# 3

# Random Variables and Probability Distributions

## CHAPTER OUTLINE

## LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Determine probabilities for discrete random variables from probability mass functions and for continuous random variables from probability density functions, and use cumulative distribution functions in both cases.
2. Calculate means and variances for discrete and continuous random variables.
3. Understand the assumptions for each of the probability distributions presented.
4. Select an appropriate probability distribution to calculate probabilities in specific applications.
5. Use the table (or software) for the cumulative distribution function of a standard normal distribution to calculate probabilities.
6. Approximate probabilities for binomial and Poisson distributions.
7. Interpret and calculate covariances and correlations between random variables.
8. Calculate means and variances for linear combinations of random variables.
9. Approximate means and variances for general functions of several random variables.
10. Understand statistics and the central limit theorem.

# 3-1 Introduction

## ☐ Experiment

•The measurement of current in a thin copper wire(细铜丝中的电流) is an example of an experiment.

•However, the results might differ slightly in day-to-day replicates（重复试验） of the measurement because of small variations in variables that are not controlled in our experiment—

- changes in ambient temperatures(周围温度的改变),
- slight variations in gauge（测量仪器的变化）
- and small impurities in the chemical composition of the wire（化学成分的杂质不同），
- current source drifts(电源不稳定), and so forth.

## ☐ Random

•In some cases, the random variations that we experience are small enough, relative to our experimental goals, that they can be ignored.

•However, the variation is almost always present and its magnitude(大小，量纲) can be large enough that the important conclusions from the experiment are not obvious.

# 3-1 Introduction

## ☐ Random experiment

- An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.
  - ✓ **We might select one part from a day's production and very accurately measure a dimensional length.**
  - ✓ **Although we hope that the manufacturing operation produces identical parts consistently, in practice there are often small variations in the actual measured lengths due to many causes—vibrations(振动), temperature fluctuations(温度波动), operator differences(操作员不同), calibrations（刻度） of equipment and gauges, cutting tool wear, bearing wear, and raw material（原材料）changes.**
  - ✓ **Even the measurement procedure can produce variations in the final results.**

when magnified 100 times

- ■ No matter how carefully our experiment is designed and conducted, variations often occur.
- ■ Our goal is to understand, quantify, and model the type of variations that we often encounter.
- ➢ When we incorporate the variation into our thinking and analyses, we can make informed judgments from our results that are not invalidated by the variation.把变异结合到思考和分析中，从由于变异而无效的结果中做出有价值的判断。

# 3-1 Introduction

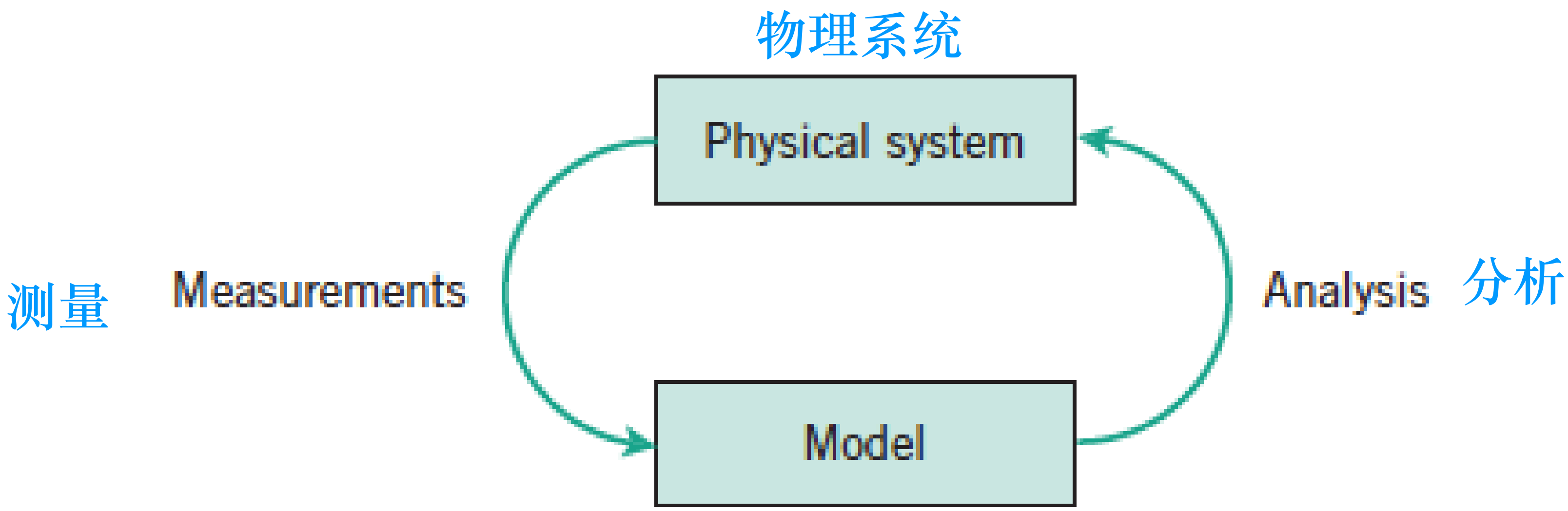



Figure 3-1 Continuous iteration between model and physical system. 模型与物理系统之间的连续迭代

In the above process, one item is missing. What is it?

Vocabulary： iteration迭代 algorithm 算法

# 3-1 Introduction

A model that combines variables that cannot be controlled (noise) into a control variable to produce the output of a system

把不能控制的变量（噪声）结合到控制变量中来产生系统输出结果的模型

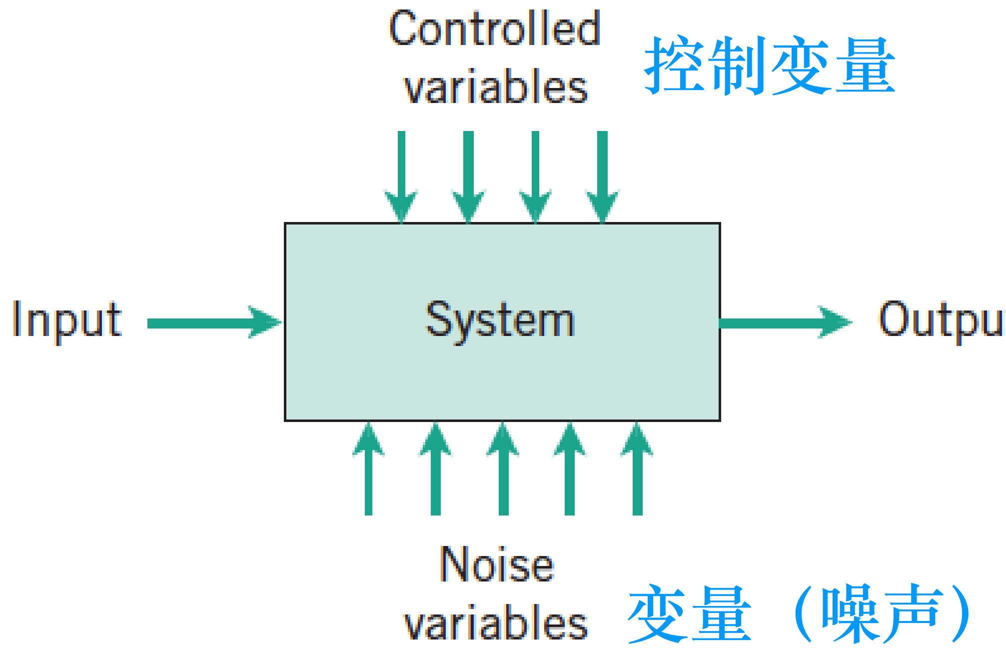



Figure 3-2 Noise variables affect the transformation of inputs to outputs.

由于存在噪声，同样的控制变量设置并不能得到相同的系统输出

# 3-1 Introduction

- For the example of measuring current in a copper wire, our model for the system might simply be Ohm’s law - a suitable approximation.
- However, if the variations are large relative to the intended use of the device under study, we might need to extend our model to include the variation.

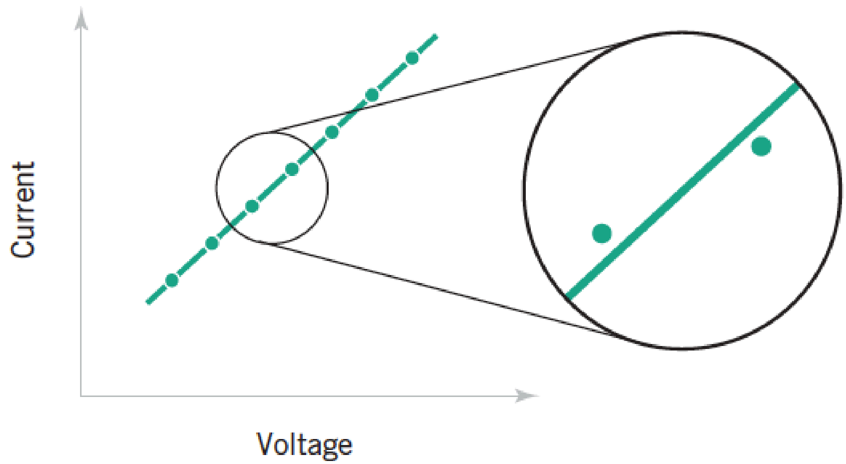
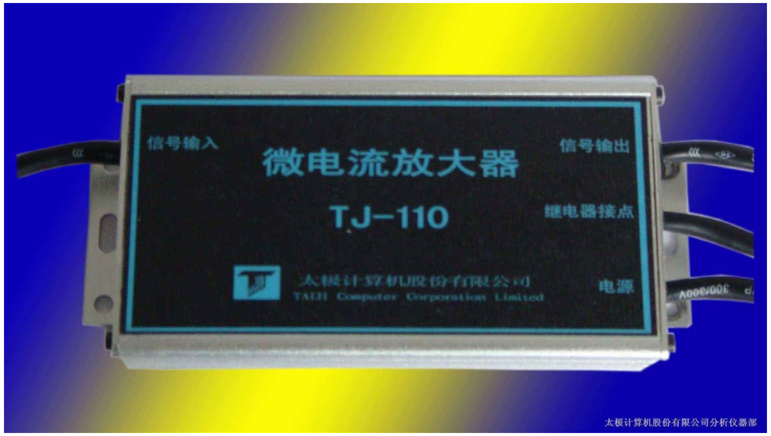



Figure 3-3 A closer examination of the system identifies deviations from the model.




An amplifier

- It is often difficult to speculate （推测） on the magnitude of the variations without empirical measurements.
- With sufficient measurements, however, we can approximate the magnitude of the variation and consider its effect on the performance of other devices, such as amplifiers, in the circuit.凭借足够的测量数据，可以估算出该变异的幅度，并考量其对电路中其他器件（如放大器）性能的影响。

# 3-2 Random Variables

- In an experiment, a measurement is usually denoted by a variable such as $X$.

- In a random experiment, a variable whose measured value can change (from one replicate重复 of the experiment to another) is referred to as a **random variable**.

# 3-2 Random Variables

A **random variable** is a numerical variable whose measured value can change from one replicate of the experiment to another.

随机变量是一个数值变量，它的测量值在重复的实验过程中是变化的。

A **discrete** random variable is a random variable with a finite (or countably infinite) set of real numbers for its range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples of **continuous** random variables:

electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:

number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error

表面的刮痕数，检验的1000个零件中有缺陷的比例，错误接收的发射比特数

# 3-3 Probability

- Used to quantify likelihood or chance
- Used to represent risk or uncertainty in engineering applications
- Can be interpreted as our **degree of belief** or **relative frequency**

# 3-3 Probability

- Probability statements describe the likelihood that particular values occur.
- The likelihood is quantified by assigning a number from the interval [0, 1] to the set of values (or a percentage from 0 to 100%).
- Higher numbers indicate that the set of values is more likely.

# 3-3 Probability

- The probability of a result can be interpreted as our subjective probability, or degree of belief, that the result will occur. (主观概率或者相信程度)
- Different individuals will no doubt assign different probabilities to the same result.
- Another interpretation of probability can be based on repeated replicates of the random experiment. The probability of a result is interpreted as the proportion of times the result will occur in repeated replicates of the random experiment.
- For example, if we assign probability 0.25 to the result that a part length is between 10.8 and 11.2 millimeters, we might interpret this assignment as follows.
- If we repeatedly manufacture parts (replicate the random experiment an infinite number of times), 25% of them will have lengths in this interval. This example provides a relative frequency interpretation of probability.
- The proportion, or relative frequency, of repeated replicates that fall in the interval will be 0.25. Note that this interpretation uses **a long-run proportion**, the proportion from an infinite number of replicates. With a small number of replicates, the proportion of lengths that actually fall in the interval might differ from 0.25.

# 3-3 Probability

- A probability is usually expressed in terms of a random variable.
- For the part length example, $X$ denotes the part length and the probability statement can be written in either of the following forms

$$P(X \in [10.8, 11.2]) = 0.25 \qquad \text{or} \qquad P(10.8 \leq X \leq 11.2) = 0.25$$

- Both equations state that the probability that the random variable $X$ assumes a value in [10.8, 11.2] is 0.25.

# 3-3 Probability

## Complement of an Event事件的补集

• Given a set $E$, the complement of $E$ is the set of elements that are not in $E$. The **complement** is denoted as $E'$.

## Mutually Exclusive Events互不相容的事件

• The sets $E_1$, $E_2$,...,$E_k$ are **mutually exclusive** if the intersection of any pair is empty. That is, each element is in one and only one of the sets $E_1$, $E_2$,..., $E_k$.

# 3-3 Probability

## Probability Properties

1. $P(X \in R) = 1$, where $R$ is the set of real numbers. 实数集
2. $0 \leq P(X \in E) \leq 1$ for any set $E$. (3-1)
3. If $E_1, E_2, \ldots, E_k$ are mutually exclusive sets, 互不相容
   $P(X \in E_1 \cup E_2 \cup \ldots \cup E_k) = P(X \in E_1) + \cdots + P(X \in E_k)$.

# 3-3 Probability

## Events

• A measured value is not always obtained from an experiment. Sometimes, the result is only classified (into one of several possible categories分成了几种可能的类).

• These categories are often referred to as **events**.

## Illustrations

- The current measurement might only be recorded as ***low, medium,* or *high***
- A manufactured electronic component might be classified only as **defective or not**
- Either a message is sent through a network or not.

# 3-4 Continuous Random Variables

## 3-4.1 Probability Density Function

• The **probability distribution** or simply **distribution** of a random variable *X* is a description of the set of the probabilities associated with the possible values for *X*.

➢ 概率分布是把概率的集合和X的可能取值联系起来的一种描述。

➢ 一个随机变量的概率分布可以用多种方式来具体表现。

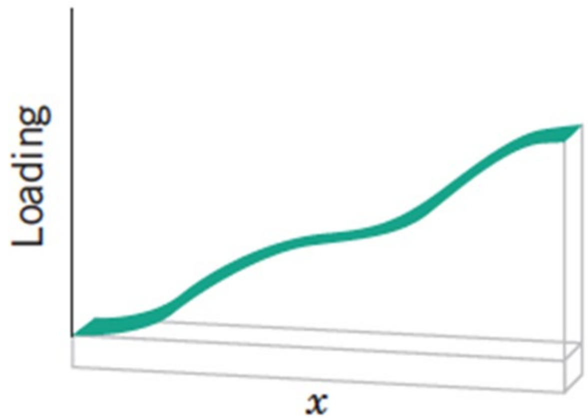



Figure 3-5 Density function of a loading on a long, thin beam.

细长横梁上载荷的密度函数

# 3-4 Continuous Random Variables

- For example, consider the density of a loading on a long, thin beam as shown in Fig. 3-5.

- For any point ***x*** along the beam, the density can be described by a function (in grams/cm).

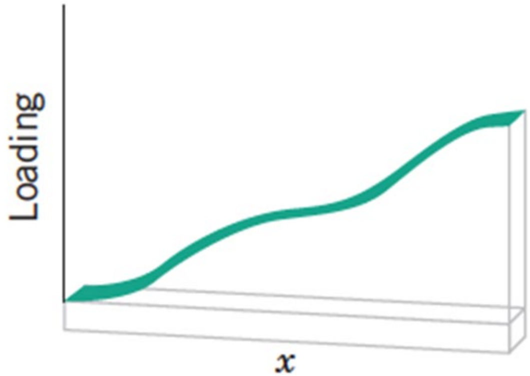



Figure 3-5 Density function of a loading on a long, thin beam.

- Intervals with large loadings correspond to large values for the function. The total loading between points *a* and *b* is determined as the integral of the density function from *a* to *b*.

- This integral is the area under the density function over this interval, and it can be loosely interpreted as the sum of all the loadings over this interval.

# 3-4 Continuous Random Variables

## 3-4.1 Probability Density Function

The **probability density function** (or pdf) $f(x)$ of a continuous random variable $X$ is used to determine probabilities as follows:

$$P(a < X < b) = \int_a^b f(x)\,dx \qquad (3\text{-}2)$$

The properties of the pdf are

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)dx = 1$

# 3-4 Continuous Random Variables

## 3-4.1 Probability Density Function

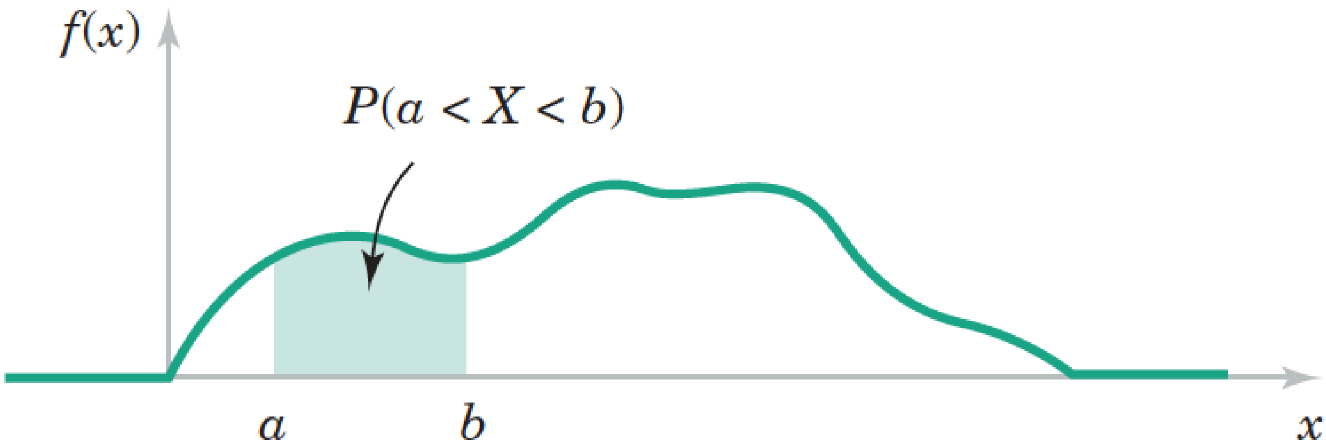



Figure 3-6 Probability determined from the area under $f(x)$.

The probability is determined by the area under the curve of the function $f(x)$ enclosed with the coordinate axis

由函数f(x)的曲线下与坐标轴围成的面积确定的概率

# 3-4 Continuous Random Variables

## 3-4.1 Probability Density Function

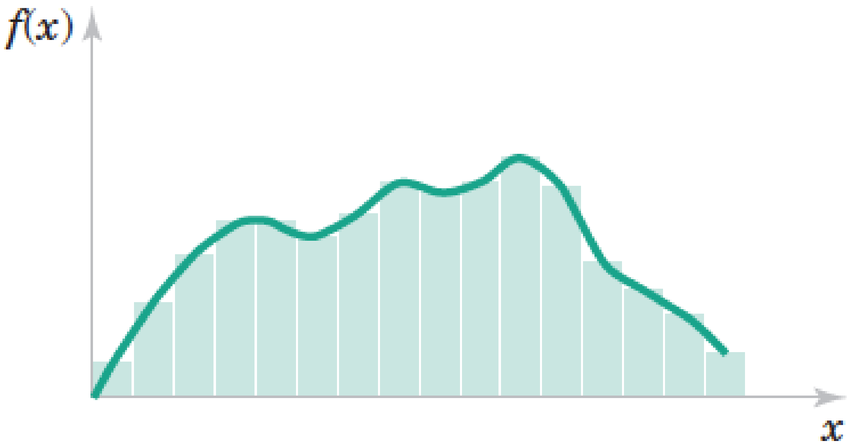



Figure 3-7 A histogram approximates a probability density function. The area of each bar equals the relative frequency of the interval. The area under $f(x)$ over any interval equals the probability of the interval.

直方图近似表示概率密度函数，每一个块的面积等于这个区间的频率。每一个区间上f(x)下的面积等于这个区间的概率。

# 3-4 Continuous Random Variables

## 3-4.1 Probability Density Function

If $X$ is a continuous random variable, for any $x_1$ and $x_2$,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

$$P(X = x) = 0$$

# 3-4 Continuous Random Variables

## 电线中的电流

EXAMPLE 3-2
Current in a Wire

Define the random variable and distribution.

Write the probability statement.

Compute the probability.

细铜丝中测出的电流　　毫安表

Let the continuous random variable $X$ denote the current measured in a thin copper wire in milliamperes. Assume that the range of $X$ is [0, 20 mA], and assume that the probability density function of $X$ is $f(x) = 0.05$ for $0 \leq x \leq 20$. What is the probability that a current measurement is less than 10 milliamperes?

