle G(e^{-j\omega})} \end{aligned}$$

Clearly, $\left|G(e^{-j\omega})\right|$ must equal unity to map the unit circle of the Z-plane onto the unit circle of the z-plane.

(b) Consider one allpass term in the product, and note that α_k is real.

$$Z^{-1} = \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

The inside of the unit circle of the Z-plane is

$$0 \le |Z| < 1$$

Or equivalently,

$$1 < \left|Z^{-1}\right| < \infty$$

Substituting the allpass term for Z^{-1} gives

$$\begin{aligned} 1 &< \left|\frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}\right| \\ (1 - \alpha_k z^{-1})(1 - \alpha_k z^{*-1}) &< (z^{-1} - \alpha_k)(z^{*-1} - \alpha_k) \\ 1 - \alpha_k z^{-1} - \alpha_k z^{*-1} + \alpha_k^2 z^{-1} z^{*-1} &< z^{-1} z^{*-1} - \alpha_k z^{*-1} - \alpha_k z^{-1} + \alpha_k^2 \\ (1 - \alpha_k^2) &< z^{-1} z^{*-1}(1 - \alpha_k^2) \end{aligned}$$

If $(1 - \alpha_k^2) < 0$, then

$$\begin{aligned} 1 &> z^{-1} z^{*-1} \\ 1 &> \frac{1}{|z|^2} \\ |z| &> 1 \end{aligned}$$

The inside of the unit circle of the Z-plane maps to the *outside* of the unit circle of the z-plane. This is not the desired result. However, if $(1 - \alpha_k^2) > 0$, then

$$\begin{aligned} 1 &< z^{-1} z^{*-1} \\ 1 &< \frac{1}{|z|^2} \\ |z| &< 1 \end{aligned}$$

The inside of the unit circle of the Z-plane maps to the *inside* of the unit circle of the z-plane. This is the desired result. Thus, for condition 2 to be satisfied,

$$\begin{aligned} 1 - \alpha_k^2 &> 0 \\ |\alpha_k|^2 &< 1 \\ |\alpha_k| &< 1 \end{aligned}$$

This condition holds for the general case as well since the general case is just a product of the simpler allpass terms.

(c) First, it is shown that $G(z^{-1})$ produces the desired mapping for some value of α. Starting with $G(z^{-1})$,

$$
\begin{aligned}
Z^{-1} &= \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \\
e^{-j\theta} &= \frac{e^{-j\omega}-\alpha}{1-\alpha e^{-j\omega}} \\
e^{-j\theta}-\alpha e^{-j\theta}e^{-j\omega} &= e^{-j\omega}-\alpha \\
e^{-j\omega}(1+\alpha e^{-j\theta}) &= e^{-j\theta}+\alpha \\
e^{-j\omega} &= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}} \\
&= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}}\cdot\frac{1+\alpha e^{j\theta}}{1+\alpha e^{j\theta}} \\
&= \frac{e^{-j\theta}+2\alpha+\alpha^2 e^{j\theta}}{1+2\alpha\cos\theta+\alpha^2}
\end{aligned}
$$

Using Euler's formula,

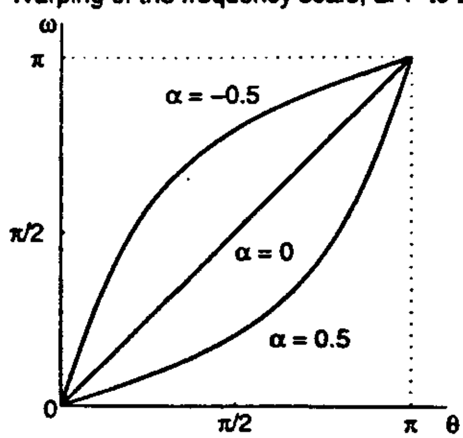
$$
\begin{aligned}
e^{-j\omega} &= \frac{\cos\theta - j\sin\theta + 2\alpha + \alpha^2\cos\theta + j\alpha^2\sin\theta}{1+2\alpha\cos\theta+\alpha^2} \\
&= \frac{2\alpha + (1+\alpha^2)\cos\theta + j[(\alpha^2-1)\sin\theta]}{1+2\alpha\cos\theta+\alpha^2}
\end{aligned}
$$

Noting that $-\omega = \tan^{-1}\left[\frac{Im\{\cdot\}}{Re\{\cdot\}}\right]$,

$$
\begin{aligned}
-\omega &= \tan^{-1}\left[\frac{(\alpha^2-1)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right] \\
\omega &= \tan^{-1}\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right]
\end{aligned}
$$

This relationship is plotted in the figure below for different values of α. Although a warping of the frequency scale is evident in the figure, (except when $\alpha = 0$, which corresponds to $Z^{-1} = z^{-1}$), if the original system has a piecewise-constant lowpass frequency response with cutoff frequency θ_p, then the transformed system will likewise have a similar lowpass response with cutoff frequency ω_p determined by the choice of α.

