Başkent University
MAT 151 – Calculus I
Fall 2005 – Midterm Exam

November 14, 2005

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **8 Q**uestions, ANSWER **6 Q**uestions only;
- In the table below, indicate clearly the **Q**uestion that you select and its **Part**, if any exists;
- Do not ask any question to anybody other than your instructor.

Q							**Total**
Part							
Grade							

GOOD LUCK!

Q1 (15 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) A man is drinking soda through a straw from a conical cup 8 inches deep and 4 inches in diameter at the top. At the instant the soda is 5 inches deep, he is drinking at the rate of 3 cubic inches per second. How fast is the level of the soda dropping at that time?

(B) A man standing 3 feet from the base of a lamp post casts a shadow of 4 feet long. If the man is 6 feet tall and walks away from the lamp post at a speed of 400 feet per minute, at what rate will his shadow lengthen?

<u>Q2</u> Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Sketch $y = f(x) = \dfrac{x^2}{x-1}$ in detail (make a summary table).

(B) (15 pts) Sketch $y = f(x) = x^2\left(1 - \dfrac{x}{6}\right)$ in detail (make a summary table).

Q3 (15 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) Evaluate $\lim\limits_{x \to \pi/2} (\tan x - \sec x)$

(i) by using L'Hopital's Rule
(ii) without using L'Hopital's Rule.

(B) Find the following limits. **Do NOT USE L'Hôpital's Rule.**

(i) $\lim\limits_{x \to 2^-} \dfrac{(x-2)\sqrt{2x}}{|x-2|}$, **(ii)** $\lim\limits_{x \to \infty} \dfrac{x^2 + \operatorname{arccot} x}{4x^2 + \operatorname{arccsc} x}$, **(iii)** $\lim\limits_{x \to -\infty} \dfrac{x^{5/3} + x^{4/3} - 7}{x^{8/5} + 3x}$.

Q4 Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Evaluate any TWO of the following limits

(i) $\lim_{x\to 0}(\cos x)^{\cot x}$, **(ii)** $\lim_{x\to(\pi/2)^-}(1+\cos x)^{\tan x}$, **(iii)** $\lim_{x\to 0^+}(e^x-1)^x$.

(B) (15 pts) Find $\lim_{x\to\infty} \ln x \cdot \operatorname{arccsc} x$.

Q5 (15 pts) Given $x^2 - xy + y^3 = 1$.

(i) Show that the tangent lines to the above curve at the points $(1,\ 1)$ and $(1,\ 0)$ are perpendicular.

(ii) Write equations for the tangent and normal lines at $(1,\ 1)$.

<u>Q6</u> (20 pts) Sketch roughly the graphs of the following functions

(i) $y = \mathrm{e}^{x^2}$, **(ii)** $y = \left(\frac{1}{2}\right)^x$, **(iii)** $y = \operatorname{arcsec} x$, **(iv)** $y = \arctan(2x)$, **(v)** $y = \log_{(1/2)} x$.

Q7 (20 pts) Find the derivatives of the following functions

(i) $y = (\sin x)^{\ln x}$, **(ii)** $y = \operatorname{arcsec}(x^2)$, **(iii)** $y = \arctan(\tan x)$ for $-\pi/2 < x < \pi/2$,
(iv) $y = \arccos(\log_3 x)$, **(v)** $y = 3^{\sin x}$.

Q8 (15 pts) Show that the function $f(x) = x^3 - 3x^2 + x + 1$ assumes the values $1/2$ and $3/2$ in the interval $[-1, 1]$.

BAŞKENT UNIVERSITY
FACULTY OF ENGINEERING
MAT 151 – CALCULUS I
2004 –2005
SPRING SEMESTER
MID-TERM EXAM

April 12, 2005

Student ID	Name / Surname	Department

DURATION : 100 minutes.

NOTES:

- **CHECK that there are 6 questions on 5 sheets of paper;**
- **Each question is out of 20 points;**
- **SELECT and SOLVE ONLY 5 (Five) questions out of the 6 (Six) given questions;**
- **In case you solve all the 6 questions, we will discard The Question for which you will score the best;**
- **In the table below, INDICATE clearly which questions you SELECT;**
- **Write your derivations clearly. NO CREDIT will be given for answers not supported by work and not explained clearly.**
- **DO NOT ASK any question to anybody other than your instructors.**
- **DO NOT USE an electronic calculator.**
- **Mobile phones must be CLOSED during the examination.**
- **During the FIRST 30 minutes of examination, nobody can leave the room.**

QUESTION						TOTAL
GRADE	/20	/20	/20	/20	/20	/100

GOOD LUCK...

Q-1) (20 points) Find the following limits;

$i)$ **(10 points)** $\lim\limits_{x\to\infty} \dfrac{2x}{\cot\left(\dfrac{4}{x}\right)}$ **; Do NOT USE l'Hopital's Rule**

$ii)$ **(10 points)** $\lim\limits_{x\to 0} \left(e^{x}+x\right)^{1/2x}$ **; Solve by USING l'Hopital's Rule**

Q-2) (20 points) Find the following derivatives;

$i)$ **(5 points)** $f(x) = \sec\sqrt{x}\,.\tan\left(\frac{1}{x}\right)$

$ii)$ **(5 points)** $f(x) = \sec^2[\tan x]$

$iii)$ **(5 points)** $f(x) = \ln\left(x^2\right) + \tan^2\left(e^{2/x}\right)$

$iv)$ **(5 points)** $f(x) = \sin^2\left[(\cos x)^{\ln(1-x)}\right]$

Q-3) (20 points) Given the point $P_o(x_o, \pi/2)$ on the curve (C) of equation

$$2.x.y + \pi.\sin y = 2.\pi$$

i) **(5 points)** Find the value of x_o

ii) **(15 points)** Find an equation for the normal line to the curve (C) at the point P_o evaluated in part (i).

Q-4) (20 points) Given the function f(x) defined by;

$$f(x)=\begin{cases} x^2+a & ; \; x<3 \\ b.x & ; \; x\geq 3 \end{cases}$$

What are the values of the constants "***a***" and "***b***", that make f(x) differentiable on its domain of definition?

Q-5) (20 points) Let $y = f(x) = x\sqrt{8 - x^2}$;

i) (2 points) Find the domain of definition of *f(x)*.

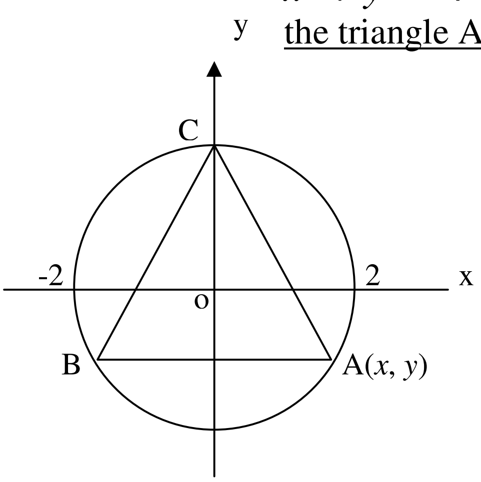
ii) (5 points) Find any critical point and point of inflection of *f(x)*.

iii) (5 points) Determine the intervals on which *f(x)* is increasing/decreasing and the intervals on which *f(x)* is concave up/concave down.

iv) (4 points) Determine any local and/or absolute extreme value of *f(x)*.

v) (4 points) Sketch the graph of *f(x)* by using the above results.

Q-6) (20 points) Find the largest area of the isosceles triangle (CA=CB), with one vertex (C) on the y-axis, that can be inscribed in the circle $x^2 + y^2 = 4$ as shown in figure below. **Hint:** Write the area of the triangle ABC as a function of y.



Başkent University
MAT 151 – Calculus I
Fall 2006 – Midterm Exam

November 22, 2006

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- In the table below, indicate clearly the **Part** of the **Q**uestion that you choose to solve, if any exists;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	Total
Part	X		X				
Grade							

GOOD LUCK!

<u>Q1</u> (15 pts) Sketch <u>roughly</u> the graphs of <u>any four</u> of the following functions.

(i) $f(x) = \arccos x$, **(ii)** $f(x) = \operatorname{arccsc} x$, **(iii)** $f(x) = \log_{(2/3)} x$,

(iv) $f(x) = 3^{x^2}$, **(v)** $f(x) = \arctan |x|$.

<u>Q2</u> Answer <u>only one</u> part (**Part (A)**, **Part (B)**, or **Part (C)**).

Sketch in <u>full detail</u> the graph of

(A) (20 pts) $f(x) = \dfrac{x^2}{x+1}$,

(B) (20 pts) $f(x) = \dfrac{x^2-4}{x^2-1}$,

(C) (15 pts) $f(x) = 2x^4 - 4x^2 + 1$.

<u>Q3</u> (20 pts) Use Logarithmic Differentiation to find the derivatives of <u>any two</u> of the following functions.

(i) $f(x) = (\arctan x)^{\ln x}$, **(ii)** $f(x) = (\sin x)^{\operatorname{arcsec} x}$,

(iii) $f(x) = (\ln x)^{\operatorname{arccsc} x}$, **(iv)** $f(x) = (3^x + \sin x)^{\arccos x}$.

<u>Q4</u> (15 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) Find <u>any two</u> of the following limits

(i) $\lim\limits_{x\to 0}\dfrac{x\arcsin x}{x-\sin x}$, **(ii)** $\lim\limits_{x\to(\pi/2)^-}(1+\cos x)^{\tan x}$, **(iii)** $\lim\limits_{x\to\infty}(\ln x)^{1/x}$.

(B) Find the following limits.

(i) $\lim\limits_{x\to 0}\dfrac{\arcsin(1-x)\cdot\arccos\,x^2}{\operatorname{arccot}\,x+\operatorname{arcsec}(1/|x|)}$, **(ii)** $\lim\limits_{x\to 0}\dfrac{x\sin(1/x)\cos(1/x)}{5+\sqrt{x+1}}$,

(iii) $\lim\limits_{x\to -3}\left(\dfrac{x+3}{x^3+27}\right)^{1/3}$ (Do **NOT USE** L'Hôpital's Rule).

<u>Q5</u> (20 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) **(i)** Water is poured into a conical paper cup at the rate of 2 cubic inches per second. If the cup is 6 inches tall and the top of the cup has a radius of 4 inches, how fast is the water level rising when the water is 4 inches deep?

(ii) An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the x-axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

(B) **(i)** A ladder 13 feet long is leaning against a wall. If the base of the ladder is pulled away from the wall at the rate of 0.5 feet per second, how fast will the top of the ladder drop when the base is 5 feet from the wall?

(ii) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm.

Q6 Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (10 pts) Consider the function $f(x) = x^4 - 2x^2 + 1$.
(i) Use the Second Derivative Test to find the local extreme values of $f(x)$.
(ii) Does it have any point of inflection? Explain!
(iii) What are the absolute extreme values of $f(x)$ in the interval [0, 2]? Explain!

(B) (15 pts)
(i) Find the point(s) on the curve $x^2 + xy + y^2 = 1$ at which the tangent line is parallel to the line $y - x = 5$.
(ii) Find equations for the tangent and normal lines at such a point.

Başkent University
MAT 151 – Calculus I
Spring 2007 – Midterm Exam

April 07, 2007

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	Total
Grade							

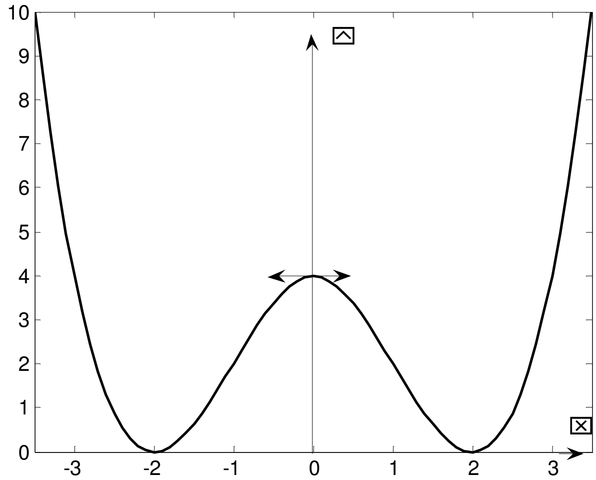
G O O D   L U C K !

Q4 (20 pts) Sketch roughly the graphs of the following functions.

(i) $f(x) = \operatorname{arccot} x$, **(ii)** $f(x) = |2 - |x||$, **(iii)** $f(x) = \ln(|2 - |x||)$,
(iv) $f(x) = x^2 - 4x + 7$, **(v)** $f(x) = \sin(2x)$.

<u>Q2</u> (20 pts) Sketch in <u>full detail</u> the graph of the following function. Make a summary table and determine any local or absolute extreme value and point of inflection.

$f(x) = |x|\, x^2 - 3x^2 + 4$.



Q3 (15 pts) Use Logarithmic Differentiation to find the derivatives of the following functions.

(i) $f(x) = (\operatorname{arccot} x)^{\ln x}$, **(ii)** $f(x) = (\sec x)^{\operatorname{arcsec} x}$, **(iii)** $f(x) = (\ln x)^{\ln(1/x)}$.

