EEM 323

ELECTROMAGNETIC WAVE THEORY II

PLANE ELECTROMAGNETIC WAVE PROPAGATION – 1

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Önemli not: Ders notlarındaki şekillerin hazırlanmasında internet ortamından faydalanılmıştır. Özellikle belirtilmeyen tüm şekil, tablo, eşitlik ve denklemler vb. "D. K, Fundamentals of Engineering Electromagnetics, Addison-Wesley Inc." ile "D. K, Field and Wave Electromagnetics, Mc-Graw Hill Inc." kitabından taranarak elde edilmiştir. Alıntıların kaynağına kolay ulaşılabilmesi maksadıyla numarası ve altyazıları da gösterilmektedir.

DERS KİTABI

[1] David Keun Cheng, *Fundamentals of Engineering Electromagnetics*, Addison-Wesley Publishing, Inc., 1993.
veya David Keun Cheng, Çeviri: Adnan Köksal, Birsen Saka, *Mühendislik Elektromanyetiğinin Temelleri – Fundamentals of Engineering Electromagnetics*, Palme Yayınları.

KAYNAK / YARDIMCI KİTAPLAR:

[2] David Keun Cheng, *Field and Wave Electromagnetics*, Addison-Wesley Publishing, Inc. *veya* David Keun Cheng, Çeviri: Mithat İdemen, *Elektromanyetik Alan Teorisinin Temelleri – Field and Wave Electromagnetics*, Literatür Yayıncılık.

[3] Stanley V. Marshall, Richard E. DuBroff, Gabriel G. Skitek, *Electromagnetic Concepts and Applications*, Dördüncü Basım, Prentice Hall International, Inc., 1996.

[4] Joseph A. Edminister, Elektromanyetik, 2. Baskıdan çeviri, Çevirenler: M. Timur Aydemir, E. Afacan, K. C. Nakipoğlu, Schaum's Outlines, McGraw Hill Inc., Nobel Yayın Dağıtım, Ankara, 2000.

TIME-HARMONIC FIELDS

Plane wave propagation in the z direction:

Let us start by analyzing the scalar wave equation in simple media (linear, isotropic, source-free and lossless).

In cartesian coordinates

$$\nabla^2 \mathbf{E} = \left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2\right)\mathbf{E}$$

Helmholtz wave equation in lossless and simple media

$$\nabla^2 \mathbf{E} + \mathrm{k}^2\, \mathbf{E} = \left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 + \mathrm{k}^2\right)\mathbf{E} = 0$$

Assume

$$\mathbf{E} = \boldsymbol{a}_x E_x \quad \text{and} \quad \left(\partial^2/\partial x^2\right)\mathbf{E} = 0, \quad \left(\partial^2/\partial y^2\right)\mathbf{E} = 0.$$

EM wave equation simplifies to

$$\left(\mathrm{d}^2/\mathrm{d}z^2 + \mathrm{k}^2\right)E_x = 0$$

The solution is an x polarized plane wave propagating in the z direction.

$$\begin{aligned} E_x(z) &= E_x^+(z) + E_x^-(z) \\ &= E_0^+(z)e^{-jkz} + E_0^-(z)e^{+jkz} \end{aligned}$$

Plane wave solution (taking the +z travelling wave only)

$$\boldsymbol{E}(z,t) = \boldsymbol{a}_x\, Re\left\{E_0^+(z)e^{-jkz}e^{j\omega t}\right\}$$
$$= \boldsymbol{a}_x\, E_0^+\, cos(\omega t - kz)$$

EM wave travelling in positive z direction with linear x polarization.

[Ex. 8–1, D. K. Cheng, Fundamentals of Engineering Electromagnetics, Addison-Wesley Inc. 1993]

TRANSVERSE ELECTROMAGNETIC WAVES

Plane wave propagation in arbitrary direction:

Our analysis of the wave equation for the z – travelling plane wave is also valid for a plane wave travelling in an arbitrary direction defined by

$$\boxed{\boldsymbol{R} = \alpha\, \boldsymbol{a}_x + \beta\, \boldsymbol{a}_y + \gamma\, \boldsymbol{a}_z}$$

where

$$|\boldsymbol{R}|^2 = \alpha\, \boldsymbol{a}_x + \beta\, \boldsymbol{a}_y + \gamma\, \boldsymbol{a}_z \quad \text{where} \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

and where

$\boldsymbol{R}$ is **the unit direction vector**,

α, β and γ are called the **three direction cosines.**

[D. K. Cheng, Fundamentals of Engineering Electromagnetics]

Unit direction vector and wave normal to a phase front of **a uniform plane wave**.