Warping of the frequency scale, LPF to LPF



Next, an equation for α is found in terms of θ_p and ω_p. Starting with $G(z^{-1})$,

$$\begin{aligned}
Z^{-1} &= \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \\
e^{-j\theta_p} &= \frac{e^{-j\omega_p}-\alpha}{1-\alpha e^{-j\omega_p}} \\
e^{-j\theta_p} - \alpha e^{-j\theta_p}e^{-j\omega_p} &= e^{-j\omega_p}-\alpha \\
e^{-j\theta_p} - e^{-j\omega_p} &= \alpha(e^{-j(\theta_p+\omega_p)}-1) \\
\alpha &= \frac{e^{-j\theta_p}-e^{-j\omega_p}}{e^{-j(\theta_p+\omega_p)}-1} \\
&= \frac{e^{-j(\theta_p+\omega_p)/2}(e^{-j(\theta_p-\omega_p)/2}-e^{j(\theta_p-\omega_p)/2})}{e^{-j(\theta_p+\omega_p)/2}(e^{-j(\theta_p+\omega_p)/2}-e^{j(\theta_p+\omega_p)/2})} \\
&= \frac{-2j\sin[(\theta_p-\omega_p)/2]}{-2j\sin[(\theta_p+\omega_p)/2]} \\
&= \frac{\sin[(\theta_p-\omega_p)/2]}{\sin[(\theta_p+\omega_p)/2]}
\end{aligned}$$

(d) Using the equation for ω found in part c, with $\theta_p = \pi/2$,

$$\omega_p = \tan^{-1}\left[\frac{1-\alpha^2}{2\alpha}\right]$$

(i)

$$\begin{aligned}
\omega_p &= \tan^{-1}\left[\frac{1-(-0.2679)^2}{2(-0.2679)}\right] \\
&= \tan^{-1}\left(\frac{0.9282}{-0.5358}\right) \\
&= 2\pi/3
\end{aligned}$$

(ii)

$$\begin{aligned}
\omega_p &= \tan^{-1}\left[\frac{1-(0)^2}{2(0)}\right] \\
&= \tan^{-1}(\infty) \\
&= \pi/2
\end{aligned}$$

(iii)

$$\begin{aligned}
\omega_p &= \tan^{-1}\left[\frac{1-(0.4142)^2}{2(0.4142)}\right] \\
&= \tan^{-1}(1) \\
&= \pi/4
\end{aligned}$$

(e) The first-order allpass system

$$G(z^{-1}) = -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$$

satisfies the criteria that the unit circle in the Z-plane maps to the unit circle in the z-plane, and that $\theta = 0$ maps to $\omega = \pi$. Next, α is found in terms of θ_p and ω_p.

$$Z^{-1} = -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$$

$$\begin{aligned}
e^{-j\theta_p} &= -\frac{e^{-j\omega_p}+\alpha}{1+\alpha e^{-j\omega_p}} \\
-e^{-j\theta_p}-\alpha e^{-j(\omega_p+\theta)} &= e^{-j\omega_p}+\alpha \\
\alpha(1+e^{-j(\omega_p+\theta_p)}) &= -e^{-j\theta_p}-e^{-j\omega_p} \\
\alpha &= -\frac{e^{-j\theta_p}+e^{-j\omega_p}}{1+e^{-j(\omega_p+\theta_p)}} \\
&= -\frac{e^{-j(\omega_p+\theta_p)/2}(e^{-j(-\omega_p+\theta_p)/2}+e^{-j(\omega_p-\theta_p)/2})}{e^{-j(\omega_p+\theta_p)/2}(e^{j(\omega_p+\theta_p)/2}+e^{-j(\omega_p+\theta_p)/2})} \\
&= -\frac{\cos[(\omega_p-\theta_p)/2]}{\cos[(\omega_p+\theta_p)/2]}
\end{aligned}$$

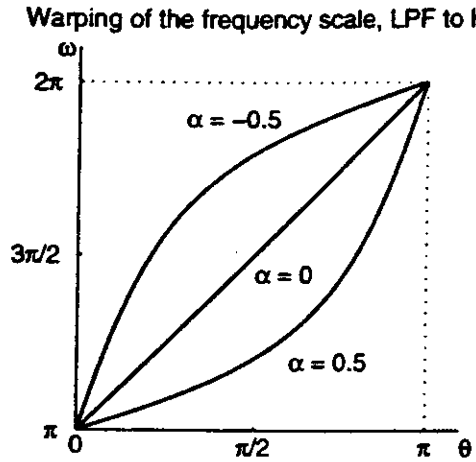
(f) First, an equation for ω is found in terms of θ and α.