<u>Q1</u> (20 pts) Find the following limits

(i) $\lim\limits_{x\to 0}\left(e^x + x\right)^{2/x}$, **(ii)** $\lim\limits_{x\to(\pi/2)^-}\left[1+\cos(2x)\right]^{\cos x}$, **(iii)** $\lim\limits_{x\to\infty}(\ln x)^{1/x}$.

Do **NOT USE** L'Hôpital's Rule for **(iv)** and **(v)**.

(iv) $\lim\limits_{x\to 0}\dfrac{e^x - 1}{\tan(2x)}$, **(v)** $\lim\limits_{x\to-\infty}\left(\dfrac{\sqrt{x^2+4}+\sqrt{|x|}}{\sqrt[5]{x^5+3}+x}\right)$.

Q6 (15 pts) Find the minimum and maximum distances from the point $P(1, \sqrt{3})$ to the semicircle $y = \sqrt{16 - x^2}$. Solve the question by optimization.

<u>Q5</u> (15 pts) Let (C) be the curve of equation $2y^2 + yx = x^2 + 2$. **(i)** Find equations for the tangent and normal lines to the curve (C) at the point $P(1,\ 1)$. **(ii)** Is the graph of (C) concave up or concave down at $P(1,\ 1)$? Explain!

Başkent University
MAT 151 - Calculus I
Fall 2005 - Final Exam

January 02, 2006

Student ID	Name / Surname	Department

Duration: 120 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **9** Questions, ANSWER **7** Questions only;
- In the table below, write clearly the Question that you select and its Part, if any exists;
- Do not ask any question to anybody other than your instructor.

Q								
Part								**Total**
Grade								

GOOD LUCK!

Q1 (20 pts). Find ANY TWO of the following limits.

$$(\iota)\ \lim_{x\to 1^+}\left(x-1\right)^{x-1}\ , \quad (\iota\iota)\ \lim_{x\to\infty}\sqrt{x}\sin\left(\frac{1}{\sqrt{x}}\right)\ , \quad (\iota\iota\iota)\ \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)\ .$$

Q2 Answer either **Part (A)** or **Part (B)**, but NOT both.

The region $\mathcal{R}$ under the graph of $y = f(x)$ is revolved about the x-axis. Sketch the solid of revolution and find its volume.

(A) (15 pts) $f(x) = \sqrt{|\ln x|}\,, \quad 0 < x \leq 1\,.$

(B) (20 pts) $f(x) = \sqrt{\dfrac{\ln x}{x}}\,, \quad 1 \leq x < \infty\,.$

Q3 Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Is the improper integral convergent or divergent? Explain!

$$\int_1^\infty \frac{x}{(x^7+1)^{1/3}}\,\mathrm{d}x\,.$$

(B) (20 pts) Is the improper integral convergent or divergent? Explain!

$$\int_1^\infty \frac{\sqrt{x}+1}{\sqrt{x(x-1)+1}}\,\mathrm{d}x\,.$$

Q4 Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Find the hight and radius of the largest right circular cylinder that can be put into a sphere of radius $\sqrt{3}$.

(B) (15 pts) What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3.

Q5 (20 pts) Use the method indicated to find ANY TWO of the following integrals.

(ι) $\displaystyle\int \frac{\sec^2(1+\ln x)}{x}\,\mathrm{d}x$ (Substitution)

$(\iota\iota)$ $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+36}}$ (Trigonometric Substitution)

$(\iota\iota\iota)$ $\displaystyle\int \frac{2x^2+1}{x^3+1}\,\mathrm{d}x$ (Partial Fractions)

$(\iota\upsilon)$ $\displaystyle\int x\arctan x\,\mathrm{d}x$ (Integration by Parts)

Q6 (30 pts) The region $\mathcal{R}$ bounded by the curves $y = x^2$, $y = \sqrt{x}$ is revolved about the y-axis. SKETCH the solid of revolution and set up an integral for the volume by using

(ι) The Washer Method
($\iota\iota$) The Shell Method
($\iota\iota\iota$) Calculate the volume.

Q7 (20 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

Decide if the following improper integral converges or diverges:
(ι) by using the Direct Comparison Test
$(\iota\iota)$ by direct computation.

$$(\mathbf{A}) \quad \int_1^{\infty} \frac{2x}{1+x^4}\,\mathrm{d}x \ . \qquad\qquad (\mathbf{B}) \quad \int_1^{\infty} \frac{\ln\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x \ .$$

Q8 (20 pts) Sketch a rough graph and find the derivative.

(ι) $y = \operatorname{arcsec} x$, $(\iota\iota)$ $y = \operatorname{arccot} x$, $(\iota\iota\iota)$ $y = \tanh x$,

$(\iota\upsilon)$ $y = \log_{(2/5)} x$, (υ) $y = 3^x$.

Q9 (15 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) Find the area of the region $\mathcal{R}$ bounded by the curves $x = 2y^2 - 4\,,\ x = y^2\,.$ SKETCH $\mathcal{R}$

(B) Find the circumference of a circle of radius r .

Başkent University
MAT 151 – Calculus I
Spring Semester 2005 – Final Exam

June 01, 2005

Student ID	Name/Surname	Department

Duration: 110 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **9** Questions. Answer **7** Questions **only**;
- In the table below, indicate clearly the **7 Q**uestions that you select;
- Do not ask any question to anybody other than your instructor.

Q								**Total**
Points	**15**	**15**	**15**	**15**	**15**	**15**	**15**	
Grade								

G O O D L U C K !

<u>Q1</u> The region R in the first quadrant bounded by the curves $y = 2 - x^2$, $y = x^2$, and the line $x = 0$ is revolved about the line $x = 1$ to generate a solid. Find the volume of the solide by the Shell Method. Sketch R and the solid region D.

Q2 The region R bounded by the curve $y = x^2 + 1$ and the line $y = x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solide by the Washer Method. Sketch R and the solid region D.

Q3 Evaluate the following integral.

$$\int \frac{\cos y \, dy}{\sin^2 y + 6 \sin y + 13}$$

<u>Q4</u> Evaluate the following integral.

$$\int \frac{dx}{x^2\,(x^2+1)^{1/2}}$$

<u>Q5</u> Find the following limits.

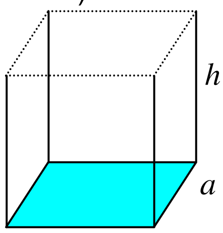
i) $\lim\limits_{x \div 1^{+}} \dfrac{(2\,x)^{1/2}\,(x-1)}{|x-1|}$, **ii)** $\lim\limits_{x \div 0^{+}} (\sin x)^{\tan x}$

Q6 Evaluate the following integral.

$$\int_{2}^{\infty} \frac{3\,\mathrm{d}x}{x^2 - x}$$

<u>Q7</u> Evaluate the following integral $\displaystyle\int_0^{\pi/3} x \tan^2 x \, dx$

<u>Q8</u> A box with a <u>square</u> base, rectangular sides, and <u>open</u> top is to contain 8000 cm^3 of space. The material for its base costs \$0.02 per cm^2 and that for its sides costs \$0.01 per cm^2. Determine the dimensions of the box (a and h) so that the <u>cost</u> of materials is a <u>minimum</u>. **(Hint:** Write the total cost C of the box as a function of a).



Q9 Let $y = f(x) = -x^4 + 8x^3$. Determine the intervals on which $f(x)$ is increasing / decreasing and the intervals on which $f(x)$ is concave up / concave down. Determine any local and absolute extreme value of $f(x)$. Sketch the graph of $f(x)$.

Başkent University
MAT 151 – Calculus I
Fall 2006 – Final Exam

January 15, 2007

Student ID	Name / Surname	Department

Duration: 115 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **8 Q**uestions but solve **7 Q**uestions only;
- In the table below, indicate clearly the **Q**uestion that you choose to solve and its **Part**, if any exists;
- Do not ask any question to anybody other than your instructor.

Q								Total
Part								
Grade								

GOOD LUCK!

<u>Q1</u> (20 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) Find the following limits.

(i) $\lim\limits_{x\to 1^+}\left(\dfrac{x+1}{2}\right)^{1/\operatorname{arcsec} x}$,   **(ii)** $\lim\limits_{x\to\infty}\left(\sqrt{x^4+5x^2+3}-x^2\right)$.

(B) Find the following limits.

(i) $\lim\limits_{x\to 0}(1+\sin x)^{\csc x}$,   **(ii)** $\lim\limits_{x\to\infty}(\ln x)^{1/x}$.

Q2 Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (25 pts) Calculate ANY TWO of the following improper integrals.

(i) $\displaystyle\int_1^{\infty} \frac{3x-1}{x^3-x^2}\,\mathrm{d}x$, **(ii)** $\displaystyle\int_0^{1} (\ln x)^2\,\mathrm{d}x$,

(iii) The area of the region that lies under the graph of $y = \dfrac{x^2}{(1-9x^2)^{3/2}}$, $0 \le x < 1/3$. Is it finite?

(B) (15 pts) Use a Comparison Test to determine if ONE of the given improper integrals is convergent or divergent.

(i) $\displaystyle\int_1^{\infty} \frac{\sqrt{x} + \operatorname{arcsec} x + \ln x}{x^2 \arctan x + x^{3/2}}\,\mathrm{d}x$ OR **(ii)** $\displaystyle\int_1^{\infty} \frac{\mathrm{e}^x + 2^x \sin x}{3^x + x^3 + \operatorname{arccsc} x}\,\mathrm{d}x$

<u>Q3</u> (20 pts)

(i) Sketch ROUGHLY the graphs of: **(a)** $y = \operatorname{arcsec} x$, **(b)** $y = e^{1/x}$,
(c) $y = \log_{(1/2)} (x-1)$, **(d)** $y = \arcsin |x|$.

(ii) Sketch IN FULL detail the graph of $y = \dfrac{x^2 - x - 2}{x-1}$.

<u>Q4</u> (25 pts) The region R bounded by the curve $y = x^2 + 1$ and the lines $y = 1$, and $x = 1$ is revolved about the line $x = 2$. SKETCH the solid of revolution and find its VOLUME by

(i) The Shell Method , **(ii)** The Washer Method .

Q5 (20 pts) A square-based, open-top rectangular tank of volume 500 ft^3 is to be built by welding thin stainless steel plates together along their edges. Find the dimensions for the base and height that will make the tank weigh as little as possible.

<u>Q6</u> (15 pts) Find the arclength of $y = \dfrac{e^x + e^{-x}}{2}$, $0 \le x \le 1$.

Q7 (15 pts) Find the derivatives of ANY TWO of the following functions.

(i) $y = (\arctan x)^{(e^x)}$, **(ii)** $y = (\tan x)^{\sec x}$, **(iii)** $y = (x)^{\operatorname{arccsc} x}$.

<u>Q8</u> (25 pts) Use the method indicated to find ANY TWO of the following integrals.

(i) $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,\mathrm{d}x$, (Trigonometric Substitution)

(ii) $\displaystyle\int \frac{2x^2+x+1}{x^3+x^2+x}\,\mathrm{d}x$, (Partial Fractions)

(iii) $\displaystyle\int x^3\cos(x^2)\,\mathrm{d}x$, (Integration by Parts).

Başkent University
MAT 151 – Calculus I
Spring 2007 – Final Exam

June 04, 2007

Student ID	Name / Surname	Department

Duration: 115 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **7 Q**uestions;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	7	Total
Grade								

GOOD LUCK!

Q1 (20 pts) Find the following limits.

(i) $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^{3x}$, **(ii)** $\lim\limits_{x\to(\pi/2)^-}(1+\cos x)^{\sec x}$, **(iii)** $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 2x}{\cos x-\cos 3x}$.

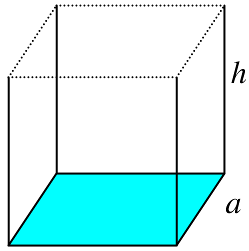
Q2 (25 pts)

(i) Calculate $\int_{1}^{\infty} \frac{\ln x}{x^2}\,\mathrm{d}x$, **(ii)** Calculate $\int_{0}^{\pi/2} \frac{\tan x}{\sqrt{\sec x}}\,\mathrm{d}x$, **(iii)** Use a Comparison Test to determine if the given improper integral is convergent or divergent: $\int_{2}^{\infty} \frac{x^3 + x^2 \sin x}{x^5 + \mathrm{e}^{-x} + \arctan x}\,\mathrm{d}x$.

Q3 (15 pts) Sketch IN FULL detail the graph of $y = \dfrac{x^2 - 2x + 1}{x - 2}$. Make a summary table.

Q4 (25 pts) The region R in the first quadrant bounded by the curve $y = x^2$ and the line $y = 3x$, is revolved about the line $x = 3$. SKETCH the solid of revolution and find its VOLUME by

(i) The Shell Method , **(ii)** The Washer Method .



<u>Q5</u> (15 pts) A box with a square base, rectangular sides, and open top is to contain 2000 cm^3 of space. The material for its base costs 0.3 YTL per cm^2 and that for its sides costs 0.6 YTL per cm^2. Determine the dimensions a and h of the box so that the cost of materials is a minimum.

<u>Q6</u> (20 pts)

(i) Find the derivative of $y = (\arcsin x)^{\sec x}$, **(ii)** Find the derivative of $y = x\sqrt{x+\sqrt{x}}$.