**Solution.** The random variable is the current measurement with distribution given by $f(x)$. The pdf is shown in Fig. 3-8. It is assumed that $f(x) = 0$ wherever it is not specifically defined. The probability requested is indicated by the shaded area in Fig. 3-8.

$$P(X < 10) = \int_0^{10} f(x)\,dx = 0.5$$

$$P(5 < X < 10) = \int_5^{10} f(x)\,dx = 0.25$$

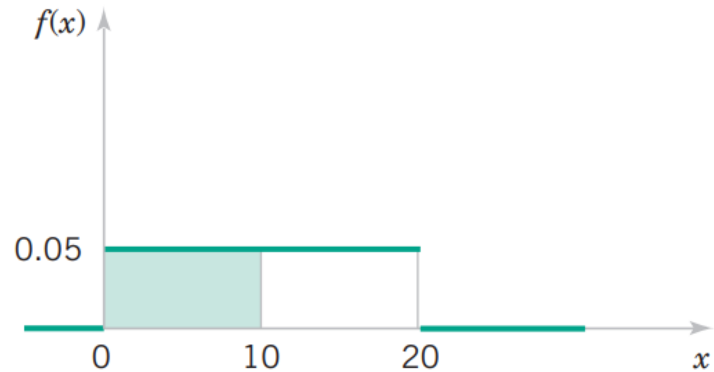
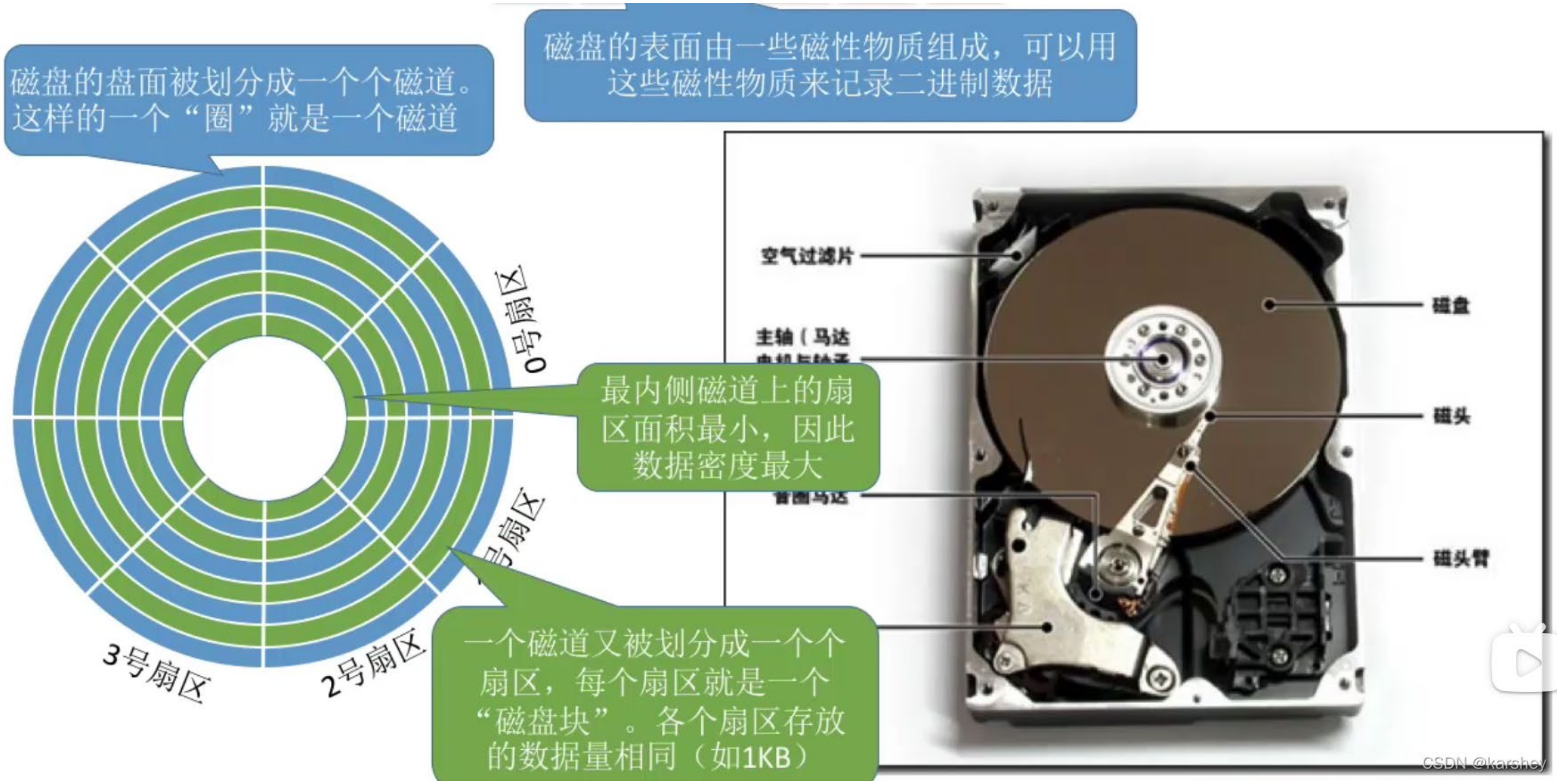
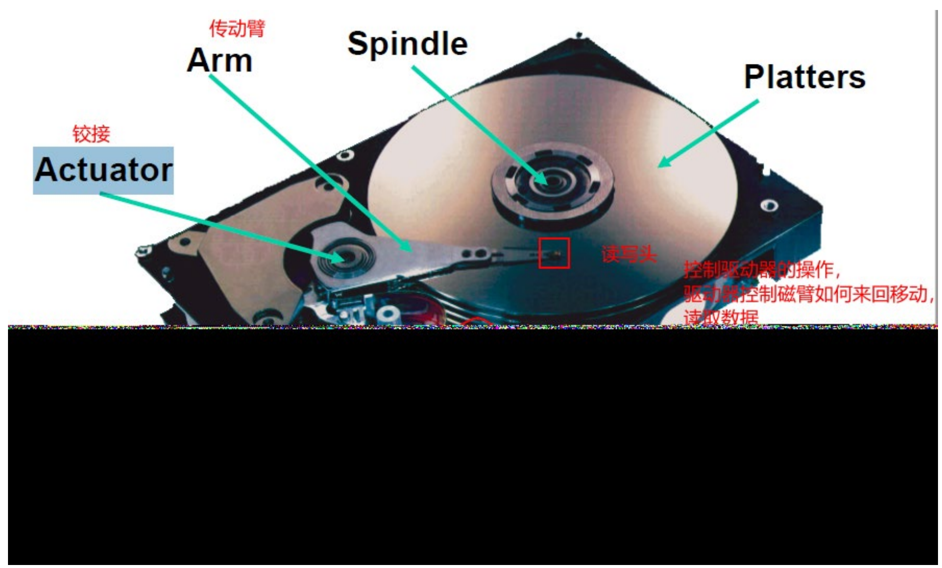



Figure 3-8 Probability density function for Example 3-2.




Magnetic disk



# 3-4 Continuous Random Variables

磁盘上的裂缝

用千分尺测出磁盘上磁道的起点到第一个裂缝的距离

disk

Let the continuous random variable $X$ denote the distance in micrometers from the start of a track on a magnetic disk until the first flaw. Historical data show that the distribution of $X$ can be modeled by a pdf $f(x) = \dfrac{1}{2000} e^{-x/2000}$, $x \geq 0$. For what proportion of disks is the distance to the first flaw greater than 1000 micrometers?

指数分布常用于 “首次事件发生的等待时间”场景（如设备故障时间、无缺陷距离等）

**Solution.** The density function and the requested probability are shown in Fig. 3-9. Now,

$$P(X > 1000) = \int_{1000}^{\infty} f(x)\,dx = \int_{1000}^{\infty} \frac{e^{-x/2000}}{2000}\,dx = -e^{-x/2000}\Big|_{1000}^{\infty} = e^{-1/2} = 0.607$$

What proportion of parts is between 1000 and 2000 micrometers?

**Solution.** Now,

$$P(1000 < X < 2000) = \int_{1000}^{2000} f(x)\,dx = -e^{-x/2000}\Big|_{1000}^{2000} = e^{-1/2} - e^{-1} = 0.239$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(X < 1000) = 1 - P(X > 1000) = 1 - 0.607 = 0.393$.

# 3-4 Continuous Random Variables

## EXAMPLE 3-3

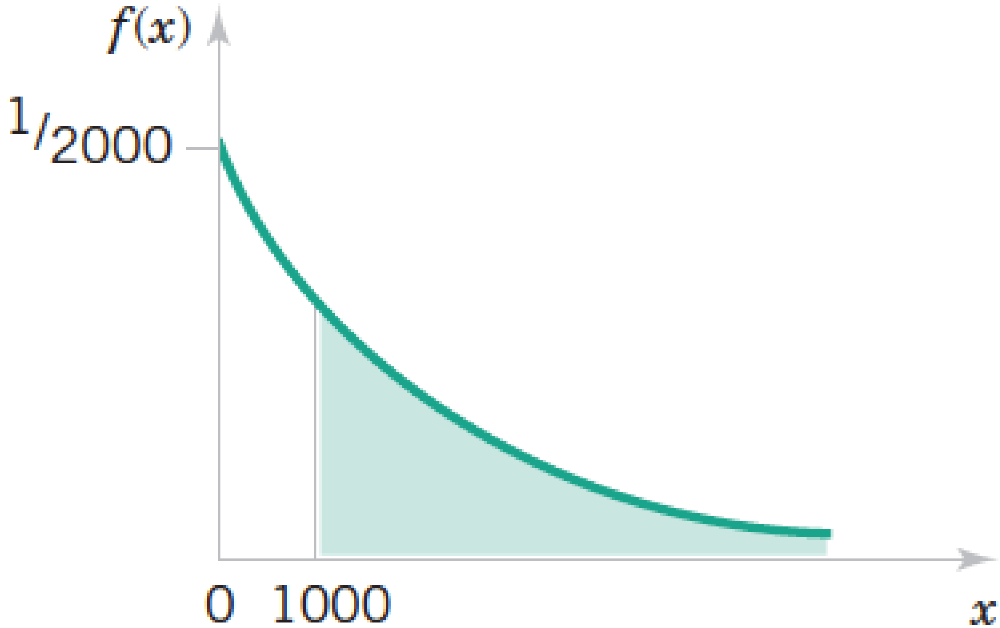



Figure 3-9 Probability density function for Example 3-3.

# 3-4 Continuous Random Variables

## 3-4.2 Cumulative Distribution Function

The **cumulative distribution function** (or cdf) of a continuous random variable $X$ with probability density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$$

for $-\infty < x < \infty$.

# 3-4 Continuous Random Variables

EXAMPLE 3-4
Flaw on a Magnetic Disk Distribution Function

Consider the distance to flaws in Example 3-3 with pdf

$$f(x) = \frac{1}{2000} \exp(-x/2000)$$

for $x \geq 0$. The cdf is determined from

$$F(x) = \int_0^x \frac{1}{2000} \exp(-u/2000)\, du = 1 - \exp(-x/2000)$$

for $x \geq 0$. It can be checked that $\frac{d}{dx} F(x) = f(x)$.

# 3-4 Continuous Random Variables

EXAMPLE 3-3

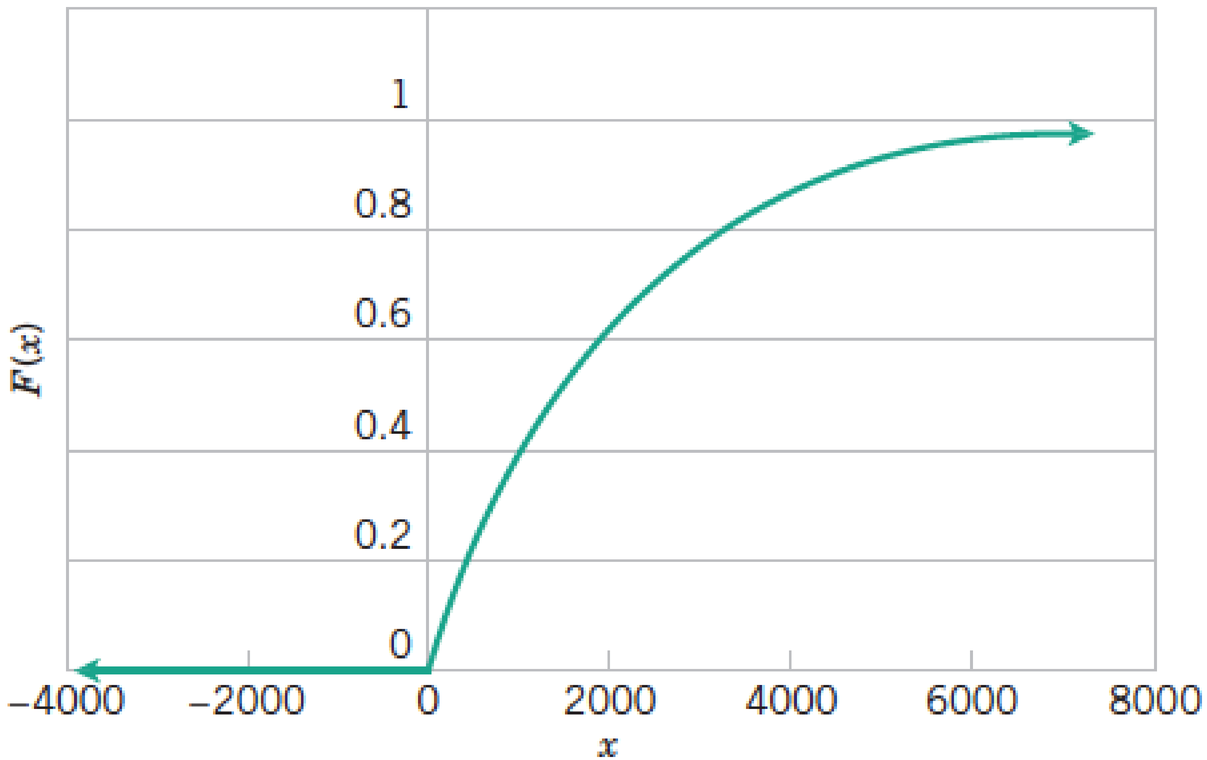



Figure 3-10 Cumulative distribution function for Example 3-4.

# 3-4 Continuous Random Variables

## EXAMPLE 3-3

**Solution.** The random variable is the distance until the first surface flaw with distribution given by $F(x)$. The requested probability is

$$P(X < 1000) = F(1000) = 1 - \exp\left(-\frac{1}{2}\right) = 0.393$$

Determine the probability that the distance until the first surface flaw exceeds 2000 micrometers.

**Solution.** Now we use

$$\begin{aligned} P(2000 < X) &= 1 - P(X \leq 2000) = 1 - F(2000) = 1 - [1 - \exp(-1)] \\ &= \exp(-1) = 0.368 \end{aligned}$$

Determine the probability that the distance is between 1000 and 2000 micrometers.

**Solution.** The requested probability is

$$\begin{aligned} P(1000 < X < 2000) &= F(2000) - F(1000) = 1 - \exp(-1) - [1 - \exp(-0.5)] \\ &= \exp(-0.5) - \exp(-1) = 0.239 \end{aligned}$$

# 3-4 Continuous Random Variables

## 3-4.3 Mean and Variance

Suppose $X$ is a continuous random variable with pdf $f(x)$. The **mean** or **expected value** of $X$, denoted as $\mu$ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad (3\text{-}3)$$

The **variance** of $X$, denoted as $V(X)$ or $\sigma^2$, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = E(X^2) - \mu^2$$

The **standard deviation** of $X$ is $\sigma$.

# 3-4 Continuous Random Variables

For the distance to a flaw in Example 3-2, the mean of $X$ is

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x \frac{e^{-x/2000}}{2000}\,dx$$

A table of integrals or integration by parts can be used to show that

$$E(X) = -xe^{-x/2000}\Big|_0^\infty + \int_0^\infty e^{-x/2000}\,dx = 0 - 2000\,e^{-x/2000}\Big|_0^\infty = 2000$$

The variance of $X$ is

$$V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_0^\infty (x-2000)^2 \frac{e^{-x/2000}}{2000}\,dx$$

A table of integrals or integration by parts can be applied twice to show that

$$V(X) = 2000^2 = 4{,}000{,}000$$

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

Undoubtedly, the most widely used model for the distribution of a random variable is a **normal distribution**.

- **Gaussian distribution 高斯分布**

**□ Inverse Gaussian Distribution 逆高斯分布**

又称Wald 分布，是一种连续概率分布，常用于描述非负随机变量的“首次通过时间”、“返程时间”，其右偏特性和尾部行为对建模 ″首次通过时间″、″生存分析″ 和 ″金融回报″ 等领域的现象至关重要。

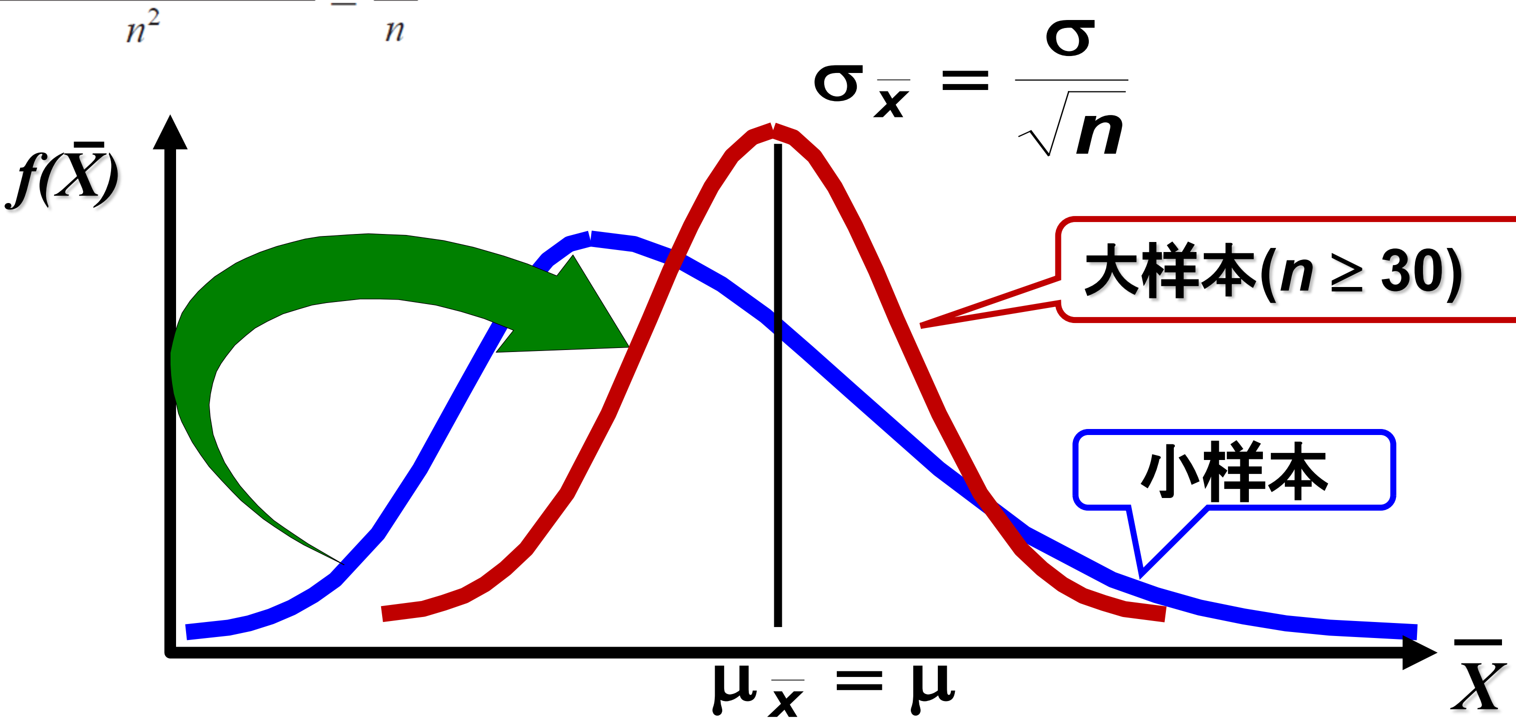
- **Central limit theorem 中心极限定理**

# 3-5 Important Continuous Distributions

The **central limit theorem** states that if you take sufficiently large samples from a population, the samples' means will be **normally distributed**, even if the population isn't normally distributed.