$$\begin{aligned}
Z^{-1} &= -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}} \\
e^{-j\theta} &= -\frac{e^{-j\omega}+\alpha}{1+\alpha e^{-j\omega}} \\
-e^{-j\theta}-\alpha e^{-j(\omega+\theta)} &= e^{-j\omega}+\alpha \\
e^{-j\omega}(1+\alpha e^{-j\theta}) &= -e^{-j\theta}-\alpha \\
-e^{-j\omega} &= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}} \\
e^{-j(\omega-\pi)} &= \frac{e^{-j\theta}+\alpha}{1+\alpha e^{-j\theta}}\cdot\frac{1+\alpha e^{j\theta}}{1+\alpha e^{j\theta}} \\
&= \frac{e^{-j\theta}+2\alpha+\alpha^2 e^{j\theta}}{1+2\alpha\cos\theta+\alpha^2} \\
&= \frac{\cos\theta-j\sin\theta+2\alpha+\alpha^2\cos\theta+j\alpha^2\sin\theta}{1+2\alpha\cos\theta+\alpha^2} \\
&= \frac{\cos\theta+2\alpha+\alpha^2\cos\theta+j(-\sin\theta+\alpha^2\sin\theta)}{1+2\alpha\cos\theta+\alpha^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\omega+\pi &= \tan^{-1}\left[\frac{(\alpha^2-1)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right] \\
\omega &= \tan^{-1}\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right]+\pi
\end{aligned}$$

Note that this lowpass to highpass expression is the similar to the lowpass to lowpass expression for ω found in part (c). The only difference is the additive π term, which shifts the lowpass filter into a highpass filter. The frequency warping is plotted below.



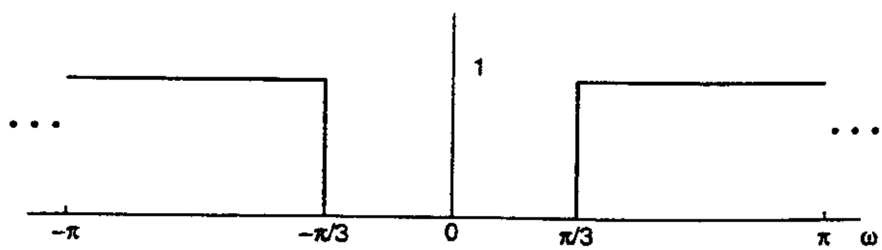
For $\theta_p = \pi/2$, this becomes

$$\omega = \tan^{-1}\left[\frac{(1-\alpha^2)}{2\alpha}\right] + \pi$$

(i)

$$\begin{aligned}\omega_p &= \tan^{-1}\left[\frac{1-(-0.2679)^2}{2(-0.2679)}\right] + \pi \\ &= \tan^{-1}\left(\frac{0.9282}{-0.5358}\right) + \pi \\ &= 2\pi/3 + \pi \\ &= 5\pi/3\end{aligned}$$

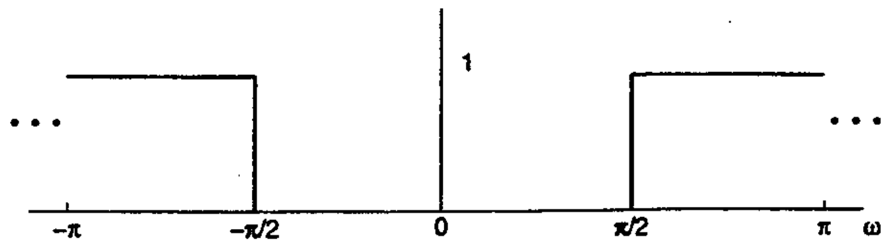
The right edge of the low pass filter gets warped to $5\pi/3$, which is equivalent to $-\pi/3$. The frequency response of this filter appears below.



(ii)

$$\begin{aligned}\omega_p &= \tan^{-1}\left[\frac{1-(0)^2}{2(0)}\right] + \pi \\ &= \tan^{-1}(\infty) + \pi \\ &= \pi/2 + \pi \\ &= 3\pi/2\end{aligned}$$

The right edge of the low pass filter gets warped to $3\pi/2$, which is equivalent to $-\pi/2$. The frequency response of this filter appears below.



(iii)

$$\begin{aligned}\omega_p &= \tan^{-1}\left[\frac{1-(0.4142)^2}{2(0.4142)}\right]+\pi\\ &= \tan^{-1}\left(\frac{0.8284}{0.8284}\right)+\pi\\ &= \pi/4+\pi\\ &= 5\pi/4\end{aligned}$$

The right edge of the low pass filter gets warped to $5\pi/4$, which is equivalent to $-3\pi/4$. The frequency response of this filter appears below.