(iii) Find the arclength of $y = -\ln(\cos x)$ for $0 \le x \le \pi/4$.

Q7 (30 pts) Evaluate the following integrals.

(i) $\displaystyle\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$, **(ii)** $\displaystyle\int \sec^4 x\,\mathrm{d}x$, **(iii)** $\displaystyle\int \frac{3x^2-5x+10}{(x-4)(x^2+3)}\,\mathrm{d}x$, **(iv)** $\displaystyle\int \sinh^2(3x)\,\mathrm{d}x$.

Başkent University
MAT 151 – Calculus I
Spring 2007 – Midterm Exam

April 07, 2007

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	Total
Grade							

GOOD LUCK!

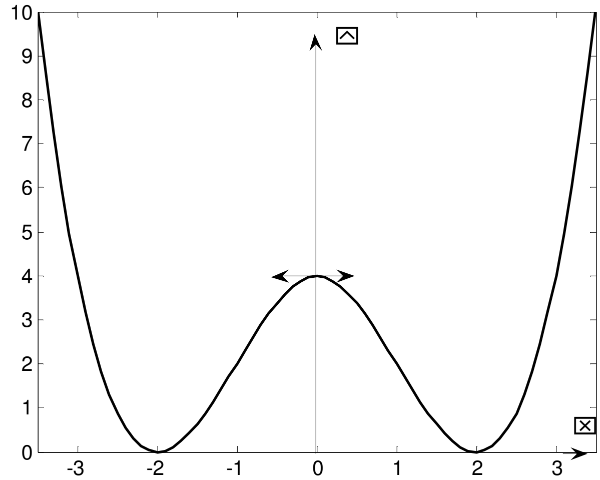
<u>Q4</u> (20 pts) Sketch <u>roughly</u> the graphs of the following functions.

(i) $f(x) = \operatorname{arccot} x$, **(ii)** $f(x) = |2 - |x||$, **(iii)** $f(x) = \ln(|2 - |x||)$,
(iv) $f(x) = x^2 - 4x + 7$, **(v)** $f(x) = \sin(2x)$.

Q2 (20 pts) Sketch in full detail the graph of the following function. Make a summary table and determine any local or absolute extreme value and point of inflection.

$f(x) = |x|\,x^2 - 3x^2 + 4$.

Answer:



Q3 (15 pts) Use Logarithmic Differentiation to find the derivatives of the following functions.

(i) $f(x) = (\operatorname{arccot} x)^{\ln x}$, **(ii)** $f(x) = (\sec x)^{\operatorname{arcsec} x}$, **(iii)** $f(x) = (\ln x)^{\ln(1/x)}$.

<u>Q1</u> (20 pts) Find the following limits

(i) $\lim\limits_{x\to 0}\left(e^x + x\right)^{2/x}$, **(ii)** $\lim\limits_{x\to(\pi/2)^-}\left[1+\cos(2x)\right]^{\cos x}$, **(iii)** $\lim\limits_{x\to\infty}(\ln x)^{1/x}$.

Do **NOT USE** L'Hôpital's Rule for **(iv)** and **(v)**.

(iv) $\lim\limits_{x\to 0}\dfrac{e^x - 1}{\tan(2x)}$, **(v)** $\lim\limits_{x\to-\infty}\left(\dfrac{\sqrt{x^2+4}+\sqrt{|x|}}{\sqrt[5]{x^5+3}+x}\right)$.

<u>Q6</u> (15 pts) Find the minimum and maximum distances from the point $P(1, \sqrt{3})$ to the semicircle $y = \sqrt{16 - x^2}$. Solve the question by optimization.

<u>Q5</u> (15 pts) Let (C) be the curve of equation $2y^2 + yx = x^2 + 2$. **(i)** Find equations for the tangent and normal lines to the curve (C) at the point $P(1,\ 1)$. **(ii)** Is the graph of (C) concave up or concave down at $P(1,\ 1)$? Explain!

Başkent University
MAT 151 – Calculus I
Fall 2007 – Midterm Exam

November 23, 2007

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- In the table below, indicate clearly the **Part** of the **Q**uestion that you choose to solve, if any exists;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	Total
Part							
Grade							

GOOD LUCK!

Q1. Answer either **Part (A)** or **Part (B)**, but NOT both.

Find the dimensions (radius and height) of the cylinder of largest volume that can be inscribed in the upper half of the sphere $x^2 + y^2 + z^2 = 9$ if

(A) (15 pts) the cylinder is in the horizontal position,

(B) (20 pts) the cylinder is in the vertical position. In this case also determine whether or not the values found for the dimensions make the total area of the cylinder maximum. Explain!

Q2. Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) A man $6\ \mathrm{ft}$ tall walks at a speed of $4\ \mathrm{ft/sec}$ toward a street lamp which is $14\ \mathrm{ft}$ above the ground. How fast is the length of the man's shadow decreasing (at any time t)?

(B) (15 pts) A point is moving in circular orbit $x^2 + y^2 = 25$. As it passes through the point $(3, 4)$, its y-coordinate is decreasing at the rate of 2 units per second. How is the x-coordinate changing?

Q3. (20 pts) Sketch <u>roughly</u> the graphs of <u>any four</u> of the following functions.

(i) $f(x) = \arcsin x$, **(ii)** $f(x) = \operatorname{arcsec} x$, **(iii)** $f(x) = \log_{(5/7)} |x|$,

(iv) $f(x) = 3^{1/x}$, **(v)** $f(x) = \csc x$.

Q4. Answer <u>only one</u> part (**Part (A)**, **Part (B)**, or **Part (C)**).

Sketch in <u>full detail</u> the graph of the function $y = f(x)$, and write down an equation for the tangent line at the given point P_0.

(A) (15 pts) $y = f(x) = x^3 - 6x^2 + 9x - 3$, $P_0(1, 1)$.

(B) (15 pts) $y = f(x) = \dfrac{x^2 - 4}{x + 1}$, $P_0(2, 0)$.

(C) (20 pts) $y = f(x) = x^x$, $x > 0$, $P_0(2, 4)$.

Q5. Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Find the derivatives ($\mathrm{d}y/\mathrm{d}x$ or $\mathrm{d}f/\mathrm{d}x$) of the following equation and functions.
(i) $y = (\cos y)^{(x^2)}$, **(ii)** $f(x) = (\ln x)^{\arctan x}$, **(iii)** $f(x) = (\cot x)^x$.

(B) (15 pts) Find **(i)** $\int \frac{3\sin x \cos x}{\sqrt{1+3\sin^2 x}}\mathrm{d}x$ and **(ii)** $\int \frac{1}{x\sqrt{9x^2-1}}\mathrm{d}x$, $(x > 1)$.

Q6. (24 pts)

(i) Find <u>any two</u> of the following limits.

(a) $\lim\limits_{x\to 0}\left(\dfrac{\arccos x \cdot \arctan x}{\arcsin x \cdot \ln(x+1)}\right)$, (b) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{2x}$, (c) $\lim\limits_{x\to 1^-}(1-x)^{\ln x}$.

(ii) Find the following limits. (DO NOT USE L'HÔPITAL'S RULE)

(d) $\lim\limits_{x\to 1}\left(\dfrac{x^4-1}{x^3-1}\right)$, (e) $\lim\limits_{x\to 0}\left(\dfrac{1-\cos x}{x\sin x}\right)$.

Başkent University
MAT 151 – Calculus I
Fall 2007 – Final Exam

January 07, 2008

Student ID	Name / Surname	Department

Duration: 120 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **8 Q**uestions;
- In the table below, indicate clearly the **Part** of the **Q**uestion that you choose to solve, if any exists;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	7	8	Total
Part									
Grade									

GOOD LUCK!

Q1. (20 pts) Find the area of the region

(i) bounded by the curves $x = 2y^2 - 4$ and $x = y^2$.
(ii) the circle of radius r.

Q2. (20 pts) Find any <u>two</u> of

(i) $\lim\limits_{x \to 0^+} \left(\dfrac{\sin x}{x} \right)^{1/x}$, **(ii)** $\lim\limits_{x \to 0^+} (\ln x \cdot \arctan x)$,

(iii) $\lim\limits_{x \to \infty} (x \cdot \operatorname{arccsc} x)$, **(iv)** $\lim\limits_{x \to 0^+} (2^{1/x} \cdot \sin x)$.

(Hint: Make use of the fact that $\lim\limits_{x \to 0^+} \dfrac{\sin x}{x} = 1$.)

Q3. (24 pts) Answer <u>any three</u> of the following improper integrals:

(i) $\displaystyle\int_2^{\infty}\left(\operatorname{arccsc} x-\frac{1}{\sqrt{x^2-1}}\right)\mathrm{d}x$, **(ii)** $\displaystyle\int_1^{\infty}\frac{\ln(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$, **(iii)** $\displaystyle\int_1^{\infty}\frac{x^2}{1+x^6}\,\mathrm{d}x$.

(iv) Use the Limit Comparison Test to decide if the following integral converges or diverges:

$$\int_1^{\infty}\frac{\sqrt{x^2+1}+\sqrt{x}\arctan x}{\left[x^5+x^2\ln(x^2)+x^3\operatorname{arccsc} x\right]^{1/3}}\,\mathrm{d}x.$$

Q4. Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (15 pts) A rectangle is to be inscribed into a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

(B) (20 pts) A man is drinking cola through a straw from a conical cup, 8 inches deep and 4 inches in diameter at the top. At the instant the cola is 5 inches deep, he is drinking at the rate of 3 cubic inches per second. How fast is the level of the cola dropping at that time?

Q5. Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (15 pts) Find the volume of the sphere of radius a. Sketch the solid of revolution.

(B) (25 pts) Let (R) denotes the region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 1$. Use the method indicated to find the volume of revolution if (R) is revolved about
(i) the x-axis (the Cylindrical Shells Method), Sketch the solid of revolution.
(ii) the line $y = 2$ (the Method of Washers), Sketch the solid of revolution.

Q6. (25 pts)

(i) Find the circumference of a circle of radius a.
(ii) Let $y = f(x)$ be a smooth positive function on $[a, b]$. Then the area of the surface generated by revolving the graph of $f(x)$ about the x-axis is given by

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\,\mathrm{d}x\,.$$

Use this formula to find the surface area of a sphere of radius r.

Q7. (24 pts) Find the following integrals

(i) $\int e^x \sin x \, dx$,

(ii) $\int \left(\frac{2x-1}{x^2+x} + \frac{x}{x^2+x+1} \right) dx$,

(iii) $\int \frac{1}{x\sqrt{x^2+9}} \, dx$,

(iv) $\int \sec^4 x \, dx$.

Q8. (25 pts)

(i) Show that the graphs of $f(x) = x^4 - 5x^2$ and $g(x) = 2x^3 - 4x^2 + 6$ intersect between $x = 0$ and $x = 3$.

(ii) Sketch <u>roughly</u> the graphs of <u>any four</u> of the following functions:

(a) $y = \coth(x^2)$, **(b)** $y = \arctan(1/|x|)$, **(c)** $y = \operatorname{csch} x$,

(d) $y = x + \dfrac{1}{x-1} + \dfrac{1}{x+1}$, **(e)** $y = 3^{-x^2}$.

Başkent University
MAT 151 – Calculus I
Summer Session 2007 – Final Exam

August 09, 2007

Student ID	Name / Surname	Department

Duration: 115 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **7 Q**uestions;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	7	Total
Grade								

GOOD LUCK!

Q1 (20 pts) Find the following limits.

(i) $\lim_{x\to\frac{\pi}{2}} \dfrac{\int_0^{\cos x} \sin(t^2)\,\mathrm{d}t}{(x-\pi/2)^2}$, **(ii)** $\lim_{x\to\infty} \left(2^x + x\right)^{1/x}$.

(i) $\lim_{x\to\frac{\pi}{2}} \dfrac{\int_0^{\cos x} \sin(t^2)\,\mathrm{d}t}{(x-\pi/2)^2} \to \dfrac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\frac{\pi}{2}} \frac{\int_0^{\cos x} \sin(t^2)\,\mathrm{d}t}{(x-\pi/2)^2} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x\cdot\sin(\cos^2 x)}{2(x-\pi/2)} \to \frac{0}{0}$$

$$= \left(\lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{2}\right)\times\left(\lim_{x\to\frac{\pi}{2}} \frac{\sin(\cos^2 x)}{(x-\pi/2)}\right) = -(1/2)\times\lim_{x\to\frac{\pi}{2}} \frac{-2\sin(2x)\cdot\cos(\cos^2 x)}{1}$$

$$= -(1/2)\times 0 = 0$$

(ii) $\lim_{x\to\infty} \left(2^x + x\right)^{1/x} \to \infty^0$. Set $F(x) = \left(2^x + x\right)^{1/x} \Rightarrow \ln F(x) = \dfrac{\ln(2^x + x)}{x} \to \dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \ln F(x) = \lim_{x\to\infty} \frac{\left((\ln 2)2^x + 1\right)/(2^x + x)}{1} = \lim_{x\to\infty} \frac{\left((\ln 2)2^x + 1\right)}{(2^x + x)} = \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{\left((\ln 2)2^x + 1\right)}$$

$$= \lim_{x\to\infty} \frac{(\ln 2)^2}{(\ln 2) + 2^{-x}} = \ln 2 \Rightarrow \lim_{x\to\infty} F(x) = \mathrm{e}^{\ln 2} = 2$$

Q2 (25 pts)

(i) Calculate $\int_1^{\infty} \dfrac{\ln x}{x^3}\,\mathrm{d}x$, **(ii)** Calculate $\int_0^{\pi/2} \dfrac{\cot x}{\sqrt{\csc x}}\,\mathrm{d}x$, **(iii)** Use a Comparison Test to determine if the given improper integral is convergent or divergent: $\int_{10}^{\infty} \dfrac{2x^3 + x\cos x}{3x^5 + 7\mathrm{e}^{-x} + \operatorname{arccot} x}\,\mathrm{d}x$.