**从均值为 $\mu$，方差为 $\sigma^2$ 的一个任意总体中抽取容量为 $n$ 的样本，当 $n$ 充分大时，样本均值的抽样分布近似服从均值为 $\mu$、方差为 $\sigma^2/n$ 的正态分布。**

$$V(\bar{X}) = \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

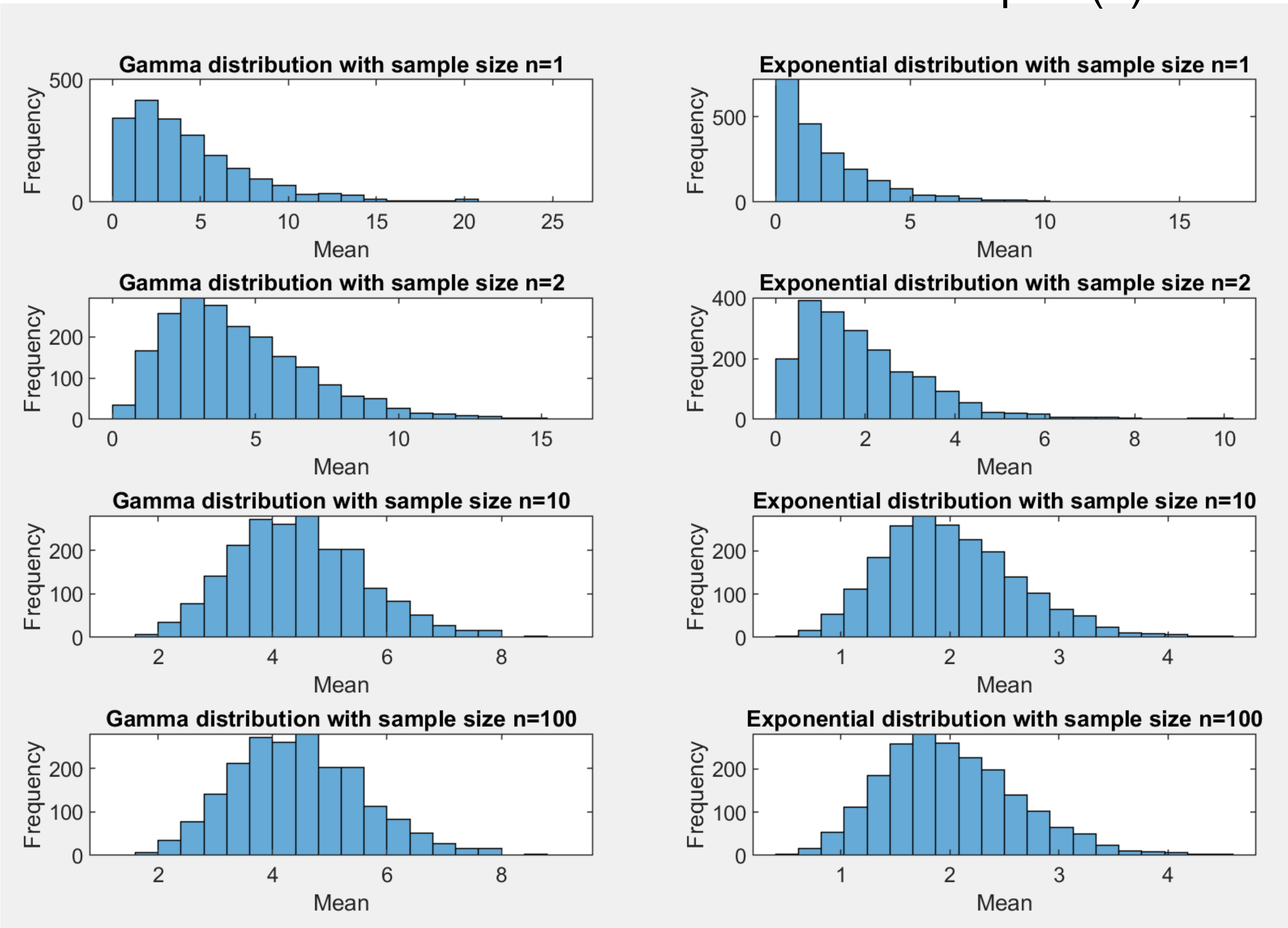


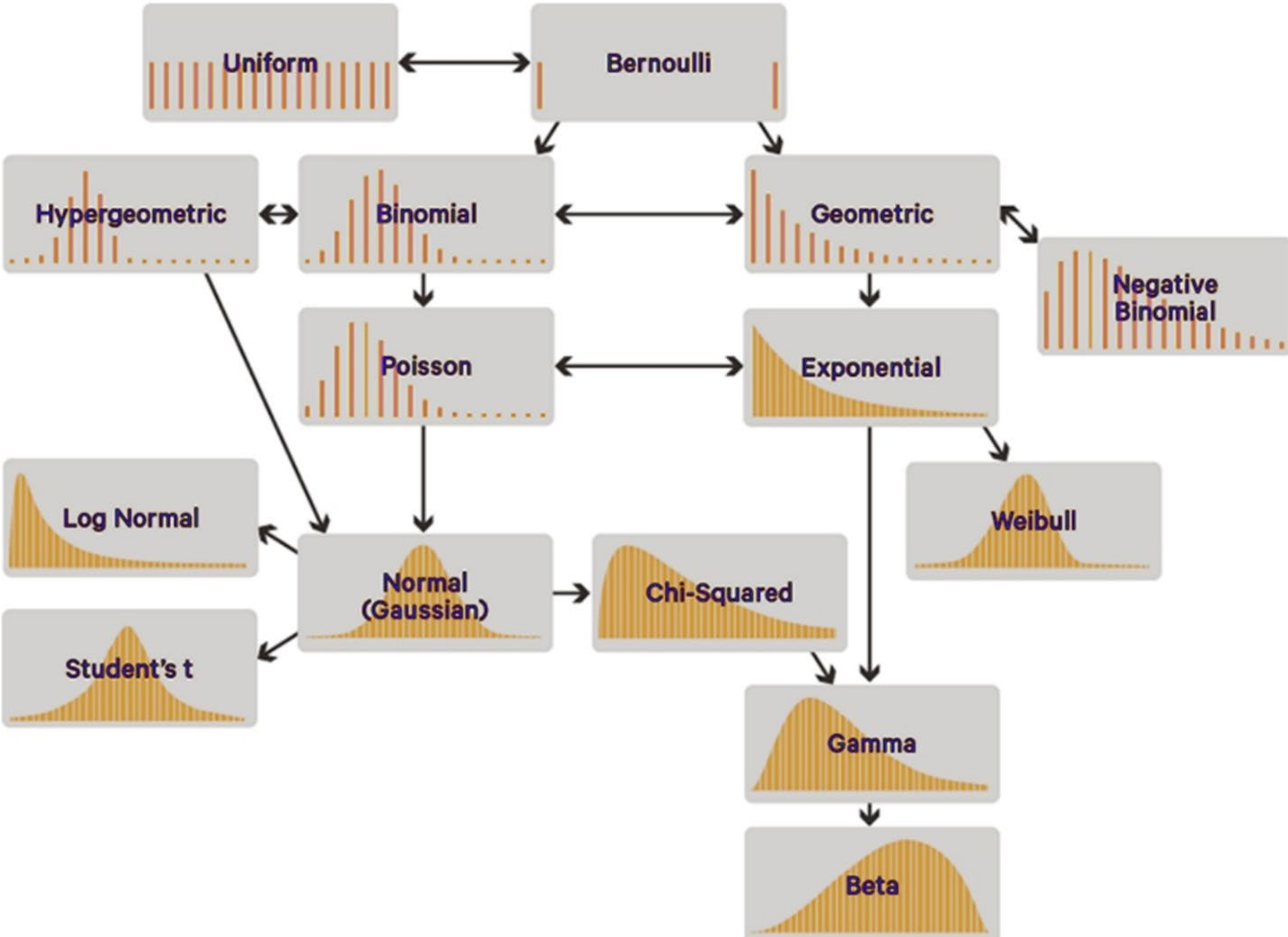
gamrnd(1.5,3) wblrnd(1.5,3)

**中心极限定理:用样本来估计总体。样本平均值呈正态分布；样本容量越大，结果就越接近正态分布**

gamrnd(1.5,3)

exprnd(2)




Uniform
Bernoulli
Hypergeometric
Binomial
Geometric
Negative Binomial
Poisson
Exponential
Log Normal
Weibull
Normal (Gaussian)
Chi-Squared
Student's t
Gamma
Beta

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

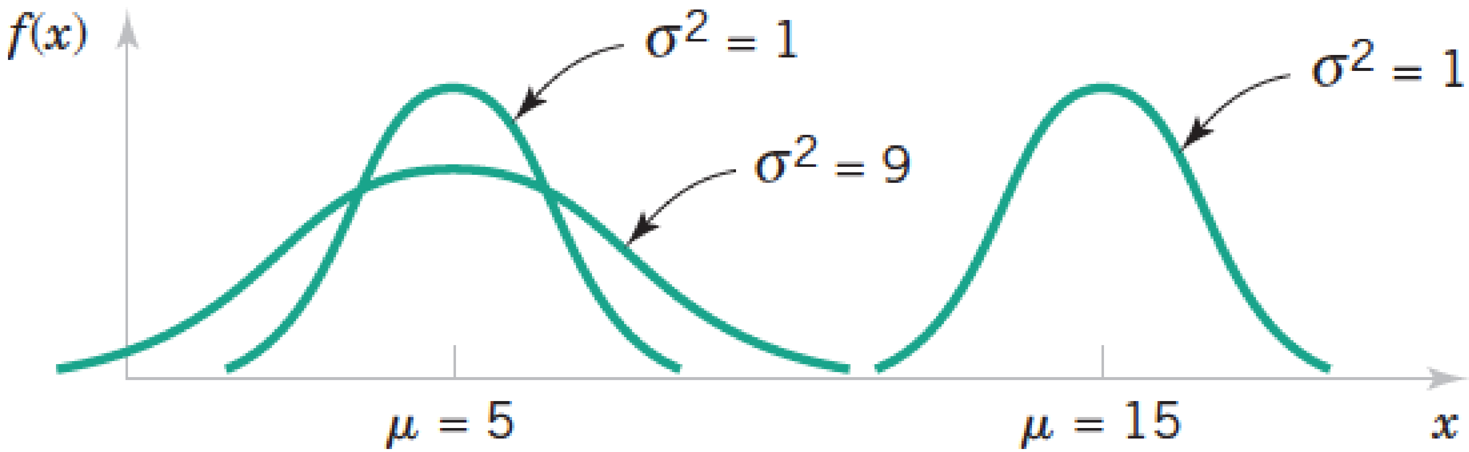



Figure 3-11 Normal probability density functions for selected values of the parameters $\mu$ and $\sigma^2$.

均值决定了概率密度函数的中心值，方差决定了宽度

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

A random variable $X$ with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \text{ for } -\infty < x < \infty \tag{3-4}$$

has a **normal distribution** (and is called a **normal random variable**) with parameters $\mu$ and $\sigma$, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2$$

The mean and variance of the normal distribution are derived at the end of this section.

# 3-5 Important Continuous Distributions

**EXAMPLE 3-7**
Current in a Wire: Normal Distribution

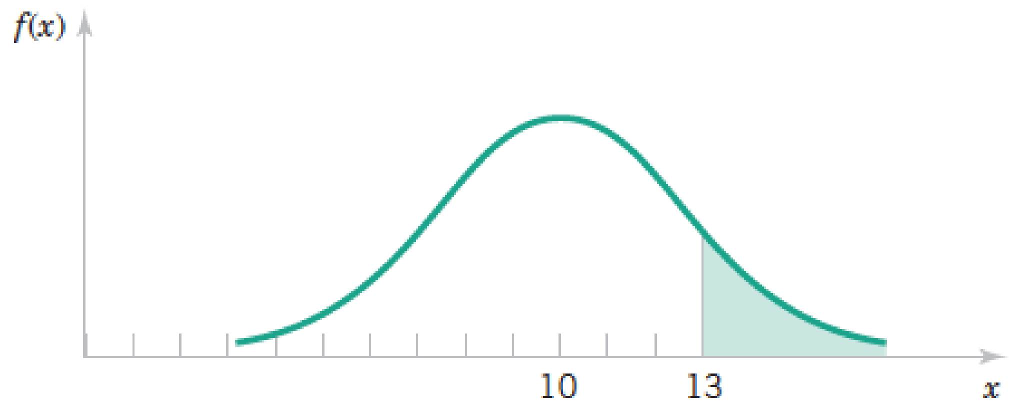



Figure 3-12 Probability that $X > 13$ for a normal random variable with $\mu = 10$ and $\sigma^2 = 4$ in Example 3-7.

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 milliamperes$^2$. What is the probability that a measurement exceeds 13 milliamperes?

**Solution.** Let $X$ denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. This probability is shown as the shaded area under the normal probability density function in Fig. 3-12. Unfortunately, there is no closed-form expression for the integral of a normal pdf, and probabilities based on the normal distribution are typically found numerically or from a table (which we will introduce later).

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

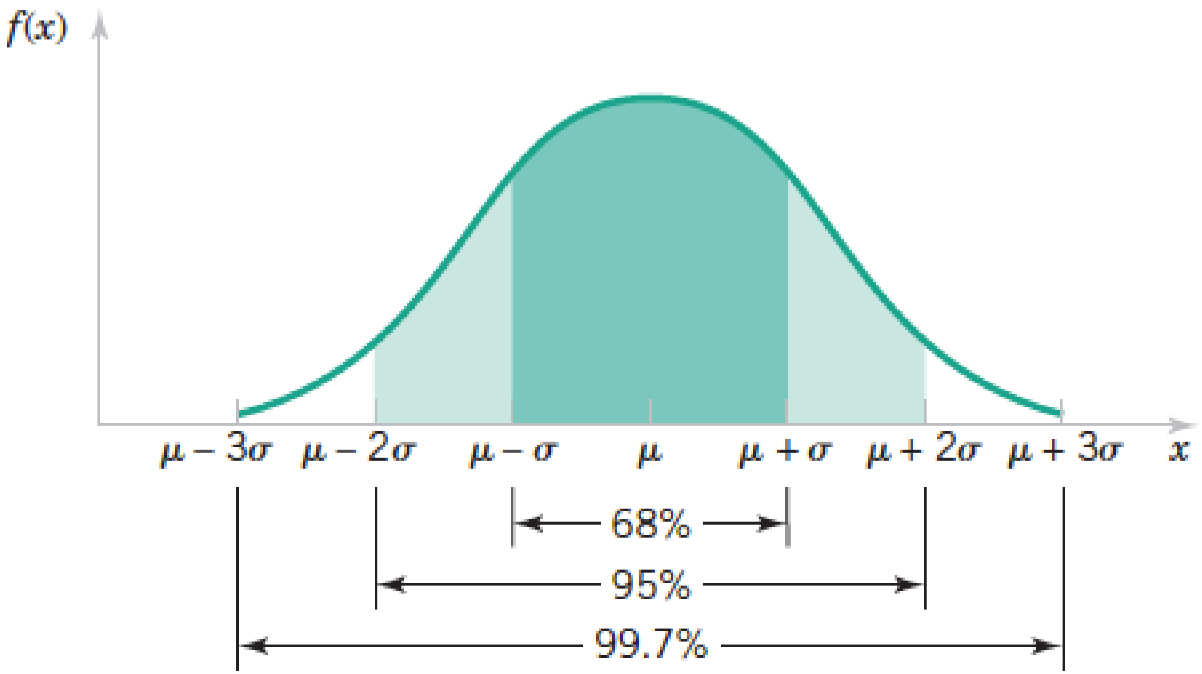



Figure 3-13 Probabilities associated with a normal distribution.

Also, from the symmetry of $f(x)$, $P(X > \mu) = P(X < \mu) = 0.5$.

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **standard normal** random variable. A standard normal random variable is denoted as $Z$.

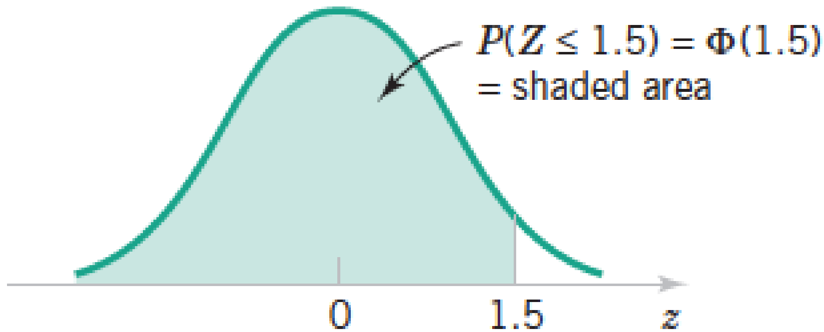
The function

$$\Phi(z) = P(Z \leq z)$$

is used to denote the **cumulative distribution function** of a standard normal random variable. A table (or computer software) is required because the probability cannot be calculated in general by elementary methods.

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution




| $z$ | 0.00 | 0.01 | 0.02 | 0.03 |
|---|---|---|---|---|
| 0 | 0.50000 | 0.50399 | 0.50398 | 0.51197 |
| ⋮ | | ⋮ | | |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 |

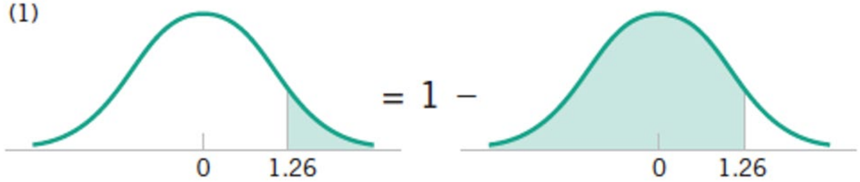
**Figure 3-14** Standard normal probability density function.

Assume that $Z$ is a standard normal random variable. Appendix A Table I provides probabilities of the form $P(Z \le z)$. The use of Table I to find $P(Z \le 1.5)$ is illustrated in Fig. 3-14. We read down the $z$ column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00 to be 0.93319.

The column headings refer to the hundredth's digit of the value of $z$ in $P(Z \le z)$. For example, $P(Z \le 1.53)$ is found by reading down the $z$ column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699. ■

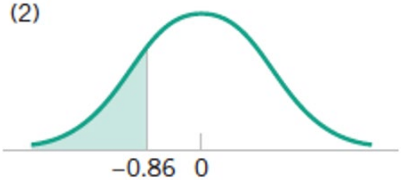
# 3-5 Important Continuous Distributions

(1)




$P(Z > 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.89616 = 0.10384$

(2)




$P(Z < -0.86) = 0.19490$

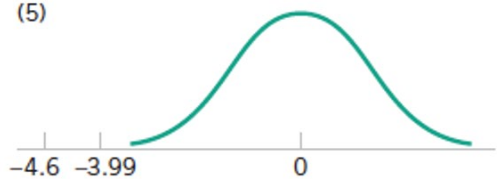
(3)


$P(Z > -1.37) = P(Z < 1.37) = 0.91465$

(4)


$P(-1.25 < Z < 0.37) = 0.64431 - 0.10565 = 0.53866$

(5)

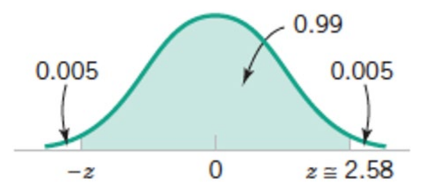



$P(Z \leq -4.6)$

(6)


$P(Z > z) = 0.05$

(7)




$P(-z < Z < z) = 0.99.$

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

If $X$ is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, $Z$ is a **standard normal** random variable.

- Creating a new random variable by this transformation is referred to as standardizing(标准化).
- The random variable Z represents the distance of X from its mean in terms of standard deviations.
- It is the key step in calculating a probability for an arbitrary normal random variable.

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

Suppose $X$ is a normal random variable with mean $\mu$ and variance $\sigma^2$. Then

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) \qquad (3\text{-}5)$$

where

$Z$ is a **standard normal** random variable, and

$z = (x-\mu)/\sigma$ is the **$z$-value** obtained by **standardizing** $x$.

The probability is obtained by entering **Appendix A Table I** with $z = (x-\mu)/\sigma$.

# 3-5 Important Continuous Distributions

## 3-5.1 Normal Distribution

**EXAMPLE 3-10**
**Current in a Wire: Normal Distribution Probability**

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 milliamperes2. What is the probability that a measurement will exceed 13 milliamperes?

Solution. Let X denote the current in milliamperes. The requested probability can be represented as P(X > 13).

Let $Z = (X - 10)/2$. The relationship between several values of $X$ and the transformed values of $Z$ are shown in Fig. 3-16. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Table I,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

The calculation can be written more simply as

$$P(X > 13) = P\left(\frac{X - 10}{2} > \frac{13 - 10}{2}\right) = P(Z > 1.5) = 0.06681$$

# 3-5 Important Continuous Distributions

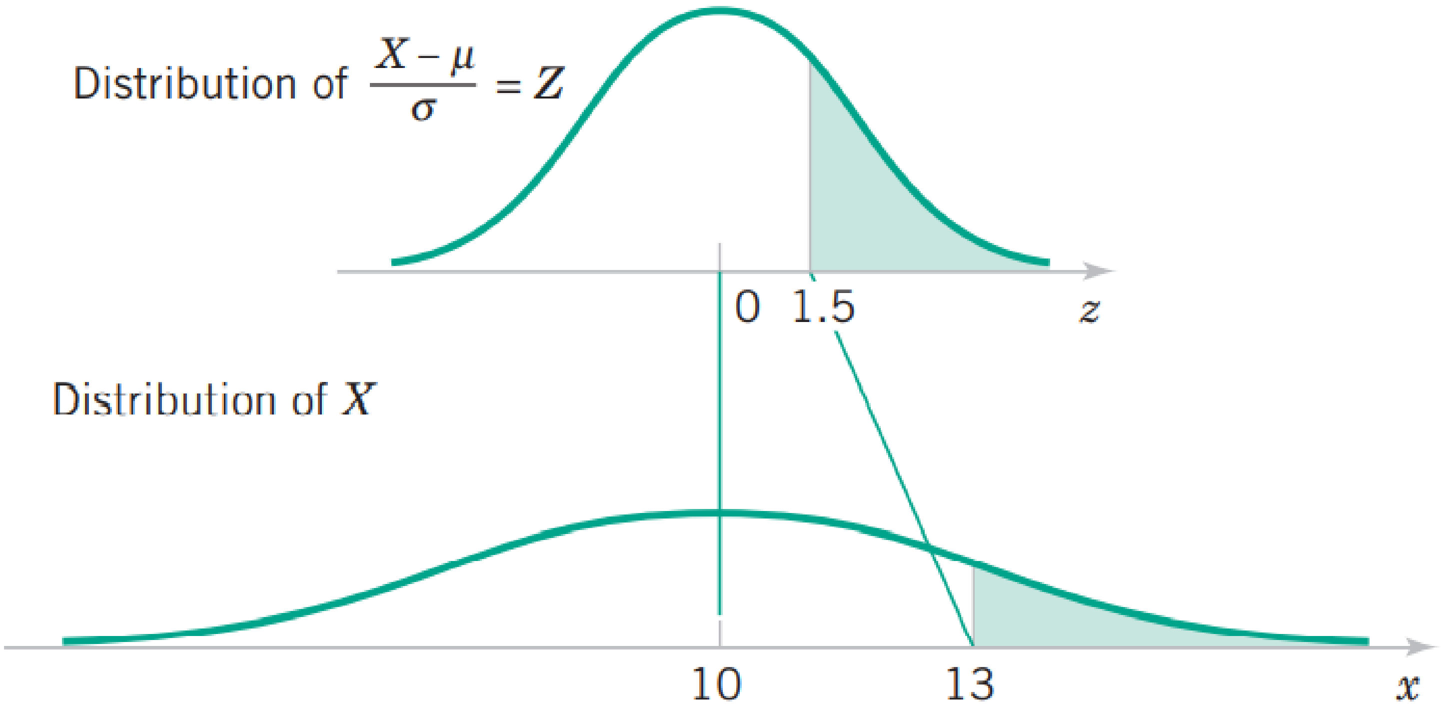



Figure 3-16 Standardizing a normal random variable.