TRANSVERSE ELECTROMAGNETIC WAVES
Uniform phasefronts

The keyword now we would like to examine is **uniform**

What if the plane wave is not uniform?

$\rightarrow$ Wavefront will not be planar.

How can we define a planar surface?

[R. D. Guenter, Modern Optics]

It is defined by the equation (Note that D. K. Cheng prefers the notation R and R. D. Guenter prefers **r** instead)

$\boldsymbol{r} \bullet \boldsymbol{n} = s$ ($\boldsymbol{n}$ will be placed by $\boldsymbol{a}_s$ in the following slides).

Note that s is the distance between two points on two different wavefronts.

- Wavefront is a surface where the phase of wave at those points are equal to some value $s\,(2\pi/\lambda) = ks$ (given in radians).
    - Note that for s = multiples of λ then the phase is the same multiples of 2π
- $S(\boldsymbol{r})$ function defines the wavefront of wave
- The outward propagating wave can now be defined by

$$\boxed{\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{a}_p\, Re\{E_0^+(\boldsymbol{r})e^{-jk(\boldsymbol{r}\bullet\boldsymbol{n})}e^{j\omega t}\}\\ &= \boldsymbol{a}_p\, E_0^+\, cos(\omega t - S(\boldsymbol{r}))\end{aligned}}$$

for some polarization vector $\boldsymbol{a}_p$ perpendicular to $\boldsymbol{n}$.

[R. D. Guenter, Modern Optics]

$$\boxed{\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{a}_p\, E_0^+\, cos(\omega t - S(\boldsymbol{r}))}$$

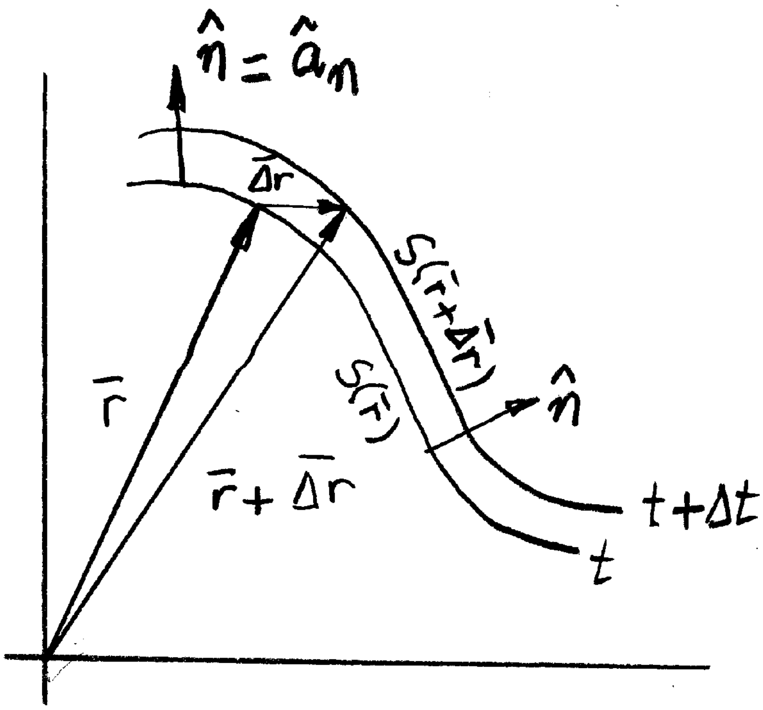
Now we are ready to calculate **the phase velocity** of the wavefront.

When wave propagates

- Time parameter t increases
- Recalling the equation $\boldsymbol{E}(z,t) = \boldsymbol{a}_x\, E_0^+\, cos(\omega t - kz)$
  wave propagates along the z – axis
- Then, distance from the reference z increases
- Wave propagates along the z – axis in time.