(i) Integration by parts leads to $\int \frac{\ln x}{x^3}\,\mathrm{d}x = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$.

$$\int_1^{\infty} \frac{\ln x}{x^3}\,\mathrm{d}x = \lim_{p\to\infty}\int_1^{p} \frac{\ln x}{x^3}\,\mathrm{d}x = -\lim_{p\to\infty}\left(\frac{\ln p}{2p^2} + \frac{1}{4p^2} - \frac{1}{4}\right) = -\lim_{p\to\infty}\frac{\ln p}{2p^2} + \frac{1}{4} = -\lim_{p\to\infty}\frac{1}{4p^2} + \frac{1}{4} = \frac{1}{4}$$

(ii) $\int \frac{\cot x}{\sqrt{\csc x}}\,\mathrm{d}x = \int \frac{\csc x \cot x}{(\csc x)^{3/2}}\,\mathrm{d}x = -\int (\csc x)^{-3/2}\,\mathrm{d}(\csc x) = \frac{2}{\sqrt{\csc x}} + C = 2\sqrt{\sin x} + C$.

$$\int_0^{\pi/2} \frac{\cot x}{\sqrt{\csc x}}\,\mathrm{d}x = \lim_{p\to 0^+}\int_p^{\pi/2} \frac{\cot x}{\sqrt{\csc x}}\,\mathrm{d}x = 2\lim_{p\to 0^+}(1 - \sqrt{\sin p}) = 2.$$

(iii) Use the Limit Comparison Test. Set $f(x) = \frac{2x^3 + x\cos x}{3x^5 + 7e^{-x} + \operatorname{arccot} x}$ and let $g(x) = \frac{1}{x^2}$. Next, Find:

$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{2x^5 + x^3\cos x}{3x^5 + 7e^{-x} + \operatorname{arccot} x} = \lim_{x\to\infty}\frac{2 + (\cos x / x^2)}{3 + (e^{-x}/x^5) + (\operatorname{arccot} x / x^5)} = \frac{2}{3}$. Hence, the integrals $\int_2^{\infty} f(x)\,\mathrm{d}x$ and $\int_2^{\infty} g(x)\,\mathrm{d}x$ both converge or both diverge. However, the integral

$\int_{10}^{\infty} g(x)\,\mathrm{d}x = \int_{10}^{\infty}\frac{1}{x^2}\,\mathrm{d}x = \lim_{p\to\infty}\int_{10}^{p}\frac{1}{x^2}\,\mathrm{d}x = -\lim_{p\to\infty}\left(\frac{1}{p} - \frac{1}{10}\right) = \frac{1}{10}$ converges. By the Limit Comparison Test, the integral $\int_{10}^{\infty}\frac{2x^3 + x\cos x}{3x^5 + 7e^{-x} + \operatorname{arccot} x}\,\mathrm{d}x$ converges too.

Q3 (15 pts) Sketch IN FULL detail the graph of $y = \frac{x^2 - 6x + 9}{x - 4}$. Make a summary table and determine any local or absolute extreme value and point of inflection.

$D = (-\infty, 4) \cup (4, +\infty)$. $y = \frac{x^2 - 6x + 9}{x - 4} = \frac{(x-3)^2}{x-4} = x - 2 + \frac{1}{x-4}$. Hence, $y = x - 2$ is an oblique asymptote and $x = 4$ is a vertical one.

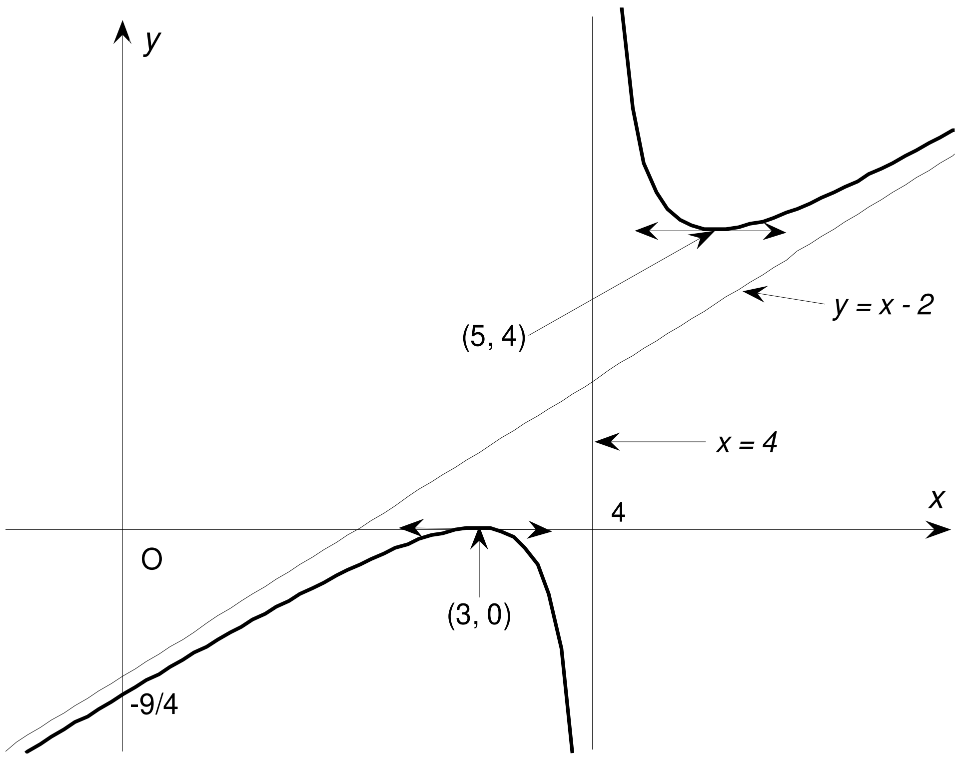
$$\lim_{x\to+\infty} y = +\infty, \quad \lim_{x\to-\infty} y = -\infty, \quad \lim_{x\to 2^+} y = +\infty, \quad \lim_{x\to 2^-} y = -\infty.$$

$$y' = 1 - \frac{1}{(x-4)^2} = \frac{(x-5)(x-3)}{(x-4)^2}$$; the Critical Points are $x = 3$ & $x = 5$ ($y(3) = 0$ & $y(5) = 4$).

$$y'' = +\frac{2}{(x-4)^3}.$$

x	$-\infty$ 3^-	3^+ 4^-	4^+ 5^-	5^+ $+\infty$
y'	+	− Undefined	Undefined −	+
y''	−	−	+	+
y	$-\infty \nearrow 0$ C.D.	$0 \searrow -\infty$ C.D.	$+\infty \searrow 4$ C.U.	$4 \nearrow +\infty$ C.U.

$y(3) = 0$ is a local maximum & $y(5) = 4$ is a local minimum.



Q4 (25 pts) The region R in the first quadrant bounded by the curve $y = x^2 + 1$ and the line $y = 3x + 1$, is revolved about the line $x = 3$. SKETCH the solid of revolution and find its VOLUME by

(i) The Shell Method , **(ii)** The Washer Method .

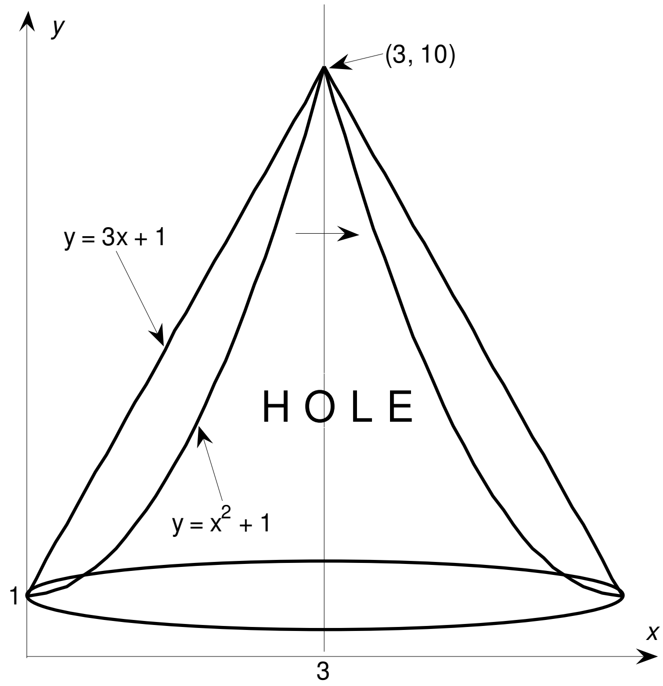
(i) The Shell Method: $R(x) = 3 - x, \quad H(x) = 3x - x^2$:

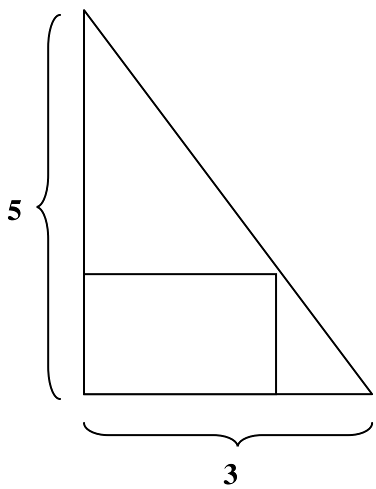
$$V = 2\pi\int_0^3 (3-x)(3x-x^2)\,\mathrm{d}x = 2\pi\int_0^3 (3-u)u^2\,\mathrm{d}x = \frac{27\pi}{2}\ \text{units}^3.\ (\text{where } u = 3 - x)$$

(ii) The Washer Method: Inner Radius $R_1(y) = 3 - \sqrt{y-1}$, Outer Radius $R_2(y) = 3 - \dfrac{y-1}{3}$:

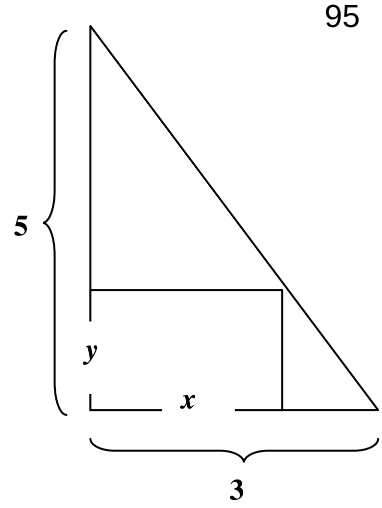
$$V = \pi\int_1^{10}\left[(3-\frac{y-1}{3})^2 - (3-\sqrt{y-1})^2\right]\mathrm{d}y = \pi\int_0^9\left(\frac{u^2}{9} - 3u + 6u^{1/2}\right)\mathrm{d}u = \frac{27\pi}{2}\ \text{units}^3,$$

where we have set $u = y - 1$.





Q5 (15 pts) A rectangle is to be cut from a right-angle triangle with sides 3 cm and 5 cm, so that one side of the rectangle is parallel to the side of the triangle of length 3 cm. What are the dimensions of the rectangle with the largest possible area?



Let x & y be the sides of the recatngle. The area of the rectangle is:

$A = x\,y$. In order to eliminate x & y, we use the similarity between the right-angle triangles of sides (3, 5) & (x, $5 - y$). We have:
$\dfrac{x}{3} = \dfrac{5-y}{5}$ or $x = \dfrac{3}{5}(5-y)$. Multiplying both sides of the last equation by y we obtain A as a function of y:
$A = \dfrac{3}{5}(5-y)y = \dfrac{3}{5}(5y - y^2)$, with $0 < y < 5$. The derivative $\mathrm{d}A/\mathrm{d}y = (3/5)(5 - 2y)$ vanishes for $y = 5/2$ providing one critical point in the interval $(0, 5)$ at which the second derivative $\mathrm{d}^2A/\mathrm{d}y^2 = -6/5$ is negative. Hence, the area has a local maximum at $y = 5/2$ and the dimensions of the corresponding rectangle are $x \times y = (3/2)\times(5/2)$.

<u>Q6</u> (20 pts)

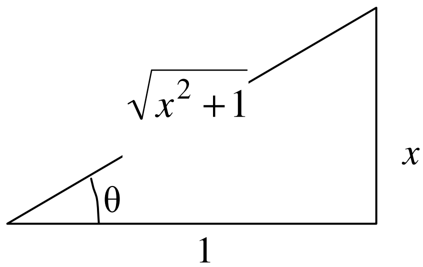
(i) Find the derivative of $y = (\operatorname{arcsec} x)^{\cot x}$,

(ii) Find the arclength of $y = \ln x$ for $\sqrt{3} \le x \le 2\sqrt{2}$.