# 3-5 Important Continuous Distributions

Determine the value for which the probability that a current measurement is below this value is 0.98.

Define the random variable and distribution.

Write the probability statement.

Compute the probability.

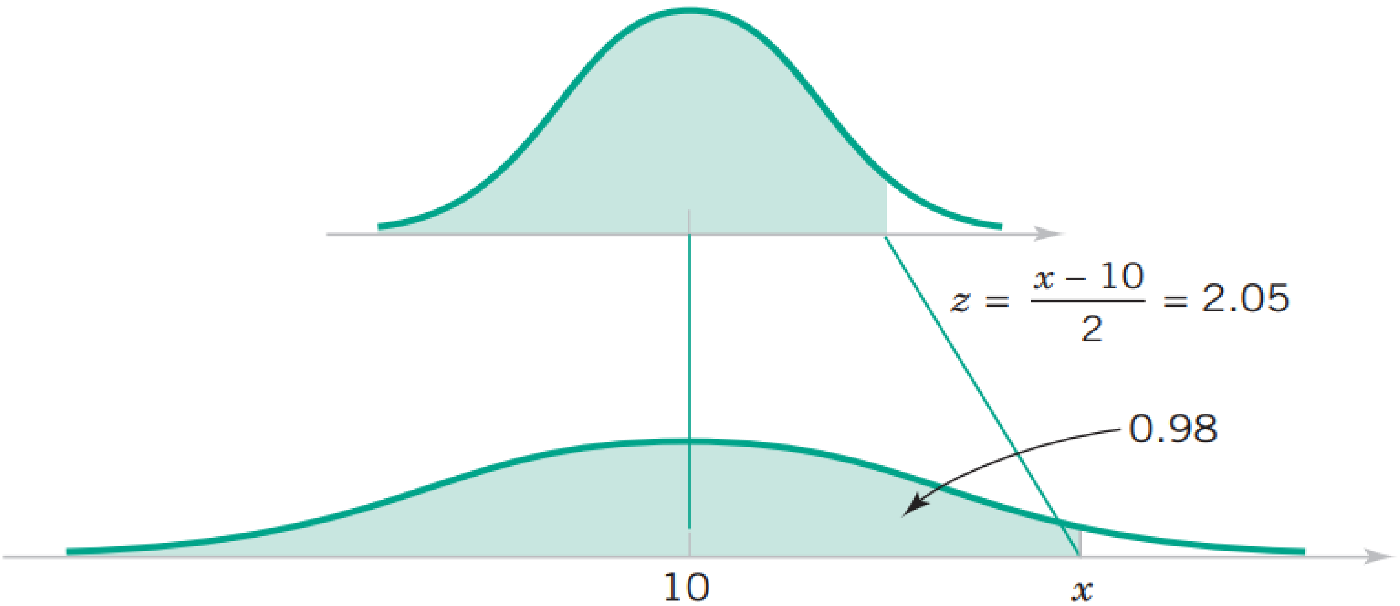



**Figure 3-17** Determining the value of $x$ to meet a specified probability, Example 3-11.

# 3-5 Important Continuous Distributions

**EXAMPLE 3-13**
Diameter of a Shaft

The diameter of a shaft in a storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are $0.2500 \pm 0.0015$ inch. What proportion of shafts conforms to specifications?

**Solution.** Let $X$ denote the shaft diameter in inches. The requested probability is shown in Fig. 3-19 and

$$\begin{aligned} P(0.2485 < X < 0.2515) &= P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right) \\ &= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6) \\ &= 0.91924 - 0.0000 = 0.91924 \end{aligned}$$

# 3-5 Important Continuous Distributions

**EXAMPLE 3-13**

Most of the nonconforming shafts are too large, because the process mean is located very near to the upper specification limit. If the process is centered so that the process mean is equal to the target value of 0.2500,

$$P(0.2485 < X < 0.2515) = P\left(\frac{0.2485 - 0.2500}{0.0005} < Z < \frac{0.2515 - 0.2500}{0.0005}\right)$$
$$= P(-3 < Z < 3) = P(Z < 3) - P(Z < -3)$$
$$= 0.99865 - 0.00135 = 0.9973$$

By recentering the process, the yield is increased to approximately 99.73%.

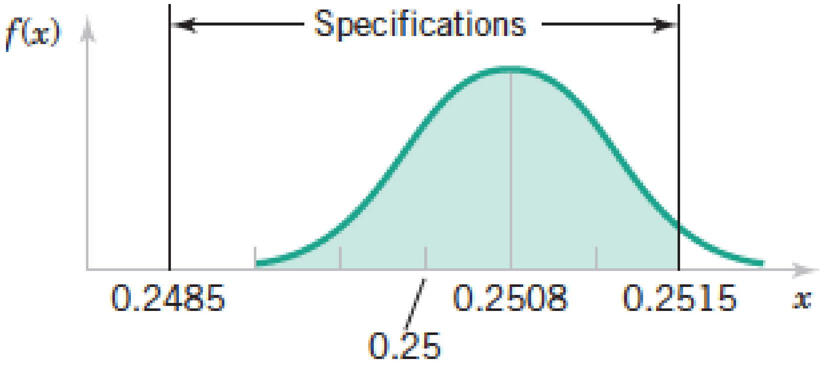



Figure 3-19 Distribution for Example 3-13.

# 3-5 Important Continuous Distributions

## 3-5.2 Lognormal Distribution

$$F(x) = P[X \le x] = P[\exp(W) \le x] = P[W \le \ln(x)]$$

$$= P\left[Z \le \frac{\ln(x) - \theta}{\omega}\right] = \Phi\left[\frac{\ln(x) - \theta}{\omega}\right]$$

Let $W$ have a normal distribution with mean $\theta$ and variance $\omega^2$; then $X = \exp(W)$ is a **lognormal random variable** with probability density function

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln(x) - \theta)^2}{2\omega^2}\right] \qquad 0 < x < \infty \tag{3-6}$$

The mean and variance of $X$ are

$$E(X) = e^{\theta + \omega^2/2} \quad \text{and} \quad V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1) \tag{3-7}$$

如果事件是互相影响的，那么则不能用普通的正态分布来研究，而要使用对数正态分布。

# 3-5 Important Continuous Distributions

## 3-5.2 Lognormal Distribution

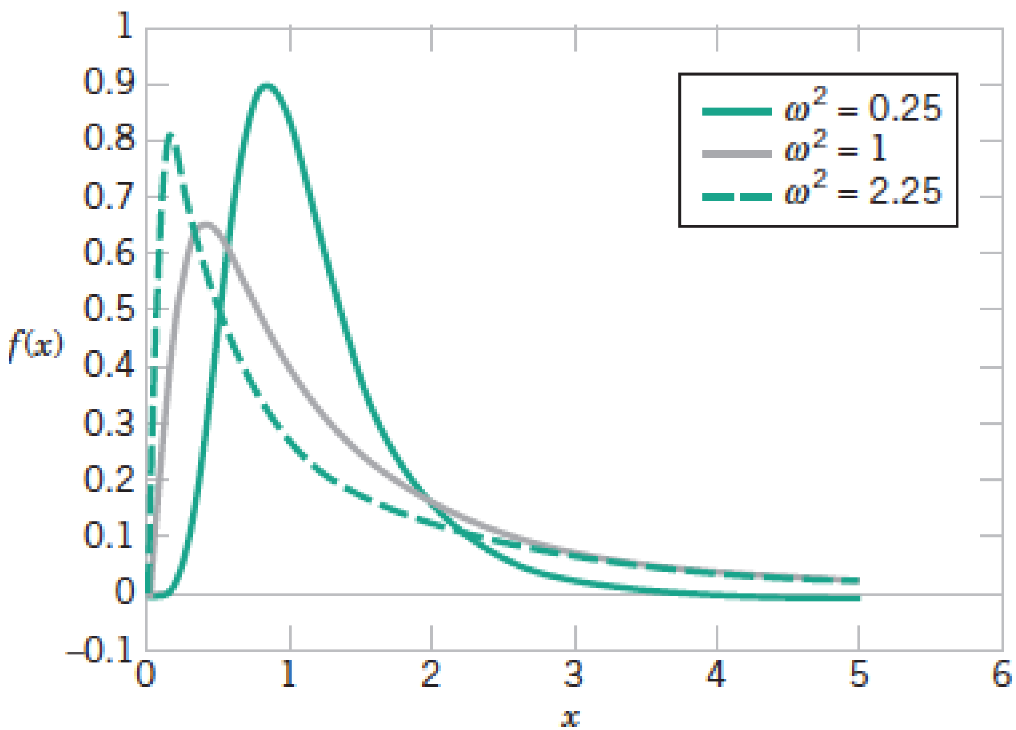



Figure 3-20 Lognormal probability density functions with $\theta = 0$ for selected values of $\omega^2$.

对数正态缺乏对称性，当大于1时，其乘积增长很快，所以会出现长尾效应long tail effect。

# 3-5 Important Continuous Distributions

## 3-5.3 Gamma Distribution

The **gamma function** is

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx, \qquad \text{for } r > 0 \tag{3-8}$$

It can be shown that the integral in the definition of $\Gamma(r)$ is finite. Furthermore, by using integration by parts it can be shown that

$$\Gamma(r) = (r - 1)\Gamma(r - 1)$$

# 3-5 Important Continuous Distributions

## 3-5.3 Gamma Distribution

The random variable $X$ with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0 \tag{3-9}$$

is a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$. The mean and variance are

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \tag{3-10}$$

λ:Scale parameter尺度参数
γ:Shape parameter形状参数

# 3-5 Important Continuous Distributions

## 3-5.3 Gamma Distribution

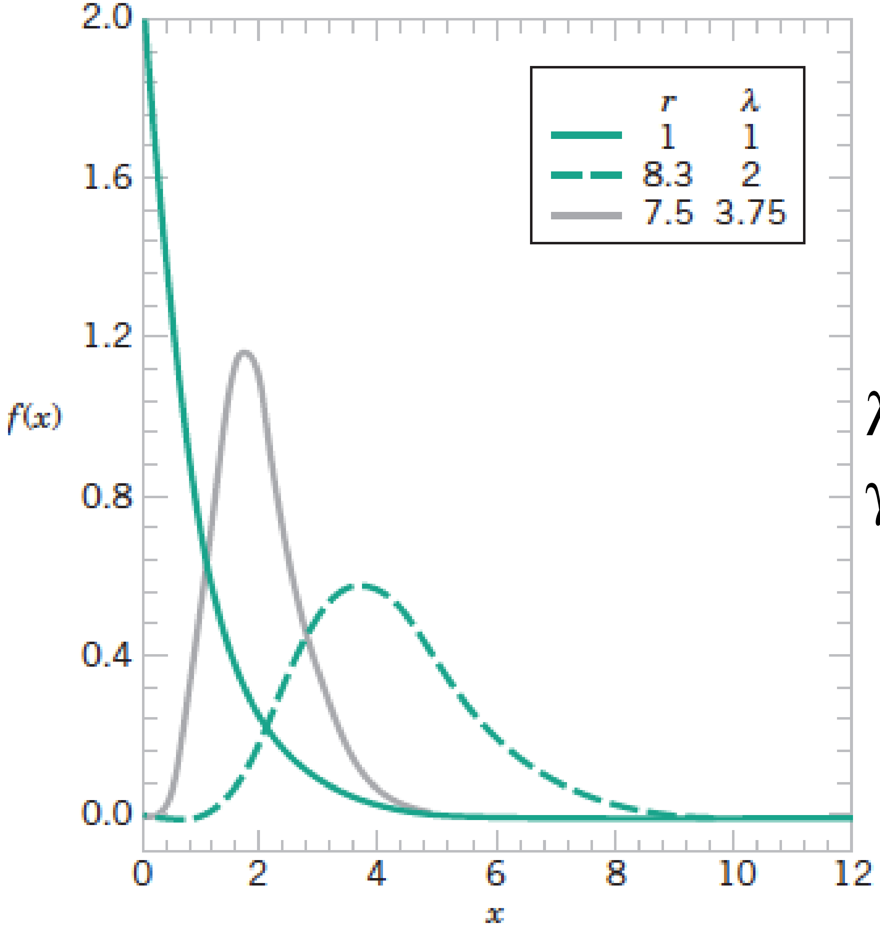



Figure 3-21 Gamma probability density functions for selected values of λ and *r*.

λ:Scale parameter尺度参数

γ:Shape parameter形状参数

# 3-5 Important Continuous Distributions

## 3-5.4 Weibull Distribution

The random variable $X$ with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0 \tag{3-11}$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

# 3-5 Important Continuous Distributions

## 3-5.4 Weibull Distribution

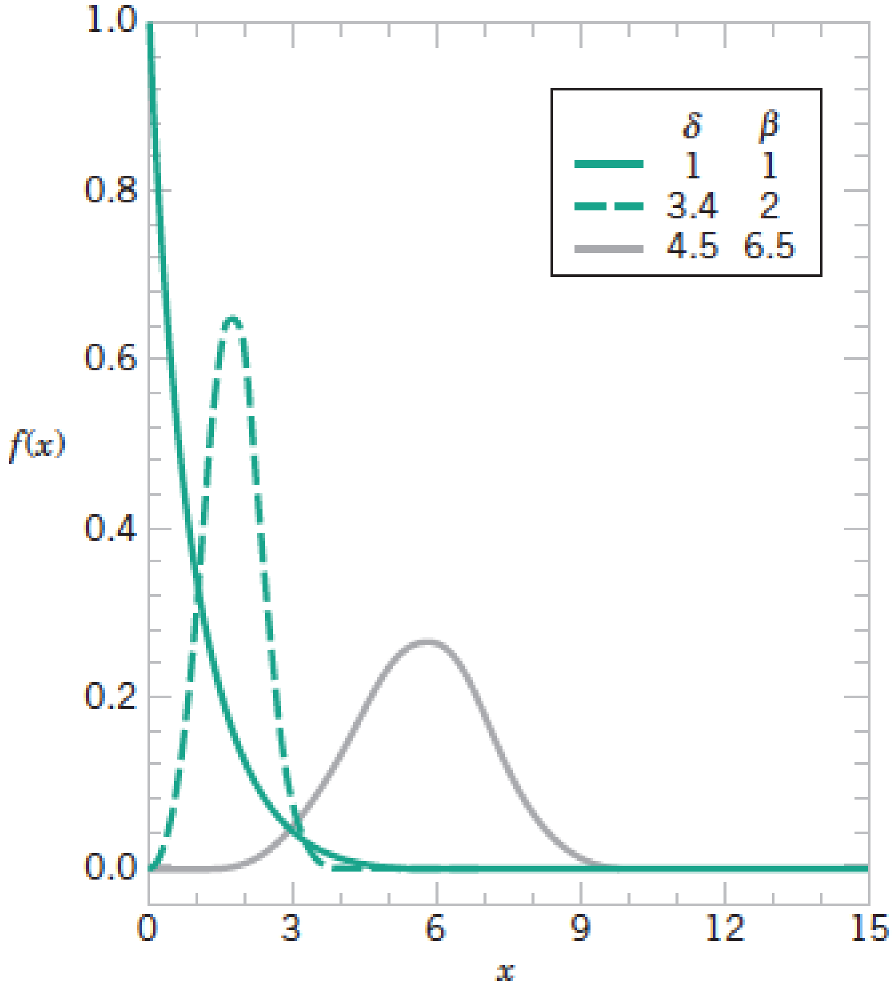



Figure 3-22 Weibull probability density functions for selected values of δ and β.

# 3-5 Important Continuous Distributions

## 3-5.4 Weibull Distribution

If $X$ has a Weibull distribution with parameters $\delta$ and $\beta$, the cumulative distribution function of $X$ is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

The mean and variance of the Weibull distribution are as follows.

If $X$ has a Weibull distribution with parameters $\delta$ and $\beta$,

$$\mu = \delta\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad \sigma^2 = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \qquad (3\text{-}12)$$

# 3-6 Probability Plots

## 3-6.1 Normal Probability Plots正态概率图

- How do we know if a normal distribution is a reasonable model for data?

- **Probability plotting** is a graphical method for determining whether sample data conform to a hypothesized **假设** distribution based on a subjective**主观** visual**视觉** examination of the data.

- Probability plotting typically uses special graph paper, known as **probability paper**, that has been designed for the hypothesized distribution. Probability paper is widely available for the normal, lognormal, Weibull, and various chi-square and gamma distributions.

# 3-6 Probability Plots

## 3-6.1 Normal Probability Plots

➢**排序**: To construct a probability plot, the observations in the sample are first ranked from smallest to largest.
176, 191, 214, 220, 205, 192, 201, 190, 183, 185

➢**计算累计频率：** Calculate ($j$-0.5)/$n$, where j is the order according to $x_{(j)}$

➢**画图** : ordered observations as $x$-axis, ($j$-0.5)/$n$ as $y$-axis

| $j$ | $x_{(j)}$ | $(j-0.5)/10$ |
|---|---|---|
| 1 | 176 | 0.05 |
| 2 | 183 | 0.15 |
| 3 | 185 | 0.25 |
| 4 | 190 | 0.35 |
| 5 | 191 | 0.45 |
| 6 | 192 | 0.55 |
| 7 | 201 | 0.65 |
| 8 | 205 | 0.75 |
| 9 | 214 | 0.85 |
| 10 | 220 | 0.95 |

# 3-6 Probability Plots

## 3-6.1 Normal Probability Plots

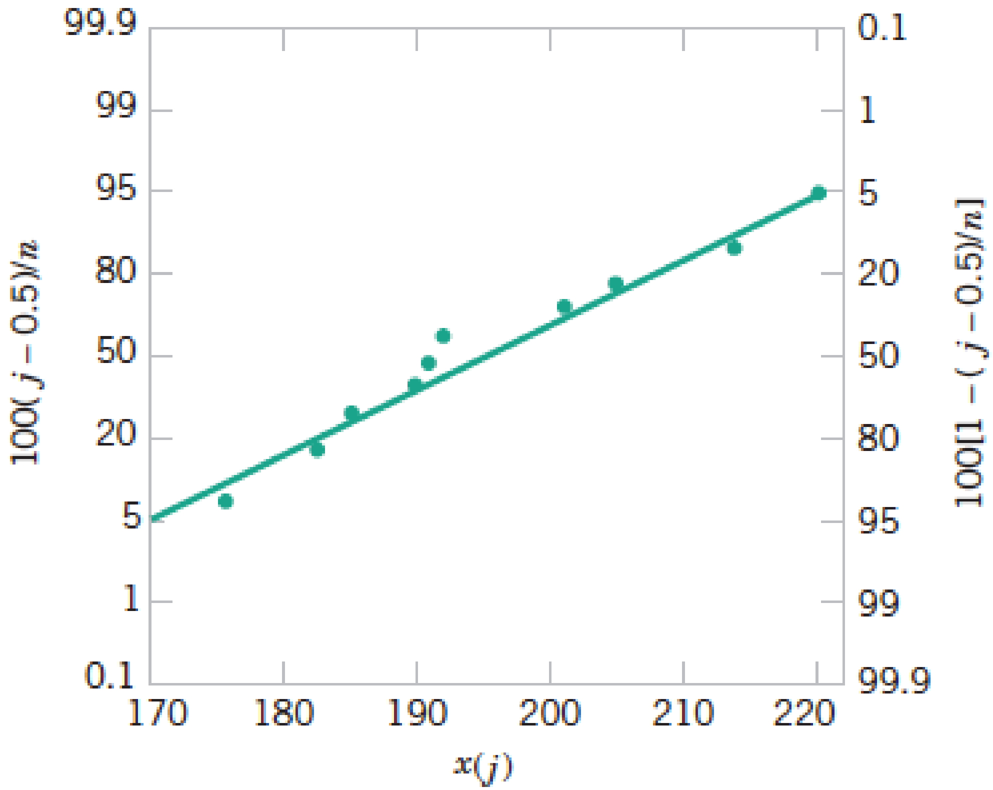



Figure 3-24 Normal probability plot for the battery life.

| $j$ | $x_{(j)}$ | $(j-0.5)/10$ |
|---|---|---|
| 1 | 176 | 0.05 |
| 2 | 183 | 0.15 |
| 3 | 185 | 0.25 |
| 4 | 190 | 0.35 |
| 5 | 191 | 0.45 |
| 6 | 192 | 0.55 |
| 7 | 201 | 0.65 |
| 8 | 205 | 0.75 |
| 9 | 214 | 0.85 |
| 10 | 220 | 0.95 |

Matlab函数:normplot

A good rule of thumb（**经验法则**）is to draw the line approximately between the 25th and 75th percentile points.