Question: What happens to $\left(\omega t - S(\boldsymbol{r})\right)$?

Now let us assume that we are at a point on the wave stationary on some wavefront of constant phase, propagating with the wave !..

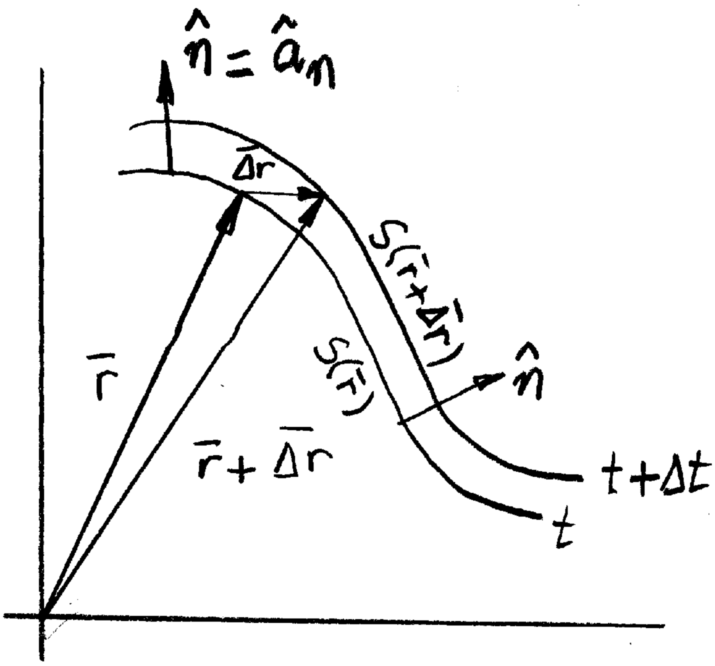


- Scalar time parameter t increases to $t + \Delta t$
- Distance vector parameter $\boldsymbol{r}$ increases to $\boldsymbol{r} + \Delta\boldsymbol{r}$

Note that the unit vector a_s is in the direction of maximum increase of phase.

Phase velocity:

Note that we are stationary on the wavefront travelling along the wave with the same velocity of increasing phase.



Then, the phase in

, the cosine term;

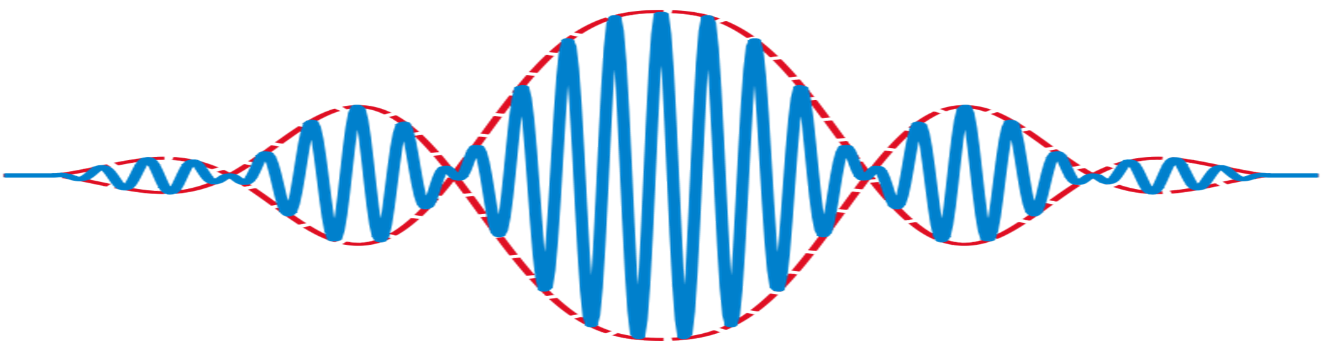
should be constant.

A note on the definition of Phase 'velocity'

Phase velocity is clearly a scalar

In physics, speed is scalar and velocity is vector.

But, **phase velocity is a scalar** !..