(i) $\ln y = \cot x \cdot \ln(\operatorname{arcsec} x) \Rightarrow \dfrac{y'}{y} = -\csc^2 x \ln(\operatorname{arcsec} x) + \cot x \dfrac{1}{|x|\sqrt{x^2-1}\operatorname{arcsec} x}$

$$\Rightarrow y' = \left(-\csc^2 x \ln(\operatorname{arcsec} x) + \cot x \frac{1}{|x|\sqrt{x^2-1}\operatorname{arcsec} x}\right)(\operatorname{arcsec} x)^{\cot x}.$$

(ii) $y' = 1/x$. The arclength is given by:



$$L = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+(1/x)^2}\,\mathrm{d}x = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x}\,\mathrm{d}x.$$

Set $x = \tan\theta \to \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$ & $\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$ (since $x > 0$ implies $\tan\theta > 0$, hence θ is in the first quadrant and consequently $\sec\theta > 0$). The indefinite integral writes:

$$\int \frac{\sqrt{x^2+1}}{x}\mathrm{d}x = \int \frac{\sec^3\theta}{\tan\theta}\mathrm{d}\theta = \int \frac{1}{\sin\theta\cos^2\theta}\mathrm{d}\theta = \int \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos^2\theta}\mathrm{d}\theta = \int\left(\frac{\sin\theta}{\cos^2\theta} + \csc\theta\right)\mathrm{d}\theta$$

$$= \int(\sec\theta\tan\theta + \csc\theta)\mathrm{d}\theta = \sec\theta + \ln|\csc\theta - \cot\theta| + C$$

From the figure associated with the substitution $x = \tan\theta$, we have $\csc\theta = \sqrt{x^2+1}/x$, $\cot\theta = 1/x$, with $\sec\theta = \sqrt{x^2+1}$, the indefinite integral writes:

$$\int \frac{\sqrt{x^2+1}}{x}\,\mathrm{d}x = \sqrt{x^2+1} + \ln\left(\frac{\sqrt{x^2+1}-1}{x}\right) + C \text{ and}$$

$$L = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+(1/x)^2}\,\mathrm{d}x = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{x^2+1}}{x}\,\mathrm{d}x = \left\{\sqrt{x^2+1} + \ln\left(\frac{\sqrt{x^2+1}-1}{x}\right)\right\}\Bigg|_{\sqrt{3}}^{\sqrt{8}} = 1 + \frac{1}{2}\ln\left(\frac{3}{2}\right) \text{ units.}$$

Q7 (30 pts) Evaluate the following integrals.

(i) $\displaystyle\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$, **(ii)** $\displaystyle\int \csc^4 x\,\mathrm{d}x$, **(iii)** $\displaystyle\int \frac{x^2+5x+2}{(x-4)(x^2+3)}\,\mathrm{d}x$, **(iv)** $\displaystyle\int \cosh^2(4x)\,\mathrm{d}x$.

(i) Set $x = (2/5)\sec\theta \rightarrow \mathrm{d}x = (2/5)\sec\theta\tan\theta\,\mathrm{d}\theta$. Since $2/5 \le x \le 4/5 \Rightarrow 0 < \theta < \pi/2$.

$$\sqrt{25x^2-4} = 2\sqrt{\sec^2\theta - 1} = 2|\tan\theta| = 2\tan\theta.$$

$$\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = \int \frac{2\tan\theta}{(2/5)\sec\theta}(2/5)\sec\theta\tan\theta\,\mathrm{d}\theta = 2\int \tan^2\theta\,\mathrm{d}\theta = 2\int(\sec^2\theta - 1)\,\mathrm{d}\theta$$

$$= 2(\tan\theta - \theta) + C = \sqrt{25x^2-4} - 2\arccos\left(\frac{2}{5x}\right) + C$$

$$\Rightarrow \int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = \left[\sqrt{25x^2-4} - 2\arccos\left(\frac{2}{5x}\right)\right]\Bigg|_{2/5}^{4/5} = 2\sqrt{3} - \frac{2\pi}{3}.$$

(ii) Integration by parts leads to

$$\int \csc^4 x\,\mathrm{d}x = -\csc^2 x\cot x - 2\int \csc^2 x\,\underbrace{\cot^2 x}_{(\csc^2 x - 1)}\,\mathrm{d}x = -\csc^2 x\cot x - 2\int \csc^4 x\,\mathrm{d}x - 2\cot x$$

$$3\int \csc^4 x\,\mathrm{d}x = -\csc^2 x\cot x - 2\cot x \Rightarrow \int \csc^4 x\,\mathrm{d}x = -\left(\frac{\csc^2 x + 2}{3}\right)\cot x + C.$$

Or, directly:

$$\int \csc^4 x\,\mathrm{d}x = \int \csc^2 x \underbrace{\csc^2 x}_{(\cot^2 x+1)}\,\mathrm{d}x = \underbrace{\int \csc^2 x \cot^2 x\,\mathrm{d}x}_{\text{set: } u=\cot x} + \int \csc^2 x\,\mathrm{d}x = -\int u^2\,\mathrm{d}u - \cot x$$

$$= -\frac{\cot^3 x}{3} - \cot x + \overline{C}$$

(iii) $\dfrac{x^2+5x+2}{(x-4)(x^2+3)} = \dfrac{A}{x-4} + \dfrac{Bx+C}{x^2+3} = \dfrac{2}{x-4} - \dfrac{x-1}{x^2+3}$, with $A = 2$, $B = -1$ & $C = 1$

$$\Rightarrow \int \frac{3x^2-5x+10}{(x-4)(x^2+3)}\,\mathrm{d}x = 2\int \frac{1}{x-4}\,\mathrm{d}x - \frac{1}{2}\int \frac{2x}{x^2+3}\,\mathrm{d}x + \int \frac{1}{x^2+(\sqrt{3})^2}\,\mathrm{d}x$$

$$= 2\ln|x-4| - (1/2)\ln(x^2+3) + (1/\sqrt{3})\arctan(x/\sqrt{3}) + C$$

(iv) $\displaystyle\int \cosh^2(4x)\,\mathrm{d}x = \frac{1}{2}\int [\cosh(8x)+1]\,\mathrm{d}x = \frac{1}{2}\left[\frac{\sinh(8x)}{8} + x\right] + C.$

Başkent University
MAT 151 – Calculus I
Summer Session 2007 – Midterm Exam

July 19, 2007

Student ID	Name / Surname	Department

Duration: 100 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	Total
Grade							

GOOD LUCK!

Q1 (20 pts) Find the following limits

(i) $\lim\limits_{x\to 0}\left(e^x-2x\right)^{4/x}$, **(ii)** $\lim\limits_{x\to\pi^+}\left[1+\sin(3x)\right]^{\cot x}$, **(iii)** $\lim\limits_{x\to\infty}(\ln x)^{3/x}$.

Do **NOT USE** L'Hôpital's Rule for **(iv)** and **(v)**.

(iv) $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{\sin(5x)}$, **(v)** $\lim\limits_{x\to-\infty}\left(\dfrac{\sqrt{x^2+4}+3\sqrt{|x|}}{\sqrt[7]{x^7+9}+x}\right)$.

(i) $\lim\limits_{x\to 0}\left(e^x-2x\right)^{4/x}\to 1^\infty$. Set $F(x)=\left(e^x-2x\right)^{4/x}\Rightarrow \ln F(x)=\dfrac{\ln(e^x-2x)}{x}\to\dfrac{0}{0}$.

$$\lim_{x\to 0}\ln F(x)=4\lim_{x\to 0}\frac{(e^x-2)/(e^x-2x)}{1}=-4\Rightarrow\lim_{x\to 0}F(x)=e^{-4}.$$

(ii) $\lim\limits_{x\to\pi^+}\left[1+\sin(3x)\right]^{\cot x}\to 1^\infty$. Set $F(x)=\left[1+\sin(3x)\right]^{\cot x}\Rightarrow \ln F(x)=\dfrac{\ln\left[1+\sin(3x)\right]}{\tan x}\to\dfrac{0}{0}$.

$$\lim_{x\to\pi^+}\ln F(x)=\lim_{x\to\pi^+}\frac{3\cos(3x)/(1+\sin(3x))}{\sec^2 x}=-3\Rightarrow\lim_{x\to\pi^+}F(x)=e^{-3}.$$

(iii) $\lim\limits_{x\to\infty}(\ln x)^{3/x}\to\infty^0$. Set $F(x)=(\ln x)^{3/x}\Rightarrow \ln F(x)=3\dfrac{\ln(\ln x)}{x}\to\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\ln F(x)=3\lim_{x\to\infty}\frac{1}{x\ln x}=0\Rightarrow\lim_{x\to 0}F(x)=e^0=1.$$

(iv)

$$\lim_{x\to 0}\frac{e^{3x}-1}{\sin 5x}=\left(\lim_{x\to 0}\frac{e^{3x}-e^0}{x}\right)\Bigg/\left(\lim_{x\to 0}\frac{\sin 5x-\sin 0}{x}\right)=\left.(e^{3x})'\right|_{x=0}\Big/\left.(\sin 5x)'\right|_{x=0}=3e^0\big/5\cos 0=3/5$$

(v)
$$\lim_{x\to-\infty}\left(\frac{\sqrt{x^2+4}+3\sqrt{|x|}}{\sqrt[7]{x^7+9}+x}\right)=\left(\lim_{x\to-\infty}\frac{|x|}{x}\right)\left(\lim_{x\to-\infty}\frac{\left(\sqrt{1+\dfrac{4}{x^2}}+\dfrac{3}{\sqrt{|x|}}\right)}{\left(1+\dfrac{9}{x^7}\right)^{1/7}+1}\right)=-1\times\frac{1}{1+1}=-\frac{1}{2}.$$

Q2 (20 pts) Sketch in <u>full detail</u> the graph of the following function. Make a summary table and determine any local or absolute extreme value and point of inflection.

$f(x) = x^3/3 - (|x|x)/2$.

$D = (-\infty, +\infty)$.

$f(-x) = (-x)^3/3 - [|-x|(-x)]/2 = -x^3/3 + (|x|x)/2 = -f(x)$ for all x in D. $f(x)$ is an odd function and its graph is symmetric with repect to the origin.

$$\lim_{x\to+\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

For $x > 0$: $f(x) = x^3/3 - x^2/2 = (x^2/3)(x - 3/2)$

$f'(x) = x^2 - x = x(x-1)$; the critical points are $x = 0$ & $x = 1$ ($f(0) = 0$ & $f(1) = -1/6$).

$f''(x) = 2x - 1$; the function has a point of inflection at $(1/2, -1/12)$.

For $x < 0$: $f(x) = x^3/3 + x^2/2 = (x^2/3)(x + 3/2)$

$f'(x) = x^2 + x = x(x+1)$

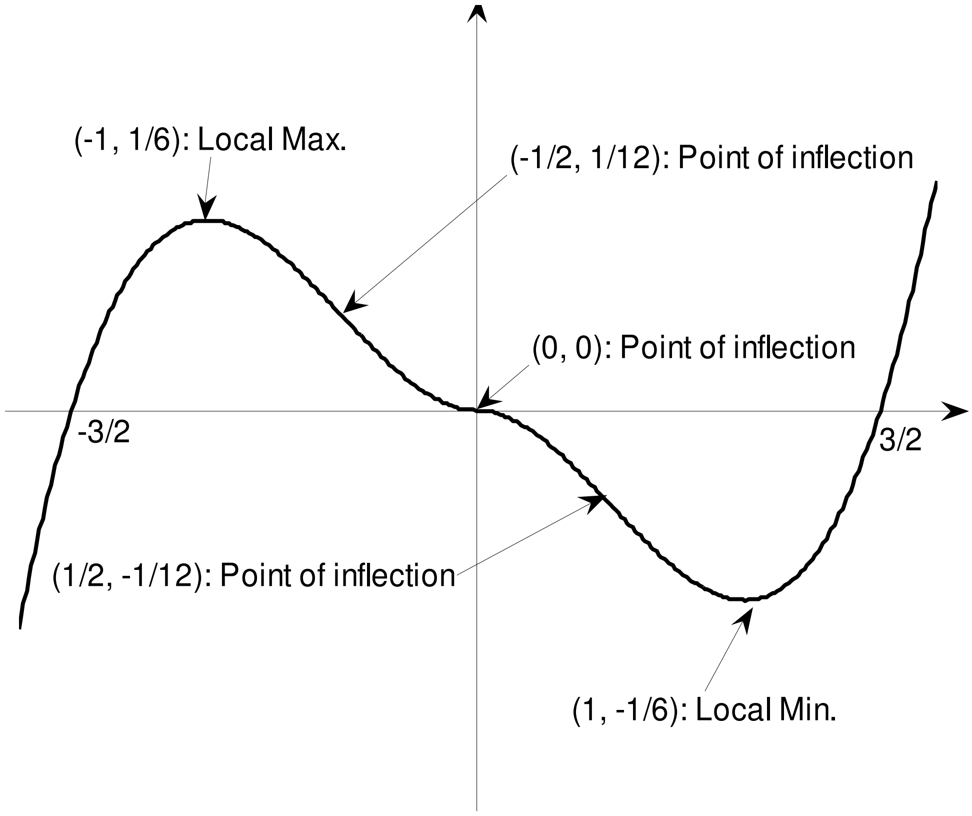
$f''(x) = 2x + 1$.