# 3-6 Probability Plots

## 3-6.1 Normal Probability Plots

$$\frac{j - 0.5}{n} = P(Z \le z_j) = \Phi(z_j)$$

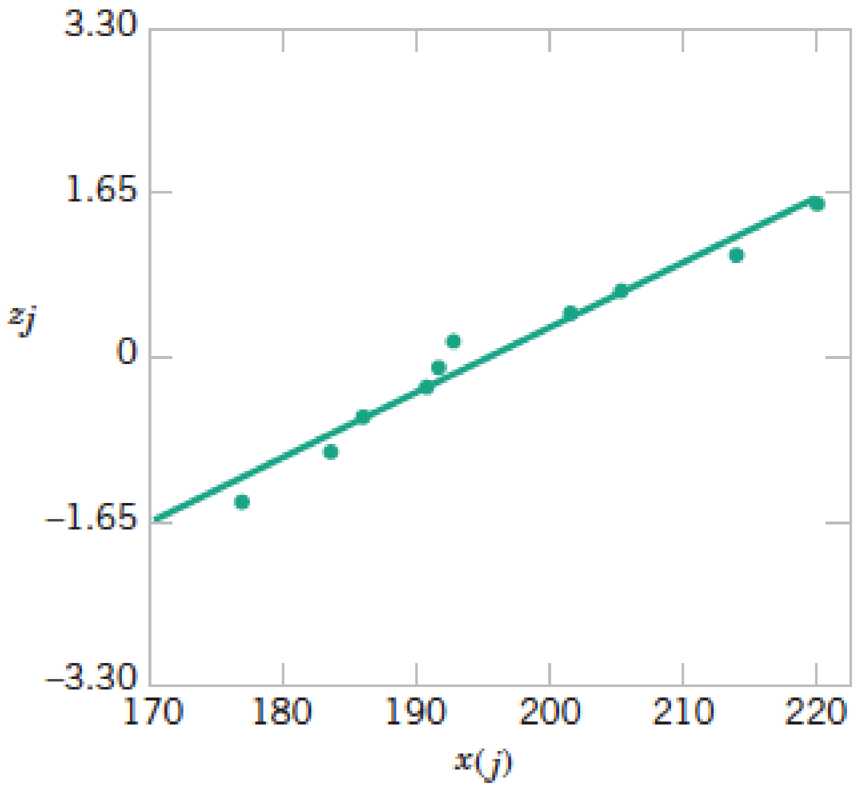



Figure 3-25 Normal probability plot obtained from standardized normal scores.

| $j$ | $x_{(j)}$ | $(j - 0.5)/10$ | $z_j$ |
|---|---|---|---|
| 1 | 176 | 0.05 | −1.64 |
| 2 | 183 | 0.15 | −1.04 |
| 3 | 185 | 0.25 | −0.67 |
| 4 | 190 | 0.35 | −0.39 |
| 5 | 191 | 0.45 | −0.13 |
| 6 | 192 | 0.55 | 0.13 |
| 7 | 201 | 0.65 | 0.39 |
| 8 | 205 | 0.75 | 0.67 |
| 9 | 214 | 0.85 | 1.04 |
| 10 | 220 | 0.95 | 1.64 |

# 3-6 Probability Plots

## 3-6.2 Other Probability Plots

Table 3-1 Crack Length (mm) for an Aluminum Alloy

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 81 | 98 | 291 | 101 | 98 | 118 | 158 | 197 | 139 | 249 |
| 249 | 135 | 223 | 205 | 80 | 177 | 82 | 64 | 137 | 149 |
| 117 | 149 | 127 | 115 | 198 | 342 | 83 | 34 | 342 | 185 |
| 227 | 225 | 185 | 240 | 161 | 197 | 98 | 65 | 144 | 151 |
| 134 | 59 | 181 | 151 | 240 | 146 | 104 | 100 | 215 | 200 |

# 3-6 Probability Plots

## 3-6.2 Other Probability Plots

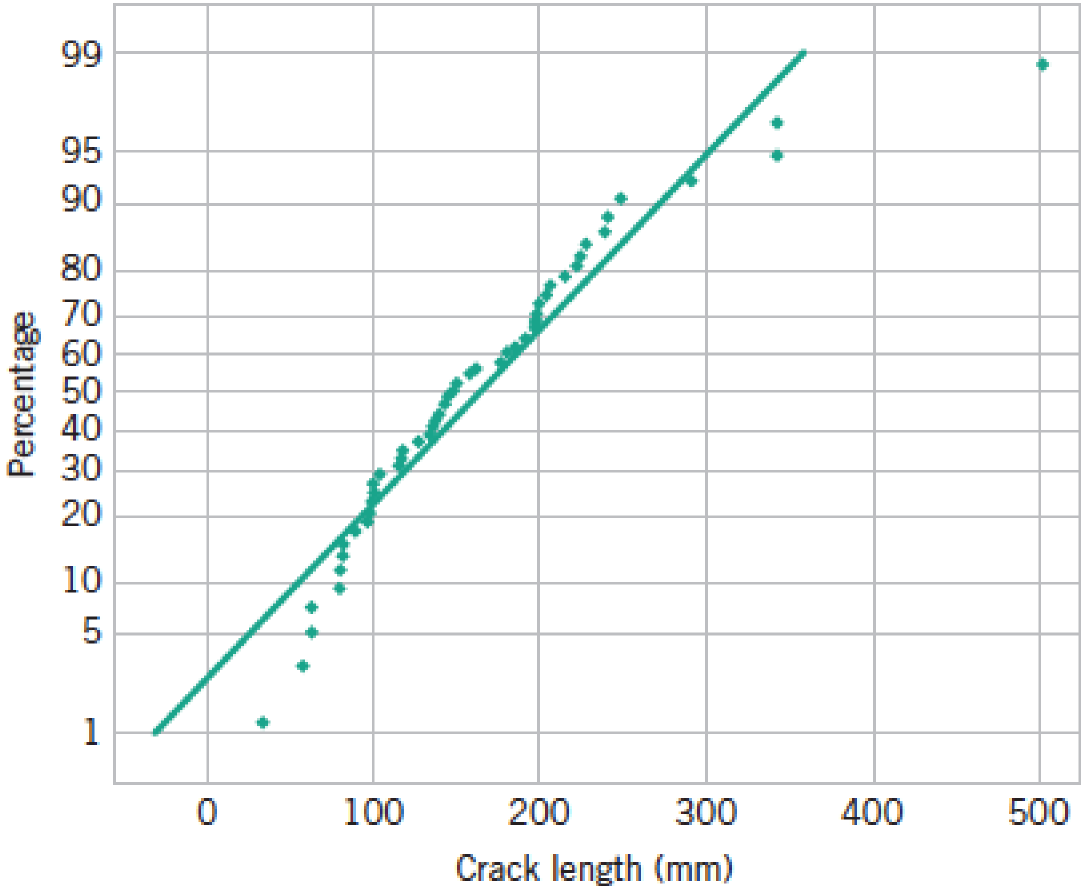



Figure 3-26 Normal probability plot for the crack-length data in Table 3-1.

# 3-6 Probability Plots

## 3-6.2 Other Probability Plots

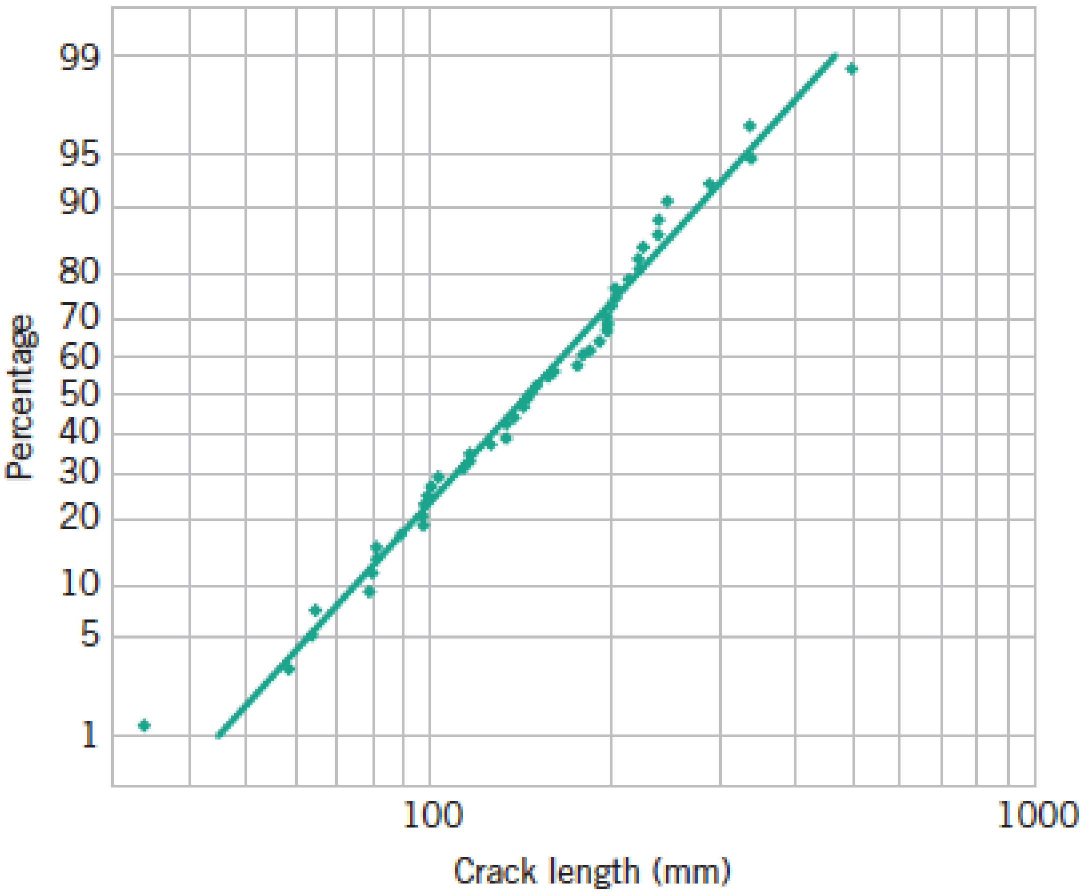



Figure 3-27 Lognormal probability plot for the crack-length data in Table 3-1.

# 3-6 Probability Plots

## 3-6.2 Other Probability Plots

$$y = \ln(-\ln[1 - F(t)])$$

$$x = \ln(t)$$

Matlab命令 : wblplot

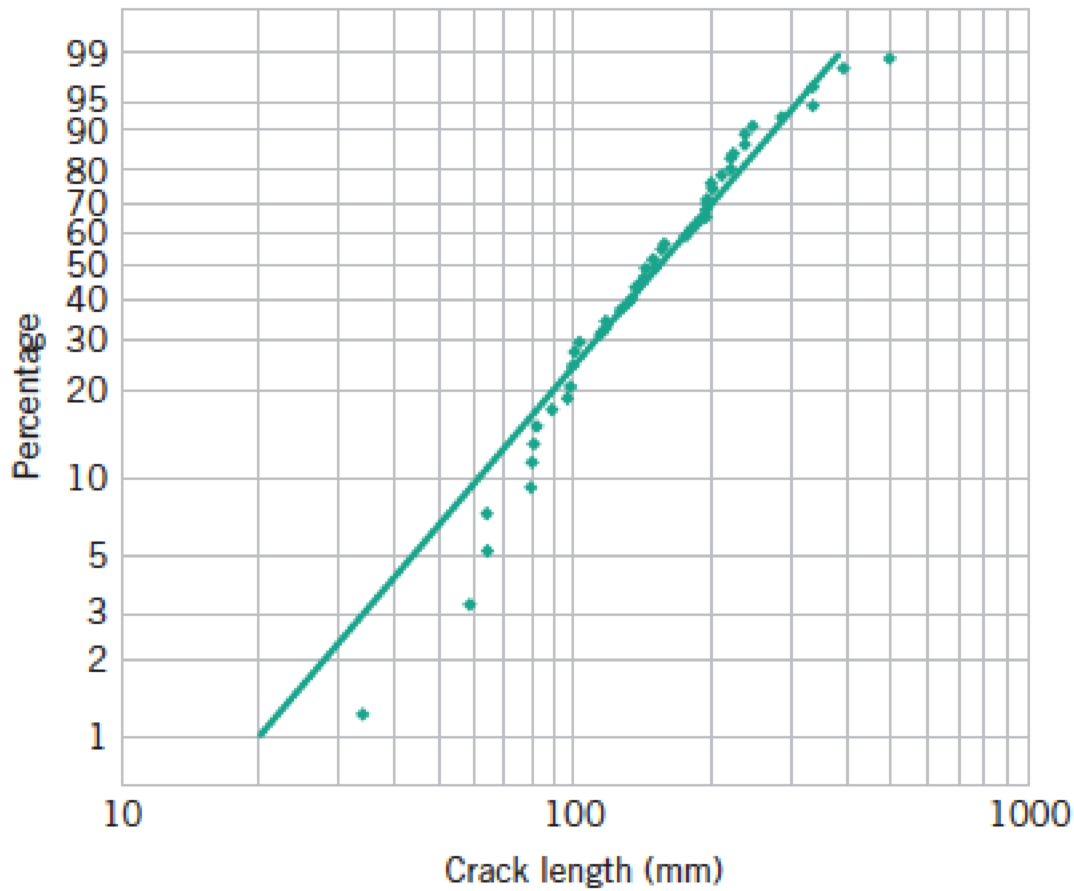



Figure 3-28 Weibull probability plot for the crack-length data in Table 3-1.

# 3-7 Discrete Random Variables

- Only measurements at **discrete** points are possible

**EXAMPLE 3-20**
Semiconductor Wafer Contamination

The analysis of the surface of a semiconductor wafer records the number of particles of contamination that exceed a certain size. Define the random variable $X$ to equal the number of particles of contamination.

The possible values of $X$ are integers from 0 up to some large value that represents the maximum number of these particles that can be found on one of the wafers. If this maximum number is very large, it might be convenient to assume that any integer from zero to $\infty$ is possible.

# 3-7 Discrete Random Variables

## 3-7.1 Probability Mass Function

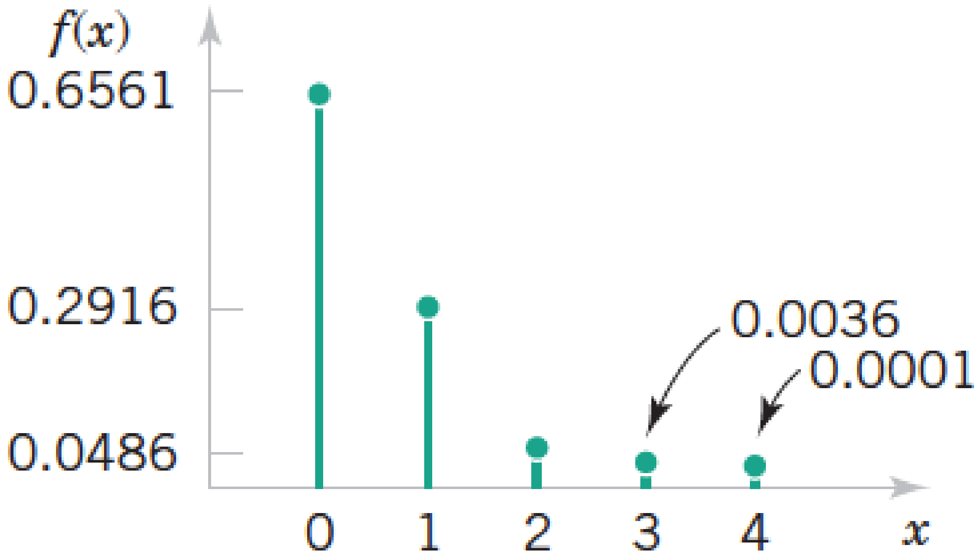



Figure 3-29 Probability distribution for $X$ in Example 3-21.

# 3-7 Discrete Random Variables

## 3-7.1 Probability Mass Function

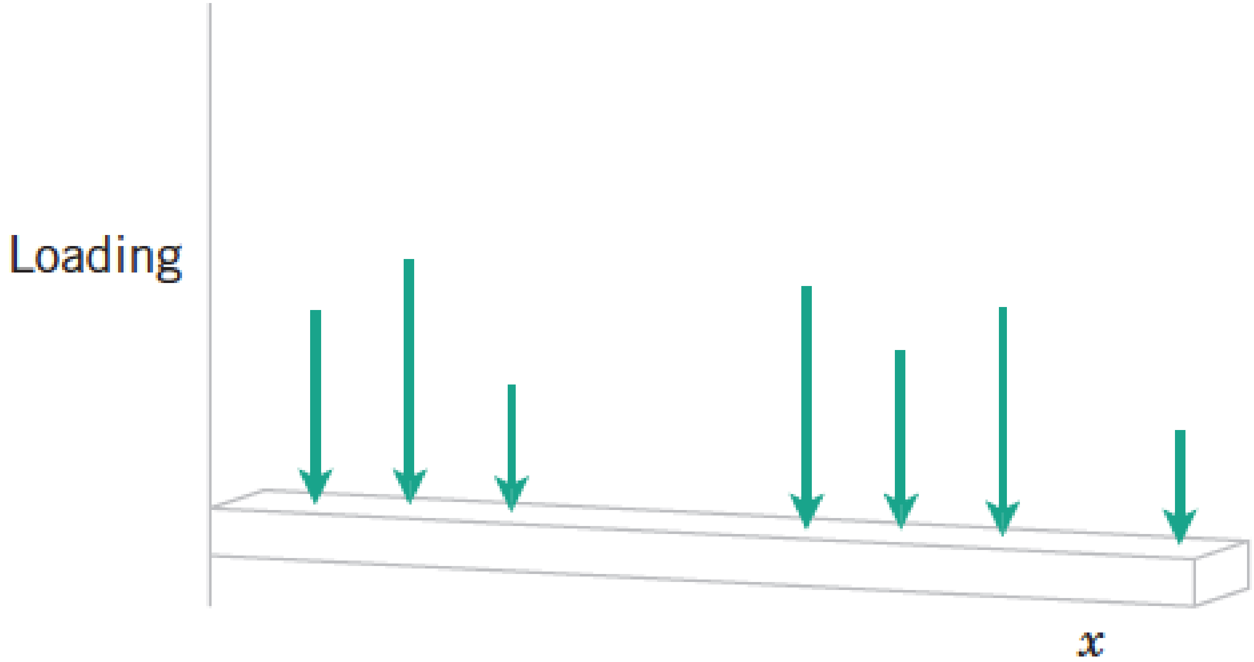



Figure 3-30 Loadings at discrete points on a long, thin beam.

# 3-7 Discrete Random Variables

## 3-7.1 Probability Mass Function

For a discrete random variable $X$ with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** (or pmf) is

$$f(x_i) = P(X = x_i) \tag{3-13}$$

# 3-7 Discrete Random Variables

## 3-7.2 Cumulative Distribution Function

The **cumulative distribution function** of a discrete random variable $X$ is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

# 3-7 Discrete Random Variables

## 3-7.2 Cumulative Distribution Function

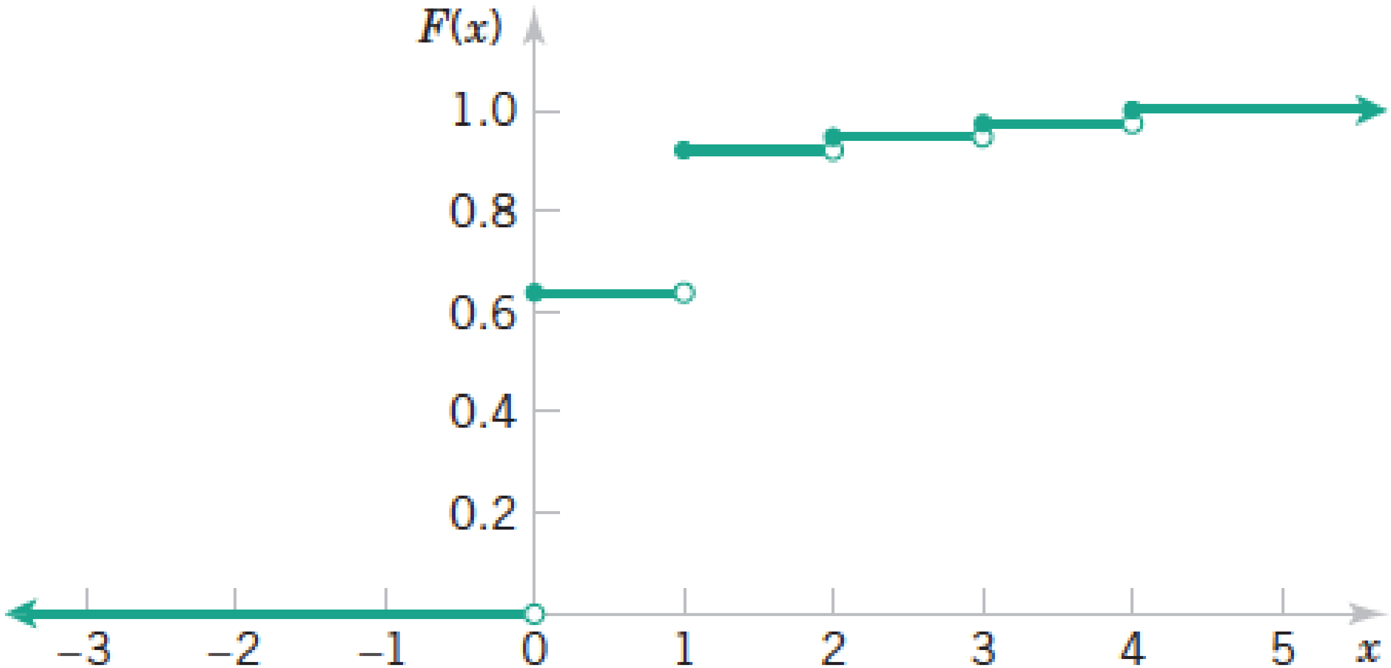



Figure 3-31 Cumulative distribution function for $x$ in Example 3-22.

# 3-7 Discrete Random Variables

## 3-7.3 Mean and Variance

Let the possible values of the random variable $X$ be denoted as $x_1, x_2, \ldots, x_n$. The pmf of $X$ is $f(x)$, so $f(x_i) = P(X = x_i)$.

The **mean** or **expected value** of the discrete random variable $X$, denoted as $\mu$ or $E(X)$, is

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \tag{3-14}$$

The **variance** of $X$, denoted as $\sigma^2$ or $V(X)$, is

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$$

The **standard deviation** of $X$ is $\sigma$.

# 3-7 Discrete Random Variables

## 3-7.3 Mean and Variance

EXAMPLE 3-24

Product Revenue

Two new product designs are to be compared on the basis of revenue potential. Marketing feels that the revenue from design A can be predicted quite accurately to be \$3 million. The revenue potential of design B is more difficult to assess. Marketing concludes that there is a probability of 0.3 that the revenue from design B will be \$7 million, but there is a 0.7 probability that the revenue will be only \$2 million. Which design would you choose?