Constant phase (travelling along with the wave),

Phase at $(\boldsymbol{r}, t)$ should be equal to the phase at

$(\boldsymbol{r} + \Delta\boldsymbol{r}, t + \Delta \mathrm{t})$

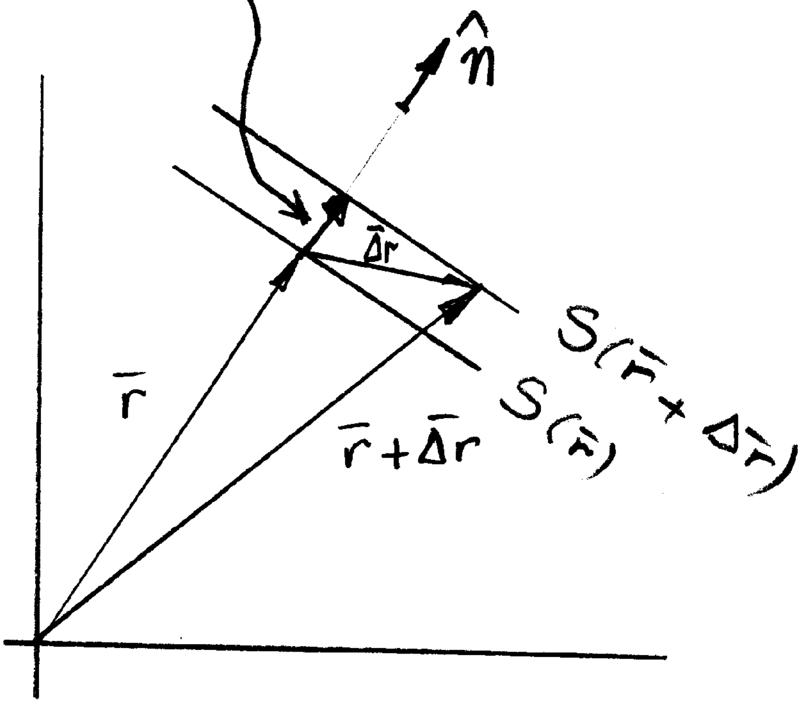
$$\boxed{\omega t - S(\boldsymbol{r}) = \omega t + \omega\Delta t - S(\boldsymbol{r} + \Delta\boldsymbol{r})}$$

Phase at time t = Phase at time $t+\Delta t$

$$\boxed{\omega\Delta t - [S(\boldsymbol{r} + \Delta\boldsymbol{r}) - S(\boldsymbol{r})\,] = 0}$$

$$\Delta\mathbf{r} \bullet \boldsymbol{a}_s = \Delta s$$

$$\underbrace{\bar{\Delta} r \cdot \hat{n}} = \Delta s$$



RAY APPROXIMATION FOR PLANE EM WAVES

To be able to visualize vectors clearly, let us be interested in the instantaneous values by assuming limiting case $\Delta t \rightarrow 0$ such that it is also true that $\Delta \mathbf{r} \rightarrow 0$.

Then, wavefront at point $\mathbf{r}$, will be almost planar as shown in Figure.

Define, $\Delta S(\mathbf{r}) = S(\boldsymbol{r} + \Delta \boldsymbol{r}) - S(\boldsymbol{r})$ as the phase difference between wavefronts in radians. And, $\Delta \mathrm{s}$ is the physical distance between wavefronts in meters.

Note that $\Delta S(\mathbf{r})$ = the directional derivative of the scalar field function $S(\boldsymbol{r})$ in the direction defined by the unit vector $\boldsymbol{n}$.

Recalling the definition (see D. K. Cheng's textbook)

$$\boxed{\Delta S(\mathbf{r}) = grad(S(\mathbf{r})) \bullet (\boldsymbol{a_s})\mathrm{ds} = (\nabla \mathrm{S}(\mathbf{r}) \bullet \boldsymbol{a_s})\ \mathrm{ds}}$$

For the case $\Delta t \rightarrow 0$ and $\Delta \mathbf{r} \rightarrow 0$, we can rename