Hence, $f'(0) = 0$ and $f''(0)$ does not exist. Since there is a change in the concavity around $x = 0$, the function has another point of inflection at $(0, 0)$.

The other critical points and points of inflection are obtained by symmetry. Here is the summary table for $x \geq 0$:

x	0		$1/2$		1		$+\infty$
y'	0	$-$		$-$		$+$	
y''	undefined	$-$		$+$		$+$	
y	0	C.D. ↘	$-1/12$	C.U. ↘	$-1/6$	C.U. ↗	$+\infty$

The function has 3 points of inflection, one local minimum and one local maximum as shown in the graph below. The x-intercepts are $-3/2$ and $3/2$ and the y-intercept is 0.


(-1, 1/6): Local Max.
(-1/2, 1/12): Point of inflection
(0, 0): Point of inflection
-3/2
3/2
(1/2, -1/12): Point of inflection
(1, -1/6): Local Min.

<u>Q3</u> (15 pts) Use Logarithmic Differentiation to find the derivatives of the following functions.

(i) $f(x) = (\operatorname{arccot} x)^{\tan x}$, **(ii)** $f(x) = (\csc x)^{\operatorname{arcsec} x}$, **(iii)** $f(x) = (\ln x)^{\ln(1/x)}$.

(i) $\ln f = \tan x \cdot \ln(\operatorname{arccot} x) \Rightarrow \dfrac{f'}{f} = \sec^2 x \ln(\operatorname{arccot} x) - \dfrac{\tan x}{(1+x^2)\operatorname{arccot} x}$

$$\Rightarrow f' = \left(\sec^2 x \ln(\operatorname{arccot} x) - \frac{\tan x}{(1+x^2)\operatorname{arccot} x} \right) (\operatorname{arccot} x)^{\tan x}.$$

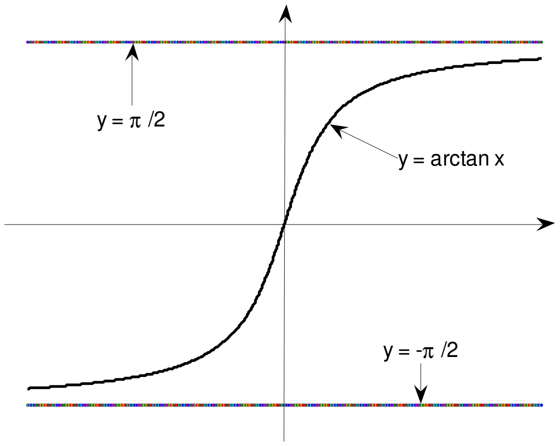
(ii) $f' = \left(\dfrac{\ln(\csc x)}{|x|\sqrt{x^2-1}} - \operatorname{arcsec} x \cot x \right) (\operatorname{arccot} x)^{\tan x}.$

(iii) $f' = -\left(\dfrac{\ln(\ln x)}{x} + \dfrac{1}{x} \right) (\ln x)^{\ln(1/x)}.$

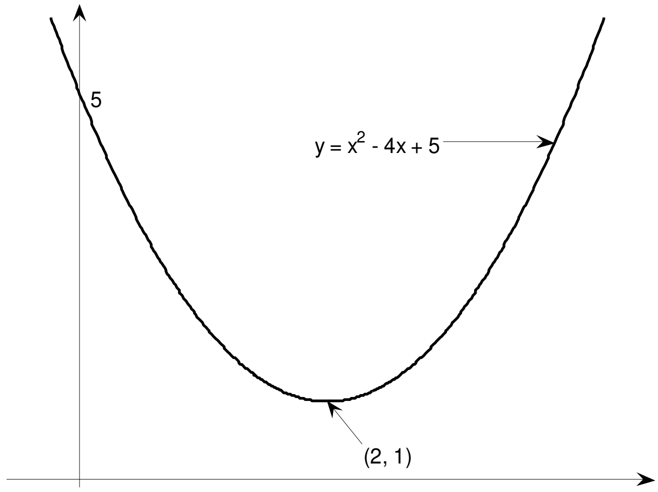
Q4 (20 pts) Sketch <u>roughly</u> the graphs of the following functions.

(i) $f(x) = \arctan x$, **(ii)** $f(x) = |2 - |x-1||$, **(iii)** $f(x) = \ln[3/(2-x)]$,
(iv) $f(x) = x^2 - 4x + 5$, **(v)** $f(x) = \sin(3x)$.

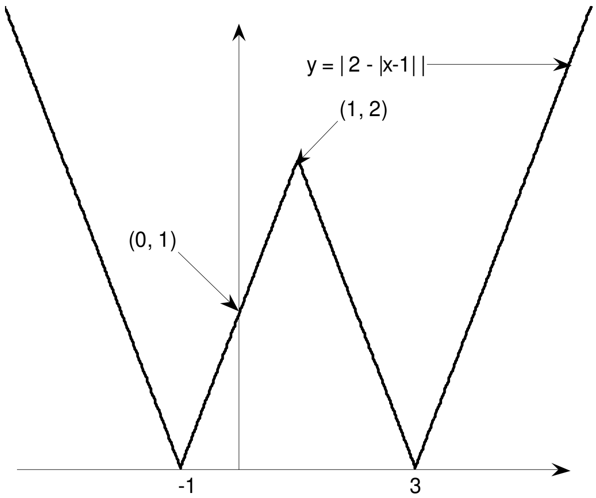
(i)



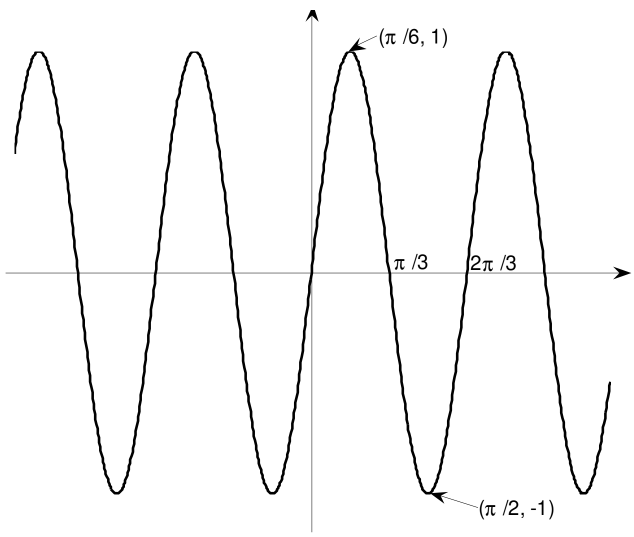
(iv)



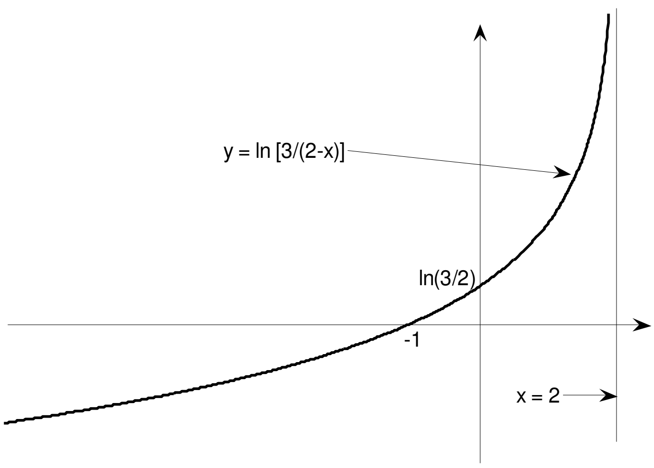
(ii)



(v)



(iii)



Q5 (15 pts) Let (C) be the curve of equation $2y^2 + yx = x^2 + 2$. **(i)** Find equations for the tangent lines to the curve (C) at the points $P_1(0, 1)$ and $P_2(0, -1)$. **(ii)** Is the graph of (C) concave up or concave down at $P_1(0, 1)$ and $P_2(0, -1)$? Explain!

By implicit differentiation we obtain: $4yy' + y'x + y = 2x$ (1). Hence, $y' = (2x - y)/(x + 4y)$ (2). From Eq. (2) one sees that the tangent lines at $P_1(0, 1)$ and $P_2(0, -1)$ have the same slope, say, $-1/4$. The tangent line at the point $P_1(0, 1)$ has the equation $y = -(x/4) + 1$ and that at the point $P_2(0, -1)$ has the equation $y = -(x/4) - 1$.

The concavity of a graph is determined by the sign of the second derivative. From Eq. (1), it is straighforward to obtain the second derivative y'' again by implicit differentiation: $(x + 4y)\, y'' + (1 + 4y')\, y' = 2$. Hence, $y'' = [2 - (1 + 4y')\, y']/(x + 4y)$. At the point $P_1(0, 1)$, with $x = 0$, $y = 1$ & $y' = -1/4$, we get $y'' = 1/2$ so the graph is concave up at P_1. At the point $P_2(0, -1)$, with $x = 0$, $y = -1$ & $y' = -1/4$, we get $y'' = -1/2$ so the graph is concave down at P_2.

Q6 (15 pts) Find the minimum distance from the point $P(3, 0)$ to the curve of equation $y = x^2$. Determine the point on $y = x^2$ that is closest to $P(3, 0)$.
(**Hint:** $(4x^3 + 2x - 6)$ is divisible by $(x - 1)$).

Let (x, x^2) be a point on the parabola $y = x^2$. The square of the distance from $(3, 0)$ to (x, x^2) is the function $f(x)$ given by:

$f(x) = (x - 3)^2 + (x^2 - 0)^2 = x^4 + x^2 - 6x + 9$. $f'(x) = 4x^3 + 2x - 6 = 2(x - 1)(2x^2 + 2x + 3)$.
Since $(2x^2 + 2x + 3) = 2[x + (1/2)]^2 + (5/2) > 0$ for all x, $x = 1$ is the unique critical point. With $f''(x) = 12x^2 + 2$ we have $f''(1) = 14 > 0$. By the Second-Derivative Test, $f(x)$ has a local minimum at $x = 1$. Hence, the minimum distance from the point $P(3, 0)$ to the curve of equation $y = x^2$ is $\sqrt{f(1)} = \sqrt{5}$. The closest point on the parabola $y = x^2$ to the point $P(3, 0)$ is the point $(1, 1)$.

Başkent University
MAT 151 – Calculus I
Fall 2008 – Midterm Exam

November 17, 2008

Student ID	Name / Surname	Department

Duration: 100 minutes.

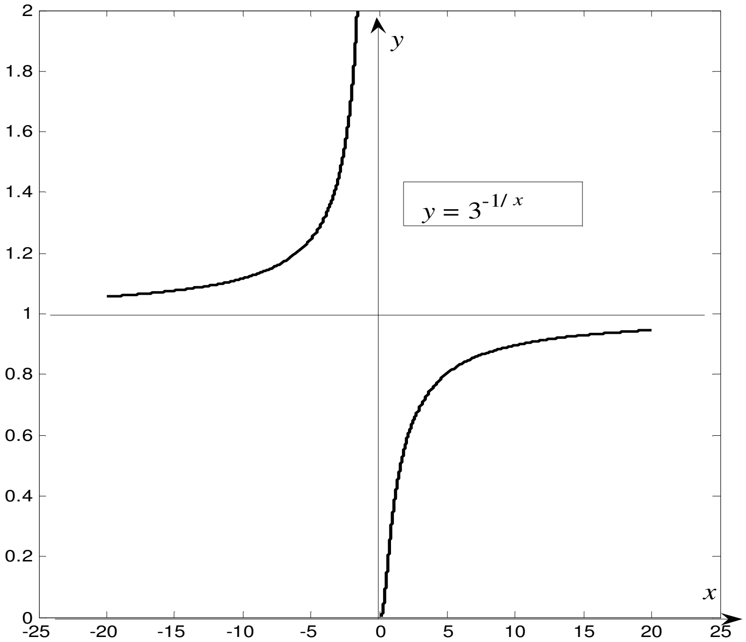
- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **6 Q**uestions;
- In the table below, indicate clearly the **Part** of the **Q**uestion that you choose to solve, if any exists;
- Do not ask any question to anybody other than your instructor.

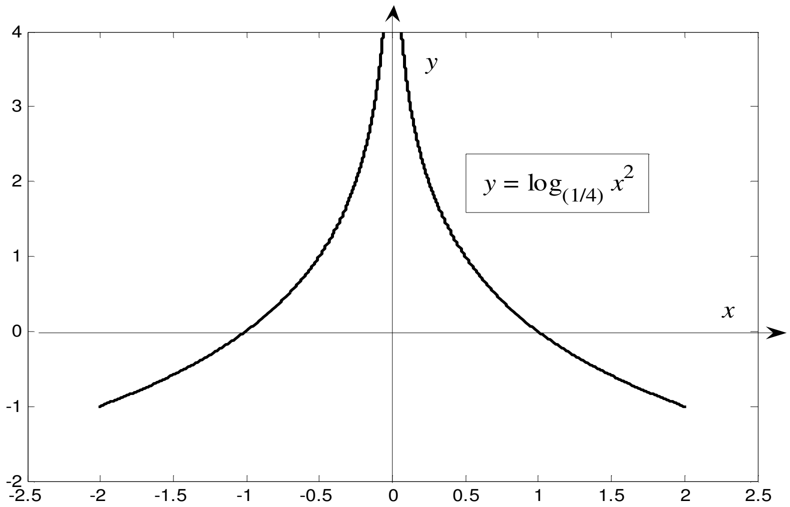
Q	1	2	3	4	5	6	Total
Part	~~ ~~	~~ ~~	~~ ~~	~~ ~~			
Grade							

GOOD LUCK!