**Solution.** Let $X$ denote the revenue from design A. Because there is no uncertainty in the revenue from design A, we can model the distribution of the random variable $X$ as \$3 million with probability one. Therefore, $E(X) = \$3$ million.

# 3-7 Discrete Random Variables

## 3-7.3 Mean and Variance

EXAMPLE 3-21

Let $Y$ denote the revenue from design B. The expected value of $Y$ in millions of dollars is

$$E(Y) = \$7(0.3) + \$2(0.7) = \$3.5$$

Because $E(Y)$ exceeds $E(X)$, we might choose design B. However, the variability of the result from design B is larger. That is,

$$\sigma^2 = (7 - 3.5)^2(0.3) + (2 - 3.5)^2(0.7) = 5.25 \text{ millions of dollars squared}$$

and.

$$\sigma = \sqrt{5.25} = 2.29 \text{ millions of dollars}$$

# 3-8 Binomial Distribution

- A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli trial**.

- It is usually assumed that the trials that constitute the random experiment are **independent**. This implies that the outcome from one trial has no effect on the outcome to be obtained from any other trial.

- Furthermore, it is often reasonable to assume that the **probability of a success on each trial is constant**.

# 3-8 Binomial Distribution

- Consider the following random experiments and random variables.
  - Flip a coin 10 times. Let $X$ = the number of heads obtained.
  - Of all bits transmitted through a digital transmission channel, 10% are received in error. Let $X$ = the number of bits in error in the next 4 bits transmitted.

Do they meet the following criteria:

1. Does the experiment consist of **Bernoulli trials**?
2. **Are the trials that constitute the random experiment are independent?**
3. **Is probability of a success on each trial is constant?**

# 3-8 Binomial Distribution

A random experiment consisting of $n$ repeated trials such that

1. the trials are independent,
2. each trial results in only two possible outcomes, labeled as *success* and *failure*, and
3. the probability of a success on each trial, denoted as $p$, remains constant

is called a *binomial experiment*.

The random variable $X$ that equals the number of trials that result in a *success* has a **binomial distribution** with parameters $p$ and $n$ where $0 \leq p \leq 1$ and $n = \{1, 2, 3, \ldots\}$. The pmf of $X$ is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \tag{3-15}$$

# 3-8 Binomial Distribution

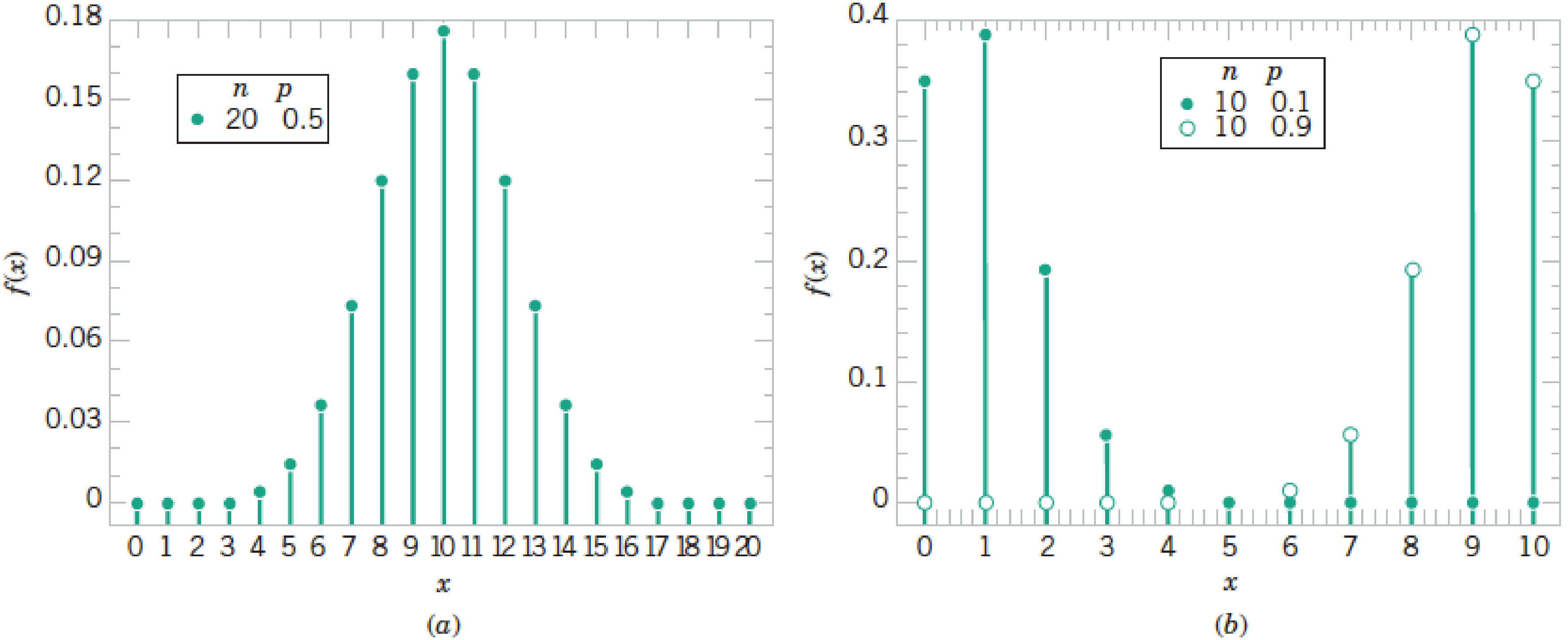



Figure 3-32 Binomial distribution for selected values of $n$ and $p$.

# 3-8 Binomial Distribution

If $X$ is a binomial random variable with parameters $p$ and $n$,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1 - p)$$

**EXAMPLE 3-28**

**Bit Transmission Errors: Binomial Mean and Variance**

For the number of transmitted bits received in error in Example 3-21, $n = 4$ and $p = 0.1$ so

$$E(X) = 4(0.1) = 0.4$$

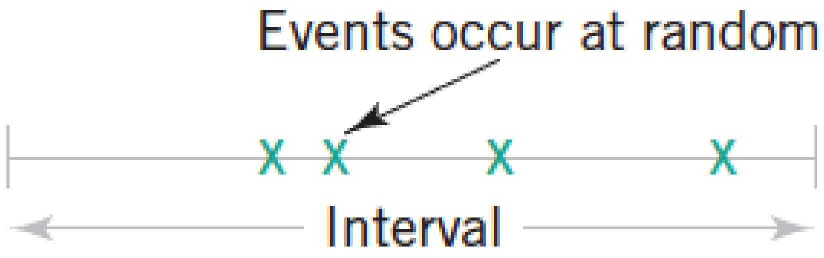
The variance of the number of defective bits is

$$V(X) = 4(0.1)(0.9) = 0.36$$

These results match those that were calculated directly from the probabilities in Example 3-23.

# 3-9 Poisson Process

Figure 3-33 In a Poission process, events occur at random in an interval.



# 3-9 Poisson Process

## 3-9.1 Poisson Distribution

**EXAMPLE 3-29**

**Limit of Bit Errors**

Consider the transmission of $n$ bits over a digital communication channel. Let the random variable $X$ equal the number of bits in error. When the probability that a bit is in error is constant and the transmissions are independent, $X$ has a binomial distribution. Let $p$ denote the probability that a bit is in error. Then $E(X) = pn$. Now suppose that the number of bits transmitted increases and the probability of an error decreases exactly enough that $pn$ remains equal to a constant—say, $\lambda$. That is, $n$ increases and $p$ decreases accordingly, such that $E(X)$ remains constant. Then

$\lambda = np$

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Also, because the number of bits transmitted tends to infinity, the number of errors can equal any nonnegative integer. Therefore, the possible values for $X$ are the integers from zero to infinity.

# 3-9 Poisson Process

## 3-9.1 Poisson Distribution

In general, consider an interval $T$ of real numbers partitioned into subintervals of small length $\Delta t$ and assume that as $\Delta t$ tends to zero,

(1) the probability of more than one event in a subinterval tends to zero,

(2) the probability of one event in a subinterval tends to $\lambda \Delta t / T$,

(3) the event in each subinterval is independent of other subintervals.

A random experiment with these properties is called a **Poisson process.**

# 3-9 Poisson Process

## 3-9.1 Poisson Distribution

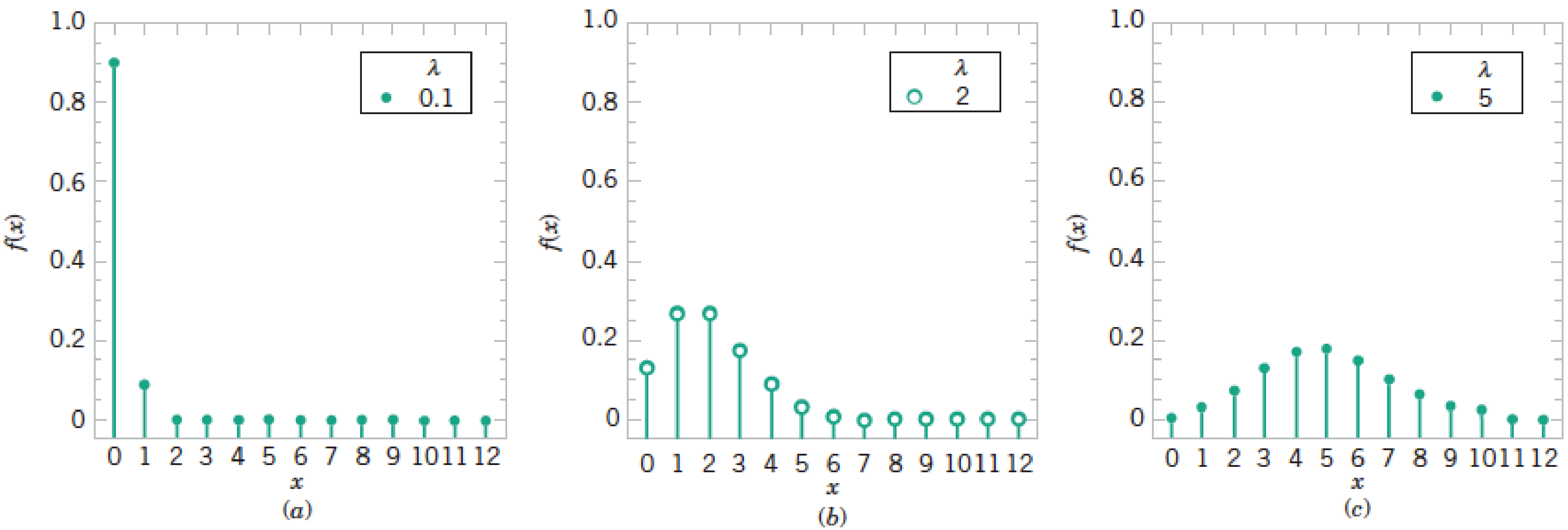



Figure 3-34 Poisson distribution for selected values of the parameter λ.

# 3-9 Poisson Process

## 3-9.1 Poisson Distribution

**EXAMPLE 3-32**
**Contamination on Optical Disks**

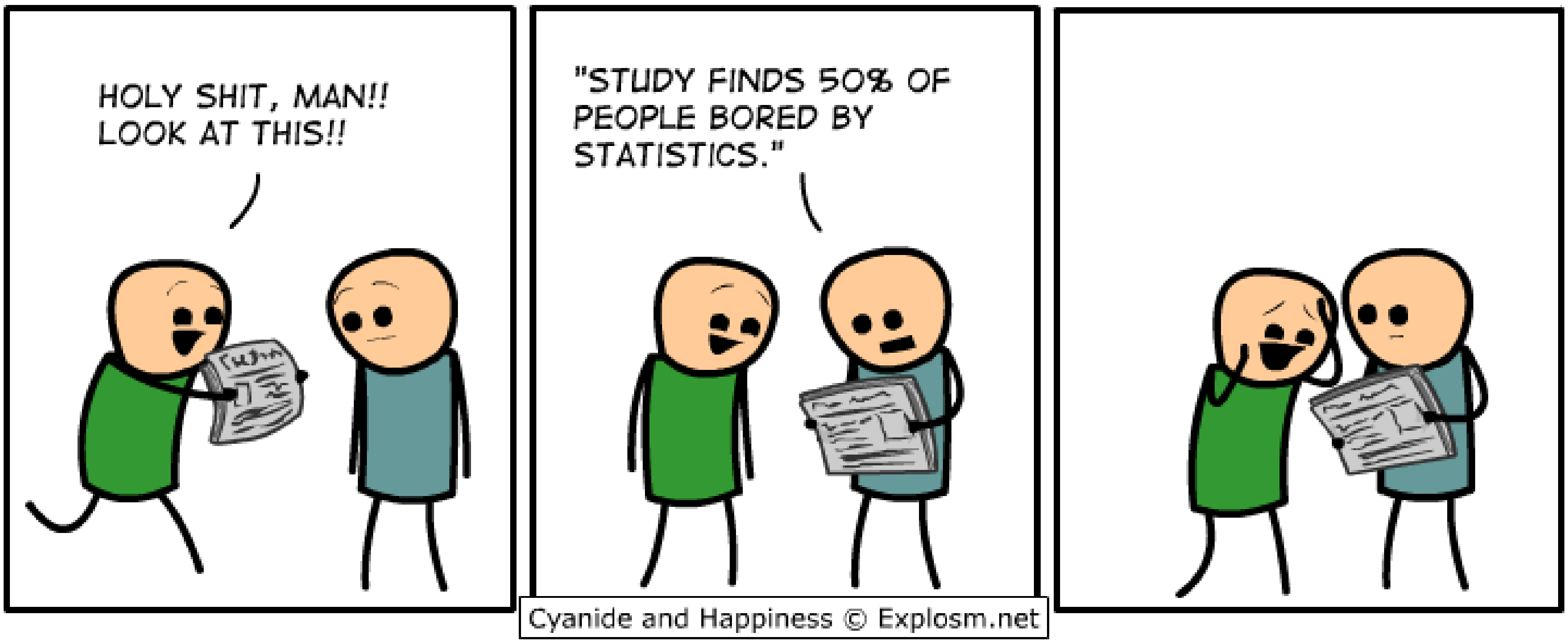
Contamination is a problem in the manufacture of optical storage disks. The number of particles of contamination that occur on an optical disk has a Poisson distribution, and the average number of particles per centimeter squared of media surface is 0.1. The area of a disk under study is 100 squared centimeters. Determine the probability that 12 particles occur in the area of a disk under study.

**Solution.** Let $X$ denote the number of particles in the area of a disk under study. Because the mean number of particles is 0.1 particles per cm$^2$,

$$\begin{aligned} E(X) &= 100 \text{ cm}^2 \times 0.1 \text{ particles/cm}^2 \\ &= 10 \text{ particles} \end{aligned}$$

Therefore,

$$P(X = 12) = \frac{e^{-10}10^{12}}{12!} = 0.095$$


HOLY SHIT, MAN!! LOOK AT THIS!!
"STUDY FINDS 50% OF PEOPLE BORED BY STATISTICS."
Cyanide and Happiness © Explosm.net

# 3-9 Poisson Process

## 3-9.1 Poisson Distribution

EXAMPLE 3-32

Determine the probability that zero particles occur in the area of the disk under study.

**Solution.** Now, $P(X = 0) = e^{-10} = 4.54 \times 10^{-5}$.

Determine the probability that 12 or fewer particles occur in the area of a disk under study.

**Solution.** This probability is

$$
\begin{aligned}
P(X \le 12) &= P(X = 0) + P(X = 1) + \cdots + P(X = 12) \\
&= \sum_{i=0}^{12} \frac{e^{-10} 10^i}{i!}
\end{aligned}
$$

Because this sum is tedious to compute, many computer programs calculate cumulative Poisson probabilities. From Minitab, we obtain $P(X \le 12) = 0.7916$.

# 3-9 Poisson Process

## 3-9.2 Exponential Distribution

- The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire. The distance between flaws is another random variable that is often of interest.
- Let the random variable $X$ denote the *length* from any starting point on the wire until a flaw is detected.
- As you might expect, the distribution of $X$ can be obtained from knowledge of the distribution of the number of flaws. The key to the relationship is the following concept:

  The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters—simple, but sufficient for an analysis of the distribution of $X$.

# 3-9 Poisson Process

## 3-9.2 Exponential Distribution

The random variable $X$ that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ has an **exponential distribution** with parameter $\lambda$. The pdf of $X$ is

$$f(x) = \lambda e^{-\lambda x}, \qquad \text{for } 0 \leq x < \infty \tag{3-19}$$

The mean and variance of $X$ are

$$E(X) = \frac{1}{\lambda} \quad \text{and} \quad V(X) = \frac{1}{\lambda^2} \tag{3-20}$$

# 3-9 Poisson Process

## 3-9.2 Exponential Distribution

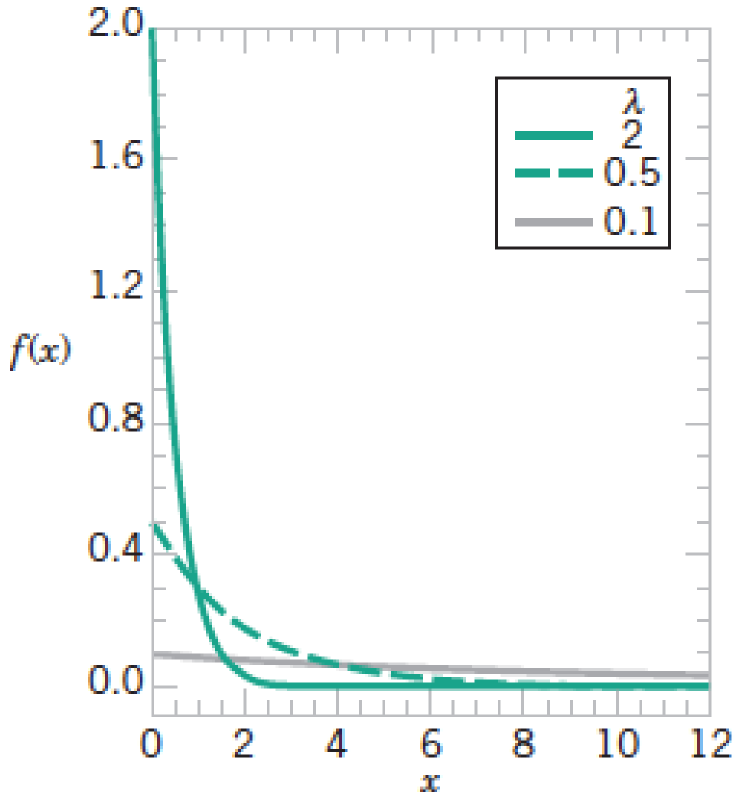



Figure 3-35 Probability density function of an exponential random variable for selected values of λ.

# 3-9 Poisson Process

## 3-9.2 Exponential Distribution

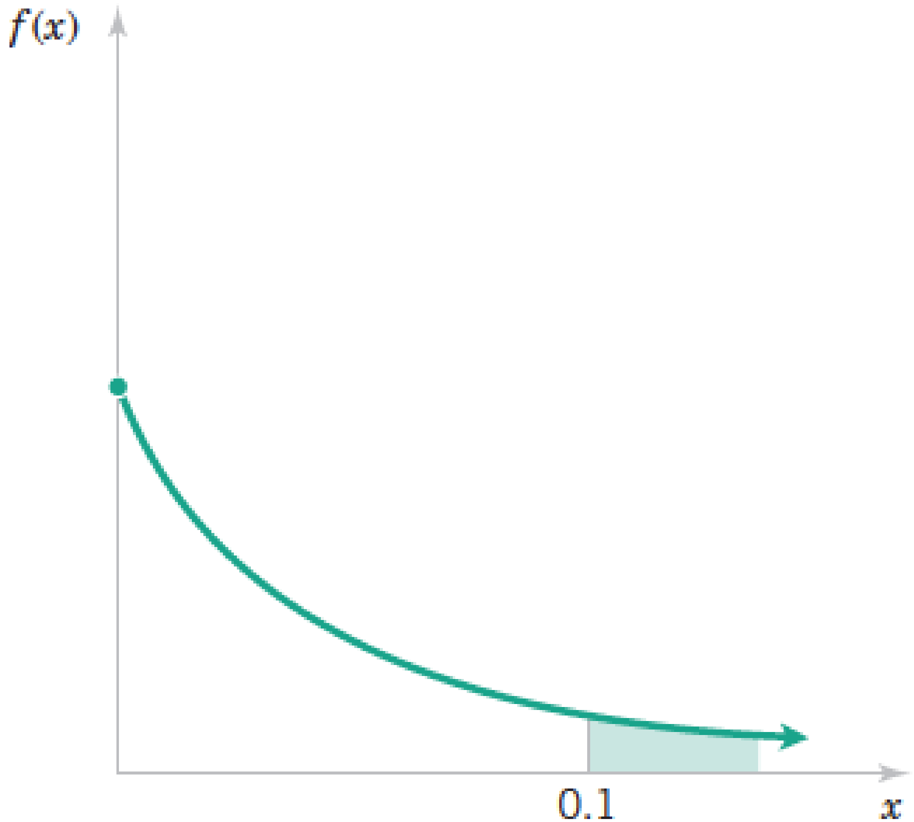



Figure 3-36 Probability for the exponential distribution in Example 3-33.

# 3-9 Poisson Process

## 3-9.2 Exponential Distribution

• The exponential distribution is often used in reliability studies as the model for the time until failure of a device.

• For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours.

•The **lack of memory property** of the exponential distribution implies that the ***device does not wear out***. The lifetime of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable.

• However, the lifetime of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution that does not lack memory.

# 3-10 Normal Approximation to the Binomial and Poisson Distributions

## Normal Approximation to the Binomial

If $X$ is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{3-21}$$

is approximately a standard normal random variable. Consequently, probabilities computed from $Z$ can be used to approximate probabilities for $X$.