$\Delta t \rightarrow dt$ and $\Delta \mathbf{r} \rightarrow d\mathbf{r}$

Then, we have

$$\omega\,\mathrm{dt} - [S(\boldsymbol{r} + \boldsymbol{dr}) - S(\boldsymbol{r})\,]\mathrm{ds} = 0$$

$$\omega\,dt - (\nabla \mathrm{S}(\mathbf{r}) \bullet \boldsymbol{a_s})\mathrm{ds} = 0$$

and since n points in the same direction with $\nabla \mathrm{S}(\mathbf{r})$ (simply $\nabla \mathrm{S}$)

$$\boldsymbol{a}_s = \frac{\nabla \mathrm{S}}{|\nabla \mathrm{S}|} \quad \rightarrow \quad \nabla \mathrm{S} \bullet \boldsymbol{a}_s = |\nabla \mathrm{S}|$$

Recalling, phase velocity

$$v_p = ds/dt$$

Phase velocity: $$v_p = \frac{\omega}{|\nabla \mathrm{S}(\mathbf{r})|}$$

We now know that

Phase Velocity is maximum in the direction of $\nabla \mathrm{S}(\mathbf{r})$

or namely, $\boldsymbol{a}_s$.

But what is its amplitude; $|\nabla \mathrm{S}(\mathbf{r})|$?

We can substitute our wave solution (locally planar wave structure)

$$\boldsymbol{E} = \boldsymbol{a}_E\, A\, e^{(jks)}$$

into Helmholtz equation

Related notes from the Book:

[R. D. Guenter, Modern Optics, pp.131–132]

[R. D. Guenter, Modern Optics, pp.131–132]

Helmholtz equation relates $\nabla S(\mathbf{r})$ with the index of refraction n

$$|\nabla S(\mathbf{r})|^2 = (k/k_0)^2 = n^2$$

$$|\nabla S(\mathbf{r})| = n$$

Its direction is along the unit vector $\boldsymbol{a_s}$, then

Eikonal Equation: $\nabla S(\mathbf{r}) = n\, \boldsymbol{a_s}$

It states that,

- **equal phase surfaces form wavefronts,**
- **wavefronts are perpendicular to the direction of propagation**
- **accurate for large wavenumbers**

Eikonal equation:

$\mathbf{r}$ r is the position vector of the wavefront

$\boldsymbol{a_s}$ is the unit vector in the direction of propagation

If n varies in space,

then the dielectric constant ε is a function of position

then the **rays are curved**.

When is k large? Recall $k = 2\pi/\lambda$

If k is large

Then, wavelength λ is very small

Then, frequency is f is very large

For $\lambda \to \infty$ (limit of Helmholtz equation)

Then electromagnetic wave can accurately be assumed as **rays**.

Wave propagation is local and acts as small **plane waves**, as first proposed by Christiaan Huygens.

Eikonal equation is mostly accure for lasers.

Fermat's Principle:

Remember the Eikonal Equation:

$$\nabla \mathrm{S}(\mathbf{r}) = n\, \boldsymbol{a_s}$$

The time T that a point of the electromagnetic wave needs to cover a path between the points a and b is given by:

$$T = \int_a^b dt = \frac{1}{c}\int_a^b \frac{c}{v}\frac{ds}{dt}dt = \frac{1}{c}\int_a^b n ds$$

c is the speed of light in vacuum, ds is an infinitesimal displacement along the ray, $v=ds/dt$ the speed of light in a medium and $n = c/v$ the refractive index of that medium.

The optical path length of a ray from a point A to a point B is defined by:

The optical path length: $S = \int_a^b n ds$

and it is related to

The travel time: $T = S/c$

The optical path length is a purely geometrical quantity since time is not considered in its calculation.

French mathematician Pierre de Fermat (1601 – 1665):

"An extremum in the light travel time between two points a and b is equivalent to an extremum of the optical path length between those two points."

The historical form proposed by French mathematician Pierre de Fermat is incomplete.

A complete modern statement of the variational Fermat principle is that;

"The optical length of the path followed by light between two fixed points, a and b, is an extremum. The optical length is defined as the physical length multiplied by the refractive index of the material."

[Int: wikipedia.com]

[R. Marques, F. Martin, and M. Sorolla. Metamaterials with Negative Parameters. Wiley, 2008]