Q1. (12 pts) Sketch ROUGHLY the graphs of the following functions.

(i) $f(x) = 3^{-1/x}$, **(ii)** $f(x) = \operatorname{arccsc} x$, **(iii)** $f(x) = \log_{(1/4)} x^2$, **(iv)** $y = \dfrac{x-1}{x+1}$.

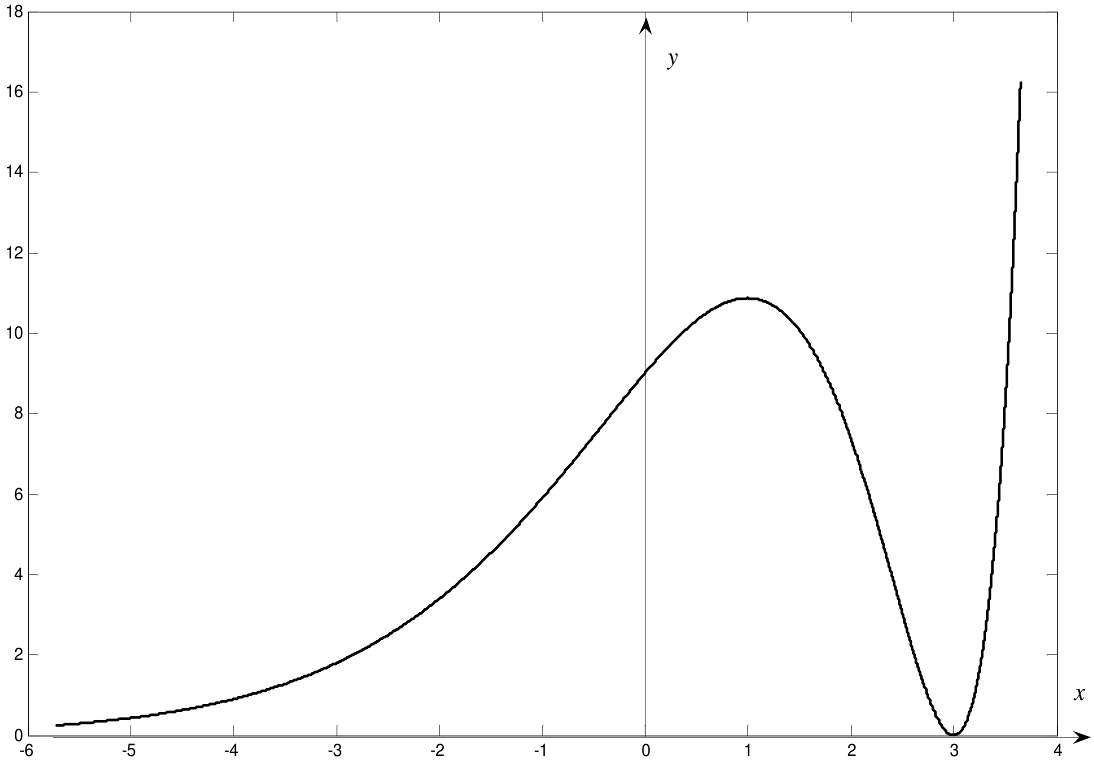




Q2. (25 pts)

(i) Sketch the graph of $y = e^x (x-3)^2$ in FULL DETAIL and write down equations for the tangent and normal lines at the point $(2, e^2)$.

(ii) Find the asymptotes of: **(a)** $y = f(x) = \dfrac{x^3 + x}{x^2 - 1}$, **(b)** $y = g(x) = \dfrac{x^2}{x(x+1)}$.



(i) $y = e^x (x-3)^2$

$D_f = (-\infty, \infty)$

$\lim\limits_{x\to\infty} f(x) = \infty$,

$$\lim_{x\to-\infty} f(x) \to 0\times\infty \Rightarrow \lim_{x\to-\infty} f(x) = \frac{(x-3)^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2(x-3)}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{\infty} = 0^+ \Rightarrow$$

$y = 0$ is a horizontal asymptote.

$y' = e^x (x-3)(x-1)$ → two critical points: $x = 3$, $x = 1$

$y'' = e^x (x - \sqrt{2} - 1)(x + \sqrt{2} - 1)$ → two points of inflection at: $x = 1 + \sqrt{2}$, $x = 1 - \sqrt{2}$

Q3. (20 pts)

Find ANY TWO of the following limits (i) – (iv):

(i) $\lim\limits_{x\to 0^+} x^{\sin x}$, **(ii)** $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x^2}\right)^{x^2}$, **(iii)** $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\cot x\right)$, **(iv)** $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}+\ln x\right)$.

Find ANY TWO of the following limits (DO NOT USE L'HÔPITAL'S RULE):

(v) $\lim\limits_{x\to\infty}\dfrac{\sin x\tan(1/x)}{x\{\sqrt{1+(1/x)}-1\}}$, **(vi)** $\lim\limits_{x\to\infty} x\left[\cos\left(\dfrac{1}{x}\right)-1\right]\arctan x$,

(vii) $\lim\limits_{x\to 1^+}\left(\dfrac{x^3-x^2+x-1}{x-1}\right)\operatorname{arccot} x$.

(i) $\lim\limits_{x\to 0^+} x^{\sin x}=1$, **(ii)** $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x^2}\right)^{x^2}=\dfrac{1}{\mathrm{e}}$, **(iii)** $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\cot x\right)=0$,

(iv)

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{(-1/x^2)}=-\lim_{x\to 0^+} x=0\Rightarrow \lim_{x\to 0^+}\left(\frac{1}{x}+\ln x\right)=\lim_{x\to 0^+}\left(\frac{1+x\ln x}{x}\right)=+\infty$$

(v) Set $u = 1/x$,

$$\lim_{x\to\infty}\frac{\sin x\tan(1/x)}{x\{\sqrt{1+(1/x)}-1\}}=\underbrace{\left(\lim_{x\to\infty}\frac{\sin x}{x}\right)}_{0\text{ (By the Sandwich Th.)}}\underbrace{\left(\lim_{u\to 0^+}\frac{\tan u}{u}\right)}_{1}\underbrace{\left(\lim_{u\to 0^+}\frac{u}{\sqrt{u+1}-1}\right)}_{2}=0$$

(vi) Set $u = 1/x$, $\lim\limits_{x\to\infty} x\left[\cos\left(\dfrac{1}{x}\right)-1\right]\arctan x=\underbrace{\left(\lim\limits_{u\to 0^+}\dfrac{\cos u-1}{u}\right)}_{0}\underbrace{\left(\lim\limits_{x\to\infty}\arctan x\right)}_{\pi/2}=0$

(vii) $\lim\limits_{x\to 1^+}\left(\dfrac{x^3-x^2+x-1}{x-1}\right)\operatorname{arccot} x=\underbrace{\left(\lim\limits_{x\to 1^+}(x^2+1)\right)}_{2}\underbrace{\left(\lim\limits_{x\to 1^+}\operatorname{arccot} x\right)}_{\pi/4}=\pi/2$.

Q4. (18 pts) Find $\mathrm{d}y/\mathrm{d}x$ if $y = f(x)$ is
(i) $y = \operatorname{arcsec} x$, **(ii)** $y = 2^{\sin x}$, **(iii)** $y = (\tan x)^x$.

Find the following integrals: **(iv)** $\int \csc x \cot x \,\mathrm{d}x$, **(v)** $\int x\, 5^{x^2}\,\mathrm{d}x$, **(vi)** $\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$.

Q5. Answer only ONE of the following three parts:

(A) (15 pts) We are asked to design a one-liter oil can shaped like a right-circular cylinder. What dimensions will use the least material?

(B) (15 pts) Find the largest volume of the right-circular cone that can be inscribed in a sphere of radius 3. (Recall that the volume of the cone is $V = (1/3)\times(\text{base area})\times(\text{height})$.)

(C) (20 pts) A conical paper cup 4 inches across the top and 6 inches deep leaks water at a rate of 1 cubic inch per minute. At what rate does the level of the water drop
(i) When the water is 2 inches deep?
(ii) When the cup is half full?

(A) The weight of the material is proportional to its total area A. With $\pi r^2 h = 1$, we obtain: $A(r) = 2\pi r^2 + 2\pi rh = 2\pi r^2 + (2/r),\ r \in (0,\infty)$. $\mathrm{d}A/\mathrm{d}r = 4\pi r - (2/r^2)$. A has its absolute minimum at $r = 1/(2\pi)^{1/3}$, $h = (2\pi)^{2/3}/\pi$.

(B) The relation between the radius and the height of the cone is $h/\sqrt{r^2+h^2} = \sqrt{r^2+h^2}/6 \Rightarrow r^2 = 6h - h^2$, where $6 = 2\times 3$ is the diameter of the sphere. The volume of the cone is then $V(h) = \pi r^2 h/3 = \pi(6h^2 - h^3)/3,\ h \in [0,6]$. V has its absolute maximum at $h = 4,\ r = 2\sqrt{2}$ with $V_{\max} = 32\pi/3$.

Q6. (20 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) **(i)** Find the range of $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 1$, $x \in [-3,2]$.

(ii) Is the range R_f of f an interval? If so, what are its endpoints? Expalin!

(B) Let $f(x) = \begin{cases} \frac{1-\cos x}{x}, & x \neq 0 \\ 2 \quad , & x = 0 \end{cases}$.

Is it possible to <u>redefine</u> $f(x)$ at $x = 0$ so that it becomes differentiable there? And if it is, can you find $f\,'(0)$ after redefining f at $x = 0$?

(A) **(i)** The given function is continuous on the closed domain $[-3,2]$, so it assumes there an absolute maximum and an absolute minimum values. Its derivative is $f'(x) = (x+2)(x-1)$.

Evaluating $f(-3) = 5/2$, $f(-3) = 5/3$, $f(-2) = 13/3$, $f(1) = -1/6$. Hence, the range is: $R_f = [-1/6, 13/3]$. By the intermediate Value Theorem, f assumes any value between $-1/6$ and $13/3$, so R_f is an interval.

(B) Since $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = 0$, we set $f(0) = 0$ to make $f(x)$ continuous at $x = 0$. With this new value of $f(0)$ we evaluate the limit:

$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{\sin h}{2h} = \frac{1}{2}$. Since the limit exists, $f\,'(0) = 1/2$.

Başkent University
MAT 151 – Calculus I
Fall 2008 – Final Exam

January 09, 2009

Student ID	Name / Surname	Department

Duration: 120 minutes.

- Write your derivations clearly. No CREDIT will be given for answers not supported by work;
- Do not use an electronic calculator;
- There are **7 Q**uestions;
- In the table below, indicate clearly the **Part** of the **Q**uestion that you choose to solve, if any exists;
- Do not ask any question to anybody other than your instructor.

Q	1	2	3	4	5	6	7	Total
Part								
Grade								

GOOD LUCK!

Q1. (25 pts)

(i) Use the Limit Comparison Test to decide if the following improper integral converges or diverges: $\displaystyle\int_1^\infty \frac{x+\sin x\cdot \operatorname{arcsec} x}{x^3+x+\ln x}\,\mathrm{d}x$.

(ii) Decide by direct computation if the following improper integrals converge or diverge:

(a) $\displaystyle\int_0^1 \ln x\,\mathrm{d}x$ **(b)** $\displaystyle\int_0^\infty \frac{x}{x^4+1}\,\mathrm{d}x$.

(i) Let $f=\dfrac{x+\sin x\cdot \operatorname{arcsec} x}{x^3+x+\ln x}$ and choose $g=\dfrac{1}{x^2}$. Then $f>0$ and $g>0\ \forall\ x\geq 1$, and

$\displaystyle\lim_{x\to\infty}\frac{f}{g}=\lim_{x\to\infty}\frac{1+\dfrac{\sin x\cdot\operatorname{arcsec} x}{x}}{1+\dfrac{1}{x^2}+\dfrac{\ln x}{x^3}}=1>0$. Hence, since $\displaystyle\int_1^\infty g\,\mathrm{d}x=\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent

$\Rightarrow \displaystyle\int_1^\infty f\,\mathrm{d}x$ is convergent.

(ii) (a) $\displaystyle\int_0^1 \ln x\,\mathrm{d}x=\lim_{p\to 0^+}\left(\int_p^1 \ln x\,\mathrm{d}x\right)=\lim_{p\to 0^+}\left(x\ln x-x\Big|_p^1\right)=\lim_{p\to 0^+}(-1-p\ln p+p)=-1$

with $\displaystyle\lim_{p\to 0^+} p\ln p=\lim_{p\to 0^+}\frac{\ln p}{1/p}=\lim_{p\to 0^+}\frac{1/p}{-1/p^2}=\lim_{p\to 0^+} p=0$.