# 3-10 Normal Approximation to the Binomial and Poisson Distributions

## Normal Approximation to the Binomial

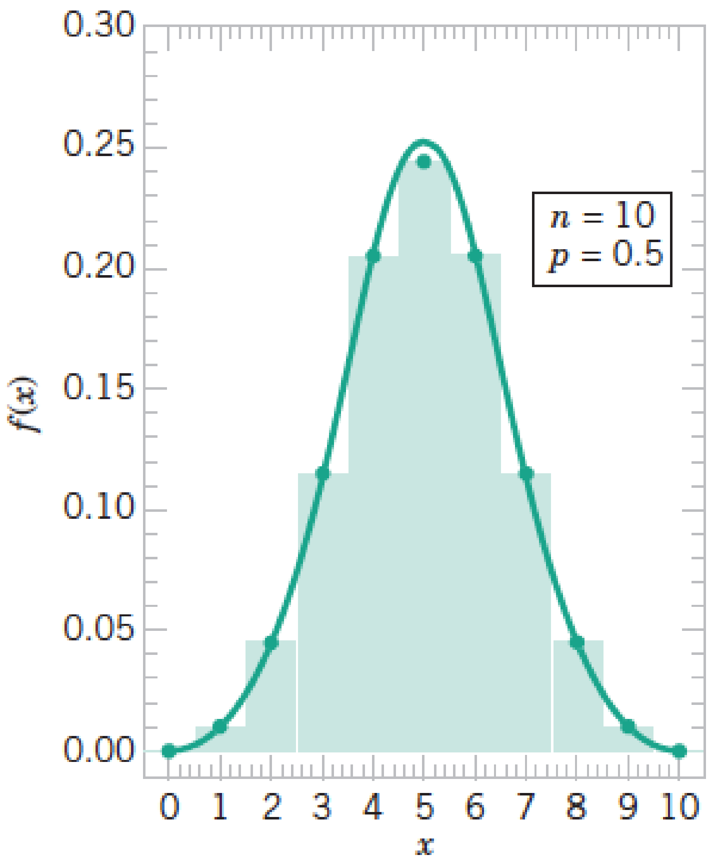



Figure 3-37 Normal approximation to the binomial distribution.

# 3-10 Normal Approximation to the Binomial and Poisson Distributions

## Normal Approximation to the Binomial

**EXAMPLE 3-32**

Again consider the transmission of bits in the previous example. To judge how well the normal approximation works, assume that only $n = 50$ bits are to be transmitted and that the probability of an error is $p = 0.1$. The exact probability that 2 or fewer errors occur is

$$P(X \leq 2) = \binom{50}{0} 0.9^{50} + \binom{50}{1} 0.1(0.9^{49}) + \binom{50}{2} 0.1^2(0.9^{48}) = 0.11$$

Based on the normal approximation,

$$P(X \leq 2) = P\left(\frac{X - 5}{\sqrt{50(0.1)(0.9)}} < \frac{2.5 - 5}{\sqrt{50(0.1)(0.9)}}\right) \cong P(Z < -1.18) = 0.12$$

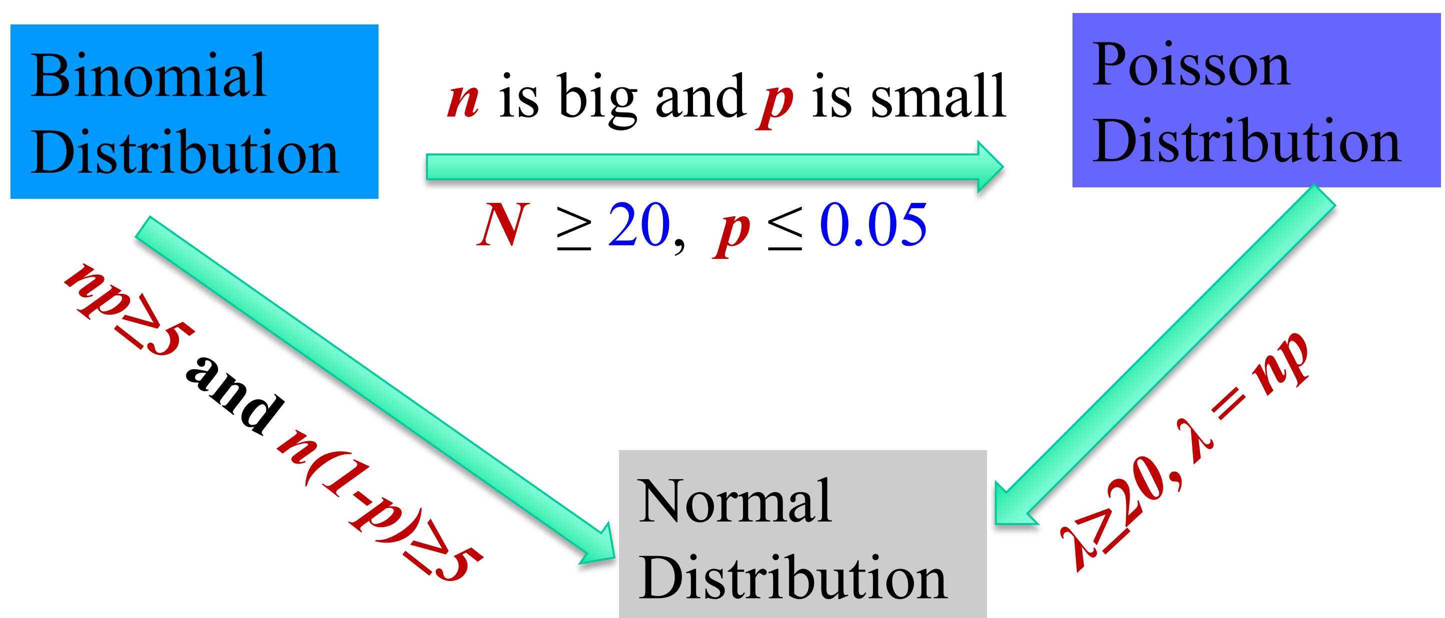
# 3-10 Normal Approximation to the Binomial and Poisson Distributions

## Normal Approximation to the Poisson

If $X$ is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \qquad (3\text{-}22)$$

is approximately a standard normal random variable.



# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

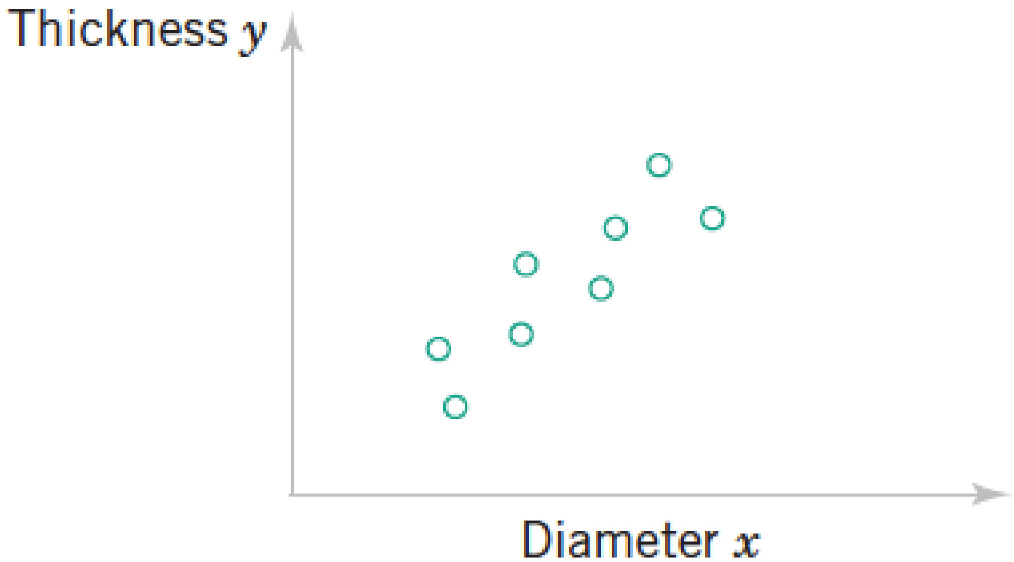



Figure 3-39 Scatter diagram of diameter and thickness measurements.

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

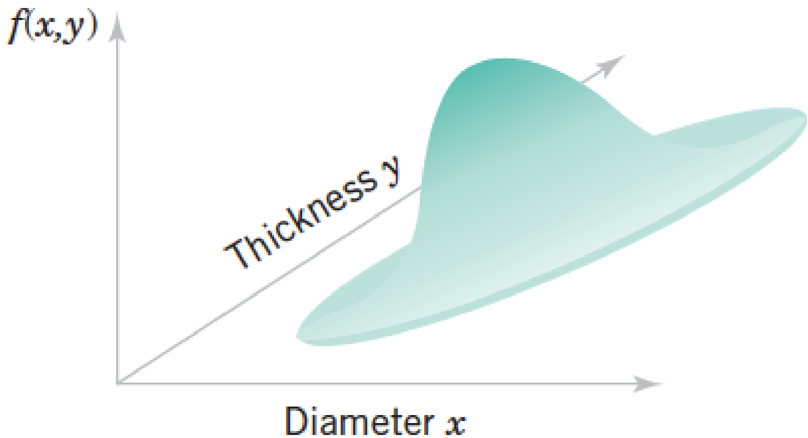



Figure 3-40 Joint probability density function of $x$ and $y$.

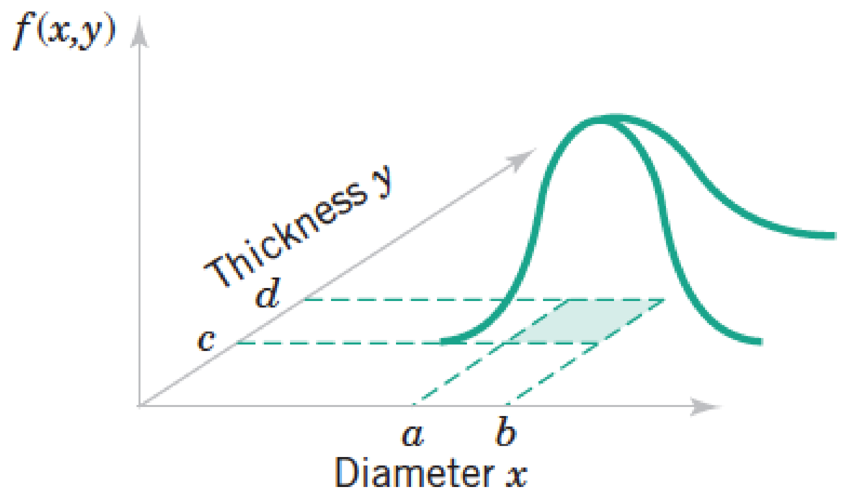



Figure 3-41 Probability of a region is the volume enclosed by $f(x, y)$ over the region.

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

$$P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Theorem If X and Y have a continuous joint distribution with joint p.d.f f then the marginal p.d.f. f_1 of X is

$$\mathrm{f}_1(\mathrm{x}) = \int_{-\infty}^{\infty} \mathrm{f}(\mathrm{x}, \mathrm{y})\mathrm{dy} \text{ for } -\infty < \mathrm{x} < \infty$$

Similarly,the marginal p.d.f.f_2 of Y is

$$\mathrm{f}_2(\mathrm{y}) = \int_{-\infty}^{\infty} \mathrm{f}(\mathrm{x}, \mathrm{y})\mathrm{dx} \text{ for } -\infty < \mathrm{y} < \infty$$

Marginal : 边际，边缘

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

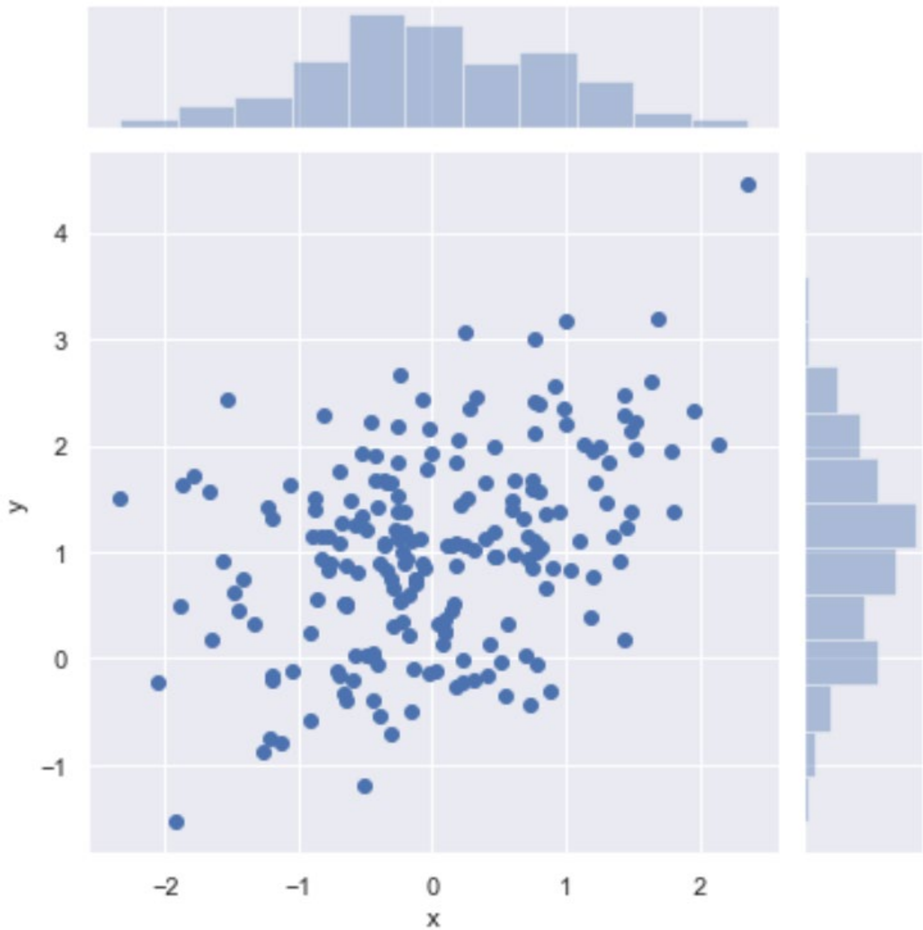
$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)dydx$$

$$F(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(x, y)dydx$$

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

```
sns.jointplot(x="x", y="y", data=df);
```



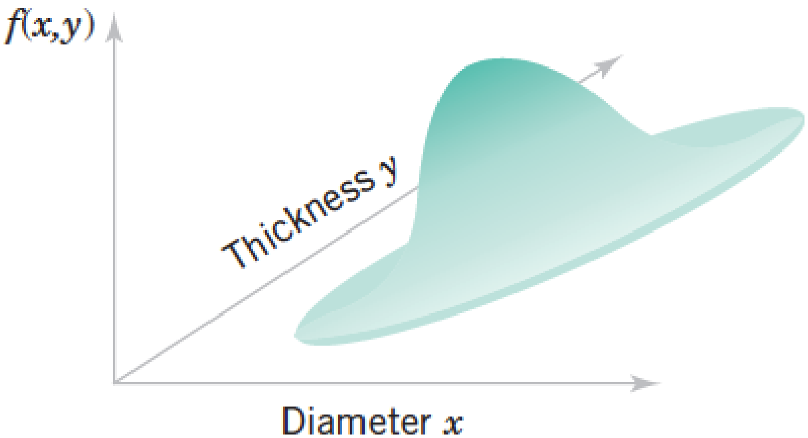



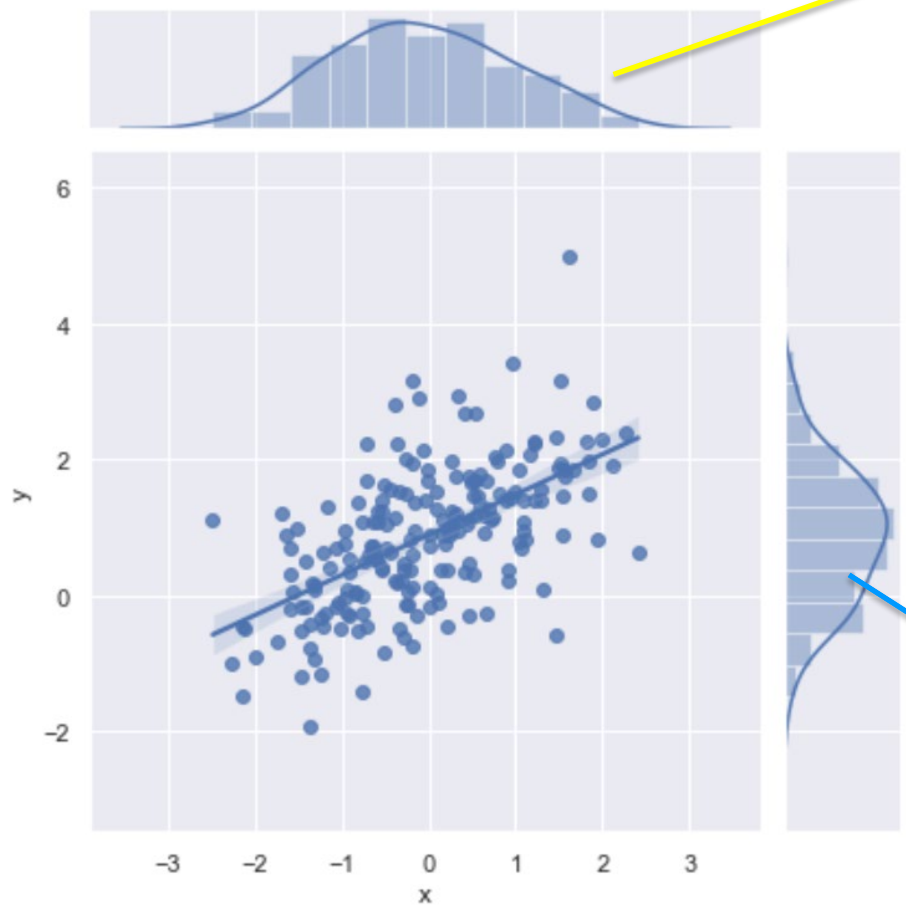

Figure 3-40 Joint probability density function of $x$ and $y$.

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

```
sns.jointplot(x="x", y="y", data=df, kind='reg')
```

Marginal distribution of $x$



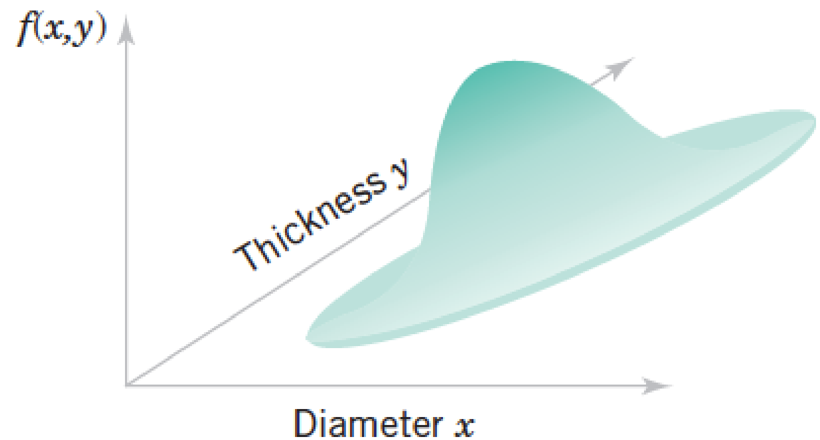



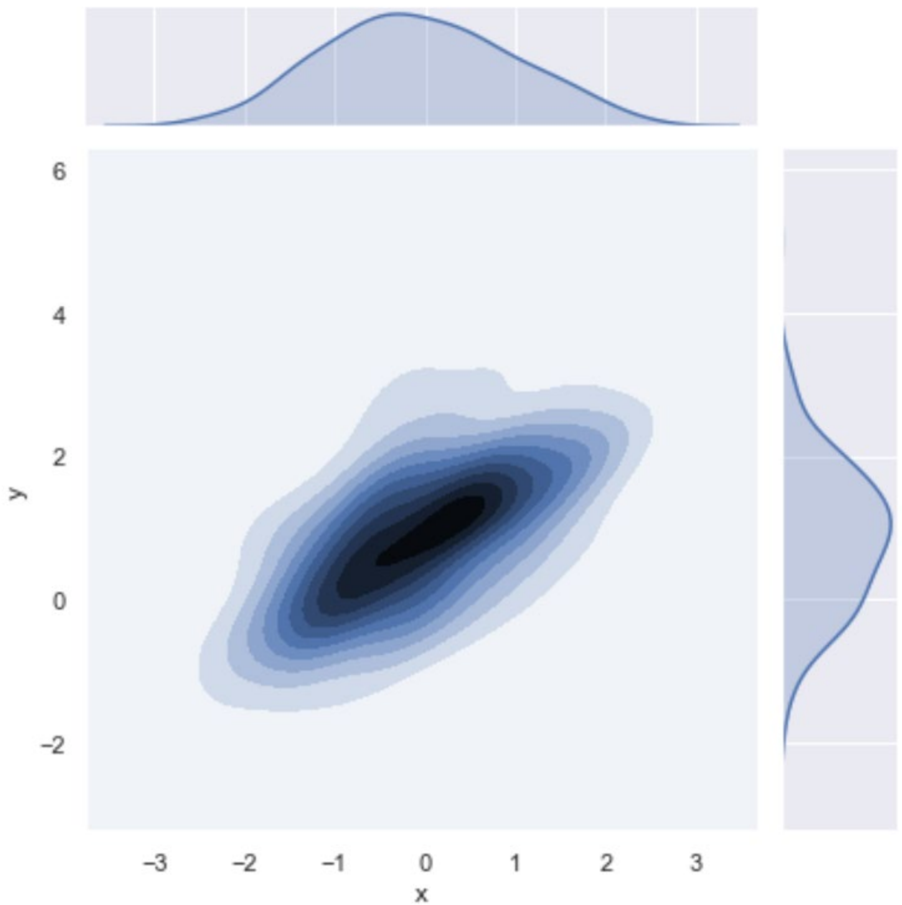

Figure 3-40 Joint probability density function of $x$ and $y$.

Marginal distribution of $y$

# 3-11 More Than One Random Variable and Independence

## 3-11.1 Joint Distributions

```
sns.jointplot(x="x", y="y", data=df, kind='kde')
```



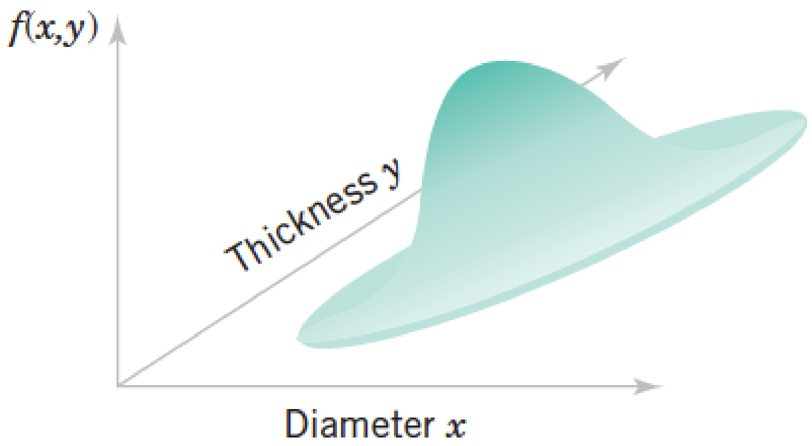




Figure 3-40 Joint probability density function of $x$ and $y$.

# 3-11 More Than One Random Variable and Independence

## 3-11.2 Independence

The random variables $X_1, X_2, \ldots, X_n$ are **independent** if

$$P(X_1 \in E_1, X_2 \in E_2, \ldots, X_n \in E_n) = P(X_1 \in E_1)P(X_2 \in E_2) \cdots P(X_n \in E_n)$$

for *any* sets $E_1, E_2, \ldots, E_n$.

# 3-11 More Than One Random Variable and Independence

## 3-11.2 Independence

**EXAMPLE 3-37**
Optical Drive Diameters

In Example 3-13, the probability that a diameter meets specifications was determined to be 0.919. What is the probability that 10 diameters all meet specifications, assuming that the diameters are independent?

**Solution.** Denote the diameter of the first shaft as $X_1$, the diameter of the second shaft as $X_2$, and so forth, so that the diameter of the tenth shaft is denoted as $X_{10}$. The probability that all shafts meet specifications can be written as

$$P(0.2485 < X_1 < 0.2515, 0.2485 < X_2 < 0.2515, \ldots, 0.2485 < X_{10} < 0.2515)$$

In this example, the only set of interest is

$$E_1 = (0.2485, 0.2515)$$

With respect to the notation used in the definition of independence,

$$E_1 = E_2 = \cdots = E_{10}$$

# 3-11 More Than One Random Variable and Independence

## 3-11.2 Independence

EXAMPLE 3-37

Recall the relative frequency interpretation of probability. The proportion of times that shaft 1 is expected to meet the specifications is 0.919, the proportion of times that shaft 2 is expected to meet the specifications is 0.919, and so forth. If the random variables are independent, the proportion of times in which we measure 10 shafts that we expect all to meet the specifications is

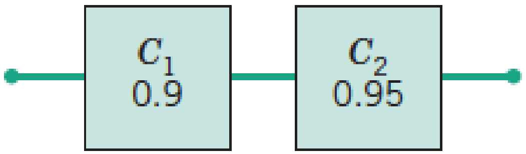
$$P(0.2485 < X_1 < 0.2515, 0.2485 < X_2 < 0.2515, \ldots, 0.2485 < X_{10} < 0.2515)$$
$$= P(0.2485 < X_1 < 0.2515) \times P(0.2485 < X_2 < 0.2515) \times \cdots \times P(0.2485 < X_{10} < 0.2515) = 0.919^{10} = 0.430$$

# 3-11 More Than One Random Variable and Independence

## 3-11.2 Independence

**EXAMPLE 3-39**
Series System

The system shown here operates only if there is a path of functional components from left to right. The probability that each component functions is shown in the diagram. Assume that the components function or fail independently. What is the probability that the system operates?




**Solution.** Let $C_1$ and $C_2$ denote the events that components 1 and 2 are functional, respectively. For the system to operate, both components must be functional. The probaility that the system operates is

$$P(C_1, C_2) = P(C_1)P(C_2) = (0.9)(0.95) = 0.855$$

Note that the probability that the system operates is smaller than the probability that any component operates. This system fails whenever *any* component fails. A system of this type is called a **series system.**

# 3-12 Functions of Random Variables

$$Y = X + c$$

$$E(Y) = E(X) + c = \mu + c \tag{3-23}$$

$$V(Y) = V(X) + 0 = \sigma^2 \tag{3-24}$$

$$Y = cX$$

$$E(Y) = E(cX) = cE(X) = c\mu \tag{3-25}$$

$$V(Y) = V(cX) = c^2V(X) = c^2\sigma^2 \tag{3-26}$$

# 3-12 Functions of Random Variables

## 3-12.1 Linear Combinations of Independent Random Variables

The mean and variance of the linear function of **independent** random variables are

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \tag{3-27}$$

and

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 \tag{3-28}$$

# 3-12 Functions of Random Variables

## 3-12.1 Linear Combinations of Independent Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent, normally distributed random variables with means $E(X_i) = \mu_i$, $i = 1, 2, \ldots, n$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$. Then the linear function

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

is normally distributed with mean

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$$

and variance

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2$$

# 3-12 Functions of Random Variables

## 3-12.1 Linear Combinations of Independent Random Variables

EXAMPLE 3-43

Perimeter of a Molded Part: Normal Distribution 周长

Once again, consider the manufactured part described previously. Now suppose that the length $X_1$ and the width $X_2$ are normally and independently distributed with $\mu_1 = 2$ centimeters, $\sigma_1 = 0.1$ centimeter, $\mu_5 =$ 5 centimeters, and $\sigma_2 = 0.2$ centimeter. In the previous example we determined that the mean and variance of the perimeter of the part $Y = 2X_1 + 2X_2$ were $E(Y) =$ centimeters and $V(Y) =$ square centimeter, respectively. Determine the probability that the perimeter of the part exceeds 14.5 centimeters.

**Solution.** From the above result, $Y$ is also a normally distributed random variable, so we may calculate the desired probability as follows:

$$P(Y > 14.5) = \qquad = \qquad =$$

Therefore, the probability is 0.13 that the perimeter of the part exceeds 14.5 centimeters.

## 3-12.2 What If the Random Variables Are Not Independent?

The correlation between two random variables $X_1$ and $X_2$ is

$$\rho_{X_1X_2} = \frac{E(X_1X_2) - \mu_1\mu_2}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\sigma_1^2\sigma_2^2}} \tag{3-29}$$

with $-1 \le \rho_{X_1X_2} \le +1$, and $\rho_{X_1X_2}$ is usually called the **correlation coefficient.**

$$r = \frac{\sum_{i=1}^{n}\left(\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)\right)}{\sqrt{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}}$$

# 3-12 Functions of Random Variables

## 3-12.2 What If the Random Variables Are Not Independent?

Let $X_1, X_2, \ldots, X_n$ be random variables with means $E(X_i) = \mu_i$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$, and covariances $\text{Cov}(X_1, X_2)$, $i, j = 1, 2, \ldots, n$ with $i < j$. Then the mean of the linear combination

$$Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

is

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \tag{3-30}$$

and the variance is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 + 2\sum_{i<j}\sum c_i c_j \text{Cov}(X_i, X_j) \tag{3-31}$$

# 3-12 Functions of Random Variables

## 3-12.3 What If the Function Is Nonlinear?

X is a random variable and Y=$h$(X), where $h$ is a continuous function in R.

If $X$ has mean $\mu_X$ and variance $\sigma_X^2$, the approximate mean and variance of $Y$ can be computed using the following result:

$$E(Y) = \mu_Y \simeq h(\mu_X) \tag{3-32}$$

$$V(Y) = \sigma_Y^2 \simeq \left(\frac{dh}{dX}\right)^2 \sigma_X^2 \tag{3-33}$$

where the derivative $dh/dX$ is evaluated at $\mu_X$.

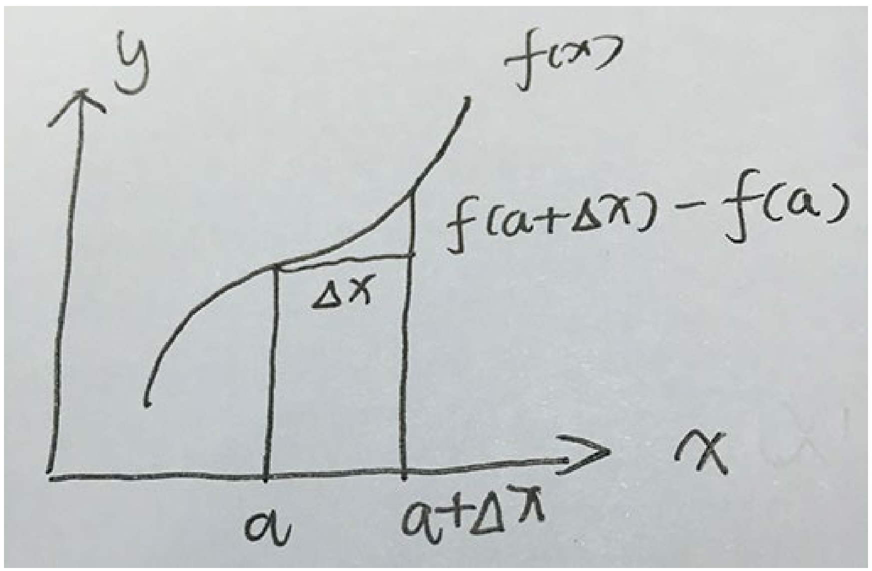
Engineers usually call equation 3-33 the **transmission of error** or **propagation of error formula.**

误差传播

## Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$




$$\frac{f(a+\Delta x) - f(a)}{\Delta x} = \tan\alpha$$

$$f'(a) = \tan\alpha$$

$$f(a+\Delta x) = f(a) + f'(a)\Delta x$$

# 3-12 Functions of Random Variables

## 3-12.3 What If the Function Is Nonlinear?

Equations 3-32 and 3-33 are developed by approximating the nonlinear function $h$ with a linear function. The linear approximation is found by using a first-order Taylor series. Assuming that $h(X)$ is differentiable, the first-order Taylor series approximation for $Y = h(X)$ around the point $\mu_X$ is

$$Y \simeq h(\mu_X) + \frac{dh}{dX}(X - \mu_X) \qquad (3\text{-}34)$$

Now $dh/dX$ is a constant when it is evaluated at $\mu_X$, $h(\mu_X)$ is a constant, and $E(X) = \mu_X$, so when we take the expected value of $Y$, the second term in equation 3-34 is zero and consequently

$$E(Y) \simeq h(\mu_X)$$

The approximate variance of $Y$ is

$$V(Y) \simeq V[h(\mu_x)] + V\left[\frac{dh}{dX}(X - \mu_X)\right] = \left(\frac{dh}{dX}\right)^2 \sigma_X^2$$

**delta method**

# 3-12 Functions of Random Variables

## 3-12.3 What If the Function Is Nonlinear?

**EXAMPLE 3-44**

Power in a Circuit

The power $P$ dissipated by the resistance $R$ in an electrical circuit is given by $P = I^2R$ where $I$, the current, is a random variable with mean $\mu_I = 20$ amperes and standard deviation $\sigma_I = 0.1$ amperes. The resistance $R = 80$ ohms is a constant. We want to find the approximate mean and standard deviation of the power. In this problem the function $h = I^2R$, so taking the derivative $dh/dI = 2IR = 2I(80)$ and applying the equations 3-32 and 3-33, we find that the approximate mean power is

$$E(P) = \mu_P \simeq h(\mu_I) = \mu_I^2 R = 20^2(80) = 32{,}000 \text{ watts}$$

and the approximate variance of power is

$$V(P) = \sigma_P^2 \simeq \left(\frac{dh}{dI}\right)^2 \sigma_I^2 = [2(20)(80)]^2 0.1^2 = 102{,}400 \text{ square watts}$$

So the standard deviation of the power is $\sigma_p \simeq 320$ watts. Remember that the derivative $dh/dI$ is evaluated at $\mu_I = 20$ amperes. ■

# 3-12 Functions of Random Variables

## 3-12.3 What If the Function Is Nonlinear?

Sometimes the variable $Y$ is a nonlinear function of several random variables, say,

$$Y = h(X_1, X_2, \ldots, X_n) \tag{3-35}$$

where $X_1, X_2, \ldots, X_n$ are assumed to be independent random variables with means $E(X_i) = \mu_i$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$. The delta method can be used to find approximate expressions for the mean and variance of $Y$. The first-order Taylor series expansion of equation 3-35 is

$$\begin{aligned} Y &\simeq h(\mu_1, \mu_2, \ldots, \mu_n) + \frac{\partial h}{\partial X_1}(X_1 - \mu_1) + \frac{\partial h}{\partial X_2}(X_2 - \mu_2) + \cdots + \frac{\partial h}{\partial X_n}(X_n - \mu_n) \\ &= h(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} \frac{\partial h}{\partial X_i}(X_i - \mu_i) \end{aligned} \tag{3-36}$$

# 3-12 Functions of Random Variables

## 3-12.3 What If the Function Is Nonlinear?

Let

$$Y = h(X_1, X_2, \ldots, X_n)$$

for independent random variables $X_i$, $i = 1,2, \ldots, n$, each with mean $\mu_i$ and variance $\sigma_i^2$, the approximate mean and variance of $Y$ are

$$E(Y) = \mu_Y \simeq h(\mu_1, \mu_2, \ldots, \mu_n) \tag{3-37}$$

$$V(Y) = \sigma_Y^2 \simeq \sum_{i=1}^{n} \left(\frac{\partial h}{\partial X_i}\right)^2 \sigma_i^2 \tag{3-38}$$

where the partial derivatives $\partial h/\partial X_i$ are evaluated at $\mu_1, \mu_2, \ldots, \mu_n$.

# 3-13 Random Samples, Statistics, and The Central Limit Theorem

Independent random variables $X_1, X_2, \ldots, X_n$ with the same distribution are called a **random sample.**

A **statistic** is a function of the random variables in a random sample.

The probability distribution of a statistic is called its **sampling distribution.**

# 3-13 Random Samples, Statistics, and The Central Limit Theorem

The probability distribution of a statistic is called its **sampling distribution.**

Linear functions of normally and independently distributed random variables

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$E(\bar{X}) = \frac{\mu + \mu + \cdots + \mu}{n} = \mu$$

$$V(\bar{X}) = \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

# 3-13 Random Samples, Statistics, and The Central Limit Theorem

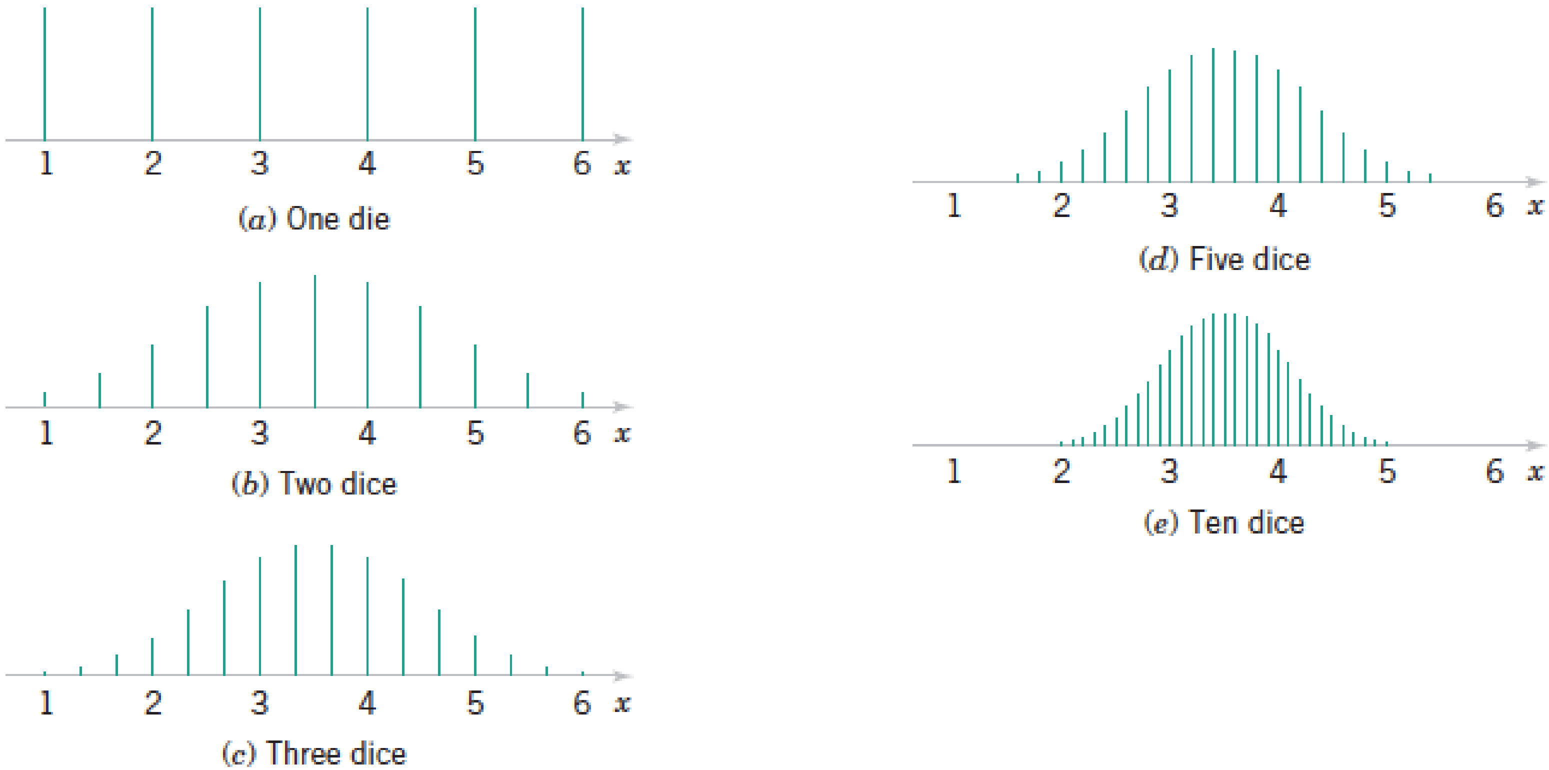
## Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size $n$ taken from a population with mean $\mu$ and variance $\sigma^2$, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \tag{3-39}$$

as $n \to \infty$, is the standard normal distribution.

# 3-13 Random Samples, Statistics, and The Central Limit Theorem




(a) One die

(b) Two dice

(c) Three dice

(d) Five dice

(e) Ten dice

投骰子的平均得分

# 3-13 Random Samples, Statistics, and The Central Limit Theorem

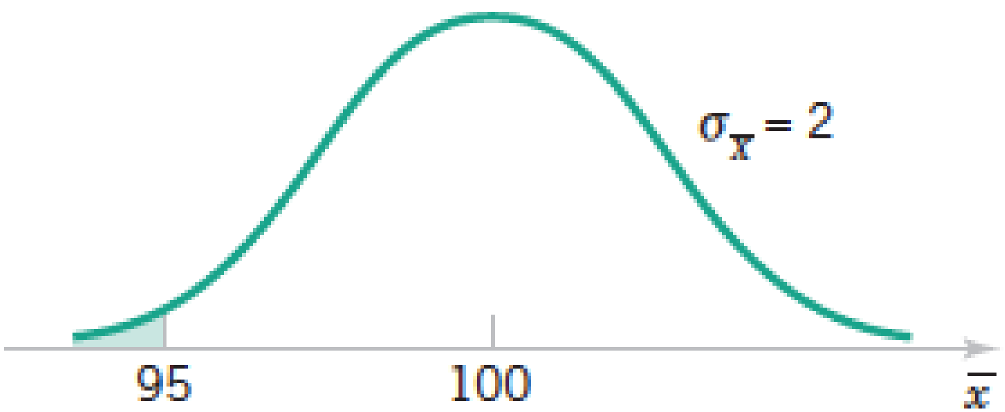



Figure 3-44 Probability density function of average resistance.

# 3-13 Random Samples, Statistics, and The Central Limit Theorem

**EXAMPLE 3-48**

**Average Resistance**

An electronics company manufactures resistors that have a mean resistance of 100 Ω and a standard deviation of 10 Ω. Find the probability that a random sample of $n = 25$ resistors will have an average resistance less than 95 Ω.

Note that the sampling distribution of $\overline{X}$ is approximately normal, with mean $\mu_{\overline{X}} = 100\ \Omega$ and a standard deviation of

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$$

Therefore, the desired probability corresponds to the shaded area in Fig. 3-44. Standardizing the point $\overline{X} = 95$ in Fig. 3-44, we find that

$$z = \frac{95 - 100}{2} = -2.5$$

and, therefore,

$$P(\overline{X} < 95) = P(Z < -2.5) = 0.0062$$

## IMPORTANT TERMS AND CONCEPTS

- Beta distribution
- Binomial distribution
- Central limit theorem
- Continuity correction
- Continuous random variable
- Cumulative distribution function
- Delta method
- Discrete random variable
- Event
- Exponential distribution
- Gamma distribution
- Independence
- Joint probability distribution
- Lognormal distribution
- Mean of a random variable
- Normal approximations to binomial and Poisson distributions
- Normal distribution
- Normal probability plot
- Poisson distribution
- Poisson process
- Probability
- Probability density function
- Probability distribution
- Probability mass function
- Probability plots
- Propagation of error
- Random experiment
- Random sample
- Random variable
- Sampling distribution
- Standard deviation of a random variable
- Standard normal distribution
- Statistic
- Variance of a random variable
- Weibull distribution