(ii) (b) Set $u=x^2\Rightarrow \mathrm{d}u=2x\,\mathrm{d}x\Rightarrow$

$$\int_0^\infty \frac{x}{x^4+1}\,\mathrm{d}x=\frac{1}{2}\lim_{p\to 0^+}\left(\int_0^p \frac{1}{u^2+1}\,\mathrm{d}u\right)=\frac{1}{2}\lim_{p\to 0^+}\arctan\Big|_0^p=\frac{\pi}{4}.$$

Q2. (25 pts) Find the following integrals. Answer ANY THREE.

(i) $\displaystyle\int_0^{\pi/4} \tan^4 x \, dx$, **(ii)** $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2} \, dx$, **(iii)** $\displaystyle\int \frac{x}{x^2+x+1} \, dx$,

(iv) $\displaystyle\int_{-\pi/4}^{0} 2x\sqrt{1-\cos(4x)} \, dx$, **(v)** $\displaystyle\int x\sec^2 x \, dx$, **(vi)** $\displaystyle\int \left(\frac{e^{\sqrt{x}}}{\sqrt{x}} + \frac{2x+1}{x^2-1} \right) dx$.

(i) $$\int \tan^4 x \, dx = \underbrace{\int \tan^2 x \sec^2 x \, dx}_{u=\tan x \Rightarrow du=\sec^2 x\, dx} - \int (\sec^2 x - 1)\, dx = \frac{\tan^3 x}{3} - \tan x + x + C$$

$$\int_0^{\pi/4} \tan^4 x \, dx = \frac{\pi}{4} - \frac{2}{3}.$$

(ii) Let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \, d\theta \Rightarrow$

$$\int \frac{\sqrt{4-x^2}}{x^2}\, dx = \int \cot^2\theta \, d\theta == \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{4-x^2}}{x} - \arcsin\left(\frac{x}{2}\right) + C$$

(iii) $$\int \frac{x}{x^2+x+1}\, dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\, dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\, dx$$

$$= \frac{\ln(x^2+x+1)}{2} - \frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} + C.$$

(iv) $$\int_{-\pi/4}^{0} 2x\sqrt{\underbrace{1-\cos(4x)}_{2\sin^2(2x)}}\, dx = 2\sqrt{2}\int_{-\pi/4}^{0} x\,|\sin(2x)|\, dx = -2\sqrt{2}\int_{-\pi/4}^{0} x\sin(2x)\, dx$$

$$\underbrace{\int x\sin(2x)\, dx}_{\textit{Integration by Parts}} = -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C \Rightarrow \int_{-\pi/4}^{0} 2x\sqrt{1-\cos(4x)}\, dx = -\frac{\sqrt{2}}{2}.$$

(v) $$\int \underbrace{x}_{u} \underbrace{\sec^2 x\, dx}_{dv} = x\tan x - \int \tan x\, dx = x\tan x + \ln|\cos x| + C.$$

(vi) $$\int \left(\frac{e^{\sqrt{x}}}{\underbrace{\sqrt{x}}_{u=\sqrt{x}}} + \frac{2x+1}{x^2-1} \right) dx = 2e^{\sqrt{x}} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Q3. (15 pts) Find the following limits. Answer ANY THREE.

(i) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$, **(ii)** $\lim\limits_{x\to\infty}\dfrac{\operatorname{arcsec} x}{\arctan x}$,

(iii) $\lim\limits_{x\to 1}(1-\ln x)^{4/(x-1)}$, **(iv)** $\lim\limits_{x\to 0}(x\ln x)\cdot\left(\dfrac{\sin(x^2)}{x^2}\right)$.

(i) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)=\lim\limits_{x\to 0}\dfrac{\sin x - x}{x\sin x}=\cdots=0$.

(ii) $\lim\limits_{x\to\infty}\dfrac{\operatorname{arcsec} x}{\arctan x}=\dfrac{\pi/2}{\pi/2}=1$.

(iii) $F=(1-\ln x)^{4/(x-1)} \Rightarrow \lim\limits_{x\to 1}\ln F=\lim\limits_{x\to 1}\dfrac{4\ln(1-\ln x)}{x-1}=-4\lim\limits_{x\to 1}\dfrac{1/x}{1-\ln x}=-4 \Rightarrow$

$\lim\limits_{x\to 1}F=\mathrm{e}^{-4}$.

(iv) $\lim\limits_{x\to 0}(x\ln x)\cdot\left(\dfrac{\sin(x^2)}{x^2}\right)=0$.

Q4. Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) (20 pts) Find the arc length of the curve (C): $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$, which is the closed curve $x^{2/3} + y^{2/3} = 1$.

(B) (15 pts) Find the arc length of the circle $x^2 + y^2 = a^2$.

(A) $L = 4\int_0^1 \sqrt{1+y'^2}\,\mathrm{d}x$. We find y' by Implicit Differentiation:

$$(2/3)x^{-1/3} + (2/3)y^{-1/3}\,y' = 0 \Rightarrow y' = -\,y^{1/3}/x^{1/3} \Rightarrow$$

$$\sqrt{1+y'^2} = \sqrt{1+\frac{y^{2/3}}{x^{2/3}}} = \sqrt{1+\frac{1-x^{2/3}}{x^{2/3}}} = \sqrt{\frac{1}{x^{2/3}}} = x^{-1/3} \Rightarrow$$

$$L = 4\int_0^1 \sqrt{1+y'^2}\,\mathrm{d}x = 4\int_0^1 x^{-1/3}\,\mathrm{d}x = 4\left.\frac{x^{2/3}}{2/3}\right|_0^1 = \frac{12}{2}(1-0) = 6.$$

(B) $L = 4\int_0^a \sqrt{1+y'^2}\,\mathrm{d}x$. With $y = \sqrt{a^2 - x^2}$ we have $y' = -x/\sqrt{a^2-x^2}$.

$$\sqrt{1+y'^2} = \sqrt{1+\frac{x^2}{a^2-x^2}} = \frac{1}{\sqrt{1-(x/a)^2}} \Rightarrow$$

$$L = 4\int_0^a \sqrt{1+y'^2}\,\mathrm{d}x = 4\int_0^a \underbrace{\frac{1}{\sqrt{1-(x/a)^2}}}_{u=x/a}\,\mathrm{d}x = 4a\int_0^1 \underbrace{\frac{1}{\sqrt{1-u^2}}}_{u=x/a}\,\mathrm{d}u = 4a\lim_{p\to 1^-}\int_0^p \underbrace{\frac{1}{\sqrt{1-u^2}}}_{u=x/a}\,\mathrm{d}u$$

$$= 4a\lim_{p\to 1^-}\int_0^p \underbrace{\frac{1}{\sqrt{1-u^2}}}_{u=x/a}\,\mathrm{d}u = 4a\lim_{p\to 1^-}\left(\arcsin u\Big|_0^p\right) = 4a(\pi/2) = 2\pi a.$$

Q5. (25 pts) Answer either **Part (A)** or **Part (B)**, but NOT both.

(A) The region under the graph of $y = \dfrac{\sqrt{\ln x}}{x}$, $x \geq 1$, is revolved about the x-axis. Sketch the solid of revolution and find its volume.

(B) Let (R_1) be the region bounded by the curve $y = -x^2 + x$ and the x-axis; (R_2) be the region bounded by $y = \sqrt{x}$, the y-axis and the line $y = 1$.
Sketch the solid of revolution and find its volume
(i) by the Washer (Slicing) Method if (R_1) is revolved about the x-axis,
(ii) by the Shell Method if (R_2) is revolved about the y-axis.

(A) $V = \pi \displaystyle\int_0^\infty \frac{\ln x}{x^2}\,\mathrm{d}x = \pi \lim_{p\to\infty} \int_0^p \underbrace{\ln x}_{u}\, \underbrace{\frac{\mathrm{d}x}{x^2}}_{\mathrm{d}v} = -\pi \lim_{p\to\infty}\left(\frac{\ln p}{p} + \frac{1}{p} - 1\right) = \pi$

$\left(\lim_{p\to\infty} \dfrac{\ln p}{p} = \lim_{p\to\infty} \dfrac{1/p}{1} = 0\right)$.

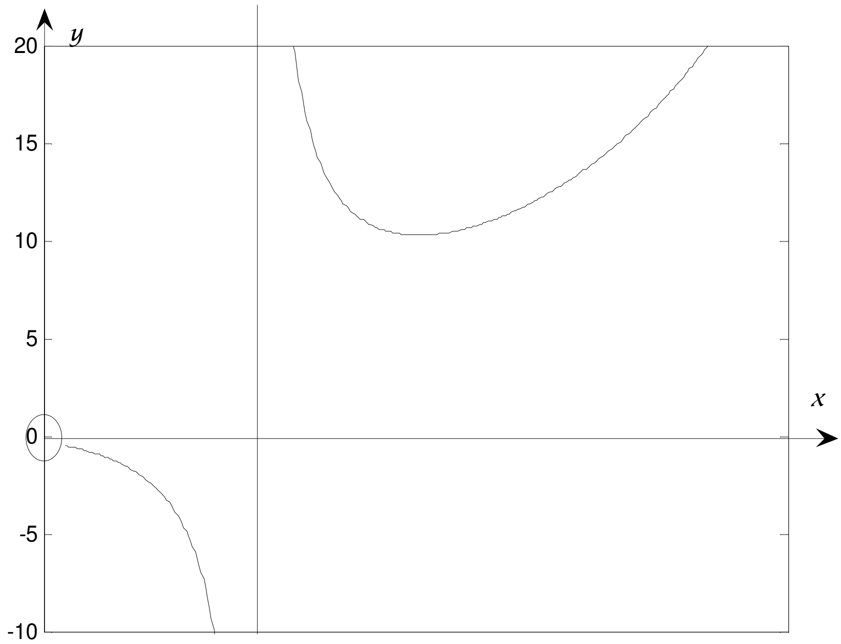
(B) (i) $V = \pi \displaystyle\int_0^1 (-x^2 + x)^2\,\mathrm{d}x = \frac{\pi}{30}$.

(B) (ii) $V = 2\pi \displaystyle\int_0^1 x(1 - \sqrt{x})\,\mathrm{d}x = \frac{\pi}{5}$.

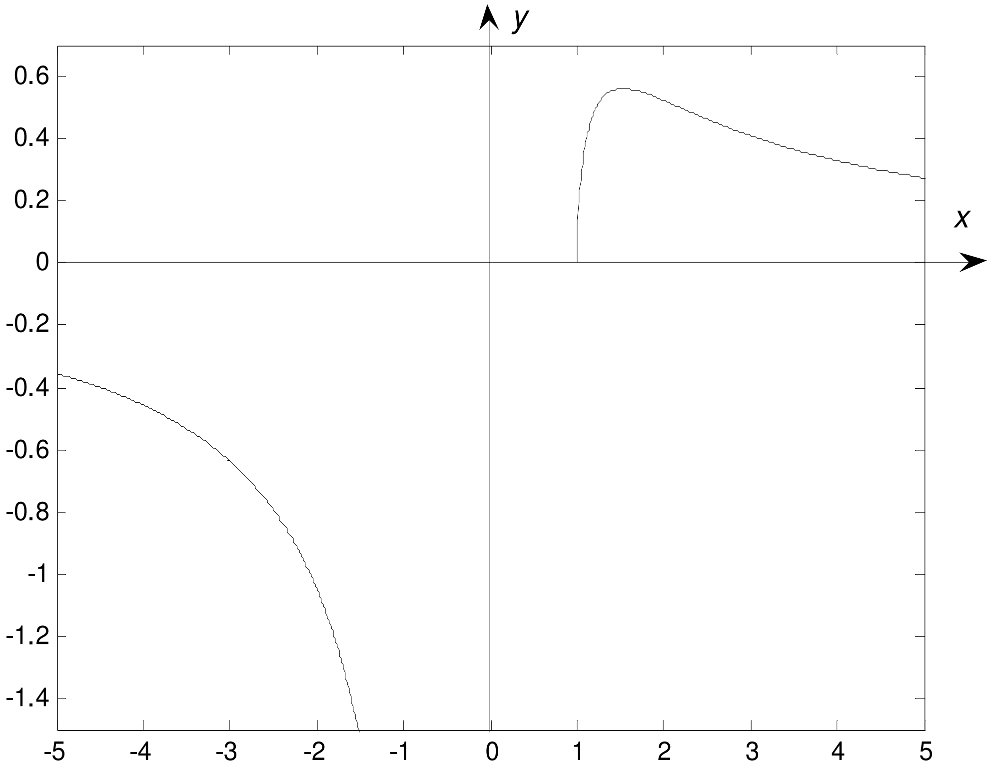
Q6. (15 pts) Sketch <u>roughly</u> the graphs of <u>any four</u> of the following functions:

(i) $y = \operatorname{arccot} x$, **(ii)** $y = \dfrac{1}{x} + x$, **(iii)** $y = \dfrac{e^x}{\ln x}$, **(iv)** $y = \sin|x|$, **(v)** $y = \dfrac{\operatorname{arcsec} x}{x}$.

(iii)



(v)



Q7. (15 pts) Find $\mathrm{d}y/\mathrm{d}x$ if y is

(i) $y = x^2 \cdot 2^{\cos x}$, **(ii)** $y = (x^2 + x + 1)^{x^2}$, **(iii)** $y = \dfrac{\operatorname{arcsec} x}{x^2 + x}$.